- Let A be the set of un-bribed voters who awarded p at most $\alpha_m - 1$ points. If $|A| \le \epsilon^{-1}N$, simply bribe all voters in A and let them move p to the top position. p is now ranked top by all voters and received the maximum score obtainable and by the co-winner assumption she wins.
- Otherwise, let $B \subseteq A$ be the set un-bribed voters which gave p at most $(1-\epsilon)\alpha_m$ points. If $|B| \ge N$ we are done: we can just bribe N of them and let them move p to the top position, thus decreasing g by $\epsilon\alpha_m N$ points.

If the two above cases do not hold, then it holds that $|A| > \epsilon^{-1}N$, but $|B| < N$. In words, there are at most $|B| < N$ voters who gave p at most $(1-\epsilon)\alpha_m$ points, and $|A \setminus B| > 5N$ voters who gave p more than $(1-\epsilon)\alpha_m$ points. For the time being, we bribe the voter-set B using the method described in the latter item; we will shortly bribe another N voters as well.

Assume we have bribed B; then by now all voters have given p more than $(1-\epsilon)\alpha_m$ points. In other words, p's current score is at least $(1-\epsilon)\alpha_m n$. Now randomly pick N voters from $A \setminus B$. Let them all put p in the top position, and rank all other candidates randomly, that is, the ranking of all other candidates will be determined by a random permutation. Now let c be some candidate and define r such that $s'(c) = r\alpha_m n$. In words, c has received $r\alpha_m$ points from each voter *on average*. Now assume for a moment we first *delete* the N voters we bribe, and only then *re-add* the voters with their new ballots.

When we delete N voters, c loses $r\alpha_m N$ points in expectation. Formally, let X_c be the number of points c had actually lost. Then $\mathbb{E}[X_c] = r\alpha_m N$. We want to make sure that c will lose *approximately* $r\alpha_m N$ points. However—as it is many times the case—we are afraid that X_c will diverge too much from $\mathbb{E}[X_c]$. To analyze that, note that we can treat X_c as a sum of independent random variables $X_{v,c}$, where

$$X_{v,c} = \begin{cases} \alpha_{j(v,c)} & \text{if } v \text{ is chosen to be bribed;} \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 6, we get that $X_c \in [r\alpha_m N \pm R_1(\lambda, \alpha_m, r\alpha_m N)] \subseteq [r\alpha_m N \pm R_1(\lambda, \alpha_m, \alpha_m N)]$ with failure probability at most $\mathcal{N}^{-\lambda}$.

When we re-add the bribed voters according to our scheme, c receives a score in $[\bar{\alpha}N \pm R_1(\lambda, \alpha_m, \alpha_m N)]$ points with failure probability at most $\mathcal{N}^{-\lambda}$—again, by a similar application of Corollary 6.

Summing up, after the entire bribery process, c had lost at least $(r\alpha_m - \bar{\alpha})N - 2R_1(\lambda, \alpha_m, \alpha_m N)$ points with failure probability at most $2\mathcal{N}^{-\lambda}$. Using the union-bound, the same can be made to hold for all m candidates simultaneously with failure probability at most $2\mathcal{N}^{-\lambda+1}$.

We can now split to cases; candidates with $r \ge 1 - 4\epsilon$ lost at least $\epsilon\alpha_m N$ points, assuming that $2R_1(\lambda, \alpha_m, \alpha_m N) \le \epsilon\alpha_m N$ (as it is asymptotically; otherwise the entire input is constant-sized). Candidates with $r < 1 - 4\epsilon$ might have gained points in the process, however the number of points gained in the process is bounded by the number of points awarded in the voter re-addition stage. Since the number of these awarded points is at most $\bar{\alpha}N + R_1(\lambda, \alpha_m, \alpha_m N)$, each such candidate c now have score of at most $s''(c) \le (1-4\epsilon)\alpha_m n + \bar{\alpha}N + R_1(\lambda, \alpha_m, \alpha_m N)$. However, since $N \le \epsilon n$ (follows by the fact that $n \ge |A| > \epsilon^{-1}N$), and $R_1(\lambda, \alpha_m, \alpha_m N) \le \epsilon\alpha_m N < \epsilon\alpha_m n$,

$$\begin{aligned} s''(c) &\le (1-4\epsilon)\alpha_m n + \bar{\alpha}N + R_1(\lambda, \alpha_m, \alpha_m N) \\ &\le (1-4\epsilon)\alpha_m n + \bar{\alpha}\epsilon n + \epsilon\alpha_m n \\ &< (1-4\epsilon)\alpha_m n + \alpha_m \epsilon n + \epsilon\alpha_m n \\ &= (1-2\epsilon)\alpha_m n \le s'(p) - \epsilon\alpha_m n\ . \end{aligned}$$

We conclude that after this process, every candidate either lost $\epsilon\alpha_m N$ points, or gained points, but in that case never surpassed $s'(p) - \epsilon\alpha_m n \le s''(p) - \epsilon\alpha_m n$. The amount of voters we have bribed is $|B| + N \le 2N < \epsilon^{-1}N$. The lemma thus follows. □

With Lemmas 13 and 14, we have just shown that for many types of $\boldsymbol{\alpha}$, the ratio between a margin to the number of bribed voters needed in order to close the margin is $O(\alpha_m)$. This leads to the following:

Lemma 15. *Assuming that Lemma 12 did not fail, then besides the $\tilde{k} = k^\star + f$ voters we have already bribed, with failure probability at most $\lceil R_4/(\epsilon\alpha_m \ln^{1+\delta}\mathcal{N})\rceil \cdot 2\mathcal{N}^{-\lambda+1}$, it holds that at most $f' = \epsilon^{-2}R_4/\alpha_m + \epsilon^{-1}\ln^{1+\delta}\mathcal{N}$ additional voters are needed to be bribed in order for p to win, for some constant $\epsilon > 0$.*

Proof. By repeatedly applying the algorithm in the constructive proof of either Lemma 13 or Lemma 14, until p wins. For constant scoring rules the analysis is straightforward. For non-concentrated scoring rules, since every batch of $\epsilon^{-1}N = \epsilon^{-1}\ln^{1+\delta}\mathcal{N}$ bribed voters decrease the margin by at least $\epsilon\alpha_m N$ points, at most $f' = \lceil R_4/(\epsilon\alpha_m N)\rceil \cdot \epsilon^{-1}N$ bribed voters are needed.

As for the failure probability, we can be conservative and require that each of the $\lceil R_4/(\epsilon\alpha_m N)\rceil$ iterations will succeed; using the union-bound, the probability any of the iterations will fail is at most $\lceil R_4/(\epsilon\alpha_m N)\rceil \cdot 2\mathcal{N}^{-\lambda+1}$. □

We are now ready to complete the proof for Theorem 1.

Proof of Theorem 1. Let $\bar{k}$ be the number of voters bribed by an optimal strategy, and notice that $k^\star \le \bar{k}$, since the LP is a relaxation of the original problem. Following the above discussion, we had bribed overall $k^\star + f + f' \le \bar{k} + f + f'$ voters. For the sake of brevity, and since our concern is order of magnitude analysis, we will only loosely bound both the approximation factor $f + f'$ and the failure probability.

Since R_4 can be loosely bounded by $41\lambda^2\alpha_m(\bar{k}+1)^{1/2}\ln^2\mathcal{N}$, then $f + f'$ is bounded by $43\lambda^2\epsilon^{-2}(\bar{k}+1)^{1/2}\ln^2\mathcal{N} = \widetilde{O}(\sqrt{\bar{k}})$. As for the failure probability, we require both Lemmas 12 and 15 to succeed; the probability any of them would fail is at most $6\mathcal{N}^{-\lambda+1} + \lceil R_4/(\epsilon\alpha_m N)\rceil \cdot 2\mathcal{N}^{-\lambda+1} \le (48\lambda^2\epsilon^{-1}\ln\mathcal{N}) \cdot (\bar{k}+1)^{1/2} 2\mathcal{N}^{-\lambda+1} = \widetilde{O}(\bar{k})/\mathcal{N}^{\lambda-1}$. Setting $\lambda = 3$ will thus provide at most $1/\Omega(\mathcal{N})$ failure probability, since $\bar{k} \le n$.

By running the algorithm a linear number of times, and choosing the run yielding minimal number of bribed voters, the failure probability becomes exponentially-small, while the runtime stays polynomial. □

This work was supported by the Israel Science Foundation, under Grant No. 1488/14 and Grant No. 1394/16.