Discrete Geometry: First Assignment

§1. PRESENTATION: REVIEW ON LINEAR ALGEBRA

PROBLEM 1. Let V be a vector space over a field $\mathbb{F}$ and let $v_1, \ldots, v_n$ be vectors in V. Prove that $S := \mathbf{LS}(v_1, \ldots, v_n)$ is a subspace of V.

SOLUTION FOR PROBLEM 1. Note that by taking all the scalars in the linear combination to be 0 we establish that $\mathbf{0} \in S$. Let $u, v \in S$, it suffice to show that $\alpha u + \beta v \in S$. Indeed,

$$\alpha u + \beta v = \alpha \sum_{i=1}^{n} \gamma_i v_i + \beta \sum_{i=1}^{n} \delta_i v_i = \sum_{i=1}^{n} \left(\underbrace{(\alpha \gamma_i + \beta \delta_i)}_{\in \mathbb{F}} v_i \right) \in S.$$

Note that the first inequality follows directly from the definition of **LS**, the second equality yields from elementary algebra, and the third set membership follows (again) from the definition of **LS**. ■

PROBLEM 2. Is a single vector v_1 **LI** or **LD** (according to Definition 2)?

SOLUTION FOR PROBLEM 2. Note that v_1 is linearly dependent if there exists $\alpha_1 \neq 0$ such that

$$\alpha_1 v_1 = \mathbf{0} \implies \alpha_1 = 0.$$

Therefore, if $v_1 = \mathbf{0}$ then v_1 is linearly dependent. Otherwise, it is linearly independent. ■

§2. PRESENTATION: AFFINE SPACES

DEFINITION 1. *Points $p_1, \ldots, p_k$ are* Affinely Dependent *if one of them is an affine combination of the others.*

DEFINITION 2. *Points $p_1, \ldots, p_k$ are* Affinely Dependent *if there exists scalars $\alpha_1, \ldots, \alpha_k$, not all zero, such that*

$$\alpha_1 p_1 + \ldots + \alpha_k p_k = \mathbf{0} \text{ and } \alpha_1 + \ldots + \alpha_k = 0. \tag{2.1}$$

PROBLEM 3. Prove that the Definition 1 and Definition 2 are equivalent.

SOLUTION FOR PROBLEM 3.

Definition 1 $\implies$ Definition 2. Assume w.l.o.g.[1] that p_1 is affinely dependent of the others.

[1] Indeed, one can reorder the elements such that it is holds.

That is,

$$p_1=\sum_{i=2}^{k}\alpha_i p_i,\text{ and }\sum_{i=2}^{k}\alpha_i=1 \iff p_1-\sum_{i=2}^{k}\alpha_i p_i=\mathbf{0},\text{ and }1-\sum_{i=2}^{k}\alpha_i=1-1=0.$$

It follows that the Definition 2 is satisfied.

Definition 2 $\implies$ Definition 1. Suppose that Definition 2 is true, and let $\alpha_1,\ldots,\alpha_k$ satisfy (2.1). As the premise in this definition asserts that not all the coefficient are zero we can assume that $\alpha_1\neq 0$ and therefore dividing both parts of (2.1) by α_1 yields

$$p_1+\frac{\alpha_2}{\alpha_1}p_2+\frac{\alpha_3}{\alpha_1}p_3+\ldots+\frac{\alpha_k}{\alpha_1}p_k=\mathbf{0}\implies p_1=-\left(\frac{\alpha_2}{\alpha_1}p_2+\frac{\alpha_3}{\alpha_1}p_3+\ldots+\frac{\alpha_k}{\alpha_1}p_k\right),$$

and moreover

$$1+\frac{\alpha_2}{\alpha_1}+\frac{\alpha_3}{\alpha_1}+\ldots+\frac{\alpha_k}{\alpha_1}=0\implies -\left(\frac{\alpha_2}{\alpha_1}+\frac{\alpha_3}{\alpha_1}+\ldots+\frac{\alpha_k}{\alpha_1}\right)=1.$$

It follows that Definition 1 is satisfied and hence the claim follows.

■

PROBLEM 4. Prove that $d+2$ points in $\mathbb{R}^d$ are always **AD**.

LEMMA 5. *A set of $k\geq 2$ points $p_1,\ldots,p_k\in\mathbb{R}^n$ is affinely dependent iff $p_2-p_1,\ldots,p_k-p_1$ are linearly dependent.*

PROOF OF LEMMA 5. Note that $p_1=\alpha_2p_2+\ldots\alpha_kp_k$ where $\alpha_2+\ldots+\alpha_k=1$ holds iff

$$\underbrace{(\alpha_2+\ldots+\alpha_k)}_{=1}p_1=\alpha_2p_2+\ldots\alpha_kp_k \iff \alpha_2\cdot(p_2-p_1)+\alpha_3(p_3-p_1)+\ldots+\alpha_k(p_k-p_1)=\mathbf{0}.$$

As $\alpha_2+\ldots+\alpha_k=1$ it follows that not all the scalars are zero and therefore the claim follows. ■

SOLUTION FOR PROBLEM 4. A well-known theorem in linear algebra states that every $d+1$ vectors in $\mathbb{R}^d$ are linearly dependent. From Lemma 5 it follows that every $d+2$ points in $\mathbb{R}^d$ are affinely dependent. Indeed, the r.h.s.[2] of Lemma 5 is a set of $d+1$ vectors in $\mathbb{R}^d$ and therefore it is linearly dependent which implies that the set of $d+2$ points in the l.h.s. is affinely dependent. ■

PROBLEM 6. Prove that Affinely Dependent points are "unnecessary" for Affine Combination. That is, if $q\in\mathbb{R}^d$ is Affine Combination of $p_1,\ldots,p_k\in\mathbb{R}^d$ and p_1 is an Affine combination of $p_2,\ldots,p_k$. Then, q is an Affine Combination of $p_2,\ldots,p_k$.

[2] Right-hand side with respect to the if and only if logical statement.

SOLUTION FOR PROBLEM 6. It follows from the definition of affine combination that

$$p_1 = \alpha_2 p_2 + \ldots \alpha_k p_k \qquad , \alpha_2 + \ldots + \alpha_k = 1 \tag{2.2}$$
$$q = \beta_1 p_1 + \ldots + \beta_k p_k \qquad , \beta_1 + \ldots + \beta_k = 1. \tag{2.3}$$

Substitute l.h.s. of (2.2) in (2.3) yields

$$\begin{aligned} q &= \beta_1 \underbrace{(\alpha_2 p_2 + \ldots \alpha_k p_k)}_{p_1} + \beta_2 p_2 + \ldots + \beta_k p_k \\ &= (\beta_1 \alpha_2 + \beta_2)\, p_2 + (\beta_1 \alpha_3 + \beta_3)\, p_3 + \ldots + (\beta_1 \alpha_k + \beta_k)\, p_k = \sum_{i=2}^{k} (\beta_1 \alpha_i + \beta_i)\, p_i \end{aligned}$$

In order to show that the last term is indeed an affine combination it remains to prove that the coefficients of the p_i's sums up to 1. To that end, note that

$$\sum_{i=2}^{k} (\beta_1 \alpha_i + \beta_i) = \beta_1 \sum_{i=2}^{k} \alpha_i + \sum_{i=2}^{k} \beta_i = \beta_1 \cdot 1 + (1 - \beta_1) = 1,$$

where the second inequality follows from (2.2) and (2.3). ∎

PROBLEM 7. Assume F is an affine subspace of $\mathbb{R}^d$. Show that F is affinely closed.

PROOF OF PROBLEM 7. Let $p_1, \ldots, p_k \in F$ and let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ such that $\alpha_1 + \ldots + \alpha_k = 1$. We show that $\sum_{i=1}^{k} \alpha_i p_i \in F$. The definition of affine subspace establishes the existence of $w \in F$ and a linear subspace F' such that $F = \{v + w | v \in F'\}$. Hence for every $p'_1, \ldots, p'_k \in F'$ it holds that $\sum_{i=1}^{k} \alpha'_i p'_i \in F'$. In particular, fixing $p'_i = p_i - w$ for every $1 \leq i \leq k$ yields that

$$\sum_{i=1}^{k} \alpha_i p'_i = \sum_{i=1}^{k} \alpha_i (p_i - w) = \sum_{i=1}^{k} \alpha_i p_i - \sum_{i=1}^{k} \alpha_i w = \sum_{i=1}^{k} \alpha_i p_i - w \cdot \underbrace{\sum_{i=1}^{k} \alpha_i}_{=1} = \sum_{i=1}^{k} \alpha_i p_i - w.$$

It follows that $\sum_{i=1}^{k} \alpha_i p_i - w \in F'$ and therefore

$$\sum_{i=1}^{k} \alpha_i p_i = \underbrace{\left(\sum_{i=1}^{k} \alpha_i p_i - w \right)}_{\in F'} + w \in F.$$

∎