§1. A little notes about set theory axioms based on [1]

§ 1.1 subsection title.

Axiom 1.1.1 **Axiom of extension**

Two sets are equal if and only if they have the same elements.

Axiom 1.1.2 **Axiom of specification**

To every set A and to every condition $S(x)$ there corresponds a set B whose elements are exactly those elements of A for which $S(X)$ holds.

It is an immediate consequence of the axiom of extension that the axiom of specification determines the set B uniquely.

The following assumption will take a temporary role only.

There exists a set.

An immediate consequence is the existence of a set without any element at all. Indeed, let $\{x \in A \colon x \neq x\}$ (or any other false condition). By the axiom of extension there could be only one such set. This set will be called the empty set, and we will denote it by $\emptyset$. $\emptyset \subseteq A$, for every set A. To see this assume towards a contradiction that the claim is false, that is, there exists an element in $\emptyset$ such that it is not satisfies the condition of A. As $\emptyset$ has no elements a contradiction is obtained.

All we know so far is that there exists one set and it is the empty set.

The following question arises: Are there enough sets to ensure that every set is an element of some set? Is it true that for any two sets there is a third one that they both belong to? What about three sets, four sets, any number of sets? We need a new principle of set construction to resolve such questions. The following principle is a good beginning.

Axiom 1.1.3 **Axiom of pairing**

For any two sets there exists a set that they both belong to.

Axiom 1.1.4 **Axiom of pairing - equivalent formulation**

For any two sets there exists a set that contains both of them and nothing else.

The axiom of pairing ensures that every set is an element of some set and that any two sets are simultaneously elements of some one and the same set.

Axiom 1.1.5 **Unions And Intersections**

For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

Here are some easily proved facts about the unions of pairs:

$$A \cup \emptyset = A \qquad ,$$

$$A \cup B = B \cup A \qquad \text{commutativity,}$$

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad \text{associativity,}$$

$$A \cup A = A \qquad \text{idempotence,}$$

$$A \subseteq B \iff A \cup B = B \qquad .$$

Axiom 1.1.6 **Axiom of power set**

For each set there exists a collection of sets that contains among its elements all the subsets of the given set.

Axiom 1.1.7 **Axiom of power set-equivalent formulation**

Let A be a set, then there exists a set

$$\mathcal{P}(A) \text{ such that } X \in \mathcal{P} \iff X \subseteq A.$$

It should be noted that, if A is infinite, there is no hope to find, for each subset of A, a formula âĂIJdescribing itâĂİ, because $\mathcal{P}(A)$ is uncountable. This is however not a problem: the axiom tells you that you have a âĂIJcontainerâĂİ for all subsets of A; when you prove that a set B is a subset of A, then you know it belongs to $\mathcal{P}(A)$; and conversely, if you pick $B \in \mathcal{P}(A)$, you know $B \subseteq A$.

The real purpose of the axiom is that the subsets of a set form a set. In particular, for instance, the equivalence relations on a set form a set that can be isolated from $\mathcal{P}(A \times A)$ using a suitable predicate and the axiom of separation.

This axiom aims to declare the existence of the set of subsets.

We defined $X \subseteq S$ iff every $x \in X$ implies $x \in S$. We can prove that $S, \emptyset \subseteq S$. This implies that subsets indeed exists. However, when we asks our-self: "How many subsets there are in total?", we don't know.

Lets summarize:

- Subsets exists, because set exists.
- Any subset of a set S, will belong to $\mathcal{P}(S)$, because that's simply how we defined $\mathcal{P}(S)$.
- The Power Set Axiom states: Henceforth, we will all agree that we consider $\mathcal{P}(S)$ to be a set for every set S.

Axiom 1.1.8 **Axiom of infinity**

There exists a set containing 0 and containing the successor of each of its elements.

§ 1.2 Why do we need the axiom of infinity?. Since the other axioms of ZFC cannot prove that any infinite set exists. The way this is done is roughly the following steps:

1. Remember a set of axioms Σ is inconsistent if for any sentence A the axioms leas to a proof of $A \wedge \neg A$. This can be written as $\Sigma \vdash A \wedge \neg A \rightarrow \neg\mathrm{Con}(\Sigma)$.
2. If Inf is the statement: "an infinite set exists", then $\neg$Inf is the statement "no infinite set exists", therefore, Inf is true and $\neg$Inf is false.

3. If we don't need the axiom of infinity, then with the other axioms $\mathrm{ZFC}^\star \coloneqq \mathrm{ZFC} - \mathrm{Inf}$, we should be able to prove Inf as a theorem. In other words, we'll posit that $\mathrm{ZFC}^\star \vdash \mathrm{Inf}$.

4. We assume that ZFC is consistent, hence every subset is consistent, therefore, $\mathrm{ZFC}^\star$ is consistent.

5. We then add $\neg\mathrm{Inf}$ as an axiom to $\mathrm{ZFC}^\star$,which will be denoted as ZFC^+.

6. By showing that $(\mathrm{ZFC} - \mathrm{Inf}) + \neg\mathrm{Inf}$ has a model (a set in which all the axioms are true when quantifiers range only over the elements of the set), we can prove the relative consistency $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC}^+)$. In other words, we're basically just proving ZFC^+ is consistent, but we need to explicit that this proof assume ZFC consistent.

7. The model we want is HF, the set of all hereditary finite sets. I'll leave it to you to verify all the axioms of ZFC^+ holds in this set. But the important point is that $\mathrm{HF} \models \mathrm{ZFC}^+$, and out relative consistency is proven. (This follows from Godel's completeness theorem).

8. We are assuming that $\mathrm{ZFC}^\star \vdash \mathrm{Inf}$, but because ZFC^+ is an extension of $\mathrm{ZFC}^\star$ it must also be the case that $\mathrm{ZFC}^+ \vdash \mathrm{Inf}$. But then we have $\mathrm{ZFC}^+ \vdash \mathrm{Inf} \wedge \neq \mathrm{Inf}$ and is thus inconsistent, a contradiction. Therefore, we must conclude that our hypothesis $\mathrm{ZFC}^\star \vdash \mathrm{Inf}$ is false and there is no proof of Inf from the other axioms of ZFC. As a result Inf must be taken as an axiom in order to be able to prove that any infinite set exists.

§ 1.3 Another reason:. A finitist, who rejects the axiom of infinity, will be denied both the Dedekind and Cauchy constructions of real numbers. The problem is that the reals are uncountable (by Cantor's theorem), and in a finitist's universe, the universe itself is countable. Let

1. $V_0 \coloneqq \varnothing$.

2. $V_{n+1} \coloneqq \mathscr{P}(V_n)$ where $\mathscr{P}$ denotes the power set.

The universe if we accept the negation of the axiom of infinity is $V_\Omega = \bigcup_{x\in\Omega}$, a countable union of at most countable sets. That is, there are no real number which implies a contradiction.

Axiom 1.3.1 **Axiom of choice**

The Cartesian product of a non-empty family of nonempty sets is non-empty.

Put in other words, if $\{X_i\}$ is a family of non-empty sets indexed by a nonempty set I, then there exists a family $\{x_i\}_{i\in I}$ such that $x_i \in X_i$ for each $i \in I$.

§2. Order

Relation - Antisymmetric

A relation R in a set X is called anti-symmetric if, for every x and y in X, the simultaneous validity of $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$.

Partial order

A partial order in a set X is reflexive, anti-symmetric and transitive relation in X.

The justification for the qualifying "partial" is that some questions about order may be left unanswered. If for every $x, y \in X$ either $x \leq y$ or $y \leq x$ the $\leq$ is called total (or simple, or linear) order.

§ 2.1 Examples of partial order:.

1. Inclusion (partial and not linear): for every X, the relation $\subseteq$ is a partial order in the power set $\mathcal{P}(X)$.
2. Less than or equal in the set of naturals number. This is a linear order.
3. For function let $f \colon X \to Y$. Let

$$F \coloneqq \{g \colon A \to B \text{ s.t. } A \subseteq X \text{ and } B \subseteq Y\}.$$

Finally let

$$R = \{(f, g) \colon \mathrm{dom}(f) \subseteq \mathrm{dom}(g) \text{ and } f(x) = g(x) \text{ for all } x \in \mathrm{dom}(f)\}.$$

In other words, $(f, g) \in R$ means that f is a restriction of g, or equivalently, that g is an extension of f. If we recall that functions are, after all, subsets of the Cartesian product $X \times Y$, we recognize that $(f, g) \in R$ means the same as $f \subseteq g$; extension is a special case of inclusion.

References

[1] P. R. Halmos. *Naive set theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1974. Reprint of the 1960 edition.