Module-1 : Classical Mechanics

Scalar fields and Vector fields

Field: A field is a physical quantity which is assigned to every point in space (or, more generally, space-time). If a field is independent of time is called stationary or steady state field.

Scalar field: A scalar quantity, smoothly assigned to each point of a certain region of space, is called a scalar field.

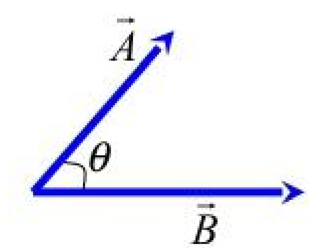
Example: time, temperature, volume, density, mass, energy etc.

One can denote scalar field as ϕ (*x, y, z*) which can be represented as different scalar surface. And since the scalar field has a definite value at each point in space it is independent of the coordinates

Vector field: A vector field is a function which has both a **magnitude** and a **direction in space** and it can vary at different space points.

Example: electric fields, magnetic fields, gravitation fields etc.

Dot Product of Vectors



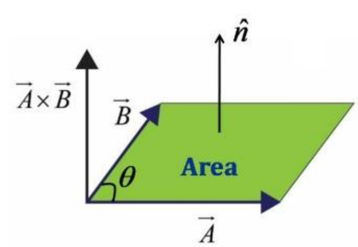
$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|Cos\theta = |\vec{B}||\vec{A}|Cos\theta$$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$$

$$\vec{A}.\vec{B} = \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right).\left(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}\right)$$
$$= A_1B_1 + A_2B_2 + A_3B_3$$

Cross Product of Vectors



$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$

$$\vec{B}\times\vec{A} = |\vec{A}||\vec{B}|\sin\theta(-\hat{n})$$

$$\vec{A}\times\vec{B} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} = \hat{i}[A_2B_3 - A_3B_2] - \hat{j}[A_1B_3 - A_3B_1] + \hat{k}[A_1B_2 - A_2B_1]$$

Concept of Gradient

Let $\vec{A} = \vec{\nabla}\Phi$ everywhere in a region of space, defined by $a_1 \le x \le a_2$ $b_1 \le x \le b_2$ and $c_1 \le x \le c_2$, where $\Phi\,(x, y, z)$ is a Scalar field and has continuous derivatives in the region.

$$\vec{\nabla}\Phi = \left(\hat{i}\frac{\partial}{dx} + \hat{j}\frac{\partial}{dy} + \hat{k}\frac{\partial}{dz} \right) \Phi\,(x, y, z) = \hat{i}\frac{\partial \Phi}{dx} + \hat{j}\frac{\partial \Phi}{dy} + \hat{k}\frac{\partial \Phi}{dz}$$

Mathematical analysis of Constrained Motion

Constraint Equation

Holonomic constraint

[Constraints can be expressed as an equation connecting position of the particles and the time.

$f(q_1, q_2, q_3, \ldots\ldots q_n, t) = 0$,

where $\{q_1, q_2, q_3, \ldots\ldots q_n\}$ are n generalized coordinates. They are independent of velocity of the particles.]

Scleronomic constraints

[Constraints which do not explicitly depend on time.]

Rheonomic constraints

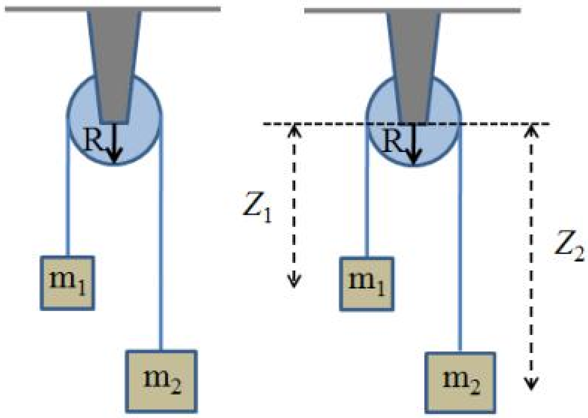
[Constraints which explicitly depend on time.]

Non-Holonomic constraint

[Constraints can not be expressed as an proper equation connecting position of the particles and the time.

$f(q_1, q_2, q_3, \ldots\ldots q_n) \ge 0$.

They are dependent of velocity of the particles.]



Pulley-Mass system:

l = length of the inextensible cable

Z_1 and Z_2 = Displacements of m_1 and m_2 at any instant of time from fixed dotted line passing through the center of the pulley

Constraint Equation :

$$Z_1 + \pi R + Z_2 = l$$

$$\Rightarrow Z_1 + Z_2 = \text{Constant}$$

Differentiating w.r.t time,

$$\frac{dZ_1}{dt} + \frac{dZ_2}{dt} = 0$$

$$\Rightarrow v_1 + v_2 = 0$$

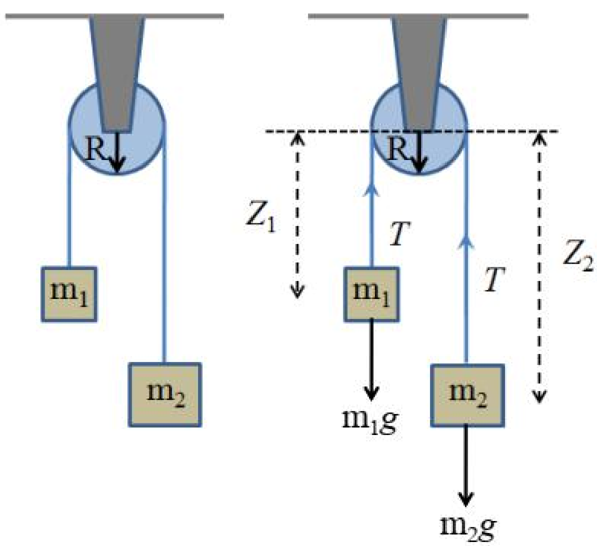
$$\Rightarrow v_2 = -v_1 \;(\text{velocity})$$

Differentiating w.r.t time,

$$\frac{dv_1}{dt} + \frac{dv_2}{dt} = 0$$

$$\Rightarrow a_1 + a_2 = 0$$

$$\Rightarrow a_2 = -a_1 \;(\text{acceleration})$$



Pulley-Mass system:

Equation of Motion:

$$m_1 a_1 = T - m_1 g$$

$$+ \quad m_2 a_2 = m_2 g - T$$

$$m_1 a_1 + m_2 a_2 = (m_2 - m_1) g$$

As magnitude of accelerations of m_1 and m_2 are same,

$$|a_2| = |-a_1|$$

$$\boxed{a_1 = -\frac{(m_2 - m_1)}{(m_2 + m_1)} g \qquad a_2 = \frac{(m_2 - m_1)}{(m_2 + m_1)} g}$$

Questions:

1. Particles P and Q, of masses 0.6 kg and 0.2 Kg respectively, are attached to the ends of a light inextensible string, which passes over a smooth fixed peg. The particles are held at rest with the string taut. Both particles are at a height of 0.9 m above the ground as shown in Fig. 1.

   The system is released and each of the particles moves vertically.

   Find (a) the acceleration of P and the tension in the string before P reaches the ground and

   (b) the time taken for P to reach the ground. (g = 10m/s^2). **(10) [BL5]**

   [*Hints:* Use the given formula above.]

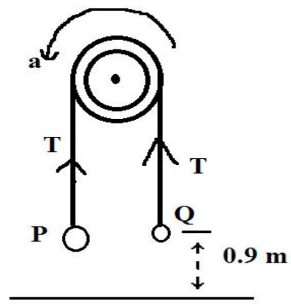


Fig.1

2. A block of mass 2 kg is at rest on a horizontal floor. The coefficient of friction between the block and the floor is μ. A force of magnitude 12 N acts on the block at an angle α to the horizontal, where tan α = ¾. When the applied force acts downward as in Fig. 2a, the block remains at rest.

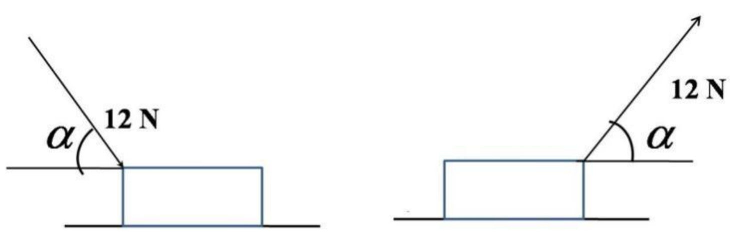


Fig. 2a Fig. 2b

   (a) Show that $\mu \geq 6/17$

   (b) When the applied force acts upwards as shown in Fig. 2b, the block slides along the floor. Find another inequality for μ. **(10) [BL5]**

   [*Hints:* Use the friction related formula.]

3. Consider a system of four particles in *x-y* plane. Of these, two particles each of mass m are located at (-1, 1) and (1, -1). The remaining two particles each of mass $2m$ are located at (1, 1) and (-1, -1). Analyze the *xy*- component of the moment of inertia of this

system of particles. **(5) [BL4]**

[*Hints:* Use the concept of moment of inertia.]

4. If **A** and **B** are irrotational, analyze that **A** × **B** is solenoidal. **(5) [BL4]**

[*Hints:* Use the concept of divergence and curl of a vector.]

5. A particle of mass *m* moves in the X-Y plane and the position of the particle is given by $\vec{r} = \hat{\imath}\ a\ Cos\ \omega t + \hat{\jmath}\ b\ Sin\ \omega t$ where *a, b* and *ω* are constants.

Evaluate that

(a) The force acting on the particle is always directed to towards the origin.

(b) Calculate the torque and angular momentum about the origin.

(c) Is the force field conservative? **(10) [BL5]**

Module-2: Oscillations

Equation of motion of free vibration:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Solution:$\boldsymbol{x = C\ Sin\ (\omega t + \varphi)}$

Equation of motion of damped vibration:

$$\frac{d^2x}{dt^2} = -2b\frac{dx}{dt} - \omega^2 x$$

Solution: $\boldsymbol{x = e^{-bt}\left(A\, e^{\sqrt{b^2-\omega^2}t} + B\, e^{-\sqrt{b^2-\omega^2}t}\right)}$

Case 1 (Low damping b <ω):$\boldsymbol{x = Ce^{-bt}\, Sin\left(\sqrt{b^2-\omega^2}t + \varphi\right)}$, **where** $\boldsymbol{C = \sqrt{A^2+B^2}}$

Case 2 (Critical damping b ~ ω):$\boldsymbol{x = e^{-bt}[(A+B) + (A-B)t]}$

Case 3 (Large damping b >ω):$\boldsymbol{x = e^{-bt}\left(A\, e^{\sqrt{b^2-\omega^2}t} + B\, e^{-\sqrt{b^2-\omega^2}t}\right)}$

Equation of motion of forced vibration:

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\, e^{ipt}$$

$\boldsymbol{p}$ is frequency of external periodic force.