Linear Programming

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§1. Linear Programming Introduction

§ 1.1 Definition.

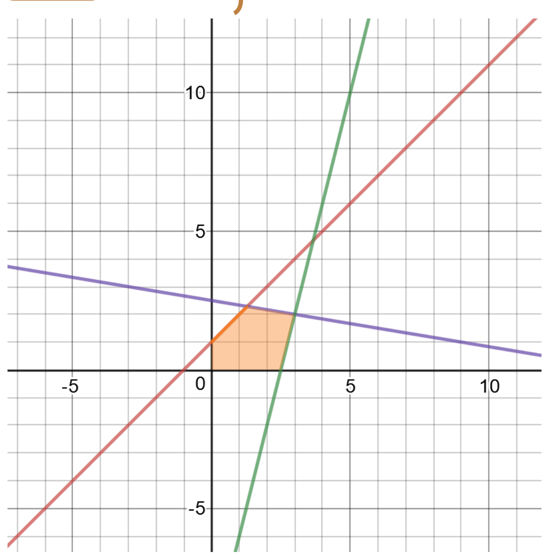
By linear programming we mean an optimization problem of the following form:

$$\max_{x\in\mathbb{R}^n} \left\{c^T x : \ Ax \le b\right\} \tag{1}$$

Where $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, and $c \in \mathbb{R}^n, b \in \mathbb{R}^n$ are fixed.

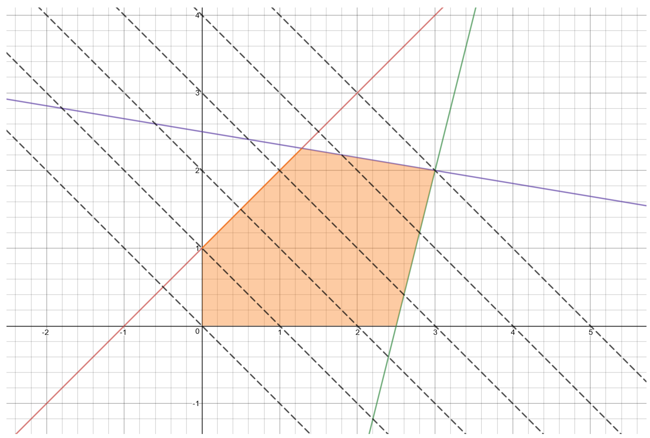
§ 1.2 The Geometric Prespective.

$$\begin{array}{ll}
\max \ \boxed{x_1 + x_2} & \left.\right\} \text{The Objective Function} \\
\boxed{\begin{array}{l} x_2 - x_1 \le 1 \\ x_1 + 6x_2 \le 15 \\ 4x_1 - x_2 \le 10 \end{array}} & \left.\vphantom{\begin{array}{l} x \\ x \\ x \end{array}}\right\} \text{Linear Constraints} \\
\boxed{\begin{array}{l} x_1 \ge 0 \\ x_2 \ge 0 \end{array}} & \left.\vphantom{\begin{array}{l} x \\ x \end{array}}\right\} \text{Non-negativity Constraints}
\end{array} \tag{2}$$



Definition 3. *A feasible solution is a vector* $\mathbf{x}$ *which satisfied the linear constraints and the non-negativity constrains.*

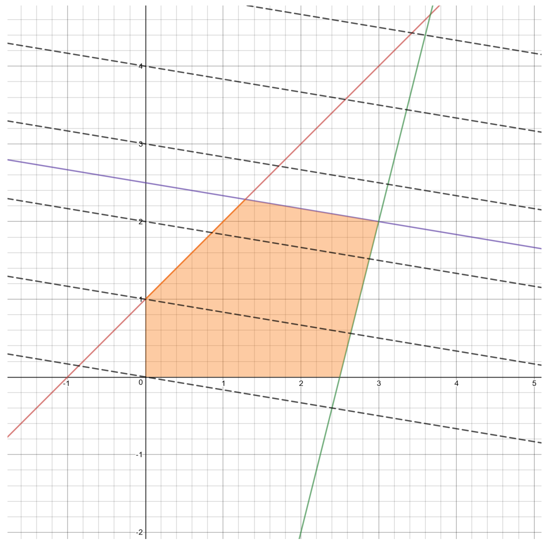
Every point in the orange domain is a feasible solution, but which point is an optimal solution? At first glance we shall use an auxiliary line that depicts the objective function:



We can note that the optimal solution is $(3, 2)$ which turns out to settle on the corner of the polygon. **Is optimal solution always lay on the corner?** Lets consider a similar linear programming problem:

$$
\begin{aligned}
\max \ & \left.\boxed{\tfrac{x_1}{6} + x_2}\right\} \text{The Objective Function} \\
& \left.\boxed{\begin{aligned} x_2 - x_1 &\le 1 \\ x_1 + 6x_2 &\le 15 \\ 4x_1 - x_2 &\le 10 \end{aligned}}\right\} \text{Linear Constraints} \\
& \left.\boxed{\begin{aligned} x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}}\right\} \text{Non-negativity Constraints}
\end{aligned}
\tag{4}
$$

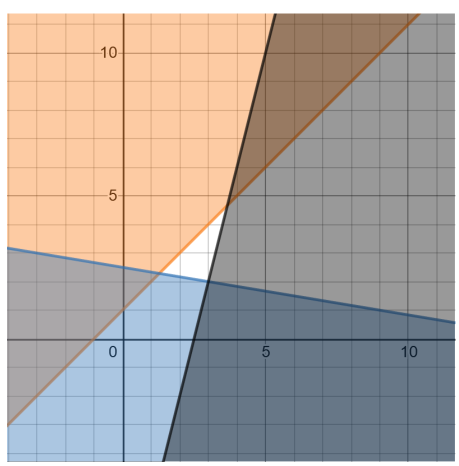
As one of the constraints is bounding from above the objective function it turns out that every point at this segment of the line inside the feasible domain is an optimal solution, so the answer in **no**, but nevertheless one can note that there is a corner which is an optimal solution.



§ 1.3 Infesibility. From our geometrical view, it is clear that a linear programming program may have no solution, in which case we say it is infeasible. For example, consider the following linear programming program:

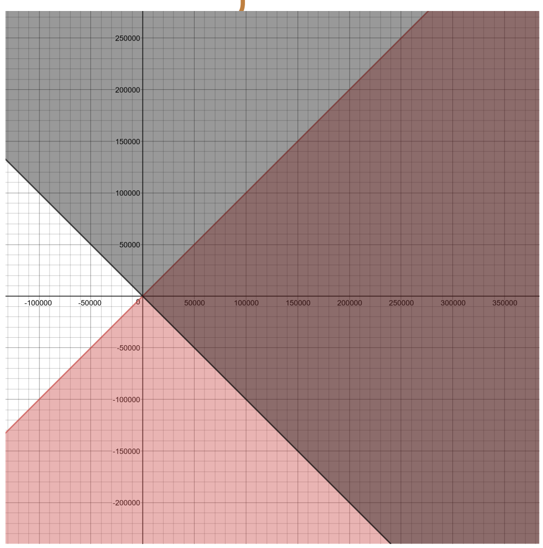
$$
\begin{aligned}
&\max \ \boxed{x_1 + x_2} && \left.\right\} \text{The Objective Function} \\
&\boxed{\begin{aligned} & x_2 - x_1 \geq 1 \\ & x_1 + 6x_2 \leq 15 \\ & -4x_1 + x_2 \leq -10 \end{aligned}} && \left.\vphantom{\begin{aligned} a \\ a \\ a \end{aligned}}\right\} \text{Linear Constraints} \\
&\boxed{\begin{aligned} & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}} && \left.\vphantom{\begin{aligned} a \\ a \end{aligned}}\right\} \text{Non-negativity Constraints}
\end{aligned}
\tag{5}
$$

Which yield the an infeasible linear program illustrated in the following figure:



§ 1.4 Unboundedness. A linear programming program is feasible but has no optimal solution is said to be unbounded:

$$
\begin{array}{rl}
\max \ \boxed{x_1 + x_2} & \left.\vphantom{\begin{array}{c}a\\b\end{array}}\right\} \text{The Objective Function} \\[1em]
\boxed{\begin{array}{l} x_2 - x_1 \le 0 \\ -x_2 - x_1 \le 0 \end{array}} & \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} \text{Linear Constraints} \\[1em]
\boxed{\begin{array}{l} x_1 \ge 0 \\ x_2 \ge 0 \end{array}} & \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} \text{Non-negativity Constraints}
\end{array}
\tag{6}
$$



Algorithmically we are interested in the following problem:

> Let A, b, c be define as above our goal is to decide whether the linear programming problem is infeasible or unbounded, if not find an optimal solution
>
> $$\max_{x \in \mathbb{R}^n} \left\{ c^T x : \ Ax \le b \right\}$$

§2. Equivalent Forms

We defined the Linear Programming problem as $\max_{x \in \mathbb{R}^n} \left\{ c^T x : \ Ax \le b \right\}$, but all of the examples seen so far also included non-negativity constraints that are missing from the definition. Moreover, an inevitable question arises: What about minimization?

Lemma 7. *All of the following are equivalent in the sense that those are all reducible to* *one another.*

$$\max_{x\in\mathbb{R}^n}\left\{c^Tx:\ Ax\le b\right\} \tag{8}$$
$$\max_{x\in\mathbb{R}^n}\left\{c^Tx:\ x\ge 0,\ Ax\le b\right\} \tag{9}$$
$$\max_{x\in\mathbb{R}^n}\left\{c^Tx:\ x\ge 0,\ Ax= b\right\} \tag{10}$$
$$\min_{x\in\mathbb{R}^n}\left\{c^Tx:\ Ax\ge b\right\} \tag{11}$$
$$\min_{x\in\mathbb{R}^n}\left\{c^Tx:\ x\ge 0,\ Ax\ge b\right\} \tag{12}$$
$$\min_{x\in\mathbb{R}^n}\left\{c^Tx:\ x\ge 0,\ Ax= b\right\} \tag{13}$$

Remark 14. Naturally, we do not mean that c, b, A are identical when we make these transitions.

§ 2.1 Equational Forms. Forms (10) and (13) above are referred to as equational forms due to the "equation" $Ax = b$, nevertheless note that $x \ge 0$ is not given with equality.

§2.1.1 Transition recipe from form (8) to form (10). We aim to preform the following transition:

$$\max_{x\in\mathbb{R}^n}\left\{c^Tx:\ Ax\le b\right\} \overset{\text{Transition}}{\Longrightarrow} \max_{x\in\mathbb{R}^n}\left\{c^Tx:\ x\ge 0,\ Ax= b\right\}$$

We need a non-negativity constraint for x_1. In order to assert this constrain we recall that any integer can be expressed as the difference of two natural numbers. Thus we can define $x_1 \coloneqq y_1 - z_1,\ y_1\ge 0,\ z_1\ge 0$

$$\begin{aligned} \max\ & \boxed{3x_1 - 2x_2} \\ & \boxed{2x_1 - x_2 \le 4} \\ & \boxed{x_1 + 3x_2 \ge 5} \\ & \boxed{x_2 \ge 0} \end{aligned}$$

Introduce a slack variable x_3 which yields $2x_1 - x_2 + x_3 = 4,\ x_3 \ge 0$

Oppose the inequality direction
$-x_1 - 3x_2 \le -5$
Introduce another slack variable x_4
$-x_1 - 3x_2 + x_4 = -5,\ x_4 \ge 0$

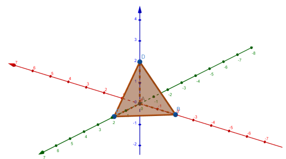
We Obtain

$$\begin{aligned} \max\ & \boxed{3y_1 - 3z_1 - 2x_2} && \text{The Objective Function} \\ \text{s.b.t}\ & \boxed{\begin{aligned} &2y_1 - 2z_1 - x_2 + x_3 = 4 \\ &-y_1 + z_1 - 3x_2 + x_4 = -5 \end{aligned}} && \text{Linear Constraints} \\ & \boxed{\begin{aligned} &y_1 \ge 0, \quad y_2 \ge 0 \\ &x_3 \ge 0, \quad x_4 \ge 0 \end{aligned}} && \text{Non-negativity Constraints} \end{aligned}$$

§3. Detecting Solubility in equational form

The equational form has the benefit of having the simplest geometry enabling simpler discussion as the solubility of the associated linear programming problem.

§ 3.1 Geometry of equational form in 3 variables. We restrict ourselves to 3 variables. The following example is aimed to provide an intuition to the following message: The feasible solutions that interest us the most are the ones who lie at the corner.



Remark 15. Why be interested in basic solutions? These are easiest to recognize.

Problem 1. How to define these solutions when we have n variables? In $\mathbb{R}^n, \ n > 3$, we will not have a picture.

Definition 16. *If A in an $m \times n$ matrix whose columns are indexed by $[n]$ and $I \subseteq [n]$ then A_I denotes the matrix obtained from A by restricting A only to the columns indexed by I.*

Definition 17. *Let A be an $m \times n$ matrix with $\mathbf{rk}\, A = m$. A basic feasible solution for*

$$\max\{c^T x : \ x \geq 0, \ Ax = b\}$$

is a feasible solution $x \in \mathbb{R}^n$ for which there exists $B \subseteq [n]$, and $|B| = m$ such that

- *The columns of A indexed by B are independent.*
- *$x_j = 0$ for all $j \notin B$.*

If $x_j = 0$ for some $j \in B$ than x is called degenerate.

Remark 18. Recall that the columns of a matrix are independent if and only if the matrix is non-singular. Hence we could replace those requirements.

Definition 19. *Once B is fixed then:*

- *$x_j, \ j \in B$ is called a* basic variable.
- *$x_j, \ j \notin B$ is called a nonbasic variable.*

§4. Characterization of basic feasible solutions

Remark 20. We consider $\{c^T x : \ x \geq 0, \ Ax = b\}$ where $\mathbf{rk}\, A = m$.

Lemma 4.1: Characterization of basic feasible solutions

A feasible solution x is basic if and only if the columns of A_k, $k \coloneqq \{j \in [n] : \; x_j > 0\}$ are independent.

Lemma 4.2

For all $B \subseteq [n]$ such that $|B| = m$ and A_B is non-singular, there exists **at most one** $x \in \mathbb{R}^n$ such that $x_j = 0$ for all $j \notin B$.

§ 4.1 Basic, Feasibility & Optimality.

§4.1.1 Main Message. The sole obstacles for $\max\{c^T x : \; x \geq 0, \; Ax = b\}$ to have an optimal solution are infeasibility and unboundedness.

Every feasible bounded linear programming problem has an optimal solution.

Theorem 4.1: Feasibility & Boundedness $\implies \exists$ optimal

Let $A_{m \times n}$ be a matrix with $\mathbf{rk}\, A = m$.

- If the linear programming $\max\{c^T x : \; x \geq 0, \; Ax = b\}$ has a feasible solution and the objective function is bounded, then this linear programming problem has an optimal solution.

- If an optimal solution exists then there is a basic feasible solution that is optimal.

This gives as the following idea: Search for an optimal solution among all basic feasible solutions, as there are at most $\binom{n}{m}$ basic feasible solution, so this is possible.

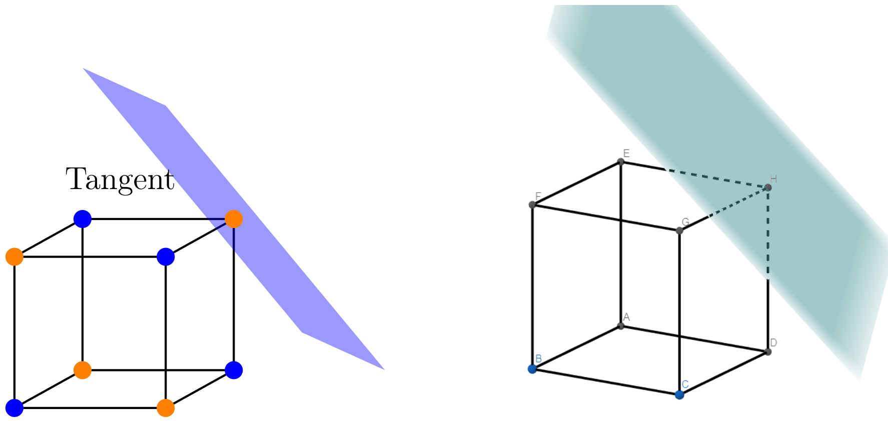
§5. The Geometry Behind Linear Programming - Abridged

We would like to generalize the above case:

- P - the edgy ball of Linear Programming .

- $d^T x = d_1 x_1 + d_2 x_2 + d_3 x_3$ - in 3D this is almost a plane.

- $v \coloneqq (v_1, v_2, v_3)$ is a corner of P, if we can find d and r for which $d_1 x_1 + d_2 x_2 + d_3 x_3 = r$ i.e. the plane meets P only at v.



(a) (b)

Figure 1

Theorem 5.1: Vertex $\iff$ Basic solution

Let P be the set of feasible solutions of $\max\{c^T x : \ x \geq 0, \ Ax = b\}$, and let $v \in P$, then v is a vertex of $P \iff v$ is a basic solution of the Linear Programming .