Discrete Geometry – Third Assignment

§1. Lower Envelopes

Lemma 1.1. *Let f, g be monotonically increasing functions. Then $f \circ g$ is monotonically increasing.*

Proof of Lemma 1.1. Let $x \le y$. As g is monotonically increasing it follows that $g(x) \le g(y)$. Moreover, as f is monotonically increasing it follows that $f\left(g(x)\right) \le f\left(g(y)\right)$. ■

Problem 1. For every c there exists an n_0 such that for all $n \ge n_0$ we have that $A_k(n) \ge A^{(c)}_{k-1}(n)$.

Solution for Problem 1. It suffice to show that $A^{(c)}_{k-1}(n) = o(A_k(n))$. Note that

$$\begin{aligned}\lim_{n\to\infty} \frac{A_k(n)}{A^{(c)}_{k-1}(n)} &= \lim_{n\to\infty} \frac{A_{k-1}(A_k(n-1))}{A^{(c)}_{k-1}(n)} = \lim_{n\to\infty} \frac{A_{k-1}(A_{k-1}(A_k(n-2)))}{A^{(c)}_{k-1}(n)} = \cdots = \lim_{n\to\infty} \frac{A^{(c)}_{k-1}(A_k(n-c))}{A^{(c)}_{k-1}(n)} \\ &\ge \lim_{n\to\infty} \frac{2^c \cdot A_k(n-c)}{2^c \cdot n} = \lim_{n\to\infty} \frac{A_k(n-c)}{n} \ge \lim_{n\to\infty} \frac{2(n-c)}{n} = \lim_{n\to\infty} n - \frac{2c}{n} = \infty,\end{aligned}$$

where the first and the second inequalities hold due to the monotonicity of Ackermann's function. Indeed, $A_1(n) = 2n$ and therefore monotonically increasing. In addition, Lemma 1.1 implies that the composition of monotonically increasing functions is also monotonically increasing. ■

Problem 2. Let $A(n) = A_n(3)$. Prove that for every $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$ it holds that $A(n) \ge A_k(n)$.

Solution for Problem 2. Let n_0 satisfy $A_{n_0}(2) \ge n_0, k$. It then follows that for every $n \ge n_0$ it holds that

$$A_n(3) = A_{n-1}(A_n(2)) \ge A_{n-1}(n) \ge A_k(n),$$

Where both inequality follows from the chose of n_0. Note that Corollary A.5 implies that $f_k := A_k(n)$ is unbounded and therefore such choice for n_0 is possible. ■

Problem 3. In the lecture we defined $x_k(m) := e_k(m)/n_k(m)$. We claimed that $x_k(m) \ge k/2$ and the proof was by induction on k and for each k by induction on m.

1. Complete the base case where $k = 1$.
2. Complete the base case where $m = 1$.
3. Check the details.

Solution for Problem 3.

1. Applying to the definition of $S_1(m)$, we have that the length of the lower envelope is m. That is, $e_1(m) = m$. In addition, the number of fans is defined to be 1, i.e., $f_1(m) = 1$. It

follows that
$$x_1(m) = \frac{e_k(m)}{n_k(m)} = \frac{e_k(m)}{f_k(m) \cdot m} = \frac{m}{m} = 1 \geq \frac{1}{2} = \frac{k}{2}.$$

2. Following the recurrences relation that was stated in the class we have the following relations:
$$f_k(1) = 2f_{k-1}(2) \qquad \text{and} \qquad e_k(1) = e_{k-1}(2) + 2f_{k-1}(2).$$
Therefore,
$$x_k(1) = \frac{e_k(1)}{n_k(1)} = \frac{e_k(1)}{f_k(1)} = \frac{e_{k-1}(2) + 2f_{k-1}(2)}{2f_{k-1}(2)} = \frac{e_{k-1}(2)}{n_{k-1}(2)} + 1 = x_{k-1}(2) + 1.$$
The induction hypothesis implies that $x_k(1) \geq k/2$.

3. The general case where $k, m \geq 2$. The way that $S_k(m)$ was constructed implies that $f_k(m) = f_{k-1}(M) \cdot M$ where $M = f_k(m-1)$ is the number of fans in S'. We note that $M - 1 \geq M/2$ as $M \geq 2$. Applying to the recurrence relation that was proven in the class we have
$$e_k(m) = e_k(m-1) \cdot f_{k-1}(M) + e_{k-1}(M) + (M-1) \cdot f_{k-1}(M).$$
To see this note that we have $f_{k-1}(M)$ copies of S' each contributes $e_k(m-1)$ to the length. In addition, we should consider the contribution of S^*, that is, $e_{k-1}(M)$. Last but not least, one should take into account the $M - 1$ additional parts gained from this clever (yet tedious) construction at each fun. It follows that
$$\begin{aligned} e_k(m) &= e_{k-1}(M) + f_{k-1}(M)\left(e_k(m-1) + M - 1\right) \\ &\geq e_{k-1}(M) + f_{k-1}(M)\left(e_k(m-1) + M/2\right). \end{aligned}$$
Dividing both sides by $n_k(m)$ yields:
$$\begin{aligned} x_k(m) &\geq \frac{e_{k-1}(M) + f_{k-1}(M)\left(e_k(m-1) + M/2\right)}{n_k} \\ &\geq \frac{e_{k-1}(M) + f_{k-1}(M)\left(e_k(m-1) + M/2\right)}{m \cdot f_k(m)} \\ &\geq \frac{e_{k-1}(M) + f_{k-1}(M)\left(e_k(m-1) + M/2\right)}{mM \cdot f_{k-1}(M)} \\ &\geq \frac{e_k(m-1) + M/2}{mM} + \frac{e_{k-1}(M)}{m \cdot n_k} \\ &\geq \frac{e_k(m-1)}{mM} \cdot \frac{n_k(m-1)}{n_k(m)} + \frac{1}{2m} + \frac{x_{k-1}(M)}{m} \\ &\geq \frac{e_k(m-1)}{mM} \cdot \frac{(m-1) \cdot M}{n_k(m-1)} + \frac{1}{m} \cdot \left(\frac{1}{2} + x_{k-1}(M)\right) \\ &\geq \frac{e_k(m-1)}{n_k(m-1)} \cdot \frac{m-1}{m} + \frac{1}{m} \cdot \left(\frac{1}{2} + x_{k-1}(M)\right) \\ &\geq \left(1 - \frac{1}{m}\right) \cdot x_k\,(m-1) + \frac{1}{m} \cdot \left(\frac{1}{2} + x_{k-1}\,(M)\right). \end{aligned}$$

The induction hypothesis now implies that $x_k(m-1) \geq k/2$ and that $x_{k-1}(M) \geq (k-1)/2$. Concluding that

$$\left(1-\frac{1}{m}\right)\cdot\frac{k}{2}+\frac{1}{m}\left(\frac{1}{2}+\frac{k}{2}-\frac{1}{2}\right) \geq \frac{k}{2}\left(1-\frac{1}{m}+\frac{1}{m}\right)=\frac{k}{2},$$

as claimed.

$\blacksquare$

PROBLEM 4. Prove that if d, D are given in advance, and if r is chose large enough, then $x_{23} < x_{12}$.

SOLUTION FOR PROBLEM 4. Let ℓ_1, ℓ_2, ℓ_3 be the three segment from left to right as describe in Figure 1. Assume without loss of generality that the bottom left corner of the left segment (denoted by ℓ_1) is $p_1 = (0,0)$ as describe in Figure 1.

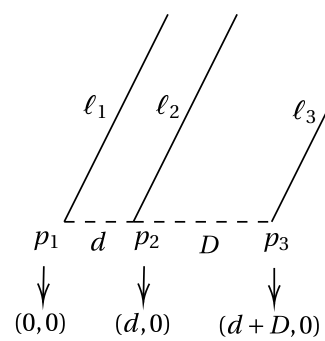


Figure 1: A schematic illustration.

As we saw in the lecture, we can assume that the slopes of the lines are $1+\varepsilon m_1, 1+\varepsilon m_2, 1+\varepsilon m_3$, where $\varepsilon > 0$. We start by finding the lines:

$$\ell_1:\ y_1(x) = (1+\varepsilon m_1)\,x, \quad \ell_2:\ y_2(x) = (1+\varepsilon m_2)\cdot(x-d), \quad \ell_3:\ y_3(x) = (1+\varepsilon m_3)\cdot(x-(d+D)).$$

Now, we find the intersection between the points that satisfies ℓ_1 and ℓ_2:

$$\begin{aligned}
y_1(x) &= y_2(x) \\
(1+\varepsilon m_1)\,x &= (1+\varepsilon m_2)\cdot(x-d) \\
x\,(1+\varepsilon m_2-(1+\varepsilon m_1)) &= d\cdot(1+\varepsilon m_2) \\
x\varepsilon\,(m_2-m_1) &= d\cdot(1+\varepsilon m_2) \\
&\boxed{x_{12} = \frac{d\cdot(1+\varepsilon m_2)}{\varepsilon\cdot(m_2-m_1)}}
\end{aligned}$$

Similarly we find x_{23}, namely, the intersection of ℓ_2, ℓ_3:

$$
\begin{aligned}
(1+\varepsilon m_2)\cdot(x-d) &= (1+\varepsilon m_3)\cdot(x-(d+D)) \\
x\,(1+\varepsilon m_2-(1+\varepsilon m_3)) &= d\cdot(1+\varepsilon m_2-(1+\varepsilon m_3)) - D\cdot(1+\varepsilon m_3) \\
\varepsilon x\,(m_2-m_3) &= \varepsilon d\,(m_2-m_3) - D\,(1+\varepsilon m_3) \\
&\boxed{x_{23} = d + D\cdot\frac{(1+\varepsilon m_3)}{\varepsilon\,(m_3-m_2)}}
\end{aligned}
$$

As the question ask for satisfying $x_{23} < x_{12}$ we have the constrain:

$$
\begin{aligned}
x_{23} = d + \frac{D\cdot(1+\varepsilon m_3)}{\varepsilon\,(m_3-m_2)} &< \frac{d\cdot(1+\varepsilon m_2)}{\varepsilon\cdot(m_2-m_1)} = x_{12} \\
d\,(m_3-m_2)\,[\varepsilon\,(m_2-m_1)-(1+\varepsilon m_2)] &< -D\cdot(1+\varepsilon m_3)\,(m_2-m_1) \\
D\cdot(1+\varepsilon m_3)\,(m_2-m_1) &< d\,(m_3-m_2)\,[(1+\varepsilon m_2)-\varepsilon\,(m_2-m_1)]
\end{aligned}
$$

Reordering,

$$
\begin{aligned}
\frac{D}{d} &< \frac{(m_3-m_2)\,[(1+\varepsilon m_2)-\varepsilon\,(m_2-m_1)]}{(1+\varepsilon m_3)\,(m_2-m_1)} \\
&= (m_3-m_2)\cdot\left(\frac{1+\varepsilon m_2}{(1+\varepsilon m_3)\,(m_2-m_1)} - \frac{\varepsilon}{1+\varepsilon m_3}\right) \\
&< \frac{m_3-m_2}{m_2-m_1}\cdot\underbrace{\frac{1+\varepsilon m_2}{1+\varepsilon m_3}}_{\le 1} \\
&\le \frac{m_3-m_2}{m_2-m_1} \\
&= r
\end{aligned}
$$

To see the last equality note that

$$
\frac{m_3-m_2}{m_2-m_1} = \frac{\frac{m_3-m_2}{m_2}}{\frac{m_2-m_1}{m_2}} = \frac{r-1}{1-\frac{1}{r}} = \frac{r-1}{\frac{r-1}{r}} = r.
$$

We established the following relation:

$$
\boxed{r\,(d,D) \ge \frac{D}{d}}
$$

■

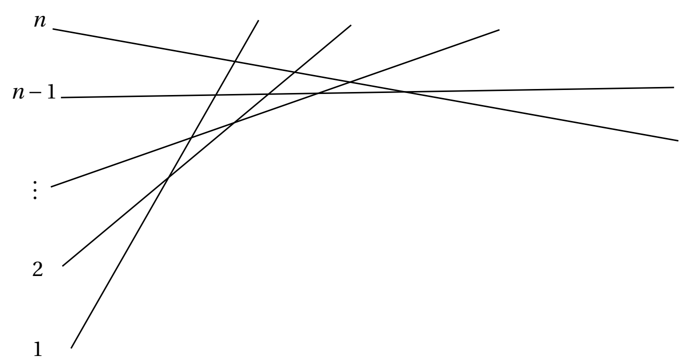
Let $\lambda_s(n)$ denote the maximum length of a Davenport-Schinzel sequence of order s.

PROBLEM 5. Prove that:

1. $\lambda_1(n) = n$.
2. $\lambda_2(n) = 2n - 1$.
   - **HINT (Upper-Bound).** Let z be the last symbol to appear for the first time. Show that z can appear only one. Delete z and use induction on n.

SOLUTION FOR PROBLEM 5.

1. We start by showing that $\lambda_1(n) \geq n$. Consider the following Davenport-Schinzel sequence of order 1 of length n: $\langle 1, 2, 3, \ldots, n \rangle$.

   The upper bound is trivial. The appearance of any additional segment would form an $a \ldots b \ldots a$ for some $a, b \in \Sigma = \{1, \ldots, n\}$, and hence negates the definition of Davenport-Schinzel sequence of order 1.

2. **Upper-Bound.** As the hint suggests, we prove the claim by induction on n. For $n = 1$ then there is only one element, and indeed $\lambda_2(1) = 2 \cdot 1 - 1 = 1$. Assuming $\lambda_2(n-1) = 2n - 3$ and let A be a Davenport-Schinzel sequence of order 2. Let z be the last symbol to appear for the first time and let y be the successive element in the sequence. We show that z appears only once in A. Assume towards a contradiction that the claim is false, then A cannot be a Davenport-Schinzel sequence of order 2 as it forms (see Figure 2) the configuration $(\cdots y \cdots zy \cdots y \cdots)$ which violates the definition of Davenport-Schinzel sequence of order 2.

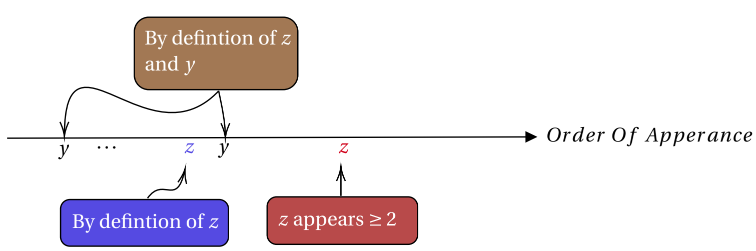
   

   Figure 2: The assumption that z at least twice yields the configuration $y \to z \to y \to z$.

   Removing the single appearance of z from U can yield a violation of the definition of Davenport-Schinzel sequence if the two neighbours of z share the same symbol. In this

case, choose one of the neighbours arbitrarily and remove also the chosen neighbour. The resulting sequence is a Davenport-Schinzel sequence of order 2 with $n-1$ segments (cause we remove the segment z). It follows by the induction hypothesis that the maximum length of A' is $2(n-1)$, and by the way A' was formed, we have that A' is shorter by at most 2 elements from A. Hence, the maximum length of A is at most $2n-1$, as claimed.
Lower-Bound. The sequence $(1,2,\ldots,n-1,n,n-1,\ldots,2,1)$ is a Davenport-Schinzel sequence of order 2 with n segments of length $2n-1$.

■

§2. TVERBERG'S THEOREM

PROBLEM 6. Write that last part of the proof formally (and not just with an example).

Proof. We adopt the notation from the slides and note that for every $1 \le i \le N+1$ it holds that $\sum_{j=1}^{r} \hat{q_{i,j}} = \hat{0}$. As we want to show that $\hat{0} \in \mathbf{conv}(\hat{q_{i,r}},\ldots,\hat{q_{i,r}})$, we normalize the last equation by dividing by r and driving that there exists a convex combination of $\hat{q_{i,r}},\ldots,\hat{q_{i,r}}$ which forms $\hat{0}$. The colorful Carathéodory Theorem implies that there exists a sequence $j_1,\ldots,j_{N+1}$ such that $\hat{0} \in \mathbf{conv}\left(\hat{q}_{1,j_1},\ldots,\hat{q}_{N+1,j_{N+1}}\right)$. Hence there exists scalars $\alpha_1,\ldots,\alpha_{N+1} \ge 0$ such that $\sum_{i=1}^{N+1} \alpha_i = 1$ and $\hat{0} = \sum_{i=1}^{N+1} \alpha_i \hat{q}_{i,j_i}$. As vector are equal iff the entries are element-wise equals we have

$$0 = \sum_{i:\ j_i=k} \alpha_i q_i + \sum_{i:\ j_i=r} \alpha_i \cdot (-q_i) \iff \sum_{i:\ j_i=r} \alpha_i q_i = \sum_{i:\ j_i=k} \alpha_i q_i,$$

for every $1 \le k \le r-1$. Recall that each $q_i = (p_i, 1)$ and hence the sum of the coefficient at each side of the equation sums to the same quantity, say λ. Hence, dividing all equations by λ yields a point $\rho := \frac{1}{\lambda} \cdot \sum_{i:\ j_i=r} \alpha_i q_i$ such that ρ can be expressed as r convex combinations, say $\mathbf{conv}(\mathcal{G}_1),\ldots,\mathbf{conv}(\mathcal{G}_r)$ where $\cup_{i=1}^{r} \mathcal{G}_i = \mathcal{U} := \{p_1,\ldots,p_{N+1}\}$, i.e. $\mathcal{G}_1,\ldots\mathcal{G}_r$ is a partition of $\mathcal{U}$.

■

PROBLEM 7. Let $p_1,\ldots,p_{d+1} \in \mathbb{R}^d$ form the vertices of a simplex. Let q be the center of the simplex (i.e., $q = (p_1 + \ldots + p_{d+1})/(d+1)$). Replace each p_i by a tiny cloud of $r-1$ points. What are **all** the Tverberg partitions of the set? And how many Tverberg partitions are there?

Proof. Note that the number of points is $(r-1)\cdot(d+1)+1$, that is a Tverberg's number. For the first set in the partition there are $(r-1)^{d+1}$ options, since in each cloud there are $(r-1)$ option and there are $(d+1)$ clouds. Once it has been chosen, the second set in the partition has $(r-2)^{d+1}$ as at each cloud one vertex has been discarded cause repetition is not allowed in a partition. Finally, the last set has $1^{d+1} = 1$ options. The element q is also taken as a set in the partition, i.e., the set $\{q\}$ is a set in the partition. Therefore, we have

$$(r-1)^{d+1} \cdot (r-2)^{d+1} \cdot \ldots 2^{d+1} \cdot 1^{d+1} = (1 \cdot 2 \cdot \ldots \cdot r-1)^{d+1} = ((r-1)!)^{d+1} \text{ Options.}$$

■

§A. The Ackermann's Hierarchy

Lemma A.1. *Prove that $A_k(n+1) \geq A_k(n) + 2$ whenever $k, n \in \mathbb{N}$.*

Corollary A.2. *$A_k(n) \geq 2n$ whenever $k, n \in \mathbb{N}$.*

Proof of Lemma A.1. We prove it using induction over $(k,n) \in \mathbb{N} \times \mathbb{N}$, where $(a,b) \prec (c,d)$ if either $a < c$, or $a = c$ and $b < d$. Observe that the claim holds for $k, n = 0$ as $A_0(1) = 2 \geq 0 + 2 = A_0(0) + 2$. Assume the claim holds for every $(k', n') \prec (k,n)$ and show for (k,n). Note that

$$A_k(n+1) = A_{k-1}(A_k(n)) \geq 2 \cdot A_k(n) \geq A_k(n) + 2n \geq A_k(n) + 2.$$

■

Corollary A.3. *Prove that $A_k(n+1) > A_k(n)$ whenever $k, n \in \mathbb{N}$.*

Lemma A.4. *Prove that $A_{k+1}(n) \geq A_k(n)$ whenever $m, n \in \mathbb{N}$.*

Proof of Lemma A.4. Note that $A_1(0) = 0 = A_0(0)$. Assume the claim holds for all $(k', n') \prec (k,n)$ and show for (k,n). To that end, note

$$A_{k+1}(n) = A_k(A_{k+1}(n-1)) \geq A_k(2(n-1)) \geq A_k(n),$$

where the first inequality is due to Lemma A.1. ■

Corollary A.5. *Fix $n \in \mathbb{N}$. Then $f_k = A_k(n)$ is unbounded.*

Proof of Corollary A.5. In the proof of Lemma A.4 we showed that:

$$A_{k+1}(n) = A_k(A_{k+1}(n-1)) \geq A_k(2(n-1)).$$

Continue this fashion one obtains:

$$A_{k+1}(n) \geq A_k(2(n-1)) \geq A_{k-1}\left(2^2(n-1)\right) \geq \ldots \geq A_{k-k'}\left(2^{k'}(n-1)\right) \geq A_1\left(2^k(n-1)\right) = 2^{k+1}(n-1).$$

Clearly, as the last expression in the right hand side tends to infinity as k tends to infinity the claim follows. ■

References

[1] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.