Dual Solution with Cross-Good Congestion

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1 Solution with Cross-Good Congestion

Consider the immobile labor case with cross-good congestion, using a CES aggregator across flows

$$Q_{jk} = \left(\sum_n m^n (Q_{jk}^n)^\eta\right)^{\frac{1}{\eta}} \quad \text{with } \eta > 1 \text{ to guarantee convexity,}$$

and labor the only production factor $F_j^n(L_j^n) = z_j^n(L_j^n)^a$. The Lagrangian is

$$\begin{aligned}
\mathcal{L} = \sum_j \omega_j L_j U(c_j, h_j) - \sum_j P_j^D \left[c_j L_j + \sum_{k \in N(j)} \delta_{jk}^\tau I_{jk}^{-\gamma} \left(\sum_{n=1}^N m^n (Q_{jk}^n)^\eta\right)^{\frac{1+\beta}{\eta}} - \left(\sum_n (D_j^n)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \right] \\
- \sum_j \sum_n P_j^n \left[D_j^n + \sum_{k \in N(j)} Q_{jk}^n - z_j^n (L_j^n)^a - \sum_{i \in N(j)} Q_{ij}^n \right] \\
- \sum_j W_j \left[\sum_n L_j^n - L_j \right] + \sum_{j,k,n} \zeta_{jkn}^Q Q_{jk}^n + \sum_{j,n} \zeta_{jn}^L L_j^n + \sum_{j,n} \zeta_{jn}^C D_j^n + \sum_j \zeta_j^c c_j.
\end{aligned}$$

The decision variables are c_j, D_j^n, Q_{jk}^n, and L_j^n. The FOC's are

$$[c_j] \qquad \omega_j L_j U'(c_j, h_j) + \zeta_j^c = P_j^D L_j$$

$$[D_j^n] \qquad P_j^D (D_j^n)^{\frac{-1}{\sigma}} \left(\sum_n (D_j^n)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} + \zeta_{jn}^C = P_j^n$$

$$[Q_{jk}^n] \qquad P_k^n - P_j^n + \zeta_{jkn}^Q = P_j^D (1+\beta) \delta_{jk}^\tau m^n \left(\sum_n m^n (Q_{jk}^n)^\eta\right)^{\frac{1+\beta}{\eta}-1} (Q_{jk}^n)^{\eta-1} I_{jk}^{-\gamma}$$

$$[L_j^n] \qquad a P_j^n z_j^n (L_j^n)^{a-1} + \zeta_{jn}^L = W_j$$

Complementary slackness implies $\zeta_j^c = \zeta_{jn}^C = \zeta_{jkn}^Q = \zeta_{jn}^L = 0$. The FOC for D_j^n can be simplified

$$P_j^D (D_j^n)^{\frac{-1}{\sigma}} D_j^{\frac{1}{\sigma}} = P_j^n \quad \Rightarrow \quad D_j^n = D_j \left(\frac{P_j^n}{P_j^D}\right)^{-\sigma},$$

and, using the definition of D_j as a CES aggregate of D_j^n yields

$$D_j = \left(\sum_n \left(D_j \left(\frac{P_j^n}{P_j^D}\right)^{-\sigma}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \quad \Rightarrow \quad P_j^D = \left(\sum_n (P_j^n)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{1}$$

The FOC for c_j yields

$$c_j = U'^{-1}\left(\frac{P_j^D}{\omega_j},\ h_j\right).$$

The flow constraint yields

$$Q_{jk}^n = \left[\frac{P_k^n - P_j^n}{P_j^D(1+\beta)\delta_{jk}^\tau m^n (Q_{jk})^{1+\beta-\eta} I_{jk}^{-\gamma}}\right]^{\frac{1}{\eta-1}},$$

from which we can recover an expression for Q_{jk}

$$\begin{aligned}
Q_{jk} &= \left(\sum_n m^n (Q_{jk}^n)^\eta\right)^{\frac{1}{\eta}} \\
&= \left(\sum_n m^n \left[\frac{P_k^n - P_j^n}{P_j^D(1+\beta)\delta_{jk}^\tau m^n (Q_{jk})^{1+\beta-\eta} I_{jk}^{-\gamma}}\right]^{\frac{\eta}{\eta-1}}\right)^{\frac{1}{\eta}} \\
\Rightarrow Q_{jk} &= \left(\sum_n m^n \left[\frac{P_k^n - P_j^n}{P_j^D(1+\beta)\delta_{jk}^\tau m^n I_{jk}^{-\gamma}}\right]^{\frac{\eta}{\eta-1}}\right)^{\frac{\eta-1}{\eta\beta}}.
\end{aligned}$$

The resource constraint yields that the bundle of tradeable goods (pre-transport cost), D_j, equals consumption plus transport costs

$$D_j = c_j L_j + \sum_{k\in N(j)} \delta_{jk}^\tau I_{jk}^{-\gamma} Q_{jk}^{1+\beta}.$$

Finally, the FOC for L_j^n yields

$$L_j^n = \left(\frac{W_j}{a P_j^n z_j^n}\right)^{\frac{1}{a-1}}.$$

Now $L_j = \sum_n L_j^n$, and assuming perfect substitutability of labor between sectors such that $W_j = W_k = W\ \forall j,k$,

$$\frac{L_j^n}{L_j} = \frac{\left(P_j^n z_j^n\right)^{\frac{1}{1-a}}}{\sum_{n'}\left(P_j^{n'} z_j^{n'}\right)^{\frac{1}{1-a}}} \quad \Rightarrow \quad L_j^n = \frac{\left(P_j^n z_j^n\right)^{\frac{1}{1-a}}}{\sum_{n'}\left(P_j^{n'} z_j^{n'}\right)^{\frac{1}{1-a}}} L_j.$$

Thus, all decision variables can be solved as a function of prices P_j^n.

2 Dual Solution

The dual representation of the problem involves solving

$$\inf_\lambda \sup_{\mathbf{x}} \mathcal{L}(\mathbf{x}(\lambda), \lambda),$$

i.e., minimizing the lagrangian with decision variables $\mathbf{x} = \{c_j, D_j^n, Q_{jk}^n, L_j^n\}$ expressed as a function of the lagrange multipliers $\lambda = \{P_j^n\}$. Then, because $\mathbf{x}(\lambda)$ is the optimal solution of the primal problem and its gradient is zero, the Envelope Theorem applies and the gradient of the Lagrangian is simply the negative constraint

$$\Delta_\lambda \mathcal{L}(\mathbf{x}(\lambda), \lambda) = -\left(D_j^n(\lambda) + \sum_{k\in N(j)} Q_{jk}^n(\lambda) - z_j^n (L_j^n(\lambda))^a - \sum_{i\in N(j)} Q_{ij}^n(\lambda)\right) \ \forall j, n.$$

To characterize the optimum also requires the Hessian matrix of second derivatives

$$\mathbf{H}_\lambda = \Delta_{\lambda\lambda}^2 \mathcal{L}(\mathbf{x}(\lambda), \lambda),$$

which needs to be positive (semi-)definite for a minimum.

To derive the Hessian Matrix analytically, I first obtain an explicit expression for $Q^n_{ij}(\lambda)$

$$
\begin{aligned}
Q^n_{jk} &= \left[\frac{P^n_k - P^n_j}{P^D_j(1+\beta)\delta^\tau_{jk} m^n I^{-\gamma}_{jk}}\right]^{\frac{1}{\eta-1}} \underbrace{\left(\sum_{n'} m^{n'}\left[\frac{P^{n'}_k - P^{n'}_j}{P^D_j(1+\beta)\delta^\tau_{jk} m^{n'} I^{-\gamma}_{jk}}\right]^{\frac{\eta}{\eta-1}}\right)^{\frac{\eta-\beta-1}{\eta\beta}}}_{Q^{(\eta-\beta-1)/(\eta-1)}_{jk}} \\
&= \left[\frac{P^n_k - P^n_j}{P^D_j(1+\beta)\delta^\tau_{jk} m^n I^{-\gamma}_{jk}}\right]^{\frac{1}{\eta-1}} \underbrace{[P^D_j(1+\beta)\delta^\tau_{jk} I^{-\gamma}_{jk}]}_{PK0}{}^{\frac{\eta-\beta-1}{-(\eta-1)\beta}} \underbrace{\left(\sum_{n'} m^{n'}\left[\frac{P^{n'}_k - P^{n'}_j}{m^{n'}}\right]^{\frac{\eta}{\eta-1}}\right)^{\frac{\eta-\beta-1}{\eta\beta}}}_{\left[Q_{jk} PK0^{1/\beta}\right]^{(\eta-\beta-1)/(\eta-1)}} \\
&= \underbrace{[(1+\beta)\delta^\tau_{jk} I^{-\gamma}_{jk}]^{\frac{-1}{\beta}}}_{K} \underbrace{(P^D_j)^{\frac{-1}{\beta}}}_{P} \underbrace{\left[\frac{P^n_k - P^n_j}{m^n}\right]^{\frac{1}{\eta-1}}}_{A} \underbrace{\left(\sum_{n'} m^{n'}\left[\frac{P^{n'}_k - P^{n'}_j}{m^{n'}}\right]^{\frac{\eta}{\eta-1}}\right)^{\frac{\eta-\beta-1}{\eta\beta}}}_{B=\left[Q_{jk} PK0^{1/\beta}\right]^{(\eta-\beta-1)/(\eta-1)}}.
\end{aligned}
$$

We must then consider four derivatives of Q^n_{jk} w.r.t. $P^{n'}_k$, P^n_k, $P^{n'}_j$ and P^n_j, respectively. The general from of the derivative amounts to a triple product rule

$$
\partial Q^n_{jk}/\partial P^a_b = K(\underbrace{P'AB}_{0 \text{ for } k} + \underbrace{PA'B}_{0 \text{ for } n'} + PAB')
$$

with derivative terms

$$
\begin{aligned}
P'(P^{n'}_j) &= \frac{-1}{\beta}(P^{n'}_j)^{-\sigma}(P^D_j)^{\sigma-\frac{1+\beta}{\beta}} = \frac{-1}{\beta}(P^n_j)^{-\sigma}P^{1+\beta-\beta\sigma} \\
A'(P^n_j) &= -\frac{1}{\eta-1}\frac{1}{m^n}A^{2-\eta} \quad (= 0 \text{ for } P^{n'}_j) \\
B'(P^{n'}_j) &= -\frac{\eta-\beta-1}{(\eta-1)\beta}A(n')B^{\frac{\eta-\beta-1-\eta\beta}{\eta-\beta-1}} = -\frac{\eta-\beta-1}{(\eta-1)\beta}A(n')\left[Q_{jk}PK0^{1/\beta}\right]^{1-\frac{\beta(1+\eta)}{\eta-1}},
\end{aligned}
$$

where $A'(P^n_k) = -A'(P^n_j)$ and $B'(P^{n'}_k) = -B'(P^{n'}_j)$, and the n derivatives are equal to the n' ones except for A'. The derivative terms can then be simplified as factors of Q^n_{jk} as follows

$$
\begin{aligned}
KP'AB &= -\frac{1}{\beta}(P^{n'}_j)^{-\sigma}(P^D_j)^{\sigma-1}Q^n_{jk} \quad (= 0 \text{ for } k \text{ derivatives}) \\
KPA'B &= -\frac{Q^n_{jk}}{(\eta-1)(P^n_k - P^n_j)} \qquad (= 0 \text{ for } n', \text{ opposite sign for } k \text{ derivatives}) \\
KPAB' &= -\frac{\eta-\beta-1}{(\eta-1)\beta}A(n')\left(PK0\; Q^\beta_{jk}\right)^{\frac{-\eta}{\eta-1}} Q^n_{jk} \quad (\text{opposite sign for } k \text{ derivatives}).
\end{aligned}
$$

Collecting common factors, the complete derivative is

$$
Q^{n\prime}_{jk} = -\frac{Q^n_{jk}}{(\eta-1)\beta}\left[\underbrace{\frac{\eta-1}{P^D_j}\left(\frac{P^{n'}_j}{P^D_j}\right)^{-\sigma}}_{0 \text{ for } k} + \underbrace{\frac{\beta}{P^n_k - P^n_j}}_{0 \text{ for } n',\ - \text{ for } k} + \underbrace{\left(\frac{(\eta-\beta-1)^{\eta-1}(P^{n'}_k - P^{n'}_j)}{m^{n'}\left(P^D_j(1+\beta)\delta^\tau_{jk}I^{-\gamma}_{jk}Q^\beta_{jk}\right)^\eta}\right)^{\frac{1}{\eta-1}}}_{- \text{ for } k}\right].
$$

For $D^n_j(\lambda)$, combining equations yields

$$
D^n_j(\lambda) = \left(\underbrace{c_j(\lambda)L_j}_{C} + \underbrace{\sum_{k\in N(j)} \delta^\tau_{jk} I^{-\gamma}_{jk} Q_{jk}(\lambda)^{1+\beta}}_{T}\right)\underbrace{\left(\frac{P^n_j}{P^D_j(\lambda)}\right)^{-\sigma}}_{G}
$$

Again we must then consider derivatives w.r.t. $P_k^{n'}$, P_k^n, $P_j^{n'}$ and P_j^n, where for off-diagonal ($k \neq j$) prices $C' = G' = 0$. The general form is

$$\partial D_j^n / \partial P_b^a = (\underbrace{C'}_{0 \text{ for } k} + T')G + \underbrace{(C+T)G'}_{0 \text{ for } k}$$

with derivatives

$$C'(P_j^{n'}) = \omega_j^{-1} \underbrace{(P_j^{n'})^{-\sigma}(P_j^D)^{\sigma}}_{G \text{ for } n} U'^{-1\prime}(P_j^D/\omega_j, h_j) Lj$$

and

$$G'(P_j^{n'}) = \sigma P_j^n (P_j^{n'})^{-\sigma}(P_j^D)^{\sigma-2} G^{\frac{\sigma+1}{\sigma}} = \sigma(P_j^n P_j^{n'})^{-\sigma}(P_j^D)^{2\sigma-1}$$

$$G'(P_j^n) = -\sigma \frac{P_j^D - (P_j^n)^{1-\sigma}(P_j^D)^{\sigma}}{(P_j^D)^2} G^{\frac{\sigma+1}{\sigma}} = G'(P_j^{n'}) - \frac{\sigma}{P_j^D} G^{\frac{\sigma+1}{\sigma}} = G'(P_j^{n'}) - \frac{\sigma(P_j^D)^{\sigma}}{(P_j^n)^{\sigma+1}}.$$

To differentiate the T term, lets again unpack it a bit

$$T = \sum_{k \in N(j)} \delta_{jk}^{\tau} I_{jk}^{-\gamma} \left(\sum_n m^n \left[\frac{P_k^n - P_j^n}{P_j^D(1+\beta)\delta_{jk}^{\tau} m^n I_{jk}^{-\gamma}} \right]^{\frac{\eta}{\eta-1}} \right)^{\frac{(\eta-1)(1+\beta)}{\eta\beta}}$$

$$= \sum_{k \in N(j)} \delta_{jk}^{\tau} I_{jk}^{-\gamma} [\underbrace{P_j^D(1+\beta)\delta_{jk}^{\tau} I_{jk}^{-\gamma}}_{PK0}]^{\frac{1+\beta}{-\beta}} \left(\sum_n m^n \left[\frac{P_k^n - P_j^n}{m^n} \right]^{\frac{\eta}{\eta-1}} \right)^{\frac{\eta-\beta-1}{\eta\beta}+1}$$

$$= \sum_{k \in N(j)} \underbrace{\frac{[(1+\beta)\delta_{jk}^{\tau} I_{jk}^{-\gamma}]^{\frac{-1}{\beta}}}{1+\beta}}_{K/(1+\beta)} \underbrace{(P_j^D)^{\frac{1+\beta}{-\beta}}}_{P^{1+\beta}} \underbrace{\left(\sum_n m^n \left[\frac{P_k^n - P_j^n}{m^n} \right]^{\frac{\eta}{\eta-1}} \right)^{\frac{\eta-\beta-1}{\eta\beta}+1}}_{B^{\frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta}}}.$$

The general form of the derivative is

$$T' = \sum_{k \in N(j)} \frac{K}{1+\beta} \left[\underbrace{(1+\beta)P'P^{\beta} B^{\frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta}}}_{0 \text{ for } k} + \frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta} P^{1+\beta} B' B^{\frac{-\eta\beta}{\eta-\beta-1+\eta\beta}} \right]$$

$$T' = \underbrace{(1+\beta)P'P^{-1}T}_{0 \text{ for } k} + \frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta} \sum_{k \in N(j)} \delta_{jk}^{\tau} I_{jk}^{-\gamma} Q_{jk}^{1+\beta} B' B^{-1}$$

$$T' = \underbrace{-\frac{1+\beta}{\beta}(P_j^{n'})^{-\sigma}(P_j^D)^{\sigma-1}T}_{0 \text{ for } k} + \frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta} \frac{\eta-\beta-1}{-(\eta-1)\beta} \sum_{k \in N(j)} \delta_{jk}^{\tau} I_{jk}^{-\gamma} Q_{jk}^{1+\beta} A(n') \left(PK0 \; Q_{jk}^{\beta} \right)^{\frac{-\eta}{\eta-1}}$$

$$T' = \sum_{k \in N(j)} \delta_{jk}^{\tau} I_{jk}^{-\gamma} Q_{jk}^{1+\beta} \left[\underbrace{-\frac{1+\beta}{\beta}(P_j^n)^{-\sigma}(P_j^D)^{\sigma-1}}_{0 \text{ for } k} + \frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta} \frac{\eta-\beta-1}{-(\eta-1)\beta} A(n') \left(PK0 \; Q_{jk}^{\beta} \right)^{\frac{-\eta}{\eta-1}} \right]$$

$$T' = \sum_{k \in N(j)} \frac{\delta_{jk}^{\tau} Q_{jk}^{1+\beta}}{I_{jk}^{\gamma} Q_{jk}^n} \left[(1+\beta) \underbrace{KP'AB}_{0 \text{ for } k} + \frac{\eta-\beta-1}{\eta-\beta-1+\eta\beta} \underbrace{KPAB'}_{- \text{ for } k} \right].$$

Thus, we can compute them along with the Q_{jk}^n derivatives above, noting that $B'(P_k^n) = -B'(P_j^n)$.

Finally, we have the production function $z_j^n(L_j^n(\lambda))^a$, where the off-diagonal ($k \neq j$) derivatives are 0, and the diagonal ones are

$$L_j^{n\prime}(P_j^n) = \frac{\psi - \Psi}{\psi} \frac{a(z_j^n)^2}{a-1} \frac{\psi^{2a}}{\Psi^{a+1}} L_j^a \quad \text{and} \quad L_j^{n\prime}(P_j^{n'}) = \frac{a z_j^n z_j^{n'}}{a-1} \frac{\psi^a \psi'^a}{\Psi^{a+1}} L_j^a,$$

with $\psi = \left(P_j^n z_j^n\right)^{\frac{1}{1-a}}$, $\psi' = \left(P_j^{n'} z_j^{n'}\right)^{\frac{1}{1-a}}$ and $\Psi = \sum_{n'} \left(P_j^{n'} z_j^{n'}\right)^{\frac{1}{1-a}}$.

As a numerical note, there should not be negative flows caused by negative price differentials $P_k^n - P_j^n$. Thus, such terms need to be replaced with $max(P_k^n - P_j^n, 0)$ when coding this up.