Operators & Observables

In quantum mechanics, measurable physical quantities—position, momentum, energy, angular momentum—are not numbers you simply plug in; they are represented by **operators** acting on quantum states. Understanding operators and their mathematical properties is crucial for predicting and interpreting measurement outcomes.

2.1 Observables as Operators

An **observable** is any measurable physical quantity. In the formalism of quantum mechanics, each observable corresponds to a **linear Hermitian operator** A^\hat{A}A^ acting on the Hilbert space of states. When this operator acts on a state |ψ⟩\psi\rangle|ψ⟩, the resulting vector gives information about how |ψ⟩\psi\rangle|ψ⟩ behaves under that measurement.

For example:

- Position operator in one dimension: x^ψ(x)=xψ(x)\hat{x}\psi(x)=x\psi(x)x^ψ(x)=xψ(x).
- Momentum operator: p^=−iℏddx\hat{p}=-i\hbar\frac{d}{dx}p^=−iℏdxd.
- Hamiltonian (energy): H^=−ℏ22md2dx2+V(x)\hat{H}=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)H^=−2mℏ2dx2d2+V(x).

2.2 Hermitian (Self-Adjoint) Operators

Physical observables must yield real measurement outcomes. Mathematically, this requires that the operator A^\hat{A}A^ be **Hermitian** (also called self-adjoint). This means:

⟨ϕ|A^ψ⟩=⟨A^ϕ|ψ⟩for all |ϕ⟩,|ψ⟩.\langle \phi | \hat{A}\psi \rangle = \langle \hat{A}\phi | \psi \rangle \quad\text{for all }|\phi\rangle,|\psi\rangle.⟨ϕ|A^ψ⟩=⟨A^ϕ|ψ⟩for all |ϕ⟩,|ψ⟩.

Hermitian operators have two crucial properties:

1. Their **eigenvalues are real**—ensuring real measurement results.
2. Their **eigenvectors form a complete basis**—allowing any state to be expanded in eigenstates of the observable.

2.3 Eigenvalues and Eigenstates

If A^|ϕ⟩=a|ϕ⟩\hat{A}|\phi\rangle=a|\phi\rangleA^|ϕ⟩=a|ϕ⟩, then |ϕ⟩\phi\rangle|ϕ⟩ is an **eigenstate** of A^\hat{A}A^ with **eigenvalue** aaa. Measurement of the observable AAA in state |ϕ⟩\phi\rangle|ϕ⟩ always yields aaa with 100% probability.

For a general state |ψ⟩\psi\rangle|ψ⟩, which can be written as a superposition of eigenstates,

|ψ⟩=∑ncn|ϕn⟩,|\psi\rangle=\sum_n c_n|\phi_n\rangle,|ψ⟩=n∑cn|ϕn⟩,

the probability of obtaining ana_nan is |cn|2|c_n|^2|cn|2. After the measurement, the system collapses to the eigenstate |ϕn⟩\phi_n\rangle|ϕn⟩.

2.4 Expectation Values

For an observable A^\hat{A}A^, the **expectation value** in state |ψ⟩\psi\rangle|ψ⟩ is:

⟨A⟩ψ=⟨ψ|A^|ψ⟩.\langle A\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle.⟨A⟩ψ=⟨ψ|A^|ψ⟩.

This is the average value you would get if you repeated the measurement many times on identically prepared systems. In position space,

$$\langle A\rangle=\int \psi^*(x)\,\hat{A}\,\psi(x)\,dx.$$

2.5 Commutators and Compatibility

Two operators $\hat{A}$ and $\hat{B}$ are said to **commute** if $[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}=0$. If they commute, they have a common set of eigenstates, and the corresponding observables can be measured simultaneously with definite values.

If they **do not commute**, there is an intrinsic limit to the precision with which you can know their values simultaneously. The most famous case is position and momentum:

$$[\hat{x},\hat{p}]=i\hbar.$$

This leads directly to the **Heisenberg uncertainty principle**:

$$\Delta x \,\Delta p \ge \frac{\hbar}{2}.$$

2.6 Projection Operators and Measurement

A measurement outcome corresponds to a **projection operator** $\hat{P}_a$ onto the eigenspace associated with eigenvalue a. The probability of obtaining a in state $|\psi\rangle$ is:

$$P(a)=\langle\psi|\hat{P}_a|\psi\rangle.$$

After the measurement, the state collapses to $\hat{P}_a|\psi\rangle/\sqrt{P(a)}$.

2.7 The Hamiltonian as Generator of Time Evolution

Among all operators, the **Hamiltonian** $\hat{H}$ is special. It represents the total energy and generates time evolution through the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle.$$

The time-evolution operator is:

$$\hat{U}(t)=e^{-i\hat{H}t/\hbar},$$

which is **unitary** (preserves normalization).

2.8 Angular Momentum and Spin Operators

Angular momentum in quantum mechanics is described by operators $\hat{L}_x,\hat{L}_y,\hat{L}_z$ satisfying commutation relations:

$$[\hat{L}_x,\hat{L}_y]=i\hbar\hat{L}_z\quad\text{(cyclic permutations)}.$$

These lead to quantized angular momentum values. Similarly, intrinsic spin is described by **Pauli matrices** $\hat{S}_x,\hat{S}_y,\hat{S}_z$, which also obey analogous commutation rules but correspond to a fixed spin magnitude (e.g., $\hbar/2$ for electrons).

2.9 Ladder Operators

Some operators are conveniently handled using **ladder (raising and lowering) operators**. For instance, in the quantum harmonic oscillator, you define:

a^=mω2ℏx^+i2mℏωp^,a^†=(Hermitian conjugate),\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\hat{x}+\frac{i}{\sqrt{2m\hbar\omega}}\hat{p}, \quad \hat{a}^\dagger=\text{(Hermitian conjugate)},a^=2ℏmωx^+2mℏωip^,a^†=(Hermitian conjugate),

which satisfy [a^,a^†]=1[\hat{a},\hat{a}^\dagger]=1[a^,a^†]=1.
These ladder operators make it easy to compute eigenstates and eigenvalues of the Hamiltonian without solving differential equations directly.

2.10 Unitary and Symmetry Operators

Beyond observables, there are **unitary operators** representing symmetries (rotations, translations, parity). They preserve inner products and probabilities. For example, the translation operator shifts the wavefunction in space, and its generator is the momentum operator. Symmetries often lead to conservation laws via **Noether's theorem**.

2.11 Summary of Key Points

- Every observable corresponds to a **Hermitian operator**.
- **Eigenvalues** are possible measurement outcomes; **eigenstates** give definite outcomes.
- The **expectation value** gives the average measurement result.
- **Commutators** determine which observables can be simultaneously known.
- The **Hamiltonian** generates time evolution.
- **Angular momentum and spin operators** exhibit intrinsic quantization.
- **Ladder operators** simplify certain problems.
- **Unitary operators** represent symmetry transformations and conserve probability.

Operators and observables form the backbone of quantum mechanics: they tell you how to calculate measurable quantities, how different quantities relate, and how symmetries and conservation laws emerge in the quantum world.