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Fundamental Principles of Counting

Enumeration, or counting, may strike one as an obvious process that a student learns when first studying arithmetic. But then, it seems, very little attention is paid to further development in counting as the student turns to "more difficult" areas in mathematics, such as algebra, geometry, trigonometry, and calculus. Consequently, this first chapter should provide some warning about the seriousness and difficulty of "mere" counting.

Enumeration does not end with arithmetic. It also has applications in such areas as coding theory, probability and statistics, and in the analysis of algorithms. Later chapters will offer some specific examples of these applications.

As we enter this fascinating field of mathematics, we shall come upon many problems that are very simple to state but somewhat "sticky" to solve. Thus, be sure to learn and understand the basic formulas — but do *not* rely on them too heavily. For without an analysis of each problem, a mere knowledge of formulas is next to useless. Instead, welcome the challenge to solve unusual problems or those that are different from problems you have encountered in the past. Seek solutions based on your own scrutiny, regardless of whether it reproduces what the author provides. There are often several ways to solve a given problem.

1.1 The Rules of Sum and Product

Our study of discrete and combinatorial mathematics begins with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic principles. We want to develop the ability to "decompose" such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, taking note of the principles being used. This is the approach we shall follow here.

Our first principle of counting can be stated as follows:

The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Note that when we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

EXAMPLE 1.1

A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one or the other of these two subjects.

EXAMPLE 1.2

The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java, and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

EXAMPLE 1.3

The computer science instructor of Example 1.2 has two colleagues. One of these colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If n denotes the maximum number of different books on this topic that this instructor can borrow from them, then $5 \leq n \leq 8$, for here both colleagues *may* own copies of the same textbook(s).

EXAMPLE 1.4

Suppose a university representative is to be chosen either from 200 teaching or 300 non-teaching employees, then there are $200 + 300 = 500$ possible ways to choose this representative.

Extension of Sum Rule: If tasks $T_1, T_2, \ldots, T_m$ can be done in $n_1, n_2, \ldots, n_m$ ways respectively and no two of these tasks can be performed at the same time, then the number of ways to do *one* of these tasks is $n_1 + n_2 + \cdots + n_m$.

EXAMPLE 1.5

If a student can choose a project either 20 from mathematics or 35 from computer science or 15 from engineering, then the student can choose a project in $20 + 35 + 15 = 70$ ways.

The following example introduces our second principle of counting.

EXAMPLE 1.6

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee A on Monday, and then with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in $5 \times 7 = 35$ ways.

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

EXAMPLE 1.7

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways.

EXAMPLE 1.8

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3{,}276{,}000$ different possible plates.

b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6{,}760{,}000$ different license plates are possible.

c) If repetitions are allowed, as in part **(b)**, how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

EXAMPLE 1.9

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a *bit* — that is, one of the *binary digits* 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its *address*. For some computers, such as embedded microcontrollers (as found in the ignition system for an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a *byte*. Using the rule of product, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.

A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" (such as the PICmicro microcontroller) contain thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there are $256 \times 256 = 2^8 \times 2^8 = 2^{16} = 65{,}536$ available addresses that could be used to identify cells in the main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is presently used in the Pentium[†] processor, where there are as many as $2^8 \times 2^8 \times 2^8 \times 2^8 = 2^{32} = 4{,}294{,}967{,}296$ addresses for use in identifying the cells in main memory. When a programmer deals with the UltraSPARC[‡] or Itanium[§] processors, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises $8 \times 8 = 64$ bits, and there are $2^{64} = 18{,}446{,}744{,}073{,}709{,}551{,}616$ possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

[†] Pentium (R) is a registered trademark of the Intel Corporation.
[‡] The UltraSPARC processor is manufactured by Sun (R) Microsystems, Inc.
[§] Itanium (TM) is a trademark of the Intel Corporation.

At times it is necessary to combine several different counting principles in the solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

EXAMPLE 1.10

At the AWL corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch—either a muffin and a hot beverage or a sandwich and a cold beverage.

By the rule of product, there are $6 \times 2 = 12$ ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are $8 \times 3 = 24$ possibilities for a sandwich and cold beverage. So by the rule of sum, there are $12 + 24 = 36$ ways in which Carl can purchase Ms. Dodd's lunch.

EXAMPLE 1.11

A tourist can travel from Hyderabad to Tirupati in four ways (by plane, train, bus or taxi). He can then travel from Tirupati to Tirumala hills in five ways (by RTC bus, taxi, rope way, motorcycle or walk). Then the tourist can travel from Hyderabad to Tirumala hills in $4 \times 5 = 20$ ways.

Extension of Product Rule: Suppose a procedure consists of performing tasks $T_1, T_2, \ldots, T_m$ in that order. Suppose task T_i can be performed in n_i ways *after* the tasks $T_1, T_2, \ldots, T_{i-1}$ are performed, then the number of ways the procedure can be executed in the designated order is $n_1, n_2, n_3, \ldots, n_m$.

EXAMPLE 1.12

"Charmas" brand shirt available in 12 colors, has a male and female version. It comes in four sizes for each sex, comes in three makes of economy, standard and luxury. Then the number of different types of shirts produced are $12 \times 2 \times 4 \times 3 = 288$.

EXAMPLE 1.13

Application of both Product and Sum Rule: A hotel offers 12 kinds of sweets, 10 kinds of hot tiffins and 5 kinds of beverages (hot tea, hot coffee, juice, coke, ice cream). The breakfast consists of a sweet and a hot beverage or a hot tiffin and a cold beverage. Then the number of ways the breakfast can be ordered is $12 \times 2 + 10 \times 3 = 24 + 30 = 54$ ways.

1.2 Permutations

Continuing to examine applications of the rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called *permutations* when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

EXAMPLE 1.14

In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is *arrangement*, which designates the importance of *order*. If A, B, C, . . . , I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are three such different arrangements, even though the first two involve the same five students.

To answer this question, we consider the positions and possible numbers of students we can choose in order to fill each position. The filling of a position is a stage of our procedure.

10	$\times$	9	$\times$	8	$\times$	7	$\times$	6
1st position		**2nd position**		**3rd position**		**4th position**		**5th position**

Each of the 10 students can occupy the 1st position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the 2nd position. Continuing in this way, we find only six students to select from in order to fill the 5th and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled from right to left—namely, $6 \times 7 \times 8 \times 9 \times 10$. If the 3rd position is filled first, the 1st position second, the 4th position third, the 5th position fourth, and the 2nd position fifth, then the answer is $9 \times 6 \times 10 \times 8 \times 7$, still the same value, 30,240.

As in Example 1.14, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

Definition 1.1 For an integer $n \geq 0$, *n factorial* (denoted $n!$) is defined by

$$0! = 1,$$
$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \quad \text{for} \quad n \geq 1.$$

One finds that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. In addition, for each $n \geq 0$, $(n+1)! = (n+1)(n!)$.

Before we proceed any further, let us try to get a somewhat better appreciation for how fast $n!$ grows. We can calculate that $10! = 3{,}628{,}800$, and it just so happens that this is exactly the number of *seconds* in six *weeks*. Consequently, 11! exceeds the number of seconds in one *year*, 12! exceeds the number in 12 years, and 13! surpasses the number of seconds in a *century*.

If we make use of the factorial notation, the answer in Example 1.14 can be expressed in the following more compact form:

$$10 \times 9 \times 8 \times 7 \times 6 = 10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{5!}.$$

Definition 1.2 Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

Starting with the letters a, b, c, there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size-2 permutations: ab, ba, ac, ca, bc, cb.

If there are n distinct objects and r is an integer, with $1 \le r \le n$, then by the rule of product, the number of permutations of size r for the n objects are

$$P(n, r) = \underset{\text{1st position}}{n} \times \underset{\text{2nd position}}{(n-1)} \times \underset{\text{3rd position}}{(n-2)} \times \cdots \times \underset{r\text{th position}}{(n-r+1)}$$

$$= (n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-r)!}.$$

For $r = 0$, $P(n, 0) = 1 = n!/(n-0)!$, so $P(n, r) = n!/(n-r)!$ holds for all $0 \le r \le n$. A special case of this result is Example 1.14, where $n = 10$, $r = 5$, and $P(10, 5) = 30{,}240$. When permuting all of the n objects in the collection, we have $r = n$ and find that $P(n, n) = n!/0! = n!$.

Note, for example, that if $n \ge 2$, then $P(n, 2) = n!/(n-2)! = n(n-1)$. When $n > 3$ one finds that $P(n, n-3) = n!/[n-(n-3)]! = n!/3! = (n)(n-1)(n-2)\cdots(5)(4)$.

The number of permutations of size r, where $0 \le r \le n$, from a collection of n objects, is $P(n, r) = n!/(n-r)!$. (Remember that $P(n, r)$ counts (linear) arrangements in which the objects *cannot* be repeated.) However, if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \ge 0$.

EXAMPLE 1.15 The number of *words* of three distinct letters formed from the letters of the word "JNTU" is $P(4, 3) = 4!/(4-3)! = 24$. If repetitions are allowed, the number of possible six-letter sequence is $4^6 = 4096$.

EXAMPLE 1.16 In how many ways can eight men and eight women be seated in a row if **(a)** any person may sit next to any other **(b)** men and women must occupy alternate seats **(c)** generalize this result for n men and n women.

Here eight men and eight women are 16 indistinguishable objects.

a) The number of permutations 16 chosen from 16 objects is $P(16,16) = 16! = 20922789890000$.

b) Here men and women are distinct (different)

i)

M	W	M	W	M	W	M	W	M	W	M	W	M	W	M	W
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Man sitting first: the number of ways is $8!\,8!$

ii)

W	M	W	M	W	M	W	M	W	M	W	M	W	M	W	M
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Woman sitting first: $8!\,8!$

Thus the number of ways men and women occupy alternatively is $8!\,8! + 8!\,8! = 2(8!)^2$

c) Any person may sit: $(2n)!$

Men and women sit alternatively: $2(n!)^2$

EXAMPLE 1.17

A committee of eight is to be formed from 16 men and 10 women. In how many ways can the committee be formed if **(a)** there are no restrictions **(b)** there must be 4 men and 4 women **(c)** there should be an even number of women **(d)** more women than men **(e)** at least 6 men.

a) No distinction between men and women. Problem is to choose 8 out of a set of 26 persons. So the number of ways 8 are chosen out of 26 is $C(26, 8) = 26!/8!(18!) = 2.480721325 \times 10^{17}$. 15 62275

b) First stage choose 4 men out of 16 given by $C(16, 4)$. Second stage choose 4 women out of 10 in $C(10, 4)$ ways. Using product rule, the number of ways in which the committee consisting of 4 men and 4 women is $C(16, 4)C(10, 4) = 1{,}820 \times 210 = 382{,}200$.

c) If $2i$ even number of women are chosen, then the remaining $8 - 2i$ members of the committee should be men. By product rule, $C(10, 2i)C(16, 8 - 2i)$. Then the total number of ways is

$$\sum_{i=1}^{4} \binom{10}{2i}\binom{16}{8-2i}$$

d) Since the strength of the committee is 8, there should be 5 or more women so that women outnumber men. Using product rule, the number of ways is

$$\sum_{i=5}^{8} \binom{10}{i}\binom{16}{8-i}$$

e) When the number of men is 6 or more we get by a similar argument, the number of ways as

$$\sum_{i=6}^{8} \binom{16}{i}\binom{10}{8-i}$$

EXAMPLE 1.18

The number of permutations of the letters in the word COMPUTER is $8!$. If only five of the letters are used, the number of permutations (of size 5) is $P(8, 5) = 8!/(8-5)! = 8!/3! = 6720$. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} \doteq 6.872 \times 10^{10}$.†

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†The symbol "$\doteq$" is read "is approximately equal to."

Table 1.1

(a)				(b)							
A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

EXAMPLE 1.19

Unlike Example 1.18, the number of (linear) arrangements of the four letters in BALL is 12, not $4!$ $(= 24)$. The reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in Table 1.1(a).

If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have $4! = 24$ permutations. These are listed in Table 1.1(b). Table 1.1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a *pair* of permutations with distinct L's. Consequently,

$$2 \times (\text{Number of arrangements of the letters B, A, L, L})$$
$$= (\text{Number of permutations of the symbols B, A, } L_1, L_2),$$

and the answer to the original problem of finding all the arrangements of the four letters in BALL is $4!/2 = 12$.

EXAMPLE 1.20

Using the idea developed in Example 1.19, we now consider the arrangements of all nine letters in DATABASES.

There are $3! = 6$ arrangements with the A's distinguished for each arrangement in which the A's are not distinguished. For example, $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, and $DA_3TA_2BA_1SES$ all correspond to DATABASES, when we remove the subscripts on the A's. In addition, to the arrangement $DA_1TA_2BA_3SES$ there corresponds the pair of permutations $DA_1TA_2BA_3S_1ES_2$ and $DA_1TA_2BA_3S_2ES_1$, when the S's are distinguished. Consequently,

$$(2!)(3!)(\text{Number of arrangements of the letters in DATABASES})$$
$$= (\text{Number of permutations of the symbols D, } A_1, \text{T, } A_2, \text{B, } A_3, S_1, \text{E, } S_2),$$

so the number of arrangements of the nine letters in DATABASES is $9!/(2!\,3!) = 30{,}240$.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinatorial formulas.

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, . . . , and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \cdots + n_r = n$, then there are $\frac{n!}{n_1!\,n_2!\cdots n_r!}$ (linear) arrangements of the given n objects.

EXAMPLE 1.21

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

possible arrangements. Among these are

$$\frac{7!}{3!\,1!\,1!\,1!\,1!} = 840$$

in which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

EXAMPLE 1.22

Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these paths.

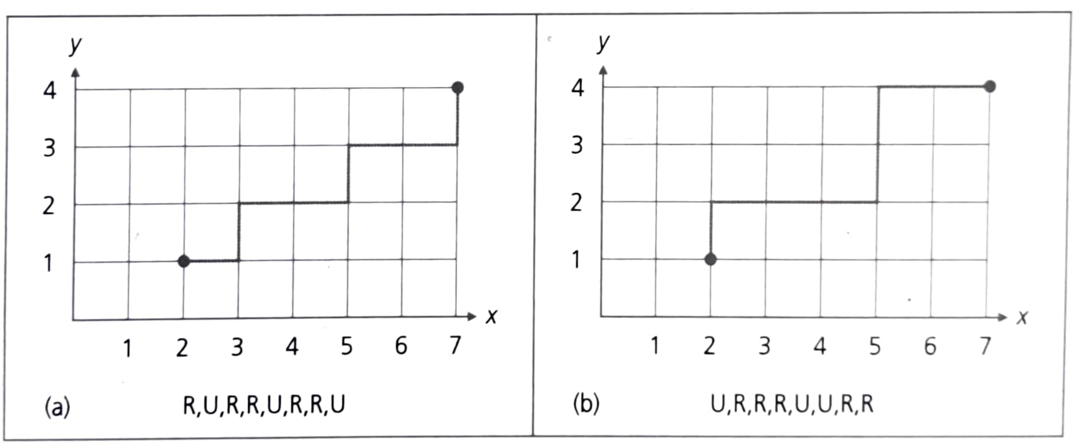


Figure 1.1

Beneath each path in Fig. 1.1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. In general, the overall trip from (2, 1) to (7, 4) requires $7 - 2 = 5$ horizontal moves to the right and $4 - 1 = 3$ vertical moves upward. Consequently, each path corresponds to a list of five R's and three U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is $8!/(5!\,3!) = 56$.

EXAMPLE 1.23

We now do something a bit more abstract and prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer. Because our argument relies on counting, it is an example of a *combinatorial proof*.

Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$. The number of ways in which we can arrange all of these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!\,2!\cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

Finally, we will apply what has been developed so far to a situation in which the arrangements are no longer linear.

EXAMPLE 1.24

If six people, designated as A, B, . . . , F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 1.2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]

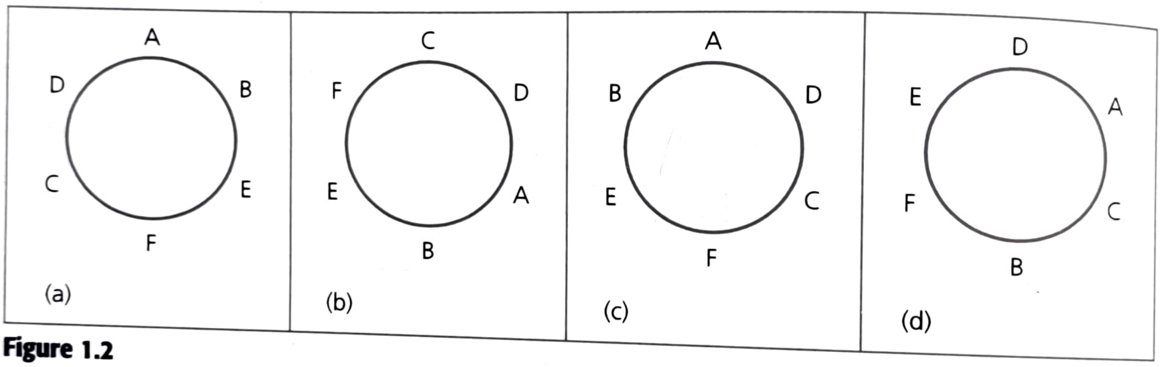


Figure 1.2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2(a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements — BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements, we have $6 \times$ (Number of circular arrangements of A, B, . . . , F) = (Number of linear arrangements of A, B, . . . , F) = 6!.

Consequently, there are $6!/6 = 5! = 120$ arrangements of A, B, . . . , F around the circular table.

EXAMPLE 1.25

Suppose now that the six people of Example 1.24 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.24 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Using B, C, . . . , F to fill

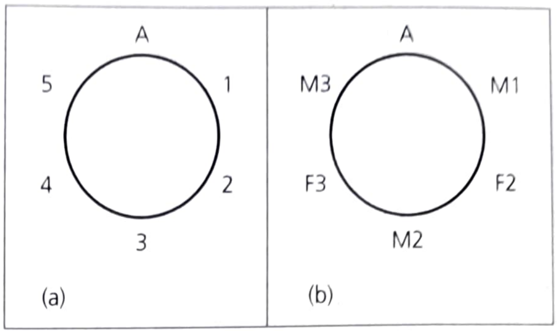


Figure 1.3

these five positions is the problem of permuting B, C, . . . , F in a linear manner, and this can be done in $5! = 120$ ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways in which these six people can be arranged with no two men or women seated next to each other.

EXERCISES 1.1 AND 1.2

1. During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board.

a) If the president is to be one of these candidates, how many possibilities are there for the eventual winner?

b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

c) Which counting principle is used in part **(a)**? in part **(b)**?

2. Answer part **(c)** of Example 1.8.

3. Buick automobiles come in four models, 12 colors, three engine sizes, and two transmission types. **(a)** How many distinct Buicks can be manufactured? **(b)** If one of the available colors is blue, how many different blue Buicks can be manufactured?

4. The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members).

a) How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?

b) Three members of the board of directors are physicians. How many slates from part **(a)** have **(i)** a physician nominated for the presidency? **(ii)** exactly one physician appearing on the slate? **(iii)** at least one physician appearing on the slate?

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the get-away car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

6. To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a \$5 entrance fee and has a chance to win one of the different-sized trophies that are to be awarded to the first eight runners who finish.

a) If 30 people enter the race, in how many ways will it be possible to award the trophies?

b) If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

7. A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?

8. Matthew works as a computer operator at a small university. One evening he finds that 12 computer programs have been submitted earlier that day for batch processing. In how many ways can Matthew order the processing of these programs if **(a)** there are no restrictions? **(b)** he considers four of the programs higher in priority than the other eight and wants to process those four first? **(c)** he first separates the programs into four of top priority, five of lesser priority, and three of least priority, and he wishes to process the 12 programs in such a way that the top-priority programs are processed first and the three programs of least priority are processed last?

9. Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium, or large containers of the following beverages: coffee (black, with cream, with sugar, or with cream and sugar), tea (plain, with cream, with sugar, with cream and sugar, with lemon, or with lemon and sugar), hot cocoa, and orange juice. When Carol comes to Patter's, in how many ways can she order

a) one bakery item and one medium-sized beverage for herself?

b) one bakery item and one container of coffee for herself and one muffin and one container of tea for her boss, Ms. Didio?

c) one piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

10. Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

11. Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown in Fig. 1.4.

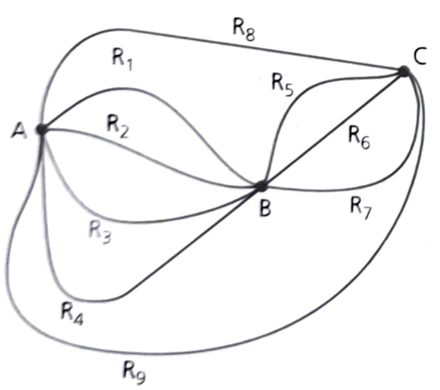


Figure 1.4

a) In how many ways can Linda travel from town A to town C?

b) How many different round trips can Linda travel from town A to town C and back to town A?

c) How many of the round trips in part **(b)** are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.)

12. List all the permutations for the letters a, c, t.

13. a) How many permutations are there for the eight letters a, c, f, g, i, t, w, x?

b) Consider the permutations in part **(a)**. How many start with the letter t? How many start with the letter t and end with the letter c?

14. Evaluate each of the following.

a) $P(7, 2)$ **b)** $P(8, 4)$ **c)** $P(10, 7)$ **d)** $P(12, 3)$

15. In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

16. An alphabet of 40 symbols is used for transmitting messages in a communication system. How many distinct messages (lists of symbols) of 25 symbols can the transmitter generate if symbols can be repeated in the message? How many if 10 of the 40 symbols can appear only as the first and/or last symbols of the message, the other 30 symbols can appear anywhere, and repetitions of all symbols are allowed?

17. In the Internet each network interface of a computer is assigned one, or more, Internet addresses. The nature of these Internet addresses is dependent on network size. For the Internet Standard regarding reserved network numbers (STD 2), each address is a 32-bit string which falls into one of the following three classes. **(1)** A class A address, used for the largest networks, begins with a 0 which is then followed by a seven-bit *network number*, and then a 24-bit *local address*. However, one is restricted from using the network numbers of all 0's or all 1's and the local addresses of all 0's or all 1's. **(2)** The class B address is meant for an intermediate-sized network. This address starts with the two-bit string 10, which is followed by a 14-bit network number and then a 16-bit local address. But the local addresses of all 0's or all 1's are not permitted. **(3)** Class C addresses are used for the smallest networks. These addresses consist of the three-bit string 110, followed by a 21-bit network number, and then an eight-bit local address. Once again the local addresses of all 0's or all 1's are excluded. How many different addresses of each class are available on the Internet, for this Internet Standard?

18. Morgan is considering the purchase of a low-end computer system. After some careful investigating, she finds that there are seven basic systems (each consisting of a monitor, CPU, keyboard, and mouse) that meet her requirements. Furthermore, she also plans to buy one of four modems, one of three CD ROM drives, and one of six printers. (Here each peripheral device of a given type, such as the modem, is compatible with all seven basic systems.) In how many ways can Morgan configure her low-end computer system?

19. A computer science professor has seven different programming books on a bookshelf. Three of the books deal with C++, the other four with Java. In how many ways can the professor arrange these books on the shelf **(a)** if there are no restrictions? **(b)** if the languages should alternate? **(c)** if all the C++ books must be next to each other? **(d)** if all the C++ books must be next to each other and all the Java books must be next to each other?

20. Over the Internet, data are transmitted in structured blocks of bits called *datagrams*.

a) In how many ways can the letters in DATAGRAM be arranged?

b) For the arrangements of part **(a)**, how many have all three A's together?

21. a) How many arrangements are there of all the letters in SOCIOLOGICAL?

b) In how many of the arrangements in part **(a)** are A and G adjacent?

c) In how many of the arrangements in part **(a)** are all the vowels adjacent?

22. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

23. Twelve clay targets (identical in shape) are arranged in four hanging columns, as shown in Fig. 1.5. There are four red targets in the first column, three white ones in the second column, two green targets in the third column, and three blue ones in the fourth column. To join her college drill team, Deborah must break all 12 of these targets (using her pistol and only 12 bullets) and in so doing must always break the existing target at the bottom of a column. Under these conditions, in how many different orders can Deborah shoot down (and break) the 12 targets?

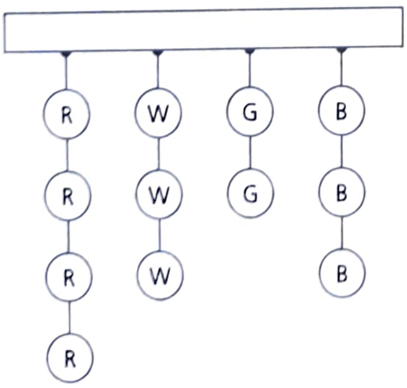


Figure 1.5

24. Show that for all integers $n, r \geq 0$, if $n + 1 > r$, then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r).$$

25. Find the value(s) of n in each of the following: **(a)** $P(n, 2) = 90$, **(b)** $P(n, 3) = 3P(n, 2)$, and **(c)** $2P(n, 2) + 50 = P(2n, 2)$.

26. How many different paths in the xy-plane are there from (0, 0) to (7, 7) if a path proceeds one step at a time by going either one space to the right (R) or one space upward (U)? How many such paths are there from (2, 7) to (9, 14)? Can any general statement be made that incorporates these two results?

27. a) How many distinct paths are there from $(-1, 2, 0)$ to $(1, 3, 7)$ in Euclidean three-space if each move is one of the following types?

$$(\text{H}): (x, y, z) \rightarrow (x+1, y, z);$$
$$(\text{V}): (x, y, z) \rightarrow (x, y+1, z);$$
$$(\text{A}): (x, y, z) \rightarrow (x, y, z+1)$$

b) How many such paths are there from $(1, 0, 5)$ to $(8, 1, 7)$?

c) Generalize the results in parts **(a)** and **(b)**.

28. a) Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
   counter := counter + 1
for j := 5 to 10 do
   counter := counter + 2
for k := 15 downto 8 do
   counter := counter + 3
```

b) Which counting principle is at play in part **(a)**?

29. Consider the following program segment where i, j, and k are integer variables.

```
for i := 1 to 12 do
   for j := 5 to 10 do
      for k := 15 downto 8 do
         print (i - j)*k
```

a) How many times is the **print** statement executed?

b) Which counting principle is used in part **(a)**?

30. A sequence of letters of the form abcba, where the expression is unchanged upon reversing order, is an example of a *palindrome* (of five letters). **(a)** If a letter may appear more than twice, how many palindromes of five letters are there? of six letters? **(b)** Repeat part **(a)** under the condition that no letter appears more than twice.

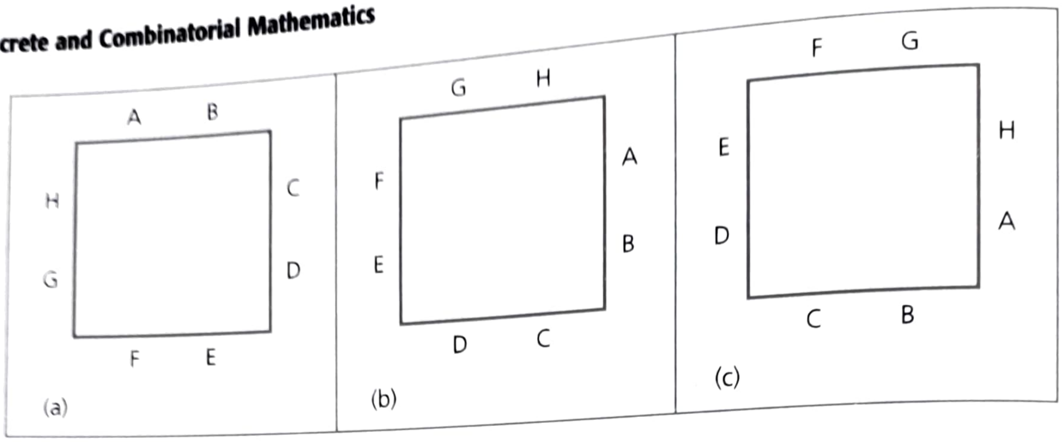


Figure 1.6

31. Determine the number of six-digit integers (no leading zeros) in which **(a)** no digit may be repeated; **(b)** digits may be repeated. Answer parts **(a)** and **(b)** with the extra condition that the six-digit integer is **(i)** even; **(ii)** divisible by 5; **(iii)** divisible by 4.

32. a) Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $n!/(3!)^k$ is an integer.

b) Generalize the result of part **(a)**.

33. a) In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part **(a)** if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

34. How many distinct four-digit integers can one make from the digits 1, 3, 3, 7, 7, and 8?

35. a) In how many ways can seven people be arranged about a circular table?

b) If two of the people insist on sitting next to each other, how many arrangements are possible?

36. a) In how many ways can eight people, denoted A, B, . . . , H be seated about the square table shown in Fig. 1.6, where Figs. 1.6(a) and 1.6(b) are considered the same but are distinct from Fig. 1.6(c)?

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other?

37. Sixteen people are to be seated at two circular tables, one of which seats 10 while the other seats six. How many different seating arrangements are possible?

38. A committee of 15 — nine women and six men — is to be seated at a circular table (with 15 seats). In how many ways can the seats be assigned so that no two men are seated next to each other?

39. Write a computer program (or develop an algorithm) to determine whether there is a three-digit integer abc $(= 100a + 10b + c)$ where $abc = a! + b! + c!$.

1.3 Combinations: The Binomial Theorem

The standard deck of playing cards consists of 52 cards comprising four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards: ace, 2, 3, . . . , 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine of clubs), KD (king of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer important, then the six permutations AH–9C–KD, AH–KD–9C, 9C–AH–KD, 9C–KD–AH, KD–9C–AH, and KD–AH–9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, *with no reference to order*, corresponds to 3! permutations of three cards. In equation form this translates into

$$\begin{aligned}(3!) \times (\text{Number of selections of size 3 from a deck of 52}) &\\ &= \text{Number of permutations of size 3 for the 52 cards}\\ &= P(52, 3) = \frac{52!}{49!}.\end{aligned}$$

Consequently, three cards can be drawn, without replacement, from a standard deck in $52!/(3!\,49!) = 22{,}100$ ways.

If we start with n distinct objects, each *selection*, or *combination*, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \le r \le n.$$

In addition to $C(n, r)$ the symbol $\binom{n}{r}$ is also frequently used. Both $C(n, r)$ and $\binom{n}{r}$ are sometimes read "n choose r." Note that for all $n \ge 0$, $C(n, 0) = C(n, n) = 1$. Further, for all $n \ge 1$, $C(n, 1) = C(n, n-1) = n$. When $0 \le n < r$, then $C(n, r) = \binom{n}{r} = 0$.

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem. When order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

EXAMPLE 1.26

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in $C(20, 11) = \binom{20}{11} = 20!/(11!\,9!) = 167{,}960$ ways. However, once the 11 arrive, how she arranges them around her rectangular dining table is an arrangement problem. Unfortunately, no part of the theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

EXAMPLE 1.27

Lynn and Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the five numbers (between 1 and 49 inclusive). This she can do in $\binom{49}{5}$ ways (since matching does *not* involve order). Meanwhile Patti selects the powerball — here there are $\binom{42}{1}$ possibilities. Consequently, by the rule of product, Lynn and Patti can select the six numbers for their PowerBall ticket in $\binom{49}{5}\binom{42}{1} = 80{,}089{,}128$ ways.

EXAMPLE 1.28

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in

$$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways.}$$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3} = 10$ ways, and the other four questions can be selected in $\binom{5}{4} = 5$ ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

i) The student answers three of the first five questions and four of the last five: By the rule of product this can happen in $\binom{5}{3}\binom{5}{4} = 10 \times 5 = 50$ ways, as in part (b).

ii) Four of the first five questions and three of the last five questions are selected by the student: This can come about in $\binom{5}{4}\binom{5}{3} = 5 \times 10 = 50$ ways—again by the rule of product.

iii) The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that this last case can occur in $\binom{5}{5}\binom{5}{2} = 1 \times 10 = 10$ ways.

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 50 + 10 = 110$ selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

EXAMPLE 1.29

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in $\binom{53}{9} = 4{,}431{,}613{,}550$ ways.

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in $\binom{50}{6} = 15{,}890{,}700$ ways.

c) For a certain tournament the team must comprise four juniors and five seniors. The teacher can select the four juniors in $\binom{28}{4}$ ways. For each of these selections she has $\binom{25}{5}$ ways to choose the five seniors. Consequently, by the rule of product, she can select her team in $\binom{28}{4}\binom{25}{5} = 1{,}087{,}836{,}750$ ways for this particular tournament.

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following example demonstrates this.

EXAMPLE 1.30

The gym teacher of Example 1.29 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four teams? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled in $\binom{36}{9}$ ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \left(\frac{36!}{9!\,27!}\right)\left(\frac{27!}{9!\,18!}\right)\left(\frac{18!}{9!\,9!}\right)\left(\frac{9!}{9!\,0!}\right)$$
$$= \frac{36!}{9!\,9!\,9!\,9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

b) For an alternative solution, consider the 36 students lined up as follows:

1st student	2nd student	3rd student	. . .	35th student	36th student

To select the four teams, we must distribute nine A's, nine B's, nine C's, and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!\,9!\,9!\,9!}, \quad \text{as in part (a).}$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

EXAMPLE 1.31

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow \text{E} \uparrow \text{E} \uparrow \text{S} \uparrow \text{T} \uparrow \text{L} \uparrow \text{L} \uparrow \text{S} \uparrow \text{H} \uparrow$$

Three of these locations can be selected in $\binom{9}{3} = 84$ ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are $5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of a list of $n+1$ terms like $a_m, a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, where m and n are integers and $n \geq 0$. This notation is called the *Sigma notation* because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{m+n} = \sum_{i=m}^{m+n} a_i.$$

Here, the letter i is called the *index* of the summation, and this index accounts for all integers starting with the *lower limit* m and continuing on up to (and including) the *upper limit* $m+n$.

We may use this notation as follows.

1) $\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j$, for there is nothing special about the letter i.

2) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2$, because $0^2 = 0$.

3) $\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + \cdots + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$.

4) $\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34) = 2(7+8+9+10) = 2\sum_{i=7}^{10} i$.

5) $\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$.

6) $\sum_{i=1}^{5} a = a + a + a + a + a = 5a$.

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 1.28 as

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = \sum_{i=3}^{5} \binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4} \binom{5}{7-j}\binom{5}{j}.$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this book.

EXAMPLE 1.32

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called *strings*, made up from a prescribed *alphabet* of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of *length* 2. Among the 27 strings of length 3 are 000, 012, 202, and 110.

In general, if n is any positive integer, then by the rule of product there are 3^n strings of length n for the alphabet 0, 1, and 2. If $x = x_1x_2x_3 \cdots x_n$ is one of these strings, we define the *weight* of x, denoted $\mathrm{wt}(x)$, by $\mathrm{wt}(x) = x_1 + x_2 + x_3 + \cdots + x_n$. For example, $\mathrm{wt}(12) = 3$ and $\mathrm{wt}(22) = 4$ for the case where $n = 2$; $\mathrm{wt}(101) = 2$, $\mathrm{wt}(210) = 3$, and $\mathrm{wt}(222) = 6$ for $n = 3$.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string x contains no 1's, then each of the 10 locations in x can be filled with either 0 or 2, and by the rule of product there are 2^{10} such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}2^8$ strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table 1.2.

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is $2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10} = \sum_{n=0}^{5} \binom{10}{2n}2^{10-2n}$.

Table 1.2

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Often we must be careful of *overcounting*—a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

EXAMPLE 1.33

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

i) ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.

ii) five of spades, seven of spades, ten of spades, seven of diamonds, and the king of diamonds.

iii) two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in the deck that are not clubs. Consequently, she can make her selection in $\binom{39}{5}$ ways.

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were *not* counted in part (**a**). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

of all five-card hands contain at least one club.

c) Can we obtain the result in part **(b)** in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1}\binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

Something here is definitely *wrong*! This answer is larger than that in part **(b)** by more than one million hands. Did we make a mistake in part **(b)**? Or is something wrong with our present reasoning?

For example, suppose that Ellen first selects

the three of clubs

and then selects

the five of clubs,
king of clubs,
seven of hearts, and
jack of spades.

If, however, she first selects

the five of clubs

and then selects

the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,

is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first selects

the king of clubs

and then follows this by selecting

the three of clubs,
five of clubs,
seven of hearts, and
jack of spades

is not different from the other two selections mentioned earlier.

Consequently, this approach is *wrong* because we are overcounting — by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 1.3. From the results in Table 1.3 we see, for example, that there are $\binom{13}{2}\binom{39}{3}$ five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are $\binom{13}{3}\binom{39}{2}$ such hands.

Since no two of the cases in Table 1.3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}$$
$$=\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$
$$=(13)(82,251)+(78)(9139)+(286)(741)+(715)(39)+(1287)(1)$$
$$=2,023,203.$$

Table 1.3

Number of Clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Nonclubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

We shall close this section with three results related to the concept of combinations.

First we note that for integers n, r, with $n \geq r \geq 0$, $\binom{n}{r} = \binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size r from a collection of n distinct objects, the selection process leaves behind $n - r$ objects. Consequently, $\binom{n}{r} = \binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size r (objects chosen) and the selections of size $n - r$ (objects left behind). An example of this correspondence is shown in Table 1.4, where $n = 5$, $r = 2$, and the distinct objects are 1, 2, 3, 4, and 5. This type of correspondence will be more formally defined in Chapter 5 and used in other counting situations.

Our second result is a theorem from our past experience in algebra.

THEOREM 1.1 *The Binomial Theorem.* If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

Before considering the general proof, we examine a special case. If $n = 4$, the coefficient of x^2y^2 in the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)}\ \underset{\text{4th factor}}{(x+y)}$$

is the number of ways in which we can select two x's from the four x's, one of which is available in each factor. (Although the x's are the same in appearance, we distinguish them as the x in the first factor, the x in the second factor, . . . , and the x in the fourth factor. Also, we note that when we select two x's, we use two factors, leaving us with two other factors from which we can select the two y's that are needed.) For example, among the possibilities, we can select (1) x from the first two factors and y from the last two or (2) x from the first and third factors and y from the second and fourth. Table 1.5 summarizes the six possible selections.

Consequently, the coefficient of x^2y^2 in the expansion of $(x + y)^4$ is $\binom{4}{2} = 6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Table 1.4

Selections of Size $r = 2$ (Objects Chosen)		Selections of Size $n - r = 3$ (Objects Left Behind)	
1. 1, 2	6. 2, 4	1. 3, 4, 5	6. 1, 3, 5
2. 1, 3	7. 2, 5	2. 2, 4, 5	7. 1, 3, 4
3. 1, 4	8. 3, 4	3. 2, 3, 5	8. 1, 2, 5
4. 1, 5	9. 3, 5	4. 2, 3, 4	9. 1, 2, 4
5. 2, 3	10. 4, 5	5. 1, 4, 5	10. 1, 2, 3

Table 1.5

Factors Selected for x		Factors Selected for y	
(1)	1, 2	(1)	3, 4
(2)	1, 3	(2)	2, 4
(3)	1, 4	(3)	2, 3
(4)	2, 3	(4)	1, 4
(5)	2, 4	(5)	1, 3
(6)	3, 4	(6)	1, 2

Now we turn to the proof of the general case.

Proof: In the expansion of the product

$$\underset{\text{1st factor}}{(x+y)}\ \underset{\text{2nd factor}}{(x+y)}\ \underset{\text{3rd factor}}{(x+y)} \cdots \underset{n\text{th factor}}{(x+y)}$$

the coefficient of $x^k y^{n-k}$, where $0 \le k \le n$, is the number of different ways in which we can select k x's [and consequently $(n-k)$ y's] from the n available factors. (One way, for example, is to choose x from the first k factors and y from the last $n-k$ factors.) The total number of such selections of size k from a collection of size n is $C(n, k) = \binom{n}{k}$, and from this the binomial theorem follows.

In view of this theorem, $\binom{n}{k}$ is often referred to as a *binomial coefficient*. Notice that it is also possible to express the result of Theorem 1.1 as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}.$$

EXAMPLE 1.34

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and $\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2$.

COROLLARY 1.1 For each integer $n > 0$,

a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$, and

b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$.

Proof: Part **(a)** follows from the binomial theorem when we set $x = y = 1$. When $x = -1$ and $y = 1$, part **(b)** results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem.*

THEOREM 1.2 For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

How many terms will be there in the expansion?

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \cdots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, ..., and x_t from n_t of the last $n - n_1 - n_2 - n_3 - \cdots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.30, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\cdots-n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this product is equal to

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_t!},$$

which is also written as

$$\binom{n}{n_1, n_2, n_3, \ldots, n_t}$$

and is called a *multinomial coefficient.* (When $t = 2$ this reduces to a binomial coefficient.)

EXAMPLE 1.35

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = 35$.

b) Suppose we need to know the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$. If we replace a by v, $2b$ by w, $-3c$ by x, $2d$ by y, and 5 by z, then we can apply the multinomial theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4} = 302{,}702{,}400$. But $\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4 = \binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4(a^2b^3c^2d^5) = 435{,}891{,}456{,}000{,}000\,a^2b^3c^2d^5$.

EXERCISES 1.3

1. Calculate $\binom{6}{2}$ and check your answer by listing all the selections of size 2 that can be made from the letters a, b, c, d, e, and f.

2. Facing a four-hour bus trip back to college, Diane decides to take along five magazines from the 12 that her sister Ann Marie has recently acquired. In how many ways can Diane make her selection?

3. Evaluate each of the following.

a) $C(10, 4)$ **b)** $\binom{12}{7}$ **c)** $C(14, 12)$ **d)** $\binom{15}{10}$

4. In the Braille system a symbol, such as a lowercase letter, punctuation mark, suffix, and so on, is given by raising at least one of the dots in the six-dot arrangement shown in part **(a)** of Fig. 1.7. (The six Braille positions are labeled in this part of the figure.) For example, in part **(b)** of the figure the dots in positions 1 and 4 are raised and this six-dot arrangement represents the letter c. In parts **(c)** and **(d)** of the figure we have the representations for the letters m and t, respectively. The definite article "the" is shown in part **(e)** of the figure, while part **(f)** contains the form for the suffix "ow." Finally, the semicolon, ;, is given by the six-dot arrangement in part **(g)**, where the dots at positions 2 and 3 are raised.

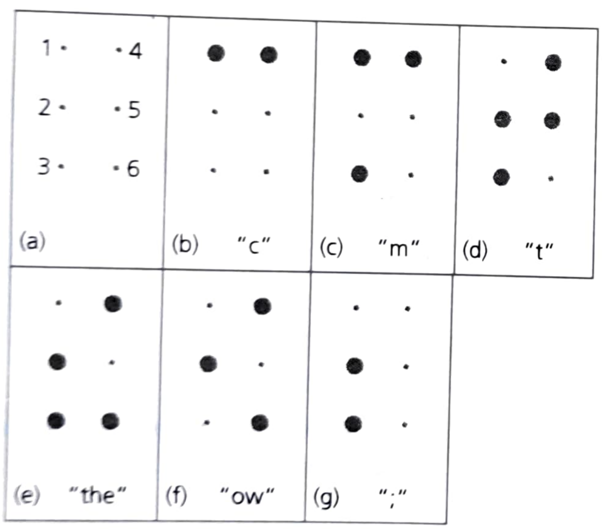


Figure 1.7

a) How many different symbols can we represent in the Braille system?

b) How many symbols have exactly three raised dots?

c) How many symbols have an even number of raised dots?

5. a) How many *permutations* of size 3 can one produce with the letters m, r, a, f, and t?

b) List all the *combinations* of size 3 that result for the letters m, r, a, f, and t.

6. If n is a positive integer and $n > 1$, prove that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

7. A committee of 12 is to be selected from 10 men and 10 women. In how many ways can the selection be carried out if **(a)** there are no restrictions? **(b)** there must be six men and six women? **(c)** there must be an even number of women? **(d)** there must be more women than men? **(e)** there must be at least eight men?

8. In how many ways can a gambler draw five cards from a standard deck and get **(a)** a flush (five cards of the same suit)? **(b)** four aces? **(c)** four of a kind? **(d)** three aces and two jacks? **(e)** three aces and a pair? **(f)** a full house (three of a kind and a pair)? **(g)** three of a kind? **(h)** two pairs?

9. How many bytes contain **(a)** exactly two 1's; **(b)** exactly four 1's; **(c)** exactly six 1's; **(d)** at least six 1's?

10. How many ways are there to pick a five-person basketball team from 12 possible players? How many selections include the weakest and the strongest players?

11. A student is to answer seven out of 10 questions on an examination. In how many ways can he make his selection if **(a)** there are no restrictions? **(b)** he must answer the first two questions? **(c)** he must answer at least four of the first six questions?

12. In how many ways can 12 different books be distributed among four children so that **(a)** each child gets three books? **(b)** the two oldest children get four books each and the two youngest get two books each?

13. How many arrangements of the letters in MISSISSIPPI have no consecutive S's?

14. A gym coach must select 11 seniors to play on a football team. If he can make his selection in 12,376 ways, how many seniors are eligible to play?

15. a) Fifteen points, no three of which are collinear, are given on a plane. How many lines do they determine?

b) Twenty-five points, no four of which are coplanar, are given in space. How many triangles do they determine? How many planes? How many tetrahedra (pyramidlike solids with four triangular faces)?

16. Determine the value of each of the following summations.

a) $\sum_{i=1}^{6}(i^2 + 1)$ **b)** $\sum_{j=-2}^{2}(j^3 - 1)$ **c)** $\sum_{i=0}^{10}[1 + (-1)^i]$

d) $\sum_{k=n}^{2n}(-1)^k$, where n is an odd positive integer

e) $\sum_{i=1}^{6} i(-1)^i$

17. Express each of the following using the summation (or Sigma) notation. In parts **(a)**, **(d)**, and **(e)**, n denotes a positive integer.

a) $\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}, \quad n \geq 2$

b) $1+4+9+16+25+36+49$

c) $1^3-2^3+3^3-4^3+5^3-6^3+7^3$

d) $\frac{1}{n}+\frac{2}{n+1}+\frac{3}{n+2}+\cdots+\frac{n+1}{2n}$

e) $n-\left(\frac{n+1}{2!}\right)+\left(\frac{n+2}{4!}\right)-\left(\frac{n+3}{6!}\right)+\cdots$
$+(-1)^n\left(\frac{2n}{(2n)!}\right)$

18. For the strings of length 10 in Example 1.32, how many have **(a)** four 0's, three 1's, and three 2's; **(b)** at least eight 1's; **(c)** weight 4?

19. Consider the collection of all strings of length 10 made up from the alphabet 0, 1, 2, and 3. How many of these strings have weight 3? How many have weight 4? How many have even weight?

20. In the three parts of Fig. 1.8, eight points are equally spaced and marked on the circumference of a given circle.

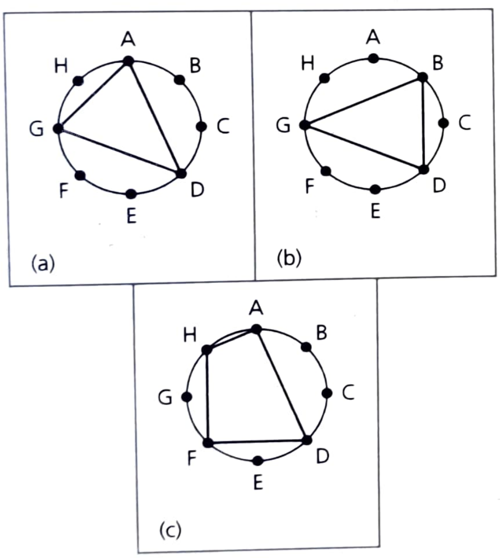


Figure 1.8

a) For parts **(a)** and **(b)** of Fig. 1.8 we have two different (though congruent) triangles. These two triangles (distinguished by their vertices) result from two selections of size 3 from the vertices A, B, C, D, E, F, G, H. How many different (whether congruent or not) triangles can we inscribe in the circle in this way?

b) How many different quadrilaterals can we inscribe in the circle, using the marked vertices? [One such quadrilateral appears in part **(c)** of Fig. 1.8.]

c) How many different polygons of three or more sides can we inscribe in the given circle by using three or more of the marked vertices?

21. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

22. a) In the complete expansion of $(a+b+c+d)\cdot(e+f+g+h)(u+v+w+x+y+z)$ one obtains the sum of terms such as agw, cfx, and dgv. How many such terms appear in this complete expansion?

b) Which of the following terms do *not* appear in the complete expansion from part **(a)**?

i) afx **ii)** bvx **iii)** chz
iv) cgw **v)** egu **vi)** dfz

23. Determine the coefficient of x^9y^3 in the expansions of **(a)** $(x+y)^{12}$, **(b)** $(x+2y)^{12}$, and **(c)** $(2x-3y)^{12}$.

24. Complete the details in the proof of the multinomial theorem.

25. Determine the coefficient of

a) xyz^2 in $(x+y+z)^4$

b) xyz^2 in $(w+x+y+z)^4$

c) xyz^2 in $(2x-y-z)^4$

d) xyz^{-2} in $(x-2y+3z^{-1})^4$

e) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

26. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of **(a)** $(w+x+y+z+1)^{10}$, **(b)** $(2w-x+3y+z-2)^{12}$, and **(c)** $(v+w-2x+y+5z+3)^{12}$.

27. Determine the sum of all the coefficients in the expansions of

a) $(x+y)^3$ **b)** $(x+y)^{10}$ **c)** $(x+y+z)^{10}$

d) $(w+x+y+z)^5$

e) $(2s-3t+5u+6v-11w+3x+2y)^{10}$

28. For any positive integer n determine

a) $\displaystyle\sum_{i=0}^{n}\frac{1}{i!(n-i)!}$ **b)** $\displaystyle\sum_{i=0}^{n}\frac{(-1)^i}{i!(n-i)!}$

29. Show that for all positive integers m and n,

$$n\binom{m+n}{m}=(m+1)\binom{m+n}{m+1}.$$

30. With n a positive integer, evaluate the sum

$$\binom{n}{0}+2\binom{n}{1}+2^2\binom{n}{2}+\cdots+2^k\binom{n}{k}+\cdots+2^n\binom{n}{n}.$$

31. For x a real number and n a positive integer, show that

a) $1=(1+x)^n-\binom{n}{1}x^1(1+x)^{n-1}$
$+\binom{n}{2}x^2(1+x)^{n-2}-\cdots+(-1)^n\binom{n}{n}x^n$

8

The Principle of Inclusion and Exclusion

We now return to the topic of enumeration as we investigate the *Principle of Inclusion and Exclusion*. Extending the ideas in the counting problems on Venn diagrams in Chapter 3, this principle will assist us in establishing the formula we conjectured in Section 5.3 for the number of onto functions $f: A \to B$, where A, B are finite (nonempty) sets. Other applications of this principle will demonstrate its versatile nature in combinatorial mathematics.

8.1 The Principle of Inclusion and Exclusion

In this section we develop some notation for stating this new counting principle. Then we establish the principle by a combinatorial argument. Following this, a wide range of examples demonstrate how this principle may be applied.

We shall motivate the Principle of Inclusion and Exclusion with a series of three examples, the first two of which will be reminiscent of the work we did with counting and Venn diagrams in Section 3.3.

EXAMPLE 8.1

Let S represent the set of 100 students enrolled in the freshman engineering program at Central College. Then $|S| = 100$. Now let c_1, c_2 denote the following conditions (or properties) satisfied by some of the elements of S:

c_1: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Freshman Composition.

c_2: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Economics.

Suppose that 35 of these 100 students are enrolled in Freshman Composition and that 30 of them are enrolled in Introduction to Economics. We shall denote this by

$$N(c_1) = 35 \quad \text{and} \quad N(c_2) = 30.$$

If nine of these 100 students are enrolled in both Freshman Composition and Introduction to Economics then we write $N(c_1c_2) = 9$.

Further, of these 100 students, there are $100 - 35 = 65$ who are *not* taking Freshman Composition. Denoting $|S|$ by N, we can designate this by writing $N(\overline{c}_1) = N - N(c_1)$. In a similar way we designate that there are $N(\overline{c}_2) = N - N(c_2) = 100 - 30 = 70$ of these students who are not taking Introduction to Economics. The number who *are* taking Freshman Composition and who are *not* taking Introduction to Economics is $N(c_1\overline{c}_2) = N(c_1) - N(c_1c_2) = 35 - 9 = 26$. Likewise, of these 100 students, there are $N(\overline{c}_1c_2) = N(c_2) - N(c_1c_2) = 30 - 9 = 21$ who are enrolled in Introduction to Economics but not in Freshman Composition. Of particular interest are those students (from among these 100 freshmen) who are taking neither Freshman Composition nor Introduction to Economics — that is, they are *not* taking Freshman Composition and they are also *not* taking Introduction to Economics. Their number is $N(\overline{c}_1\overline{c}_2)$. And since $N(\overline{c}_1) = N(\overline{c}_1c_2) + N(\overline{c}_1\overline{c}_2)$, we learn that $N(\overline{c}_1\overline{c}_2) = N(\overline{c}_1) - N(\overline{c}_1c_2) = 65 - 21 = 44$.

The preceding observations also demonstrate that

$$\begin{aligned} N(\overline{c}_1\overline{c}_2) &= N(\overline{c}_1) - N(\overline{c}_1c_2) = [N - N(c_1)] - [N(c_2) - N(c_1c_2)] \\ &= N - N(c_1) - N(c_2) + N(c_1c_2) = N - [N(c_1) + N(c_2)] + N(c_1c_2) \\ &= 100 - [35 + 30] + 9 = 44, \quad \text{as we saw above.} \end{aligned}$$

From the Venn diagram in Fig. 8.1, we see that if $N(c_1)$ denotes the number of elements of S in the left-hand circle and $N(c_2)$ denotes the number in the right-hand circle, then $N(c_1c_2)$ is the number of these elements from S in the overlap, while $N(\overline{c}_1\overline{c}_2)$ counts those elements of S that are outside the union of these two circles. Consequently, we see once again — this time from the figure — that

$$N(\overline{c}_1\overline{c}_2) = N - [N(c_1) + N(c_2)] + N(c_1c_2),$$

where the last term is added on because it was eliminated twice in the term $[N(c_1) + N(c_2)]$. (Also, at this point, the reader may wish to look back at the second formula following Example 3.26 to find the same result presented with a different notation.)

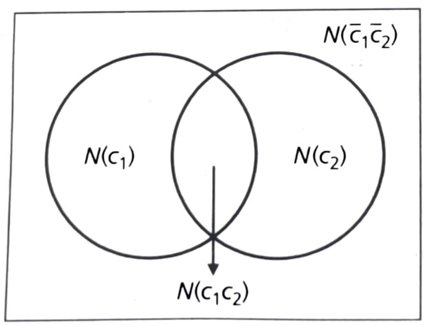


Figure 8.1

[Before we advance to our next example where we will introduce a third condition, let us note that $N(\overline{c}_1\overline{c}_2)$ is *not* the same as $N(\overline{c_1c_2})$. For $N(\overline{c_1c_2}) = N - N(c_1c_2) = 100 - 9 = 91$, in this example, while $N(\overline{c}_1\overline{c}_2) = 44$, as we learned earlier. However, $N(\overline{c}_1 \text{ or } \overline{c}_2) = N(\overline{c_1c_2}) = 91 = 65 + 70 - 44 = N(\overline{c}_1) + N(\overline{c}_2) - N(\overline{c}_1\overline{c}_2)$.]

EXAMPLE 8.2

We start with the same 100 students as in Example 8.1 and the same conditions c_1, c_2, but now we consider a third condition, given as follows:

c_3: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Fundamentals of Computer Programming.

It is still the case that $N(c_1) = 35$, $N(c_2) = 30$, and $N(c_1c_2) = 9$, but now we are also given that $N(c_3) = 30$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$ (that is, there are five of these 100 freshmen who are taking Freshman Composition, Introduction to Economics, and Fundamentals of Computer Programming). Looking to Fig. 8.2, we learn that

$$N(\overline{c}_1\overline{c}_2\overline{c}_3) = N - [N(c_1) + N(c_2) + N(c_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3).$$

So here we have $N(\overline{c}_1\overline{c}_2\overline{c}_3) = 100 - [35 + 30 + 30] + [9 + 11 + 10] - 5 = 30$. That is, out of these 100 students there are 30 who are *not* enrolled in any of the courses: (i) Freshman Composition; (ii) Introduction to Economics; or (iii) Fundamentals of Computer Programming.

[We also learn here that $N(\overline{c}_3) = 70 = 100 - 30 = N - N(c_3)$, $N(\overline{c}_1\overline{c}_3) = 46 = 100 - [35 + 30] + 11 = N - [N(c_1) + N(c_3)] + N(c_1c_3)$, and $N(\overline{c}_2\overline{c}_3) = 50 = 100 - [30 + 30] + 10 = N - [N(c_2) + N(c_3)] + N(c_2c_3)$. Furthermore, we note the similarity here with the result for $|\overline{A} \cap \overline{B} \cap \overline{C}|$ given in the second formula following Example 3.27.]

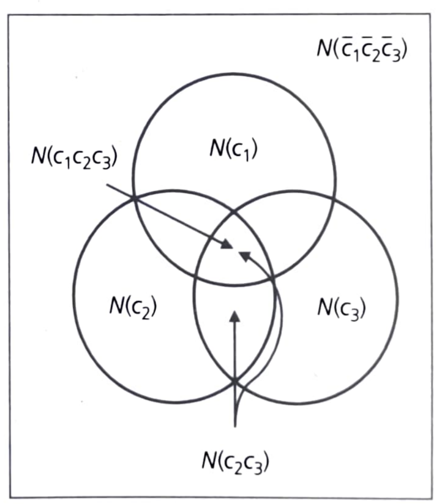


Figure 8.2

EXAMPLE 8.3

Based on the results in the previous two examples we may now feel that for a given finite set S (with $|S| = N$) and four conditions c_1, c_2, c_3, c_4 we should have

$$\begin{aligned} N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = N &- [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \qquad (*) \\ &+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] \\ &+ N(c_1c_2c_3c_4). \end{aligned}$$

To show that this is the case we consider an arbitrary element x from S and show that it is counted the same number of times on both sides of the above equation.

0) If x satisfies none of the four conditions, then it is counted once on the left side of Eq. (*) [in $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$], and once on the right side of Eq. (*) [in N].

1) If x satisfies only one of the conditions, say c_1, then it is not counted at all on the left side of Eq. (*). But on the right side of Eq. (*), x is counted once in N and once in $N(c_1)$, for a total of $1 - 1 = 0$ times.

2) Now suppose that x satisfies conditions c_2, c_4 but does *not* satisfy conditions c_1, c_3. Once again x is not counted on the left side of Eq. (*). For the right side of Eq. (*), x is counted once in N, once in each of $N(c_2)$ and $N(c_4)$, and then once in $N(c_2c_4)$, totaling $1 - [1+1] + 1 = 1 - \binom{2}{1} + \binom{2}{2} = 0$ times.

3) Continuing with the case for three conditions, we'll suppose here that x satisfies conditions c_1, c_2, and c_4, but *not* c_3. As in the previous two cases, x is not counted on the left side of Eq. (*). On the right side of Eq. (*), x is counted once in N, once in each of $N(c_1)$, $N(c_2)$, and $N(c_4)$, once in each of $N(c_1c_2)$, $N(c_1c_4)$, and $N(c_2c_4)$, and, finally, once in $N(c_1c_2c_4)$. So on the right side of Eq. (*), x is counted $1 - [1+1+1] + [1+1+1] - 1 = 1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0$ times, in total.

4) Finally, if x satisfies all four of the conditions c_1, c_2, c_3, c_4, then once again it is not counted on the left side of Eq. (*). On the right side of Eq. (*), x is counted once for each of the 16 terms on the right side of this equation—for a total of $1 - [1+1+1+1] + [1+1+1+1+1+1] - [1+1+1+1] + 1 = 1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$ times.

Consequently, from these preceding five cases we have shown that the two sides of Eq. (*) count the same elements from S, and this provides a combinatorial proof for the formula for $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

So now we shall reconsider the situation in Example 8.2 and introduce a fourth condition as follows:

c_4: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Design.

We already know that $N(c_1) = 35$, $N(c_2) = 30$, $N(c_3) = 30$, $N(c_1c_2) = 9$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$. If $N(c_4) = 41$, $N(c_1c_4) = 13$, $N(c_2c_4) = 14$, $N(c_3c_4) = 10$, $N(c_1c_2c_4) = 6$, $N(c_1c_3c_4) = 6$, $N(c_2c_3c_4) = 6$, and $N(c_1c_2c_3c_4) = 4$, then, using the equation we derived above, it follows that $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = 100 - [35+30+30+41] + [9+11+13+10+14+10] - [5+6+6+6] + 4 = 100 - 136 + 67 - 23 + 4 = 12$. Thus, of the 100 students in the freshman engineering program at Central College, there are 12 who are not taking any of the four courses: Freshman Composition, Introduction to Economics, Fundamentals of Computer Programming, or Introduction to Design.

If we are interested in the number (from these 100 students) who are taking Freshman Composition, but none of the other three courses, then we should want to compute $N(c_1\overline{c}_2\overline{c}_3\overline{c}_4)$. To do so we start by observing that

$$N(\overline{c}_2\overline{c}_3\overline{c}_4) = N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) + N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4),$$

which can be established by an argument similar to the one above for $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$. This then leads us to

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4).$$

Using the result in Example 8.2 we find that

$$\begin{aligned}N(\overline{c}_2\overline{c}_3\overline{c}_4) &= N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &\quad - N(c_2c_3c_4) \\ &= 100 - [30 + 30 + 41] + [10 + 14 + 10] - 6 = 27, \text{ and}\end{aligned}$$

$$N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = 27 - 12 = 15.$$

So there are 15 students in this set of 100 who are taking Freshman Composition, but none of the other courses: Introduction to Economics, Fundamentals of Computer Programming, or Introduction to Design.

Further, we also observe that

$$\begin{aligned}N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) &= N(\overline{c}_2\overline{c}_3\overline{c}_4) - N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) \\ &= \{N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &\quad - N(c_2c_3c_4)\} - \{N - [N(c_1) + N(c_2) + N(c_3) + N(c_4)] \\ &\quad + [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] \\ &\quad - [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\}, \text{ or}\end{aligned}$$

$$\begin{aligned}N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) &= N(c_1) - [N(c_1c_2) + N(c_1c_3) + N(c_1c_4)] \\ &\quad + [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4)] - N(c_1c_2c_3c_4).\end{aligned}$$

So here $N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) = 35 - [9 + 11 + 13] + [5 + 6 + 6] - 4 = 35 - 33 + 17 - 4 = 15$, as we found above.

Having seen the results in Examples 8.1, 8.2, and 8.3, now it is time for us to generalize these results and establish the Principle of Inclusion and Exclusion. To do so we once again let S be a set with $|S| = N$, and we let $c_1, c_2, \ldots, c_t$ be a collection of t conditions or properties—each of which may be satisfied by some of the elements of S. Some elements of S may satisfy more than one of the conditions, whereas others may not satisfy any of them. For all $1 \le i \le t$, $N(c_i)$ will denote the number of elements in S that satisfy condition c_i. (Elements of S are counted here when they satisfy only condition c_i, as well as when they satisfy c_i and other conditions c_j, for $j \ne i$.) For all $i, j \in \{1, 2, 3, \ldots, t\}$ where $i \ne j$, $N(c_ic_j)$ will denote the number of elements in S that satisfy both of the conditions c_i, c_j, and perhaps some others. [$N(c_ic_j)$ does *not* count the elements of S that satisfy *only* c_i, c_j.] Continuing, if $1 \le i, j, k \le t$ are three distinct integers, then $N(c_ic_jc_k)$ denotes the number of elements in S satisfying, perhaps among others, each of the conditions c_i, c_j, and c_k.

For each $1 \le i \le t$, $N(\overline{c}_i) = N - N(c_i)$ denotes the number of elements in S that do not satisfy condition c_i. If $1 \le i, j \le t$ with $i \ne j$, $N(\overline{c}_i\overline{c}_j)$ = the number of elements in S that do not satisfy either of the conditions c_i or c_j. [This is *not* the same as $N(\overline{c_ic_j})$, as we observed at the end of Example 8.1.]

With the necessary preliminaries now in hand we state the following theorem.

THEOREM 8.1 *The Principle of Inclusion and Exclusion.* Consider a set S, with $|S| = N$, and conditions c_i, $1 \le i \le t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy *none* of the conditions c_i, $1 \le i \le t$, is denoted by $\overline{N} = N(\overline{c}_1\overline{c}_2\overline{c}_3 \cdots \overline{c}_t)$ where

$$\begin{aligned}\overline{N} = N &- [N(c_1) + N(c_2) + N(c_3) + \cdots + N(c_t)] \\ &+ [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)] \\ &- [N(c_1c_2c_3) + N(c_1c_2c_4) + \cdots + N(c_1c_2c_t) + N(c_1c_3c_4) + \cdots \\ &+ N(c_1c_3c_t) + \cdots + N(c_{t-2}c_{t-1}c_t)] + \cdots + (-1)^t N(c_1c_2c_3 \cdots c_t),\end{aligned} \tag{1}$$

or

$$\overline{N} = N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_ic_j) - \sum_{1\le i<j<k\le t} N(c_ic_jc_k) + \cdots + (-1)^t N(c_1c_2c_3\cdots c_t). \tag{2}$$

Proof: Although this result can be established by applying the Principle of Mathematical Induction to the number t of conditions, we shall give a combinatorial proof. The argument will be reminiscent of the ideas we saw in Example 8.3 in establishing the formula for $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

For each $x \in S$ we show that x contributes the same count, either 0 or 1, to each side of Eq. (2).

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms in Eq. (2). Consequently, x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies *exactly* r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

1) One time in N.

2) r times in $\sum_{1\le i\le t} N(c_i)$. (Once for each of the r conditions.)

3) $\binom{r}{2}$ times in $\sum_{1\le i<j\le t} N(c_ic_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

4) $\binom{r}{3}$ times in $\sum_{1\le i<j<k\le t} N(c_ic_jc_k)$. (Why?)

. . . .

$(r + 1)$ $\binom{r}{r} = 1$ time in $\sum N(c_{i_1}c_{i_2}\cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r\binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times},$$

by the binomial theorem. Therefore, the two sides of Eq. (2) count the same elements from S, and the equality is verified.

An immediate corollary of this principle is given as follows:

COROLLARY 8.1 Under the hypotheses of Theorem 8.1, the number of elements in S that satisfy at least one of the conditions c_i, where $1 \leq i \leq t$, is given by $N(c_1 \text{ or } c_2 \text{ or } \ldots \text{ or } c_t) = N - \overline{N}$.

Before solving some examples, we examine some further notation for simplifying the statement of Theorem 8.1.

We write

$$S_0 = N,$$
$$S_1 = [N(c_1) + N(c_2) + \cdots + N(c_t)],$$
$$S_2 = [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_t) + N(c_2c_3) + \cdots + N(c_{t-1}c_t)],$$

and, in general,

$$S_k = \sum N(c_{i_1}c_{i_2}\cdots c_{i_k}),\ 1 \leq k \leq t,$$

where the summation is taken over all selections of size k from the collection of t conditions. Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t.$$

Now let us look at how this principle is used to solve certain enumeration problems.

EXAMPLE 8.4 Determine the number of positive integers n where $1 \leq n \leq 100$ and n is *not* divisible by 2, 3, or 5.

Here $S = \{1, 2, 3, \ldots, 100\}$ and $N = 100$. For $n \in S$, n satisfies

a) condition c_1 if n is divisible by 2,

b) condition c_2 if n is divisible by 3, and

c) condition c_3 if n is divisible by 5.

Then the answer to this problem is $N(\overline{c}_1\overline{c}_2\overline{c}_3)$.

As in Section 5.2 we use the notation $\lfloor r \rfloor$ to denote the greatest integer less than or equal to r, for any real number r. This function proves to be helpful in this problem as we find that

$N(c_1) = \lfloor 100/2 \rfloor = 50$ [since the 50 ($= \lfloor 100/2 \rfloor$) positive integers 2, 4, 6, 8, ..., 96, 98 ($= 2 \cdot 49$), 100 ($= 2 \cdot 50$) are divisible by 2];

$N(c_2) = \lfloor 100/3 \rfloor = \lfloor 33\ 1/3 \rfloor = 33$ [since the 33 ($= \lfloor 100/3 \rfloor$) positive integers 3, 6, 9, 12, ..., 96 ($= 3 \cdot 32$), 99 ($= 3 \cdot 33$) are divisible by 3];

$N(c_3) = \lfloor 100/5 \rfloor = 20$;

$N(c_1c_2) = \lfloor 100/6 \rfloor = 16$ [since there are 16 ($= \lfloor 100/6 \rfloor$) elements in S that are divisible by both 2 and 3—hence divisible by lcm(2, 3) $= 2 \cdot 3 = 6$];

$N(c_1c_3) = \lfloor 100/10 \rfloor = 10$;

$N(c_2c_3) = \lfloor 100/15 \rfloor = 6$; and

$N(c_1c_2c_3) = \lfloor 100/30 \rfloor = 3$.

Applying the Principle of Inclusion and Exclusion, we find that

$$\begin{aligned} N(\overline{c}_1\overline{c}_2\overline{c}_3) &= S_0 - S_1 + S_2 - S_3 = N - [N(c_1) + N(c_2) + N(c_3)] \\ &\quad + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3) \\ &= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 = 26. \end{aligned}$$

(These 26 numbers are 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, and 97.)

EXAMPLE 8.5

In Chapter 1 we found the number of nonnegative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 18$. We now answer the same question with the extra restriction that $x_i \le 7$, for all $1 \le i \le 4$.

Here S is the set of solutions of $x_1 + x_2 + x_3 + x_4 = 18$, with $0 \le x_i$ for all $1 \le i \le 4$. So $|S| = N = S_0 = \binom{4+18-1}{18} = \binom{21}{18}$.

We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i, where $1 \le i \le 4$, if $x_i > 7$ (or $x_i \ge 8$). The answer to the problem is then $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

Here by symmetry $N(c_1) = N(c_2) = N(c_3) = N(c_4)$. To compute $N(c_1)$, we consider the integer solutions for $x_1 + x_2 + x_3 + x_4 = 10$, with each $x_i \ge 0$ for all $1 \le i \le 4$. Then we add 8 to the value of x_1 and get the solutions of $x_1 + x_2 + x_3 + x_4 = 18$ that satisfy condition c_1. Hence $N(c_i) = \binom{4+10-1}{10} = \binom{13}{10}$, for each $1 \le i \le 4$, and $S_1 = \binom{4}{1}\binom{13}{10}$.

Likewise, $N(c_1c_2)$ is the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 2$, where $x_i \ge 0$ for all $1 \le i \le 4$. So $N(c_1c_2) = \binom{4+2-1}{2} = \binom{5}{2}$, and $S_2 = \binom{4}{2}\binom{5}{2}$.

Since $N(c_ic_jc_k) = 0$ for every selection of three conditions, and $N(c_1c_2c_3c_4) = 0$, we have

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246.$$

So of the 1330 nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 18$, only 246 of them satisfy $x_i \le 7$ for each $1 \le i \le 4$.

Our next example establishes the formula conjectured in Section 5.3 for counting onto functions.

EXAMPLE 8.6

For finite sets A, B, where $|A| = m \ge n = |B|$, let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$, and $S =$ the set of all functions $f: A \to B$. Then $N = S_0 = |S| = n^m$.

For all $1 \le i \le n$, let c_i denote the condition on S where a function $f: A \to B$ satisfies c_i if b_i is *not* in the range of f. (Note the difference between c_i here and c_i in Examples 8.4 and 8.5.) Then $N(\overline{c}_i)$ is the number of functions in S that have b_i in their range, and $N(\overline{c}_1\overline{c}_2 \cdots \overline{c}_n)$ counts the number of onto functions $f: A \to B$.

For all $1 \le i \le n$, $N(c_i) = (n-1)^m$, because each element of B, except b_i, can be used as the second component of an ordered pair for a function $f: A \to B$, whose range does not include b_i. Likewise, for all $1 \le i < j \le n$, there are $(n-2)^m$ functions $f: A \to B$ whose range contains neither b_i nor b_j. From these observations we have $S_1 = [N(c_1) + N(c_2) + \cdots + N(c_n)] = n(n-1)^m = \binom{n}{1}(n-1)^m$, and $S_2 = [N(c_1c_2) + N(c_1c_3) + \cdots + N(c_1c_n)$

$+ N(c_2c_3) + \cdots + N(c_2c_n) + \cdots + N(c_{n-1}c_n)] = \binom{n}{2}(n-2)^m$. In general, for each $1 \le k \le n$,

$$S_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} N(c_{i_1}c_{i_2}\cdots c_{i_k}) = \binom{n}{k}(n-k)^m.$$

It then follows by the Principle of Inclusion and Exclusion that the number of onto functions from A to B is

$$\begin{aligned} N(\overline{c}_1\overline{c}_2\overline{c}_3\cdots\overline{c}_n) &= S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n \\ &= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m \\ &\quad + \cdots + (-1)^n(n-n)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m. \end{aligned}$$

Before we finish discussing this example, let us note that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m$$

can also be evaluated even if $m < n$. Furthermore, for $m < n$, the expression

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\cdots\overline{c}_n)$$

still counts the number of functions $f\colon A \to B$, where $|A| = m$, $|B| = n$, and each element of B is in the range of f. But now this number is 0.

For example, suppose that $m = 3 < 7 = n$. Then $N(\overline{c}_1\overline{c}_2\overline{c}_3\cdots\overline{c}_7)$ counts the number of onto functions $f\colon A \to B$ for $|A| = 3$ and $|B| = 7$. We know this number is 0, and we also find that

$$\begin{aligned} \sum_{i=0}^{7}(-1)^i\binom{7}{7-i}(7-i)^3 &= \binom{7}{7}7^3 - \binom{7}{6}6^3 + \binom{7}{5}5^3 - \binom{7}{4}4^3 + \binom{7}{3}3^3 - \binom{7}{2}2^3 + \binom{7}{1}1^3 - \binom{7}{0}0^3 \\ &= 343 - 1512 + 2625 - 2240 + 945 - 168 + 7 - 0 = 0. \end{aligned}$$

Hence, for all $m, n \in \mathbf{Z}^+$, if $m < n$, then

$$\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m = 0.$$

We now solve a problem similar to those in Chapter 3 that dealt with Venn diagrams.

EXAMPLE 8.7

In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns *car*, *dog*, *pun*, or *byte* occurs?

Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$ For each $1 \le i \le 4$, a permutation in S is said to satisfy condition c_i if the permutation contains the pattern *car*, *dog*, *pun*, or *byte*, respectively.

In order to compute $N(c_1)$, for example, we count the number of ways the 24 symbols *car*, $b, d, e, f, \ldots, p, q, s, t, \ldots, x, y, z$ can be permuted. So $N(c_1) = 24!$, and in a similar way we obtain

$$N(c_2) = N(c_3) = 24!, \quad \text{while } N(c_4) = 23!$$

For $N(c_1c_2)$ we deal with the 22 symbols *car*, *dog*, $b, e, f, h, i, \ldots, m, n, p, q, s, t, \ldots, x, y, z$, which can be permuted in 22! ways. Hence $N(c_1c_2) = 22!$, and comparable calculations give

$$N(c_1c_3) = N(c_2c_3) = 22!, \qquad N(c_ic_4) = 21!, \quad i \neq 4.$$

Furthermore,

$$N(c_1c_2c_3) = 20!, \qquad N(c_ic_jc_4) = 19!, \qquad 1 \leq i < j \leq 3,$$
$$N(c_1c_2c_3c_4) = 17!$$

So the number of permutations in S that contain none of the given patterns is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$$

Our next example deals with a number theory problem.

EXAMPLE 8.8

For $n \in \mathbf{Z}^+$, $n \geq 2$, let $\phi(n)$ be the number of positive integers m, where $1 \leq m < n$ and $\gcd(m, n) = 1$—that is, m, n are relatively prime. This function is known as *Euler's phi function*, and it arises in several situations in abstract algebra involving enumeration. We find that $\phi(2) = 1$, $\phi(3) = 2$, $\phi(4) = 2$, $\phi(5) = 4$, and $\phi(6) = 2$. For each prime p, $\phi(p) = p - 1$. We would like to derive a formula for $\phi(n)$ that is related to n so that we need not make a case-by-case comparison for each m, $1 \leq m < n$, against the integer n.

The derivation of our formula will use the Principle of Inclusion and Exclusion as in Example 8.4. We proceed as follows: For $n \geq 2$, use the Fundamental Theorem of Arithmetic to write $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, where $p_1, p_2, \ldots, p_t$ are distinct primes and $e_i \geq 1$, for all $1 \leq i \leq t$. We consider the case where $t = 4$. This will be enough to demonstrate the general idea.

With $S = \{1, 2, 3, \ldots, n\}$, we have $N = S_0 = |S| = n$, and for each $1 \leq i \leq 4$ we say that $k \in S$ satisfies condition c_i if k is divisible by p_i. For $1 \leq k < n$, $\gcd(k, n) = 1$ if k is not divisible by any of the primes p_i, where $1 \leq i \leq 4$. Hence $\phi(n) = N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

For each $1 \leq i \leq 4$, we have $N(c_i) = n/p_i$; $N(c_ic_j) = n/(p_ip_j)$, for all $1 \leq i < j \leq 4$. Also, $N(c_ic_jc_\ell) = n/(p_ip_jp_\ell)$, for all $1 \leq i < j < \ell \leq 4$, and $N(c_1c_2c_3c_4) = n/(p_1p_2p_3p_4)$. So

$$\begin{aligned}
\phi(n) &= S_0 - S_1 + S_2 - S_3 + S_4 \\
&= n - \left[\frac{n}{p_1} + \cdots + \frac{n}{p_4}\right] + \left[\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \cdots + \frac{n}{p_3p_4}\right] \\
&\quad - \left[\frac{n}{p_1p_2p_3} + \cdots + \frac{n}{p_2p_3p_4}\right] + \frac{n}{p_1p_2p_3p_4}
\end{aligned}$$

$$= n\left[1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_4}\right) + \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + \cdots + \frac{1}{p_3p_4}\right)\right.$$
$$\left. - \left(\frac{1}{p_1p_2p_3} + \cdots + \frac{1}{p_2p_3p_4}\right) + \frac{1}{p_1p_2p_3p_4}\right]$$
$$= \frac{n}{p_1p_2p_3p_4}[p_1p_2p_3p_4 - (p_2p_3p_4 + p_1p_3p_4 + p_1p_2p_4 + p_1p_2p_3)$$
$$+ (p_3p_4 + p_2p_4 + p_2p_3 + p_1p_4 + p_1p_3 + p_1p_2)$$
$$- (p_4 + p_3 + p_2 + p_1) + 1]$$
$$= \frac{n}{p_1p_2p_3p_4}[(p_1-1)(p_2-1)(p_3-1)(p_4-1)]$$
$$= n\left[\frac{p_1-1}{p_1}\cdot\frac{p_2-1}{p_2}\cdot\frac{p_3-1}{p_3}\cdot\frac{p_4-1}{p_4}\right] = n\prod_{i=1}^{4}\left(1-\frac{1}{p_i}\right).$$

In general, $\phi(n) = n\prod_{p|n}(1-(1/p))$, where the product is taken over all primes p dividing n. When $n = p$, a prime, $\phi(n) = \phi(p) = p\,[1-(1/p)] = p-1$, as we observed earlier. If $n = 23{,}100$, for example, we find that

$$\begin{aligned}\phi(23{,}100) &= \phi(2^2\cdot 3\cdot 5^2\cdot 7\cdot 11)\\ &= (23{,}100)(1-(1/2))(1-(1/3))(1-(1/5))(1-(1/7))(1-(1/11))\\ &= 4800.\end{aligned}$$

The Euler phi function has many interesting properties. We shall investigate some of them in the exercises for this section and in the Supplementary Exercises.

The next example provides another encounter with the circular arrangements introduced in Chapter 1.

EXAMPLE 8.9

Six married couples are to be seated at a circular table. In how many ways can they arrange themselves so that no wife sits next to her husband? (Here, as in Example 1.24, two seating arrangements are considered the same if one is a rotation of the other.)

For $1 \le i \le 6$, we let c_i denote the condition where a seating arrangement has couple i seated next to each other.

To determine $N(c_1)$, for instance, we consider arranging 11 distinct objects — namely, couple 1 (considered as one object) and the other 10 people. Eleven distinct objects can be arranged around a circular table in $(11-1)! = 10!$ ways. However, here $N(c_1) = 2(10!)$, where the 2 takes into account whether the wife in couple 1 is seated to the left or right of her husband. Similarly, $N(c_i) = 2(10!)$, for $2 \le i \le 6$, and $S_1 = \binom{6}{1}2(10!)$.

Continuing, let us now compute $N(c_ic_j)$, for $1 \le i < j \le 6$. Here we are arranging 10 distinct objects — couple i (considered as one object), couple j (likewise considered as one object), and the other eight people. Ten distinct objects can be arranged around a circular table in $(10-1)! = 9!$ ways. So here $N(c_ic_j) = 2^2(9!)$ because there are two ways for the wife in couple i to be seated next to her husband, and two ways for the wife in couple j to be seated next to her husband. Consequently, $S_2 = \binom{6}{2}2^2(9!)$.

Similar reasoning shows us that

$N(c_1c_2c_3) = 2^3(8!)$, $S_3 = \binom{6}{3}2^3(8!)$ $\qquad$ $N(c_1c_2c_3c_4) = 2^4(7!)$, $S_4 = \binom{6}{4}2^4(7!)$

$N(c_1c_2c_3c_4c_5) = 2^5(6!)$, $S_5 = \binom{6}{5}2^5(6!)$ $\qquad$ $N(c_1c_2c_3c_4c_5c_6) = 2^6(5!)$, $S_6 = \binom{6}{6}2^6(5!)$.

With S_0 (the total number of arrangements of the 12 people) $= (12-1)! = 11!$, we find that the number of arrangements where no couple is seated side by side is

$$\begin{aligned} N(\overline{c}_1\overline{c}_2 \cdots \overline{c}_6) &= \sum_{i=0}^{6}(-1)^i S_i = \sum_{i=0}^{6}(-1)^i\binom{6}{i}2^i(11-i)! \\ &= 39{,}916{,}800 - 43{,}545{,}600 + 21{,}772{,}800 - 6{,}451{,}200 \\ &\quad + 1{,}209{,}600 - 138{,}240 + 7680 \\ &= 12{,}771{,}840. \end{aligned}$$

Our final example recalls some of the graph theory we studied in Chapter 7.

EXAMPLE 8.10

In a certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?

Calling the villages a, b, c, d, and e, we seek the number of loop-free undirected graphs on these vertices, where no vertex is isolated. Consequently, we want to count situations such as those illustrated in parts (a) and (b) of Fig. 8.3, but not situations such as those shown in parts (c) and (d).

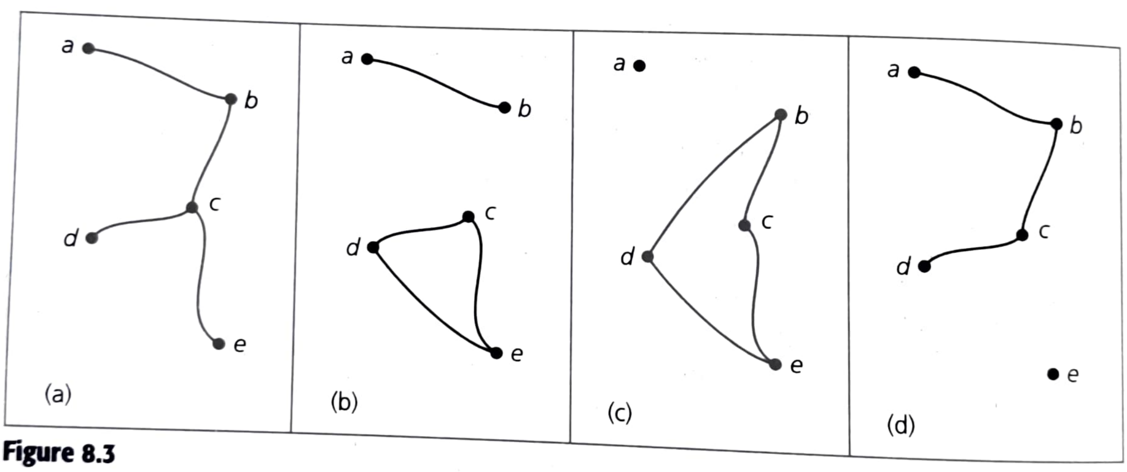


Figure 8.3

Let S be the set of loop-free undirected graphs G on $V = \{a, b, c, d, e\}$. Then $N = S_0 = |S| = 2^{10}$ because there are $\binom{5}{2} = 10$ possible two-way roads for these five villages, and each road can be either included or excluded.

For each $1 \leq i \leq 5$, let c_i be the condition that a system of these roads isolates village a, b, c, d, and e, respectively. Then the answer to the problem is $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5)$.

For condition c_1 village a is isolated, so we consider the six edges (roads) $\{b, c\}$, $\{b, d\}$, $\{b, e\}$, $\{c, d\}$, $\{c, e\}$, $\{d, e\}$. With two choices for each edge—namely, put the edge in the graph or leave the edge out—we find that $N(c_1) = 2^6$. Then by symmetry $N(c_i) = 2^6$ for all $2 \leq i \leq 5$, so $S_1 = \binom{5}{1}2^6$.

When villages a and b are to be isolated, each of the edges $\{c, d\}$, $\{d, e\}$, $\{c, e\}$ may be put in or left out of our graph. This results in 2^3 possibilities, so $N(c_1c_2) = 2^3$, and $S_2 = \binom{5}{2}2^3$.

Similar arguments tell us that $N(c_1c_2c_3) = 2^1$ and $S_3 = \binom{5}{3}2^1$; $N(c_1c_2c_3c_4) = 2^0$ and $S_4 = \binom{5}{4}2^0$; and $N(c_1c_2c_3c_4c_5) = 2^0$ and $S_5 = \binom{5}{5}2^0$.

Consequently,

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5) = 2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768.$$

EXAMPLE 8.11

Find the number of integers between 1 and 10,000 inclusive, which are divisible by none of 5, 6 or 8.

Let P_1 be the property that an integer is divisible by 5, P_2 the property that an integer is divisible by 6, P_3 the property that an integer is divisible by 8. Let A be the set consisting of the first 10,000 integers. Let A_i be the set consisting of those integers in A with property P_i, for $i = 1, 2, 3$. The problem is to find the number of integers in $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$. Now $|A_1| = \lfloor\frac{10,000}{5}\rfloor = 2000$, $|A_2| = \lfloor\frac{10,000}{6}\rfloor = 1666$, $|A_3| = \lfloor\frac{10,000}{8}\rfloor = 1250$. Integers in the set $A_1 \cap A_2$ are divisible by both 5 and 6. Note that an integer is divisible by both 5 and 6 if it is divisible by their lcm $\{5, 6\} = 30$. Also lcm $\{5, 8\} = 40$, lcm $\{6, 8\} = 24$. Then $|A_1 \cap A_2| = \lfloor\frac{10,000}{30}\rfloor = 333$, $|A_1 \cap A_3| = \lfloor\frac{10,000}{40}\rfloor = 250$, $|A_2 \cap A_3| = \lfloor\frac{10,000}{24}\rfloor = 416$. Also $|A_1 \cap A_2 \cap A_3| = \lfloor\frac{10,000}{120}\rfloor = 83$, since lcm $\{5, 6, 8\} = 120$. Now by Principle of Inclusion–Exclusion, the number of integers between 1 and 10,000 that are divisible by none of 5, 6 and 8 equals

$$\begin{aligned}|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |A| - (|A_1| + |A_2| + A_3|) + (|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|)\\ &\quad -|A_1 \cap A_2 \cap A_3|\\ &= 10,000 - (2000 + 1666 + 1250) + (333 + 250 + 416) - 83 = 6000.\end{aligned}$$

EXAMPLE 8.12

Determine the number of permutations of the letters $J, N, U, I, S, G, R, E, A, T$ such that none of the words *JNU IS* and *GREAT* occur as consecutive letters (that is, permutations such as *JNTUISGREAT*, *ISJNUGREAT*, *UNJGREATSI* etc are not allowed).

Let A be the set of all permutations of the 10 letters given. Let P_1 be the property that a permutation in A contains the word *JNU* as consecutive letters, let P_2 be the property that a permutation contains the word *IS* and let P_3 be the property that a permutation contains the word *GREAT*. Let A_i be the set of those permutations in A satisfying the property P_i for $i = 1, 2, 3$. The problem is to find the number of permutations in $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$. Now $|A| = 10! = 3,628,800$. The set A_1 contains the permutations of the 8 symbols *JNU*, I, S, G, R, E, A, T so $|A_1| = 8!$. Similarly A_2 contains permutations of the 9 symbols J, N, U, *IS*, G, R, E, A, T so $|A_2| = 9! = 362,880$. Similarly A_3 contains permutations of the 6 symbols J, N, U, I, S, *GREAT* so $|A_3| = 6! = 720$. Also $|A_1 \cap A_2| = 7! = 5040$, since $A_1 \cap A_2$ contains permutations of the 7 symbols *JNU*, *IS*, G, R, E, A, T. Similarly $|A_1 \cap A_3| = 4! = 24$, since $A_1 \cap A_3$ contains permutations of the 4 symbols *JNU*, I, S, *GREAT*. Also $|A_2 \cap A_3| = 5! = 120$, since $A_2 \cap A_3$ contains permutations of the 5 symbols J, N, U, *IS*, *GREAT*. Finally $|A_1 \cap A_2 \cap A_3| = 3! = 6$, since $A_1 \cap A_2 \cap A_3$ contains the permutations of the three symbols *JNU*, *IS*, *GREAT*. Using the Principle of Inclusion–Exclusion we have

$$\begin{aligned}|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= 3,628,800 - (40,320 + 362,880 + 720)\\ &\quad + (5040 + 24 + 120) - 6\\ &= 3,230,058.\end{aligned}$$

EXAMPLE 8.13

During a 12-week conference of mathematics, the vice-chancellor (V.C.) met his seven friends from college. During the conference, V.C. met each friend at lunch 35 times, every pair of them 16 times, every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once. If he had lunch every day during the 84 days of conference, did he ever have lunch alone.

For $1 \leq i \leq 7$, let c_i denote the situation where the ith friend had lunch with V.C. Then $N = 12$ weeks $\times$ 7 days $= 84$ days $= S_0$. Since V.C. met each friend at lunch 35 times, $N(c_i) = 35$ for any i. Since there are $\binom{7}{1}$ ways of one friend out of seven, we have $S_1 = \binom{7}{1}35$. Also since V.C. met every pair 16 times, we get $S_2 = \binom{7}{2}16$. Similarly $S_3 = \binom{7}{3}8$, $S_4 = \binom{7}{4}4$, $S_5 = \binom{7}{5}2$, $S_6 = \binom{7}{6}1$, $S_7 = \binom{7}{7}0$. Thus by Principle of Inclusion–Exclusion, V.C. not having lunch with any of the seven friends on any of the 84 days

$$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5 + S_6 - S_7$$
$$= 84 - 7(35) + 21(16) - 35(8) + 35(4) - 21(2) + 7.1 - 1.0$$
$$= 0$$

Consequently during the 12-week conference V.C. always had a company (of his friends from college) at lunch on any day.

EXERCISES 8.1

1. Let S be a finite set with $|S| = N$ and let c_1, c_2, c_3, c_4 be four conditions, each of which may be satisfied by one or more of the elements of S. Prove that $N(\overline{c}_2\overline{c}_3\overline{c}_4) = N(c_1\overline{c}_2\overline{c}_3\overline{c}_4) + N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

2. Establish the Principle of Inclusion and Exclusion by applying the Principle of Mathematical Induction to the number t of conditions.

3. Of the 100 students in Example 8.3, how many are taking **(a)** Fundamentals of Computer Programming but none of the other three courses; **(b)** Fundamentals of Computer Programming and Introduction to Economics but neither of the other two courses?

4. Annually, the 65 members of the maintenance staff sponsor a "Christmas in July" picnic for the 400 summer employees at their company. For these 65 people, 21 bring hot dogs, 35 bring fried chicken, 28 bring salads, 32 bring desserts, 13 bring hot dogs and fried chicken, 10 bring hot dogs and salads, 9 bring hot dogs and desserts, 12 bring fried chicken and salads, 17 bring fried chicken and desserts, 14 bring salads and desserts, 4 bring hot dogs, fried chicken, and salads, 6 bring hot dogs, fried chicken, and desserts, 5 bring hot dogs, salads, and desserts, 7 bring fried chicken, salads, and desserts, and 2 bring all four food items. Those (of the 65) who do not bring any of these four food items are responsible for setting up and cleaning up for the picnic. How many of the 65 maintenance staff will **(a)** help to set up and clean up for the picnic? **(b)** bring only hot dogs? **(c)** bring exactly one food item?

5. Determine the number of positive integers n, $1 \leq n \leq 2000$, that are

a) not divisible by 2, 3, or 5

b) not divisible by 2, 3, 5, or 7

c) not divisible by 2, 3, or 5, but are divisible by 7

6. Determine how many integer solutions there are to $x_1 + x_2 + x_3 + x_4 = 19$, if

a) $0 \leq x_i$ for all $1 \leq i \leq 4$

b) $0 \leq x_i < 8$ for all $1 \leq i \leq 4$

c) $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 7$, $3 \leq x_4 \leq 8$

7. In how many ways can one arrange all of the letters in the word INFORMATION so that no pair of consecutive letters occurs more than once? [Here we want to count arrangements such as IINNOOFRMTA and FORTMAIINON but not INFORINMOTA (where "IN" occurs twice) or NORTFNOIAMI (where "NO" occurs twice).]

8. Determine the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 19$ where $-5 \leq x_i \leq 10$ for all $1 \leq i \leq 4$.

9. Determine the number of positive integers x where $x \leq 9{,}999{,}999$ and the sum of the digits in x equals 31.

10. Professor Bailey has just completed writing the final examination for his course in advanced engineering mathematics. This examination has 12 questions, whose total value is to be 200 points. In how many ways can Professor Bailey assign the 200 points if each question must count for at least 10, but not more than 25, points and the point value for each question is to be a multiple of 5?

a) When there are exactly two pairs of consecutive identical letters, the number of arrangements in set that satisfies exactly $m = 2$ of the conditions c_1, c_2, c_3, c_4 is given by

$$\begin{aligned} E_2 &= S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 \\ &= 544{,}320 - 3(80{,}640) + 6(5040) \\ &= 332{,}640. \end{aligned}$$

b) The number of arrangements in set that satisfy at least m, that is, $m \geq 2$ conditions is

$$\begin{aligned} L_2 &= S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 \\ &= 544{,}320 - 2(80{,}640) + 3(5040) \\ &= 398{,}160. \end{aligned}$$

EXERCISES 8.2

1. For the situation in Examples 8.13 and 8.14 compute E_i for $0 \leq i \leq 5$ and show that $\sum_{i=0}^{5} E_i = N = |S|$.

2. a) In how many ways can the letters in ARRANGEMENT be arranged so that there are exactly two pairs of consecutive identical letters? at least two pairs of consecutive identical letters?

   b) Answer part (a), replacing two with three.

3. In how many ways can one arrange the letters in CORRESPONDENTS so that (a) there is no pair of consecutive identical letters? (b) there are exactly two pairs of consecutive identical letters? (c) there are at least three pairs of consecutive identical letters?

4. Let $A = \{1, 2, 3, \ldots, 10\}$, and $B = \{1, 2, 3, \ldots, 7\}$. How many functions $f: A \to B$ satisfy $|f(A)| = 4$? How many have $|f(A)| \leq 4$?

5. In how many ways can one distribute ten distinct prizes among four students with exactly two students getting nothing? How many ways have at least two students getting nothing?

6. Zelma is having a luncheon for herself and nine of the women in her tennis league. On the morning of the luncheon she places name cards at the ten places on her table and then leaves to run a last-minute errand. Her husband, Herbert, comes home from his morning tennis match and unfortunately leaves the back door open. A gust of wind scatters the ten name cards. In how many ways can Herbert replace the ten cards at the places on the table so that exactly four of the ten women will be seated where Zelma had wanted them? In how many ways will at least four of them be seated where they were supposed to be?

7. If 13 cards are dealt from a standard deck of 52, what is the probability that these 13 cards include (a) at least one card from each suit? (b) exactly one void (for example, no clubs)? (c) exactly two voids?

8. The following provides an outline for proving Corollary 8.2. Fill in the needed details.

   a) First note that $E_t = L_t = S_t$.

   b) What is E_{t-1}, and how are L_t and L_{t-1} related?

   c) Show that $L_{t-1} = S_{t-1} - \binom{t-1}{t-2}S_t$.

   d) For all $1 \leq m \leq t-1$, how are L_m, L_{m+1}, and E_m related?

   e) Using the results in steps (a) through (d), establish the corollary by a backward type of induction.

8.3 Derangements: Nothing Is in Its Right Place

In elementary calculus the Maclaurin series for the exponential function is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

so

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots.$$

To five places, $e^{-1} = 0.36788$ and $1 - 1 + (1/2!) - (1/3!) + \cdots - (1/7!) \doteq 0.36786$. Consequently, for all $k \in \mathbf{Z}^+$, if $k \geq 7$, then $\sum_{n=0}^{k}((-1)^n)/n!$ is a very good approximation to e^{-1}.

We find these ideas helpful in working some of the following examples.

EXAMPLE 8.16

While at the racetrack, Ralph bets on each of the ten horses in a race to come in according to how they are favored. In how many ways can they reach the finish line so that he loses all of his bets?

Removing the words *horses* and *racetrack* from the problem, we really want to know in how many ways we can arrange the numbers $1, 2, 3, \ldots, 10$ so that 1 is not in first place (its natural position), 2 is not in second place (its natural position), ..., and 10 is not in tenth place (its natural position). These arrangements are called the *derangements* of $1, 2, 3, \ldots, 10$.

The Principle of Inclusion and Exclusion provides the key to calculating the number of derangements. For each $1 \leq i \leq 10$, an arrangement of $1, 2, 3, \ldots, 10$ is said to satisfy condition c_i if integer i is in the ith place. We obtain the number of derangements, denoted by d_{10}, as follows:

$$\begin{aligned} d_{10} &= N(\overline{c}_1\overline{c}_2\overline{c}_3 \cdots \overline{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \cdots + \binom{10}{10}0! \\ &= 10!\left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \cdots + \binom{10}{10}(0!/10!)\right] \\ &= 10![1 - 1 + (1/2!) - (1/3!) + \cdots + (1/10!)] \doteq (10!)(e^{-1}). \end{aligned}$$

The sample space here consists of the $10!$ ways the horses can finish. So the *probability* that Ralph will lose every bet is approximately $(10!)(e^{-1})/(10!) = e^{-1}$. This probability remains (more or less) the same if the number of horses in the race is $11, 12, \ldots$. On the other hand, for n horses, where $n \geq 10$, the probability that our gambler wins at least one of his bets is approximately $1 - e^{-1} \doteq 0.63212$.

EXAMPLE 8.17

The number of derangements of $1, 2, 3, 4$ is

$$\begin{aligned} d_4 &= 4![1 - 1 + (1/2!) - (1/3!) + (1/4!)] \\ &= 4![(1/2!) - (1/3!) + (1/4!)] = (4)(3) - 4 + 1 = 9. \end{aligned}$$

These nine derangements are

2143	3142	4123
2341	3412	4312
2413	3421	4321.

Among the $24 - 9 = 15$ permutations of $1, 2, 3, 4$ that are *not* derangements one finds 1234, 2314, 3241, 1342, 2431, and 2314.

EXAMPLE 8.18

Peggy has seven books to review for the C–H Company, so she hires seven people to review them. She wants two reviews per book, so the first week she gives each person one book to read and then redistributes the books at the start of the second week. In how many ways can she make these two distributions so that she gets two reviews (by different people) of each book?

She can distribute the books in 7! ways the first week. Numbering both the books and the reviewers (for the first week) as 1, 2, . . . , 7, for the second distribution she must arrange these numbers so that none of them is in its natural position. This she can do in d_7 ways. By the rule of product, she can make the two distributions in $(7!)d_7 \doteq (7!)^2(e^{-1})$ ways.

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right], \qquad \text{for } n > 1.$$

For example

$$D_2 = 2!\left[1 - \frac{1}{1!} + \frac{1}{2!}\right] = 1$$

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2$$

$$D_4 = 9, \quad D_5 = 44, \quad D_6 = 265, \quad D_7 = 1854.$$

EXAMPLE 8.19

A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters **(a)** no letter is put into the correct envelope **(b)** exactly 1 letter is put into the correct envelope **(c)** exactly 98 letters are put into the correct envelope **(d)** exactly 99 letters are put into the correct envelope **(e)** all letters are put into the correct envelopes?

The probability of derangement of n objects is $\frac{D_n}{n!}$. For example

Probability of derangement

n:	2	3	4	5	6	7
$\frac{D_n}{n!}$:	0.5	0.333	0.375	0.3667	0.36806	0.36786

a) The probability of no letter being put in the correct envelope is $\frac{D_n}{100!}$ because the number of favorable cases (the derangements) is D_n and the total number of favorable cases is $n! = 100!$.

b) When exactly 1 letter is put correctly, the number of derangements for the remaining 99 letters is D_{99}. This can happen in $\binom{100}{1}$ or $\binom{100}{99}$ ways. So the probability is

$$\frac{\binom{100}{1}D_{99}}{100!} = \frac{100D_{99}}{100!}$$

c) When exactly 98 letters are put into the right (correct) envelope, the number of derangements for the remaining 2 is $D_2 = 1$. This can happen in $\binom{100}{98} = \binom{100}{2}$ ways.

The required probability is

$$\frac{\binom{100}{2} \cdot D_2}{100!} = \frac{\binom{100}{2}}{100!}$$

d) When exactly 99 letters are put in the correct envelope, it is not possible to misplace the remaining one letter. This is also put in the right envelope. Thus the probability is zero (It is an impossible event).

e) When all letters are put in the correct envelope, this can happen only once out of the total 100! cases. Thus the required probability is $\frac{1}{100!}$.

EXAMPLE 8.20

a) List all the derangements of the numbers 1, 2, 3, 4, 5 where the first three numbers are 1, 2, 3 in some order.

b) List all the derangements of the numbers 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, 3 in some order.

a) When 1, 2, 3 are in some order, there are only two derangements
(i) 23,154 and **(ii)** 31,254
(other examples include 21,354 and 32,154)

b) There are only four such derangements. For example, one such set is
(i) 231,546 **(ii)** 312,546 **(iii)** 231,645 **(iv)** 312,645
(other examples include **(i)** 213,546 **(ii)** 321,546 **(iii)** 213,645 **(iv)** 321,645)

EXERCISES 8.3

1. In how many ways can the integers 1, 2, 3, . . . , 10 be arranged in a line so that no even integer is in its natural position?

2. a) List all the derangements of 1, 2, 3, 4, 5 where the first three numbers are 1, 2, and 3, in some order.

b) List all the derangements of 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, and 3, in some order.

3. How many derangements are there for 1, 2, 3, 4, 5?

4. How many permutations of 1, 2, 3, 4, 5, 6, 7 are not derangements?

5. a) Let $A = \{1, 2, 3, \ldots, 7\}$. A function $f: A \to A$ is said to have a *fixed point* if for some $x \in A$, $f(x) = x$. How many one-to-one functions $f: A \to A$ have at least one fixed point?

b) In how many ways can we devise a secret code by assigning to each letter of the alphabet a different letter to represent it?

6. How many derangements of 1, 2, 3, 4, 5, 6, 7, 8 start with **(a)** 1, 2, 3, and 4, in some order? **(b)** 5, 6, 7, and 8, in some order?

7. For the positive integers 1, 2, 3, . . . , $n-1$, n, there are 11,660 derangements where 1, 2, 3, 4, and 5 appear in the first five positions. What is the value of n?

8. Four applicants for a job are to be interviewed for 30 minutes each: 15 minutes with each of supervisors Nancy and Yolanda. (The interviews are in separate rooms, and interview starts at 9:00 A.M.) **(a)** In how many ways can these interviews be scheduled during a 1-hour period? **(b)** One applicant, named Josephine, arrives at 9:00 A.M. What is the probability that she will have her two interviews one after the other? **(c)** Regina, another applicant, arrives at 9:00 A.M. and hopes to be finished in time to leave by 9:50 A.M. for another appointment. What is the probability that Regina will be able to leave on time?

9. In how many ways can Mrs. Ford distribute ten distinct books to her ten children (one book to each child) and then collect and redistribute the books so that each child has the opportunity to peruse two different books?

10. a) When n balls, numbered 1, 2, 3, . . . , n are taken in succession from a container, a *rencontre* occurs if the mth ball withdrawn is numbered m, for some $1 \leq m \leq n$. Find the probability of getting **(i)** no rencontres; **(ii)** (exactly) one rencontre, **(iii)** at least one rencontre; and **(iv)** r rencontres, where $1 \leq r \leq n$.

b) Approximate the answers to the questions in part **(a)**.

11. Ten women attend a business luncheon. Each woman checks her coat and attaché case. Upon leaving, each woman is given a coat and case at random. **(a)** In how many ways can the coats and cases be distributed so that no woman gets either of her possessions? **(b)** In how many ways can they be distributed so that no woman gets back both of her possessions?

12. Ms. Pezzulo teaches geometry and then biology to a class of 12 advanced students in a classroom that has only 12 desks. In how many ways can she assign the students to these desks so that **(a)** no student is seated at the same desk for both classes? **(b)** there are exactly six students each of whom occupies the same desk for both classes?

13. Give a combinatorial argument to verify that for all $n \in \mathbf{Z}^+$,

$$n! = \binom{n}{0}d_0 + \binom{n}{1}d_1 + \binom{n}{2}d_2 + \cdots + \binom{n}{n}d_n = \sum_{k=0}^{n}\binom{n}{k}d_k.$$

(For each $1 \le k \le n$, $d_k =$ the number of derangements of $1, 2, 3, \ldots, k$; $d_0 = 1$.)

14. a) In how many ways can the integers $1, 2, 3, \ldots, n$ be arranged in a line so that none of the patterns $12, 23, 34, \ldots, (n-1)n$ occurs?

b) Show that the result in part **(a)** equals $d_{n-1} + d_n$. ($d_n =$ the number of derangements of $1, 2, 3, \ldots, n$.)

15. Answer part **(a)** of Exercise 14 if the numbers are arranged in a circle, and, as we count clockwise about the circle, none of the patterns $12, 23, 34, \ldots, (n-1)n, n1$ occurs.

16. What is the probability that the gambler in Example 8.16 wins **(a)** (exactly) five of his bets? **(b)** at least five of his bets?

8.4 Rook Polynomials

Consider the six-square "chessboard" shown in Fig. 8.6 (*Note*: The shaded squares are not part of the chessboard.) In chess a piece called a *rook* or *castle* is allowed at one turn to be moved horizontally or vertically over as many unoccupied spaces as one wishes. Here a rook in square 3 of the figure could be moved in one turn to squares 1, 2, or 4. A rook at square 5 could be moved to square 6 or square 2 (even though there is no square between squares 5 and 2).

For $k \in \mathbf{Z}^+$ we want to determine the number of ways in which k rooks can be placed on the unshaded squares of this chessboard so that no two of them can take each other—that is, no two of them are in the same row or column of the chessboard. This number is denoted by r_k, or by $r_k(C)$ if we wish to stress that we are working on a particular chessboard C.

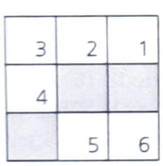


Figure 8.6

For any chessboard, r_1 is the number of squares on the board. Here $r_1 = 6$. Two nontaking rooks can be placed at the following pairs of positions: $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$, $\{4,5\}$, and $\{4,6\}$, so $r_2 = 8$. Continuing, we find that $r_3 = 2$, using the locations $\{1,4,5\}$ and $\{2,4,6\}$; $r_k = 0$, for $k \ge 4$.

With $r_0 = 1$, the *rook polynomial*, $r(C, x)$, for the chessboard in Fig. 8.6 is defined as $r(C, x) = 1 + 6x + 8x^2 + 2x^3$. For each $k \ge 0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.

What we have done here (using a case-by-case analysis) soon proves tedious. As the size of the board increases, we have to consider cases wherein numbers such as r_4 and r_5 are nonzero. Consequently, we shall now make some observations that will allow us to make use of small boards and somehow break up a large board into smaller *subboards*.

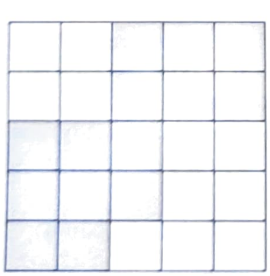


Figure 8.7

The chessboard C in Fig. 8.7 is made up of 11 unshaded squares. We note that C consists of a 2×2 subboard C_1 located in the upper left corner and a seven-square subboard C_2 located in the lower right corner. These subboards are *disjoint* because they have no squares in the same row or column of C.

Calculating as we did for our first chessboard, here we find

$$r(C_1, x) = 1 + 4x + 2x^2, \qquad r(C_2, x) = 1 + 7x + 10x^2 + 2x^3,$$
$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) \cdot r(C_2, x).$$

Hence $r(C, x) = r(C_1, x) \cdot r(C_2, x)$. But did this occur by luck or is something happening here that we should examine more closely? For example, to obtain r_3 for C, we need to know in how many ways three nontaking rooks can be placed on board C. These fall into three cases:

a) All three rooks are on subboard C_2 (and none is on C_1): $(2)(1) = 2$ ways.

b) Two rooks are on subboard C_2 and one is on C_1: $(10)(4) = 40$ ways.

c) One rook is on subboard C_2 and two are on C_1: $(7)(2) = 14$ ways.

Consequently, three nontaking rooks can be placed on board C in $(2)(1) + (10)(4) + (7)(2) = 56$ ways. Here we see that 56 arises just as the coefficient of x^3 does in the product $r(C_1, x) \cdot r(C_2, x)$.

In general, if C is a chessboard made up of *pairwise* disjoint subboards $C_1, C_2, \ldots, C_n$, then $r(C, x) = r(C_1, x)r(C_2, x) \cdots r(C_n, x)$.

The last result for this section demonstrates the type of principle we have seen in other results in combinatorial and discrete mathematics: Given a large chessboard, break it into smaller subboards whose rook polynomials can be determined by inspection.

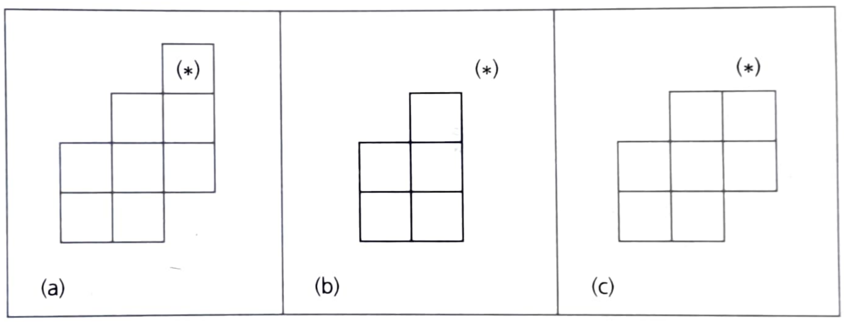


Figure 8.8

Consider chessboard C in Fig. 8.8(a). For $k \geq 1$, suppose we wish to place k nontaking rooks on C. For each square of C, such as the one designated by $(*)$, there are two possibilities to examine.

a) Place one rook on the designated square. Then we remove, as possible locations for the other $k - 1$ rooks, all other squares of C in the same row or column as the designated square. We use C_s to denote the remaining smaller subboard [seen in Fig. 8.8(b)].

b) We do not use the designated square at all. The k rooks are placed on the subboard C_e [C with the one designated square eliminated—as shown in Fig. 8.8(c)].

Since these two cases are all-inclusive and mutually disjoint,

$$r_k(C) = r_{k-1}(C_s) + r_k(C_e).$$

From this we see that

$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k. \tag{1}$$

If n is the number of squares in the chessboard (here n is 8), then Eq. (1) is valid for all $1 \le k \le n$, and we write

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k. \tag{2}$$

For Eq. (2) we realize that the summations may stop before $k = n$. We have seen cases, as in Fig. 8.6, where r_n and some prior r_k's are 0. The summations start at $k = 1$, for otherwise we could find ourselves with the term $r_{-1}(C_s)x^0$ in the first summand on the right-hand side of Eq. (2).

Equation (2) may be rewritten as

$$\sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k \tag{3}$$

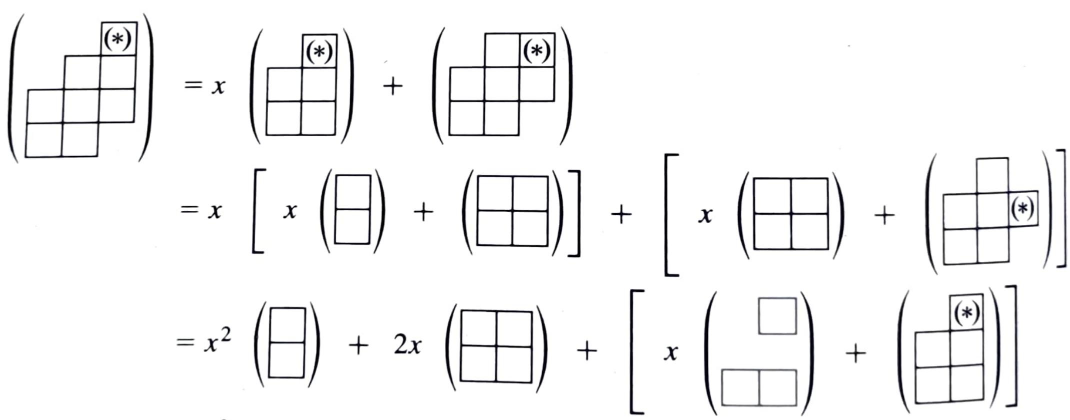
or

$$1 + \sum_{k=1}^{n} r_k(C)x^k = x \cdot r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k + 1,$$

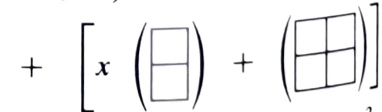
from which it follows that

$$r(C, x) = x \cdot r(C_s, x) + r(C_e, x). \tag{4}$$

We now use this final equation to determine the rook polynomial for the chessboard shown in part (a) of Fig. 8.8. Each time the idea in Eq. (4) is used, we mark the special square we are using with (*). Parentheses are placed about each chessboard to denote the rook polynomial of the board.



$$= x^2(1 + 2x) + 2x(1 + 4x + 2x^2) + x(1 + 3x + x^2)$$



$$= 3x + 12x^2 + 7x^3 + x(1 + 2x) + (1 + 4x + 2x^2) = 1 + 8x + 16x^2 + 7x^3.$$