CS 0368-4246: Combinatorial Methods in Algorithms (Spring 2025) August 19, 2025

Final Exam (B)

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Instructions (Please Read Carefully)

- This exam consists of **four** questions. The number of points for each question and sub-question is indicated alongside it. The total available points add up to **105**, but the final score will be capped at **100**.
- You may use theorems or lemmas that were proved in class, provided you cite them exactly as stated. Any modifications of these results—or use of results from homework—must be fully re-proven.
- Write your answers clearly and in an organized manner in the provided exam booklet. Clearly label each answer with the corresponding question number, and indicate whether each page is a draft or a final answer.
- Unless explicitly stated otherwise, all graphs are assumed to be undirected and unweighted, and all notation follows that used in class.
- Unless explicitly stated otherwise, algorithms with an expected run-time or that succeed with high probability are sufficient.

Exam Questions

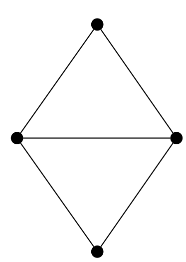

Figure 1: The theta graph θ.

1. *(25 points)* Let θ denote the *theta graph*, defined as a 4-cycle C_4 together with an additional edge connecting two non-adjacent vertices on the cycle; see Figure 1.

   (a) *(15 points)* Design an algorithm that, given a graph G, checks whether G contains θ as a subgraph. Aim for the fastest running time in terms of n. There is no need to prove optimality.

   (b) *(10 points)* Determine, up to constant factors, the extremal number $\mathrm{ex}(\theta, n)$. That is, give *asymptotically* matching upper and lower bounds.

2. *(25 points)* A graph G is said to be a *disjoint union of at most two cliques* if there exists a (possibly trivial) partition $V = V_1 \uplus V_2$ such that both V_1 and V_2 induce cliques, and there are no edges between them. Design a property testing algorithm that, given query access to the adjacency matrix of a graph G, distinguishes with probability at least $9/10$ between: (1) G is a disjoint union of at most two cliques, and (2) G is ε-far from having this property (i.e., at least εn^2 edge insertions or deletions are required to make G such a union). Your algorithm should make a number of queries that depends only on ε (and not on n).

3. *(30 points)* In this question we will construct spanners efficiently, with guidance.

   (a) *(10 points)* You can use an algorithm that computes a $(\varphi,\ \varphi{\cdot}\log^2 n{\cdot}m)$-expander decomposition in $m^{1+o(1)}$ time as a black-box. Show that for any graph G we can find, in $m^{1+o(1)}$ time, a disjoint edge partition to graphs (edge sets) $G_1,\ldots,G_{\log m}$ such that each graph G_i is the disjoint union of expanders with conductance $\Omega(1/\mathrm{polylog}(n))$.

   (b) *(10 points)* Let G be a φ-expander, show that you can construct in $O(m+n)$ time a tree in G which is a $O(\frac{\log n}{\varphi})$ spanner of G (i.e., every edge is replaced by a path of at most that length).

   (c) *(10 points)* Show that for any graph G we can in $m^{1+o(1)}$ time construct a subgraph with $O(n \cdot \mathrm{polylog}(n))$ edges that is a $O(\mathrm{polylog}(n))$-spanner of G.

4. *(25 points)* In this question we examine graphs that show the tightness of theorems from class.

   (a) *(15 points)* Compute the conductance $\varphi(K_n)$ and the second smallest eigenvalue $\lambda_2(N_{K_n})$ of the normalized Laplacian of the complete graph K_n. Compute the asymptotic behavior of each as $n \to \infty$, and deduce that one direction of Cheeger's inequality is asymptotically tight.

   (b) *(10 points)* Identify the minimum cuts in the cycle graph C_n and determine how many such cuts exist. Conclude that a lemma discussed in class (bounding the number of distinct minimum cuts in a graph) is tight.