Algorithm 2: Computing $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)]$

1: $\mathcal{L}(0, \{0\}_{i\in[n]}) = 0, \mathcal{L}(0, \{q_i\}) = 1$ ($q_i \neq 0\ \forall i \in [n]$)
2: **for** $t = 0$ to n **do**
3: **if** $\exists i \in [n]$ s.t. $\pi(t) = \pi(i)$ **then**
4: $\mathcal{L}(t, \{q_i\}_{i\in[n]}) = p_t\mathcal{L}(t-1, \{q_i - 1\}_{i\in\pi(t)} \cup \{q_j\}_{j\in[n]\setminus\pi(t)}) + (1-p_t)\mathcal{L}(t-1, \{q_i\}_{i\in[n]})$
5: **else**
6: $\mathcal{L}(t, \{q_i\}_{i\in[n]}) = \mathcal{L}(t-1, \{q_i\}_{i\in[n]})$
7: **end if**
8: **end for**
9: **return** $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)] = \mathcal{L}(n, \{s_i\}_{i\in[n]})$

Proof. (*Sketch*) In Appendix J, we show a reduction from X3C to NS-∃CCAUP that adapts the reduction from X3C presented in Theorem 9 of (**?**). By Lemma **??**, deciding $Pr[\mathcal{NS}_{\mathcal{G}'}] > 0$ is #P-hard. The proof for perfect-∃CCAUP similarly adapts the reduction presented in Theorem 18 of (**?**), and is thus differed to Appendix J along with the proof for deciding $Pr[\mathcal{PF}_{\mathcal{G}'}] > 0$. The same reduction can be used for SCS. □

Boehmer and Elkind (**?**) show that finding a Nash-stable partition can be done in poly-time if each agent approves at most one coalition size. Despite the negative result in Theorem **??**, we prove that such a condition further enables us to compute both $Pr[\mathcal{NS}_{\mathcal{G}'}(\pi)]$ and $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)]$ in polynomial-time for AHGs.

Theorem 8. *If each agent $i \in N$ solely approves coalitions in $\mathcal{N}_i^+ = \{C \in \mathcal{N}_i : |C| = s_i\}$ for some $s_i \in \mathbb{N}$, then $Pr[\mathcal{NS}_{\mathcal{G}'}(\pi)]$ and $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)]$ are poly-time computable.*

Proof. (*Sketch*) Note that $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)] = Pr[\cap_{i\in N}|\pi(i) \cap I| = s_i]$. If $|\pi(i)| < s_i$ for some $i \in N$, then $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)] = 0$. Thus, we hereafter assume that $|\pi(i)| \geq s_i$. Let $\mathbb{1}_i$ be the indicator for the event that agent i participates in the game. For integers $t \leq n$ and $0 \leq q_i \leq n$ ($i \in [n]$), let $\mathcal{L}(t, \{q_i\}_{i\in[n]}) = Pr[\cap_{i\in N} \sum_{j\in\pi(i)\cap I\cap[t]} \mathbb{1}_j = q_i]$. In Appendix K, we show that it can be computed in poly-time via the dynamic program in Algorithm **??**, and so does the probability that π is perfect. For computing $Pr[\mathcal{NS}_{\mathcal{G}'}(\pi)]$, if there is no agent i with $s_i = 1$, then the grand coalition is NS. Thus, we hereafter assume that there exists at least one such agent. For each $j \in [n]$, let $N_j = \{i \in N : s_i = j\}$, and let $\ell = \max\{i | N_j = \emptyset \forall j \in [i]\}$. Boehmer and Elkind (**?**) prove that for each $j \in [\ell]$ all agents in N_j need to be in coalitions of size j in every NS outcome. Noting that $Pr[\mathcal{NS}_{\mathcal{G}'}(\pi)] = Pr[\cap_{j\in[\ell]} \cap_{i\in N_j} |\pi(i) \cap I| = j]$, the proof thus follows from arguments similar to the previous one. □

Candidate Intervals

Assuming the agent set can be placed in the natural ordering, each agent i only approves *candidate intervals* (CIs) $[a, b]$ of agents (with $i \in [a, b]$). Such a restriction was termed by Elkind and Lackner (**?**), and applied to DHGs by (**?**). Opposed to the negative result for k-lists with $k \geq 2$ (Theorems **??** and **??**), we prove that:

Theorem 9. $Pr[\mathcal{WO}^{\mathcal{C}}_{\mathcal{G}'}(C)]$, $Pr[\mathcal{WO}^{\Pi}_{\mathcal{G}'}(\pi)]$ *and* $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)]$ *are computable in poly-time for CIs.*

Proof. For an integer $0 \leq j \leq n$, let $\mathcal{SW}^\star(j)$ and $\mathcal{SW}_I(j)$ be the maximum social welfare and the maximum coalitional social welfare over all coalitions (resp.) obtainable in the subgame restricted to the random agent set $I \cap [j]$. Each agent approves all originally approved coalitions S such that $S \subseteq I \cap [j] \cup \{0\}$. Note that $\mathcal{SW}^\star(0) = 0$. Let $\#[t, j]$ be the number of agents that approve the interval $[t, j]$ in the subgame. Similar to Theorem 10 in (**?**), we infer that $\mathcal{SW}^\star(j) = \max_{t\in[j]}\{\mathcal{SW}^\star(j-1) + \#[t, j]\}$, which can thus be computed in polynomial time via dynamic programming. Noting that $Pr[\mathcal{WO}^{\Pi}_{\mathcal{G}'}(\pi)] = Pr[\mathcal{SW}_I(\pi) = \mathcal{SW}^\star(n)]$ and that $Pr[\mathcal{SW}_I(\pi) = q]$ can be computed via dynamic programming (by arguments similar to the proof of Theorem **??**, which are thus deferred to Appendix L), we infer that $Pr[\mathcal{WO}^{\Pi}_{\mathcal{G}'}(\pi)]$ is in FP for CIs. By substituting $\mathcal{SW}^\star(j)$ with $\mathcal{SW}_I(j)$, we obtain that $Pr[\mathcal{WO}^{\mathcal{C}}_{\mathcal{G}'}(C)]$ is also in FP for CIs. Since a partition π is perfect iff $\mathcal{SW}^\star(\pi) = n$, the proof for $Pr[\mathcal{PF}_{\mathcal{G}'}(\pi)]$ readily follows. □

Roommate Games

In this section, we consider a restriction of hedonic games where agents only approve coalitions of size at most 2 (**?**). The bipartite case of roommate games (RGs) is referred to as *marriage* games (**?**). See (**?**) for a survey on both types of games. Finding perfect partitions is easy by Theorem 7 in (**?**). Moreover, a core stable matching can be computed efficiently for marriage games (**?**). However, we obtain negative results for both concepts in *non-dichotomous* marriage games and other classes of hedonic games.

Theorem 10. *In marriage games, Perfect/Nash-stable-∃CCAUP(-m) and deciding $Pr[\mathcal{NS}_{\mathcal{G}'}], Pr[\mathcal{PF}_{\mathcal{G}'}] > 0$ are NPC. The same applies to* roommates*,* Representation by Individually Rational Lists of Coalitions (RIRLC) *(**?**),* additively separable hedonic games *(**?**),* B-hedonic games *and* W-hedonic games *(Aziz et al. (**?**)).*

Proof. (*Sketch*) In Appendix M, we show a reduction from MinMaxMatch, the problem of finding a maximal matching with size $\leq m$, which is known to be NP-complete even for subdivision graphs (**?**). The proof constitutes an adaptation of Theorem 1 in (**?**). The same reduction can be used for perfection. By Corollary 1 in (**?**), we further deduce the result for the other classes of hedonic games. □

Perfect Partitions in Majority Games

Thus far, we analyzed the ***probability*** that either coalitions or partitions satisfy a solution concept. In contrast, we herein explore their ***average*** performance in *majority games* (**?**). This class can be seen as a dichotomization of *fractional hedonic games* (**?**). Formally, let $G = (N, E)$ be an undirected graph, where each agent corresponds to a vertex and an edge between two agents depicts a (mutual) friendship between them. Let $G_I = (I, E^I)$ be the random subgraph of G induced by I. Letting $d_i^I(C)$ be agent i's degree in the subgraph of G_I induced by a coalition C, agent i approves C if $d_i^I(C) \geq \frac{|C|}{2}$, i.e., if i is connected to at least $\frac{|C|}{2}$ of the vertices in C. First, we characterize the distribution of both the coalitions and social welfare of *perfect* partitions (Theorems **??**-**??**), on which we elaborate in Appendices N–O. Then, we discuss the performance of perfect outcomes by providing upper and lower bounds on their *Price of Anarchy* (**?**). By Theorem 14 in (**?**), without loss of generality, a perfect partition consists of edges and triangles. Thus, we let M_n^I and T_n^I be the random variables which represent the number of edges and triangles in G_I (resp.). Let $\mathbb{T}$ be the set of all triplets (i, j, k) $(i < j < k)$ that form a triangle in G. Accordingly, the following theorem fully characterizes the social welfare of a *perfect* partition π, as well as the coalitions comprising π, for various values of $(p_i)_{i \in N}$.

Theorem 11. *For each $i \in N$ and $n \in \mathbb{N}$, let $p_i(n) = \frac{q_i(n)}{n}$ for some $q_i : \mathbb{N} \to \mathbb{R}$, $q^{\max}(n) = \max_{i \in N} q_i(n)$ and $q^{\min}(n) = \min_{i \in N} q_i(n)$. Given a perfect partition π, we infer:* **(1)** A perfect partition comprises of singletons w.h.p. (with high probability): *If $q^{\max}(n) \to 0$ as $n \to \infty$, then $T_n^I = 0$, $M_n^I = 0$ and $\mathcal{SW}_I(\pi) = 0$;* **(2)** Triangles and edges reside in perfect partitions a.s. (almost surely): *If $|\mathbb{T}| = 1$ and $\frac{n}{q^{\min}(n)} \to 1$ as $n \to \infty$, then $T_n^I \geq 1$ a.s. Otherwise, if $|\mathbb{T}| \geq 2$ and $q^{\min}(n) \to \infty$ as $n \to \infty$, then $T_n^I \geq 1$ and $M_n^I \geq 1$ a.s.;* **(3)** *If $\frac{q^{\max}(n)}{q^{\min}(n)} \to 1$ as $n \to \infty$, then $\frac{q^{\min}(n)}{n} \leq \mathbb{E}[|I|] \leq q^{\max}(n)$, thus yielding that $|I| \geq 1$ (i.e.,* at least one agent remains*) a.s.;* **(4)** *If $q_i(n) \equiv c_i$ for $c_i > 0$ $\forall i$, then $\mathbb{E}[\mathcal{SW}_I(\pi)] \leq c_{\max}^2$, where $q^{\max}(n) \equiv c_{\max}$.*

Further, we can model agents' *uncertainty about their mutual friendships*. Formally, let $(p_{ij})_{i,j \in N} \in [0, 1]^{n \times n}$ with $p_{ij} = p_{ji}$ for every $i, j \in N$. Let $\mathcal{E} \subseteq N \times N$ be a random variable, where $(i, j) \in \mathcal{E}$ with probability p_{ij} and different pairs of indices are independent, thus yielding a *Erdös-Rényi random graph* $\tilde{G} = (N, \mathcal{E})$ (**?**) whose set of edges is $\mathcal{E}$. The majority game on the resulting random graph satisfies 1-3 in Theorem **??** with minor adjustments (See Appendix O.1), yet gives rise to an additional property which extends property 4 (proved in Appendix O.2):

Theorem 12. *Let $p_{ij}(n) = c/n$ for some constant $c > 0$. Let π be a perfect partition. Then, T_n^I converges in distribution to a Poisson random variable with parameter $c^3/6$, $\mathbb{E}[M_n^I] = (n-1)c/2$ and $\mathbb{E}[\mathcal{SW}(\pi)] \leq (n-1)c$.*

Let $\mathbb{P}_I$ be the set of all perfect partitions for a random set of players I and let $\pi^\star$ be an welfare-optimal partition. Inspired by the *Price of Anarchy* (**?**), we put forth the *Price of Perfection* (**PP**) of a RDHG $\mathcal{G}'$, defined as the worst-case ratio between the social welfare of $\pi^\star$ and that of a perfect partition, i.e., $\text{PP}(\mathcal{G}') = \max_{\pi \in \mathbb{P}_I} \frac{\mathcal{SW}_I(\pi^\star)}{\mathcal{SW}_I(\pi)}$. Similarly, we define the *Expected Price of Perfection* (**EPP**) by $\text{EPP}(\mathcal{G}') = \max_{\pi \in \mathbb{P}_I} \frac{\mathbb{E}[\mathcal{SW}_I(\pi^\star)]}{\mathbb{E}[\mathcal{SW}_I(\pi)]}$. Using Theorem **??**, we devise upper and lower bounds on both variants of the price of perfection, where Corollary **??** is clearly a direct outcome of (3)-(4) in Theorem **??**.

Lemma 3. *Under the assumptions of Theorem **??** and: (1) in Theorem **??**, $PP(\mathcal{G}') = EPP(\mathcal{G}') = 0$ w.h.p.; (2) in Theorem **??**, $PP(\mathcal{G}') \leq |I|/2$ a.s.; (3)-(4) in Theorem **??**, $EPP(\mathcal{G}') \leq q^{\max}(n) = c_{\max}/2$.*

Proof. For (1), the claim clearly stems since $M_n^I = 0$ w.h.p. For (2), $\pi^\star$ clearly satisfies $\mathcal{SW}_I(\pi^\star) \leq |I|$. If we were to consider each connected component of G separately, we may assume that G is connected and does not consists of any isolated vertices. Hence, if there exists a perfect partition in G, then a perfect partition consisting of edges and triangles exists (Theorem 14 in (**?**)). However, G_I might contain isolated vertices, even if G does not. Since $M_n^I \geq 1$ a.s., we infer that $\mathcal{SW}_I(\pi) \geq 2$ at the very least, thus yielding that $\text{PP}(\mathcal{G}') \leq |I|/2$. For (3), we observe that $\mathbb{E}[\mathcal{SW}_I(\pi^\star)] \leq \mathbb{E}[|I|] \leq q^{\max}(n)$. Combined with the proof for (2), we conclude that $\text{EPP}(\mathcal{G}') \leq q^{\max}(n)$. □

Corollary 2. *Under the assumptions of (3)-(4) in Theorem **??**, if $\mathcal{SW}(\pi^\star) \geq 1$, then $EPP(\mathcal{G}') \geq 1/c_{\max}^2$. Alternately, if $\mathcal{SW}_I(\pi^\star) = |I|$, we infer that $EPP(\mathcal{G}') \geq \frac{c_{\min}}{nc_{\max}^2}$.*

Proof. The first lower bound is a direct outcome of (4) in Theorem **??**. For the second part, from (3) in Theorem **??** we infer that $\frac{q^{\min}(n)}{n} \leq \mathbb{E}[\mathcal{SW}_I(\pi^\star)] \leq q^{\max}(n)$. Combined with (4) in Theorem **??**, we conclude the desired bounds. □

Conclusions and Future Work

Our work contributes significantly to the study of hedonic games, as the first one to explore the complexity of probabilistically inferring solution concepts in uncertain domains. The main complexity results are summarized in Table **??**. Our study opens the way for many future works, including the investigation of other classes of hedonic games and other solution concepts. Further, our probabilistic setting arises several intriguing questions, among those: For an outcome satisfying a solution concept β, what is the maximum number of players whose withdrawal from the game still preserves β in the outcome induced by the remaining players? Another direction is *robustness* (**?**): A probabilistic withdrawal of players upon an outcome satisfying a solution concept β (e.g., stability) should preserve β.

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