COURSE : RELIABILITY ANALYSIS

COURSE CODE : 18MA6IERLA

MODULE – 2

Reliability : The reliability of a device is considered high if it had repeatedly performed its function with success and low if it had tended to fail in repeated trials.

The reliability of a system is defined as the probability of performing the intended function over a given period of time under specified operating conditions.

The above definition can be broken do on into four parts:

i) Probability
ii) Intended function
iii) Given period of time
iv) Specified operating conditions

Probability: Because, the reliability is a Probability, the reliability of system R_j is governed by the equation $0 \leq R_s \leq 1$. The equality sign hold good in case of equipment called one shot equipment.

Intended function: It is also defined to as the successful operation.

Example-1: As an example, let us consider the building up of voltage by a dc shunt generator. For some reasons, let us assume that the voltage is not build up.

We say that the dc shunt generator has failed to do its job. The failure in this contest doesn't imply any physical failure, but only the operational failure.

Example-2: Lightning arrester : The lightning arrestor should burst in the event of occurrence of a lightning stroke. On the occurrence of a lightning stroke, if the lightning arrestor bursts, there is the physical failure or damage but operationally it is successful.

On the other hand, if it doesn't burst there is no physical failure, but yet there is an operational failure and we say that the lightning arrestor has failed (operationally).

Given period of time: Any component has some useful life period, within which time the component should operate successfully. For example, a power transformer has a useful life of at least 20 to 25 years.

If, it fails within this time period, then the instrument is said to be unreliable and if it fails after its useful life period, then we say it is reliable.

Specified operating or environmental conditions: Any equipment is supposed to perform its duty satisfactorily under contain specified operating condition such as temperature, humidity, pressure and altitude.

Though an equipment is able to perform its duty satisfactorily in a cold country yet it may fail when used under hot climatic conditions.

The four elements discussed above are critical in determining the reliability of a system or product. As reliability is an inherent characteristics of design, it is essential that these elements be adequately considered at program inception . The purpose of reliability design is to ensure that the operating objectives are met. Until recently, this purpose was usually not taken seriously (except in the military services and in aerospace), owing to the expensive nature of reliability design. This shows that reliability can no longer be ignored in our modern, complex world.

Defects:

There are two types of defects:

- Quality defects
- Reliability defects

Quality is a major ingredient of reliability, it is important to understand its effects. Quality of a device /system is the degree of conformance to applicable specifications and workmanship standards. It is not concerned with he elements of time and environment. An equipment which has undergone all quality tests may not necessarily be more reliable. Quality is associated with the manufacture whereas reliability is primarily associated with the design.

There are three classification of quality defects:

- Critical defects
- Major defects
- Minor defects

Critical defects will always affect the products usability
Major defects may or probably will, affect usability
Minor defects usually cosmetic defects will not affect the usability of the product. Cosmetic defects may affect the saleability of the product, but not its usability.

Reliability defects on the other hand include a time frame, reliability effects are associated with failures, or the inability of the part of continues to perform its intended function.

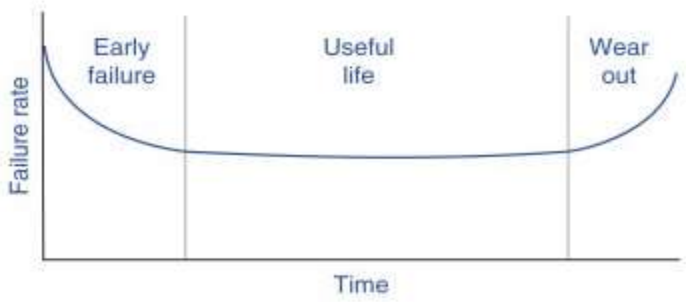
There are two classifications of reliability defects:

- Catastrophic defects
- Wear-out

Catastrophic defects usually occur randomly, they are the chance defects that cause a sudden failure to occur without warning. These defects are usually quite costly, since they can cause emergency shut-downs of systems at the most inappropriate times.
Wear-out failure occur slowly and frequently signal when failure is imminent. This type of slow degradation of quality is often an indication for replacement before a major failure can occur.

BATHTUB CURVE

The typical shape for failure rate curves, Known as bathtub curve.



The first part of curve, known as early failure period. Early failures are those which occur in the early life of a system operation. The concept of "warranty" is based on the concept of early failures. These failures are primarily due to manufacturing defects, such as weak parts, bad assembly, etc. Since the defective units are eliminated during the initial failure period, this period is known as debugging or burn-in period.

The long, fairly flat portion of the curve is called the chance failure. Chance failures are predominant during the actual working of the system. Chance failures occur at random, irregularly and unexpectedly

The final part of the curve, where the failure rate is increasing is known as the wear-out failure. Wear-out failures are caused due to aging or wearing out of components. These failure occur if the system is not maintained properly or not maintained at all and the frequency of such failures increases rapidly with time.

Causes of failures

The enhancement of system reliability, it is necessary that the, design engineer understands the causes of failure. The specific causes of failures of components or system can be many. Some are Known and other are unknown due to the complexity of the system and its environment. A few causes are given below

- Poor design of component/system
- Wrong manufacturing techniques
- Complexity of equipment/system
- Poor maintenance policies
- Human errors

Various Phases in equipment's life

There are four clearly definable phases I the creation of a equipment namely

- Concept and definition
- Design and development
- Manufacturing and installation
- Operation and maintenance

Quality and maintainability have been shown as the activities aimed at imposing the reliability of the equipment during manufacturing and installation, operation and maintenance phase of life. Operation and maintenance phase where in the equipment is operated throughout its useful service life. During this phase the essential repair and maintenance actions are taken and the performance of the equipment is monitored.

A series of managerial and technical tasks are essential throughout all four phases to achieve a desired reliability of the equipment.

PROBABILITY THEORY

Random variables (discrete & continuous), probability density function, cumulative density function. Probability distributions- binomial & poison distributions; exponential & normal distributions.

Random variables

Introduction: In a random experiment, the outcomes (results) are governed by chance mechanism and the sample space S of such a random experiment consists of all outcomes of the experiment. When the outcomes of the sample space are non-numeric, they can be quantified by assigning a real number to every event of the sample space. This assignment rule, known as the random variable or stochastic variable. In other words a random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by X,Y,Z…….The set of all real number of a random variable X is called the range of X.

Example-1 While tossing a coin, suppose that the value 1 is associated for the outcome' head' and 0 for the outcome 'tail'. The sample space S=$\{H, T\}$and if X is the random variable then
X (H) =1and X(T)=0, Range of X = $\{0,1\}$

Example-2 A pair of fair dice is tossed. The sample space S consists of the 36 ordered pair (a,b)where a and b can be any integers between 1 and 6, that is S={(1,1),(1,2),………(6,6)}
Let X assign to each point (a,b) the maximum of its numbers, that is, X(a,b)=max(a,b). For example(1,1)=1,X(3,4)=4,X(5,2)=5

Then x is a random variable where any number between 1 and 6 could occur, and no other number can occur. Thus the range space of X = {1, 2, 3, 4, 5, 6}

Let Y assign to each point (a,b) the sum of its numbers, that is Y(a,b)=a+b.For example Y(1,1)=2, Y(3,4)=7, Y(6,3)=9. Then Y is a random variable where any number between 2 and 12 could occur, and no other number can occur. Thus the range space ={2,3,4,5,6,7,8,9,10,11,12}

Discrete random variables

Definition: a random variable is said to be discrete random variable if it's set of possible outcomes, the sample space S, is countable (finite or an unending sequence with a many elements as there are whole numbers).

Example:
1) Tossing a coin and observing the outcome.
2) Tossing a coin and observing the number of heads turning up.
3) Throwing a 'die' and observing the number of the face.

Continuous random variables

Definition: A random variable is said to be continuous random variables if sample space S contains infinite number of values.

Example:
1) Weight of articles.
2) Length of nails produced by a machine.
3) Observing the pointer on a speedometer/voltmeter.
4) Conducting a survey on the life of electric bulbs.

Generally counting problems corresponds to discrete random variables and measuring problems lead to continuous random variables.

PROBABILITY DISTRIBUTIONS
Probability distribution is the theoretical counterpart of frequency distribution, and plays an important role in the theoretical study of populations.

Discrete probability distribution:

Definition: If for each value x_i of a discrete random variable X, we assign number p(x_i) such that
i) p(x_i) ≥ 0, ii)$\sum_i \mathrm{p}(x_i) = 1$ then the function p(x) is called a probability function. If the probability that X takes the values x_i isp_i, then P(X=x_i) $= p_i$or p(x_i).The set of values [x_i, p(x_i)] is called a discrete probability distribution of the discrete random variable X. The function P(X) is called the probability density function (p.d.f) or the probability mass function (p.m.f)

Cumulative distribution function

The distribution function f(x) is defined by f$(x) = P(X \leq x) = \sum_{i=1}^{x} \mathrm{p}(x_i)$, x being an integer is called the cumulative distribution function(c.d.f)

The mean and variance of the discrete probability distribution

Mean (μ) $or\ expectation E(X) \ = \sum_i x_i\, p(x_i)$
Variance (V) = $\sum_i (x_i - \mu)^2\ p(x_i)$= $\sum_i x_i{}^2\, p(x_i) - [x_i p(x_i)]^2$=$\sum_i x_i{}^2\, p(x_i) - \mu^2$
Standard deviation $(\sigma) = \sqrt{V}$

Problems1: Determine the discrete probability distribution, expectation, variance, standard deviation of a discrete random variable X which denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once.

Solution: The total number of cases are 6x6=36. The minimum number could be 1,2,3,4,5,6i.e.,X(s)=X(a,b)=min$\{a, b\}$. the number 6 will appear only in one case (6,6),so P(6)=P(X=6)=P($\{6{,}6\}$=1/36

For minimum 5, favourable cases are (5,5),(5,6),(6,5) so P(5)=P(X=5)=3/36 For minimum 4, favourable cases are(4,4,(4,5),(4,6),(5,4)(6,4) so P(4)=P(X=4)=5/36

For minimum 3, favourable cases are(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3) so P(3)=P(X=3)=7/36

For minimum 2, favourable cases are (2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),(6,2) so P(2)=P(X=2)=9/36

For minimum 1, favourable cases are
(1,1)(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)
so P(1)=P(X=1)=11/36

Thus required discrete probability distribution

X=x_i	1	2	3	4	5	6
p(x_i)	11/36	9/36	7/36	5/36	3/36	1/36

Mean = $\sum_i x_i\, p(x_i)$=1x11/36+2x9/36+3x7/36+4x5/36+5x3/36+6x1/36=2.5

Variance (V)=$\sum_i {x_i}^2\, p(x_i) - \mu^2$
=1x11/36+4x9/36+9x7/36+16x5/36+25x3/36+36x1/36$-(2.5)^2$
=1.9745
Standard deviation=$\sqrt{V}$=1.4

Example 2: The random variable X has the following probability mass function

X	0	1	2	3	4	5
P(X)	K	3K	5K	7K	9K	11K

i) find K ii) find P(X<3) iii) find P(3<X≤ 5)(June-July2011)

Solution: If X is a discrete random variable then $\sum_i P(x_i) = 1$
⇒ K+3K+5K+7K+9K+11K =1
⇒ 36K=1
⇒ K = 1/36

ii) P(X<3) = P(X=0)+P(X=1)+P(X=2)
= K+3K+5K = 9K = 9/36=1/4

iii) P(3<X≤ 5) = P(X=4)+P(X=5) =9K+11K= 20K= 20/36=5/9

Example 3: The probability distribution of a finite random variable X is given by the following table:

X_i	-2	-1	0	1	2	3
p(X_i)	0.1	k	0.2	2k	0.3	k

i) Find the value of K and calculate the mean and variance.
ii) EvaluateP(X<1). (July2006)

Solution: If X is a discrete random variable then $\sum_i P(x_i) = 1$
⇒0.1+K+0.2+2k+0.3+k =1
⇒ 0.6+4K=1
⇒ 4k=1-0.6=0.4
⇒ K = 0.1

Mean $(\mu) = \sum_i x_i\, p(x_i)$ =-2x0.1+-1x0.1+0x0.2+1x0.2+2x0.3+3x0.1
=0.8

Variance (V) = $\sum_i {x_i}^2\, p(x_i) - \mu^2$
= 4x0.1+1x0.1+0x0.2+1x0.2+4x0.3+9x0.1$-(0.8)^2$
=2.16

ii)P(X<1) = P(X=-2)+P(X=-1)+P(X=0)
=0.1+0.1+0.2
=0.4

Example 4:A random variable X has the following probability function for various values of x

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

i) Findk ii) Evaluate P(x<6) and P(3<x≤ 6) Also find the probability distribution and the cumulative distribution function of X

Solution: If X is a discrete random variable then $\sum_i P(x_i) = 1$and P(X)≥ 0
$\Rightarrow$ 0+k+2k+2k+3k+k^2+$2k^2$+$7k^2$+k=1
$\Rightarrow$10k^2+9k$-$1=0
$\Rightarrow$ k=1/10 andk=-1

Ifk=-1 the second condition fails and hence k$\neq -1$ $\therefore$ k=1/10
Hence the probability distribution is as follows.

x	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

P(x<6)=P(0)+P(1)+P(2)+P(3)+P(4)+P(5)
=0+0.1+0.2+0.2+0.3+0.01=0.81

P(3<x≤ 6)=P(4)+P(5)+P(6)
=0.3+0.01+0.02=0.33

Cumulative distribution function of X is as follows.

x	0	1	2	3	4	5	6	7
f(x)	0	0+0.1 =0.1	0.1+0.2 =0.3	0.3+0.2 0.5	0.5+0.3 0.8	0.8+0.01 0.81	0.81+0.02 0.83	0.83+0.17 1

In discrete probability distribution we are going to study
Binomial distribution
Poisson distribution.

Binomial Distribution

i) The random experiment is performed repeatedly a finite and fixed numbers of times. In other wards n, the number of trials is finite and fixed

ii) The outcome of each trial may be classified into two mutually disjoint categories, called success (the occurrence of the event) and failure (the non-occurrence of the event).

iii) All the trials are independent i.e the result of any trial, is not affected in any way by the preceding trials and doesn't affect the result of succeeding trials.

iv)The probability of success (happening of an event) in any trial is p and is constant for each trial q = 1- p, is then termed as the probability of failure and is constant for each trial.

For example, if we toss a fair coins n times (which is fixed and finite) then the outcome of any trial is one of the mutually exclusive events viz head (success) and tail (failure). Further, all the trials are independent, since the result of any throw of a coin does not affect and is not affected by the result of other throws. Moreover, the probability of success (head) in any trial is ½, which is constant for each trial. Hence the coin tossing problems will give rise to Binomial distribution

More precisely, we expect a binomial distribution under the following conditions

i) n, the number of trials is finite

ii) Trials are independent

iii) P, the probability of success is constant for each trial. Then q=1-p, is the probability of failure in any trial.

$$p(x) = nC_r\, p^x q^{n-x}$$

iv) Probability function of Binomial distribution: Consider the probability distribution table

x	0	1	2	3	.....	n
P(x)	q^n	$nC_1\, pq^{n-1}$	$nC_2\, p^2q^{n-2}$	$nC_3\, p^3q^{n-3}$		p^n

Where n is a given positive integer, p is a real number such that 0 ≤ p < 1 and q = 1-p.

The probability function for this distribution is denoted by b(n,p,x) given by; x=0,1,2,….n
$b(n,p,x) = nC_r\, p^x q^{n-x}$

This probability function is called Binomial probability function and corresponding distribution is called Binomial distribution. b(n,p,x) > 0 for x=0,1,2,….n

Mean(μ)=$\sum_{x=0}^{n} xP(x)$
Mean(μ)=np
Variance(V)=$\sum_{x=0}^{n} x^2P(x) - \mu^2$
Variance(V)=npq
Standard deviation(σ)=$\sqrt{V}$
SD(σ)=$\sqrt{npq}$

Problem :

1) Let X be a binomially distributed random variable based on 6 repetitions of an experiment. If p =0.3, evaluate the following probabilities i) P(X≤ 3) ii) P(X > 4)

Solution : Given P-0.3 and n = 6 Hence q=1-P = 0.7 and
b(n,P,x) = b (6,0.3.x) = $6C_x\, 0.3^x 0.7^{6-x}$= $P(x)$ say
$i)\ In\ this\ case\ X \leq\ 3\ Hence\ X\ can\ take\ values$ 0,1,2 and 3.
Therefore $P(X \leq\ 3)\ =\ P(0)\ +\ P(1)\ +\ P(2) + P(3)$
= $(0.7)^6\ + 6C_1\,(0.3)0.7^5 + 6C_2\, 0.3^2 0.7^4\ + 6C_3\, 0.3^3 0.7^3$
ii) In this case X > 4 Hence x takes values 5 and 6
$Therefore\ P(X >\ 4)\ =\ P(5) + P(6)\ =\ 6C_5\, 0.3^5 0.7^5 +\ (0.3)^6$

2) The probability that a pen manufactured by a company will be defective is 0.1. If 12 such pens are selected at random, find the probability that
i) Exactly two pens will be defective
ii) At most two pens will be defective
iii) None will be defective

Solution: Let the probability that a pen manufactured is defective = p.
Then p=0,1 q=1-p = 0.9 and n =12
Hence $b(n,p,x)\ = 12C_x\, 0.1^x 0.9^{12-x}\ \ =\ P(x)\ say$
i) Probability that exactly two pens will be defective = $P(X = 2) = 12C_2\, 0.1^2 0.9^{10} = 0.2301.$
ii)Probability that at most 2 pens will be defective =$P(X \leq 2)$
$= P(0) + P(1) + P(2) =\ (0.9)^{12}\ +\ 12C_1\, 0.1^1 0.9^{11} + 12C_2\, 0.1^2 0.9^{10}$
iii) Probability that none of the pens will be defective =$P(X = 0) = P(0) = (0.9)^{12} = 0.2824295.$

POISSON DISTRIBUTION

Poisson distribution is the discrete probability distribution of discrete random variable X, which has no upper bound. It is defined for non-negative values of x as follows: P(x)= $\frac{m^x e^{-m}}{x!}$ for x=0,1,2,3…… Here m>0 is called the parameter of the distribution. In binomial distribution the number successes out of total definite number of n trials is determined, whereas in Poisson distribution the number of successes at a random point time and space is determined.

Poisson distribution is suitable for 'rare' events for which the probability of occurrence p is very small and the of trials n is very large. Also binomial distribution can be approximated by Poisson distribution when n→ ∞ and p→ 0 such that m= np= constant.

Example of rare events:

i) Number of printing mistake per page.
ii) Number of accidents on a highway.
iii) Number of bad cheques at a bank.
iv) Number of defectives in a production center.

We have in case of binomial distribution, the probability of x successes out of n trials,

$P(x) = n_{c_x} p^x q^{n-x}$

$=\frac{n(n-1)(n-2)\ldots\ldots\ldots n-(x-1)}{x!} p^x q^{n-x}$

$=\frac{n.n\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right)\ldots\ldots\ldots n.\left(1-\frac{x-1}{n}\right)}{x!} p^x q^{n-x}$

$=\frac{n^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\left(1-\frac{x-1}{n}\right)}{x!} p^x q^{n-x}$

$= \frac{(np)^x\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\ldots\left(1-\frac{x-1}{n}\right)}{x!q^x} q^n$

But np =m; $q^n = (1-p)^n = (1-\frac{m}{n})^n = \left\{(1-\frac{m}{n})^{-n/m}\right\}^{-m}$ denoting $-\frac{m}{n} = k$

We have, $q^n = \left\{(1+k)^{1/k}\right\}^{-m} \to e^{-m}$ as n→ ∞ or $k \to 0$

Further $q^x = (1-p)^x \to 1$ for a fixed x as p→ 0.

Also the factor $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots..\left(1-\frac{x-1}{n}\right)$ will also tend to 1 as n→ ∞

Thus P(x)= $\frac{m^x e^{-m}}{x!}$

This is known as the Poisson distribution of the random variable. P(x) is called Poisson probability function and x is called a Poisson variant.

The distribution of probabilities for x=0,1,2,3….. is as follows.

x	0	1	2	3	.....
P(x)	e^{-m}	$\frac{me^{-m}}{1!}$	$\frac{m^2e^{-m}}{2!}$	$\frac{m^3e^{-m}}{3!}$	

We have P(x)≥ 0 and

$$\sum_{x=0}^{\infty} P(x) = e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2e^{-m}}{2!} + \frac{m^3e^{-m}}{3!} + \cdots \ldots ..$$

$$= e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= e^{-m}e^{m} = 1$$

Hence P(x) is a probability function.

Mean and standard deviation of the Poisson distribution

Mean(μ)=$\sum_{x=0}^{\infty} xP(x)$

$$=\sum_{x=0}^{\infty} x\frac{m^x e^{-m}}{x!}$$

$$=\sum_{x=1}^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$=me^{-m}\sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= me^{-m}e^{m}$$

Mean(μ) =m

Standard deviation(σ)=$\sqrt{V}$

Variance(V)=$\sum_{x=0}^{\infty} x^2P(x) - \mu^2$.........(1)

Consider$\sum_{x=0}^{\infty} x^2P(x) = \sum_{x=0}^{\infty}[x(x-1) + x]\frac{m^x e^{-m}}{x!}$

$$= \sum_{x=2}^{\infty} \frac{m^x e^{-m}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$=m^2e^{-m}\sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$=m^2e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\} + m$$

$$=m^2e^{-m}e^{m} + m$$

$\sum_{x=0}^{\infty} x^2P(x) = m^2 + m$

Equation (1) implies

Variance(V)=$m^2 + m - m^2$

=m

∴ Standard deviation(σ)=$\sqrt{m}$

Mean and variance are equal in Poisson distribution.

Problem1: The probability that individual suffers a bad reaction from an injection is 0.001. Find the probability that out of 2000 individuals i) more than 2 ii) exactly 3 will get bad reaction

Solution: As the probability of occurrence is very small, this follows Poisson distribution and we have

P(x)= $\frac{m^x e^{-m}}{x!}$

Mean=m=np=2000x0.001=2

i)P(x>2)=1-P(x≤ 2)

=1 – [P(x=0)+P(x=1)+P(x=2)]

= 1- $e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!}\right\}$

=1 - $e^{-2}[1 + 2 + 2]$=0.3233

ii)P(x=3)=$\frac{2^3 e^{-2}}{3!}$= 0.1804

Problem2:2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains i)no defective fuse ii)3 or more defective fuses.(July-2007)

Solution: By data probability of defective fuse=2/100=0.02

Mean=m=np=200x0.02=4

Poisson distribution P(x)= $\frac{m^x e^{-m}}{x!}$

=$\frac{4^x e^{-4}}{x!}$

i)P(x=0)= e^{-4}=0.0183

ii)P(x$\geq 3) = 1 - P(x < 3)$

=1- [P(X=0)+P(x=1)+P(x=2)]

=1 –e^{-4} [1+$\frac{4}{1!} + \frac{4^2}{2!}$]

=0.7621

Problem3: There **is** a chance that 5% of the pages of a book contain typographical errors. If 100 pages of the book are chosen at random, find the probability that 2 of the pages contain typographical errors, using i)Binomial distribution ii)Poisson distribution.

Solution: i) Binomial distribution

The probability that a chosen page contains typographical errors is given as p=5%=0.05, q =1-0.05=0.95, n=100

$P(x) = n_{c_x} p^x q^{n-x} \quad = 100_{c_x}(0.05)^x(0.95)^{100-x}$

P(x=2)= $100_{c_2}(0.05)^2(0.95)^{98}$= 0.081

ii) Poisson distribution.

Mean=m=np=100x0.05=5

P(x)= $\frac{m^x e^{-m}}{x!}$

P(x=2)=$\frac{5^2 e^{-5}}{2!}$=0.084

Problem4: If x is a Poisson variant such that P(x=1)=0.2P(x=2).find the mean & evaluate P(x=0)

Solution: For the Poisson distribution, the p.d.f

P(x)= $\frac{m^x e^{-m}}{x!}$

P(x=1)=$e^{-m}m$

P(x=2)=$\frac{m^2 e^{-m}}{2!}$

By data P(x=1)=0.2P(x=2)

=0.2$\frac{m^2 e^{-m}}{2!}$

This implies m=10

∴p.d.f P(x) =$\frac{10^x e^{-10}}{x!}$

P(x=0)=e^{-10}

Continuous probability distribution:

The number of events are infinitely large the probability that a specific event will occur is practically zero for this reason continuous probability statement must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval. In this context we need a continuous probability function which is defined as follows.

Definition: If for every x belonging to the range of a continuous random variable X, we assign a real number P(x) satisfying the conditions

i) P$(x) \geq 0$

ii) $\int_{-\infty}^{\infty} \mathrm{P}(x)\mathrm{dx} = 1$ then P(x) is called a Continuous probability function or probability density function(p.d.f).If (a,b) is a subinterval of the range space of X then the probability that x lies in the (a,b) is defined to be the interval of P(x) between a and b. i.e., P(a≤x≤b) = $\int_{a}^{b} \mathrm{P}(x)\mathrm{dx}$

Cumulative distribution function

If X is a continuous random variable with probability density function P(x) then the function f(x) is defined by f$(x) = P(X \leq x) = \int_{-\infty}^{x} P(x)\mathrm{dx}$ is called the cumulative distribution function(c.d.f)of X

The mean and variance of the continuous probability distribution

Mean (μ) or Expectation E(X) =$\int_{-\infty}^{\infty} x.\,p(x)dx$

Variance (V)= $\int_{-\infty}^{\infty} (x_i - \mu)^2.\,p(x)dx$=$\int_{-\infty}^{\infty} (x_i)^2.\,p(x)dx - \mu^2$

Example 1:A random variable X has the density function P(x)=$\begin{cases} kx^2 & for \quad -3 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$ find k. Also find P(x$\leq$ 2) and P(x>1)

Solution: If X is a continuous random variable then i) P$(x) \geq 0$

ii) $\int_{-\infty}^{\infty} \mathrm{P}(x)\mathrm{dx} = 1$

That is $\int_{-3}^{3} kx^2\mathrm{dx} = 1$

$$\Rightarrow \left[\frac{kx^3}{3}\right]_{-3}^{3} = 1$$

⇒ k=1/18

P(x$\leq$ 2)=$\int_{-3}^{2} \frac{x^2}{18}\mathrm{dx}$ = $\frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2}$=$\frac{35}{54}$

P(x>1) =$\int_{1}^{3} \frac{x^2}{18}\mathrm{dx}$ = $\frac{1}{18}\left[\frac{x^3}{3}\right]_{1}^{3}$ =$\frac{26}{54}$= $\frac{13}{27}$

Example 2: The daily consumption of electric power (in millions of kW-hours) is a random variable having the p.d.f P(x)= $\begin{cases} \frac{1}{9}xe^{-\frac{x}{3}} & x > 0 \\ 0 & x \le 0 \end{cases}$ if the total production is 12million kW-hours, determine the probability that there is power cut (shortage) on any given day.

Solution: Probability that the power consumed is between 0to12 is P(0$\le x \le 12) =$ $\int_0^{12} \mathrm{P}(x)\mathrm{dx} = \int_0^{12} \frac{1}{9}xe^{-\frac{x}{3}}\mathrm{dx} = \left[-\frac{x}{3}e^{-\frac{x}{3}} - e^{-\frac{x}{3}}\right]_0^{12} = 1 - 5e^{-4}$

Power supply is inadequate if daily consumption exceeds 12million kW,i.e.,
P(x>12)=1$-$ P(0$\le x \le 12$)= 1$-[1 - 5e^{-4}]$=$5e^{-4}$=0.0915781

Example 3: Find the mean and variance of p.d.f. f(x)=$\begin{cases} \frac{1}{4}e^{-\frac{x}{4}} & for \quad x > 0 \\ 0 & elsewhere \end{cases}$

Solution: Mean =$\int_{-\infty}^{\infty} x.f(x)dx = \int_0^{\infty} \mathrm{xf}(x)\mathrm{dx}$=$\int_0^{\infty} \frac{1}{4}\ xe^{-\frac{x}{4}}\mathrm{dx}$

=$\frac{1}{4}\left[\frac{xe^{-\frac{x}{4}}}{-\frac{1}{4}} - \frac{e^{-\frac{x}{4}}}{\frac{1}{16}}\right]_0^{\infty}$=$-4(0-1)$=4

Variance(V) = $\int_{-\infty}^{\infty}(x_i)^2.f(x)dx - \mu^2 = \int_0^{\infty} \mathrm{x}^2\,\frac{1}{4}\ xe^{-\frac{x}{4}}\mathrm{dx} - 16$

= $\frac{1}{4}\left[\frac{\mathrm{x}^2e^{-\frac{x}{4}}}{-\frac{1}{4}} - 2x\frac{e^{-\frac{x}{4}}}{\frac{1}{16}} + 2\frac{e^{-\frac{x}{4}}}{-\frac{1}{64}}\right]_0^{\infty} - 16$

=32-16=16

In continuous probability distribution we study

- **Uniform Distribution**
- **Exponential Distribution.**
- **Normal Distribution.**
- **Weibull Distribution**

Uniform Distribution

A random variable X having the range of some finite interval $a \le x \le b$ is said to have uniform distribution if its probability density function is constant within the range .$f(x) = \begin{cases} c & a \le x \le b \\ o & otherwise \end{cases}$

Since $\int_a^b f(x)\,dx = \int_a^b c\,dx = 1$ it follows that

$$= c\ \ x]_a^b = 1$$

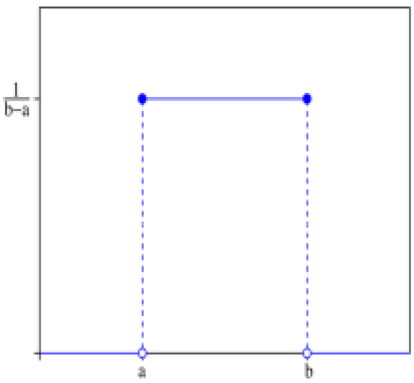
$$c(b-a) = 1$$

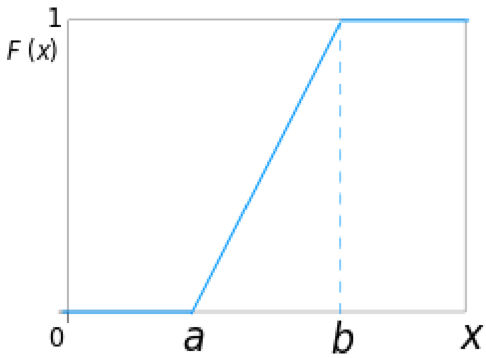
$$c = \frac{1}{b-a}$$

And therefore $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ o & otherwise \end{cases}$

The corresponding distribution function is

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$





Mean $(\mu) = \frac{b+a}{2}$

Variance(v)=) $= \frac{(b-a)^2}{12}$

Problem

1). A uniform distribution has P.D.F $f(x) = \begin{cases} \frac{1}{4} & 2 < x < 6 \\ o & otherwise \end{cases}$. Find its mean and variance Also find (i) $p(x \ge 1)$ $(ii) p(x < 4)$ $(iii) p(x > 5)$

Solution : $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ o & otherwise \end{cases}$

a=2 b=6

Mean $(\mu) = \frac{b+a}{2} \ = \frac{6+2}{2} = 4$

Variance(v)= $) = \frac{(b-a)^2}{12} = \frac{(6-2)^2}{12} = \frac{4}{3}$

(i)$p(x \geq 1) = 1 - p(x < 1)$

$= 1 - 0 = 1$

(ii) $p(x < 4) = \int_2^4 f(x)dx = \int_2^4 \frac{1}{4} dx = \frac{1}{4} x]_2^4 = \frac{1}{4}(4-2) = \frac{2}{4} = \frac{1}{2}$

(iii) $p(x > 5) = \int_5^6 f(x)dx = \int_5^6 \frac{1}{4} dx = \frac{1}{4} x]_5^6 = \frac{1}{4}(6-5) = \frac{1}{4}$

2) A random variable x is uniformly distributed over the interval $-1 < x < 1$ Find $(i)p\left(x < \frac{1}{2}\right) \quad (ii)p(\left|x - \frac{1}{2}\right| > \frac{1}{4}).\ f(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ o & otherwise \end{cases}$

Solution : $p\left(x < \frac{1}{2}\right) = \int_{-1}^{\frac{1}{2}} f(x)dx = \int_{-1}^{\frac{1}{2}} \frac{1}{2} dx = \frac{1}{2} x]_{-1}^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{2} — 1\right)$

$$= \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$$

(ii) $p\left(\left|x - \frac{1}{2}\right| > \frac{1}{4}\right) = 1 - p(\left|x - \frac{1}{2}\right| < \frac{1}{4})$

$= 1 - p\left(-\frac{1}{4} < x - \frac{1}{2} < \frac{1}{4}\right)$

$=1 - p\left(\frac{1}{4} < x < \frac{3}{4}\right)$

$$1 - \int_{\frac{1}{4}}^{\frac{3}{4}} f(x)dx = 1 - \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{1}{2} dx = 1 - \frac{x}{2}]_{\frac{1}{4}}^{\frac{3}{4}} = 1 - \frac{1}{2}\left(\frac{3}{4} - \frac{1}{4}\right) = 1 - \frac{1}{2}\left(\frac{2}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

EXPONENTIAL DISTRIBUTION

Many scientific experiments involve the measurement of the duration of time X between an initial point of time and the occurrence of some phenomenon of interest. For Example X is the life time of a light bulb which is turned on left until it burns out. The continuous random variable X having the probability density function f(x) $=\begin{cases} \boldsymbol{\alpha} e^{-\alpha x} & for \quad x > 0 \\ 0 & elsewhere \end{cases}$, where $\alpha > 0$ is known as the exponential distribution. Here the only parameter of the distribution is α

Example of random variables modelled as exponential are

i) Duration of telephone calls
ii) Time require for repair of a component
iii) Service time at a server in a queue

Mean and standard deviation of the exponential distribution

Mean $(\mu) = \int_{-\infty}^{\infty} x.f(x)dx = \int_0^{\infty} x.\boldsymbol{\alpha} e^{-\alpha x}dx = \boldsymbol{\alpha}\int_0^{\infty} x.e^{-\alpha x}dx$

$= \boldsymbol{\alpha}\left[x.\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}} - 1\frac{e^{-\alpha x}}{\boldsymbol{\alpha}^2}\right]_0^{\infty}$

$= \boldsymbol{\alpha}\left[0 - \frac{1}{\boldsymbol{\alpha}^2}(0-1)\right] = \frac{1}{\boldsymbol{\alpha}}$

Mean $(\mu) = \frac{1}{\boldsymbol{\alpha}}$

Standard deviation (σ)=$\sqrt{V}$

Variance (V) = $\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

$= \boldsymbol{\alpha}\int_0^{\infty}(x-\mu)^2 e^{-\alpha x}dx$

$= \boldsymbol{\alpha}\left[(x-\mu)^2.\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}} - 2(x-\mu)\frac{e^{-\alpha x}}{\boldsymbol{\alpha}^2} + 2\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}^3}\right]_0^{\infty}$

$=\boldsymbol{\alpha}\left[(0-\mu^2)\frac{1}{-\boldsymbol{\alpha}} - 2([0-(-\mu)]\frac{1}{\boldsymbol{\alpha}^2} - 2\frac{1}{\boldsymbol{\alpha}^3}(0-1)\right]$

$= \boldsymbol{\alpha}\left[\frac{\mu^2}{\boldsymbol{\alpha}} - \frac{2\mu}{\boldsymbol{\alpha}^2} + \frac{2}{\boldsymbol{\alpha}^3}\right]$

$= \boldsymbol{\alpha}\left[\frac{1}{\boldsymbol{\alpha}^3} - \frac{2}{\boldsymbol{\alpha}^3} + \frac{2}{\boldsymbol{\alpha}^3}\right] \quad = \frac{1}{\boldsymbol{\alpha}^2}$

Standard deviation (σ)=$\sqrt{V}$=$\sqrt{\frac{1}{\boldsymbol{\alpha}^2}}$

Mean Standard deviation is equal in exponential transformation

Problem1: In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for i) less than 10 minutes ii) 10 minutes or more iii) between 10minutes and 12 minutes (Dec.06/jan07)

Solution: The p.d.f of the exponential distribution is given by f(x)=$\alpha e^{-\alpha x}, x > 0$ and mean = 1/ α

By data $1/\alpha = 5 \quad \therefore \alpha = 1/5$ and hence f(x) =$\frac{1}{5} e^{-\frac{x}{5}}$

i) P(x<10)=$\int_0^{10} \frac{1}{5} e^{-\frac{x}{5}} dx = -\left[e^{-\frac{x}{5}}\right]_0^{10}$ =1- e^2=0.8647

ii) P(x≥ $10) = \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx$=$-\left[e^{-\frac{x}{5}}\right]_{10}^{\infty} = e^2 = 0.1353$

iii) P(10<x<12)=$\int_{10}^{12} \frac{1}{5} e^{-\frac{x}{5}} dx$=$-\left[e^{-\frac{x}{5}}\right]_{10}^{12}$=-($e^{\frac{12}{5}} - e^{-2}$)=0.0446

Problem2: The sale per day in a shop is exponentially distributed with average sale amounting to Rs.100 and net profit is 8%. Find the probability that the net profit exceeds Rs.30 on a day.

Solution: Let x be the random variable of the sale in the shop. Since x is an exponential variant the p.d.f f(x)= $\alpha e^{-\alpha x}, x > 0$ mean = 1/ α=100

$\Rightarrow \alpha = 0.01$ hence f(x)= $0.01e^{-0.01x}, x > 0$

Let A be the amount for which profit is 8%

$\Rightarrow$ A.$\frac{8}{100} = 30 \therefore A = 375$

Probability of profit exceeding Rs.30 = 1- Prod(profit$\leq Rs.30$)

=1-Prob(sale$s \leq Rs.375$)

$$=1 - \int_0^{375} (0.01)e^{-0.01x}\ dx$$

$$=1+[e^{-0.01x}]_0^{375} = e^{-3.75}$$

The probability that the net profit exceeds Rs.30 on a day is $e^{-3.75}$

Problem3: Let the mileage (in thousands of miles) of a particular tyre be random variable x having p.d.f $\begin{cases} \frac{1}{20} e^{-x/20} & for\ \ x > 0 \\ 0 & elsewhere \end{cases}$ find the probability that i) at most 10,000miles ii)any where from 16,000 to 24,000miles iii) at least 30,000miles.iv)Find the mean and the variance of the given p.d.f.

Solution: By data $\alpha = 1/20$

i)P(x$\leq 10) = \int_0^{10} f(x)dx = \int_0^{10} \frac{1}{20} e^{-x/20} dx$.=$\left[\frac{1}{20} e^{-x/20} \frac{-20}{1}\right]_0^{10}$=1-$e^{-1/2}$=0.3934

ii)P(16$\leq x \leq 24) = \int_{16}^{24} f(x)dx = \int_{16}^{24} \frac{1}{20} e^{-x/20} dx.$ =-$\left[\frac{1}{20} e^{-x/20} \frac{-20}{1}\right]_{16}^{24}$

=$e^{-4/5}$-$e^{-6/5}$ =0.148

iii) P(x$\geq 30) = \int_{30}^{\infty} f(x)dx = \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\, dx = \left[\frac{1}{20} e^{-x/20} \frac{-20}{1}\right]_{30}^{\infty} = e^{-3/2}$=0.2231

iv) Mean(μ) $= \frac{1}{\alpha} = 20$

Variance(V)=$\frac{1}{\alpha^2} = 20^2$

NORMAL DISTRIBUTION

Normal distribution is the probability distribution of continuous random variable X, known as normal random variable or normal variate it is given by f(x)=$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2}/2\sigma^2$ where -$\infty < x < \infty$, -$\infty < \mu < \infty$ & σ>0. Is known as normal distribution.

$\int_{-\infty}^{\infty} f(x)dx$ =$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2}/2\sigma^2$dx

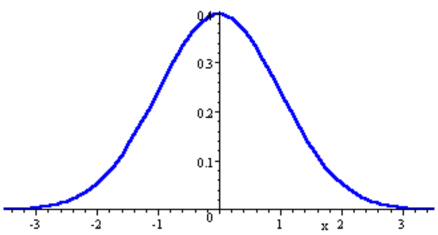
μ & σ are the two parameters of the normal distribution is also known as Gaussian distribution.

This distribution is most important , simple, useful & is the corner stone of modern statistics because sampling distribution ' t ', F, χ^2 tend to be normal for large samples & it is applicable in statistical quality control in industry.

PROPERTIES OF NORMAL DISTRIBUTION

(i)The graph of the normal distribution y=f(x) in the XY-plane is known as normal curve. Normal curve is symmetric about y axis, it is bell shaped the mean, median,& mode coincide & therefore normal curve is unimodal.

Normal curve is asymptotic to both positive & negative x-axis.



(ii)Area under the curve is unity.

(iii)Probability that the continuous random variable lies between a&b is denoted by P(a$\leq x \leq b$)& is given by $\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2}/2\sigma^2 \ dx$………(i)

Since (1) depends on the two parameters μ & σ we get different normal curves for different values of μ & σ& it is impracticable task to plot all such normal curves.Instead by introducing Z=$\frac{(x-\mu)}{\sigma}$.The RHS integral in(1) becomes independent of the two parameters μ & σhere Z is known standard variate.

(iv) Change of scale from x-axis to z-axis

P(a$\leq x \leq b$)=$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2}/2\sigma^2 \ dx$

P($z_1 \leq z \leq z_2$)=$\int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2}/2 \ \sigma dz$

=$\int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2}/2 \ dz$……………………(2)

Where z_1=$\frac{a-\mu}{\sigma}$, z_2=$\frac{b-\mu}{\sigma}$

(v) Error function or probability integral is defined as P(Z)=$\frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2}/2 \ dz$…..(3)

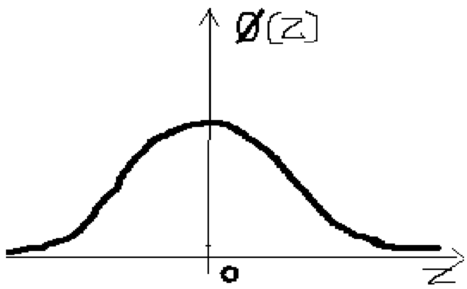
Now (2) can be written by using (3) as

P(a$\leq x \leq b$)= P($z_1 \leq z \leq z_2$)=$\int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{-z^2}/2 \ dz$…………(4)

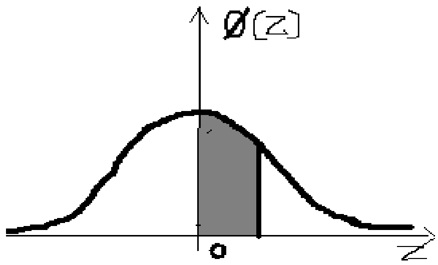
=P(z_1) $- P(z_2)$

Normal distribution f(x) transformed by the standard variable Z is given by F(Z) =$\frac{1}{\sqrt{2\pi}} e^{-z^2}/2$ with mean 0 & standard deviation 1 is known as standard normal distribution & its normal curve as standard normal curve. The probability integral (3) is tabulated for various values of Z varying from 0 to 3.9 & is known as normal table. Thus the entries in the normal table gives the area under the normal curve between the ordinates z=0 to z. Since normal curve is symmetric about y-axis the area from 0 to –z is same as the area from 0 to z. For this reason, normal table is tabulated only for positive values of z.

The integral in the RHS of (4) geometrically represents the area bounded by the standard normal curve F(Z) between z=z_1& z=z_2. Further in particular if $z_1 = 0$ $we\ have\ \emptyset(Z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2}$/2dz. This represents the area under the



standard normal curve z=0 to z.



Note:

1. $\int_{-\infty}^{\infty} \emptyset(Z)dz = 1 \Rightarrow$P(-$\infty < z \leq \infty) = 1$
2. $\int_{-\infty}^{0} \emptyset(Z)dz = \int_{0}^{\infty} \emptyset(Z)dz = 1/2 \Rightarrow$P(-$\infty \leq z \leq 0) = P(0 \leq z \leq \infty) = 1/2$
3. P(-$\infty < z < z_1$)=P(-$\infty < z \leq 0$)+P(0$\leq z < z_1$)=0.5+$\emptyset(z_1)$
4. P(z>z_2)=0.5-$\emptyset(z_2)$

MEAN & STANDARD DEVIATION OF THE NORMAL DISTRIBUTION

Mean $(\mu) = \int_{-\infty}^{\infty}$ x f(x) dx

$=\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-(x-\mu)^2}/2\sigma^2 \ dx$

Putting t=$\frac{(x-\mu)}{\sigma\sqrt{2}}$ or x= $\mu + \sigma t\sqrt{2}$, we have dx= $\sigma\sqrt{2}$dt

t also varies from -∞ to ∞

mean=$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu + \sigma t\sqrt{2}e^{-t^2} \ \sigma\sqrt{2}$dt

$=\frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt + \sigma\frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} te^{-t^2}\, dt$

$=\frac{2\,\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\, dt + \sigma\frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} te^{-t^2}\, dt$

The second integral is 0 by standard property since te^{-t^2} is an odd function.

By gamma function $\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$

Hence mean=$\frac{2\,\mu}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} + 0 = \mu$

Hence the mean of the normal distribution is equal to mean of the given distribution.

Standard deviation $\sigma = \sqrt{V}$

Variance (V)= $\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

=$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2}/2\sigma^2\ dx$

Substituting t=$\frac{x-\mu}{\sqrt{2\pi}}$, x= $\mu + \ \sigma t\sqrt{2}$, we have dx= $\sigma\sqrt{2}$dt

t also varies from -∞ to ∞

Variance (V)=$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2}\ \sigma\sqrt{2}$dt

=$\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2 e^{-t^2} dt$

=$\frac{2\sigma^2}{\sqrt{\pi}} 2\int_0^{\infty} t^2 e^{-t^2} dt$

=$\frac{2\sigma^2}{\sqrt{\pi}}\int_0^{\infty} t\,(2te^{-t^2})dt$

Taking u=t, v=2te^{-t^2}

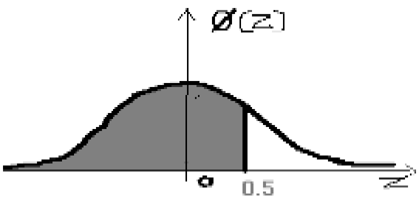
$\int uv$dt=u$\int v$dt-$\iint v$dt . $u'dt$

Variance (V)=$\frac{2\sigma^2}{\sqrt{\pi}}\{[te^{-t^2}]_0^\infty - \int_0^\infty -e^{-t^2}dt\}$

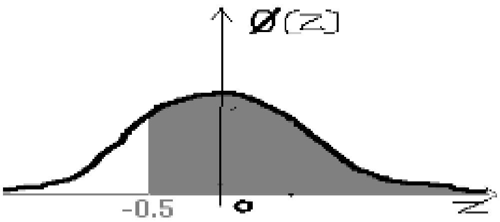
Variance (V)=$\frac{2\sigma^2}{\sqrt{\pi}}$[0+$\int_0^\infty e^{-t^2}dt$]=$\frac{2\sigma^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}$=$\sigma^2$

The variance/standard deviation of the normal distribution is equals to the variance of the given distribution.

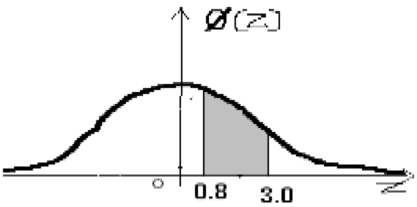
Area under standard normal curve to the left of z=1.5



Area under standard normal curve to the right of z=-0.5



Area under standard normal curve between z=0.8 to 3



Problem1: Find the following probabilities for the standard normal distribution with the help of normal probability table

a) P(-0.5$\leq z \leq 1.1$) b) P(z ≥ 0.60) c) P(z ≤ 0.75) d)P(0.2$\leq z \leq 1.4$)

Solution:

a) P(-0.5$\leq z \leq 1.1$) = P(-0.5$\leq z \leq 0) + \mathrm{P}(0 \leq z \leq 1.1)$

= P(0$\leq z \leq 0.5$)+ $\mathrm{P}(0 \leq z \leq 1.1)$

=$\emptyset(0.5) + \emptyset(1.1)$

=0.1915+0.3643

=0.5558

b) P(z ≥ 0.60) = P(z$\geq 0) - P(z \leq 0.60)$

= 0.5 $-\emptyset(0.60)$

=0.5 -0.2258 =0.2742

c) P(z ≤ 0.75) = P(z ≤ 0) + P(0$\leq$z ≤ 0.75)

=0.5 $+ \emptyset(0.75)$

=0.5 + 0.2422=0.7422

d)P(0.2$\leq z \leq 1.4$) = P(0$\leq z \leq 1.4) - \mathrm{d})\mathrm{P}(0 \leq z \leq 0.2)$

=$\emptyset(1.4) - \emptyset(0.2)$

=0.4192 -0.0793

=0.3399

Problem2: Assuming that the diameters of 1000 brass plugs taken consecutively from a machine form a normal distribution with mean 0.7515cm & standard deviation 0.002cm How many of the plugs are likely to rejected if the approved number is 0.752$\pm$0.004cm?

Solution*:* Let x represent the bras plugs, by data mean μ= 0.7515cm &
S.D $\sigma = 0.002$
We have standard normal variant(s.n.v) z= $\frac{x-\mu}{\sigma}$ =$\frac{x-0.7515}{0.002}$
Now 0.752+0.004=0.756
⇒ x =0.756 so z = $\frac{0.756-0.7515}{0.002}$=2.25
P(z>2.25)=P(0$\leq z \leq \infty) - P(0 \leq z \leq 2.25)$=0.5 $-\emptyset(2.25)$=0.5−0.4878=0.0122………….(1)
Now 0.752-0.004=0.748
⇒ x =0.748so z = $\frac{0.748-0.7515}{0.002}$=-1.75
P(z<1.75)=0.5−$\emptyset(1.75)$=0.5 – 0.4599=0.0401………(2)

Equation(1)+ Equation(2)
P(2.25<z<1.75)=0.0122+0.041=0.0523
For 1000 brass plugs 1000x0.0523=52.3=52
⇒52 plugs are rejected

Problem3: x is normal random variable with 30 as mean & S.D 5. Find the probabilities that i)26$\leq x \leq 40$ ii) x≥ 45 iii)$|x-30| \leq 5$ (May-June 2010)

Solution: We have s.n.v z= $\frac{x-\mu}{\sigma}$ =$\frac{x-30}{5}$
i)26$\leq x \leq 40$
x=26, z = $\frac{26-30}{5}$ =-0.8, x=40, z= $\frac{40-30}{5}$ =2
i.e., we shall find P(-0.8$\leq z \leq 2)$=P(-0.8$\leq z \leq 0)$ + P(0$\leq z \leq 2)$
= $\emptyset(0.8) + \emptyset(2)$=0.2881+0.4772=0.7653
ii) x≥ 45
z = $\frac{45-30}{5} = 3$
i.e., we shall find P(z$\geq 3) = P(0 \leq z \leq \infty) - P(0 \leq z \leq 3)$
=0.5 - $\emptyset(3)$=0.5 – 0.4987=0.0013
iii)$|x-30| \leq 5$⇒-5 $\leq x-30 \leq 5$⇒ 25$\leq x \leq 35$
x=25, z = $\frac{25-30}{5}$=-1, x=35, z = $\frac{35-30}{5}$=1
P(-1$\leq z \leq 1)$=P(-1$\leq z \leq 0) + P(0 \leq z \leq 1)$=$\emptyset(1) + \emptyset(1)$=2$\emptyset(1)$ = 2x0.3413=0.6826

Problem4: Find the mean and standard deviation of an examination in which grades 70 and 88 corresponding to standard scores of -0.6 and 1.4 respectively.

Solution: s.n.v z= $\frac{x-\mu}{\sigma}$

Hence -0.6 =$\frac{70-\mu}{\sigma}$ so $\mu - 0.6\sigma = 70$

1.4=$\frac{88-\mu}{\sigma}$ so $\mu + 1.4\sigma = 88$

By solving $\mu = 75.4, \sigma = 9$ are the mean and standard deviation.

Problem5: In a test of electric bulbs , it was found that the life time of a particular brand is distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs find the number of bulbs that are likely to last for i) more than 2100 hours, ii) less than 1950 hours iii) between 1900 to 2100 hours.

Solution: By data $\mu = 2000, \sigma = 60$

We have s.n.v z= $\frac{x-\mu}{\sigma} = \text{z} = \frac{x-2000}{60}$

i)To findP(x>2100)=

If x=2100, z=$\frac{2100-2000}{60} = 1.67$

P(z>1.67)=P(z$\geq 0) - P(0 < z < 1.67)$=0.5 - $\emptyset(1.67)$=0.5 – 0.4525=0.0475

∴ number of bulbs that are likely to last for more than 2100 hours is

2500x0.0475 = 118.75 ≈ 119

ii) To find P(x< 1950)

If x =1950, z=$\frac{1950-2000}{60} = -0.83$

P(z<-0.83)=P(z>0.83)=P(z$\geq 0) - P(0 < z < 0.83)$= 0.5 - $\emptyset(0.83)$=0.5 – 0.2967 = 0.2033

∴ number of bulbs that are likely to last for less than 1950 hours is

2500x0.2033= 508.25≈ 508

iii) To find P(1900<x<2100)

If x =1900, z=$\frac{1900-2000}{60} = -1.67$ and if x= 2100, z=$\frac{2100-2000}{60} = 1.67$

P(-1.67<Z<1.67)=2P(0<z<1.67)

= 2$\emptyset(1.67)$= 2x 0.4525= 0.905

∴ number of bulbs that are likely to last between 1900and 2100 hours is

2500 x 0.905=2262.5≈ 2263

and $\boldsymbol{\mu_{\bar{X}_A - \bar{X}_B} = \mu_{\bar{X}_A} - \mu_{\bar{X}_B}}$ and

$$\boldsymbol{\sigma_{\bar{X}_A-\bar{X}_B}} = \sqrt{\boldsymbol{\sigma^2}_{\bar{X}_A} + \boldsymbol{\sigma^2}_{\bar{X}_B}} = \sqrt{\frac{\sigma^2{}_1}{n_1} + \frac{\sigma_2{}^2}{n_2}}$$

In paritcular case the statistics S_1 and S_2 have the proportion of success $\mathcal{P}_1$ and $\mathcal{P}_2$ then $\boldsymbol{\mu_{\mathcal{P}_1-\mathcal{P}_2} = \mu_{\mathcal{P}_1} - \mu_{\mathcal{P}_2} = p_1 - p_2}$ **and**

$$\boldsymbol{\sigma_{\mathcal{P}_1-\mathcal{P}_2}} = \sqrt{\boldsymbol{\sigma^2}_{\mathcal{P}_1} + \boldsymbol{\sigma^2}_{\mathcal{P}_2}} = \sqrt{\frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2}}$$

Where the normal variate for the difference in means is $\mathbf{Z} = \frac{(\bar{X}_A-\bar{X}_B) - (\mu_{\bar{X}_A-\bar{X}_B})}{\sigma_{\bar{X}_A-\bar{X}_B}}$

And the normal variate for the sampling distribution of the differences of proportions is given by is $\mathbf{Z} = \frac{(\mathcal{P}_1 - \mathcal{P}_2) - \mu_{\mathcal{P}_1-\mathcal{P}_2}}{\sigma_{\mathcal{P}_1-\mathcal{P}_2}}$

1. Two brands A and B of cables have mean breaking strengths of 4000 and 4500 and standard deviations of 300 and 200 respectively. If 100 cables of brand a and 50 cables of brand B are tested, what is the probability that mean breaking strength of brand will be
   a. at least 600 more than that of A.
   b. at least 450 more more than that of A

   $\mu_{\bar{X}_A} = 4000$, $\mu_{\bar{X}_B} = 4500$, $\sigma_{\bar{X}_A} = 300$, $\sigma_{\bar{X}_B} = 200$, $n_1=100$, $n_2 = 50$

$$\mu_{\bar{X}_B-\bar{X}_A} = \mu_{\bar{X}_B} - \mu_{\bar{X}_A} = 4500 - 4000 = 500$$

$$\sigma_{\bar{X}_B-\bar{X}_A} = \sqrt{\sigma^2{}_{\bar{X}_A} + \sigma^2{}_{\bar{X}_B}} = 41.23$$

$$Z = \frac{(\bar{X}_B-\bar{X}_A) - (\mu_{\bar{X}_B-\bar{X}_A})}{\sigma_{\bar{X}_B-\bar{X}_A}}$$

$$= \frac{(\bar{X}_B-\bar{X}_A) - (\mu_{\bar{X}_B-\bar{X}_A})}{\sigma_{\bar{X}_B-\bar{X}_A}} = \frac{(\bar{X}_B-\bar{X}_A) - 500}{41.23}$$

i) $P((\bar{X}_B - \bar{X}_A) \geq 600) = P(Z \geq 2.43) = P(Z > 0) - \emptyset(2.43) = 0.0075.$

ii) $P((\bar{X}_B - \bar{X}_A) \geq 450) = P(Z \geq -1.21)$

$$= P(-1.21 < Z < 0) + P(Z \geq 0) = \emptyset(1.21) + 0.5$$

= 0.8859

2 . Two friends A and B play a game of "heads and tails" each tossing a coin 50 times. A will win the game if he tosses 3 or more heads than B, Otherwise B wins. Determine the probability that A win.

$\mathcal{P}_A$ and $\mathcal{P}_B$ are the proportions of heads obtained by A and B respectively. The probability of getting a head = p = 0.5 for both A and B.

The number of tosses made by A & B are $n_A = n_B = 50$

$$\mu_{\mathcal{P}_A-\mathcal{P}_b} = \mu_{\mathcal{P}_A} - \mu_{\mathcal{P}_B} = p_A - p_B = 0$$

$$\sigma_{\mathcal{P}_A-\mathcal{P}_B} = \sqrt{\sigma^2{}_{\mathcal{P}_A} + \sigma^2{}_{\mathcal{P}_B}} = \sqrt{\frac{p_A q_A}{n_A} + \frac{p_B q_B}{n_B}} = 0.$$

$$Z = \frac{(\mathcal{P}_A - \mathcal{P}_B) - \mu_{\mathcal{P}_A - \mathcal{P}_B}}{\sigma_{\mathcal{P}_A - \mathcal{P}_B}} = \frac{(\mathcal{P}_A - \mathcal{P}_B)}{0.1} = \frac{(\frac{5}{8})}{0.1} = 0.6$$

$$P((\mathcal{P}_A - \mathcal{P}_B) > 0.6) = P(Z > 0.6) = P(Z > 0) - P(0 < Z < 0.6)$$
$$= 0.5 - \emptyset(0.6) = 0.2742$$

Weibull Distribution

One of the most widely used distributions in reliability engineering is "**Weibull Distribution**". It is a kind of versatile distribution that can take the values from the other distributions using the parameter called the shape parameter.

Weibull Distribution Definition

The **Weibull Distribution** is a continuous probability distribution used to analyse life data, model failure times and access product reliability. It can also fit a huge range of data from many other fields like economics, hydrology, biology, engineering sciences. It is an extreme value of probability distribution which is frequently used to model the reliability, survival, wind speeds and other data.

The only reason to use Weibull distribution is because of its flexibility. Because it can simulate various distributions like normal and exponential distributions. Weibull's distribution reliability is measured with the help of parameters. The two versions of Weibull probability density function(pdf) are

- Two parameter pdf
- Three parameter pdf

Weibull Distribution Formulas

The formula general Weibull Distribution for **three-parameter pdf** is given as

$$f(x) = \begin{cases} \frac{\alpha}{\beta}\left(\frac{x-\mu}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x-\mu}{\beta}\right)^{\alpha}}, & x > \mu;\ \alpha, \beta > 0 \\ 0 & otherwise \end{cases}$$

Where,

- α is the **shape parameter**, also called as the Weibull slope or the threshold parameter.
- $\beta$is the **scale parameter**, also called the characteristic life parameter.
- μ is the **location parameter**, also called the waiting time parameter or sometimes the shift parameter.

Standard Weibull Distribution

The standard Weibull distribution is derived, when $\mu = 0$ and $\beta = 1$the formula is reduced and it becomes

$$f(x) = \begin{cases} \alpha(x)^{\alpha-1}e^{-(x)^{\alpha}}, & x > 0;\ \alpha, > 0 \\ 0 & otherwise \end{cases}$$

This is the pdf of standard weibull distribution

And cumulative distribution function is

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

Reliability Function is $R(x) = e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$

Two-Parameter Weibull Distribution

The formula is practically similar to the three parameters Weibull, except that μ isn't included:

$$f(x) = \begin{cases} \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, & x > 0;\ \alpha, \beta > 0 \\ 0 & otherwise \end{cases}$$

The failure rate is determined by the value of the shape parameter α

- If $\alpha < 1$,then the **failure rate decreases with time**
- If $\alpha = 1$, then the **failure rate is constant**
- If $\alpha > 1$, the **failure rate increases with time**

Mean of Two-parameter Weibull Distribution

$$E[x] = \mu = \beta \Gamma\left(\frac{1}{\alpha} + 1\right).$$

Variance of Two-parameter Weibull Distribution

$$\mathrm{V(X)} = \beta^2 [\Gamma\left(\frac{2}{\alpha} + 1\right) - \left(\Gamma\left(\frac{1}{\alpha} + 1\right)\right)^{\mathbf{2}}]$$

Weibull Distribution Reliability

The Weibull distribution is mostly used in reliability analysis and life data analysis because of its ability to adapt to different situations. Depending upon the parameter values, this distribution is used to model the variety of behaviours for a particular function.
The probability density function usually describes the distribution function. The parameters in the distribution control the shape, scale and location of the probability density function. Several methods are used to measure the reliability of the data. But the Weibull distribution method is one of the best methods to analyse life data.

1. The lifetime X (in hundreds of hours) of a certain type of vacuum tube has a Weibull distribution with parameters $\alpha = 2$ and β=3. Compute the following:

   $(i) E[X] and\ V[X]$ $\quad (\ ii) P(X \leq 6)$ $\quad (iii) P(1.8 \leq X \leq 6)$ $\quad (iv) P(X \geq 3)$

Solution : Let X denote the lifetime (in hundreds of hours) of vacuum tube. Give that $X \sim W(\alpha, \beta)\ where\ \alpha = 2\ and\ \beta = 3$

Using above formula of two parameter weibull distribution

The probability density function of X is

$$f(x: \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha - 1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} x > 0;\ \alpha, \beta > 0$$

The distribution function of X is

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$$

Mean and variance of X

$$E[x] = \beta \Gamma\left(\frac{1}{\alpha} + 1\right).$$

$$=3\,\Gamma\left(\frac{1}{2}+1\right)=3\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\sqrt{\pi}$$

$$=2.6587$$

$$V(X)=\beta^2[\Gamma\left(\frac{2}{\alpha}+1\right)-\left(\Gamma\left(\frac{1}{\alpha}+1\right)\right)^{\mathbf{2}}]$$

$$V(X)=3^2[\Gamma\left(\frac{2}{2}+1\right)-\left(\Gamma\left(\frac{1}{2}+1\right)\right)^{\mathbf{2}}]$$

$$V(X)=9[\Gamma(2)-\left(\Gamma\left(\frac{3}{2}\right)\right)^{\mathbf{2}}]$$

$$V(X)=9[\mathbf{1}-\left(\frac{\sqrt{\pi}}{\mathbf{2}}\right)^{\mathbf{2}}]=1.9318$$

ii) $P(X\le 6)=F(6)=1-e^{-\left(\frac{6}{3}\right)^2}=1-e^{-4}=0.9817$

$(iii)P(1.8\le X\le 6)=F(6)-F(1.8)$

$$=\left[1-e^{-\left(\frac{6}{3}\right)^2}\right]-[1-e^{-\left(\frac{1.8}{3}\right)^2}]$$

$$=e^{-(0.6)^2}-e^{-(2)^2}=e^{-(0.36)}-e^{-(4)}=0.6977-0.0183=0.6794$$

$(iv)P(X\ge 3)=1-P(X<3)=1-F(3)$

$$=1-\left[1-e^{-\left(\frac{3}{3}\right)^2}\right]=e^{-(1)^2}=0.3679$$

2. Weibull distribution describes failure times of mechanical component .value of scale parameter $\beta$is 1000r and value of shape parameter α is 2. Determine the component reliability for a 250 hrs operational period .Assume that the component starts operating at time t=0.

Solution: Reliability of mechanical component is given by $R(x)=e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$

Putting the values of $\beta=1000hr\ \ \alpha=2\ and\ \ x=250hr$

$$R(250)=e^{-\left(\frac{250}{1000}\right)^2}=0.9394$$

Thus the mechanical component for the specified time is 0.9394

The cumulative distribution function is

$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} = 1 - 0.9394 = 0.0606$