section, we thus lift proportionality axioms from the multi-winner voting realm to hedonic games.

For evaluating and formulating new variants of proportionality that are suitable for *simple* and *symmetric* ASHGs, we assume the number of agents n is *known*. First, we define the notion of *coalitional cohesiveness*, which extends the notion of cohesiveness in elections (**?**) to hedonic games. Cohesiveness dictates that the preferences within each subgroup of any coalition are sufficiently aligned. Recall that F_i is agent i's neighborhood in the social network at time t. Given an integer $\alpha \geq 2$, for any $C \in \mathcal{C}^t$ and each $m \in [\alpha]$, we say that a subgroup $S \subseteq C$ is *m-coalitionally cohesive* (or *m-cohesive* for short) if $|S| \geq m \cdot \frac{n}{\alpha}$ (i.e., S is large enough) and $|\cap_{i \in S} F_i| \geq m$ (i.e., S's members have at least m common neighbors). We also adapt the *approximate* variant of cohesiveness explored by (**?**). Given $\eta \geq 1$, the above subgroup of agents $S \subseteq C$ is *(η, m)-cohesive* if $|S| \geq \eta \cdot m \cdot \frac{n}{\alpha}$ and $|\cap_{i \in S} F_i| \geq m$.

Next, we redefine and study notions of proportionality in the context of hedonic games, that are commonly considered in the committee elections literature, which include proportional justified representation (PJR) (**?**) and extended justified representation (EJR) (**?**), as well as their approximate versions. The extension of proportional justified representation (PJR) (**?**) to hedonic games demands that the preferences of sufficiently large and cohesive groups of agents shall not be disregarded by the resulting partition, but shall be adequately represented in each coalition. Formally, coalitional PJR is defined as follows:

Definition 1. *Given $\eta \geq 1$, the partition $\mathcal{C}^t$ at time t satisfies* η-coalitional PJR *(η-**CPJR**) if for each coalition $C \in \mathcal{C}^t$, for all $m \in [\alpha]$ and for each (η, m)-cohesive subgroup of agents $S \subseteq C$, it holds that the agents within S are jointly friends of at least m agents within C, i.e., $\sum_{i \in S} v_i(C) \geq m$.*

The stronger notion of coalitional EJR (**?**) demands that there exists at least one agent in each large enough and cohesive coalition with not just one, but several friends within the coalition. Formally:

Definition 2. *Given $\eta \geq 1$, the partition $\mathcal{C}^t$ at time t satisfies* η-coalitional EJR *(η-**CEJR**) if for each coalition $C \in \mathcal{C}^t$ it holds that for each $m \in [\alpha]$ and each (η, m)-cohesive subgroup of agents $S \subseteq C$ there exists an agent $i \in S$ that is friends with at least m agents within C, i.e., $v_i(C) \geq m$.*

If a partition satisfies 1-CPJR, then we say that it satisfies CPJR for brevity. CEJR is defined similarly. By (**?**), CPJR is implied by CEJR in the committee elections setting, and thus CEJR provides a stronger axiomatic guarantee of proportionality. Next, we analyze CPJR and CEJR when friendships are revealed, where CPJR can be easily attained under mild assumptions, while CEJR can be approximated. When friendships are *uncertain*, we show that our MPCF algorithm is *optimal* under the CLV model, yet it does not always satisfy CPJR and CEJR under full certainty.

Proportionality when Friendships are Revealed

We first study CPJR and CEJR in scenarios *without uncertainty* (i.e., friendships are revealed). In such settings, at each time t and for each agent $i \in [t]$, let $\mathcal{N}_i^t = \{C \subseteq [t] : i \in C\}$ be all possible coalitions containing agent i and let F_i be agent i's friends in the social network. Thus, agent t arrives along with v_t^t, where $v_i^t : \mathcal{N}_i^t \to \{0, 1\}$ is the valuations of agent $i \in [t]$ to the agents *who arrived until time* t. Her utility for a coalition $C \subseteq [t]$ is the number of her friends within C, i.e., $v_i(C) = |F_i \cap C|$. $\mathbf{v}^t = (v_i^t)_{i \in [t]}$ is the joint valuation of the agents *who arrived until time* t.

Achieving CPJR. CPJR can be easily satisfied by an adaption of the *method of equal shares* (MES) (**??**), which we term *coalitional MES* (**CMES**) and it executes as follows. Each agent i has an initial budget of *one* dollar. Agent i spends her money across the execution by buying any agent j she approves of (i.e., $v_i(j) = 1$). Namely, at time t, agent t is assigned to a coalition whose members that approve t have at least n/α dollars in total, where those members are then asked to jointly pay n/α dollars. If multiple coalitions satisfying this property exist, CMES uniformly assigns t to some coalition with the *least* total budget. If such a coalition does not exist, then a new coalition $\{t\}$ is created. Each agent's payment can be determined similarly to the classical MES, though the algorithm remains unaffected by the spread of n/α among the agents. As agents pay n/α dollars in total for each agent assigned to an already existing coalition, CMES assigns at most α agents to each coalition. We now prove that:

Theorem 4. *The partition returned by CMES is CPJR.*

Proof. By contradiction, we assume that there are a coalition $C \in \mathcal{C}^n$, $m_C \in [\alpha]$ and an m_C-cohesive subgroup of agents $S_C \subseteq C$ such that $|\cup_{i \in S_C} F_i \cap C| \leq m_C - 1$. At each time t where agent t is assigned to C, the agents within S_C that approve t pay exactly n/α dollars, and since $|\cup_{i \in S_C} F_i \cap C| \leq m_C - 1$, they pay at most $(m_C - 1) \cdot n/\alpha$ in total. Due to $|S_C| \geq m_C \cdot n/\alpha$, the agents within S_C have at least n/α dollars at each time instant. Hence, at each time t where $t \in \cap_{i \in S_C} F_i$, those agents have enough money for buying t. Thereby, each candidate in $\cap_{i \in S_C} F_i$ will be assigned to some existing coalition. There are *at least* m_C such agents. However, as $|\cup_{i \in S_C} F_i \cap C| \leq m_C - 1$ there are *at most* $m_C - 1$ such agents, which therefore constitutes a contradiction completing the proof. □

Achieving CEJR. We begin with an impossibility result which depicts that gaining η-CEJR is indeed much harder. Even when α-bounded partitions can only contain a *single* coalition, our problem can be viewed as a special case of online approval committee elections (**?**), where the single coalition generated by an online algorithm constitutes the winning committee. Letting $H(\alpha)$ be the α-th harmonic number (i.e., $H(\alpha) = \sum_{j=1}^{\alpha} 1/j$), by Do et al. (**?**, Theorem 5.3) we thus infer the following lower bound:

Corollary 1. *There exists no deterministic online algorithm that generates a $(1-\varepsilon)H(\alpha)$-CEJR partition for any $\varepsilon > 0$.*

Adapting the algorithm in (**?**, Section 5.2) for committee elections, we next provide the optimal *Greedy Coalitionally Cohesive* (**GCC**) scheme, i.e., GCC satisfies $H(\alpha)$-CEJR.

Informally, GCC assigns each agent i to a coalition containing a sufficiently large number of agents that are friends of i. At time t, if there exists a coalition $C \in \mathcal{C}^t$ and $S \subseteq C$ with $S \subseteq N_t^t$ and $|S| \geq H(\alpha) \cdot m \cdot n/\alpha$ s.t. each agent in S is a neighbor of less than m agents assigned to C thus far, then GCC assigns t to C. If multiple such coalitions exist, GCC uniformly assigns t to the coalition of *smallest* cardinality. If such a coalition does not exist, then a new coalition $\{t\}$ is created. We now prove that:

Theorem 5. *The partition returned by GCC is $H(\alpha)$-CEJR and each coalition contains at most α agents.*

Proof. (*Sketch*) By construction, the partition is $H(\alpha)$-CEJR. Using a budgeting argument, we prove in Appendix J that each coalition contains at most α agents. □

Next, we further illuminate on the complexity of obtaining a partition satisfying η-CEJR. We remark that GCC requires checking whether a coalition is (η, m)-cohesive, where verifying its existence is generally NP-hard. The proof in Appendix K is by reduction from Maximum k-Subset Intersection (**?**).

Theorem 6. *Checking whether there exists an m-cohesive coalition is NP-hard.*

Theorem 6 dictates that GCC cannot be executed in polynomial time, and thus achieving an *optimal* partition as stated by Corollary 1 is challenging. Hence, we supply *Sub-Coalitions by Greedy Budgeting* (**SCGB**), a polynomial-time algorithm that yields a slightly worse CEJR guarantee than GCC. Our scheme adapts the algorithm in (**?**, Section 5.3) for committee elections to hedonic games. First, let $w(\cdot)$ be the inverse function of $x \mapsto x^x$, i.e., $w(\alpha) = x$ if $\alpha = x^x$. Note that $w(\alpha) = O(\log \alpha)$ and $\log \alpha = O(w(\alpha)^2)$. Let $\beta = \lceil w(\alpha) \rceil$. SCGB independently creates β sub-coalitions generated similarly to CMES, each of size $\lfloor \alpha/\beta \rfloor$. Formally, each agent is given an initial budget of $(1, \ldots, 1) \in [0,1]^\beta$, i.e., there are β independent dollars where each one is associated with a specific possible sub-coalition. The j^{th} coin can be used for buying agents who are approved by at least $n\beta^j/\alpha$. Each agent costs $n\beta/\alpha$ dollars. At time t, we find the largest triple $j \in [\beta]$ and a coalition $C \in \mathcal{C}^t$ with $S \subseteq C$ satisfying $S \subseteq N_t^t$ (we first maximize over j and then over $|S|$), s.t. $|S| \geq n\beta^j/\alpha$ and each agent in S has at least $n\beta/(\alpha|S|)$ dollars of type j left. That is, those agents can afford to buy agent t assuming that each of them pays the same amount of money using the coins of type j. If such a triple (j, C, S) exists, then SCGB assigns t to C. If multiple such triples exist, SCGB uniformly assigns t to some coalition with the *least* total budget of type j. In both cases, each agent in S pays $n\beta/(\alpha|S|)$ dollars for t. If such a triple does not exist, then a new coalition $\{t\}$ is created. As each agent has β dollars in total, then buying each agent costs $n\beta/\alpha$ dollars and thus SCGB assigns at most α agents to each coalition. In Appendix L, we show that:

Theorem 7. *SCGB returns a $\lceil w(\alpha) \rceil^2$-CEJR partition.*

Proportionality under Uncertainty

Surprisingly, when friendships are *uncertain*, our MPCF algorithm is also *optimal* for guaranteeing CEJR and CPJR in the CLV model. That is, the probability that the partition produced by MPCF satisfies CEJR (CPJR) *dominates* the probability that the partition generated by *any* other algorithm $\mathcal{A}$ satisfies CEJR (CPJR). The proof in Appendix I stems from minor modifications of the proof for Theorem 2.

Theorem 8. *Under the CLV model, let $\mathbf{p} \in [0,1]^n$ be a weight vector and let $\mathcal{A}$ be an online algorithm for our problem. Then, $\mathbb{P}[\mathcal{A}(G_\mathbf{p})$ is CEJR$] \leq \mathbb{P}[\mathcal{A}^\star(G_\mathbf{p})$ is CEJR$]$. In fact, $\mathbb{P}[\mathcal{A}(G_\mathbf{p})$ is CPJR$] \leq \mathbb{P}[\mathcal{A}^\star(G_\mathbf{p})$ is CPJR$]$.*

Corollary 2. ***Under the CLV model, MPCF is* optimal *in terms of* both *social welfare and proportionality.***

Our strong positive result does not generalize to cases where friendships are revealed. In such settings, though MPCF is almost optimal in terms of social welfare by Remark 1, the following example shows that the partition produced by MPCF does *not* necessarily satisfy CPJR or CEJR:

Example 1. *Our example is inspired by (**?**, Theorem 3). For $\alpha \geq 3$, consider that $n = \alpha + 1$, agent $\alpha + 1$ is the only friend of agent 1 and the agents $2, \ldots, \alpha$ are all friends with each other. Then, MPCF will return the partition $\mathcal{C} = ([\alpha])$ consisting of a single coalition. Note that $\{1\}$ is a 1-cohesive coalition, yet $v_i(C) = 0$, and thus yielding that the partition $\mathcal{C}$ is neither CPJR nor CEJR.*

Conclusions and Future Work

We have explored an online variant of partitioning agents in an undirected social network into coalitions of a bounded size. Initially, we gave the first results for maximizing social welfare in online hedonic games where algorithms have access to (possibly machine-learned) predictions, capturing uncertainty. Our work also initiated the study of lifting proportionality axioms from elections to hedonic games. We first analyzed the notions of CPJR and CEJR in scenarios where friendships are revealed. When friendships are *uncertain*, our MPCF algorithm is *optimal* in terms of *both* social welfare and proportionality for a vast family of natural random graphs. Our results can be seen as evidence that predictions are a promising tool for improving algorithms in online hedonic games, even if predictions are slightly noisy.

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