Linear Differential Equation with constant coefficient

D.E is said to be linear if the dependent variable and its derivatives are of first degree.

— D.E of the form.

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = \phi(x).$$

$$\frac{d}{dx} = D;\ a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = \phi(x)$$

where $a_0, a_1, a_2 \dots a_n$ are all constants and is called a linear D.E of nth order with constant coeff.

* If $\phi(x) = 0$, then the eqn is called homogenous eqn.
* If $\phi(x) \neq 0$, then the eqn is called non homogenous eqn

Note: The solⁿ of the homogenous Linear Differential Eqn:

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

$$D^2 y + a_1 Dy + a_2 y = 0.$$

$$(D^2 + Da_1 + a_2)y = 0. \quad ; f(D) = D^2 + Da_1 + a_2.$$

$\Rightarrow$ Axiliary eqn can be found by. ~~finding the value of m.~~

$$f(m) = m^2 + am + a_2 = 0.$$

Let the roots be m_1 & m_2. [n' roots]

for homogenous { By determining the type of roots, find the complementary fn, which becomes the general eqn.

For non homogenous eqn., find Particular Integral.

$$y_p = P.I = \frac{\phi(x)}{f(D)} \quad \therefore G.S = y = y_c + y_p.$$

~~There are 3 types of roots.~~

~~(i) Real & distinct ; $m_1 \neq m_2$ $\quad y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$~~

~~(ii) Real & equal ; $m_1 = m_2$ $\quad y_c = (C_1 + C_2 x) e^{mx}$~~

~~(iii) Complex ; $m_1 = \alpha \pm i\beta$ $\quad y_c = e^{\alpha x}(C_1$~~

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⇒ Solⁿ of non homogenous Linear D.E.

Consider $f(D)y = \phi(x)$

Auxiliary eqn : $f(m) = 0$

Particular solⁿ = $\frac{\phi(x)}{f(D)} = y_p$

General solⁿ $y = y_c + y_p$

⇒ Method of finding the complimentary function.

Consider the 2nd order D.E

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0.$$

$$f(D) = D^2y + a_1 Dy + a_2 y = 0$$

$$(D^2 + a_1 D + a_2)y = 0$$

Auxiliary Eqn = $f(m) = m^2 + a_1 m + a_2 = 0$.

Case ① Real & Distinct ; $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots$

Case ② Real & Equal; $y_c = (c_1 + c_2 x + c_3 x^2)e^{mx}$

Case ③ Complex. ; Complex roots occur in conjugate pairs. $(\alpha \pm i\beta)$

$$y_c = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

$$= c_1 e^{\alpha x} \cdot e^{i\beta x} + c_2 e^{\alpha x} \cdot e^{-i\beta x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$

$$= e^{\alpha x}(c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x))$$

$$= e^{\alpha x}((c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x)$$

$$y_c = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

Note :- If $m_1, m_2, m_3 \dots m_n$ are distinct roots, then the A.E. associated with the nth order of the eqn then the general solⁿ are given by.

⇒ $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ are called 'n' arbitrary constants.

Note ⇒ ① $y_c = (c_1 + c_2 x + c_3 x^2 + \dots + c_n x^{n-1})e^{mx}$

② If $m_1 = m_2 = m_3 = m$ and the remaining roots are real & distinct

$$y_c = (c_1 + c_2 x + c_3 x^2)e^{mx} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \dots$$

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Note: ① If the root are purely imaginary $\left[x^2+1=0 \Rightarrow x=\pm i\right]$ Eg:

$$y_c = A\cos\beta x + B\sin\beta x$$

② The root $\alpha \pm i\beta$ is repeated 'n' times, then the general sol^n is given by.

Eg: $(x^2+1)(x^2+1)=0$

$x=\pm i, \pm i$

$$y_c = e^{\alpha x}\left((C_1+C_2x+C_3x^2+\dots+C_nx^{n-1})\cos\beta x + i(C_1'+C_2'x+C_3'x^2+\dots+C_n'x^{n-1})\sin\beta x\right]$$

$$y_c = ((C_1+C_2x)\cos x + (C_1'+C_2'x)\sin x)$$

Problems

① $(4D^4-8D^3-7D^2+11D+6)y=0.$

Auxiliary equation; $f(m) = (4m^4-8m^3-7m^2+11m+6)=0$ ~~$m(4m^3-8m^2-7m+$~~

Start with 1 or −1 until last digit becomes '0', first digit always '0'

$m=-1$	4	−8	−7	+11	+6
	0	−4	12	−5	−6
$m=2$	4	−12	5	6	0
	0	8	−8	−6	
	4	−4	−3	0	

$\therefore 4m^4-4m^3-3m^2=0$

$4m^2-4m-3=0.$

$4m^2-6m+2m-3=0$

$2m(2m-3)+1(2m-3)=0$

$2m-3=0 \qquad 2m+1=0$

$m_1=3/2 \qquad m_2=-1/2, -1, 2.$

$$y_c = C_1e^{3/2x}+C_2e^{-1/2x}+C_3e^{-x}+C_4e^{2x}.$$

② $(D^4+2D^3-5D^2-6D)y=0.$

Auxiliary form $f(m)=(m^4+2m^3-5m^2-6m)=0.$

$m(m^3+2m^2-5m-6)=0 \quad \therefore m=0$

−1	1	2	−5	−6
	0	−1	−1	6
	1	1	−6	0

$\therefore m^3+m^2-6m=0.$

$m^2+m-6=0$

$m^2+3m-2m-6=0$

$m(m+3)-2(m+3)=0$

$m=2,-3,0,-1.$

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$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\therefore y_c = C_1 e^{2x} + C_2 e^{-3x} + C_3 + C_4 e^{-x}$.

$(D^3 + D^2 + 4D + 4)y = 0$

A.E $\Rightarrow m^3 + m^2 + 4m + 4 = 0$

-1	1	1	4	4
	0	-1	0	-4
	1	0	4	0

$\therefore m^2 + 0m + 4 = 0$.

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i \Rightarrow m = -2i, +2i, -1$.

$y_c = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

$= A\cos 2x + B\sin 2x + C_1 e^{-x}$

$(D^4 - 2D^3 + D^2)y = 0$

A.E = ~~$D^4 - 2D^3$~~ $m^4 - 2m^3 + m^2 = 0$.

$m^2(m^2 - 2m + 1) = 0$.

$m^2 = 0 \Rightarrow m = 0, 0$.

$m^2 - 2m + 1 = 0$.

$(m-1)^2 = 0$.

$m = 1, 1, 0, 0$

$\therefore y_c = (C_1 + C_2 x)e^{0x} + (C_3 + C_4 x)e^{x}$

$= C_1 + C_2 x + (C_3 + C_4 x)e^{x}$

<u>Inverse differential operator method and particular integral</u>

$\boxed{f(D)y = \phi(x)}$

The differential operator $D = d/dx$

The integration of $\phi(x)dx$ is called an inverse operation so $1/D$ stands for the integral and is called the inverse differential operator.

$1/D$, $1/D^2$, $1/D^3$ etc stands for successive integration.

<u>Particular integral</u>

<u>Type 1:</u> $\phi(x) = e^{ax}$ [RHS is exponential]

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$f(D)y = \phi(x)$

Particular Integral $= y_p = \dfrac{\phi(x)}{f(D)} = \dfrac{e^{ax}}{f(D)}$

$= \dfrac{e^{ax}}{f(a)}$ where $f(a) \neq 0$ [Replace D by a]

If $f(a) = 0$.

$\dfrac{e^{ax}}{f(D)} = \dfrac{xe^{ax}}{f'(D)} = \dfrac{xe^{ax}}{f'(a)}$ [Replace D by a]

If $f'(a) = 0$.

$\dfrac{xe^{ax}}{f'(D)} = \dfrac{x^2e^{ax}}{f''(D)} = \dfrac{x^2e^{ax}}{f''(a)}$ [Replace D by a]

Note: $e^{\log f(x)} = f(x)$

Problems

Type 1 (1)

Solve. $(D^2-4)y = \cosh(2x-1) + 3^x$.

A.E $= f(m) = m^2 - 4 = 0$.

$m^2 = 4$.

$m = \pm 2$.

$y_c = C_1e^{2x} + C_2e^{-2x}$.

$P.I = y_p = \dfrac{\cosh(2x-1) + 3^x}{D^2-4}$.

$= \dfrac{\cosh(2x-1)}{D^2-4} + \dfrac{3^x}{D^2-4}$. $[P_1 + P_2]$

$= \dfrac{e^{2x-1} + e^{-2x+1}}{2(D^2-4)} + \dfrac{3^x}{D^2-4}$.

$P_1 = \dfrac{1}{2}\left[\dfrac{e^{2x-1}}{D^2-4} + \dfrac{e^{-(2x-1)}}{D^2-4}\right]$

$= \dfrac{1}{2}\left[\dfrac{e^{2x-1}}{2^2-4} + \dfrac{e^{-(2x-1)}}{2^2-4}\right]$ [Since $D^2-4=0$]

$= \dfrac{1}{2}\left[\dfrac{xe^{2x-1}}{2D} + \dfrac{xe^{-2x+1}}{2D}\right]$

$= \dfrac{1}{2}\left[\dfrac{xe^{2x-1}}{4} - \dfrac{xe^{-2x+1}}{4}\right]$

$= \dfrac{x}{4}\left[\dfrac{e^{2x-1} - e^{-(2x-1)}}{2}\right] = \dfrac{x}{4}\sinh(2x-1)$

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$$P_2 = \frac{3^x}{D^2-4} = \frac{e^{\log 3^x}}{D^2-4}$$

$$= \frac{e^{x\log 3}}{D^2-4}$$

$$P_2 = \frac{e^{x\log 3}}{(\log 3)^2-4}$$

$$y = y_c + P_1 + P_2$$

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}\sinh(2x-1) + \frac{e^{x\log 3}}{(\log 3)^2-4}$$

RHS is trigonometric.

Type2: The particular integral of the form $\sin ax$ or $\cos ax$. In this form, try to replace D^2 by $-a^2$. The denominator $f(-a^2) \neq 0$. ~~If it is equal to '0'~~ we get that to be in Linear form.

If $f(-a^2)=0$, then multiply x to numerator diff the denom.

Solve $(D^2-4D+4)y = \cos 2x$

$$A.E = m^2-4m+4=0$$

$$m^2-2m-2m+4=0$$

$$m(m-2)-2(m-2)=0$$

$$m = 2, 2.$$

$$y_c = (c_1+c_2x)e^{2x}$$

$$P.I = y_p = \frac{\phi(x)}{f(D)}$$

$$= \frac{\cos 2x}{D^2-4D+4} = \frac{x\cos 2x}{-4-4D+4} = \frac{-1}{4}\int \cos 2x$$

$$= \frac{-1}{8}\sin 2x$$

$$\therefore G.S = (c_1+c_2x)e^{2x} - \frac{1}{8}\sin 2x$$

$$(D^3-1)y = 3\sin 2x$$

$$A.E = m^3-1=0 \Rightarrow \cancel{m^3=1 \Rightarrow m=1,1,1}$$

$$(m-1)(m^2+m+1)=0 \quad m=1$$

$$\cancel{y_c = (c_1+c_2x+c_3x^2)e^x}$$

$$m^2+m+1=0$$

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{-1+\sqrt{1-4}}{2} \text{ or } \frac{-1-\sqrt{1-4}}{2}$$

$$\frac{-1+i\sqrt{3}}{2} \text{ or } \frac{-1-i\sqrt{3}}{2}$$

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$$y = C_1 e^{x} + e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right).$$

$$P.I = \frac{\phi(x)}{f(D)} = \frac{3\sin 2x}{D^3-1}$$

$$= \frac{3\sin 2x}{-4D-1}.$$

$$= \frac{3\sin 2x}{-(4D+1)} \times \frac{4D-1}{4D-1}.$$

$$= \frac{3\sin 2x \times (4D-1)}{-(16D^2-1)}.$$

$$= \frac{3(4D-1)\sin 2x}{65}$$

$$= \cancel{12D \sin 2x}$$

$$\frac{(12D-3)\sin 2x}{65}.$$

$$= \frac{24\cos 2x - 3\sin 2x}{17}.$$

$$G.S = y_c + y_p = C_1 e^{x} + e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\sin\frac{\sqrt{3}}{2}x\right)$$

$$+ \frac{24\cos 2x - 3\sin 2x}{17}.$$

(3) $(D^2(D^2+4)(D^2+9))y = 2\sin x/2 \cos x/2.$

A.E $\Rightarrow$ $m^2(m^2+4)(m^2+9) = 0$

$m = 0, 0.$

$m = \pm 2i$

$m = \pm 3i$

$$y_c = (C_1 + C_2 x) + (A\cos 2x + B\sin 2x) + (A\cos 3x + B\sin 3x)$$

$$P.I = \frac{\phi(x)}{f(D)} = \frac{\sin x}{D^2(D^2+4)(D^2+9)}$$

$$= \frac{\sin x}{-1(3)(8)} = \frac{-\sin x}{24}.$$

$$G.S = C_1 + C_2 x + A\cos 2x + B\sin x + A\cos 3x + B\sin 3x - \frac{\sin x}{24}.$$

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Type ② When the particular integral of the form $\phi(x)/f(D)$ where $\phi(x)$ is a polynomial in x. P.I is found by division by writing $\phi(x)$ is decreasing powers of x and $f(D)$ in ascending powers of 'D'. The division gets completed with remainder $=0$. The quotient so obtained in the process of division will be the P.I.

① $(D^2+3D+2)y = x^2+3x+1$

A.E $\Rightarrow m^2+3m+2=0$

$m^2+2m+1m+2=0$

$m=-1,-2$

$y_c = c_1e^{-1x}+c_2e^{-2x}$

$PI = \frac{\phi(x)}{f(D)} = \frac{x^2+3x+1}{D^2+3D+2}$

$= \frac{x^2}{2}$

	$x^2/2$
$2+3D+D^2$	x^2+3x+1
$(-)$	x^2+3x+1
$3D(x^2/2) = 3(\frac{2x}{2}) = 3x$	0
$D^2(x^2/2) = 2(1) = 1$	

$G.S = y_c + y_p$

$= c_1e^{-x}+c_2e^{-2x}+x^2/2$

② $(D^3-7D+6)y = x^2-x+1$

A.E $= m^3-7m+6=0$

1	1	0	-7	6
	0	1	1	-6
	1	1	-6	0

$m^2+m-6=0$

~~$m(m-6)=0 \Rightarrow m=0$~~, $m^2+3m-2m-6=0$

$m=-3,+2,1$

$y_c = c_1e^{-3x}+c_2e^{+2x}+c_3e^{x}$

$PI = \frac{\phi(x)}{f(D)} = \frac{x^2-x+1}{6-7D+D^3}$

$= \frac{x^2}{6}+\frac{2x}{9}+\frac{23}{54}$

	$x^2/6 + 2x/9 + 23/54$
$6-7D+D^3$	x^2-x+1
	$x^2-7x/3+0$
$-7(x/3) = -\frac{7x}{3}$	$4x/3+1$
$D^3(x^2/6) = 1/3$	$4x/3-14/9$
	$23/9$
	$23/9$
	0

$G.S = y_c+y_p$

$= c_1e^{-3x}+c_2e^{2x}+c_3e^{x}+\frac{x^2}{6}+\frac{2x}{9}+\frac{23}{54}$

③ $(D^2+D)y = x^2+2x+4$

④ $(D^3+3D^2+3D+1)y = 6e^{-x}+\cos 2e^{0x}$

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$(D^2+D)y = x^2+2x+4.$

A.E $= m^2+m = 0.$

$m(m+1) = 0 \Rightarrow m = 0, -1.$

$y_c = C_1 + C_2 e^{-x}$

$P.I = \frac{\phi(x)}{f(D)} = \frac{x^2+2x+4}{D+D^2} = \frac{x^3}{3} + 4x$

$$\begin{array}{r|l} & x^3/3 + 4x \\ \hline D+D^2 & x^2+2x+4 \\ & x^2+2x \\ \hline & 0+4 \\ & 4 \\ \hline & 0 \end{array}$$

$\therefore G.S = y_c + y_p = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x.$

$(D^3+3D^2+3D+1)y = 6e^{-x} + \log 2 e^{0x}$

A.E $= (m^3+3m^2+3m+1) = 0.$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$

$m^3+2m^2+m = 0.$

$m^2+2m+1 = 0.$

$m^2+m+m+1 = 0.$

$m(m+1)+1(m+1) = 0 \Rightarrow m = -1, -1, -1.$

$y_c = (C_1 + C_2 x + C_3 x^2)e^{-1x}.$

$P.I = \frac{\phi(x)}{f(D)} = \frac{6e^{-x} + \log 2 e^{0x}}{D^3+3D^2+3D+1.}$

$= \frac{6e^{-x}}{D^3+3D^2+3D+1} + \frac{\log 2 e^{0x}}{D^3+3D^2+3D+1.}$

$= \cancel{\frac{6e^{-x}}{-1+3-3}} \frac{6xe^{-x}}{(3D^2+6D+3)} + \frac{\log 2 e^{0x}}{1.}$

$= \cancel{\frac{6xe^{-x}}{3-6+3}} \frac{6x^2e^{-x}}{6D+6} + \frac{\log 2 e^{0x}}{1.}$

$= \frac{6x^3e^{-x}}{6} + \log 2 e^{0x}.$

$P.I: = x^3e^{-x} + \log 2$

$G.S = y_c + y_p = (C_1 + C_2 x + C_3 x^2)e^{-x} + x^3e^{-x} + \log 2.$

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Type 4. $\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(D+a)}$.

① $(D^2-2D+4)y = e^x\cos x$.

A.E. $= f(m) = m^2-2m+4=0$.

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-16}}{2} = \frac{2\pm i\sqrt{12}}{2} = 1\pm i\sqrt{3}$.

$y_c = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x)$.

$P.I = \frac{\phi(x)}{f(D)} = \frac{e^x\cos x}{D^2-2D+4}$.

$= \cancel{\frac{e^x\cos x}{1-2+4}}\ \frac{e^x\cos x}{(D+1)^2-2(D+1)+4}$.

$= \frac{e^x\cos x}{D^2+2D+1-2D-2+4}$

$= \frac{e^x\cos x}{D^2+3}$.

$= \frac{e^x\cos x}{-1+3}$

$P.I. = \frac{e^x\cos x}{2}$.

$G.S = y_c + y_p = e^x(A\cos\sqrt{3}x + B\sin\sqrt{3}x) + \frac{e^x\cos x}{2}$.

② $(D^2-2D+5)y = e^{2x}\sin x$.

A.E. $= f(m) = m^2-2m+5=0$.

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm\sqrt{16}\,i}{2}$

$= 1\pm i2$.

$y_c = e^x(A\cos 2x + B\sin 2x)$.

$P.I = \frac{\phi(x)}{f(D)} = \frac{e^{2x}\sin x}{D^2-2D+5}$.

$= \frac{e^{2x}\sin x}{(D+2)^2-2(D+2)+5}$

$= \frac{e^{2x}\sin x}{D^2+\cancel{4}4D-2D-\cancel{4}+5}$

$= \frac{e^{2x}\sin x}{D^2+2D+5}$.

$= \frac{e^{2x}\sin x}{-1^2+2D+5}$

$= \frac{e^{2x}\sin x}{2D+4}\times\frac{(2D-4)}{(2D-4)}$

$= \frac{e^{2x}\sin x(2D-4)}{4D^2-16}$.

$= \frac{2e^{2x}\cos x+4\sin x e^{2x}-4e^{2x}\sin}{4D^2-16}$.

$= \frac{2\cdot e^{2x}(\cos x+2\sin x)}{-20}$.

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$$G.S = e^{x}(A\cos 2x + B\sin 2x) - \frac{e^{2x}(\cos x - 2\sin x)}{10}$$

(2) $(D^3 - 3D^2 + 3D - 1)y = 60x^2e^x$.

$A.E = f(m) = m^3 - 3m^2 + 3m - 1$

1	1	-3	3	-1
	0	1	-2	1
	1	-2	1	0

$m^2 - 2m + 1 = 0$.

$m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1) = 0$

$m = 1, 1, 1$

$y_c = (c_1 + c_2x + c_3x^2)e^x$.

$$P.I = y_p = \frac{\phi(x)}{f(D)} = \frac{60x^2e^x}{(D^3 - 3D^2 + 3D - 1)}$$

$$= \frac{60x^2e^x}{(D+1)^3 - 3(D+1)^2 + 3(D+1) - 1}$$

$$= \frac{60x^2e^x}{D^3 + 1 + 3D(D+1) - 3(D^2 + 1 + 2D) + 3D + 3 - 1}$$

$$= \frac{60x^2e^x}{\cancel{D^3} + \cancel{1} + \cancel{3D^2} + \cancel{3D} - \cancel{3D^2} - \cancel{3} - \cancel{6D} + \cancel{3D} + \cancel{3} - \cancel{1}}$$

$$= \frac{60x^2e^x}{D^3}$$

$$= 60e^x\left(\frac{x^2}{D^3}\right)$$

$x^5/60$

$D^3 \,|\, x^2$

$ \,|\, x^2$

$ \,|\, 0$.

$y_p = e^x x^5$

$G.S = y_c + y_p = (c_1 + c_2x + c_3x^2)e^x + e^x x^5$

Type (5) The particular integral of the form $\frac{xf(x)}{f(D)}$, $\frac{x^n f(x)}{f(D)}$

$$\frac{xf(x)}{f(D)} = \left(x - \frac{f'(D)}{f(D)}\right)\frac{f(x)}{f(D)}$$

$$\frac{x^2 f(x)}{f(D)} = x\left(\frac{xf(x)}{f(D)}\right)$$

$$\frac{x^n \cos ax}{f(D)} = \frac{x^n e^{iax}}{f(D)} \quad \text{[only Real part]}$$

$$\frac{x^n \sin ax}{f(D)} = \frac{x^n e^{iax}}{f(D)} \quad \text{[only imaginary part]}$$

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① $(D^2+16)y = x\sin x.$

$AE = f(m) \Rightarrow m^2+16=0.$

$m = \pm i4.$

$y_c = (A\cos 4x + B\sin 4x).$

$P.I = \frac{\phi(x)}{f(D)} = \frac{x\sin x}{f(D)}$

$= \frac{x e^{ix}}{D^2+16}$

$\frac{x\sin x}{f(D)} = \left(x - \frac{2D}{D^2+16}\right)\left(\frac{\sin x}{D^2+16}\right)$

$= \left(x - \frac{2D}{D^2+16}\right)\left(\frac{\sin x}{15}\right)$

$= \frac{x\sin x}{15} - \frac{2\cos x}{15(D^2+16)}$

$= \frac{x\sin x}{15} - \frac{2\cos x}{15(15)}$

$= \frac{x\sin x}{15} - \frac{2\cos x}{225}$

$y_p = \frac{15x\sin x - 2\cos x}{225}$

$G.S = y_c + y_p = A\cos 4x + B\sin 4x + \frac{15x\sin x - 2\cos x}{225}$

② $(D^2+16)y = x\sin 3x.$ ~~$P.I = \frac{x\sin 3x}{7} + C$~~

③ $(D^2-2D+1)y = x\cos x.$

$A.E = m^2 - 2m + 1 = 0.$

$m^2 - 1m - 1m + 1 = 0.$

$m(m-1) - 1(m-1) = 0 \Rightarrow m = 1, 1.$

$y_c = (C_1 + C_2 x)e^x.$

$P.I = \frac{\phi(x)}{f(D)} = \frac{x\cos x}{D^2-2D+1} = \left[x - \frac{(2D-2)}{(D^2-2D+1)}\right]\left(\frac{\cos x}{D^2-2D+1}\right)$

$= \left[x - \frac{(2D-2)}{D^2-2D+1}\right]\left(\frac{\cos x}{-2D}\right)\left(\frac{-1}{2}\sin x\right)$

$= \left[\frac{x\cos x}{2D} + \frac{2D\cos x}{2D(D^2-2D+1)} - \frac{2\cos x}{2D(D^2-2D+1)}\right]$

$= \left(x - \frac{2}{2D}\right)\frac{\cos x}{2D} + \frac{2\sin x}{2D^3-4D^2+2D} - \frac{2\cos x}{2D^3-4D^2+2D}$

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(4) $(D^2-4D+4)y = 8x^2e^{2x}\sin x.$

A.E $\Rightarrow f(m)=0.$

$m^2-4m+4=0.$

$m^2-2m-2m+4=0 \Rightarrow m=2,2.$

$y_c = (C_1+C_2x)e^{2x}.$

$$y_p = \frac{\phi(x)}{f(D)} = \frac{8e^{2x}x^2\sin x}{(D^2-4D+4)}$$

$$= \frac{8e^{2x}(x^2\sin x).}{(D+2)^2-4(D+2)+4}$$

$$= \frac{8e^{2x}(x^2\sin x)}{D^2+4+4D-4D-8+4}$$

$$= \frac{8e^{2x}(x^2\sin x)}{D^2}$$

$$\left[\frac{x^n\sin ax}{f(D)} = \text{imaginary part of } \frac{e^{iax}x^n}{f(D)}\right]$$

$$= (8e^{2x})\text{ imaginary } \frac{e^{ix}x^2}{D^2}.$$

$$= (8e^{2x})\left[\text{imaginary } e^{ix}\frac{x^2}{(D^2+i)^2}\right]$$

$$= 8e^{2x}\left[\text{im } e^{ix}\frac{x^2}{D^2-1+2iD^2}\right]$$

$$= 8e^{2x}\left[\text{im } e^{ix}\frac{x^2}{D^2+2Di-1}\right]$$

$-1+2Di+D^2$

$$= 8e^{2x}\left[\text{im } e^{ix}(-x^2-4xi-6)\right]$$

$$= 8e^{2x}[\text{imaginary } e^{ix}(-x^2-4xi+6)]$$

$$= 8e^{2x}(\text{imaginary } (\cos x+i\sin x)((-x^2+6)-4xi))$$

$$= 8e^{2x}(\text{img}[(-x^2+6)\cos x-4xi\cos x+(-x^2+6)i\sin x+4x\sin x]$$

$$= 8e^{2x}[\text{I.P}[(-x^2+6)\cos x+4x\sin x-4xi\cos x+(-x^2+6)i\sin x]]$$

$$P.I = 8e^{2x}[(-x^2+6)\sin x-4x\cos x]$$

$$C.S = (C_1+C_2x)e^{2x}+8e^{2x}[(-x^2+6)\sin x-4x\cos x]$$

$-x^2-4xi-6$

x^2

$x^2-4xi-2$

$4xi+2$

$4xi-8-0$

$4i+2$

$4i+0$

2

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(5) Solve $(D^3+D^2-4D-4)y = 3e^{-x}-4x-6$.

A.E $= f(m) = 0$

$m^3+m^2-4m-4=0$.

-1	1	1	-4	-4
	0	-1	0	4
	1	0	-4	0

$m^2+0m-4=0$

$m = \pm 2$.

$y_c = C_1e^{2x}+C_2e^{-2x}$

$P.I = \frac{\phi(x)}{f(D)} = \frac{3e^{-x}-4x-6}{D^3+D^2-4D-4}$.

$= \frac{3e^{-x}}{(-1)^3+(-1)^2-4(-1)-4} - \frac{4x}{(-4-4D+D^2+D^3)} - \frac{6e^{0x}}{(-4-4D+D^2+D^3)}$

$= \frac{x3e^{-x}}{3D^2+2D-4} - (-x+1) - \left(\frac{6}{-4}\right)$

$= \frac{x3e^{-x}}{3-2-4} - (1-x) + \frac{6}{4}$.

$= \frac{3xe^{-x}}{-3} - 1 + x + \frac{3}{2}$.

$y_p = \frac{1}{2} + x - xe^{-x}$

$-x+1$

$-4-4D+D^2+D^3 \;\big|\; 4x$

$4x+4$

$0+4$

-4

6

$C.S = C_1e^{2x}+C_2e^{-2x} + \frac{1}{2} + x - xe^{-x}$.

(6) $(D^2-3D+2)y = 2\sin x\cos x + xe^{x}$.

A.E $\Rightarrow f(m)=0$.

$m^2-3m+2=0 \Rightarrow m = +2, +1$.

$y_c = C_1e^{x}+C_2e^{2x}$.

$P.I = \frac{\phi(x)}{f(D)} = \frac{\sin 2x + xe^{x}}{D^2-3D+2}$.

$= \frac{\sin 2x}{-4-3D+2} + \frac{xe^{x}}{(D+1)^2-3(D+1)+2}$

$= \frac{\sin 2x}{-(3D+2)} + \frac{xe^{x}}{D^2+1+2D-3D-3+2}$

$= \frac{\sin 2x}{-(3D+2)} \times \frac{(3D-2)}{(3D-2)} + \frac{xe^{x}}{D^2-D}$.

$= \frac{6\cos 2x - 2\sin 2x}{-(9D^2-4)} + \frac{xe^{x}}{D^2-D}$.

$= \cos 2x - 2\sin 2$.

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$$y_p = \frac{6\cos 2x - 2\sin 2x}{-(9(-4)-4)} + e^x\left(\frac{x}{-D+D^2}\right)$$

$$y_p = \frac{6\cos 2x - 2\sin 2x}{40} + e^x\left(\frac{-x^2-x}{2}\right)$$

$-D+D^2$	x
	$x-1$
	1
	1
	0

$$G.S = y_c + y_p$$
$$= C_1e^x + C_2e^{2x} + \frac{6\cos 2x - 2\sin 2x}{36} + e^x\left(\frac{-x^2}{2} - x\right)$$

(7) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$, given that $y(0) = 0$; $\frac{dy}{dx}(0) = 15$

$A.E = m^2 + 5m + 6 = 0.$

$m = -2, -3.$

$y_c(x) = C_1e^{-2x} + C_3e^{-3x}.$

$y(0) = 0 \Rightarrow C_1e^{-2(0)} + C_3e^{-3(0)} = 0.$

$C_1 + C_3 = 0.$

$\frac{dy}{dx}(0) = -2C_1e^{-2x} - 3C_3e^{-3x} = 15.$

$= -2C_1 - 3C_3 = 15.$

$2C_1 + 2C_2 = 0$

$-2C_1 - 3C_2 = 15$

$-C_2 = 15$

$\boxed{C_2 = -15 \Rightarrow C_1 = 15}$

$G.S = 15e^{-2x} - 15e^{-3x}.$

(8) $(D^2 - 4D + 5)y = 0$ $\quad y(0) = 1 \quad y_1(0) = 2.$

$A.E = m^2 - 4m + 5 = 0$

~~$m = 1, 5$~~ $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$y_c = C_1e^x + C_2e^{5x}.$

$y(0) = C_1 + C_2 = 1.$

$y_1(0) = C_1e^x + 5C_2e^{5x} = 2.$

$C_1 + 5C_2 =$

$= \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm 2i}{2} = 2 \pm i$

(9) $(D^2 - 3D + 2)y = 0$; $y(0) = 0$; $y_1(0) = 1.$

$y_c = e^{2x}(C_1\cos x + C_2\sin x).$

$y(0) = e^{2x}(C_1) = 0.$

~~$C_1 = 0$ & $C_2 =$~~

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(10) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 2\sin 3x$; $y(0)=0$ & $\frac{dy}{dx}(0)=0$.

$f(m) = m^2 - 9m = 0$

$m = 0, 9.$

$y_c = C_1 + C_2 e^{9x}.$

$P.I = \frac{\phi(x)}{f(D)} = \frac{2\sin 3x}{D^2 - 9D} = \frac{2\sin 3x}{-9-9D}$

$= \frac{2\sin 3x(9-9D)}{-(9+9D)(9-9D)}$

$= \frac{18\sin 3x - 54\cos 3x}{-(81-81D^2)}$

$= \frac{18\sin 3x - 54\cos 3x}{-(81+243)}$

$= \frac{18\sin 3x - 54\cos 3x}{-(324)}$

$= \frac{\sin 3x - 3\cos 3x}{-(45)}$

$y = C_1 + C_2 e^{9x} + \frac{3\cos 3x - \sin 3x}{45}$

$x = 0, y = 0.$

$0 = C_1 + C_2 + 3/45$

$C_1 + C_2 = -3/45.$

(11) Solve $(D^2+4D)y = x^2 + \cos 2x + 2^{-x}$.

$A.E = f(m) = 0.$

$m^2 + 4m = 0 \Rightarrow m(m+4) = 0 \Rightarrow m = 0, -4.$

$y_c = C_1 + C_2 e^{-4x}.$

$P.I = \frac{\phi(x)}{f(D)} = \frac{x^2 + \cos 2x + 2^{-x}}{D^2 + 4D}$

$= \frac{x^2}{4D+D^2} + \frac{\cos 2x}{-4+4D} + \frac{e^{-x\log 2}}{D^2+4D}$

$= \frac{x^3}{12} - \frac{x^2}{16} + \frac{x}{32} + \frac{\cos 2x(4D+4)}{(4D-4)(4D+4)} + \frac{e^{-x\log 2}}{(-\log 2)^2 - 4\log 2}$

$= \frac{x^3}{12} - \frac{x^2}{16} + \frac{x}{32} + \left(\frac{-8\sin 2x + 4\cos 2x}{16D^2 - 16}\right) + \frac{e^{-x\log 2}}{(-\log 2)^2 - 4\log 2}$

$= \frac{x^3}{12} - \frac{x^2}{16} + \frac{x}{32} + \frac{8\sin 2x - 4\cos 2x}{80} + \frac{e^{-x\log 2}}{(-\log 2)^2 - 4\log 2}$

$4D+D^2 \,|\, x^2$

$x^2 + x/2$

$0 - x/2$

$-x/2$

$0 + 1/8$

$1/8$

0

$x/32$

$x^3/12 - x^2/16$

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$\therefore$ G.S $= y(x) = y_c + y_p$

$$G.S = C_1 + C_2 e^{-4x} + x^3/12 - x^2/16 + x/32 + \frac{8\sin 2x - 4\cos 2x + e^{x\log 2}}{80}$$

$(\log 2)^2 + 4$

Application Problems.

① In an LCR circuit the charge (q^3) on a plane of a condenser is given by $L\frac{d^2q}{dt^2} + r\frac{dq}{dt} + \frac{q}{c} = E\sin pt$. The circuit is tuned to resonance so that $p^2 = 1/LC$. If initially the current 'i' and the charge 'q' be '0' show that the small values of $\frac{R}{L}$, the current in the circuit at time 't' is given by $\frac{Et}{2L}\sin pt$.

Sol^n

$$\left(L\frac{d^2}{dt^2} + r\frac{d}{dt} + \frac{1}{c}\right)q = E\sin pt$$

$$A.E = f(m) = Lm^2 + rm + \frac{1}{c} = 0$$

$$\text{Roots} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-r \pm \sqrt{r^2 - 4L/c}}{2L} = \frac{1}{2L}\left(-r \pm \sqrt{r^2 - \frac{4L}{c}}\right)$$

$$= \frac{-r}{2L} \pm \frac{1}{2L}\sqrt{r^2 - \frac{4L}{c}} = \frac{-r}{2L} \pm \sqrt{\frac{r^2}{4L^2} - \frac{1}{LC}}$$

$$= \frac{-r}{2L} \pm \sqrt{\frac{r^2}{4L^2} - p^2}$$

$$= \frac{-r}{2L} + \sqrt{\frac{-1}{LC}} \quad \left[\because \frac{R}{L} \text{ is small}\right]$$

$$= \frac{-r}{2L} \pm \frac{i}{\sqrt{LC}}$$

$$q_c = e^{(-r/2L)t}\left(C_1\cos\left(\frac{t}{\sqrt{LC}}\right) + C_2\sin\left(\frac{t}{\sqrt{LC}}\right)\right)$$

$$= e^{(-r/2L)t}\left(C_1\cos pt + C_2\sin pt\right)$$

As (R/L) is small $\therefore$ we will be considering the first order of RL and try to neglect $\left(\frac{R^2}{L^2}\right)$

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$$P.I = \frac{\phi(x)}{f(D)} = \frac{E\sin pt}{(LD^2+rD+1/C)}$$

$$q_c = \left(1 - \frac{rt}{2L} + \left(\frac{rt}{2L}\right)^2 + \cdots\right)\left(C_1\cos pt + C_2\sin pt\right)$$

$$q_c = \left(1 - \frac{rt}{2L}\right)\left(C_1\cos pt + C_2\sin pt\right).$$

$$P.I = \frac{\phi(x)}{f(D)} = \frac{E\sin pt}{(LD^2+rD+1/C)} = \frac{E\sin pt}{(Lp^2+rD+1/C)}$$

$$= \frac{E\sin pt}{(L(-1/LC)+rD+1/C)}$$

$$= \frac{E\sin pt}{(-1/C+rD+1/C)}$$

$$P.I = -\frac{E}{rp}\cos pt = q_p.$$

The complete solⁿ is $= q_c + q_p = q$

$$= \left(1 - \frac{rt}{2L}\right)\left(C_1\cos pt + C_2\sin pt\right) - \frac{E\cos pt}{rp} \quad \text{--- (1)}$$

Initially when $t=0$, $q=0$ & $i=0$ in eqn (1)

$$0 = C_1 - \frac{E}{rp} \Rightarrow \boxed{C_1 = \frac{E}{Rp}}$$

Diff eqn (1) w.r.t 't'

$$\frac{dq}{dt} = \left(\frac{-R}{2L}\right)\left(C_1\cos pt + C_2\sin pt\right) + \left(1 - \frac{Rt}{2L}\right)\left(-C_1p\sin pt + C_2p\cos pt\right) + \frac{E\sin pt}{rp} \quad \text{--- (2)}$$

$$dq/dt = i = 0$$

$$0 = \left(\frac{-R}{2L}\right)(C_1) + (1)(C_2p)$$

$$C_2p = C_1 \cdot \frac{R}{2L}$$

$$\boxed{C_2 = \frac{E}{Rp^2} \cdot \frac{R}{2L} = \frac{E}{2p^2L}}$$

$$q = \left(1 - \frac{rt}{2L}\right)\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) - \frac{E\cos pt}{rp}$$

$$q = \frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt - \frac{rt}{2L}$$

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Substitute C_1 & C_2 in Eqn ② ~~we get the current = $\frac{Et}{2L}\sin pt$~~

$$i = -\frac{ER}{4L^2p^2}\sin pt + \frac{Et}{2L}\sin pt - \frac{ERt}{4L^2p}\cos pt$$

small
so neglect

$$i = \frac{Et}{2L}\sin pt$$

LINEAR DIFFERENTIAL EQN WITH CONST COEFF.

⇒ Cauchy's & Legendre's Linear D.E.

$$a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = \phi(x)$$

Cauchy's Linear Eqn

$a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = \phi(x)$ is called the Cauchy's D.E of second order

Working Procedure

Step① – Substitute $z = \log x$ or $x = e^z$ in the case of Cauchy's eqn

Step② – $x\frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$, $D = \frac{d}{dz}$; $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$.

Step③ – Substitute to obtain a linear eqn with constant coeff.

Step4 : – Solve Linear D.E to obtain the sol^n $y = y_c + y_p$ in terms of 'z'

Step5 : – Substitute for 'z' & present the sol^n y in terms of 'x'

Problem:

① $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x$.

$D = \frac{d}{dz}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$; $x\frac{dy}{dx} = Dy$.

$\therefore (D(D-1) + 4D + 2)y = e^z$

A.E = $f(m) = 0 \Rightarrow m(m-1) + 4m + 2 = 0$

$m^2 + 3m + 2 = 0$

$m = -1, -2$

$y_c = C_1e^{-z} + C_2e^{-2z}$

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$$P.I = \frac{\phi(x)}{f(D)} = \frac{e^z}{D^2+3D+2} = \frac{e^z}{1+3+2} = \frac{e^z}{6}.$$

$$\therefore G.S = C_1 e^{-1z} + C_2 e^{-2z} + \frac{e^z}{6}.$$

$$G.S = C_1 (x)^{-1} + C_2 (x)^{-2} + \frac{x}{6}$$

(2) $$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\sin(\log x).$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y. \; ; \; x\frac{dy}{dx} = Dy.$$

$$(D(D-1) - D + 2)y = x\sin(\log x)$$

$$A.E = f(m) = 0.$$

$$m(m-1) - m + 2 = 0$$

$$m^2 - 2m + 2 = 0.$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

$$= \frac{2 \pm \sqrt{4-8}}{2}.$$

$$= \frac{2 \pm i\sqrt{4}}{2} = 1 \pm i$$

$$y_c = e^z(A\cos z + B\sin z).$$

$$P.I = \frac{\phi(x)}{f(D)} = \frac{e^z\sin(z)}{D^2-2D+2}$$

$$= e^z\left(\frac{\sin(z)}{(D+1)^2-2(D+1)+2}\right)$$

$$= e^z\left(\frac{\sin(z)}{D^2+1+2D-2D-2+2}\right)$$

$$= e^z\left(\frac{\sin(z)}{D^2+1}\right)$$

$$= e^z\left(\frac{\sin(z)}{2D}\right)$$

$$= \frac{e^z}{2}(-\cos z) = -\frac{ze^z\cos z}{2}$$

$$G.S = e^z(A\cos z + B\sin z) - \frac{ze^z\cos z}{2}.$$

$$= x\,(A\cos(\log x) + B\sin(\log x)) - \frac{\log x \cdot x\cos(\log x)}{2}.$$

(3) $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 2\cos^2(\log x).$$

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$$x^2\frac{d^2y}{dx^2} = D(D-1)y \; ; \; x\frac{dy}{dx} \; Dy$$

$$(D(D-1) + D + 1)y = 2\cos^2(z).$$

A.E $\Rightarrow f(m) = 0$

$$m^2 - m + m + 1 = 0$$

$$m = \pm\sqrt{-1}.$$

$$m = \pm i.$$

$$y_c = (A\cos z + B\sin z)$$

$$P.I = \frac{\phi(z)}{f(D)} = \frac{2\cos^2(z)}{D(D^2+1)}$$

$$= \frac{\cos 2z + 1}{D(D-1)}$$

$$= \frac{\cos 2z}{D^2 - D} + \frac{1}{D^2 - D}$$

$$= \frac{\cos 2z}{-(4+D)} \times \frac{(4-D)}{(4-D)} - z$$

$$= \frac{4\cos 2z + 2\sin 2z}{-(16 - D^2)} - z$$

$$= \frac{4\cos 2z + 2\sin 2z}{-(16+4)} - z$$

$$= \frac{4\cos 2z + 2\sin 2z}{-20} - z$$

$$P.I. = \frac{-2\cos 2z - \sin 2z}{20} - z$$

$$G.S = A\cos z + B\sin z - \frac{2\cos 2z - 2\sin 2z}{20} - z.$$

$-D + D^2 \;\Big|\; 1 \;\; 1 \;\; 0$

(5) $$x^2\frac{d^2y}{dx^2} - (2m-1)x\frac{dy}{dx} + (m^2+n^2)y = n^2x^m\log x.$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y \; ; \; x\frac{dy}{dx} = Dy.$$

$$\therefore (D(D-1) - (2m-1)D + (m^2+n^2))\,y = n^2 z e^{mz}.$$

$$(D^2 - \cancel{D} - 2mD + \cancel{D} + m^2 + n^2)y = n^2 e^{zm} z.$$

$$(D^2 - 2mD + m^2 + n^2)y = n^2 e^{zm} z$$

A.E $= f(k) = 0 \Rightarrow k^2 - 2mk + (m^2+n^2) = 0.$

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$$k = \frac{2m \pm \sqrt{4m^2 - 4(m^2+n^2)}}{2}$$

$$k = \frac{2m \pm 2\sqrt{-n^2}}{2}$$

$$k = \frac{m \pm in}{2}$$

$$y_c = e^{m/2 z}\left(A\cos(n/2)z + B\sin(n/2)z\right)$$

$$P.I. = \frac{\phi(z)}{f(D)} = \frac{n^2 e^{zm} z}{D^2 - 2mD + m^2 + n^2}$$

$$= e^{zm}\left(\frac{n^2 z}{(D+m)^2 - 2m(D+m) + m^2 + n^2}\right)$$

$$= e^{zm}\left(\frac{n^2 z}{D^2 + \cancel{m^2} + \cancel{2m} - \cancel{2mD} - \cancel{2m^2} + \cancel{m^2} + n^2}\right)$$

$$= n^2 e^{zm}\left(\frac{z}{D^2 + n^2}\right)$$

$$\begin{array}{r|l} n^2 + D^2 & z \\ & z + 0 \\ & 0 \end{array}$$

$$P.I = \cancel{e^{zm} \cdot \frac{z^3}{6}} \quad \boxed{P.I = e^{zm} \cdot z}$$

$$G.S = y_c + y_p = e^{\frac{mz}{2}}\left(A\cos(n/2)z + B\sin(n/2)z\right) + e^{zm} \cdot z$$

$$= e^{(n/2)\log x}\left(A\cos((n/2)\log x) + B\sin((n/2)\log x)\right) + e^{\log x^m} \cdot \log x$$

(3) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x.$

⇒ Legendre's Equation

Solution of Legendre's Equation:

$$(ax+b)^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = \phi(x).$$

is the non homogenous eqn of 2nd order.

Sustitute $z = \log(ax+b)$ or $e^z = ax+b$

⇒ $(ax+b)\frac{dy}{dx} = aDy$

⇒ $(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y.$ $\quad \boxed{D = \frac{d}{dz}}$

⇒ $(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y.$

We substitute this to obtain the linear D.E with constant coeff.

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$$G.S = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{81}\left[2ze^{2z} + \frac{20}{3}e^{z} - 17/4\right]$$

(1) $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 8x^2+4x+1$

Substitute $(ax+b)^2\frac{d^2y}{dx^2} = a^2 D(D-1)y$.

$3(3x+2)\frac{dy}{dx} = 3Dy$; $z = \log(3x+2)$, $e^z = 3x+2$.

$$\boxed{x = \frac{e^z-2}{3}}$$

$\therefore (3D(D-1)y) + 3Dy - 36y = 8\frac{(e^z-2)}{3} + 4\frac{(e^z-2)}{3} + 1$.

A.E $= f(m) = 0 \Rightarrow 3m(m-1) + 3m - 36 = 0$.

$3m^2 - 3m + 3m - 36 = 0$

$3m^2 - 36 = 0$ $\quad 9m^2 - 6m - 36 = 0$

$m = \pm 2\sqrt{3}i$ $\quad 3m^2 - 2m - 12 = 0$ $\quad m = \pm 2$

$y_c = \cancel{(A\cos(2\sqrt{3})z + B\sin(2\sqrt{3})z)}$. $C_1 e^{2z} + C_2 e^{-2z}$

$$P.I = \frac{\phi(z)}{f(D)} = \frac{8}{3}\left(\frac{e^z-2}{D^2-4}\right) + \frac{4}{3}\left(\frac{e^z-2}{3D^2-4}\right) + \frac{1}{D^2-4}$$

$$= \frac{8}{3}\left(\frac{e^z}{-3} + \frac{1}{-2}\right) + \frac{4}{3}\left(\frac{e^z}{-33} - \frac{1}{18}\right) - \frac{1}{36}.$$

$$= \frac{-12e^z}{99} - \frac{8}{3\cdot 18} - \frac{4}{3\cdot 18} - \frac{1}{36}.$$

$$= \frac{-4e^z}{33} - \frac{12}{54} - \frac{1}{36}$$

$$= \frac{-4e^z}{33} - \frac{2}{9} - \frac{1}{36}$$

$$= \frac{-4e^z}{33} - \left(\frac{8+1}{36}\right)$$

$$= \frac{-4e^z}{33} - \frac{1}{4}$$

$36 - 3D^2 \,\big|\, -1/18$

-2

$-1/18$

0

$3 \,|\, 9, 36$

$3 \,|\, 3, 12$

$1, 4$

$\therefore$ G.S $= \cancel{(A\cos(2\sqrt{3})\log(3x+2) + B\sin(2\sqrt{3})\log(3x+2))}$

$= C_1 e^{2\log x} + C_2 e^{-2\log x}$

(2) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(x+1))$

$z = \log(1+x)$; $e^z = 1 = x$.

$(1+x)^2\frac{d^2y}{dx^2} = 1D(D-1)y$ $\quad (1+x)\frac{dy}{dx} = 1Dy$.

$\therefore (D(D-1) + 1D + 1)y = 2\sin z.$

$A.E \Rightarrow f(m) = 0$

$m(m-1) + 1m + 1 = 0$

$m^2 - m + m + 1 = 0 \Rightarrow m = \pm i.$

$y_c = (A\cos z + B\sin z)$

$P.I = \frac{\phi(z)}{f(D)} = \frac{2\sin z}{D^2+1} = \frac{2\sin z}{2D} = -2\cos z.$

$G.S = y_c + y_p = A\cos z + B\sin z + \sin z.$

$= A\cos(\log(x+1)) + B\sin(\log(x+1)) - \log(1+x)\cdot\cos(\log(1+x))$

$\left((2x+1)^2\frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y\right) = x\log(2x+1).$

$z = \log(2x+1).$ or $(2x+1)\ e^z \Rightarrow x = \frac{e^z - 1}{2}.$

$(2x+1)^2 y'' = 4D(D-1)y.$

$(2x+1)y' = 2Dy.$

$\therefore 4D(D-1)y - 4Dy - 12y = \left(\frac{e^z - 1}{2}\right)z.$

$A.E = f(m) = 0 \Rightarrow 4m(m-1) - 4m - 12 = 0$

$4m^2 - 4m - 4m - 12 = 0$

$4m^2 - 8m - 12 = 0.$

$m^2 - 2m - 3 = 0.$

$m = -1, 3.$

$y_c = C_1 e^{-z} + C_2 e^{3z}.$

$P.I = \frac{\phi(z)}{f(D)} = \left(\frac{e^z - 1}{2}\right)z = \frac{1}{2}\left[\frac{e^z \cdot z}{(D^2-2D-3)} - \frac{z}{4(D^2-2D-3)}\right]$

$D^2 - 2D - 3 = \frac{1}{8}\left[\frac{e^z \cdot z}{(D+1)^2 - 2(D+1) - 3} - \left(\frac{-z}{3} + \frac{2}{9}\right)\right]$

$= \frac{1}{8}\left[\frac{e^z \cdot z}{D^2 + 1 + 2D - 2D - 2 - 3} + \frac{z}{3} - \frac{2}{9}\right]$

$= \frac{1}{8}\left[\frac{e^z z}{D^2 - 4} + \frac{z}{3} - \frac{2}{9}\right]$

$= \frac{1}{8}\left[e^z\left(\frac{-z}{4}\right) + \frac{z}{3} - \frac{2}{9}\right]$

$= -\frac{e^z z}{32} + \frac{z}{24} - \frac{2}{72}$

$-3 - 2D + D^2 \,\big|\, z$

$-2\frac{d}{dz}\left(-\frac{z}{3}\right)$ $\quad z + \frac{2}{3}$

$-\frac{2}{3}$

$-\frac{2}{3}$

0

$-2/3 + 2/9.$

$-2/4.$

$-4 + D^2 \,\big|\, z$

$\frac{z}{0}.$

$G.S = C_1 e^{-z} + C_2 e^{3z} - \frac{e^z z}{32} + \frac{z}{24} - \frac{2}{72}$

$G.S = C_1 e^{-\log(2x+1)} + C_2 e^{3\log(2x+1)} - \frac{(2x+1)\log(2x+1)}{32} + \frac{\log(2x+1)}{24} - \frac{2}{72}$

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④ $(3x-2)^2\frac{d^2y}{dx^2} - 2(3x-2)\frac{dy}{dx} = 9(3x-2)\sin(\log(3x-2))$

$z = \log(3x-2)$ & $e^z = 3x-2$.

$(3x-2)^2 D^2 = 9\cdot D(D-1)y$.

$(3x-2)D = 3Dy$.

$\therefore (9D(D-1) - 3(2D))y = 9e^z\sin(z)$

$(9D^2 - 9D - 9D)y = 9e^z\sin(z)$

$(9D^2 - 18D)y = 9e^z\sin(z)$.

A.E $= f(m) = 0$

$9m^2 - 18m = 0$

$9m(m-2) = 0$

$m = 0, 2$.

$y_c = c_1 + c_2e^{2z}$.

$P.I = \frac{\phi(z)}{f(D)} = \frac{9e^z\sin(z)}{9D^2 - 18D}$

$= \frac{e^z\sin(z)}{(D+1)^2 - 2(D+1)}$

$= \frac{e^z\sin(z)}{D^2+1+2D-2D+2}$

$= e^z\left(\frac{\sin(z)}{2D}\right)$

$= e^z\left(\frac{\sin(z)}{-2}\right)$

$P.I = \left(\frac{-e^z}{2}\right)(\sin(z))$

$G.S = y_c + y_p = c_1 + c_2e^{2z} - \frac{e^z}{2}\cdot z\sin z$.

$G.S = c_1 + c_2e^{2\log(3x-2)} - \frac{e^{\log(3x-2)}}{2}\sin(\log(3x-2))$

⑤ $(2x+1)^2\frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(2x+1)^2$.

$z = \log(2x+1)$; $(2x+1) = e^z$

$(2x+1)^2 D^2 = 4D(D-1)y$.

$(2x+1)\frac{dy}{dx} = 2Dy$.

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$4D(D-1)y - 6(2Dy) + 16y = 8e^{2z}$.

$(4D(D-1) - 12D + 16)y = 8e^{2z}$.

A.E $\Rightarrow f(m) = 0 \Rightarrow 4m(m-1) - 12m + 16 = \cancel{8e^{2z}}\ 0$

$4m^2 - 4m - 12m + 16 = \cancel{8e^{2z}}\ 0$

$4m^2 - 16m + 16 = 0$.

$\cancel{m^2 - 2m + 2 = 0}$ $4m^2 - 8m - 8m + 16 = 0$

Roots $= m = 2, 2$.

$y_c = (c_1 + c_2 z)e^{2z}$.

$P.I = \frac{\phi(z)}{f(D)} = \frac{8(e^z)^2}{4D^2 - 16D + 16} = \frac{8e^{2z}}{16 - 32} = \frac{z\,8e^{2z}}{8D - 16}$

$= \cancel{z8e^{2z}} = \frac{z^2 8e^{2z}}{8}$.

$G.S = (c_1 + c_2 z)e^{2z} + z^2 8e^{2z}$

$G.S = (c_1 + c_2 \log(2x+1))e^{2z} + (\log(2x+1))^2 e^{2\log(2x+1)}$

(6) $(2x+1)^2 y'' - 2(2x+1)y' + 2y = x\log(2x+1)$.

(7) $(2x-1)^2 y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3$.

Partial Differential Equation.

Formation of PDE by eliminating the arbitrary constants.

$f(x, y, z, a, b) = 0$, where z is a f^n of 'xy' and a & b are the arbitrary constants. Differentiate the given f^n w.r.t x & y partially and eliminate the arbitrary constant (a, b) to form the P.D.E.

Problems

(1) $z = (x^2 + a)(y^2 + b)$.

$\frac{\partial z}{\partial x} = 2x(y^2 + b)$; $\frac{\partial z}{\partial y} = 2y(x^2 + a)$.

$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 2x \cdot 2y(x^2 + a)(y^2 + b) = 4xyz$.

P.D.E $= 4xyz$.

(2) $z = (x-a)^2 + (y-b)^2$.

$\partial z/\partial x = 2(x-a)$ $\partial z/\partial y = 2(y-b)$. $\Rightarrow x - a = \frac{1}{2}\partial z/\partial x$ & $(y-b) = \frac{1}{2}\partial z/\partial y$

$\cancel{(\partial z/\partial x + \partial z/\partial y)^2 = 4[(x-a) + (y-b)]^2}$.

$z = \frac{1}{4}(\partial z/\partial x)^2 + \frac{1}{4}(\partial z/\partial y)^2$. $p = \partial z/\partial x$

$4z = p^2 + q^2$ $q = \partial z/\partial y$.

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③ $z = xy + y\sqrt{x^2-a^2+b}$

$\frac{\partial z}{\partial x} = p = y + \frac{(y)(2x)}{2\sqrt{x^2-a^2+b}}$; $\frac{\partial z}{\partial y} = q = x + \sqrt{x^2-a^2+b}$

$\sqrt{x^2-a^2+b}\, p = y\sqrt{x^2-a^2+b} + xy$; $\frac{\partial z}{\partial y} = \sqrt{x^2-a^2+b} = q-x$.

$\cancel{z = (p-y)\sqrt{x^2-a^2+b} + xy}$; $(q-x) = \sqrt{x^2-a^2+b}$

$\cancel{z = (p-y)(q-x) + y(q-x)}$

$z = p\sqrt{x^2-a^2+b}$

$z = p(q-x)$ //

④ $z = a\log\left(\frac{b(y-1)}{1-x}\right) = a\log(b(y-1)) - a\log(1-x)$

$\frac{\partial z}{\partial x} = p = \frac{a}{(1-x)}$; $\frac{\partial z}{\partial y} = q = \frac{ab}{b(y-1)}$

$\frac{p}{q} = \frac{\alpha}{(1-x)} \times \frac{(y-1)}{\alpha} = \left(\frac{y-1}{1-x}\right)$

$z = p(1-x)\log\left(b\left(\frac{p}{q}\right)\right)$

$e^{z/p(1-x)} = b\left(\frac{p}{q}\right) \Rightarrow b = \frac{e^{z/p(1-x)}}{q}\cdot p$

$z = p(1-x)\log\left(e^{z/p(1-x)}\cdot\frac{p}{q}\cdot\frac{p}{q}\right)$

⑤ Find the PDE of the family of all spheres whose center lie at a plane $z=0$ & have a constant radius 'r'.

$(x-a)^2 + (y-b)^2 + (z-0)^2 = r^2$

$(x-a)^2 + (y-b)^2 + z^2 = r^2$ — ①

$2(x-a) + 0 + 2z\frac{\partial z}{\partial x} = 0$.

$(x-a) = -z\,\partial z/\partial x$ — ②

$0 + 2(y-b) + 2z\,\partial z/\partial y = 0$

$(y-b) = -z\,\partial z/\partial y$. — ③

Substituting ② & ③ in ①;

$(-z\,\partial z/\partial x)^2 + (-z\,\partial z/\partial y)^2 + z^2 = r^2$

$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 + z^2 = r^2$

$z^2(p^2 + q^2 + 1) = r^2$ //

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Since 3 constants are present, 2 P.D.E are obtained

(6) $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. — (1)

$2x/a^2 + 2z/c^2\, \partial z/\partial x = 0$. — (2) $\Rightarrow x/a^2 = -z/c^2\, \partial z/\partial x \Rightarrow xc^2/a^2 = -z\, \partial z/\partial x$

$\cancel{2y/b^2 + 2z/c^2\, \partial z/\partial x = 0}$ — (3) $\Rightarrow y/b^2 = -z/c^2\, \partial z/\partial y \Rightarrow yc^2/b^2 = -z\, \partial z/\partial y$

$c^2/a^2 = -[\partial^2 z/\partial x^2 \cdot z + (\partial z/\partial x)^2]$. — (3)

& $\dfrac{xc^2}{a^2} = -z\dfrac{\partial z}{\partial x}$.

$\dfrac{c^2}{a^2} = \dfrac{-z}{x}\dfrac{\partial z}{\partial x}$.

$-\dfrac{z}{x}\dfrac{\partial z}{\partial x} = -\left[z\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial z}{\partial x}\right)^2\right]$

$\dfrac{z}{x}\dfrac{\partial z}{\partial x} = \left[z\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial z}{\partial x}\right)^2\right]$ — (4)

Partially differentiating (1) w.r.t. y.

$2y/b^2 + 2z/c^2\, \partial z/\partial y = 0$.

$\dfrac{y}{b^2} = \dfrac{-z}{c^2}\dfrac{\partial z}{\partial y}$.

$\dfrac{c^2}{b^2} = \dfrac{-z}{y}\dfrac{\partial z}{\partial y}$ — (5).

$\left(\dfrac{-z}{y}\dfrac{\partial z}{\partial y}\right) = \left(\dfrac{-z}{y}\right)\dfrac{\partial^2 z}{\partial y^2} + \left(\dfrac{\partial z}{\partial y}\right)^2 \cdot \cancel{\dfrac{z}{y}}$ → (6)

$\dfrac{z}{y}\dfrac{\partial z}{\partial y} = \dfrac{z}{y}\dfrac{\partial^2 z}{\partial y^2} + \left(\dfrac{\partial z}{\partial y}\right)^2 \cancel{\dfrac{z}{y}}$ —

$p = \partial z/\partial x$; $q = \partial z/\partial y$; $r = \partial^2 z/\partial x^2$; $t = \partial^2 z/\partial y^2$; $s = \partial^2 z/\partial x \partial y$.

(7) $x^2/a^2 + y^2/b^2 = 2z$.

$2x/a^2 = 2\, \partial z/\partial x \Rightarrow x/a^2 = \partial z/\partial x \Rightarrow 1/a^2 = 1/x\, \partial z/\partial x$.

$2y/b^2 = 2\, \partial z/\partial y \Rightarrow y/b^2 = \partial z/\partial y \Rightarrow 1/b^2 = 1/y\, \partial z/\partial y$.

$x\, \partial z/\partial x + y\, \partial z/\partial y = 2z$. //

(8) ⇒ Elimination of arbitrary fn & form P.D.E.

Consider a relation containing an arbitrary fn of x, y, z where z is the fn of x & y.

Here x & y are independent variable & z is the dependent variable. By Diff this relation partially w.r.t x & y for the appropriate no. of times. We try to elliminate the function. The resulting equation will be a P.D.E eqn.

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eliminate

(1) $z = y\left(f(y/x)\right) \rightarrow (1)$

$\frac{\partial z}{\partial x} = \left[y f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)\right]$

$\frac{\partial z}{\partial x} = \frac{-y^2}{x^2} f'\left(\frac{y}{x}\right)$

$f'\left(\frac{y}{x}\right) = -\frac{x^2}{y^2}\frac{\partial z}{\partial x}$

$f\left(\frac{y}{x}\right) = \frac{z}{y}$

$\frac{\partial z}{\partial y} = \frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)$

$\frac{\partial z}{\partial y} = \frac{y}{x}\cdot\left(\frac{-x^2}{y^2}\right)\frac{\partial z}{\partial x} + \frac{z}{y}$

$\frac{\partial z}{\partial y} = \frac{-x^2}{y}\frac{\partial z}{\partial x} + \frac{z}{y}$

$y\frac{\partial z}{\partial y} = -x^2\frac{\partial z}{\partial x} + z \Rightarrow z = x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

(2) $z = y^2 + 2f(1/x + \log y)$

$\frac{\partial z}{\partial x} = 2f'(1/x + \log y)(-1/x^2)$ $\quad \frac{\partial z}{\partial y} = 2y + 2f'(1/x + \log y)(1/y)$

$2f'(1/x + \log y) = -x^2\frac{\partial z}{\partial x} \Rightarrow \cancel{\frac{\partial z}{\partial y} = 2y \rightarrow \frac{x^2}{y}\frac{\partial z}{\partial x}}$

(1) $\quad$ (2) $- 2f'(1/x + \log y) = y(\partial z/\partial y - 2y)$

Equating (1) & (2)

$-x^2\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y} - 2y^2$

$y\frac{\partial z}{\partial y} = 2y^2 - x^2\frac{\partial z}{\partial x}$

(3) $\phi(xy + z^2, x + y + z) = 0$ $\qquad f(x,y) = 0 \Rightarrow y = f(x)$

$\phi(u, v) = 0 \Rightarrow u = \phi(v)$; $\quad v = x + y + z$, $u = xy + z^2$ $\qquad z(x,y)$

$v = x + y + z = \phi(xy + z^2)$

$\cancel{1 + \partial z/\partial x = \phi'(xy + z^2)y}$ $\qquad 1 + \frac{\partial z}{\partial x} = \phi'(xy + z^2)(y$

$\phi'(xy + z^2) = (1 + \partial z/\partial x)/(y + 2z\partial z/\partial x)$ (1)

$x + y + z = \phi'(xy + z^2)$

$1 + \partial z/\partial y = \phi'(xy + z^2)(x + 2z\partial z/\partial y)$

$\cancel{y + y\partial z/\partial y = 1 + \partial z/\partial x}$

$\cancel{y - 1 = \partial z/\partial x - y\partial z/\partial y}$

$1 + \frac{\partial z}{\partial y} = \cancel{\phi'(xy + z^2)}\frac{(1 + \partial z/\partial x)}{(y + 2x\partial z/\partial x)}(x + 2z\partial z/\partial y)$

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Solve the P.D.E by elliminating arbitrary f^n from $z = \phi(x+ay) + \psi(x-ay)$.

$$\frac{\partial z}{\partial x} = \phi'(x+ay) + \psi'(x-ay). \; ; \; \frac{\partial z}{\partial y} = \phi'(x+ay)a - \psi'(x-ay)a$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(x+ay) + \psi''(x-ay) \; ; \; \frac{\partial^2 z}{\partial y^2} = \phi''(x+ay)a^2 + \psi''(x-ay)a^2$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\left(\phi''(x+ay) + \psi''(x-ay)\right)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$$

Form the P.D.E eqn by elliminating the arbitrary f^n from the relation $f(x^2+y^2, z-xy) = 0$.

$u = x^2+y^2$
$v = z-xy$

$f(u,v) = 0$.

$v = f(u)$.

$$z - xy = f(x^2+y^2).$$

$$\frac{\partial z}{\partial x} - y = f'(x^2+y^2)2x. \Rightarrow \frac{\partial z}{\partial y} - x = f'(x^2+y^2)2y$$

$$\frac{1}{2x}\frac{\partial z}{\partial x} - \frac{y}{2x} = f'(x^2+y^2) \Rightarrow \frac{1}{2y}\frac{\partial z}{\partial y} - \frac{x}{2y} = f'(x^2+y^2)$$

$$\frac{1}{2x}\frac{\partial z}{\partial x} - \frac{y}{2x} = \frac{1}{2y}\frac{\partial z}{\partial y} - \frac{x}{2y}.$$

$$y\frac{\partial z}{\partial x} - y^2 = x\frac{\partial z}{\partial y} - x^2$$

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = x^2 - y^2$$

— OR —

$f(u,v) = 0$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot 2x + \frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial x} - y\right) = 2x\frac{\partial f}{\partial u} - y\frac{\partial f}{\partial v} \quad (1)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\cdot 2y + \cancel{\frac{\partial f}{\partial v}(-x)} = \frac{\partial f}{\partial v}\left(\frac{\partial z}{\partial y} - x\right) \quad (2)$$

$$J(u,v) = \begin{vmatrix} 2x & \frac{\partial z}{\partial x} - y \\ 2y & \frac{\partial z}{\partial y} - x \end{vmatrix} = 0 \Rightarrow 2x\left(\frac{\partial z}{\partial y} - x\right) - \left(\frac{\partial z}{\partial x} - y\right)2y = 0$$

$\left(1 + \frac{\partial z}{\partial x}\right)$

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→ Form the PDE by elliminating the arbitrary f^n
$z = yf(x) + x\phi(y)$, f & ϕ are arbitrary f^n.

⇒ <u>Solution by direct Integration</u>

Consider the eqn of 2nd order i.e $f(x, y, z, \partial z/\partial x, \partial^2 z/\partial x^2, \partial^2 z/\partial y^2, \partial^2 z/\partial x \partial y) = 0$

If we consider the sol^n by direct Integration, we place the constant of integration as the arbitrary f^n of the variable which is kept constant.

① ~~Solution~~ Solve $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$.

$$\int \frac{\partial^2 z}{\partial x \partial y}\, dx = \int \left(\frac{x}{y} + a\right) dx.$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y) \Rightarrow \int \frac{\partial z}{\partial y}\, dy = \int \left(\frac{x^2}{2y} + ax\right) dy.$$

$$\Rightarrow z = \frac{x^2 \log y}{2} + axy + \int f(y)\, dy + g(x)$$

$g(x)$ & $f(y)$ are arbitrary constants of x & y respectively

② $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$

$$\int \frac{\partial^3 z}{\partial x^2 \partial y}\, dx + \int 18xy^2\, dx + \int \sin(2x - y)\, dx = 0.$$

$$\int \frac{\partial^2 z}{\partial x \partial y}\, dx + \int 9x^2 y^2\, dx + \int \frac{-\cos(2x - y)}{2}\, dx = 0$$

$$\int \frac{\partial z}{\partial y}\, dy + \int 3x^3 y^2\, dy + \int \frac{-\sin(2x - y)}{4}\, dy = 0.$$

$$z + x^3 y^3 - \frac{\cos(2x - y)}{4} = \frac{x^2}{2} f(y) + x g(y) + h(x) = 0$$

③ Solve $\partial^3 z/\partial x^2 \partial y = \cos(2x + 3y)$

$$\int \frac{\partial^3 z}{\partial x^2 \partial y}\, dx = \int \cos(2x + 3y)\, dx.$$

$$\int \frac{\partial^2 z}{\partial x \partial y}\, dx = \int \frac{-\sin(2x + 3y)}{2}\, dx + \int f(y)\, dx.$$

$$\int \frac{\partial z}{\partial y}\, dy = \int \frac{+\cos(2x + 3y)}{4}\, dy + \int x f(y)\, dy + \int g(y)\, dy.$$

$$z = \frac{-\sin(2x + 3y)}{12} + xF(y) + G(y) + h(x)$$

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$z=0$ if y is an odd multiple of $\pi/2$.

$\partial^2 z/\partial x\partial y = \sin x \sin y$. for which $\partial z/\partial y = -2\sin y$ when $x=0$,

$$\int \frac{\partial^2 z}{\partial x \partial y}\, dx = \int \sin x \sin y$$

$$\int \frac{\partial z}{\partial y} = \int -\cos x \sin y\, dy + \int f(y)\, dy$$

$$z = \cos x \cos y + F(y) + g(x).$$

$$\partial z/\partial y = -\cos x \sin y + f(y) + 0.$$

$$f(y) = \partial z/\partial y + \cos x \sin y = \partial z/\partial y + \sin y = -\sin y$$

$$F(y) = z - \cos x \cos y$$

& For $y = \pi/2$.

$$z = \cos x \cos y + F(y) + g(x).$$

$$\cancel{0 = 0 + z - 0 + g(x).}$$

$$\cancel{g(x) = 0}$$

$$\therefore \cancel{z = \cos x \cos y + \cos y} + 0.$$

$$z = \cos x \cos y + \cos y + g(x).$$

$$g(x) = 0.$$

$\therefore$ Eqn $\Rightarrow$ $z = \cos x \cos y + \cos y = \cos y \cdot 2\cos^2 x/2$

(3) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y}$ subjected to the condition $\partial z/\partial x = \log_e x$ when $y = 1$ & $z = 0$ when $x = 1$.

$$\int \frac{\partial^2 z}{\partial x \partial y}\, dy = \int \frac{x}{y}\, dy.$$

$$\cancel{\int \frac{\partial z}{\partial y}\, dy = \int \frac{x^2}{2y}\, dy + \int f(y)}$$

$$\cancel{z = \frac{x^2}{2} \log y + F(y)}$$

$$\cancel{\frac{\partial z}{\partial x} = x \log y + f(y)}$$

$$\cancel{\log_e x = x \log 1 +}$$

$$\int \frac{\partial z}{\partial x}\, dx = \int x \log y\, dx + \int f(x)\, dx$$

when $y = 1$.

$$\log_e x = x \log 1 + f(x)$$

$$f(x) = \log_e x.$$

$$z = \frac{x^2}{2} \log y + x \log x - x + g(y)$$

when $x = 1$.

$$0 = \tfrac{1}{2} \log y + 0 + 1 + g(y)$$

$$g(y) = -\tfrac{1}{2} \log y + 1.$$

$$\therefore z = \frac{x^2}{2} \log y + x \log x - x - \frac{1}{2} \log y + 1.$$

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(4) $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$ subjected to the condition $z(x,0) = x^2$.
$\frac{\partial z}{\partial y}(1,y) = \cos y$

$\int \frac{\partial^2 z}{\partial x \partial y} dx = \int x^2 y \, dx$

$\int \frac{\partial z}{\partial y} dy = \int \frac{x^3}{3} y \, dy + f(y) dy$

$z = \frac{x^3 y^2}{6} + \sin y - \frac{y^2}{6} + g(x)$

when $x = 1$.
$\cos y = \frac{y}{3} + f(y)$
$f(y) = \cos y - y/3$.

when $y = 0$
$x^2 = 0 + 0 - 0 + g(x)$.
$g(x) = x^2$.

$\therefore z = \frac{x^3 y^2}{6} + \sin y - \frac{y^2}{6} + x^2$.

(5) $\frac{\partial^2 z}{\partial x^2} = xy$ subjected to the condition $\frac{\partial z}{\partial x} = \log(1+y), x=1$
& $z=0$ when $x=0$

$\int \frac{\partial^2 z}{\partial x^2} dx = \int xy \, dx$.

$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y)$.

$\int \frac{\partial z}{\partial x} dx = \int \frac{x^2 y}{2} dx + \int f(y) dx$.

when $x = 1$.
$\log(1+y) = \frac{y}{2} + f(y)$.
$f(y) = \log(1+y) - \frac{y}{2}$.

$z = \frac{x^3 y}{6} + x \log(1+y) - \frac{xy}{2} + g(xy)$.

$0 = 0 + 0 - 0 + g(y)$

Eqn is $\Rightarrow z = \frac{x^3 y}{6} + x \log(1+y) - \frac{xy}{2}$

(6) $\frac{\partial^2 u}{\partial x^2} = (x+y)$

$\Rightarrow$ ~~$\int \frac{\partial^2 u}{\partial x^2} dx$~~. Find the Partial Differential Equation of all planes which are at a constant distance of 'a' from the origin.

(A) Eqn of normal $\Rightarrow lx + my + nz = a$.
$l, m, n \Rightarrow$ directional constants.
$l^2 + m^2 + n^2 = 1$
$n^2 = 1 - l^2 - m^2 \Rightarrow n = \sqrt{1 - l^2 - m^2}$

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$$lx + my + \sqrt{1-l^2-m^2}\, z = a \quad \text{—(2)}$$

$$l + 0 + \sqrt{1-l^2-m^2}\,\frac{\partial z}{\partial x} = 0.$$

$$l = -\sqrt{1-l^2-m^2}\,\frac{\partial z}{\partial x}\, p. \quad \text{—(3)}$$

Diff w.r.t y.

$$0 + m + \sqrt{1-l^2-m^2}\, q = 0.$$

$$m = -\sqrt{1-l^2-m^2}\, q. \quad \text{—(4)}$$

$$l^2+m^2 = (1-l^2-m^2)p^2 + (1-l^2-m^2)q^2.$$

$$= p^2 - p^2l^2 - m^2p^2 + q^2 - q^2l^2 - q^2m^2$$

$$l^2+m^2 = p^2 - (l^2+m^2)p^2 + q^2 - (l^2+m^2)q^2$$

$$(l^2+m^2)(1+p^2+q^2) = p^2+q^2.$$

$$l^2+m^2 = \frac{p^2+q^2}{1+p^2+q^2}$$

$$1-(l^2+m^2) = 1 - \frac{p^2+q^2}{1+p^2+q^2} = \frac{1+p^2+q^2-p^2-q^2}{1+p^2+q^2} = \frac{1}{1+p^2+q^2}.$$

$$\therefore \cancel{\frac{p^2+q^2}{1+p^2+q^2} + \frac{1}{1+p^2+q^2} = 1.}$$

$$L = -\sqrt{\frac{1}{1+p^2+q^2}}\; p. \quad ; \quad m = -\sqrt{\frac{1}{1+p^2+q^2}}\; q.$$

$$n = \sqrt{\frac{1}{1+p^2+q^2}}.$$

$$\therefore -px\sqrt{\frac{1}{1+p^2+q^2}} - qy\sqrt{\frac{1}{1+p^2+q^2}} + z\sqrt{\frac{1}{1+p^2+q^2}} = a$$

⇒ The solution of Lagrange's Linear PDE of first order in the form.

$Pp + Qq = R$

Methods of obtaining the General sol^n.

(i) write the equation in the standard form. i.e $Pp + Qq = R$, where. P, Q & R are the f^n of (x, y, z) [Subsidary Eqn]

(ii) Form the Lagrange's Axiliary Eqn (A.E) it is of the form.

AE : $dx/P = dy/Q = dz/R$.

(iii) The nature of the sol^n to the simultanous eqn of the form

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Case(1): By using method of Grouping.

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$u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ are said to be the complete sol^n of the system of simultaneous eqn
Provided u & v are linearly independent.
i.e $u/v \neq c$. Then ~~[illegible]~~ sol^n can be written as
$\phi(u,v) = 0 \Rightarrow v = \phi(u)$.
This sol^n which does not contain an arbitary constant is called a general sol^n

Case(2) <u>The method of multipliers.</u>
Sometimes we use the multipliers l, m, n and l', m', n'
Such that AE: $dx/P = dy/Q = dz/R = \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = k_1$
$\dfrac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R} = k_2$.

<u>Method of grouping</u>
<u>Problems:</u>

(1) $p\cot x + q\cot y = \cot z$
AE $= dx/\cot x = dy/\cot y = dz/\cot z$
$\int \dfrac{dx}{\cot x} = \int \dfrac{dy}{\cot y}$.
$\log|\sec x| = \log|\sec y| + \log c_1$
$\log|\sec x| - \log|\sec y| = \log C_1$
$\log\left(\dfrac{\sec x}{\sec y}\right) = \log C_1$
$C_1 = \dfrac{\sec x}{\sec y}$

$\int \tan y\,dy = \int \tan z\,dz$
$\log|\sec y| = \log|\sec z| + \log C_2$.
$\log\left|\dfrac{\sec y}{\sec z}\right| = \log C_2$.
$C_2 = \dfrac{\sec y}{\sec z}$.

G.S $\Rightarrow \phi(\sec x/\sec y, \sec y/\sec x) = 0$.

(2) $x^2 p + y^2 q = z^2$.
AE: $dx/x^2 = dy/y^2 = dz/z^2$.
$\int dx/x^2 = \int dy/y^2$
$\dfrac{x^{-1}}{-1} = \dfrac{y^{-1}}{-1} + C_1$.
$C_1 = 1/y - 1/x$.

$\int dy/y^2 = \int dz/z^2$.
$y^{-1}/-1 = z^{-1}/-1 + C_2$.
$C_2 = 1/z - 1/y$.

G.S $\Rightarrow \phi(1/y - 1/x, 1/z - 1/y) = 0$

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(2) $(mz-ny)p + (nx-lz)q = ly - mx$.

A.E = $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$ Can't do method of grouping

$\cancel{\int \dfrac{dx}{mz-ny} = \int \dfrac{dy}{nx-lz}}$ $\dfrac{x\,dx + y\,dy + z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = k_1$.

$\cancel{\dfrac{x}{mz-ny}} = \dfrac{x\,dx + y\,dy + z\,dz}{\cancel{mxz} - \cancel{nxy} + \cancel{nxy} - \cancel{lyz} + \cancel{lyz} - \cancel{mzx}} = k_1$.

$\dfrac{x\,dx + y\,dy + z\,dz}{0} = k_1$

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = C_1$.

$x^2 + y^2 + z^2 = 2C_1 = C_1$ — (1).

$\dfrac{l\,dx + m\,dy + n\,dz}{\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{lmz} + \cancel{nly} - \cancel{mnx}} = k_2$.

$\dfrac{l\,dx + m\,dy + n\,dz}{0} = k_2$.

$l\,dx + m\,dy + n\,dz = 0$

$lx + my + nz = C_2$.

$\therefore$ G.S $\Rightarrow \phi(x^2+y^2+z^2,\ lx+my+nz) = 0$.

(3) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

A.E $\Rightarrow \dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$; use the multipliers as $\cancel{1/x, dx, 1/y}$

$\dfrac{1/x^2\,dx + 1/y^2\,dy + 1/z^2\,dz}{(y-z) + (z-x) + (x-y)} = k_1$

$\int 1/x^2\,dx + \int 1/y^2\,dy + \int 1/z^2\,dz = 0$.

$-1/x - 1/y - 1/z = C_1$. — (1).

$\dfrac{1/x\,dx + 1/y\,dy + 1/z\,dz}{xy - xz + yz - xy + zx - zy} = k_2$.

$\int 1/x\,dx + \int 1/y\,dy + \int 1/z\,dz = 0$

$\log x + \log y + \log z = \log C_1$.

$xyz = C_1$. — (2).

$\therefore$ G.S $\Rightarrow \phi(-(1/x + 1/y + 1/z),\ xyz)$

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$\frac{d(y)}{dx}$

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④ ~~$x^2(y^2-z^2)p+y^2(z^2-x^2)$~~

$$x(y^2-z^2)p+y(z^2-x^2)q = z(x^2-y^2)$$

AE $\Rightarrow$ $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2-z^2+z^2-x^2+x^2-y^2} = k_1$$

$$\int\frac{1}{x}dx + \int\frac{1}{y}dy + \int\frac{1}{z}dz = 0$$

$$\log xyz = \log c_1 \Rightarrow c_1 = xyz.$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^2y^2-z^2x^2+y^2z^2-x^2y^2+x^2z^2-y^2z^2}$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0.$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c_1$$

$$x^2+y^2+z^2 = 2c_1 = c_1$$

$$\phi(x^2+y^2+z^2, xyz) = 0.$$

→ Eliminating arbitrary f^n to form P.D.E.

→ Various possible solution of 1-Dimensional wave eqn. $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$ by the method of separation of variables.

-OR-

$\boxed{U_{tt} = c^2 U_{xx}}$

Consider; $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$U = XT$; $X(x) = X$, $T(t) = T$.

$$\frac{\partial^2(XT)}{\partial t^2} = c^2\frac{\partial^2(XT)}{\partial x^2}.$$

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}.$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}.$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = k.$$

$$\frac{d^2T}{dt^2} = kc^2T \;;\; D = \frac{d}{dt}.$$

$$(D^2-kc^2)T = 0 \quad \text{— (1)}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = k. \quad ; D = \frac{d}{dx}.$$

$$\frac{d^2X}{dx^2} = kX \Rightarrow \left(\frac{d^2X}{dx^2} - kX\right) = 0$$

$$(D^2-k)X = 0 \quad \text{— (2)}$$

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Case(i) $K=0$;

$(D^2-(0)c^2)T=0$

A.E $\Rightarrow m^2=0$

$m=0,0$

$T=(C_1+C_2t)e^{0t}$

$K=0$

$(D^2-0)X=0$

$m^2=0$

$m=0,0$

$X=(C_1+C_2x)e^{0x}$

$U=XT=(C_1'+C_2'x)(C_1+C_2t)$ → sol^n when $K=0$

Case(ii) $K=+ve=p^2$

$(D^2-p^2c^2)T=0$

A.E $=m^2-p^2c^2=0$

$m=\pm pc$

~~$T=(C_1+C_2t)e^{pct}$~~

$T=C_1'''e^{pct}+C_2'''e^{-pct}$

$(D^2-p^2)X=0$

A.E $=m^2=p^2$

$m=\pm p$

$X=C_1''e^{px}+C_2''e^{-px}$

$U=XT=(C_1'''e^{pct}+C_2'''e^{-pct})(C_1''e^{px}+C_2''e^{-px})$

Case(iii) $K=-ve=-p^2$

$(D^2+p^2c^2)T=0$

A.E $\Rightarrow m^2=-p^2c^2$

$m=\pm ipc$

$T=(A'\cos pct+B'\sin pct)$

$(D^2+p^2)X=0$

A.E $\Rightarrow m^2=-p^2$

$m=\pm ip$

$X=(A\cos p+B\sin p)$

$U=XT=(A\cos px+B\sin px)(A'\cos pct+B'\sin pct)$

The various solutions are:

$\Rightarrow$ when $U=XT$ & $K=0$

$U=(C_1'+C_2'x)(C_1+C_2t)$

$\Rightarrow$ when $U=XT$ & $K=+ve=p^2$

$U=(C_1''e^{px}+C_2''e^{-px})(C_1'''e^{pct}+C_2'''e^{-pct})$

$\Rightarrow$ when $U=XT$ & $K=-ve=-p^2$

$U=(A\cos px+B\sin px)(A'\cos pct+B'\sin pct)$.

Various possible solution of 1-D heat eqn $V_t=c^2V_{xx}$; $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$

Consider; $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$; $u=XT$

$\frac{\partial(XT)}{\partial t}=c^2\frac{\partial^2(XT)}{\partial x^2}$

$X\frac{\partial(T)}{\partial t}=c^2T\frac{\partial^2 X}{\partial x^2}$ ÷ XTc^2

$\frac{1}{c^2T}\frac{\partial T}{\partial t}=\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$

$\frac{1}{c^2T}\frac{\partial T}{\partial t}=K$ [Equating each eqn to 'K']

$\frac{\partial T}{\partial t}=Kc^2T$

$(D-Kc^2)T=0$ —— ①

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$$\frac{d^2X}{dx^2} = kX \quad \ldots$$

$$(D^2 - k)X = 0 \quad \text{—} (2)$$

Case (1): when $k = 0$

$(D - kc^2)T = 0$ $\qquad$ $(D^2 - k)X = 0$

$(D - 0)T = 0$ $\qquad$ $(D^2 - 0)X = 0$

A.E $\Rightarrow$ $m = 0$ $\qquad$ $m = 0, 0$

$T = C_1 e^{0t}$ $\qquad$ $X = C_1 e^{0x} + C_2' e^{0x}$

$U = XT = (C_1' e^{0x} + C_2' e^{0x})(C_1 e^{0t})$

Case (2): when $k = +ve = p^2$

$(D - kc^2)T = 0$ $\qquad$ $(D^2 - k)X = 0$

$(D - p^2c^2)T = 0$ $\qquad$ $(D^2 - p^2)X = 0$

A.E $\Rightarrow$ $m = p^2c^2$ $\qquad$ $m^2 = p^2$

$m = \pm p$

$T = C_1'' e^{p^2c^2t}$ $\qquad$ $X = C_1''' e^{px} + C_2''' e^{-px}$

$U = XT = (C_1''' e^{px} + C_2''' e^{-px})(C_1'' e^{p^2c^2t})$

Case (3): when $k = -ve = -p^2$

$(D^2 - kc^2)T = 0$ $\qquad$ $(D^2 - k)X = 0$

$(D + p^2c^2)T = 0$ $\qquad$ $(D^2 + p^2)X = 0$

A.E $\Rightarrow$ $m + p^2c^2 = 0$ $\qquad$ $m^2 + p^2 = 0$

$m = -p^2c^2$ $\qquad$ $m = \pm ip$

$T = (A' e^{-p^2c^2t})$ $\qquad$ $X = (A\cos px + B\sin px)$

$U = XT = (A\cos px + B\sin px)(A' e^{-p^2c^2t})$

Various sol^{n}s are:

→ when $U = XT$ & $k = 0$

$U = (C_1' + C_2')(C_1)$

→ when $U = XT$ & $k = p^2$

$U = (C_1''' e^{px} + C_2''' e^{-px})(C_1'' e^{p^2c^2t})$

→ when $U = XT$ & $k = -p^2$

$U = (A\cos px + B\sin px)(A' e^{-p^2c^2t})$.

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y, z, x

$x^2y - y^2z + zy^2 - z^2x + x^2z - xy^2$

$x^2y - y^2z + zy^2 - z^2x + x^2z - xy^2$

$(x^2-yz)p + (y^2-zx)q = z^2 - xy$

$$\frac{dx}{(x^2-yz)} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x^2-yz)-(y^2-zx)} = \frac{dy-dz}{(y^2-zx)-(z^2-xy)} = \frac{dz-dx}{(z^2-xy-x^2+yz)}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{(x^2-y^2)+zx-yz} = \frac{dy-dz}{(y^2-z^2)+(xy-zx)} = \frac{dz-dx}{(z^2-x^2)+(yz-xy)}$$

$$\frac{dx-dy}{(x^2-y^2)+z(x-y)} = \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dz-dx}{(z^2-x^2)+y(z-x)}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)}$$

Comparing

$$\frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)} = \frac{d(z-x)}{(z-x)}$$

$\log(x-y) = \log(y-z) + \log C_1$

$$\left(\frac{x-y}{y-z}\right) = C_1$$

$\log(y-z) = \log(z-x) + \log C_2$

$$C_2 = \left(\frac{y-z}{z-x}\right)$$

$$\phi\left(\frac{y-z}{z-x}, \frac{x-y}{y-z}\right) = 0$$

Application Problem - module 1

② The D.E for a circuit in which self inductance and capacitance neutralize & is of the form. $L\frac{d^2L}{dt^2} + \frac{i}{C} = 0$. Find current 'i' as function of t given that 'i' is the max current & $i=0$, when $t=0$

$L\,d^2i/dt^2 + i/C = 0$ can also be written as:

$d^2i/dt^2 + i/LC = 0$

$(D^2 + 1/LC)\,i = 0$

AE => $f(m) = 0$ => $m^2 + 1/LC = 0$

$m = \pm i/\sqrt{LC}$

$i = C_1\cos 1/\sqrt{LC}\,t + C_2\sin 1/\sqrt{LC}\,t$ —— ①

$i = 0, t = 0$ in eqn ①

$0 = C_1(1) + C_2(0) \Rightarrow \boxed{C_1 = 0}$

$I_{max} = I\sin\frac{1}{\sqrt{LC}}t$

$i = C_2\sin 1/\sqrt{LC}\,t$

comparing

$C_2 = I$

$$\boxed{i = I\sin\frac{1}{\sqrt{LC}}t}$$

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VECTOR CALCULUS

Introduction: A vector is a quantity having both magnitude & direction. The vector quantities like Force, velocity, acceleration etc. have a lot of references in physical & engineering problems.

Derivative of a vector valued f^n

Let the position vector of a point $P(x,y,z)$ in space be $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

If x, y, z are all f^n of a single parameter 't' then $\vec{r}$ is said to be the function of 't'. This is also refered to as a vector point f^n and it is denoted by

$\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

The derivative of the vector $\vec{r}(t)$ is denoted by $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{dx} + \frac{d\vec{r}}{dy} + \frac{d\vec{r}}{dz}$

$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$ and is a vector along the tangent to the curve and peak.

If 't' is a time variable then velocity $\vec{v} = \frac{d\vec{r}}{dt}$ i.e velocity of particle at time 't'

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$ i.e the rate of change of velocity $\vec{v}$ & is called the acc^n of the particle at time 't'.

Further $|\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$ → speed of P where 's' represents arc length measured from the fixed pnt on the curve on to the point 'P' on the curve.

Note :→ The component of a vector $\vec{A}$ along the given vector (direction) $\vec{D}$) is the resolved part of $\vec{A}$ is given by $\vec{A}\cdot\hat{n}$, $\hat{n}$ represents $\frac{\vec{D}}{|\vec{D}|}$

→ The component of $\vec{A}$ $\perp^{lar}$ to the vector $\vec{D}$ is unit normal vector (along the normal) and is given by.

$|\vec{A} - (\vec{A}\hat{n})\hat{n}|$

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component of vel $= \vec{v} \cdot \hat{n}$

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Problems

(1) If $x = t^2+1$, $y = 4t-3$ and $z = 2t^2-6t$ represents the ~~parabolic~~ parametric eqn of the curve the determine the

(i) unit tangent vector at any pnt on the curve.

(ii) The angle between the tangent at $t=1$ & $t=2$.

Sol:

Position vector $\Rightarrow \vec{r}(t) = (t^2+1)\hat{i} + (4t-3)\hat{j} + (2t^2-6t)\hat{k}$

Tangent $= \vec{T} = \frac{d\vec{r}}{dt} = 2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(2t)^2 + 4^2 + (4t-6)^2}$$

$$= \sqrt{4t^2 + 16 + 16t^2 + 36 - 48t}$$

$$= \sqrt{20t^2 - 48t + 52}$$

$$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{2t\hat{i} + 4\hat{j} + (4t-6)\hat{k}}{\sqrt{20t^2 - 48t + 52}}$$

$$\boxed{\text{Angle betw 2 vectors} = \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}}$$

$T(1) = 2\hat{i} + 4\hat{j} - 2\hat{k} = |T(1)| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

$T(2) = 4\hat{i} + 4\hat{j} + 2\hat{k}$ $\quad |T(2)| = \sqrt{16+16+4} = 6$

Angle between $t=1$ & $t=2$ tangent;

$$\cos\theta = \frac{8 + 16 - 4}{(2\sqrt{6})(6)} = \frac{8 + 16 - 4}{12\sqrt{6}}$$

$$\cos\theta = \frac{2 + 4 - 1}{3\sqrt{6}}$$

$$\theta = \cos^{-1}\left(\frac{2\hat{i} + 4\hat{j} - \hat{k}}{3\sqrt{6}}\right)$$

$$\theta = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)$$

(2) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t-5$ where 't' is the time. Find the component of its velocity and the acc^n in the direction of $\hat{i} - 2\hat{j} + 2\hat{k}$ at $t=1$

$\Rightarrow \vec{r} = (2t^2)\hat{i} + (t^2-4t)\hat{j} + (3t-5)\hat{k}$

$$\frac{d\vec{r}}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$$

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$Acc^n = \frac{d^2r}{dt^2} = 4\hat{i} + 2\hat{j}$

Velocity at $t=1$; $v = 4\hat{i} - 2\hat{j} + 3\hat{k}$

Acc^n at $t = 1$; $a = 4\hat{i} + 2\hat{j}$

$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

Component of velocity $= \vec{v} \cdot \hat{n}$

$= \frac{(4\hat{i} - 2\hat{j} + 3\hat{k})(\hat{i} - 2\hat{j} + 2\hat{k})}{3}$

$= \frac{4 + 4 + 6}{3} = \frac{14}{3}$

Component of acc^n $= \vec{a} \cdot \hat{n}$

$= \frac{(4\hat{i} + 2\hat{j}) \cdot (\hat{i} - 2\hat{j} + 2\hat{k})}{3}$

$= \frac{4 - 4}{3} = 0$

<u>Scalar & Vector Field Functions</u>

If every point of x, y, z of a region 'R' in space corresponds to a scalar $\phi(x,y,z)$ and is called scalar point function and we say that a scalar field ϕ is defined in Region 'R'

Eg. $\phi = xyz$; $\phi = x^2 + y^2 + z^2$

If every point x, y, z of region 'R' in space corresponds to a vector, $\vec{A}$ of xyz then $\vec{A}$ is called a vector points function defined in 'R'

Eg: $\vec{A}(x,y,z)$ $\quad \vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{A} = x^2y\hat{i} + yz\hat{j} + z\hat{k}$

<u>Operators</u>: The vector differential operator (Del (∇) or nabla and is defined by

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \sum \frac{\partial}{\partial x}\hat{i}$

The laplacian operator (∇^2) is defined by.

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \sum \frac{\partial^2}{\partial x^2}$

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Gradient, Divergence and Curl.

Gradient of a scalar field.

If a scalar $\phi(x,y,z)$ is continuously differentiable function, then the gradient of ϕ (grad ϕ) is defined to be $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$.

Grad ϕ is a vector quantity

Divergence of a vector field.

If $\vec{A}$ of (x,y,z) is continously differentiable vector function, then divergence of A (div $\vec{A}$) and it is defined as $\nabla\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

(a_1 & a_2, a_3 are functions of x, y, z).

$$\nabla\cdot\vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}.$$

Then Div. $\vec{A}$ is a scalar quantity.

Curl of a vector field.

If $\vec{A}(x,y,z)$ is continously differentiable vector f^n then curl of $\vec{A}$ (curl $\vec{A}$) can be written as

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Curl $\vec{A}$ is a vector quantity.

Note ⇒ The laplacian of a scalar f^n is a scalar quantity
⇒ Laplacian of a vector f^n is a vector quantity
⇒ If ϕ is a scalar f^n then the eqn $\nabla^2\phi = 0$ is called "Laplace Eqn"
⇒ A function which satisfies the Laplace Eqn is called "Harmonic function"

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⇒ The unit normal vector $\hat{n}$ along $\nabla\phi$ is given by:

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$, the angle betw the 2 surfaces is defined to be equal to the angle between their normals.

Eg: Let eqn of two surfaces be given by.

$\phi_1(x,y,z) = C_1 \qquad \phi_2(x,y,z) = C_2$.

(Angle betw surfaces) ⇒ $\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|} \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|} = \hat{n}_1 \cdot \hat{n}_2$

$\theta \rightarrow$ angle between the normals.

If $\theta = \frac{\pi}{2}$, then the surfaces are set to intersect each other, orthogonally.

When $\theta = \pi/2$, $\cos \pi/2 = 0$

Directional Derivatives.

If $\phi(x,y,z)$ is a scalar f^n and $\vec{d}$ is a given direction then $\nabla\phi \cdot \hat{n} \Rightarrow \hat{n} = \frac{\vec{d}}{|\vec{d}|}$ is called the directional derivative of ϕ along $\hat{n}$.

Note ⇒ Directional Derivative is maximum along (del ϕ) $\nabla\phi$ and the maximum value will be equal to $|\nabla\phi|$.

⇒ The maximum Direction Derivative of a scalar function ϕ at a point P is also called as normal derivatives of the scalar function. at P. So the normal derivative will be equal to $|\nabla\phi|$ ($|\text{Grad}\,\phi|$)

Problem

Q. If $\phi = x^3 + y^3 + z^3 - 3xyz$, find (Grad ϕ) and $|\text{Grad}\,\phi|$ at the point $P(1,-1,2)$.

Sol^n ⇒ $\text{Grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\frac{\partial\phi}{\partial x} = 3x^2 - 3yz$; $\frac{\partial\phi}{\partial y} = 3y^2 - 3xz$ $\quad \frac{\partial\phi}{\partial z} = 3z^2 - 3xy$

$\nabla\phi = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$

$\nabla\phi = (3+6)\hat{i} + (3-6)\hat{j} + (12+3)\hat{k}$ [At $P(1,-1,2)$]

$\nabla\phi = 9\hat{i} - 3\hat{j} + 15\hat{k}$

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$|\nabla\phi| = \sqrt{9^2+(-3)^2+(15)^2} = 3\sqrt{35}$

(2) Determine the unit normal vector to the surface $x^2y - 2xz + 2y^2z^4 = 10$ at $(2, 1, -1)$

$\nabla\cdot\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= (2xy - 2z)\hat{i} + (2x^2 + 4yz^4)\hat{j} + (-2x + 8y^2z^3)\hat{k}$

At $(2,1,-1)$

$\nabla\cdot\phi = (4+2)\hat{i} + (4+4)\hat{j} + (-4-8)\hat{k}$

$= 6\hat{i} + 8\hat{j} - 12\hat{k}$

$|\nabla\phi| = \sqrt{36+64+256} = 15.6$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{6\hat{i}+8\hat{j}-12\hat{k}}{15.6}$

(3) Find the angle between the surface $x^2+y^2+z^2=9$ & $x^2+y^2-z=3$ at the point $(2,-1,2)$

Sol^n: $\phi = x^2+y^2+z^2-9$ $\qquad$ $\psi = x^2+y^2-z-3$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$ $\qquad$ $\nabla\psi = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$

$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$ $\qquad$ $\nabla\psi = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\nabla\phi = 2(2)\hat{i} + 2(-1)\hat{j} + 2(2)\hat{k}$ $\qquad$ $\nabla\psi = 4\hat{i} - 2\hat{j} - 2$

$= 4\hat{i} - 2\hat{j} + 4\hat{k}$ $\qquad$ $|\nabla\psi| = \sqrt{21}$

$|\nabla\phi| = \sqrt{16+16+4}$

$= 6.$

$\cos\theta = \frac{\nabla\phi\cdot\nabla\psi}{|\nabla\phi||\nabla\psi|}$

$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})(4\hat{i}-2\hat{j}-\hat{k})}{6\sqrt{21}}$

$= \frac{16\hat{i}-4\hat{j}-4\hat{k}}{6\sqrt{21}}$

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(4) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $2\hat{i} - \hat{j} - 2\hat{k}$

Sol^n ⇒ $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$.

$\vec{d} = 2\hat{i} - \hat{j} - 2\hat{k}$

$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= 2xyz\,\hat{i} + 4z^2\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k} + 8xz\,\hat{k}$

$\nabla\phi_{(1,-2,-1)} = 2(1)(-2)(-1)\hat{i} + 4(-1)^2\hat{i} + (1)^2(-1)\hat{j} + (1)^2(-2)\hat{k} + 8(1)(-1)\hat{k}$

$\nabla\phi = 4\hat{i} + 4\hat{i} - 1\hat{j} - 2\hat{k} - 8\hat{k}$

$\nabla\phi = 8\hat{i} - 1\hat{j} - 10\hat{k}$

Directional derivative $= \nabla\phi \cdot \hat{n}$

$= (8\hat{i} - 1\hat{j} - 10\hat{k})\left(\frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}\right)$

$= \frac{16 + 1 + 20}{3} = \frac{37}{3}$

(5) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$

Sol^n: $\vec{d} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$

$= (4(8) - 6(2)(-1)^2(2))\hat{i} + (-6(2)^2(-1)(2))\hat{j} + (12(2)(4) - 3(4)(-1)^2)\hat{k}$

$= (32 - 24)\hat{i} + (48)\hat{j} + (96 - 12)\hat{k}$

$= 8\hat{i} + 48\hat{j} + 84\hat{k}$

directional derivative $= \nabla\phi \cdot \hat{n}$

$= (8\hat{i} + 48\hat{j} + 84\hat{k})\left(\frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}\right)$

$= \frac{16 - 144 + 504}{7} = \frac{376}{7}$

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(6) If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(-1,1,2)$ ~~along~~ has a maximum magnitude of $|\nabla\phi| = 32$ units in the direction $\parallel^{la}$ to y-axis. Find a, b, c

Sol^n ⇒ $\nabla\phi \cdot \hat{j} = 32$ [$\because \parallel^{la}$ to y-axis]

$$\nabla\phi_{(-1,1,2)} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= ay^2\hat{i} + 3cz^2x^2\hat{i} + 2axy\hat{j} + bz\hat{j} + by\hat{k} + 2czx^3\hat{k}$$

$$= a\hat{i} + 12c\hat{j} - 2a\hat{j} + 2b\hat{j} + b\hat{k} - 4c\hat{k}$$

$\nabla\phi \cdot \hat{j} = 32.$

$-2a + 2b = 32.$

$-a + b = 16. \quad \text{—— (1)}$

Since $\nabla\phi$ is $\parallel^{la}$ to y-axis. Consider x & $z = 0$ and solve.

$\nabla\phi \cdot \hat{i} = 0$ & $\nabla\phi \cdot \hat{k} = 0$

$a + 12c = 0$ $\quad$ $b - 4c = 0$

$a + 12c = 0$ $\quad$ $b - 4c = 0$

$3b - 12c = 0$ $\quad$ $4 - 4c = 0$

$a + 3b = 0$ $\quad$ $\boxed{c = 1.}$

$-a + b = 16$ $\quad$ $-a + b = 16.$

$4b = 16.$ $\quad$ $-a + 4 = 16$

$\boxed{b = 4.}$ $\quad$ $-a = 12 \Rightarrow \boxed{a = -12}$

(7) Find a, b, c so that the directional derivative of $\phi = ax^2y^2 + byz + cz^2x^2$ at $(1,2,-1)$ has the maximum magnitude of 64 in the direction of z-axis

$b = 24, a = 6$
$c = -8$

(8) If $\vec{F} = (2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k})$. Find $\nabla \cdot \vec{F}$ & $(\nabla \times \vec{F})$

$$\nabla \cdot \vec{F} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

$$\nabla \cdot \vec{F} = (2y^3z^4)\hat{i} + (6x^2yz^4)\hat{j} + (12x^2y^3z^2)\hat{k}$$

$$(\nabla \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^3z^4 & 6x^2yz^4 & 12x^2y^3z^2 \end{vmatrix} = \hat{i}(36x^2y^2z^2 - 24x^2yz^3) - \hat{j}(24xy^3z^2 - 8y^3z^3) + \hat{k}(12xyz^4 - 6y^2z^4)$$

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$\cancel{(\nabla\times\vec{F}) = \hat{i}(36x^2y^2z^2 - 24x^2y^2z^3)\hat{i} - (24xy^3z^2 - 8y^3z^3)\hat{j} + (12xyz^4 - 6y^2z^4}$

$$(\nabla\times\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^3 & 4x^2y^3z^3 \end{vmatrix}$$

$= \hat{i}\,(12x^2y^2z^3 - 12x^2y^2z^3) - \hat{j}\,(8xy^3z^3 - 8xy^3z^3)$
$+ \hat{k}\cdot(6xy^2z^4 - 6xy^2z^4)$
$= 0$

Q If $\vec{F} = \nabla(x^3y + y^3z + z^3x - x^2y^2z^2)$ Find the divergence & curl of at the point $(1,2,3)$

Sol^n: $\cancel{\nabla\cdot\vec{F} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}}$ $\phi = x^3y + y^3z + z^3x - x^2y^2z^2$.

$\nabla\cdot\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{k} + \frac{\partial\phi}{\partial z}\hat{k}$

$\vec{F} = (3x^2y + z^3 - 2xy^2z^2)\hat{i} + (x^3 + 3y^2z - 2x^2yz^2)\hat{j} + (y^3 + 3z^2x - 2x^2y^2z)\hat{k}$

$\text{div}\cdot\vec{F} = \frac{\partial(3x^2y + z^3 - 2xy^2z^2)}{\partial x} + \frac{\partial(x^3 + 3y^2z - 2x^2yz)}{\partial y} + \frac{\partial}{\partial z}(y^3 + 3z^2x - 2xy^2z)$

$\text{div}\cdot\vec{F} = 6xy - 2y^2z^2 + 6yz - 2x^2z^2 + 6xz - 2x^2y^2$
$= 12 - 2(4)(9) + 6(3)(2) - 2(1)(3) + 6(1)(3) - 2(1)^2(2)^2$
$= 12 - 72 + 36 - 18 + 18 - 8$
$= \cancel{12 - 36 - 6} + 10 - 32$
$= \cancel{22 - 42}$
$= \cancel{-20}$

$$\text{curl}\times\vec{F} = \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y & & \end{vmatrix}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y + z^3 - 2xy^2z^2 & x^3 + 3y^2z - 2x^2yz & y^3 + 3z^2x - 2xy^2z \end{vmatrix}$$

$\text{curl}\times\vec{F} = \hat{i}\,(3y^2 - 4x^2yz) - \hat{j}\,(3z^2 - 4xy^2z - 3z^2 + 4xy^2z)$
$+ \hat{k}\,(3x^2 - 4xyz - 3x^2 + 4xyz^2)$
$= \hat{i}\,(12 - 4(1)(2)(3)) - \hat{j}\,(3(9) - 4(1)(4)(3) - 3(9) + 4(1)(4)(3))$
$+ \hat{k}\,(3(1) - 4(1)(2)(3) - 3(1)^2 + 4(1)(2)(9))$
$= \hat{i}\,(12 - 24) - \hat{j}\,(27 - 48 - 27 + 48) + \hat{k}\,(3 - 24 - 3 + 72)$
$= \hat{i}\,(-12) + \hat{k}\,(8$ $\quad 0$

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Solinoidal & irrotational

If $\vec{F}$ is said to be solinoidal div. $\vec{F} = 0$ and irrotational when curl of $\vec{F} = 0$.

Irrotational vector field is also called as conservative or potential field.

When ~~irrotational vector f~~ $\vec{F}$ is irrotational there always exists a scalar potential ϕ such that $\nabla\phi = \vec{F}$. Then ϕ is called the scalar potential of $\vec{F}$.

Problem ① If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ then prove that $\vec{F}$.curl $\vec{F} = 0$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \hat{i}(-1-0) - \hat{j}(-1-0) + \hat{k}(0-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{F}\cdot\text{curl}\,\vec{F} = (x+y+1)(-1) + (1)(1) + (-x-y)(-1)$$

$$= \cancel{-x} - \cancel{y} - 1 + 1 + \cancel{x} + \cancel{y}$$

$$= 0$$

② Show that $\vec{F} = \dfrac{x\hat{i} + y\hat{j}}{x^2+y^2}$ is both solinoidal & irrotational.

solinoidal

$$\nabla\cdot\vec{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}$$

$$= \left[\frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}\right] + \frac{x^2+y^2 - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{x^2}+y^2 - \cancel{2x^2} + \cancel{x^2} + \cancel{y^2} - \cancel{2y^2}}{(x^2+y^2)}$$

$$= 0$$

curl

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/x^2+y^2 & y/x^2+y^2 & 0 \end{vmatrix}$$

$$= \cancel{\hat{i}\left[\left(\frac{-2y}{x^2+y^2}\right)\left(\frac{1}{x}\right)\right]} -$$

$$= \hat{i}(0-0) + \hat{j}(0-0) + \cancel{\hat{k}\left(-\frac{2xy}{(x^2+y^2)^2}\right) + \frac{2xy}{(x^2+y^2)^2}}$$

$$= 0$$

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(3) Prove that $\vec{V}$ is solinoidal & $\vec{F}$ is irrotational.

$\vec{V} = 3xy^2z^2\hat{i} + y^3z^2\hat{j} - 2y^2z^3\hat{k}$

$\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$

$$\text{div}\,\vec{V} = \frac{\partial \vec{V}}{\partial x} + \frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial z}$$

$$= 3y^2z^2 + 3y^2z^2 - 6y^2z^2$$

$$= 0$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \hat{i}(-x+x) + \hat{j}(-y+y) + \hat{k}(-z+z)$$

$$= 0$$

(4) Find the constant a & b such that $\vec{F} = (axy + z^3)\hat{i} + (3x^2-z)\hat{j} + (bxz^2-y)\hat{k}$ is irrotational. Also ϕ such that $\vec{F} = \nabla\phi$

$\nabla \times \vec{F} = 0.$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy+z^3 & 3x^2-z & bxz^2-y \end{vmatrix}$$

$$= \hat{i}(-1+1) - \hat{j}(bz^2-3z^2) + \hat{k}(6x-ax) = 0.$$

$$-(bz^2-3z^2)\hat{j} + \hat{k}(6x-ax) = 0$$

$\Rightarrow bz^2 - 3z^2 = 0. \quad \Rightarrow 6x - ax = 0$

$bz^2 = 3z^2 \qquad 6x = ax$

$\boxed{b = 3} \qquad \boxed{a = 6}$

$\vec{F} = \nabla\phi.$

Integrate $\phi = (3x^2y + xz^3)\hat{i} + (3x^2y - yz)\hat{j} + (xyz^3 - zy^2)\hat{k}$ ① ② ③

$+ f(y,z) + g(x,z) + h(x,y)$

In ① & ② ① & ③, ② & ③ take common.

$\boxed{\phi = 3x^2y + xz^3 - yz}$ — ④

Compare ① with ④, ② with ④, ③ with ④.

$f(y,z) = -zy. \quad g(x,z) = xz^3 \quad h(x,y) = 3x^2y.$

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(5) Find a, b, c such that $\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational & also find scalar potential.

$\nabla \times \bar{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$\hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$

$(c+1)\hat{i} + \hat{j}(-4+a) + \hat{k}(b-2) = 0$

$\boxed{c=-1}$ $\boxed{a=4}$ $\boxed{b=2}$

$\bar{F} = \nabla\phi$

$\frac{\partial\phi}{\partial x} = x+2y+az$

$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z)$

$\frac{\partial\phi}{\partial y} = bx-3y-z$

$\phi = 2xy - \frac{3y^2}{2} - yz + f(x,z)$

$\frac{\partial\phi}{\partial z} = 4x - y + 2z$

$\phi = 4xz - yz + z^2 + f(x,y)$

$\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2$

$f(y,z) = -\frac{3y^2}{2} - yz + z^2$

$f(x,z) = \frac{x^2}{2} + 4xz + z^2$

$f(x,y) = \frac{x^2}{2} + 2xy - \frac{3y^2}{2}$

(6) Show that the vector field $\bar{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$ is irrotational & find its scalar potential such that

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$\vec{F} = \text{Grad}\,\phi$

$\text{curl}\,\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zz & z^2-xy \end{vmatrix}$$

$= \hat{i}(-x+z) - \hat{j}(-y+y) + \hat{k}(0+z)$

$= \hat{i}(-x+z) - 0\hat{j} + z\hat{k}$.

$\vec{F} = \text{grad}\,\phi$

$\phi = \left(\frac{-x^2}{2} + 2x\right)\hat{i} + \left(\frac{z^2}{2}\right)\hat{k} + f(y,z) + g(x,y)$

$\phi = -\frac{x^2}{2} + 2x + \frac{z^2}{2}$

$f(y,z) = z^2/2$; $g(x,y) = -x^2/2 + 2x$.

Vector Integration

Vector Line Integral

A curve 'c' in space which consists of infinitely small line elements of length dr. Then the line integral of vector $\vec{A}(x,y,z)$ along the curve 'c' is defined to be the sum of scalar products of $\vec{A}$ and $d\vec{r}$ and it is represented as

$$\int_c \vec{A}\cdot d\vec{r}$$

If 'c' is a closed curve, which does not intersect anywhere the line integral around 'c' and it is denoted by integration of closed curve 'c'.

$$\oint_c \vec{A}\cdot d\vec{r}$$

If $\vec{F}$ is the force acting upon the particle. In displacing it along the curve 'c', then

$\int_c \vec{F}\cdot d\vec{r}$ represents total workdone by the force. It also represents the circulation of $\vec{F}$ about 'c' where $\vec{F}$ represents the velocity of the fluid.

$\vec{F}$ is said to be irrotational if $\oint_c \vec{F}\cdot d\vec{r} = 0$

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(1) If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ evaluate $\int \vec{F}\cdot d\vec{r}$ where 'C' is the curve in the plane $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Sol^n

$\vec{F} = 3xy\hat{i} - y^2\hat{j}$

$r = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\int \vec{F}\cdot d\vec{r} = \int 3xy\,dx - y^2 dy$ $\quad y = 2x^2$

$dy = 4x\,dx$

$= \int_0^1 6x^3 dx - (2x^2)^2 4x$

$= \int_0^1 6x^3 dx - 16x^5 dx$

$= \left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$

$= \frac{3}{2} - \frac{8}{3}$

$= \frac{9 - 16}{6}$

$= \frac{-7}{6}$

(2) Find the workdone in moving a particle $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along straight line from $(0,0,0)$ to $(2,1,3)$ and a curve $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

Sol^n: Eqn of line:

$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$

$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$

~~$dx = 2dt$ $dy = dt$ $dz = 3dt$~~

$x = 2t$; $y = t$; $z = 3t$

$\vec{F}\cdot d\vec{r} \Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{F}\cdot d\vec{r} = 3x^2 dx + (2xz - y)dy + z\,dz$

$\vec{F}\cdot d\vec{r} = (4t^2)dt + (2(2t)(3t) - t)dt + 3t(3dt)$

$\vec{F}\cdot d\vec{r} = 24t^2 dt + (12t^2 - t)dt + 9t\,dt$

When $x = 2 \Rightarrow t = 1$

$y = 1 \Rightarrow t = 1$ $\quad \therefore$ ~~t varies from 0 to 1~~

$z = 3 \Rightarrow t = 1$

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$\therefore = \int_0^1 (36t^2+8t)\,dt$

$= \left(\frac{36t^3}{3} + \frac{8t^2}{2}\right)_0^1$

$= 12+4 = 16$

Work defined by $x^2 = 4y$ & $3x^3 = 8z$ from $x=0$ to $x=2$.

$y = \frac{x^2}{4}, \quad z = \frac{3x^3}{8}$

$dy = \frac{2x\,dx}{4}, \quad dz = \frac{9x^2\,dx}{8}$

$dy = \frac{x}{2}dx, \quad dz = \frac{9}{8}x^2 dx$

$\vec{F}\cdot d\vec{r} = 3x^2dx + (2xz-y)dy + z\,dz$

$\int \vec{F}\cdot d\vec{r} = \int_0^2 3x^2dx + \left(2x\left(\frac{3x^3}{8}\right) - \frac{x^2}{4}\right)\frac{2x}{4}dx + \frac{3x^3}{8}\cdot\frac{9}{8}x^2dx$

$= \left[x^3 + \frac{3}{32}x^6 - \frac{x^4}{32} + \frac{27}{64}x^6\right]_0^2$

$= 8 + \frac{3}{8}\cdot\frac{32}{8} - \frac{16}{32} + \frac{9}{64}\cdot\frac{64}{2}$

$= 8 + \frac{12}{8} - \frac{1}{2} + \frac{9}{2} \qquad = 8 + \frac{64}{16} - \frac{16}{32} + \frac{9\cdot 64}{128}$

$= 8 + 12/8 + 8/2 \qquad 8 + \frac{9}{2} = \frac{16+9}{2} = 16$

$= \frac{80+24+40}{10}$

$= \frac{144}{10}$

Evaluate $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$, $x^2+y^2=a^2$, $z=0$.

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$

$\vec{F}\cdot d\vec{r} = \sin y\,dx + x\,dy + x\cos y\,dy$

$= d(x\sin y) + x\,dy$

$\int \vec{F}\cdot d\vec{r} = \int d(x\sin y) + \int x\,dy$

$x^2+y^2 = a^2 \Rightarrow x = a\cos\theta, \; y = a\sin\theta. \quad 0 \le \theta \le 2\pi$

$dx = -a\sin\theta\,d\theta \quad dy = a\cos\theta\,d\theta$

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$$\int_0^{\pi} \vec{F}\,d\vec{r} = \int_0^{\pi} d(a\cos\theta\sin(a\sin\theta)) + \int_0^{\pi} a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= \left[a\cos\theta \cdot \sin(a\sin\theta)\right]_0^{\pi} + \int_0^{\pi} a^2\cos^2\theta\, d\theta$$

$$= 0 + a^2\int_0^{\pi}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} d\theta + \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{2}\left[(2\pi - 0) + \frac{1}{2}(0)\right]$$

$$= \frac{2\pi a^2}{2} = \pi a^2.$$

Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where 'C' is the curve in the xy plane $y = x^2$ from $(0,0)$ to $(1,2)$

$\vec{F} = 3xy\hat{i} - y^2\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$ $\qquad y = x^2$

$\int \vec{F}\,d\vec{r} = \int 3xy\,dx - y^2 dy.$ $\qquad dy = 2x\,dx$

$$= \int_0^1 3x(x^2)dx - 2x^5 dx.$$

$$= \int 3x^3 dx - 2x^5 dx.$$

$$= \left|\frac{3x^4}{4} - \frac{2x^6}{6}\right|_0^1$$

$$= \frac{3}{4} - \frac{2}{6} = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$$

If $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve represented by $x = t, y = t^2, z = t^3$ & t lies betw $\{-1 \le t \le 1$. $\quad dx = dt \quad dy = 2t\,dt \quad dz = 3t^2 dt.$

$$\cancel{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

x

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$\vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz.$

$\int \vec{F}\cdot d\vec{r} = \int xy\,dx + yz\,dy + zx\,dz.$

$= \cancel{\int t\cdot t^2 dt}$

$= \int (t)(t^2)dt + (t^2)(t^3)2t\,dt + (t^5)(t)3t^2 dt.$

$= \left[\frac{t^4}{4} + \frac{2t^7}{7} + \frac{3t^7}{7}\right]_{-1}$

$= \frac{1}{4} + \frac{2}{7} + \frac{3}{7} - \frac{1}{4} + \frac{2}{7} + \frac{3}{7}.$

$= \frac{2}{4} + \frac{4}{7} + \frac{6}{7}.$

$= \frac{1}{2} + \frac{10}{7}$

Evaluate $\int \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\hat{i} + (x^2+y^2)\hat{j}$

(i) Path of the straight line from $(0,0)$ to $(1,0)$ and then to $(1,1)$

(ii) The straight line joining the origin & $(1,2)$.

$\vec{F} = xy\hat{i} + (x^2+y^2)\hat{j}$

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$\int \vec{F}\cdot d\vec{r} = \int xy\,dx + (x^2+y^2)dy.$

(ii) $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = t.$

$\frac{x}{1} = \frac{y}{2} = t \Rightarrow x = t \;;\; y = 2t.$ $\quad x=1 \;\; t=1$, $y=2 \;\; t=1.$

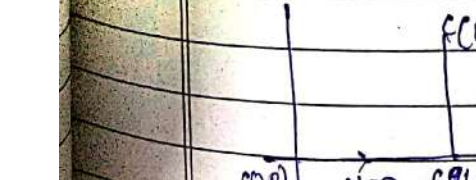
$dx = dt \quad dy = 2dt.$

$= \int_0^1 (t)(2t)dt + (t^2+4t^2)2dt.$

$= \int_0^1 2t^2 + 2t^2 + 4t^2 dt.$

$= \left(\frac{8t^3}{3}\right)_0^1 = \frac{8}{3}$

(i) $(0,0)$ to $(1,0)$ & $(1,1)$



$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}.$

Along AB $y=0$, $dy=0$

x varies from 0 to 1

$I = \int_{AB} xy\,dx + (x^2+y^2)dy$

$= \int_0^1 x^2 0\,dx = 0$

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Along BC. $\Rightarrow x = 1$ $dx = 0$; y varies from 0 to 1.

$$\int_{BC} xy\,dx + (x^2+y^2)\,dy$$

$$\int_0^1 1+y^2\,dy = \left(y+\frac{y^3}{3}\right)_0^1 = 1+\frac{1}{3}$$

$$= \frac{4}{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}$$

$$= 4/3.$$

<u>Green's Theorem in a plane.</u>

If 'R' is a closed region of the XY plane bounded by a simple closed curve 'c' and if M & N are 2 continuous f^ns of x & y having the continuous 1st order Partial diff. is given by.

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy.$$

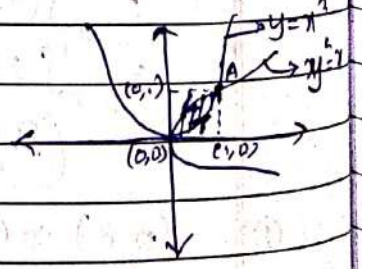
$\vec{F} = M\hat{i} + N\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}.$

$$= \iint_R \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right) dx\,dy.$$

(1) Using green's Thm in a plane for integration of 'c'.
$\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where 'c' is the boundary of the region enclosed by $y=\sqrt{x}$ & $y=x^2$.

$\Rightarrow$ $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$



$y = x^2$ & $y = \sqrt{x}$.

$x = y^2$

$x = (x^2)^2$

$x = x^4$

$x^4 - x = 0$

$x(x^3-1) = 0$

$x = 0,\ x = 1.$

$y = 0,\ y = 1.$

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By Green's Thm

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$$LHS = \oint_C M\,dx + N\,dy.$$

$$= \int_{OA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy + \int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= I_1 + I_2.$$

$$I_1 = \int_{OA} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \qquad x^2 = y.$$

$$2x\,dx = dy.$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx.$$

$$= \int_0^1 3x^2 - 8x^4\,dx + 8x^3 - 12x^4\,dx.$$

$$= \cancel{\frac{6x^4}{4}} \cancel{- \frac{16x^6}{6}} \cancel{+ \frac{8x^3}{3}} \cancel{- \frac{12x^5}{5}}$$

$$= \left| x^3 - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5} \right|_0^1$$

$$= 1 - \frac{8}{5} + \frac{8}{4} - \frac{12}{5}$$

$$= 1 + \frac{8}{3} - \frac{20}{5} = -\frac{8}{4} + 3 = \frac{-1}{3}$$

$$I_2 = \int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \qquad y^2 = x.$$

$$2y\,dy = x.$$

$$= \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int 6y^5 - 16y^3 + 4y - 6y^3.$$

$$= \left| \frac{6y^6}{6} - \frac{16y^4}{4} + \frac{4y^2}{2} - \frac{6y^4}{4} \right|_1^0$$

$$= -\left(1 - 4 + 2 - \frac{3}{2}\right) = \frac{5}{2}.$$

$$I_1 + I_2 = \frac{3}{2}.$$

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$$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

$M = (3x^2 - 8y^2)$ $\quad N = (4y - 6xy)$

$\frac{\partial M}{\partial y} = -16y$ $\quad \frac{\partial N}{\partial x} = -6y.$

x varies from 0 to 1

y varies from x^2 to $\sqrt{x}$

$$\int_0^1 \int_{y=x^2}^{\sqrt{x}} (-6y - (-16y))\,dy\,dx.$$

$$= \int_0^1 \left[5y^2\right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1 5(x - x^4)\,dx.$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{2}.$$

$\therefore$ LHS = RHS

(2) $\oint_C (xy + y^2)\,dx + x^2\,dy$; $y = x$ & $y = x^2$.

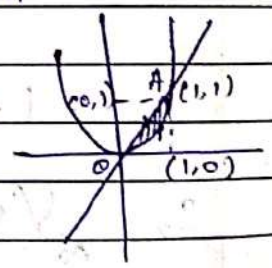
$y = x$; $y = x^2$.

$x = x^2$

$x^2 - x = 0$

$x(x-1) = 0 \Rightarrow x = 0$; $x = 1$

$y = 0$; $y = 1$.



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

x varies from $0, 1$

y — " — x^2 to x

$$= \int_0^1 \int_{x^2}^{x} \left(2x - (x + 2y)\right) dy\,dx.$$

$$= \int_0^1 \left(xy - y^2\right)_{x^2}^{x}$$

$$= \int_0^1 \cancel{x^2} - \cancel{x^2} - x^3 + x^4.$$

$$= \left(-\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = -\frac{1}{4} + \frac{1}{5} = \frac{-5+4}{20} = \frac{-1}{20}$$

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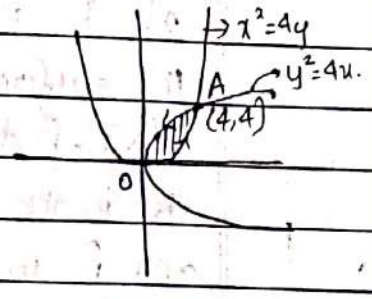
③ Find the area betn the parabolas $y^2 = 4x$, $x^2 = 4y$. with the help of Green's Thm in a plane.

Ans ⇒ The area is given by $\iint dx\,dy = \frac{1}{2}\int_C x\,dy - y\,dx$

Area enclosed by the plane curve.

$y^2 = 4x$; $x^2 = 4y$.

$x = \frac{y^2}{4}$; $\left(\frac{y^2}{4}\right)^2 = 4y$.



$\frac{y^4}{16} = 4y$.

$y^3 = 64 \Rightarrow y = 4$.

$x = y^2/4 \Rightarrow 4^2/4 = 4$.

$\frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$.

$= \frac{1}{2}\int_0^4\int_{x^2/4}^{2\sqrt{x}}(-1-1)\,dy\,dx$.

$= \frac{1}{2}\int_0^4 \left[-2y\right]_{x^2/4}^{2\sqrt{x}}\,dx$.

$= \frac{1}{2}\int_0^4\left(-4\sqrt{x} + \frac{x^2}{2}\right)dx$.

$= \frac{1}{2}\left[\frac{-4x^{3/2}}{3/2} + \frac{x^3}{6}\right]_0^4$

$= \frac{1}{2}\left|\frac{-4.8}{3/2} + \frac{64}{6}\right|$

$= \frac{1}{2}\left|\frac{-64}{3} + \frac{64}{6}\right|$

$= \frac{1}{2}\left|\frac{-128 + 64}{6}\right|$

$= \frac{1}{2}\left|\frac{-64}{6}\right|$

$= \left|\frac{-64}{12}\right| = \frac{16}{3}$ sq. units

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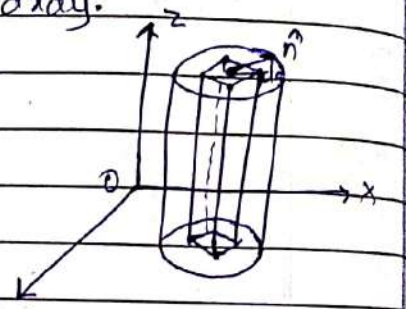
Surface & Volume Integral

The surface Integral : An integral evaluated over a surface is called surface integral. If $\hat{n}$ is the unit outward normal to surface 'S' the integral of normal component of $\vec{F}$ is given by $(\vec{F}\cdot\hat{n})$ over the surface 'S' is called the surface integral. It can be written as.

$\iint \vec{F}\cdot\hat{n}\,ds$. where ds is a small elemental area.

R is the orthonormal projection of 'S' on XOY plane and $\hat{n}$ is the unit outward normal to 'S' it should be denoted by $\hat{n}\,\hat{k}\,ds$ where $\hat{k}$ being the unit vector along z-axis. Then projection of vectorial area $\hat{n}\,ds$ on the xy plane is equal to $dxdy$.



$\hat{n}\,\hat{k}\,ds = dxdy\hat{k} \rightarrow$ xy plane

$\hat{n}\,\hat{j}\,ds = dxdz\hat{j} \rightarrow$ xz plane

$\hat{n}\,\hat{i}\,ds = dydz\hat{i} \rightarrow$ yz plane

$\hat{n}\,ds = dxdy\hat{k} + dydz\hat{j} + dxdz\hat{i}$

$\cancel{d\vec{s} = \hat{n}ds = dxdy\hat{i} + dydz\hat{j} + dxdz\hat{k}}$

$d\vec{s} = \hat{n}ds = dydz\hat{i} + dxdz\hat{j} + dxdy\hat{k}$

The volume integral. if V is the volume bounded by the surface and if $f(x,y,z)$ is the single valued f^n defined over 'V' then the volume integral of $f(x,y,z)$ over 'V' is given by. $\iiint_V f\,dv$.

If the volume is divided into some elements having sides dx, dy, dz then the volume integral is given by.

$\iiint f(x,y,z)\,dxdydz$ which can be evaluated by choosing the appropriate limits of x, y, z.

Stokes Theorem

If 'S' is the surface bounded by a simple closed curve 'c' and if $\vec{F}$ is any continuously differentiable vector function then the ~~line integral of~~

$$\oint \vec{F}\cdot d\vec{s} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

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$\oint \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds.$

① Evaluate using stokes thm $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by $x=0, x=a$ $y=0, y=b$.

W.K.T.

$\oint \vec{F}\cdot ds = \iint_S \text{curl}\vec{F}\cdot\hat{n}\, ds.$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-2y-2y) = -4y\hat{k}$$

$\text{curl}\vec{F} = -4y\,\hat{n}\,\hat{k}\, ds.$

$= \cancel{-4y\,dx\,dy}.$

$= -4y\hat{k}\,(dx\,dy\hat{k} + dy\,dz\,\hat{i} + dx\,dz\,\hat{j})$

$= -4y\,dx\,dy$

$\iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^b -4y\,dy\,dx.$

$= \int_0^a (-2y^2)_0^b\, dx$

$= \int_0^a -2b^2\,dx = (-2xb^2)_0^a = -2ab^2$

② Evaluate using stokes thm $\vec{F} = (x^2-y^2)\hat{i} + 2xy\hat{j}$ taken around a rectangle bounded by $x = \pm a, y=0, y=b$.

Sol: $\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$

$= \hat{k}(2y+2y) = 4y\hat{k}$

$\text{curl}\vec{F}\,\hat{n}\cdot ds = 4y\hat{k}\,(dx\,dy\,\hat{k} + dy\,dz\,\hat{i} + dx\,dz\,\hat{j})$

$= 4y\,dx\,dy$

$\iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds = \int_{-a}^{a}\int_0^b (4y)\,dy\,dx.$

$= \int_{-a}^{a} (2y^2)_0^b\, dx$

$= (2xb^2)_{-a}^{a}$

$= 2ab^2 + 2ab^2 = 4ab^2$

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(3) Evaluate stokes thm for the vector $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where 's' is the upper ~~half~~ surface of the sphere $x^2+y^2+z^2=1$; C is its boundary & represents the circle as $x^2+y^2=1$, $z=0$, $\vec{F}\cdot d\vec{r} = (2x-y)dx$ $(\because z=0)$

Taking to the polar coordinator $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le 2\pi$

$dx = -r\sin\theta\, d\theta$.

$\vec{F}\cdot d\vec{r} = (2r\cos\theta - r\sin\theta)(-r\sin\theta\, d\theta)$

$\therefore \int \vec{F}\, d\vec{r} = \int_0^{2\pi} (-2r^2\sin\theta\cdot\cos\theta + r^2\sin^2\theta)\, d\theta$ $[\because r=1]$

$= \int_0^{2\pi} \left(-\sin 2\theta + \left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta$

$= \left[\frac{+\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= \pi$

$= \pi$

$$\text{curl}\cdot\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & 0 & 0 \end{vmatrix}$$

$= \hat{k}\cdot(-(-1)) = \hat{k}$

$\text{curl}\vec{F}\cdot\hat{n}\cdot ds = \iint \hat{k}\cdot(dx\,dy)$

$\therefore \iint dx\,dy$ represents the area of the circle $= \pi r^2$

(4) Using strokes thm evaluate $\vec{F} = 2xy\hat{i} + (x^2-y^2)\hat{j}$ over a circle $x^2+y^2=1$, $z=0$.

$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\, ds = 0$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy & x^2-y^2 & 0 \end{vmatrix}$$

$= \hat{k}(2x-2x) = 0$

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Gauss divergence thm

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

If 'v' is the volume bounded by the surface 'S' and $\vec{F}$ is continuously differentiable vector f^n then.

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds.$$

where $\hat{n}$ is a positive unit vector drawn outward normal to 'S'.

① Evaluate using G.D.T^nm (Gauss Divergence) for the vector $\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$ taken over a rectangular parallelopiped. $0 \le x \le a$, $0 \le y \le b$ and $0 \le z \le c$.

Sol^n

$$\operatorname{div}\cdot\vec{F} = \left(\partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}\right)\left[(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy).$$

$$= 2x + 2y + 2z$$

$$= 2(x+y+z)$$

$$\int_0^a\int_0^b\int_0^c 2(x+y+z)\,dz\,dy\,dx.$$

$$= \int_0^a\int_0^b \left|2\left(2x + 2y + \frac{z^2}{2}\right)\right|_0^c$$

$$= \int_0^a\int_0^b \left(2xc + 2yc + \frac{2c^2}{2}\right) dy\,dx.$$

$$= \int_0^a \left|2xyc + \frac{2y^2}{2}c + \frac{2c^2y}{2}\right|_0^b dx.$$

$$= \int_0^a \left(2xbc + \frac{2b^2c}{2} + \frac{2bc^2}{2}\right) dx$$

$$= \left(\frac{2x^2bc}{2} + \frac{2b^2cx}{2} + \frac{2xbc^2}{2}\right)_0^a$$

$$= \frac{2a^2bc}{2} + \frac{2ab^2c}{2} + \frac{2abc^2}{2} = a^2bc + ab^2c + abc^2.$$

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Also $\iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)$.

(2) Evaluate using gauss divergence theorem for the vector $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$, S is the rectangular parallelopiped bounded by $0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3$

By G.D.T.

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds.$$

$$\operatorname{div}\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(2xy\hat{i} + yz^2\hat{j} + xz\hat{k}\right)$$

$$= 2y + z^2 + x$$

$$\int_0^2\int_0^1\int_0^3 (2y + z^2 + x)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^1 \left[2yz + \frac{z^3}{3} + zx\right]_0^3 dy\,dx.$$

$$= \int_0^2\int_0^1 (6y + 9 + 3x)\,dy\,dx$$

$$= \int_0^2 \left(3y^2 + 9y + 3xy\right)_0^1$$

$$= \int_0^2 (3 + 9 + 3x)\,dx.$$

$$= \left(3x + 9x + \frac{3x^2}{2}\right)_0^2$$

$$= 6 + 18 + 6 = 30$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 30$$

(3) Evaluate using gauss divergence Thm for the vector $\vec{F} = (x^2 - z^2)\hat{i} + 2xy\hat{j} + (y^2 + z^2)\hat{k}$. S is the surface of cube bounded by $x = 0, x = 1$, $y = 0, y = 1$, $z = 0, z = 1$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

$$\operatorname{div}\vec{F} = 2x + 2x + 2z$$

$$\int_0^1\int_0^1\int_0^1 (4x + 2z)\,dz\,dy\,dx.$$

$$= \int_0^1\int_0^1 \left(4xz + z^2\right)_0^1 dy\,dx$$

$$= \int_0^1\int_0^1 (4x + 1)\,dy\,dx$$

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$= \int_0^1 (4xy+1y)\, dx$

$= \int_0^1 (4x+1)\, dx$

$\left(\frac{4x^2}{2}+1\right)_0^1 = 2+1 = 3$

$\iint_S \vec{F}\cdot\hat{n}\, ds = 3$

(c) Evaluate using G.D.T. for $\vec{F} = (x^2-z^2)\hat{i} + 2xy\hat{j} + (y^2+z^2)\hat{k}$ over the unit cube
$(x, y, z \to 0 \text{ to } 1)$.

(d) Evaluate using G.D.T for $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ for the ~~circle~~ sphere.
$x^2+y^2+z^2 = a^2$.

By G.D.T $\Rightarrow \iint \vec{A}\cdot\hat{n}\, ds = \iiint \operatorname{div}\vec{A}\cdot dv$.

$\operatorname{div}\vec{A} = \nabla\cdot\vec{A} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)$

$= 3x^2 + 3y^2 + 3z^2$

$= 3(x^2+y^2+z^2)$

$\iiint 3(x^2+y^2+z^2)\, dx\,dy\,dz$.

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$dx\,dy\,dz = J(dr\,d\theta\,d\phi)$.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\,(+r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(-r\sin\theta\cos\phi\cos\theta)$
$-r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$.

$= r^2\sin\theta$.

$\therefore dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

r varies from $0 \to a$.
θ —— " —— $0 \to \pi$
ϕ —— " —— $0 \to 2\pi$

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$$\int_0^a\int_0^\pi\int_0^{2\pi} 3r^2 r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^a\int_0^\pi 3r^4 \sin\theta \cdot 2\pi \, d\theta\, dr$$

$$= \int_0^a \left(3r^4 (-\cos\theta)\cdot 2\pi\right)_0^\pi dr$$

$$= \int_0^a 3r^4 \cdot 2\pi(-1-1)\, dr$$

$$= \int_0^a 12\pi r^4\, dr$$

$$= +\left(\frac{12\pi r^5}{5}\right)_0^a = \frac{+12\pi a^5}{5}$$

$$\iint \vec{A}\cdot\hat{n}\, ds = \frac{12\pi a^5}{5}$$

it can be written as $\iint_S \vec{F}\cdot\hat{n}\, ds$

(6) Evaluate $\int_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ & S is the closed surface of the region in the first octant bounded by the cylinder, $y^2+z^2=9$ & the planes $x=0$, $x=2$, $y=0$ $z=0$

Sol^n ⇒ $\int_S \vec{F}\cdot\hat{n}\, ds = \iint_S \vec{F}\cdot\hat{n}\, ds$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)$$

$$= 4xy - 2y + 8xz$$

x varies from $0 \to 2$.
y varies from $0 \to 3$
z varies from $0 \to \sqrt{9-y^2}$.

$y^2+z^2=9$
$y^2+0=9$
$y=3$.
$z^2=9-y^2$
$z=\sqrt{9-y^2}$.

$$\iiint \operatorname{div}\vec{F}$$

$$= \int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\, dz\, dx\, dy$$

$$= \int_0^2\int_0^3 \left(4xyz - 2yz + 4xz^2\right)_0^{\sqrt{9-y^2}}$$

$\sqrt{9-y^2} = t$

$\frac{dt}{dy} =$

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$$\int_0^2\int_0^3 4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2)\,dx\,dy$$

$9-y^2=t$.

$-2y\,dy=dt$.

$dy=\frac{-dt}{2y}$.

$$\int_0^2\int_9^0 \left(4xy\sqrt{t} - 2\sqrt{9-t}\sqrt{t} + 4xt\right)\left(\frac{-dt}{2\sqrt{9-t}}\right)dx.$$

If $y=0$

$t=9$.

$y=3$

$t=0$.

$\sqrt{9-t}=r$

$$= \int_0^2\int_9^0 \left[2x\sqrt{t}(-dt) + \sqrt{t}\,dt - \frac{2xt}{\sqrt{9-t}}\,dt\right]dx$$

$$= \cancel{\int_0^2\left(\frac{-2xt^{3/2}}{3/2} + \frac{t^{3/2}}{3/2} - 2x\left[\frac{t}{\sqrt{t}}\right]\right)}$$

$$= \int_0^2\int_9^0 \left(\frac{-2xt^{3/2}}{3/2} + \frac{t^{3/2}}{3/2} - 2x\left(\frac{-(9-t)}{\sqrt{9-t}} + \frac{9}{\sqrt{9-t}}\right)\right)_9^0 dt$$

$$= \int_0^2 \frac{-2xt^{3/2}}{3/2} + \frac{t^{3/2}}{3/2} - 2x\left(-\sqrt{9-t} + \frac{9}{\sqrt{9-t}}\right)_9^0 dt.$$

$$= \int_0^2 -\frac{4x}{3}(0-9)^{3/2} + \frac{2}{3}\left(0-9^{3/2}\right) + 2x\left(\frac{(9-t)^{3/2}}{3/2} - \frac{9(9-t)^{1/2}}{1/2}\right)_9^0 dx$$

$= 180$

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Infinite Series

Sequence

An order set of real numbers as $u_1, u_2, u_3 \ldots u_n$ is called a sequence and it is denoted by $\{u_n\}$.

If the number of terms in a sequence is infinite, it is said to be infinite sequence, otherwise it is finite sequence. u_n is called n^{th} term of the sequence.

1) Monotonically increasing $u_{n+1} \geq u_n$ for each value of n.
2) Monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n.

3) Alternating sequence if the terms are alternatively +ve and -ve.

Ex: 1) $1, 2, 3, 4, 5 \ldots$
2) $1, 1/2, 1/3, 1/4, 1/5 \ldots$
3) $1, -2, 3, -4, 5, -6 \ldots$

Series.

$u_1 + u_2 + \ldots + u_n$ is called a series (finite) and it is denoted by $S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \ldots$

Infinite series.

If the number of terms in the series is infinitely large then it is called infinite series and denoted by

$$\Sigma u_n = \sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + \infty.$$

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A series [illegible] more than 1 [illegible] be oscillatory if $S_n \to$ [illegible] $n \to \infty$.

Geometric series

Consider a geometric series $ar^0 + ar^1 + ar^2 + \dots$ to discuss its nature for various values of r. sum of this is given by

$$S_n = a\frac{(1-r^n)}{1-r} \quad \text{if } r < 1$$

$$S_n = a\frac{(r^n-1)}{r-1} \quad \text{if } r > 1$$

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convergence of a series for the term is $\lim_{n\to\infty} U_n = 0$.

Comparision test.

If ΣU_n and ΣV_n be 2 +ve series term such that $\lim_{n\to\infty} \frac{U_n}{V_n}$ is a non-zero finite quantity, then ΣU_n and ΣV_n behave alike i.e ΣU_n and ΣV_n both will be convergence or both will be divergence

Note:- In this series ΣU_n is used to compare with ~~of~~ another series V_n whose nature is to be known, we usually

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1) $\lim_{n \to \infty} \frac{\log n}{n} = 0$

2) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

3) $\lim_{n \to \infty} n^{1/n} = 1$

4) $\lim_{n \to \infty} (n!)^{1/n} = \infty$

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$$= \lim_{n\to\infty} \left[\left(\frac{n}{2n+1}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n(2+1/n)}$$

$$= \frac{1}{2+1/\infty} = \frac{1}{2} < 1 \quad \rightarrow \text{Convergent}$$

$$1 + x/2 + x^2/3^2 + x^3/4^3 + \dots$$

$$U_n = x^{n-1}$$

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$= \frac{1}{\infty} \times x$

$0 < 1 \rightarrow$ Convergence

3) $1 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots$

$U_n = \frac{1}{n^2}$

$(U_n)^{1/n} = (1/n^2)^{1/n}$

$\lim_{n \to \infty} U_n^{1/n} = \lim_{n \to \infty} \frac{1}{n^{2/n}} = \frac{1}{\infty^0}$

$\log k = \lim_{n \to \infty}$

$\log k = \lim_{n \to \infty} \log 1 - \frac{2}{n} \log n$

$\log k = \lim_{n \to \infty} 0 - \frac{2 \cdot 1/n}{1}$

$\log k = -\frac{2 \cdot 1/\infty}{1} = 0$

$\log k = \log 1$

$k = 1$

the test fails

4. $\sum \frac{n! \, 2^n}{n^n}$

$U_n = \frac{n! \, 2^n}{n^n}$

Dr alembert ratio test $\Rightarrow U_{n+1} = \frac{(n+1)! \, 2^{n+1}}{(n+1)^{n+1}}$

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Use DAlbert's Ratio test whenever $n!$ is used.

Infinite Series

Problems

① $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots\ldots, \; U_n = \frac{x^n n^n}{n!}$

$$\sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$$

$$U_n = \frac{x^n n^n}{n!}$$

$$\therefore U_{n+1} = \frac{x^{n+1}(n+1)^{n+1}}{(n+1)!}$$

$$\underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt} \frac{\frac{x^{n+1}(n+1)^{n+1}}{(n+1)!}}{\frac{x^n n^n}{n!}}$$

$$= \underset{n\to\infty}{Lt} \frac{\cancel{x^n}.x\,\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n} n^n}$$

$$= x \underset{n\to\infty}{Lt} \frac{(n+1)^n}{n^n}$$

$$= x \underset{n\to\infty}{Lt} \left(\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right)^n$$

$$= xe \longrightarrow \boxed{\begin{array}{l}\text{if } x<1, \text{ convergent} \\ x>1 \text{ divergent}\end{array}}$$

Cauchy's root test

② $\sum \frac{1}{(1+1/n)^2}$; $U_n = \frac{1}{(1+1/n)^2}$.

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In infinite series terms can be neglected or added.

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(2) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ $\therefore U_n = \frac{1}{(\log n)^n}$

$(U_n)^{1/n} = \frac{1}{\log n}$

$\lim_{n\to\infty} \frac{1}{\log n} = \lim_{n\to\infty} \frac{0}{1/n}$

$= 0 < 1$

$\therefore$ The series fails and is convergent.

(3) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \dots + \infty.$

$\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$

$U_n = \left(\frac{n+1}{n+2}\right)^n x^n \Rightarrow (U_n)^{1/n} = \left(\frac{n+1}{n+2}\right) x$

$\cancel{\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n x}$ $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) x = \frac{\infty}{\infty}$

$x \lim_{n\to\infty} \frac{n(1+1/n)}{n(1+2/n)} = x \lim_{n\to\infty} \frac{(1+0)}{(1+0)} = x$

if $x < 1 \Rightarrow$ convergent

if $x > 1 \Rightarrow$ divergent.

(4) $\sqrt{\frac{n+1}{n^3+1}}\, x^n$; $U_n = \sqrt{\frac{n+1}{n^3+1}}\, x^n$.

$\cancel{(U_n)^{1/n} = \left(\sqrt{\frac{n+1}{n^3+1}}\right)^{1/n} x}$ $U_{n+1} = \sqrt{\frac{n+2}{(n+1)^3+1}} \cdot x^{n+1}$.

$\cancel{\lim_{n\to\infty} =}$

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{(n+2)}{(n+1)^3+1}}\, x^{n+1}}{\sqrt{\frac{n+1}{n^3+1}} \cdot x^n}$

$\lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n^3}} \frac{\sqrt{1+2/n}}{\sqrt{(1+1/n)^3+1/n}} \times x^n \cdot x\right) \Big/ \frac{\sqrt{n}}{\sqrt{n^3}} \frac{\sqrt{1+1/n}}{\sqrt{1+1/n^3}} \cdot x^n = \frac{\sqrt{1+0}}{\sqrt{1+0}} = x$

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(5) $\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$

$U_n = \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n$; $U_{n+1} = \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} \times x^{n+1}$.

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1} \cdot \frac{\sqrt{n^2+1}}{\sqrt{n} \cdot x^n}$.

$\lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\sqrt{1+1/n}}{\cancel{\sqrt{n^2}}\sqrt{(1+1/n)^2+1/n^2}} \cdot \cancel{x^n} \cdot x \cancel{\sqrt{n^2}} \frac{\sqrt{1+1/n^2}}{\cancel{\sqrt{n}}\, \cancel{x^n}}$

$= x$ //

(6) $\frac{2}{1} + \frac{(2)(5)}{5} + \frac{(2)(5)(8)}{(1)(5)(9)} + \frac{(2)(5)(8)(11)}{(1)(5)(9)(13)} \ldots\ldots$

$U_n = \frac{2 \cdot 5 \cdot 8 \cdot 11 \ldots (3n-1)}{1 \cdot 5 \cdot 9 \cdot 13 \ldots (4n-3)}$

$U_{n+1} = \frac{2 \cdot 5 \cdot 8 \cdot 11 \ldots (3n-1)(3(n+1)-1)}{1 \cdot 5 \cdot 9 \cdot 13 \ldots (4n-3)(4(n+1)-1)}$

$U_{n+1} = \frac{2 \cdot 5 \cdot 8 \cdot 11 \ldots (3n-1)(3n+2)}{1 \cdot 5 \cdot 9 \cdot 13 \ldots (4n-3)(4n+1)}$

$\underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n}$.

$= \underset{n\to\infty}{Lt} \frac{\cancel{2 \cdot 5 \cdot 8 \cdot 11 \ldots (3n-1)}(3n+2)}{\cancel{1 \cdot 5 \cdot 9 \cdot 13 \ldots (4n-3)}(4n+1)} \cdot \frac{\cancel{2 \cdot 5 \cdot 8 \cdot 11 \ldots (3n-1)}}{\cancel{1 \cdot 5 \cdot 9 \cdot 13 \ldots (4n-3)}} = \underset{n\to\infty}{Lt} \frac{3n+2}{4n+1}$

$= \underset{n\to\infty}{lt} \frac{n(3+2/n)}{n(4+1/n)} = \frac{3+2/\infty}{4+1/\infty} = \frac{3}{4} < 1$ $\therefore$ convergent

Power Series

Derivation V.I.

The series solution of Bessel's Differential Eqn

Consider the Bessel's D.E $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ —(1)

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Since the coeff of $\frac{d^2y}{dx^2}$ is $x^2 = 0$ when $x=0$

we must assume the series sol^n in the form

$y = \sum_{r=0}^{\infty} a_r x^{k+r}$ where $a_0 \neq 0$

$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$,

$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{(k+r-2)}$ → substitute in eqn (1)

$\Rightarrow x^2 \sum a_r (k+r)(k+r-1) x^{k+r-2} + x \sum a_r (k+r) x^{k+r-1} + x^2 \sum_0^{\infty} a_r x^{k+r}$
$- n^2 \sum_0^{\infty} a_r x^{k+r} = 0.$

$\Rightarrow \sum_0^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_0^{\infty} a_r (k+r) x^{k+r} + \sum_0^{\infty} a_r x^{k+r-2} - n^2 \sum a_r x^{k+r}$
$= 0.$

$\Rightarrow \sum_0^{\infty} a_r x^{k+r} [(k+r)(k+r-1) + (k+r) - n^2] + \sum_0^{\infty} a_r x^{k+r+2} = 0.$

$\Rightarrow \sum a_r x^{k+r} \{(k+r)[(k+r-1+1)] - n^2\} + \sum a_r x^{k+r-2} = 0.$

$\Rightarrow \sum a_r x^{k+r} [(k+r)^2 - n^2] + \sum a_r x^{k+r+2} = 0.$ — (3)

The above term consists of x^k, x^{k+1} & x^{k+2} and so on where the term x^k and x^{k+1} is available only in the first summation, we shall equate the coefficient of these to '0'. The coefficient of x^k is got by talking $r=0$

$a_0 x^k (k^2 - n^2) = 0$

~~coeff~~ coeff of x^k.

$a_0 (k^2 - n^2) = 0$, $\boxed{a_0 \neq 0}$

$k^2 - n^2 = 0$

$\boxed{k = \pm n.}$

We have to find coeff x^{k+1} by putting $r=1$.

$a_1 x^{k+1} ((k+1)^2 - n^2) = 0.$

$a_1 ((k+1)^2 - n^2) = 0$ $[\because a_1 = 0]$

$(k+1)^2 - n^2 = 0$

$k+1 = \pm n$; we cannot accept this value because we

have already taken $k = \pm n$

Now we will consider the coeff of x^{k+r} from eqn '3' & equate to '0'

coeff of x^{k+r} from eqn ③ is.

The — 1 st ——— 2nd term of ③ is got by replacing r by r-2.

$\therefore a_r x^{k+r}((k+r)^2 - n^2) + a_{r-2} x^{k+r}$

$a_r((k+r)^2 - n^2) + a_{r-2} = 0$

$$a_r = \frac{-a_{(r-2)}}{(k+r)^2 - n^2}$$

when $k = n$.

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{\cancel{n^2} + r^2 + 2rn - \cancel{n^2}}$$

$$a_r = \frac{-a_{r-2}}{2nr + r^2}$$

$r = 2, 3, 4, 5, \ldots$

$$a_2 = \frac{-a_0}{4n + 2^2} = \frac{-a_0}{4(n+1)}$$

$$a_3 = \frac{-a_1}{6n+9} = \frac{-0}{6n+9} = 0$$

$$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = \frac{a_0}{4(n+1)8(n+2)} = \frac{a_0}{32(n+1)(n+2)}$$

$$a_5 = \frac{-a_3}{10n+25} = 0$$

$\therefore a_3 = a_5 = a_7 = \ldots = 0$

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} = [a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \ldots]$$

$$y = x^k [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots]$$

$k = n$.

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$$y_1 = x^n \left[a_0 + 0x - \frac{a_0}{4n} x^2 + 0x^3 + \frac{a_0}{32(n+1)(n+2)} x^4 + \dots \right]$$

$$y_1 = a_0 x^n \left[1 - \left(\frac{x}{2}\right) \cdot \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \frac{1}{2(n+1)(n+2)} + \dots \right] \rightarrow (5)$$

$k = -n$.

$$y_2 = a_0 x^{-n} \left[1 - \left(\frac{x}{2}\right) \frac{1}{(-n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{2(-n+1)(-n+2)} + \dots \right] \rightarrow (6)$$

$$\cancel{a_4 = \frac{-a_2}{8n+16} = -a_2}$$

$\therefore y = C_1 y_1 + C_2 y_2$

Here C_1 & C_2 are arbitrary constants.

$$a_0 = \frac{x^n}{2^n \Gamma(n+1)}$$

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^2 \frac{1}{2(n+1)(n+2)\Gamma(n+1)} + \dots \right]$$

$\Gamma n = (n-1)\Gamma(n-1)$ $\qquad$ $\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+2) = (n+1)\Gamma(n+1)$ $\qquad$ $\Gamma(n+3) = (n+2)\Gamma(n+2)$

(1) $J_n(x) = (-1)^n J_n(x)$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \rightarrow (1)$$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \rightarrow (2)$$

$y = A J_n(x) + B J_n(x)$

③ Using the property of Γ fnc ; $\Gamma(s+1) = s!$

$(s+n)! = \Gamma(s+n+1)$

$$\Rightarrow J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n} \frac{1}{s!\,\Gamma(s+n+1)} \quad \text{—— (1)}$$

It is same as writting.

$$J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma(s+n+1)\, s!}$$

$\therefore$ Comparing this with eqn (1) we see that summation in RHS is same as $J_n(x)$. Thus

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

<u>Note</u>: The above property is easily concluded that $J_{-n}(x)$ & $J_n(x)$ are not linearly independent when 'n' is an integer, thus the general solⁿ of Bezzle's D.E $y = A J_n(x) + B J_{-n}(x)$ where 'n' is not an integer.

<u>Recurrence Relation</u>

① $2n J_n(x) = x[J_{n+1}(x) + J_{n-1}(x)]$

<u>Proof</u>: By defⁿ

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r+n} \frac{1}{\Gamma(n+r+1)\, r!} \quad \text{—— (1)}$$

$0/0$

$$2n J_n(x) = 2n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r+n} \frac{1}{\Gamma(n+r+1)\, r!}$$

$2n = 2n + 2r - 2r$

$2n = 2(n+r) - 2r$

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2n}{\Gamma(n+r+1)\, r!}$$

We shall write $2n = 2(n+r) - 2r$ and split into 2 term

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2(n+r)}{\Gamma(n+r+1)\, r!} - \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{2r}{\Gamma(n+r+1)\, r!}$$

~~then replace~~

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{2}(n+r)}{\cancel{(n+r)}\,\Gamma(n+r)\, r!} - \sum_{r=0}^{\infty} (-1)^r \frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1} \frac{2r}{\Gamma(n+r+1)\, r!}$$

$$= x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r)\, r!} - x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)(r-1)!}$$

Replace in r as $+1$ & -1, the other term remains same.

put $r-1 = s \Rightarrow r = s+1$. in the second term of the above eqn.

$$= x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1-1)\, r!} - x \sum_{r=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+2(s+1)-1} \frac{1}{\Gamma(n+s+1+1)\, s!}$$

$$\Rightarrow 2n J_n(x) = x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} + x \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s+1} \frac{1}{\Gamma(n+s+1+1)\, s!}$$

$$2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x)$$

$$2n J_n(x) = x \left[J_{n+1}(x) + J_{n-1}(x)\right].$$

(2) $J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$ where 'n' = $+^{ve}$ integer.

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \longrightarrow (1)$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{-x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^{3r+n} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n(-x) = (-1)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n(-x) = (-1)^n J_n(x) = J_{-n}(x)$$

(3) $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \longrightarrow (1)$

D w.r.t x.

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2} \frac{1}{\Gamma(n+r+1)\, r!}$$

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$$n+2r = (n+r)+r$$

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{(n+r)}{\Gamma(n+r+1)\, r!} + \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{r}{\Gamma(n+r+1)\, r!}$$

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{\cancel{(n+r)}}{\cancel{(n+r)}\,\Gamma(n+r)\, r!} + \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{r}{\Gamma(n+r+1)\, r(r-1)!}$$

$r-1 = s \Rightarrow r = s+1$ for 2nd terme

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} + \sum_{r=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+2(s+1)-1} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

(4) $$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

<u>Proof:</u> $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \rightarrow (1)$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r (2n+2r) \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \cancel{2(n+r)} \frac{x^{2n+2r-1}}{2^{n+2r}} \frac{1}{\cancel{(n+r)}\,\Gamma(n+r)\, r!}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2r-1}}{2^{n+2r-1}} \frac{1}{\Gamma(n+r)\, r!}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r x^n \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n-1+r+1)\, r!}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{\Gamma((n-1)+r+1)\, r!}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$$

$$\cancel{\frac{d}{dx}\left(x^{-n} J_n(x)\right)}$$

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(5) $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = x^{-n} J_{n+1}(x)$

Proof: $J_n = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{n+2r}} \cdot \frac{1}{\Gamma(n+r+1)\, r!}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r (2r) \frac{x^{2r-1}}{2^{n+2r}} \cdot \frac{1}{\Gamma(n+r+1)\, r\, (r-1)!}$$

$$\frac{d}{dx}\left[x^{n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \frac{x^{-n} \cdot x^{n+2r-1}}{2^{n+2r-1}} \cdot \frac{1}{\Gamma(n+r+1)(r-1)!}$$

$r-1 = s \Rightarrow s = r+1.$

$$= \sum_{r=0}^{\infty} (-1)^{r+1} \left(\frac{x}{2}\right)^{n-1+2(s+1)} \frac{1}{\Gamma(n+1+r+1)\, s!}$$

$$= x^{-n} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+1+2s} \frac{1}{\Gamma(n+1+s+1)\, s!}$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = x^{n} J_{n+1}(x)$$

(6) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \longrightarrow (1)$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r) \frac{x^{n+2r-1+1}}{2^{n+2r-1+1}} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

Replace $n+2r = 2(n+r) - n.$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r 2(n+r) \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} - \sum_{r=0}^{\infty} (-1)^r n \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+r) \left(\frac{x}{2}\right) \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{(n+r)\Gamma(n+r)\, r!} - n \sum_{r=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

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$$x J_n'(x) = x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

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* Power Series:

The Series Solution of Bessel's differential equation:

→ Consider the Bessel's D.E

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \quad -(1)$$

✓ Since the coefficient of $\frac{d^2y}{dx^2}$ is $x^2 = 0$ when $x=0$.

✓ we must assume the series soln in the form

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{when } a_0 \neq 0$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$$

Substitute this in eqn (1)

$$\Rightarrow x^2\sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} + x\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} + x^2\sum_0^{\infty} a_r x^{k+r} - n^2\sum_0^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_0^{\infty} a_r(k+r)(k+r-1)x^{k+r} + \sum_0^{\infty} a_r(k+r)(x)^{k+r} + \sum_0^{\infty} a_r x^{k+r+2} - n^2\sum a_r x^{k+r} = 0$$

$$\Rightarrow \sum_0^{\infty} a_r x^{k+r}\left[(k+r)(k+r-1) + (k+r) - n^2\right] + \sum_0^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum a_r x^{k+r}\left[(k+r)^2 - n^2\right] + \sum a_r x^{k+r+2} = 0$$

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→ The above term consist of x^k, x^{k+1}, x^{k+2} & so where the term x^k, x^{k+1} is available only the first summation. We shall equate the co-efficient of this to zero. The co-efficient of x^k is got by taking $r=0$

x^k put $r=0$

$$a_0 x^k (k^2 - n^2) = 0$$

$$a_0 (k^2 - n^2) = 0 \qquad a_0 \neq 0$$

$$k^2 - n^2 = 0$$

$$k = \pm n$$

We have to find co-efficient of x^{k+1} by putting $r=1$ from the first series:

We can't accept this value bcz we have already taken $k = \pm n$ $(r=1)$

$$a_1 x^{k+1} ((k+1)^2 - n^2) = 0$$

$$a_1 ((k+1)^2 - n^2) = 0 \qquad \because a_1 \neq 0$$

$$(k+1)^2 - n^2 = 0$$

$$k+1 = \pm n$$

Now will consider co-efficient of x^{k+r} in general & equate to zero

$$a_r x^{k+r} ((k+r)^2 - n^2) + a_{r-2} x^{k+r}$$

$$a_r ((k+r)^2 - n^2) + a_{r-2} = 0$$

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The coefficient of x^{k+r} is 2nd term of ③ is got by replacing $r \to (r-2)$

$$a_r = \frac{-a_{r-2}}{(k+r)^2 - n^2}$$

$k=n$

$$a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2} = \frac{-a_{r-2}}{n^2 + r^2 + 2nr - n^2}$$

$$a_r = \frac{-a_{r-2}}{2nr + r^2}$$

$r = 2, 3, 4, 5, \ldots$

$$a_2 = \frac{-a_0}{4n+2^2} = \frac{-a_0}{4(n+1)}$$

$$a_3 = \frac{-a_1}{6n+9} = \frac{-0}{6n+9} = 0$$

$$a_4 = \frac{-a_2}{8n+16} = \frac{-a_2}{8(n+2)} = +\frac{a_0}{4(n+1)\,8(n+2)}$$

$$a_5 = \frac{-a_3}{10n+25} = 0$$

$a_3 = a_5 = a_7 = 0$

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} = [a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} \ldots]$$

$$y = x^k [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots]$$

$k = n$

$$y_1 = x^n \left[a_0 + 0x + \frac{-a_0}{4(n+1)} x^2 + 0 \cdot x^3 + \frac{a_0}{32(n+1)(n+2)} \cdot x^4 \ldots\right]$$

$$y_1 = a_0 x^n \left[1 - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{(n+1)} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2(n+1)(n+2)} + \ldots\right] \quad -⑤$$

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(here $x \to -n$)

$$y_2 = a_0 x^{-n}\left[1 - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{(-n+1)} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2(-n+1)(-n+2)}\right] \quad -(5)$$

Therefore the complete soln of Bessel's D.E is

$$y = c_1 y_1 + c_2 y_2$$

where c_1 & c_2 are arbitrary constant.

Now

We shall standardize the soln of eqn (5) by obtaining the particular soln choosing.

$$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

Γ - gamma

$\Gamma(n+1) = n!$

& the soln can be denoted by y_1

$$y_1 = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \left(\frac{x}{2}\right)^2 \cdot \frac{1}{n+1} + \left(\frac{x}{2}\right)^4 \cdot \frac{1}{2(n+1)(n+2)} \cdots\right]$$

$$y_1 = \left(\frac{x}{2}\right)^n\left[\frac{1}{\Gamma(n+1)} - \left(\frac{x}{2}\right)^2 \frac{1}{(n+1)\Gamma(n+1)} + \left(\frac{x}{2}\right)^4 \frac{1}{2(n+1)(n+2)\Gamma(n+1)} \cdots\right]$$

by using the property of gamma function

$$\Gamma n = (n-1)\Gamma(n-1)$$

$$\Gamma(n+1) = n\Gamma n$$

$$\Gamma(n+2) = (n+1)\Gamma(n+1)$$

$$\Gamma(n+3) = (n+2)\Gamma(n+2)$$

$$y_1 = \left(\frac{x}{2}\right)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(n+s+1)\, s!}\left(\frac{x}{2}\right)^{s+2}$$

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→ This series can also be written in the form

$$y_1 = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}$$

This function is called Bessel function of 1st kind which is denoted by

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}$$

$k = -n$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{n+2r}$$

Then the general solⁿ of Bessel eqⁿ can be written as

$$y = A J_n(x) + B J_{-n}(x)$$

where A and B are arbitrary constant and n is not an integer.

The properties of Bessel function's :

① $J_{-n}(x) = (-1)^n J_n(x)$ where n is an positive integer

Proof: Here by defⁿ - $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$ — ①

$$\therefore J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\Gamma(-n+r+1)\, r!} \quad \text{— ②}$$

$\because \Gamma(-n+r+1) = \Gamma(v-(n-1))$ is valid only for $v \geq n$ as we say assume that Γ of negative quantity for $r = 0, 1, 2 \ldots (n-2)$ & Γ of zero for $r = n-1$.

The gamma function (Γ) is defined only for +ve real no. thus eqn ② and put $(r-n) = s$ i.e $r = s+n$ so that

$$J_{-n}(x) = \sum_{s=n}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{2s-n} \frac{1}{\Gamma(s+1)\,(s-n)!}$$

using the property of gamma fun. $\Gamma(s+1) = s!$

$$J_{-n}(x) = \sum_{s=n}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{2s-n} \frac{1}{s!\,\Gamma(s-n+1)}$$

It is same as writing

$$J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{2s+n} \frac{1}{\Gamma(s+n+1)\,s!}$$

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{2s+n} \cdot \frac{1}{\Gamma(s+1)\,(s+n)!}$$

Compare with eq. (1) we observe that summation is R.H.S is same as $J(x)$. Thus we prove that $J_{-n}(x) = (-1)^n \cdot J_n(x)$.

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

These are not linearly independent when n is integer

Thus the general sol^n of Bessel D.E

$$y = A J_n(x) + B J_{-n}(x)$$

when n is not an integer.

VVI

* Recurrence relation:

(1) $2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x)\right]$ —— (1)

Proof:

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

$$2n J_n(x) = 2n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\,r!}$$

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Finite Differences.

① → Forward Difference
② → Backward Difference.

Finite Differences.

Consider a function $y = f(x)$ Let x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h \ldots x_n = x_{n-1} + h$. be set of points at common interval 'h'. Let the corresponding values of $f(x)$
$y_0 = y(x_0) = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_n = f(x_n)$
The values of independent variable x is called the argument and the corresponding functional value is called entry.

Forward Difference

The first forward difference of $f(x)$ is denoted by $\Delta f(x)$ is defined as follows.

$\Delta f(x) = f(x+h) - f(x)$

The 'Δ' is called the forward difference operator.

$\Delta y_0 = y_1 - y_0$.

Second forward difference of $f(x)$ is:

$\Delta y_0^2 = \Delta y_1 - \Delta y_0$.

The first entries in the table may be Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0 \ldots \Delta^n y_0$ are called leading forward difference.

		1st	2nd	3rd		nth.
x_0	y_0	$\cancel{\Delta y_0}$				
x_1	y_1	$\cancel{\Delta y_1}$	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	
x_2	y_2	$\cancel{\Delta y_2}$	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^n y_0$
x_3	y_3	$\cancel{\Delta y_3}$	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	
x_4	y_4		Δy_3	$\Delta^2 y_{n-2}$		
$\vdots$	$\vdots$		Δy_{n-1}			
x_n	y_n					

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Backward Difference

The first backward difference of $f(x)$ is denoted by ~~backward difference of~~ $\nabla f(x)$ and is defined by.

$\nabla f(x) = f(x) - f(x-h)$.

$\nabla f(x_n) = f(x_n) - f(x_n - h)$

$\nabla y_n = y_n - y_{n-1}$

The reverse triangle is called backward difference operator

The last entries in the table i.e ∇y_n $\nabla^2 y_n$ $\nabla^3 y_n$ ... $\nabla^n y_n$ are called the backward difference.

(1) Construct a finite diffence table for the function $f(x) = x^3 + x + 1$ where x takes the value 0, 1, 2, 3, 4, 5, 6 Identify the leading forward & backward diffenence.

$x = 0 \Rightarrow f(0) = 1$.

$x = 1 \Rightarrow f(1) = 3$.

$x = 2 \Rightarrow f(2) = 11$

$x = 3 \Rightarrow f(3) = 31$

$x = 4 \Rightarrow f(4) = 69$

$x = 5 \Rightarrow f(5) = 131$

$x = 6 \Rightarrow f(6) = 223$.

x	$y = f(x)$	1st	2nd	3rd	4th.
0	1				
		2			
1	3		6		
		8		6	
2	11		12		0
		20		6	
3	31		18		0
		38		6	
4	69		24		0
		62		6	
5	131		30		
		92			
6	223				

Forward diffence : $\Delta y_0 = 2$, $\Delta^2 y_0 = 6$, $\Delta^3 y_0 = 6$, $\Delta^4 y_0 = 0$

Backward difference: $\nabla y_0 = 92$, $\nabla^2 y_0 = 30$, $\nabla^3 y_0 = 6$, $\nabla^4 y_0 = 0$.

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(2) Construct a table of differences for following data.

x	0	10	20	30	40
$f(x)$	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^2 f(20)$, $\Delta^3 f(0)$ & $\Delta^4 f(0)$.

x	$y=f(x)$	1	2	3	4
0	1				
		$0.5\Delta f_0$			
10	1.5		$0.2\Delta^2 f_0$		
		$0.7\Delta f_{10}$		$0.0\Delta^3 f_0$	
20	2.2		$0.2\Delta^2 f_{10}$		$0.4\Delta^4 f_0$
		$0.9\Delta f_{20}$		$0.4\Delta^3 f_{10}$	
30	3.1		$0.6\Delta^2 f_{10}$		
		$1.5\Delta f_{30}$			
40	4.6				

$\therefore \Delta^2 f(20) = 0.6$

$\Delta^3 f(0) = 0.0$

$\Delta^4 f(0) = 0.4$

<u>Interpolation</u> (which lies inside a range)

If $y_0, y_1, y_2 \ldots y_n$ be a set of values of an unknown <u>fn</u>: $y = f(x)$ corresponds to the value of 'x' such that $x = x_0, x_1, \ldots x_n$ the process of finding the value of y, for any given value of 'x'. Between x_0 & x_n is called interpolation

The process of finding the value of 'y' outside the range of 'x' is called "extrapolation"

<u>Interpolation formulae for equal intervals.</u>

(1) <u>Newton's forward difference Interpolation formulae.</u>

The value of $y = f(x)$ at $x = x_0 + rh$ then $y_r = x_0 + rh$ is approximately given by

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$$

$$+ \frac{r(r-1)(r-2)\ldots(r-(n-1))}{n!}\Delta^n y_0$$

where 'r' is any real no.

$r = \frac{x - x_0}{h}$, x_0 being the first term, h being the step length

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(2) Newton's backward Interpolation formulae

The value of $y = f(x)$ at $(x = x_n + rh)$ where $y_r = f(x_n + rh)$ is approximately given by.

$$y(x) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \dots$$

$$\dots + \frac{r(r+1)\dots(r+(n-1))}{n!}\nabla^n y_n$$

where 'r' is any real no.

$r = \frac{x - x_n}{h}$, x_n being the last term, h being the step length.

(3) Find $y(1.4)$ given that

x	1	2	3	4	5
y	10	26	58	112	194.

To find y at $x=1.4$, it is near to $x=1$ so we use Newton's forward Interpolation formulae

x	y	1st	2nd	3rd
1	10			
		$16\Delta y_1$		
2	26		$16\Delta^2 y_1$	
		$32\Delta y_2$		$6\Delta^3 y_1$
3	58		$22\Delta^2 y_2$	
		$54\Delta y_3$		$6\Delta^3 y_2$
4	112		$28\Delta^2 y_3$	
		$82.\Delta y_4$		
5	194.			

$r = \frac{x - x_0}{h} = \frac{1.4 - 1}{1}$

$r = 0.4$

$$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$y(1.4) = 10 + (0.4)(16) + \frac{(0.4)(-0.6)}{2}(16) + \frac{0.4(-0.6)(-1.6)}{6}(6)$$

$$= 10 + 6.4 - 2.88 - 0.4$$

$$= 14.864.$$

(4) The area of a circle 'A' corresponds to the diameter d is given.

$D(x)$	80	85	90	95	100
$A(x)$	5026	5674	6362	7088	7854.

Find the area corresponds to the diameter at 105 using appropriate interpolation formulae.

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105 is near to 100, ∴ we use Newton's backward interpolation formulae.

x	$f(x)$	1	2	3	4
80	5026				
85	5674	648	40		
90	6362	688	38	-2.	4
95	7088	726	40	2	
100	7854	766			

$$r = \frac{x - x_n}{h} = \frac{105-100}{5} = 1.$$

$$f(105) = y_0 + 766 + 40 + 2 + 4.$$

$$= 8666.$$

(5) Using forward & backward interpolation formulae find the value of $f(1)$ & $f(10)$ where

x	3	4	5	6	7	8	9
$f(x)$	4.8	8.4	14.5	23.6	36.2	52.8	73.9

x	$f(x)$	1st	2nd	3rd	4th	5th
3	4.8	3.6				
4	8.4	6.1	-2.5	0.5		
5	14.5	9.1	3.0	0.5	0	0
6	23.6	12.6	3.5	0.5	0	
7	36.2	16.6	4	0.5		
8	52.8	21.1	4.5			
9	73.9.					

To find $f(1)$ use Newton's forward difference.

ie $y_0 = 4.8$, $\Delta y_0 = 3.6$, $\Delta^2 y_0 = -2.5$ $\Delta^3 y_0 = 0.5$

$$r = \frac{x - x_0}{h} = \frac{1-3}{1} = -2.$$

$$f(1) = y_0 + r\Delta y_0 + \frac{r(r-1)\Delta^2 y_0}{2!} + \frac{r(r-1)(r-2)\Delta^3 y_0}{3!}$$

$$= 4.8 - 7.2 + \frac{6(2.5)}{2} + \frac{(-24)}{6} 0.5$$

$f(1) = 3.1$

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$r = \frac{x - x_n}{h} = 10 - 9 = 1.$

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$f(10) = y_n + r\nabla y_n + r(r+1)\nabla^2 y_n + r(r+1)(r-2)\nabla^3 y_n$

$+ 21.1.$

$= 73.9$ ~~42 2 + 27 -12~~

$f(10) =$ 100%

From the following table evaluate the no. of students who have obtained marks between 40 & 45 using forward interpolation formulae.

Marks	30-40	40-50	50-60	60-70	70-80
No of students.	31	42	51	35	31.

The given table with f(x) represents the no. of students scored less than 'x' marks.

$< 40 \longrightarrow 31$

$< 50 \longrightarrow 31 + 42 = 73.$

$< 60 \longrightarrow 124$

$< 70 \longrightarrow 159$

$< 80 \longrightarrow 190$

x	y.	1	2	3	4
40	31				
		42.			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Using N.F.D;

$y_0 = 31, \Delta y_0 = 42, \Delta^2 y_0 = 9, \Delta^3 y_0 = -25, \Delta^4 y_0 = 37.$

$r = \frac{x - x_0}{h} = \frac{45-40}{10} = 0.5.$

$f(45) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$

$= 31 + 21 + \frac{0.5(-0.5)9}{2} + \frac{0.5(-0.5)(-1.5)(-25)}{6} + \frac{(0.5)(-0.5)(-1.5)(-2.5)37}{24}$

$= 47.86$

≈ 48

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Students between 40 & 45:
$f(45) - f(40) = 48 - 31 = 17$

(7) Given the foll table find the value of $f(42)$ & $f(85)$ using forward & backward interpolation formulae.

x	40	50	60	70	80	90
$f(x)$	184	204	226	250	276	304

x	$f(x)$	1st	2nd	3rd	4th
40	184				
		20			
50	204		2		
		22		0	
60	226		2		0
		24		0	
70	250		2		
		26		0	
80	276		2		
		28			
90	304				

for $f(42)$;

$$r = \frac{x - x_0}{h} = \frac{42-40}{10} = \frac{2}{10} = \frac{1}{5} = 0.2$$

$$f(42) = y_0 + r\Delta^2 y_0 + \frac{r(r-1)}{2!}\Delta^3 y_0 + \frac{r(r-2)}{3!}\Delta^3 y_0$$

$$= 184 + (0.2)(20) + -0.32$$

$$= 187.68$$

for $f(85)$

use N.B.P

$$r = \frac{x - x_n}{h} = \frac{85-90}{10} = \frac{5}{10} = -0.5$$

$$f(85) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 \qquad 318.75$$

$$= 184 + (0.2)(20) + 0.2 \cdot (-0.8)2$$

$$= 187.68$$

(8) Find the interpolating polynomial
$f(0)=0$ $f(2)=4$ $f(4)=56$ $f(6)=204$ $f(8)=496$
$f(10)=980$
& Hence find $f(3)$, $f(5)$ & $f(7)$

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x	0	2	4	6	8	10
$f(x)$	0	4	56	204	496	980.

x	$f(x)$	1st	2nd	3rd	4th
0	0	4			
2	4	52	48	48	
4	56	148	96	48	0.
6	204	292	144	48	
8	496	484	192		
10	980.				

$r = \frac{x - x_0}{h}$

$= \frac{8-0}{2} = \frac{x}{2}$

$y_0 = 0,\ \Delta y_0 = 4,\ \Delta^2 y_0 = 48,\ \Delta^3 y_0 = 48.$

$y(x) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0.$

$= 0 + \frac{x}{2}(4) + \frac{x}{2}\left(\frac{x}{2}-1\right)\frac{48}{2} + \frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\frac{48}{6}$

$= 2x + 6x^2 - 12x + x^3 - 2x^2 + 8x - 4x^2.$

$= x^3 - 2x$

for $f(2) = (2)^3 - 2(2) = 4.$

$f(3) = (3)^3 - 2(3) = 27 - 6 = 21.$

$f(5) = (5)^3 - 2(5) = 115.$

$f(7) = 343 - 14 = 329.$

(8) The population of a town is given by a table.

Years	1951	1961	1971	1981	1991.
Population in thousand	19.96	39.65	58.81	77.21	94.61.

Calculate the population betw 1985 & 1955 for f(1985).

$r = \frac{1951 - 1955}{10} = \frac{4}{10} = \frac{2}{5} = 0.4.$

x	$f(x)$	1st	2nd	3rd	4th
1951	19.96	19.69			
1961	39.65	19.16	−0.53	−0.23	−0.01.
1971	58.81	18.4	−0.76	−0.24	
1981	77.21	17.4	−1		
1991	94.61				

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$$f(1955) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0.$$

$= 19.96 +$

Interpolation formulae for unequal intervals.

① Newton's Divided Difference.

Let $f(x_0), f(x_1), f(x_2) \ldots f(x_n)$ be the values of an unknown f^n $y = f(x)$, corresponding to the values of 'x', such that $x : x_0, x_1, x_2 \ldots x_n$ at unequal intervals

⇒ The first order divided difference.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

The second order $f(x_0, x_1, x_2) = \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

General form $= f(x_{n-2}, x_{n-1}, x_n) = \dfrac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$

The Newton's divided difference formulae:

If $f(x_0, x_1, \ldots, x_n)$ be the set of values of an unknown f^n $y = f(x)$ corresponds to the values of x, such that $x_1, x_2 \ldots x_n$ at unequal intervals then newton's formulae.

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_0) f(x_0, x_1, x_2, x_3) + \ldots +$$
$$(x - x_0)(x - x_1) \ldots (x - x_{n-1}) f(x_0, x_1, x_2 \ldots, x_n).$$

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$(x-a)(x-b)(x-c) = x^3 - x^2(a+b+c) + x(ab+bc+ca) - abc.$

$(x-a)(x-b) = x^2 - x(a+b) + ab$

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① Using Newton's divided difference formulae find $f(4)$

x	0	2	3	6
$f(x)$	-4	2	14	158

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-4			
		3		
2	2		3	
		12		1
3	14		9	
		48		
6	158			

$f(4) =$

$f(x) = -4 \quad \Delta f(x_0) = 3 \quad 2^{nd} = 3 \quad 3^{rd} = 1.$

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$
$+ (x-x_0)(x-x_1)(x-x_2)\cancel{(x-3)} f(x_0, x_1, x_2, x_3)$

$f(4) = -4 + (4)(3) + (4)(2)(3) + (4)(2)(1)(1)$
$= -4 + 12 + 24 + 8$
$= -4 + 36 + 8$
$= 40$

② Using Newton's divided formulae find $f(8)$ & $f(15)$

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

x	$f(x)$	1st	2nd	3rd
4	48			
		52	$\frac{97-52}{7-4} = 15$	$\frac{21-15}{10-4} = 1$
5	100			
		97	$\frac{202-97}{10-5} = 21$	$\frac{27-21}{11-5} = 1$
7	294			
		202	$\frac{310-202}{11-7} = 27$	$\frac{33-27}{13-7} = 1$
10	900			
		310	$\frac{409-310}{13-10} = 33$	
11	1210			
		409		
13	2028			

$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$
$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3).$
$= 48 + (x-4)52 + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$

$f(8) = 48 + (4)52 + (4)(3)(15) + (4)(3)(1)(1)$
$= 448.$

$f(15) = 48 + (11)(52) + (11)(10)(15) + (11)(10)(8)$
$= 3150.$

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(3) $f(-4)=1245$, $f(-1)=33$, $f(0)=5$, $f(2)=98$ $f(5)=1335$.
$f(3)$.

x	$f(x)$	1st	2nd.	3rd.	4th.
-4	1245				
		-404			
-1	33		94.	-14	$\frac{27}{9}=3$
		-28			
0	5		10	13	
		2			
2	98		88		
		442.			
5	1335				

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)\cancel{(x-x_2)}f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)\cancel{(x-x_3)}f(x_0,x_1,x_2,x_3)$$

$$f(3)=1245+(7)(-404)+(7)(4)(94)+(7)(4)(3)(-14)$$
$$+(7)(4)(3)(1)(3)$$
$$=\cancel{1245-2828\ 1583}\ 125$$
$$=\cancel{3291}$$

(4) constant Interpolating polynomial for the data given below by using Newton's dividing difference.

x	2	4	5	6	8	10
$f(x)$	10	96	196	350	868	1746

x	$f(x)$	1st.	2nd.	3rd.
2	10			
		43		
4	96		19.	
		100		2
5	196		27	
		154		2
6	350		35	
		259.		2.
8	868		45	
		439		
10	1746			

$$f(x)=10+(x-2)43+(x-2)(x-4)19+(x-2)(x-4)(x-5)(2)$$
$$=10+43x-86+(x^2-x(6)+8)19+(x^2-x(6)+8)(x-5)(2)$$
$$=10+43x-86+19x^2-114x+152+(x^2-6x+8)(2x-10)$$
$$=10+43x-86+19x^2-114x+152+2x^3-10x^2-12x^2-60x$$
$$+16x-180.$$
$$=2x^3-3x^2+5x-4.$$

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(9) Using Newton's divided difference find f(9) given.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202.

The Lagrange's formulae for interpolation & inverse interpolation

Statement: If $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, $y_n = f(x_n)$ then be the set of values of an unknown f^n $y = f(x)$ corresponding to the values of 'x' such that $x_0, x_1, x_2 \ldots x_n$ not necessarily be an equal interval, then

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} y_2 + \ldots\ldots$$

$$\ldots + \frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} y_n$$

Note ⇒ We can interchange the role of x & y in the formulae & the same is called lagrange's inverse interpolation formulae which helps us to find 'x' for a given 'y' & the formulae is as follows. ~~Find 'x' fo~~

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_1)\ldots(y_0-y_n)} +$$

$$\frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)} +$$

$$\frac{(y-y_0)(y-y_1)(y-y_3)\ldots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\ldots(y_2-y_n)} + \ldots\ldots$$

$$\frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-1)\ldots(y_n-y_{n-1})}$$

(1) Using lagrange's formulae, find the interpolating polynomial that approximates to the function described by the table & find $f(3)$ x_0 x_1 x_2 x_3.

x	0	1	2	5
$f(x)$	2	3	12	147

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$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)(x_3-x_0)}y_3$$

$$y = \frac{(2)(1)(-2)(4)}{(-1)(-2)(-5)} + \frac{(3)(1)(-2)(3)}{(1)(-1)(-3)} + \frac{(3)(2)(-2)(12)}{(2)(1)(-3)} + \frac{(3)(2)(1)(147)}{(5)(4)(3)}$$

$$y = \frac{-8}{5} - \frac{9}{2} + 24 + \frac{294}{20}$$

$$= \frac{-80 - 450 + 240 + 1470}{100}$$

$$= \frac{1920}{100} = \frac{96}{5} = 19.2$$

$$y = \frac{-8}{-10} - \frac{9}{2} + 24 + \frac{294}{20}$$

$$= \frac{16 - 180 + 960 + 588}{40}$$

$y = 34.6$ (35)

(2) If $y(1)=3$, $y(3)=9$, $y(4)=30$, $y(6)=132$, find the interpolation polynomial using langrange's interpolation formulae

x	1	3	4	6
y	3	9	30	132

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-3)(x-4)(x-6)(3)}{(-2)(-3)(-5)} + \frac{(x-1)(x-4)(x-6)(9)}{(2)(-1)(-3)}$$

$$+ \frac{(x-1)(x-3)(x-6)(30)}{(3)(1)(-2)} + \frac{(x-1)(x-3)(x-2)(132)}{(5)(3)(2)}$$

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$$= \frac{(x-3)(x-4)(x-6)}{10} + \frac{3(x-1)(x-4)(x-6)}{2} - (x-1)(x-3)(x-6)(5)$$

$$+ \frac{(x-1)(x-3)(x-2)(22)}{5}$$

$$= \frac{1}{10}\left[(x-3)(x-4)(x-6) + 15(x-1)(x-4)(x-6) - 50(x-1)(x-3)(x-6) + 44(x-1)(x-3)(x-2)\right]$$

$$= \frac{1}{10}\left[x^3 - x^2(13) + x(12+24+18) - 72 + 15\left(x^3 - x^2(5) + x(4+24+6) + 24\right) - 50\left(x^3 - x^2(10) + x(3+18+6) + 18\right) + 44\left(x^3 - x^2(6) + x(3+6+2) + (6)\right)\right].$$

Simplify

$$= \frac{1}{10}\left[\cancel{x^3 - 13x^2 + 54x - 72 + 15x^3 - 165x^2 +}\right.$$

$$= \frac{1}{10}\left[8x^3 - 4x^2 - 58x + 89\right]$$

(8) Using Lagrange's interpolation formulae find value of 'y' when $x=2$, from foll. data.

x	1	3	4	6
y	4	40	85	259

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(-1)(-2)(2-4)}{(-2)(-3)(-5)}4 + \frac{(1)(-2)(-4)}{(2)(-1)(-3)}40 + \frac{(1)(-1)(-4)}{(3)(1)(-2)}85$$

$$+ \frac{(1)(-1)(-2)}{(5)(3)(2)}259.$$

$$= \frac{(-8)(4)}{30} + \left(\frac{(-8)(40)}{6}\right) + \frac{(85)(4)}{(-6)} + \frac{(2)(259)}{(30)}.$$

$$= \frac{32}{30} + \frac{320}{6} - \frac{340}{6} + \frac{518}{30}$$

$$= \frac{16}{15} + \frac{160}{3} - \frac{170}{3} + \frac{259}{15}$$

$$= \frac{32 + 800 - 1700 + 518}{30} = 17.66$$

3|2, 6, 15
1, 2, 8
5, 1, 1, 3

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Simpson's 1/3rd rule → only when it is a multiple of 2 or 4.

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(4)

Numerical Integration

This is the process of obtaining approximately the value of the definite integral.

$I = \int_a^b f(x)\,dx$. without actually integrating the f^n but only using the values of 'y' at some point of 'x' equally spaced over the closed interval [a,b]

The interval [a,b] is divided into 'n' equal parts of width 'h' where $h = \left(\frac{b-a}{n}\right)$

$a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ .... $x_n = x_0 + nh = b$.

be the pnts of division.

x	x_0	x_1	x_2	x_3	x_4	....	x_n
y	y_0	y_1	y_2	y_3	y_4	....	y_n

Simpson's 1/3rd rule:

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})\right]$$

Simpson's 3/8th rule:

$$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots) + 2(y_3 + y_6 + y_n \dots + y_{n-3})\right]$$

Weddle's Rule.

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Note: ⇒ {1 5 1 6 1 5 1} → for 6 terms

{1 5 1 6 1 5 2 5 1 6 1 5 2} → for 12 terms.

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12}\right]$$

Note ⇒ when we divide the interval [a,b] into 'n' equal parts, there will be (n+1) values of 'x'

the corresponding values of y refers to the ordinates.

If there are (n+1) ordinates there will be 'n' equal division

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→ To apply simpson's $1/3^{rd}$ rule, n must be a multiple of '2'.

→ To apply simpson's $3/8^{th}$ rule, n must be a multiple of '3'.

→ To apply weddle's rule, n must be a multiple of '6'.

→ when n=6 or multiple of 6, all the rules can be applied to find the approx value of the given definite integrals.

Problems

① Evaluate integration $\int_0^6 3x^2 dx$. dividing the interval 0→6 [0,6] into 6 equal parts. by applying simpson's $1/3^{rd}$ rule, simpson's $3/8^{th}$ rule & weddle's rule.

x	0	1	2	3	4	5	6
y	0	3	12	27	48	75	108

$h = \frac{6-0}{6}$

$h = 1$

Simpson's $1/3^{rd}$ rule

$I = \frac{1}{3}[(0+108) + 4(3+27+75) + 2(12+48+108)]$

$= \frac{1}{3}[108 + 4(105) + 2(168)]$

$= \frac{1}{3}[108 + 420 + 336]$

$= \frac{1}{3}[864]$

$= 216.$

Simpsons $3/8^{th}$ rule

$I = \frac{3}{8}[(0+108) + 3(1+12+48+75) + 2(27)]$

$= \frac{3}{8}[108 + 3(136) + 2(135)]$

$= \frac{3}{8}[108 + 408 + 270]$

$= \frac{3}{8}[786]$

$= 216$

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Weddle's Rule.

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$I = \frac{3}{10}\left[0 + 15 + 12 + 162 + 48 + 375 + 108\right]$$

$$I = 216$$

(2) By using Simpson's 1/3rd rule, simpson's 3/8th rule & weddle's rule, evaluate $\int_0^{0.6} e^{-x^2}\,dx$, taking 7 ordinate

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	1	0.99	0.9601	0.939088521	0.7788	0.6977	

$h = \frac{0.6+0}{6} = 0.1$

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5) + 2(y_2+y_4+y_6)\right]$$

$$I = \frac{0.1}{3}\left[(1+0.6977) + 4(0.99+0.9601+0.7788) + 2(0.9601+0.8521)\right]$$

$$= \frac{0.1}{3}\left[1.6977 + 10.9156 + 3.6244\right]$$

$$= 0.5351$$

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right] = 0.5351$$

(3) By using Simpson's 1/3rd rule, evaluate integration $\int_1^5 f(x)\,dx$, given

x	1	2	3	4	5
$f(x)$	13	50	70	80	100

$h = \frac{y_0+y_n}{n} = \frac{105}{4} = 1$

$$I = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$= \frac{1}{3}\left[(13+100) + 4(50+80) + 2(70)\right]$$

$$= \frac{1}{3}\left[113 + 520 + 140\right]$$

$$= 257.667$$

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(4) Evaluate $\int_0^1 \frac{1}{1+x}\,dx$, dividing interval into 6 equal parts by applying simpson 3/8 rule & deduce an app value of $\log_e 2$.

x	0	0.1666	0.3333	0.5	0.666	0.8333	1.
$f(x)$	1	0.8571	0.75	0.6666	0.6	0.5454	0.5.

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3 \cancel{+y_6})\right].$$

$$= \frac{3}{8}\cdot\frac{1}{2}\left[(1+0.5) + 3(0.8571+0.75+0.6+0.5454) + 2(0.6666 \cancel{+0.5})\right]$$

$$= 0.6931$$

(5) Using Weddle's rule estimate the area bounded by the curve x axis & the line $x=1$ & $x=4$.

x	1	1.5	2	2.5	3	3.5	4.
$f(x)$	2	2.4	2.7	2.8	3	2.6	2.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = \frac{x_n - x_0}{n} = \frac{4-1}{6} = \frac{1}{2}.$$

Weddle's Rule =)

$$\int_1^4 f(x)\,dx = I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3}{20}\left[2 + 5(2.4) + 2.7 + 6(2.8) + 3 + 5(2.6) + 2.1\right]$$

$$I = 7.74$$

(6) Evaluate $\int_0^1 \frac{x}{1+x^2}\,dx$ using weddle's rule by using 6 equal parts, hence find $\log_e 2$.

$n = 6 \Rightarrow h = \frac{1-0}{6} = \frac{1}{6}.$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	0	0.1621	0.3	0.4	0.4615	0.4918	0.5.

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{1}{20}\left[0 + 0.8105 + 0.3 + 2.4 + 0.4615 + 2.459 + 0.5\right]$$

$$= 0.34655$$

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$$0.34655 = \int_0^1 \frac{x}{1+x^2}\,dx$$

$1+x^2 = t$

$2x\,dx = dt$

$\frac{dt}{2} = x\,dx$

$x=0 \quad x=1$

$t=1 \quad t=2$

$$= \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left|\log t\right|_1^2$$

$$= \frac{1}{2}\left|\log 2 - \log 1\right|$$

$$= \frac{1}{2}\left|\log 2 - 0\right|$$

$$\boxed{\log_e 2 = 0.6931}$$

4th Problem continuation

$I = 0.6931$

$$I = \int_0^1 \frac{1}{1+x}\,dx$$

$$0.6931 = \left|\log|1+x|\right|_0^1$$

$$0.6931 = \log 2 - 0$$

$$\boxed{\log 2 = 0.6931}$$

(7) $\int_0^{\pi/2} e^{\sin x}\,dx$ by simpson's 3/8th rule & weddle's rule & 7 ordinates $(n=6)$.

Note ⇒ Always Change degree to radian. if given. (change the calculator to radians & solve)

$$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$$

x	0	$\pi/12$	$2\pi/12$	$3\pi/12$	$4\pi/12$	$5\pi/12$	$\pi/2$
$f(x)$	1	1.2953	1.6487	2.0281	2.3774	2.6272	2.7182

$$I = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{3\pi}{(12)(8)}\left[1 + 2.7182 + 23.8458 + 4.0562\right]$$

$$= 3.1043$$

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$I = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$

$I = 3.1200$

(8) Evaluate $\int_0^{1.2} e^x dx$ using weddle's rule taking 6 equal sub intervals, compare the result with exact value.

Numerical ~~method~~ solution for Algebraic & Transidental eqn

Given an equation $f(x)=0$ is generally not possible to find the root 'x' such that $f(x)$ becomes '0' exactly

The eqn involving algrebraic quantities like x, x^2, x^3 etc. are called as "algebraic eqn".

Eg: (1) $x^2 - x = 40$

(2) $x^3 + x^2 + x = 0$

The eqn which involves non algebraic quantities like e^x $\log x$, $\tan x$, $\sin x$ etc are called as "transidental eqn".

Eg: (1) $xe^x - 1 = 0$

(2) $x\log x + x = 0$

(3) $x\sin x = 0$.

The numerical methods of finding approximate roots of the given roots of given eqn is a repeated type of process known as "Iteration process".

In each step, result of previous of the previous step is used & the process is carried out till we get the result to the described accuracy.

If there exists 2 values a, b such that $f(x)$ has opp. sides, $f(a) < 0$, $f(b) > 0$ [equivalent to $f(a) \cdot f(b) < 0$] then there always exists atleast one real root in the interval (a, b)

Two numerical Iteration methods:

(1) Regular Falsi Method.

(2) Newton Raphson method.

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Newton Raphson Method.

In General;

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This is called Newton Raphson Iterative formulae. The approx. real root of the given eqn and can improve its accuracy by iterative process

Problem

① Using Newton Raphson Method find a real root of $(x\sin x + \cos x) = 0$ near to $x = \pi$ [Here x is in radians]

Sol^n

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad f'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x}) = x\cos x.$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad [\text{consider } n=0]$$

$$= \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{1}{\pi} = 2.8233$$

upto 4 iterations.

$$x_2 = 2.8233 - \frac{f(2.8233)}{f'(2.8233)}$$

$$= 2.8233 + \frac{0.06623}{(-2.68148)}$$

$$= 2.8233 - 0.02469$$

$$= 2.7986$$

$$x_3 = 2.7986 - \frac{f(2.7986)}{f'(2.7986)}$$

$x_4 =$

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(2) Using Newton's method, find the real root of the eqn
$x^3-3x-5=0$

$f(0) = -5 < 0$
$f(1) = 1-3-5 = -7 < 0$ } $x_0 = 2$ $\quad f'(x) = 3x^2-3.$
$f(2) = 8-6-5 = -3 < 0$
$f(3) = 27-9-5 = 13 > 0$ } Root lies between $f(2)$ & $f(3)$ ie 2 and 3.

$x_0 = 2.$

Step (1) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2 - \frac{(x^3-3x-5)}{(3x^2-3)} = 2 + \frac{1}{9} = 2.333$

$x_1 = 2 - \left[\frac{2-(-5)}{3-3}\right] = 2 + \frac{7}{3} = \frac{6+7}{3} = \frac{13}{3}$ 2.333

Step (2) $x_2 = 2.333 - \frac{f(x_n)}{f'(x_n)}$ (2.333)

$= 2.2805.$

Step (3) $x_3 = 2.2805 - \frac{f(2.2805)}{f'(2.2805)} = 2.2790.$

Step (4) $x_4 = 2.2790 - \frac{f(2.2790)}{f'(2.2790)} = 2.2790 - \frac{(-2.36361\times10^{-4})}{12.581523}$

$= 2.2790.$

(3) Using Newton Raphson method, find the real root of the eqn, $xe^x = 2$ correct to 4 decimal places.
$xe^x - 2 = 0$
$f(0) = 0-2 = -2$ } $x_0 = 1.$ $\quad f'(x) = xe^x + e^x = e^x(x+1)$
$f(1) = 0.71828 > 0$

$x_0 = 1.$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{f(x_0)}{f'(x_0)} = 0.8679.$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{0.06725}{4.4491} = 0.8527$

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$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.8527 - \frac{f(0.8527)}{f'(0.8527)} = 0.8526$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 8.526 - \frac{f(8.526)}{f'(8.526)} = 0.8526$$

(4) Find the approximate value of the real root the eqn $e^x - 4x = 0$.

$f(0) = 1$
$f(1) = -1.2817$ } $f'(x) = e^x - 4$.
$f(2) = -0.6109$
$f(3) = 8.08553$
$x_0 = 1$.

Step 1: $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$$x_1 = x_0 - \frac{f(1)}{f'(1)}$$

$$= \cancel{1 - 0.0183} \; 0$$

Step 2: $x_2 = x_1 - \frac{f(1)}{f'(1)}$

$$= 0 - \frac{1}{-3} = 0.3333$$

Step 3: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.3572$.

Step 4: $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.3574$.

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(3) Find the value of the root, $\tan x - x = 0$, $x = 4.5$

$f(0) = 0$

$f(1) = -0.9825$

$f(2) = -1.9680$

$f(3) = -2.9475$

$x_0 = 4.5$

$f'(x) = \sec^2 x - 1$
$= \tan^2 x$

Step 1:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 4.5 - \frac{(0)}{0} = 4.4936$$

Step 2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4.4936 - \frac{f(4.4936)}{f'(4.4936)}$$

$x_2 = 4.4974$

Step 3:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

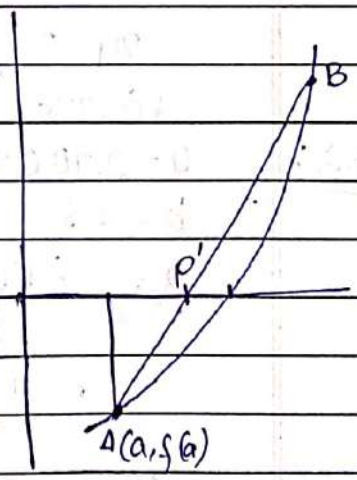
$=$

Regular Falsi Method.

$y = f(x) = 0$

$f(a) < 0$ Root lies between $[a, b]$

$f(b) > 0$



Eqn of line is given by.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{0 - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}$$

$$(x - a) = \frac{(a - b) f(a)}{f(b) - f(a)}$$

$$\boxed{x = \frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

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Problems

① Using regular Falsi method, find the real root of the eqn $x^3 - 4x - 9 = 0$.

$f(0) = -9$

$f(1) = -12$

$f(2) = -9 < 0$ ∴ Roots lie between $[2, 3]$

$f(3) = 6 > 0$ Even $a=2, b=3$ can be taken but constant

$f(4) = 39$. result won't be obtained in 3rd or 4th iteration.

$f(2.1) = -8.139$

$f(2.5) = -3.375$

$f(2.7) = -0.117$ } Root lies between $[2.7, 2.8]$

$f(2.8) = 1.752$

Step 1

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{(2.7)(1.752) - (2.8)(-0.117)}{1.752 - (-0.117)}$$

$$= \frac{4.7304 + 0.3276}{1.869}$$

$x_1 = 2.7062$

$f(2.7062) = -0.0058 < 0$

Step 2:

$a = 2.7062$ $f(a) = -0.0058 < 0$

$b = 2.8$ $f(b) = 1.752$.

$$x_2 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{4.7412 + 0.01624}{1.7578}$$

$\boxed{x_2 = 2.7065}$ [∵ x_2 is almost $= 2x_1$]

② Using regular falsi method find the roots of the eqn. $f(x) = x e^x - \cos x = 0$ that lies betw 0.4 & 0.6 carry out 4 iterations.

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$a = 0.4 \quad b = 0.6$

$f(a) = -0.3243 \quad f(b) = 0.2679.$

Step1: $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$a_1 = 0.5095 \quad f(x_1) = -0.0249 < 0.$

Step2: $x_a = 0.5095 \quad x_b = 0.6$

$f(x_a) = -0.0249 \quad f(0.6) = 0.2679.$

$x_2 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{0.1364 + 0.01494}{0.2928}$

$x_2 = 0.5171 \quad f(x_2) = -0.0019$

Step3: ~~$x_a = 0.5095$~~ $x_a = 0.5171$; $b = 0.6$

~~$f(a) =$~~ $f(x_a) = -0.0019 \quad f(b) = 0.2679.$

$x_3 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$x_3 = 0.5176$; $f(x_3) = -0.0005 < 0$

Step4: $a = 0.5176$; $b = 0.6$

$f(a) = -0.0005 \quad f(b) = 0.2679.$

$x_4 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{0.1386 + 0.0003}{0.2689}$

$x_4 = 0.5177 \quad f(x_4) = -0.00382.$

(3) Using regular Falsi Method $\tan x + \tanh x = 0$ lies between [2,3] carry out 4 iterations.

$f(a) = -1.2210$; $f(b) = 0.8525$

Step1: $x_1 = 2$; $f(x_1) = -1.2210 \quad x_2 = 3$; $f(x_2) = 0.8525.$

$x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)} = \dfrac{(2)(0.8525) - (3)(-1.2210)}{1}$

$f(2.5889) = 0.3719 = 2.5888$

Step2: $a = 2 \quad b = 2.5889,$

$f(a) = -1.2210 \quad f(b) = 0.3719$

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$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$x_2 = 2.4514$

$f(2.4514) = 0.1595$

Case(3): $a = 2 \quad f(2) = -1.2210$

$b = 2.4514 \quad f(b) = 0.1596$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0.3192 - 0.39099}{1.3806}$$

$x_3 = 2.3992 \qquad f(x_3) = 0.0661$

Case(4): $a = 2 \quad f(2) = -1.2210$

$b = 2.3992 \quad f(\cancel{0.0662.3992}) = 0.0661$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0.1322 + 2.9294}{1.2871}$$

$= 2.3787$

(4) Using regular falsi method, find the root of $\cos x = 3x - 1$ ~~cosx~~ that lies betw 0.5 & 1, here x is in radians ~~conv~~ correct to 4 decimal places.

$f(x) = \cos x - 3x + 1$

$a = 0.5 \quad f(0.5) = 0.3775 > 0$

$b = 1 \quad f(1) = -1.4596 < 0$

Case(1) $$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{-0.7298 - 0.3775}{-1.8371}$$

$x_1 = 0.6027$

$f(x_1) = 0.0157 > 0$

Case(2) $a = 0.0157 \qquad b = 1$

$f(a) = 0.6027 \qquad f(1) = -1.4596$

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base 10 — log
base e — ln

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$x_2 = \frac{-0.0229 - 0.6027}{-2.4578 + 2.0623}$

$= \cancel{0.6067}$

$x_2 = 0.6069 \quad f(x) = 0.0007$

Iter(3) $x_3 = \frac{a(f(b)) - b(f(a))}{f(b) - f(a)} = 0.6070$

$f(x_3) = 0.0003$

Iter(4) $x_4 = 0.6070$ $\quad a = 0.6070 \quad b = 1$

$f(a) = 0.0003 \quad f(b) = -1.4158$

$f(x_4) = 0.0003.$

② Using regular Falsi method, find the root of the eqn. $x \log x - 1.2 = 0$ correct to 4 decimal places.

$f(1) = \frac{-6}{5} = -1.2.$

$f(2) = -0.5979$

$f(3) = 0.2313.$

$a = 2 \quad b = 3$

$f(a) = -0.5979 \quad f(b) = 0.2313.$

Iter(1) $x_1 = 2.7210$

$f(x_1) = -0.0178$

$a = 2.7210 \quad b = 3$

$f(a) = -0.0178 \quad f(b) = 0.2313$

Iter(2) $x_2 = \cancel{2.7389}.\ 2.7400$

$f(x_2) = \cancel{0.00152}.\ -0.00152.$

Iter(3) : $x_3 = \cancel{2.5687}.\ 2.7405$

$f(x_3) = \cancel{0.1475}\ -0.00001$

Iter(4) $x_4 = 2.7406.$

$f(x_4) = \cancel{0.00402}\ \cancel{0.00}\ -4.020 \times 10^{-3}.$

⑥ Using Regular Falsi method, find app value of real root $x^3 - 3x + 4 = 0$ ~~using the method~~.

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$$x J_n'(x) = x \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n-1+2r} \frac{1}{\Gamma(n-1+r+1)\, r!} - n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = x J_{n-1}(x) - n J_n(x)$$

Problems

① $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ ② $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1)\, r!} \longrightarrow ①$$

$$J_{1/2}(x) = \left[(-1)^0 \left(\frac{x}{2}\right)^{1/2} \frac{1}{\Gamma(3/2)\,1!} + (-1)^1 \left(\frac{x}{2}\right)^{1/2+2} \frac{1}{\Gamma(5/2)\,1!} + (-1)^2 \left(\frac{x}{2}\right)^{1/2+4} \frac{1}{\Gamma(7/2)\,2!} + \dots\right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma(3/2)} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma(5/2)} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma(7/2)\,2!} + \dots\right]$$

$\Gamma(1/2) = \sqrt{\pi}$; $\Gamma(5/2) = 3/2\, \Gamma(3/2) = 3\sqrt{\pi}/4$

$\Gamma 3/2 = 1/2\, \Gamma 1/2 = \sqrt{\pi}/2$; $\Gamma(7/2) = 5/2\, \Gamma(5/2) = 15\sqrt{\pi}/8$

$$= \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{4} \frac{4}{3\sqrt{\pi}} + \frac{x^4}{16 \times 2} \times \frac{8}{15\sqrt{\pi}}\right]$$

$$= \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{3\sqrt{\pi}} + \frac{x^4}{60\sqrt{\pi}} + \dots\right]$$

$$= \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} + \dots\right] \quad \times \& \div \text{ by } 2.$$

$$= \sqrt{\frac{x}{2\pi}} \cdot \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} + \dots\right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

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$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\Gamma(n+r+1)\,r!}$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-1/2+2r}\frac{1}{\Gamma(r+1/2)\,r!}$$

Expanding $\sum_{r=0}^{\infty}$ taking $r=0,1,\ldots$

$$J_{-1/2}(x) = \left[(-1)^0\left(\frac{x}{2}\right)^{-1/2}\frac{1}{\Gamma(1/2)\,0!} + (-1)^1\left(\frac{x}{2}\right)^{-1/2+2}\frac{1}{\Gamma(3/2)\,r!} + (-1)^2\left(\frac{x}{2}\right)^{-1/2+4}\frac{1}{\Gamma(5/2)\,2!}\right]$$

$$= \left[\sqrt{\frac{2}{x}}\frac{1}{\sqrt{\pi}} + (-1)\sqrt{\frac{2}{x}}\left(\frac{x}{2}\right)^2\frac{1}{\sqrt{\pi}/2} + (-1)^2\left(\frac{x}{2}\right)^{-1/2}\left(\frac{x}{2}\right)^4\frac{1}{\Gamma(5/2)\,0!}\right]$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\sqrt{\pi}} - \frac{x^2}{4/2}\cdot\frac{2}{\sqrt{\pi}} + \frac{x^4}{16/4}\,\frac{4}{6\sqrt{\pi}} + \ldots\right]$$

$$= \sqrt{\frac{2}{x\pi}}\left[1 - \frac{x^2}{2} + \frac{x^4}{8\cdot 24} + \ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

(2) Show that $J'_{1/2}(x)\cdot J_{-1/2}(x) - J'_{-1/2}(x)\cdot J_{1/2}(x) = 2/\pi x$.

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \;;\quad J'_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x + \sin x\sqrt{\frac{2}{\pi}}\left(\frac{-1}{2}x^{-3/2}\right)$$

$$J'_{1/2}(x) = \left[\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{1}{2\pi}}\,\frac{1}{x\sqrt{x}}\sin x\right]$$

$$\Rightarrow J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \;;\quad J'_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}(-\sin x) + \cos x\sqrt{\frac{2}{\pi}}\left(\frac{-1}{2}x^{-3/2}\right)$$

$$J'_{-1/2}(x) = \left[-\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{1}{2\pi}}\,\frac{1}{x\sqrt{x}}\cos x\right]$$

$$J'_{1/2}(x)\cdot J_{-1/2}(x) - J'_{-1/2}(x)\cdot J_{1/2}(x)$$

$$= \left[\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{1}{2\pi}}\,\frac{1}{x\sqrt{x}}\sin x\right]\left[\sqrt{\frac{2}{\pi x}}\cos x\right] + \left[-\sqrt{\frac{2}{\pi x}}\sin x + \frac{\cos x}{\sqrt{2\pi x\sqrt{x}}}\right]\left[\sqrt{2/\pi x}\sin x\right]$$

$$= \sqrt{\frac{2}{\pi x}}\sqrt{\frac{2}{\pi x}}\left[(\cos^2 x + \sin^2 x)\right] = \left(\sqrt{\frac{2}{\pi x}}\right)^2 = \frac{2}{\pi x}$$

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(3) $\frac{d}{dx}\left(x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

$\frac{d}{dx}\left(x J_n(x) J_{n+1}(x)\right]$

$= x J_n(x) J_{n+1}'(x) + x J_{n+1}(x) J_n'(x) + J_n(x) J_{n+1}(x)$

$= x\left[J_n(x) J_{n+1}'(x) + J_{n+1}(x) J_n'(x)\right] + J_n(x) J_{n+1}(x)$ → (1)

Using Recurrence Relation.

$\Rightarrow x J_n'(x) = x J_{n-1}(x) - n J_n(x)$ — (2)

$x J_n'(x) = n J_n(x) - x J_{n+1}(x)$ — (3)

Replacing n by n+1 in (2);

$x J_{(n+1)}'(x) = x J_n(x) - (n+1) J_{n+1}(x)$ — (4)

Replace (3) & (4) in RHS of eqn (1);

Eqn (1) becomes

$= x J_n(x) J_{(n+1)}'(x) + x J_n'(x) J_{n+1}(x) + J_n(x) J_{n+1}(x)$

$= J_n(x)\left[x J_n(x) - (n+1) J_{n+1}(x)\right] + \left[n J_n(x) - x J_{n+1}(x)\right] J_{(n+1)}(x)$
$+ J_n(x) J_{n+1}(x)$.

$= x (J_n(x))^2 - J_n(x)(n+1) J_{n+1}(x) + n J_n(x) J_{n+1}(x) - x J_{n+1}^2(x)$
$+ J_n(x) J_{n+1}(x)$.

$= x\left[J_n^2(x) - J_{n+1}^2(x)\right]$

(4) $J_5(x)$ in terms of $J_0(x)$ & $J_1(x)$

$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$ → 1st Recurrence Relation

$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

Take $n = 1, 2, 3, 4$ & find J_5 in terms of J_0 & J_1

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$n=1$; $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$n=2$; $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

$n=3$; $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \frac{2}{x} J_1(x) - J_0(x)$$

$$= J_1(x)\left[\frac{6}{x}\left(\frac{8}{x^2} - 1\right) - \frac{2}{x}\right] - J_0(x)\left[\frac{4}{x} + 1\right]$$

$n=4$ $J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$

$$= \frac{8}{x}\left[J_1(x)\left[\frac{6}{x}\left(\frac{8}{x^2} - 1\right) - \frac{2}{x}\right] - J_0(x)\left[\frac{4}{x} + 1\right]\right]$$

$$- \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

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$n=1$; $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

$n=2$; $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$

$$J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

$n=3$; $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$= \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \frac{2}{x} J_1(x) - J_0(x)$$

$$= J_1(x)\left[\frac{6}{x}\left(\frac{8}{x^2} - 1\right) - \frac{2}{x}\right] - J_0(x)\left[\frac{4}{x} + 1\right]$$

$n=4$ $J_5(x) = \frac{8}{x} J_4(x) - J_3(x)$

$$= \frac{8}{x}\left[J_1(x)\left[\frac{6}{x}\left(\frac{8}{x^2} - 1\right) - \frac{2}{x}\right] - J_0(x)\left[\frac{4}{x} + 1\right]\right]$$

$$- \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

(5) $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

(6) $8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$

(5) Using 3rd recurrence relation;

$2J_n' = J_{n-1} - J_{n+1}$ ——— (1)

d wrt x;

$2J_n'' = J_{n-1}' - J_{n+1}'$ $\times 2$.

$4J_n'' = 2J_{n-1}' - 2J_{n+1}'$ ——— (2)

Replace n by $(n-1)$ & n by $n+1$ in (1)

~~$4J_{n+1}'' = 2J_{n-2}' - 2J_n'$~~

$2J_{n-1}' = J_{n-2} - J_n$ ——— (3) & $2J_{n+1}' = J_n - J_{n+2}$ ——— (4)

Replacing (3) & (4) in (2)

$4J_n'' = J_{n-2} - 2J_n + J_{n+2}$ ——— (5)

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Multiplying (5) by 2;

$8J_n'''$ d w.r.t x;

$4J_n''' = J_{n-2}' - 2J_n' + J_{n+2}'$

Multiplying by 2;

$8J_n''' = 2J_{n-2}' - 4J_n' + 2J_{n+2}'$

Replace n by $n-2$ & n by $n+2$ eqn (1)

$2J_{n-2}' = J_{n-3} - J_{n-1}$ — (6)

$2J_{n+2}' = J_{n+1} - J_{n+3}$ — (7)

$8J_n''' = J_{n-3} - J_{n-1} - 2[J_{n-1} - J_{n+1}] + J_{n+1} - J_{n+3}$

$8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$ //

⇒ Eqn reduces to the form of Bezzel's eqn $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$

Then sol^n will be of the form.

$y = aJ_n(\lambda x) + bJ_{-n}(\lambda x)$.

<u>The orthogonal property of Bezzel's function</u>

Define orthogonal property

If α & β are 2 distinct roots of $J_n(\alpha x) = 0$

then $\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ a^2/2\,[J_{n+1}(a\alpha)]^2 & \text{if } \alpha = \beta. \end{cases}$

$\int_0^a x J_n(\alpha x) J_n(\beta x)\, dx = 0$ if $\alpha \neq \beta$.

<u>Proof</u>: we know that $J_n(\alpha x)$ is a solution of the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0.$$

$u = J_n(\alpha x)$ $v = J_n(\beta(x))$

The associated Differential eqns are

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \quad \text{—— (1)} \times v/x$$
$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \quad \text{—— (2)} \times u/x.$$

Eqns become;

$$x^2u''\frac{v}{x} + xu'\frac{v}{x} + \alpha^2x^2\frac{uv}{x} - n^2u\cdot\frac{v}{x} = 0$$

$$x^2v''\frac{u}{x} + xv'\frac{u}{x} + \beta^2x^2v\frac{u}{x} - n^2v\cdot\frac{u}{x} = 0$$

$$xu''v + u'v + \cancel{\alpha^2x^2uv} - \frac{n^2uv}{x} = 0.$$

$$xv''u + v'u + \cancel{\beta^2x^2uv} - \frac{n^2uv}{x} = 0.$$

$$x(u''v - v''u) + (u'v - v'u) + xuv(\alpha^2 - \beta^2) = 0.$$

This can be written as

$$\frac{d}{dx}\left[x(u'v - uv')\right] = (\beta^2 - \alpha^2)xuv$$

Integrating both sides w.r.t x betw interval 0 to a

$$\int_0^a \frac{d}{dx}\left[x(u'v - uv')\right]dx = \int_0^a (\beta^2 - \alpha^2)xuv\,dx.$$

$$\cancel{x(u'v - uv')_0^a = (\beta^2 - \alpha^2)\alpha^2uv}$$

$$a(u'v - uv')_0^a = (\beta^2 - \alpha^2)\int_0^a xuv\,dx. \quad \text{—— (3)}$$

$u = J_n(\alpha x)$; $v = J_n(\beta x)$

$u' = J_n'(\alpha x)\cdot\alpha$; $v' = J_n'(\beta x)\beta.$

As a consequences of eqn (3)

$$a\left[\alpha J_n'(\alpha x)J_n\beta(x) - J_n(\alpha x)\beta J_n'(\beta x)\right]_{x=a} = (\beta^2 - \alpha^2)\int_0^a xJ_n(\alpha x)J_n(\beta x)dx$$

$$\int_0^a xJ_n(\alpha x)J_n(\beta x)dx = \frac{a}{(\beta^2 - \alpha^2)}\left[\alpha J_n'(\alpha x)J_n(\beta x) - J_n(\alpha x)\beta J_n'(\beta x)\right] \quad \text{—— (4)}$$

Since α & β are the 2 distinct roots of $J_n(\alpha x) = 0$ we have $J_n(\alpha a) = 0$ & $J_n(\beta a) = 0$ with the result of RHS of eqn (4) becomes 0 provided $(\beta^2 - \alpha^2) \neq 0$ or $\beta \neq \alpha$

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$\therefore J_{-n}(x) = (-1)^n J_n(x)$

Recurrence Relation

Thus we have

$$\int_0^a x J_n(\alpha x) J_n(\beta x) = 0$$

Problem

① Prove that $J_0'(x) = -J_1(x)$

Sol^n By the recurrence relation;

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

$n=0$ $\quad J_0'(x) = \frac{1}{2}[J_{-1}(x) - J_1(x)]$

And

$$J_{-n}(x) = (-1)^n J_n(x)$$
$$J_{-1}(x) = (-1)^1 J_1(x)$$
$$J_{-1}(x) = -J_1(x)$$

$$\therefore J_0'(x) = \frac{1}{2}[-J_1(x) - J_1(x)]$$
$$J_0'(x) = \frac{1}{2}[-2J_1(x)]$$
$$\boxed{J_0'(x) = -J_1(x)}$$

② $\frac{d}{dx}(x J_1) = x J_0$

Sol^n Using Recurrence relation;

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$n=1$; $\frac{d}{dx}[x J_1(x)] = x^1 J_0(x)$ //

Series Solution of Legendre's Differential Equation.

~~Series Solution~~ We have the Legendre's D.E. i.e

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow ①$$

The coeff of $y'' = (1-x^2)$ which is equal to $P_0(x)$ and $P_0(x) \neq 0$ at $x=0$

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