**Quantum States & Wavefunctions**

Quantum mechanics describes physical systems in a radically different way from classical mechanics. Instead of specifying a particle’s position and momentum exactly, we represent the system’s state by a mathematical object called a **quantum state**. This state encodes everything that can be known about the system and determines the probabilities of different measurement outcomes.

**1.1 The Concept of a Quantum State**

A quantum state is usually represented by a **state vector** ∣ψ⟩|\psi\rangle∣ψ⟩ in an abstract mathematical space called a **Hilbert space**. The ket ∣ψ⟩|\psi\rangle∣ψ⟩ contains all the information about the system. In the position representation, this state vector corresponds to a **wavefunction** ψ(x)\psi(x)ψ(x), which is a complex-valued function of position (and sometimes time).

Unlike a classical trajectory, a quantum state does not tell you a definite position or momentum. Instead, it gives the **probability amplitude** for each possible outcome. When you square the modulus of the wavefunction, ∣ψ(x)∣2|\psi(x)|^2∣ψ(x)∣2, you get the probability density of finding the particle at position xxx. This is the heart of the **Born rule**.

**1.2 Normalization and Probabilities**

Because probabilities must add up to one, wavefunctions must be **normalized**. In one dimension, this means

∫−∞∞∣ψ(x)∣2 dx=1.\int\_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1.∫−∞∞​∣ψ(x)∣2dx=1.

If a wavefunction is not normalized initially, you can scale it until it is. This normalization ensures that when you integrate the probability density over any interval, you get the probability of the particle being in that interval.

**1.3 Superposition Principle**

One of the most striking features of quantum mechanics is the **superposition principle**. If ψ1(x)\psi\_1(x)ψ1​(x) and ψ2(x)\psi\_2(x)ψ2​(x) are valid wavefunctions, then any linear combination

ψ(x)=c1ψ1(x)+c2ψ2(x)\psi(x) = c\_1 \psi\_1(x) + c\_2 \psi\_2(x)ψ(x)=c1​ψ1​(x)+c2​ψ2​(x)

with complex coefficients c1c\_1c1​ and c2c\_2c2​ is also a valid wavefunction (after normalization). This means a quantum system can be in a state that is “both” state 1 and state 2 until a measurement collapses it to one outcome. Interference phenomena, such as in the double-slit experiment, are direct consequences of this principle.

**1.4 The Schrödinger Equation**

The **time evolution** of a quantum state is governed by the **Schrödinger equation**, which is the quantum analog of Newton’s second law. For a nonrelativistic particle in one dimension, the time-dependent equation is:

iℏ∂∂tψ(x,t)=−ℏ22m∂2∂x2ψ(x,t)+V(x)ψ(x,t),i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t),iℏ∂t∂​ψ(x,t)=−2mℏ2​∂x2∂2​ψ(x,t)+V(x)ψ(x,t),

where mmm is the mass, V(x)V(x)V(x) the potential energy, and ℏ\hbarℏ is the reduced Planck constant.  
Solutions to this equation give ψ(x,t)\psi(x,t)ψ(x,t), the wavefunction at all times.

Often we look at the **time-independent Schrödinger equation**, which finds stationary states of definite energy EEE:

−ℏ22md2dx2ψ(x)+V(x)ψ(x)=Eψ(x).-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)= E\psi(x).−2mℏ2​dx2d2​ψ(x)+V(x)ψ(x)=Eψ(x).

Solving this eigenvalue problem gives allowed energies and corresponding stationary wavefunctions.

**1.5 Boundary Conditions and Quantization**

Because the Schrödinger equation is a differential equation, we must impose boundary conditions on ψ(x)\psi(x)ψ(x). For bound states, the wavefunction must vanish at infinity and be continuous. These conditions often lead to **quantized energy levels** — discrete values of EEE instead of a continuum. This is why atoms have discrete spectra.

**1.6 Phase and Complex Nature of Wavefunctions**

Wavefunctions are generally **complex** functions. The absolute square gives probability, but the **phase** carries crucial information about interference and time evolution. Global phases (multiplying the entire wavefunction by eiθe^{i\theta}eiθ) don’t change physical predictions, but relative phases between components of a superposition do.

**1.7 Wavepackets and Localization**

A pure plane wave ψ(x)=eikx\psi(x)=e^{ikx}ψ(x)=eikx has a perfectly defined momentum but is spread out over all space. A sharply localized particle requires a superposition of many plane waves, forming a **wavepacket**. The width of the wavepacket in position space and the spread in momentum space are inversely related — this is directly connected to the **Heisenberg uncertainty principle**.

**1.8 Measurement and Collapse**

When you measure an observable (like position or momentum), the wavefunction “collapses” to an eigenstate of that observable with a probability given by the squared amplitude. This collapse is not described by the Schrödinger equation but is a postulate of quantum mechanics. It reflects the probabilistic nature of measurement outcomes.

**1.9 Hilbert Space Formalism**

More abstractly, the space of all possible quantum states is a **Hilbert space** equipped with an inner product ⟨ϕ∣ψ⟩\langle \phi | \psi \rangle⟨ϕ∣ψ⟩. This inner product gives overlaps between states, which correspond to transition amplitudes or probabilities. Operators representing observables act on this space, and their eigenvectors form complete bases for expanding any state.

**1.10 Summary of Key Points**

* **State vector ∣ψ⟩|\psi\rangle∣ψ⟩** encodes everything about a quantum system.
* **Wavefunction ψ(x)\psi(x)ψ(x)** in position representation gives the probability amplitude.
* Probabilities come from ∣ψ∣2|\psi|^2∣ψ∣2, requiring normalization.
* **Superposition** allows linear combinations of states, leading to interference.
* **Schrödinger equation** governs time evolution.
* Boundary conditions lead to **quantized energy levels**.
* **Phase** matters for interference; global phase does not affect outcomes.
* **Wavepackets** illustrate the position-momentum tradeoff.
* Measurement outcomes are inherently **probabilistic**.

This framework of states and wavefunctions underpins everything else in quantum mechanics — operators, dynamics, perturbations, and multi-particle systems all build on this foundation.