# **3. Method**

This section details the trajectory-reconstruction procedure used to correct noisy airborne radar tracks. Specifically, we cast post-processing on a fixed time grid as a convex program whose state consists of position–velocity variables per node, constrained by linearized point-mass dynamics with constant drag and standard flat-Earth conversions for short ranges. The objective balances four quadratic criteria, sensor fit, curvature suppression, inter-node total-variation regularization, and terminal accuracy, while an impact-angle bound is imposed via a single second-order cone when active. The resulting inner problem is a QP (or QP with one SOC), solved efficiently and warm-started across grid points. Moreover, a single scalar drag parameter is selected by a light outer search on a logarithmic grid with parabolic refinement. The subsections that follow introduce the notation and operators (Section 3.1), formalize the optimization model and constraints (Section 3.2), and describe the bi-level tuning and implementation details (Section 3.3), followed by an algorithmic summary.

**Table 1.** Symbols used in this study.

| Symbol | Meaning | Units |
| --- | --- | --- |
|  | time samples, | s |
|  | step, | s |
|  | position at | m |
|  | velocity at | m s |
|  | node state | mixed |
|  | stacked decision vector | mixed |
|  | radar-derived positions | m |
|  | target reference | m |
|  | target altitude reference | m |
|  | gravitational acceleration | m s |
|  | impact angle bound | deg |
|  | drag coefficient | s |
|  | objective weights | dimensionless |
|  | latitude (rad), longitude (rad), altitude (m) | rad, rad, m |
|  | reference geodetic point | rad, rad, m |
|  | local flat-Earth coordinates | m |
|  | meridional radius of curvature | m |
|  | prime-vertical radius of curvature | m |
|  | Earth equatorial radius (WGS-84) | m |
|  | Earth eccentricity squared (WGS-84) | dimensionless |

## 3.1. Preliminaries

We estimate a physically plausible munition trajectory by post-processing noisy airborne radar tracks on a fixed time grid. The trajectory is obtained as the unique optimizer of a convex program whose constraints encode linearized point-mass dynamics with constant drag over each step, and whose objective balances sensor fit, smoothness, total variation, and terminal accuracy. A single scalar drag parameter is tuned by a light outer search.

We adopt a WGS-84 based flat-Earth approximation. Let denote latitude (rad), longitude (rad), and altitude (m). Around a reference , local coordinates are

Where

with , . For distances km and , Jacobian errors are below a meter, so the frame can be treated as locally Cartesian.

We collect radar-derived positions at times using the columns lat\_s*,* lon\_s*,* alt\_s from the input file after applying (1). Target references are and .

We collect radar-derived positions at times . Target references are and . The decision variables per node are and , with and the stacked .

Gravity is . Let extract , extract , and extract horizontal components. Difference operators implement first- and second-order finite differences on the time grid. Norms are .

## 3.2. Optimization model

*3.2.1. General form*

For , equations (1)-(4) define an affine subspace that encodes the kinematics for a fixed drag . The objective cost function is the sum of four quadratic terms:

where extracts the horizontal coordinates of the terminal state. The four terms correspond respectively to sensor fit, curvature suppression, velocity total variation, and terminal box accuracy.

We impose the following boundary and terminal constraints:

Constraint (8) enforces the maximum allowed terminal impact angle relative to vertical. It is formulated as a second-order cone in and excludes non-descending terminal velocities.

#### 

*3.2.2. Compact form*

Let encode (2)-(3) and [(6)](#eq:launch), encode (7), and encode (8) with the second-order cone . Let and represent the quadratic and linear terms of (4). The inner problem is

When the model is a QP. When and the impact angle is enforced, the model is a QP with a single SOC constraint.

*3.2.3. Algorithmic flow*

The model takes as inputs a structured dataset together with physical and algorithmic parameters. Let denote the maximum impact angle, the gravitational constant, and the search interval for the drag parameter. Additional hyperparameters include the weights and scaling factors . The goal is to produce as outputs a tuned drag , a corrected trajectory , and the corresponding optimal cost .

Given these inputs, the procedure first parses and converts the raw geodetic coordinates into SI units relative to a reference point, forming the radar track , the target reference , and the terminal altitude . A scaling step then maps positions and velocities to , ensuring well-conditioned dynamics with scaled gravity . With these scaled variables, the inner problem is assembled by constructing sparse banded matrices for the dynamic equations, boundary conditions, terminal box, and (if active) the impact-angle SOC. The Hessian is built directly from finite-difference operators corresponding to curvature and total-variation penalties.

The outer search proceeds in two phases. First, a logarithmic grid is evaluated, each instance solved efficiently by warm-starting OSQP (QP case) or ECOS (SOCP case). This yields candidate values and trajectories . Second, a parabolic refinement step identifies the best triplet, fits a quadratic in , and updates toward the analytic minimizer, discarding infeasible values if necessary. This refinement iterates until changes in both and fall below tolerance. Finally, the optimal scaled solution is mapped back to physical units via , and performance is reported in terms of terminal miss and impact angle. By construction, the procedure returns

ensuring that the reconstructed trajectory is globally optimal and consistent with both radar measurements and physical constraints.

## 3.3. Correctness

#### Here, we establish the mathematical correctness of the proposed optimization model. We proceed in three steps: strict convexity under partial observations, convexity of the feasible set and objective, uniqueness of the minimizer, and continuity of the optimal value with respect to the drag parameter . We assumed a bounded time-steps for all…

#### Lemma 1. Suppose and . Then for any nonzero sequence with and satisfying the homogeneous dynamics (2)-(3), at least one of the penalties or is strictly positive.

*Proof.* If then for all , hence is constant. Combined with and the homogeneous form of (2)-(3), constant forces and , contradicting nonzeroness. If then is affine in , that is, is constant. By (2) this constant equals . With variable , constancy of implies cannot satisfy the homogeneous version of [(3)](#eq:dyn-v) unless on the grid, again leading to . Thus at least one penalty is strictly positive.

#### Proposition 1 (Convexity). For any fixed , the feasible set of (8) is convex and the objective is convex. Hence the inner problem is a convex QP if (7) is omitted, and a convex SOCP otherwise

*Proof.* The equality set is affine. The inequalities are linear, hence convex. The impact angle constraint is , which defines a second-order cone. The objective is a sum of squared norms with nonnegative weights. Each term is convex and separable across nodes. Consequently, the Hessian is positive semidefinite. The sum of convex functions is convex, so the objective is convex on . Intersection of convex sets preserves convexity, which concludes the proof.

#### Proposition 2 (Uniqueness).

*Let . If and at least one observation is available at each time, or if and with the launch fix , then is positive definite on . Consequently, the optimizer of (8) is unique.*

*Proof.* Consider any nonzero . If , then unless all position components vanish. If all , then by (2) the corresponding velocity components must be zero as well. Hence , contradiction. Alternatively, under and fixed, Lemma 1 implies that at least one of the sums or is strictly positive for any nonzero feasible perturbation. Since these sums appear in the objective with positive weights, holds for all nonzero . Standard results on equality-constrained convex quadratic programs then yield uniqueness of the minimizer.

#### Proposition 3 (Continuity of ).

Assume that for all the feasible set of (8) is nonempty, and that the uniqueness conditions in Proposition 2 hold. Then is continuous on . Moreover, the optimizer is continuous in .

*Proof.* The data of (8) depend continuously on only through and , which are affine in by inspection of (3). The objective has a Hessian that is independent of and positive definite on by Proposition 2. Parametric convex optimization results for equality-constrained strictly convex quadratic programs assert continuity of both the optimizer and the optimal value under continuous perturbations of the data and persistence of feasibility. Therefore and are continuous in on the compact interval .

#### Guarantees.

Convexity of the inner program ensures global optimality for fixed . Proposition 2 provides sufficient conditions for uniqueness, which we satisfy in practice by setting and with the launch fix. Proposition 3 ensures that the outer tuning over is well-posed and that a simple bracketing plus parabola scheme is justified. Together these facts guarantee that the procedure returns a single corrected trajectory that is consistent with the data, the dynamics, and the terminal geometry.

The method relies on short-range flat-Earth conversion, fixed grid, and constant drag per step. If longer ranges or variable drag are required, the same structure extends with piecewise-constant drag or with a lifted state that remains convex, at the expense of a larger cone system.

# **4. Empirical Study**

This section evaluates the proposed radar-only trajectory filter on flight records containing aircraft-borne radar measurements and independent telemetry. We use two complementary streams collected during operational sorties:

* **Telemetry (truth).** Time-stamped records of the weapon state and target parameters. From raw fields (e.g., Weapon Downtrack/Crosstrack (ft), Weapon ALT (ft), Target LAT/LON/ALT), we reconstruct the geodetic path relative to the target and release point.
* **Airborne radar (observations).** Host-aircraft navigation (LAT/LON/ALT, Heading, Pitch, Roll) and radar line-of-sight (Antenna Azimuth/Elevation, Slant Range). A rigid-body chain maps the radar beam to the world frame to obtain a radar-only estimate of the weapon trajectory.

## 4.1. Experimental Setup

All radar and telemetry streams are first resampled to a fixed increment s using linear interpolation for each position component. Outliers are masked by an interquartile rule with factor (per-axis), applied to first differences before resampling. A Savitzky–Golay filter with window length (odd) and polynomial order is then applied per axis, chosen to match sensor bandwidth while preserving terminal dynamics. These hyperparameters are fixed *a priori* for all sorties.

After preprocessing, we compute and pass to the optimizer the following features: forward-difference velocities, curvature terms , total-variation increments (and the analogous velocity increments), and terminal descriptors including distance-to-target, miss-vector components, and an impact-angle proxy. For numerical conditioning, positions and velocities are normalized by scaling factors and respectively, so that and .

For each sortie, we solve the convex program described in Section 3, using the exact discrete dynamics with constant drag and, when active, a second-order cone constraint on the terminal impact angle. To capture aerodynamic variability, an outer search is performed over s using a logarithmic grid (about points) and feasibility-aware parabolic refinement. We compare three reconstruction variants:

* **Raw radar (baseline):** smoothed radar track only, without dynamic consistency.
* **QP (no cone):** full quadratic objective with linear dynamics and boundary constraints.
* **QP+SOCP (proposed):** QP with the terminal impact-angle cone and bi-level tuning of .

Unless otherwise noted, weights are fixed to , , , and . Optimization is performed with OSQP for the QP case and ECOS for the SOCP case. Both solvers operate on scaled variables with tolerances in normalized units.

## 4.2. Evaluation Metrics

Let denote the reconstructed (filtered) position and the telemetry truth at time . For sortie , we define the as , and the t as . From these errors we report:

* **CEP-50:** the 50th percentile of across sorties
* **CEP-90:** the 90th percentile of
* **LE90:** the 90th percentile of
* **RMSE:** full-trajectory accuracy:
* where is the number of aligned samples for sortie .
* **Runtime:** mean wall-clock time per inner solve and per -sweep.

## 4.3. Implementation details and reproducibility

All experiments use synchronized radar and telemetry streams, resampled and filtered as described in Section 4.1. The drag parameter is tuned using the feasibility-aware outer loop, with OSQP used for QPs and ECOS for SOCPs.

For reproducibility, each experiment logs the preprocessing hyperparameters, the scaling factors , the weight vector , and solver tolerances. The random seed, input file hash, and KKT sparsity pattern are stored with results. All reported values of are in SI units; trajectories are returned in meters and meters per second. Under these conventions, results are deterministic and reproducible given the same input and configuration.

# **5. Results**

This section evaluates the proposed optimization pipeline along four axes: experimental setup and reporting metrics, kinematic consistency (speed and acceleration), geospatial accuracy (distributional and pointwise error), and qualitative trajectory behavior. Unless noted otherwise, distances are in meters (m), speeds in meters per second (m/s), and time in seconds (s). Percentile accuracy follows the conventional circular–error–probable notation (CEP–).

**5.1. Aggregate Accuracy**

Table [1](#tab:kpi) summarizes the primary indicators for the *raw* and *optimized* trajectories. The optimization improves median and tail accuracy while keeping the motion model intact. Specifically, CEP–50 decreases from  m to  m (), and CEP–90 from  m to  m (). The optimized trajectory attains an RMSE of  m. The terminal miss distance improves from  m (raw) to  m (optimized), i.e., .

=== Enter **Table 1** here ===

*Key performance indicators (KPI). CEP and RMSE in meters (m).*

|  |  |  |
| --- | --- | --- |
| **Metric** | **Raw** | **Optimized** |
| CEP–50 | 4582.6 | 4231.2 |
| CEP–90 | 5938.5 | 5745.1 |
| Final error | 841.2 | 757.0 |
| RMSE | — | 17566.45 |

Distributional and Pointwise Behavior

The error distributions (Figure [[fig:hist]](#fig:hist)) shift left after optimization, with a visibly truncated tail—fewer extreme deviations dominate the overall RMSE. In particular, the interquartile mass contracts and the CEP markers move closer to the origin, consistent with the aggregate gains in §[1.1](#subsec:aggregate). Pointwise evolution is shown in Figure [[fig:dist-time]](#fig:dist-time). After an initial transient, the distance–to–target decays smoothly toward the terminal segment; the optimized trace remains uniformly below the raw trace in the final approach window. The final–error markers confirm the reduction.

=== Enter **Fig. 4** (error histograms) here ===

=== Enter **Fig. 5** (distance–to–target vs. time) here ===

Trajectory and Kinematic Diagnostics

Figure [[fig:panel]](#fig:panel) provides a consolidated view. The raw ground track exhibits a large off–track excursion and a sharp recovery; the optimized track remains within a narrow corridor aligned with the terminal approach. The altitude profile corroborates the same effect: the raw reconstruction contains a severe, nonphysical plunge, whereas the optimized profile realizes a smooth descent with a shallow overshoot (see also Figure [[fig:alt]](#fig:alt)). A 3D geospatial view (Figure [[fig:geo3d]](#fig:geo3d)) places these behaviors in context—optimization straightens the inbound leg and damps local oscillations near the target while preserving the global shape.

Kinematics remain well behaved. The comprehensive panels in Figure [[fig:speed-panels]](#fig:speed-panels) show stable ground–referenced motion: the mean ground speed is  m/s with only localized fluctuations near the observation boundary; the vertical–speed component is negligible outside a short transient; and the inferred acceleration is close to zero except for brief spikes co–located with that transient. For readability, Figure [[fig:gs-smoothed]](#fig:gs-smoothed) isolates the ground–speed trace with its mean reference line. Hence, geospatial gains are obtained without injecting spurious dynamics.

=== Enter **Fig. 6** (multi–panel: ground track/altitude/distance/metrics) here ===

=== Enter **Fig. 7** (altitude profile: raw vs. optimized) here ===

=== Enter **Fig. 1** (3D lon–lat–alt view) here ===