236330 - Introduction to Optimization: Homework #1

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Analytical Differentiation:

Task 1:

Find the Gradient and Hessian of:

$$f_1(x) = \varphi(Ax)$$

Where,

$$f_1 : \mathbb{R}^n \to \mathbb{R}$$
$$x \in \mathbb{R}^n$$
$$A \in \mathbb{R}^{m \times n}$$
$$\varphi : \mathbb{R}^m \to \mathbb{R}$$

Given $\nabla \varphi$ and $\nabla^2 \varphi$.

Solution:

We start by finding the gradient:

Denoting $u = Ax$. Hence $du = dAx$.

From the external definition of gradient, we know that,

$$d\varphi = \langle \nabla \varphi, du \rangle = \nabla \varphi^T du = \nabla \varphi^T A dx$$

Since $df = \nabla f^T dx$ and $d\varphi = df$ (because $f(x) = \varphi(Ax)$) we get that:

$$g = \nabla f = A^T \nabla \varphi(Ax)$$

Now for the Hessian,

$$dg = d\nabla f = dA^T \nabla \varphi(u) = A^T d\nabla \varphi(u)$$

From the external definition of Hessian we know that $d\nabla \varphi(u) = H du$.

$$A^T d\nabla \varphi(u) = A^T H(u) du = A^T H(Ax) A dx$$

Since $H = \nabla^2 \varphi$, it yields that:

$$H(x) = A^T \nabla^2 \varphi(Ax) A$$

Task 2:

Develop the Gradient and Hessian of:

$$f_2(x) = h(\varphi(x))$$

Where,

$$\begin{gathered} f_2 : \mathbb{R}^n \to \mathbb{R} \\ x \in \mathbb{R}^n \\ \varphi : \mathbb{R}^m \to \mathbb{R} \\ h : \mathbb{R} \to \mathbb{R} \end{gathered}$$

Given $\nabla\varphi,\ \nabla^2\varphi,\ h',\ h''$.

Solution:

We start by finding the gradient:

Denoting $u = \varphi(x)$. From the external definition of the gradient, we know that:

$$du = \langle \nabla\varphi(x), dx \rangle$$

$$dh = h'(u)\, du$$

Combining these two together yields:

$$dh = h'(u)\langle \nabla\varphi(x), dx\rangle = \langle h'(u)\nabla\varphi(x), dx\rangle = \langle h'(\varphi(x))\nabla\varphi(x), dx\rangle$$

Since $df = \nabla f^T dx$ we get that:

$$g = \nabla f = \left(h'(\varphi(x))\nabla\varphi(x)\right)^T \underset{(1)}{=} h'(\varphi(x))\nabla\varphi(x)$$

(1) - Since $h'(\varphi(x))$ is a scalar and $\nabla\varphi(x) = \nabla\varphi^T(x)$.

Now for the Hessian,

$$\begin{aligned}
dg &= d\nabla f = d\left(h'\left(u\right)\nabla\varphi\left(x\right)\right)\\
&= dh'\left(u\right)\nabla\varphi\left(x\right) + h'\left(u\right)d\nabla\varphi\left(x\right)\\
&= h''\left(u\right)du\nabla\varphi\left(x\right) + h'\left(u\right)\nabla^{2}\varphi\left(x\right)dx\\
&\underset{(1)}{=} h''\left(u\right)\nabla\varphi\left(x\right)du + h'\left(u\right)\nabla^{2}\varphi\left(x\right)dx\\
&= h''\left(u\right)\cdot\nabla\varphi\left(x\right)\cdot\left\langle\nabla\varphi\left(x\right),dx\right\rangle + h'\left(u\right)\nabla^{2}\varphi\left(x\right)dx\\
&= h''\left(u\right)\cdot\nabla\varphi\left(x\right)\cdot\nabla\varphi^{T}\left(x\right)dx + h'\left(u\right)\nabla^{2}\varphi\left(x\right)dx\\
&= \left(h''\left(u\right)\cdot\nabla\varphi\left(x\right)\cdot\nabla\varphi^{T}\left(x\right) + h'\left(u\right)\nabla^{2}\varphi\left(x\right)\right)\cdot dx
\end{aligned}$$

(1) - Since $du = \left\langle\nabla\varphi\left(x\right),dx\right\rangle$ is a scalar.

From the external definition of Hessian we know that $dg = Hdx$, hence:

$$H = h''\left(u\right)\cdot\nabla\varphi\left(x\right)\cdot\nabla\varphi^{T}\left(x\right) + h'\left(u\right)\nabla^{2}\varphi\left(x\right)$$

Task 3:

1. Let:

$$\varphi\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right) = sin\left(x_1x_2x_3\right)$$

The gradient of φ is:

$$\nabla\varphi\left(x_1,x_2,x_3\right) = \begin{pmatrix}\frac{\partial\varphi}{\partial x_1}\\\frac{\partial\varphi}{\partial x_1}\\\frac{\partial\varphi}{\partial x_1}\end{pmatrix} = \begin{pmatrix}x_2x_3cos\left(x_1x_2x_3\right)\\x_1x_3cos\left(x_1x_2x_3\right)\\x_1x_2cos\left(x_1x_2x_3\right)\end{pmatrix}$$

The Hessian of φ is:

$$H = \begin{pmatrix}\frac{\partial^2\varphi}{\partial x_1^2} & \frac{\partial^2\varphi}{\partial x_1\partial x_2} & \frac{\partial^2\varphi}{\partial x_1\partial x_3}\\\frac{\partial^2\varphi}{\partial x_2\partial x_1} & \frac{\partial^2\varphi}{\partial x_2^2} & \frac{\partial^2\varphi}{\partial x_2\partial x_3}\\\frac{\partial^2\varphi}{\partial x_3\partial x_1} & \frac{\partial^2\varphi}{\partial x_3\partial x_2} & \frac{\partial^2\varphi}{\partial x_3^2}\end{pmatrix}$$

Denoting $u = x_1x_2x_3$, we get:

$$H = \begin{pmatrix}-x_2^2x_3^2sin\left(u\right) & x_3\left(cos\left(u\right)-usin\left(u\right)\right) & x_2\left(cos\left(u\right)-usin\left(u\right)\right)\\x_3\left(cos\left(u\right)-usin\left(u\right)\right) & -x_1^2x_3^2sin\left(u\right) & x_1\left(cos\left(u\right)-usin\left(u\right)\right)\\x_2\left(cos\left(u\right)-usin\left(u\right)\right) & x_1\left(cos\left(u\right)-usin\left(u\right)\right) & -x_1^2x_2^2sin\left(u\right)\end{pmatrix}$$

2. Let:

$$h\left(x\right) = exp\left(x\right)$$

It is known that:

$$h(x) = h'(x) = h''(x) = exp(x)$$

TODO: Maybe add a short explanation about our implementation

Task 4:

TODO: Maybe add a short explanation about our implementation

Task 5:

Using the matrix:

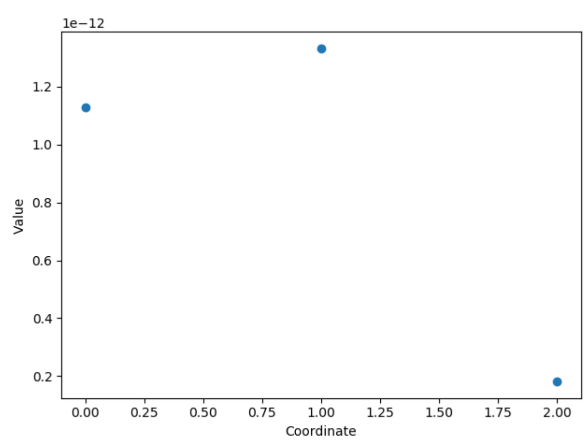
$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

And a random vector:

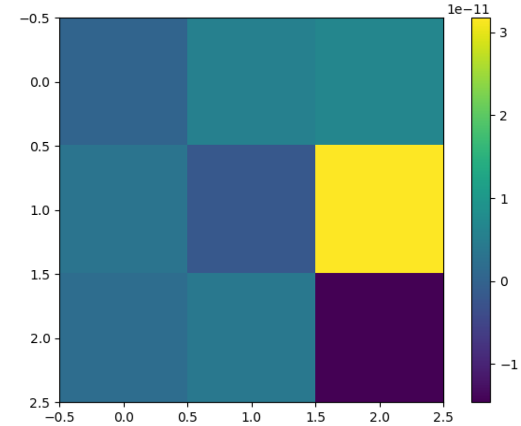
$$x = \begin{pmatrix} 0.512 \\ 0.168 \\ 0.081 \end{pmatrix}$$

We got the following results:

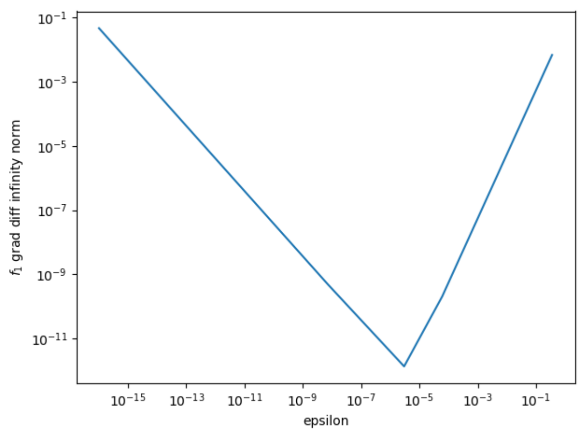
The function f_1 :
The difference between the numerical and analytical gradient:



The difference between the numerical and analytical Hessian:

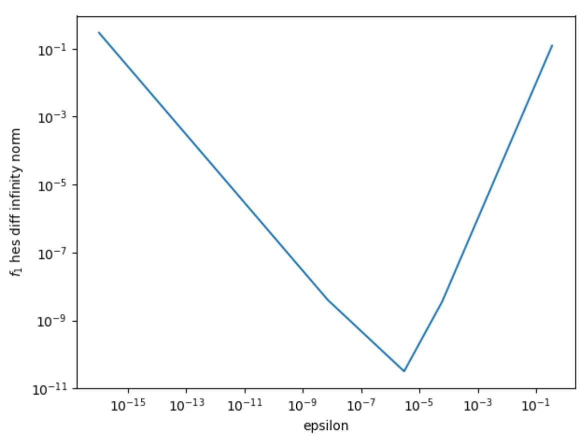


The infinity norm of the error between the numerical and analytical gradient as a function of ϵ:



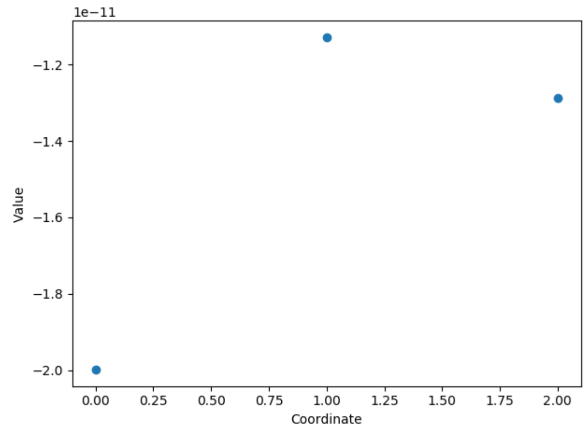
The best error rate is 1.3322×10^{-12}, for $\epsilon = 3 \times 10^{-6}$.

The infinity norm of the error between the numerical and analytical Hessian as a function of ϵ:

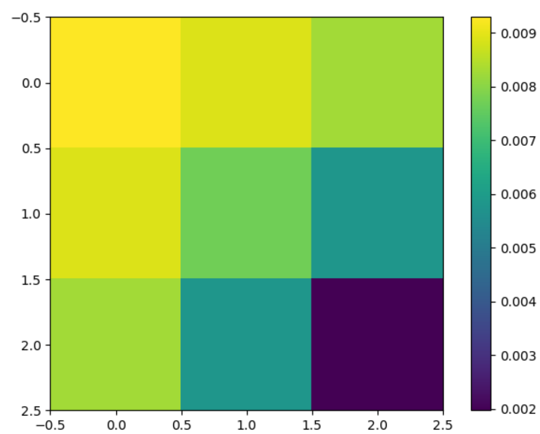


The best error rate is 3.1747×10^{-11}, for $\epsilon = 3 \times 10^{-6}$.

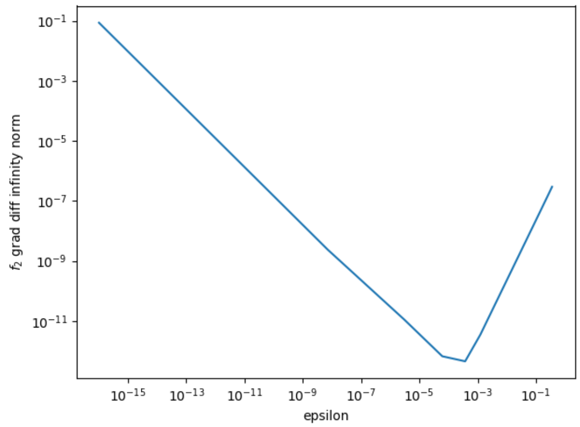
The function f_2 :
The difference between the numerical and analytical gradient:



The difference between the numerical and analytical Hessian:

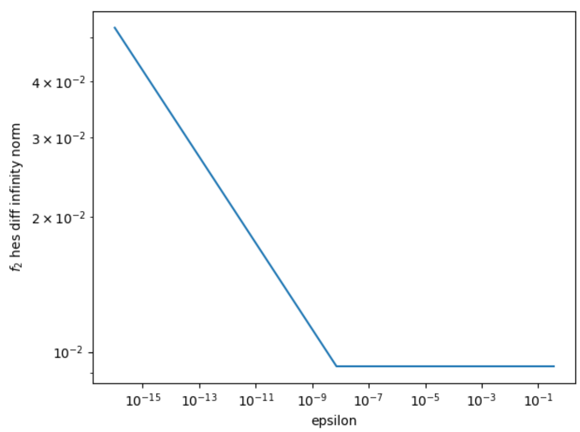


The infinity norm of the error between the numerical and analytical gradient as a function of ϵ:



The best error rate is 4.5474×10^{-13}, for $\epsilon = 0.0003718$.

The infinity norm of the error between the numerical and analytical Hessian as a function of ϵ:



The best error rate is 0.00929, for $\epsilon = 7.2547 \times 10^{-9}$.