Discrete Geometry - Second Assignment

§1. Convexity

Problem 1. Show that Helly's Theorem does not hold if we have infinitely many convex sets.

Lemma 1. *Let* $I_n = (0, 1/n)$. *Then*

$$\bigcap_{n \in \mathbb{Z}^+} I_n = \emptyset.$$

Proof of Lemma 1. Assume towards a contradiction that the claim is false. It follows that there exists an element $i \in \cap_{n \in \mathbb{Z}^+} I_n$. That is $0 < i < 1/n$ for every $n \in \mathbb{Z}^+$. Therefore $n < 1/i$, for every $n \in \mathbb{Z}^+$. A contradiction to the Archimedean property for $\mathbb{R}$. ■

Solution for Problem 1. We give a counter-example. Consider the following convex set $I := \left\{(0, 1/n) : n \in \mathbb{Z}^+\right\} \subseteq \mathbb{R}$. Note that every two sets $I_1 = (0, 1/n_1), I_2 = (0, 1/n_2)$, for some $n_1, n_2 \in \mathbb{Z}^+$, satisfies that $I_1 \cap I_2 \neq \emptyset$. Indeed, let $n_{\min} = \min(n_1, n_2)$ and notice that $1/(2n_{\min}) \in I_1 \cap I_2$. As the premise holds and Lemma 1 asserts that the consequence doesn't hold the claim follows. ■

§2. Points in convex position

Problem 2. How many steps are sufficient by the PHP?

Solution for Problem 2. We split the answer into cases:

1. If $m \geq k$, then one iteration is enough.
2. Otherwise, N iteration are enough where N is the minimum natural number that satisfies $\lceil N/c \rceil \geq k$, and c is the number of colours. That is $N = c \cdot (k-1) + 1$.

■

Solution for Problem 3. The Happy-Ending Theorem implies that any 5 points in general position form 4 points in convex position. Therefore as $k \geq 5$ it follows that P' must be in convex position. ■

§3. Incidences

Problem 4. Let $a, b \in \mathbb{R}_{\geq 0}$. Prove $a + b = \Theta(\max\{a, b\})$.

Solution for Problem 4. Without loss of generality assume that $a \geq b$. It the follows that:

$$a \leq a + b \leq 2a.$$

■

PROBLEM 5. In the class we proved the Crossing Lemma for the case that $e \geq 4v$. Here you asked to prove the Crossing Lemma for the case that $e \leq 4v$.

SOLUTION FOR PROBLEM 5. The claim is vacuously valid as the lemma asserts that the crossing number is at least

$$\frac{e^3}{64v^2} - v \leq \frac{64v^3}{64v^2} - v = v - v = 0.$$

Put in other words, the crossing number is at least zero, which is a vacuous truth. ■

PROBLEM 6. Prove that $\sqrt[3]{a+b} = \Theta\left(\sqrt[3]{a} + \sqrt[3]{b}\right)$.

SOLUTION FOR PROBLEM 6. Problem 4 asserts that is suffice to show that $\sqrt[3]{a+b} = \Theta\left(\max\left\{\sqrt[3]{a}, \sqrt[3]{b}\right\}\right)$. Therefore, assume without loss of generality that $a \geq b \geq 0$, and observe that

$$\sqrt[3]{a} \leq \sqrt[3]{a+b} \leq \sqrt[3]{2a} \leq \sqrt[3]{2} \cdot \sqrt[3]{a}.$$

The claim follows. ■

§4. POLYTOPES AND DUALITY

PROBLEM 7. Express the cube and octahedron as H-polytopes.

SOLUTION FOR PROBLEM 7. We start with the octahedron. We embrace the notation that $\boldsymbol{x} := (x, y, z)^T \in \mathbb{R}^3$ is a vector, while x is an element in the vector. Trivially,

$$\left\{\boldsymbol{x} := (x,y,z) \in \mathbb{R}^3 : \begin{array}{r} x \geq 0, x \leq 1 \\ y \geq 0, y \leq 1 \\ z \geq 0, z \leq 1 \end{array}\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \begin{array}{l} (1,0,0)^T \boldsymbol{x} \geq 1, (1,0,0)^T \boldsymbol{x} \leq 1 \\ (0,1,0)^T \boldsymbol{x} \geq 1, (0,1,0)^T \boldsymbol{x} \leq 1 \\ (0,0,1)^T \boldsymbol{x} \geq 1, (0,1,0)^T \boldsymbol{x} \leq 1 \end{array}\right\}$$

$$= \left\{\boldsymbol{x} \in \mathbb{R}^3 : \begin{array}{l} (1,0,0)^T \boldsymbol{x} \geq 1, (-1,0,0)^T \boldsymbol{x} \geq -1 \\ (0,1,0)^T \boldsymbol{x} \geq 1, (0,-1,0)^T \boldsymbol{x} \geq -1 \\ (0,0,1)^T \boldsymbol{x} \geq 1, (0,0,-1)^T \boldsymbol{x} \geq -1 \end{array}\right\}.$$

As the development of the equation is taught in high school and is pretty clear.

$$\left\{\boldsymbol{x} := (x,y,z) \in \mathbb{R}^3 : \begin{array}{r} -1 \leq x+y+z \leq 1 \\ -1 \leq x-y+z \leq 1 \\ -1 \leq x+y-z \leq 1 \\ -1 \leq x-y-z \leq 1 \end{array}\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^3 : \begin{array}{rccr} (1,1,1)^T & \boldsymbol{x} & \geq & -1 \\ -(1,1,1)^T & \boldsymbol{x} & \geq & 1 \\ (1,-1,1)^T & \boldsymbol{x} & \geq & -1 \\ -(1,-1,1)^T & \boldsymbol{x} & \geq & 1 \\ (1,1,-1)^T & \boldsymbol{x} & \geq & -1 \\ -(1,1,-1)^T & \boldsymbol{x} & \geq & 1 \\ (1,-1,-1)^T & \boldsymbol{x} & \geq & -1 \\ -(1,-1,-1)^T & \boldsymbol{x} & \geq & 1 \end{array}\right\}.$$

■

DEFINITION 1. *A* convex curve *in $\mathbb{R}^d$ is a curve that intersects every hyperplane in at most d points.*

PROBLEM 8. The *moment curve* is given by

$$\gamma = \left\{ \left(t, t^2, t^3, ..., t^d \right) : t \in \mathbb{R} \right\}.$$

Prove that the moment curve is a convex curve.

PROOF OF PROBLEM 8. Let h be an hyper-plane in $\mathbb{R}^d$ such that $\langle \boldsymbol{x}, \boldsymbol{a} \rangle = b$. Put another way, $a_1 x_1 + \ldots, a_d x_d = b$. We consider how many $\boldsymbol{x} \in \gamma$ satisfies the aforementioned equality. That is how many solution does the following equation has

$$a_1 t + a_2 t^2 + \ldots a_d t^d = b \iff a_1 t + a_2 t^2 + \ldots a_d t^d - b = 0.$$

As the fundamental theorem of algebra asserts that every polynomial of degree d has at most d roots the claim follows. ∎

PROBLEM 9. Conclude that the number of facets of the cyclic polytope is

$$\begin{cases} \binom{n-d/2}{d/2} + \binom{n-d/2-1}{d/2-1} & ,\text{if } 2|d, \\ 2 \cdot \binom{n-\lfloor d/2 \rfloor -1}{\lfloor d/2 \rfloor -1} & ,\text{otherwise.} \end{cases}$$

SOLUTION FOR PROBLEM 9. We write $p_1 \to p_2$ to denote that p_1 and p_2 are consecutive. We split the proof into cases:

d **is even.** As explained in class two configurations are possibles.

1. $p_1 \to p_2, p_3 \to p_4, \ldots, p_{d-1} \to p_d$. Therefore we gather every $p_{2k-1} \to p_{2k}$ to an element and considering the number of ways to order $n-d$ points and $d/2$ bars in a raw when ordering between bins (two bars) is not counted. From discrete mathematics it follows that the result is $\binom{(n-d)+(d/2-1)}{d/2-1} = \binom{n-d/2-1}{d/2-1}$
2. $p_2 \to p_3, p_4 \to p_5, \ldots, p_{d-1} \to p_d$. Here we have $d/2+1$ "bars", therefore, the result is $\binom{(n-d)+(d/2)}{d/2} = \binom{n-d/2}{d/2}$.

The claim follows as this is two complementary cases.

d **is odd.** As explained in class two configurations are possibles.

1. $p_1 \to p_2, p_3 \to p_4, \ldots, p_{d-2} \to p_{d-1}$.
2. $p_2 \to p_3, p_4 \to p_5, \ldots, p_{d-1} \to p_d$.

Assume $d = 2k+1$. In both cases the number of regions ("bars") is $\lfloor d/2 \rfloor + 1 = k+1$ and therefore totally there are $2 \cdot \binom{(n-2k-1)+k}{k} = 2 \cdot \binom{n-k-1}{k} = 2\binom{n-\lfloor d/2 \rfloor -1}{\lfloor d/2 \rfloor}$

∎

REFERENCES

[1] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.