Informally, GCC assigns each agent i to a coalition containing a sufficiently large number of agents that are friends of i. At time t, if there exists a coalition $C \in \mathcal{C}^t$ and $S \subseteq C$ with $S \subseteq N_t^t$ and $|S| \geq H(\alpha) \cdot m \cdot n/\alpha$ s.t. each agent in S is a neighbor of less than m agents assigned to C thus far, then GCC assigns t to C. If multiple such coalitions exist, GCC uniformly assigns t to the coalition of *smallest* cardinality. If such a coalition does not exist, then a new coalition $\{t\}$ is created. We now prove that:

Theorem 5. *The partition returned by GCC is $H(\alpha)$-CEJR and each coalition contains at most α agents.*

Proof. (*Sketch*) By construction, the partition is $H(\alpha)$-CEJR. Using a budgeting argument, we prove in Appendix J that each coalition contains at most α agents. □

Next, we further illuminate on the complexity of obtaining a partition satisfying η-CEJR. We remark that GCC requires checking whether a coalition is (η, m)-cohesive, where verifying its existence is generally NP-hard. The proof in Appendix K is by reduction from Maximum k-Subset Intersection (**?**).

Theorem 6. *Checking whether there exists an m-cohesive coalition is NP-hard.*

Theorem 6 dictates that GCC cannot be executed in polynomial time, and thus achieving an *optimal* partition as stated by Corollary 1 is challenging. Hence, we supply *Sub-Coalitions by Greedy Budgeting* (**SCGB**), a polynomial-time algorithm that yields a slightly worse CEJR guarantee than GCC. Our scheme adapts the algorithm in (**?**, Section 5.3) for committee elections to hedonic games. First, let $w(\cdot)$ be the inverse function of $x \mapsto x^x$, i.e., $w(\alpha) = x$ if $\alpha = x^x$. Note that $w(\alpha) = O(\log \alpha)$ and $\log \alpha = O(w(\alpha)^2)$. Let $\beta = \lceil w(\alpha) \rceil$. SCGB independently creates β sub-coalitions generated similarly to CMES, each of size $\lfloor \alpha/\beta \rfloor$. Formally, each agent is given an initial budget of $(1, \ldots, 1) \in [0, 1]^\beta$, i.e., there are β independent dollars where each one is associated with a specific possible sub-coalition. The j^{th} coin can be used for buying agents who are approved by at least $n\beta^j/\alpha$. Each agent costs $n\beta/\alpha$ dollars. At time t, we find the largest triple $j \in [\beta]$ and a coalition $C \in \mathcal{C}^t$ with $S \subseteq C$ satisfying $S \subseteq N_t^t$ (we first maximize over j and then over $|S|$), s.t. $|S| \geq n\beta^j/\alpha$ and each agent in S has at least $n\beta/(\alpha|S|)$ dollars of type j left. That is, those agents can afford to buy agent t assuming that each of them pays the same amount of money using the coins of type j. If such a triple (j, C, S) exists, then SCGB assigns t to C. If multiple such triples exist, SCGB uniformly assigns t to some coalition with the *least* total budget of type j. In both cases, each agent in S pays $n\beta/(\alpha|S|)$ dollars for t. If such a triple does not exist, then a new coalition $\{t\}$ is created. As each agent has β dollars in total, then buying each agent costs $n\beta/\alpha$ dollars and thus SCGB assigns at most α agents to each coalition. In Appendix L, we show that:

Theorem 7. *SCGB returns a $\lceil w(\alpha) \rceil^2$-CEJR partition.*

Proportionality under Uncertainty

Surprisingly, when friendships are *uncertain*, our MPCF algorithm is also *optimal* for guaranteeing CEJR and CPJR in the CLV model. That is, the probability that the partition produced by MPCF satisfies CEJR (CPJR) *dominates* the probability that the partition generated by *any* other algorithm $\mathcal{A}$ satisfies CEJR (CPJR). The proof in Appendix I stems from minor modifications of the proof for Theorem 2.

Theorem 8. *Under the CLV model, let $\mathbf{p} \in [0, 1]^n$ be a weight vector and let $\mathcal{A}$ be an online algorithm for our problem. Then, $\mathbb{P}[\mathcal{A}(G_\mathbf{p})$ is CEJR$] \leq \mathbb{P}[\mathcal{A}^\star(G_\mathbf{p})$ is CEJR$]$. In fact, $\mathbb{P}[\mathcal{A}(G_\mathbf{p})$ is CPJR$] \leq \mathbb{P}[\mathcal{A}^\star(G_\mathbf{p})$ is CPJR$]$.*

Corollary 2. ***Under the CLV model, MPCF is* optimal *in terms of* both *social welfare and proportionality.***

Our strong positive result does not generalize to cases where friendships are revealed. In such settings, though MPCF is almost optimal in terms of social welfare by Remark 1, the following example shows that the partition produced by MPCF does *not* necessarily satisfy CPJR or CEJR:

Example 1. *Our example is inspired by (**?**, Theorem 3). For $\alpha \geq 3$, consider that $n = \alpha + 1$, agent $\alpha + 1$ is the only friend of agent 1 and the agents $2, \ldots, \alpha$ are all friends with each other. Then, MPCF will return the partition $\mathcal{C} = ([\alpha])$ consisting of a single coalition. Note that $\{1\}$ is a 1-cohesive coalition, yet $v_i(C) = 0$, and thus yielding that the partition $\mathcal{C}$ is neither CPJR nor CEJR.*

Conclusions and Future Work

We have explored an online variant of partitioning agents in an undirected social network into coalitions of a bounded size. Initially, we gave the first results for maximizing social welfare in online hedonic games where algorithms have access to (possibly machine-learned) predictions, capturing uncertainty. Our work also initiated the study of lifting proportionality axioms from elections to hedonic games. We first analyzed the notions of CPJR and CEJR in scenarios where friendships are revealed. When friendships are *uncertain*, our MPCF algorithm is *optimal* in terms of *both* social welfare and proportionality for a vast family of natural random graphs. Our results can be seen as evidence that predictions are a promising tool for improving algorithms in online hedonic games, even if predictions are slightly noisy. Our work opens the way for many future studies. Immediate directions are exploring other classes of hedonic games in online settings and studying proportionality in general uncertain domains. It is also appealing to examine scenarios where assignments may be postponed, agents may be reassigned after each arrival, or both.

Possimus sunt quae, nihil vel quas impedit dignissimos exercitationem?Quaerat eveniet veritatis doloremque numquam rerum reprehenderit magnam ad repudiandae maxime voluptates, numquam facilis vitae id non, voluptate veritatis impedit error dolorum officiis atque similique mollitia sed facilis?Error consequuntur temporibus deleniti distinctio ut, ad sint ipsum aliquam nam consectetur officiis, qui repellat repellendus distinctio enim recusandae reiciendis eum veritatis quaerat, voluptates magnam voluptate cumque suscipit quod, dicta amet labore nisi veniam dolores explicabo quam eveniet ipsum illum?Quam ipsa consectetur similique officia quasi, ea praesentium eveniet explicabo labore nemo, quasi saepe a dolores neque laboriosam rem obcaecati, necessitatibus laboriosam nesciunt illo eaque pariatur, error quis voluptatum maxime?