MODULE 02 → **Complex Variable.**

Limit ÷ a complex valued functⁿ $f(z)$ defined in the neighbourhood of a point z_0 is said to have a limit l as $z \to z_0$ if for every $\epsilon > 0$ however small there exist a real no. δ such that $|f(z) - l| < \epsilon$ when $|z - z_0| < \delta$.

$$\boxed{\lim_{z \to z_0} f(z) = l}$$

Continuity ÷ A complex valued function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ exist and $\lim_{z \to z_0} f(z) = f(z_0)$

Differentiability ÷ A complex valued function $f(z)$ is said to be differentiable at $z = z_0$ if $\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$ exist and is unique.

This limit when exist is called the derivative of $f(z)$ at $z = z_0$ and is denoted by $f'(z_0)$.

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}.$$

Analytic function:

A complex valued function $w = f(z)$ is said to be analytic at a point $z = z_0$ if $\frac{dw}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$ exist and is unique at z_0 and in the neighbourhood of z_0.

$f(z)$ is said to be analytic in a region if it is analytic at every point of that region.

Analytic functⁿ is also called regular functⁿ or holomorphic functⁿ.

1) Find the modulus & amplitude of $\frac{(1+i)^2}{3+i}$

$$\frac{(1+i)^2}{3+i} = \frac{1 + i^2 + 2i}{3+i} = \frac{2i}{3+i} \times \frac{3-i}{3-i} = \frac{6i+2}{9+1} = \frac{6i}{10} + \frac{2}{10} = \frac{1}{5} + i\frac{3}{5}$$

$x + iy$

Amplitude $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3/5}{1/5}\right) = \tan^{-1}(3)$.

Modulus $r = \sqrt{x^2+y^2} = \sqrt{\frac{1}{25} + \frac{9}{25}} = \frac{\sqrt{10}}{5}$ //

2) P.T $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

w.k.t $e^{ix} = \cos x + i\sin x$ — ①

$e^{-ix} = \cos x - i\sin x$ — ②

① + ② ⇒ $e^{ix} + e^{-ix} = 2\cos x$ ⇒ $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

① − ② ⇒ $e^{ix} - e^{-ix} = 2i\sin x$ ⇒ $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

3) Determine the region in Z-plane represented by $1 < |z + 2i| \leq 3$.

Consider $|z + 2i| = 3$

$|x + iy + 2i| = 3$

$|x + i(y+2)| = 3$

$x^2 + (y+2)^2 = 3^2$

$(x-h)^2 + (y-k)^2 = a^2$ → eqⁿ of $\circ^{le}$

Centre $(0, -2)$ radius $= 3$

Cauchy Riemann (CR) eqⁿ in Cartesian form:

Let $f(z)$ be analytic function at the point

$z = x+iy$.

$w = f(z)$

$u+iv = f(x+iy) \longrightarrow (1)$.

Diff (1) partially w.r.t x & y.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(x+iy) \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'(x+iy)\cdot i$$

$$-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = f'(x+iy) \quad \text{—— (3)}$$

Equating Real & Imaginary parts of LHS of (2) & (3)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$u_x = v_y \qquad v_x = -u_y \quad \rightarrow \text{CR eqⁿ}$$

C.R equation in Polar eqⁿ

Let $w = f(z)$ be analytic function of $z = re^{i\theta}$.

$u+iv = f(re^{i\theta})$ —— (1)

Diff (1) partially w.r.t r & θ.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})(e^{i\theta}) \quad \text{—— (2)}.$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})(rie^{i\theta})$$

$$\frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} = f'(re^{i\theta})(e^{i\theta}) \quad \text{—— (3)}$$

Equating Real & Imaginary parts of LHS of eqⁿ (2) & (3).

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$u_r = \frac{1}{r}v_\theta \quad ; \quad v_r = \frac{-1}{r}u_\theta \longrightarrow \text{CR eqⁿ}$$

Harmonic function:

A function ϕ is said to be harmonic function if it satisfies Laplace eqⁿ $\Delta^2\phi = 0$.

Laplace eqⁿ in Cartesian form: $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

Laplace eqⁿ in Polar form: $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$.

Orthogonal: If $f(x) = u+iv$ is analytic then the family of curves $u(x,y) = C_1$ & $v(x,y) = C_2$ where C_1 & C_2 are constants then the slopes $m_1 \cdot m_2 = -1$.

Consequences of Analytic function:

1. The real and imaginary part of an analytic function are harmonic.

Converse is not true.

Ex: If $u = x^2 - y^2$ & $v = x^3 - 3xy^2$. S.T u & v are harmonic but $f(z) = u+iv$ is not analytic.

Laplace eqⁿ of u is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow 2-2=0$

Laplace eqⁿ of v is $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Rightarrow 6x-6x=0$

u & v are harmonic

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial v}{\partial x} = 3x^2 \qquad \frac{\partial^2 v}{\partial x^2} = 6x \qquad \frac{\partial v}{\partial y} = -6xy \qquad \frac{\partial^2 v}{\partial y^2} = -6x$

To say $f(z)$ is analytic, it has to satisfy.

C.R eqⁿ:-

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad$ $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$.

$2x \neq -6xy$ $\quad$ $3x^2 \neq -2y$.

C.R eqⁿ is not satisfied. $f(z)$ is not analytic.

2) If $f(z) = u+iv$ is analytic, then the family of curves $u(x,y) = C_1$ & $v(x,y) = C_2$, where C_1 & C_2 are constants intersect each other orthogonally.

Converse is not true.

Solⁿ If $u = \frac{x^2}{y}$, $y \neq 0$, $v = x^2+2y^2$. S.T u = constant v = constant are orthogonal but $f(z) = u+iv$ is not analytic.

$u = C_1$; $v = C_2$

$\frac{x^2}{y} = C_1$; $x^2 + 2y^2 = C_2$.

Diff. both sides w.r.t x, where y is function of x.

$\frac{y(2x) - x^2\, dy/dx}{y^2} = 0$; $2x + 4y\frac{dy}{dx}$

$2xy = x^2 \frac{dy}{dx}$ $\quad$ $4\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{2y}{x}$ $\quad$ $\frac{dy}{dx} = -\frac{x}{y}$.

$m_1 \cdot m_2 = \frac{2y}{\cancel{x}} \cdot \frac{-\cancel{x}}{2y} = -1$.

$\therefore$ $u = C_1$ & $v = C_2$ are orthogonal.

$\frac{\partial u}{\partial x} = \frac{\partial x}{\partial y}$ $\quad$ $\frac{\partial u}{\partial y} = \frac{-x^2}{y^2}$, $\frac{\partial v}{\partial x} = 2x$ $\quad$ $\frac{\partial v}{\partial y} = 4y$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{2x}{y} \neq 4y$

$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \Rightarrow 2x \neq \frac{x^2}{y^2}$

C.R eqⁿ are not satisfied.

$\therefore f(z)$ is NOT analytic.

Type 1 problems:- Finding the derivative of analytic function

1) If problem given is in cartesian form, $f'(z) = u_x + iv_x$

2) If problem is in polar form, $f'(z) = e^{-i\theta}(u_r + iv_r)$.

1) S.T $f(z) = \sin z$ is analytic & hence find $f'(z)$.

$w = f(z)$.

$u+iv = \sin(z)$.

$u+iv = \sin(x+iy)$.

$= \sin x \cos(iy) + \cos x \sin(iy)$.

$= \sin x \cosh y + \cos x\, i\sinh y$.

$= \sin x \cosh y + i \sinh y \cos x$.

~~$u = \sin x \cosh y$~~ $\quad$ ~~$v_x = +\sinh y \sin x$~~

~~$u_y = \sin x \sinh y$~~

$u = \sin x \cosh y$ $\quad$ $v = \sinh y \cos x$.

$u_x = \cos x \cosh y$ $\quad$ $v_x = -\sinh y \sin x$

$u_y = -\sin x \sinh y$ $\quad$ $v_y = \cosh y \cos x$.

$\because$ CR eq^n $\div$ $u_x = v_y = \cos x \cosh y$.

$v_x = u_y = -\sinh y \sin x$.

$\because$ CR eq^{n}s satisfied $f(z)$ is analytic.

$f'(z) = u_x + iv_x$.

$= \cos x \cosh y + i(-\sinh y \sin x)$

$= \cos x \cos iy - i(i \sin iy . \sin x)$

$= \cos x \cos iy + \sin x \sin iy$.

$f'(z) = \cos(x+iy)$

$f'(z) = \cos(z)$ //

2) S.T $w = \log z$ is analytic and hence find its derivative.

$w = f(z)$.

$u + iv = \log(re^{i\theta})$.

$= \log r + \log e^{i\theta}$.

$= \log r + i\theta \underbrace{\log e}_{=1}$.

$= \log r + i\theta$

$u = \log r \qquad \theta = v$.

$u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1$

CReq^{n}s $u_r = \frac{1}{r} v_\theta \qquad v_r = -\frac{1}{r} u_\theta$.

$\frac{1}{r} = \frac{1}{r} . 1 \qquad 0 = 0$.

CR eq^n is satisfied

$\therefore f(z)$ is analytic

$f'(z) = e^{-i\theta}[u_r + iv_r]$

$= e^{-i\theta}\left[\frac{1}{r} + i0\right]$

$= \frac{e^{-i\theta}}{r}$

$= \frac{1}{re^{i\theta}} = \frac{1}{z}$ //

3) $f(z) = z^n$

$u + iv = z^n$

$= (re^{i\theta})^n$.

$= r^n e^{i\theta n}$

$= r^n[\cos n\theta + i \sin n\theta]$.

$= r^n \cos n\theta + i r^n \sin n\theta$

$u = r^n \cos n\theta \qquad v = r^n \sin n\theta$.

$u_r = n r^{n-1} \cos n\theta$. $\qquad v_r = n r^{n-1} \sin n\theta$

$u_\theta = -r^n n \sin n\theta \qquad v_\theta = r^n n \cos n\theta$

$u_r = \frac{1}{r} v_\theta \quad , \quad v_r = -\frac{1}{r} u_\theta$.

$n r^{n-1} \cos n\theta = \frac{r^n n \cos n\theta}{r} \quad , \quad n r^{n-1} \sin n\theta = +\frac{r^n n \sin n\theta}{r}$

$n r^{n-1} \cos n\theta = r^{n-1} n \cos n\theta \quad , \quad n r^{n-1} \sin n\theta = r^{n-1} n \sin n\theta$

CR eq^n is satisfied.

$f(z)$ is analytic.

$$f'(z) = e^{-i\theta}\left[u_r + iu_r\right].$$
$$= e^{-i\theta}\left[n r^{n-1}\cos n\theta + i n r^{n-1}\sin n\theta\right]$$
$$= e^{-i\theta}\, n \cdot r^{n-1}\left[\cos n\theta + i\sin n\theta\right]$$
$$= e^{-i\theta}\, n\, r^{n-1}\left[e^{in\theta}\right]$$
$$= e^{i\theta(n-1)} \cdot n \cdot r^{n-1}$$
$$= n\left[r^{n-1} e^{i\theta(n-1)}\right)$$
$$f'(z) = n z^{n-1}$$

4) $\omega = \cosh z$

$$u + iv = \cos iz$$
$$= \cos i(x+iy)$$
$$= \cos(ix - y).$$
$$= \cos ix \cos y + \sin ix \sin y.$$
$$= \cosh x \cos y + i\sinh x \sin y.$$

$u = \cosh x \cos y$ $\qquad$ $v = \sinh x \sin y$.

$u_x = \sinh x \cos y$. $\qquad$ $v_x = \cosh x \sin y$

$u_y = -\cosh x \sin y$ $\qquad$ $v_y = \sinh x \cos y$.

$u_x = v_y = \sinh x \cos y$ $\qquad$ $u_y = -v_x = \cosh x \sin y$.

CR eq^{ns} is satisfied.

$\therefore f(z)$ is analytic.

$$f'(z) = u_x + iv_x.$$
$$= \sinh x \cos y + i\left[\cosh x \sin y\right].$$
$$= i\sin ix \cos y + i\cos ix \sin y$$

Type - 2 problems :-

Constructing the analytic function given real or imaginary part.

Given u or v to find u_x, u_y or v_x, v_y

Consider $f'(z) = u_x + iv_x$

If u is given replace v_x by u_y, if v is given use $v_x = u_y$ in $f'(z)$ [using CR eq^{ns}]

Put $x = z$ & $y = 0$, integrate & get $f(z)$.

In case of polar consider $f'(z) = e^{-i\theta}\left[u_r + iv_r\right]$

Use CR eq^{n} $v_r = -\frac{1}{r}u_\theta$ & for given u (or).

$u_r = \frac{1}{r}v_\theta$ for given v.

Put $r = z$ & $y = 0$, integrate get $f(z)$. This method is called M.T method.

1) Find the analytic function $f(z) = u + iv$, where $v = e^x(x\sin y + y\cos y)$ using Mille Thompson method

$$v_x = e^x(x\sin y + y\cos y) + e^x \cdot \sin y.$$
$$v_y = e^x(x\cos y + \cos y - y\sin y).$$
$$f'(z) = u_x + iv_x$$
$$= v_y + iv_x$$
$$= e^x\left[x\cos y + \cos y - y\sin y\right] + ie^x\left[x\sin y + y\cos y + \sin y\right]$$

$x \Rightarrow z$ $\quad$ $y = 0$.

$$= e^z\left[z+1\right] + ie^z\left[0\right]$$

$f'(z) = e^z\left[z+1\right]$. $\qquad$ $f(z) = \int f'(z)\,dz = \int e^z(z+1)\,dz$

$$= (z+1)e^z - e^z = ze^z + C$$

2) Find the analytic function whose real part is $\log\sqrt{x^2+y^2}$

$u = \log\sqrt{x^2+y^2}$

$u = \log(x^2+y^2)^{1/2}$

$u = \frac{1}{2}\log x^2+y^2$.

$u_x = \frac{1 \cdot \cancel{2}x}{\cancel{2}(x^2+y^2)}$ $\quad u_y = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)}$

$f'(z) = u_x + iv_x = u_x - iu_y$ $\quad [v_x = -u_y]$

$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$x = z \quad y = 0$.

$f'(z) = \frac{z}{z^2} = \frac{1}{z}$

$\int f'(z) = \int \frac{1}{z}\,dz$.

$f(z) = \log z + c$.

3) Find the analytic function $f(z) = u+iv$ where $u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$ using Millie-Thompson method.

$u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$.

$u_x = -e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\} + e^{-x}\{2x\cos y + 2y\sin y\}$

$u_y = e^{-x}\{-2y\cos y + (x^2-y^2)(-\sin y) + 2xy\cos y + 2x\sin y\}$

$f'(z) = u_x + iv_x \quad v_x = -u_y$.

$f'(z) = u_x - iu_y$.

$= e^{-x}[2x\cos y + 2y\sin y - (x^2-y^2)\cos y - 2xy\sin y]$

$- ie^{-x}[-2y\cos y - (x^2-y^2)\sin y + 2xy\cos y + 2x\sin y]$

$x = 0 \quad y = 0$.

$f'(z) = e^{-z}(2z - z^2) - ie^{-z}(0)$.

$f'(z) = e^{-z}(2z - z^2)$

$f(z) = \int (2z - z^2)e^{-z}\,dz$

$= 2z - z^2(e^{-z}) - (2-2z)e^{-z} + (-2)e^{-z} + c$

$= -2ze^{-z} + z^2e^{-z} - 2e^{-z} + 2ze^{-z} + 2e^{-z} + c$

$f(z) = z^2e^{-z} + c$ //

4) Find the analytic function where $v = -\frac{\sin\theta}{r}$ using M.T method.

$f'(z) = e^{-i\theta}[u_r + iv_r]$

$v_r = \frac{-\sin\theta}{r^2}$ $\quad v_\theta = \frac{-\cos\theta}{r}$ $\quad u_r = \frac{1}{r}v_\theta$.

$f'(z) = e^{i\theta}\left[\frac{1}{r}\left(\frac{-\cos\theta}{r}\right) + i\left(\frac{\sin\theta}{r}\right)\right]$

$r = z \quad \theta = 0$.

$f'(z) = \left(\frac{-1}{z^2} + 0\right]$

$f(z) = \int \frac{-1}{z^2}\,dz = \frac{1}{z} + c$ //

5) $u = r^2 \cos 2\theta$

$u_r = 2r \cos 2\theta$ $\qquad u_\theta = -2r^2 \sin 2\theta$.

$v_r = -\frac{1}{r} u_\theta = 2r \sin 2\theta$.

$f'(z) = e^{-i\theta}\left[2r\cos\theta + 2\sin 2\theta\right]$

$r = z \quad \theta = 0$.

$f'(z) = 2z$

$f(z) = \int 2z\,dz = z^2 + c$ //

Type 3

Finding the conjugate Harmonic function & analytic function.

Given u, find u_x, u_y.

Consider the CR eqn $u_x = v_y$ & $v_x = -u_y$.

We get 2 non-homogeneous PDE & solve it by direct integration method.

Write the common answer for v by choosing the constants.

$f(z) = u + iv$ is an analytic function, we get this by replacing $x = z$ & $y = 0$.

Same procedure for polar replace $r = z$ & $\theta = 0$.

1) Show that $u = e^x \cos y + xy$ is Harmonic. Find its Harmonic conjugate & also find the corresponding analytic function.

$u = e^x \cos y + xy$

$\frac{\partial u}{\partial x} = e^x \cos y + y$ $\qquad \frac{\partial u}{\partial y} = -e^x \sin y + x$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$ $\qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$.

L.E $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$= e^x \cos y - e^x \cos y = 0$

L.E is satisfied.

u is Harmonic.

To find conjugate

$u_x = e^x \cos y + y$ $\qquad u_y = -e^x \sin y + x$.

Consider CR eqn $u_x = v_y$ & $v_x = -u_y$.

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y + y$.

Integrate w.r.t y.

$v = \int (e^x \cos y + y)\,dy + f(x)$

$v = e^x \sin y + \frac{y^2}{2} + f(x)$.

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} = e^x \sin y - x$.

$\int$ w.r.t x.

$v = \int (e^x \sin y + x)\,dx + g(y)$.

$= e^x \sin y - \frac{x^2}{2} + g(y)$. By choosing $g(y) = \frac{y^2}{2}$ & $f(x) = \frac{-x^2}{2}$ we get as

Let $f(x) = -\frac{x^2}{2}$ & $g(y) = \frac{y^2}{2}$

$$\boxed{v = e^x \sin y + \frac{y^2}{2} - \frac{x^2}{2}}$$

$f(z) = u + iv$

$= e^x \cos y + xy + i\left[e^x \sin y + \frac{y^2}{2} - \frac{x^2}{2}\right]$

$x = z \quad y = 0$.

$f(z) = e^z + i\left(\frac{-z}{2}\right)^2$ //

$f(z) = u + iv$

$f(z) = e^x \cos y + xy + i\left[e^x \sin y + \frac{y^2}{2} - \frac{x^2}{2}\right]$

$x \to z \quad y \to 0.$

$$\boxed{f(z) = e^z + i\left[\frac{-z^2}{2}\right]}$$

→ S.T $v = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1$ is Harmonic & Find its Harmonic conjugate.

$v = x^3 - 3xy^2 - 3x^2 + 3y^2 + 1$

$v_x = 3x^2 - 3y^2 - 6x \qquad v_y = -6xy + 6y$

$v_x^2 = 6x - 6 \qquad v_y^2 = -6x + 6.$

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x - 6 - 6x + 6 = 0.$

L.E is Harmonic.

C.R eq^n :- $u_x = v_y$ & $\cancel{v_x =}\ u_y = -v_x.$

$\frac{\partial u}{\partial x} = -6xy + 6y \qquad \frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 6x.$

$\int$ w.r.t x $\qquad$ & w.r.t y.

$u = \int(-3x^2 + 3y^2 + 6x)dy + f(x)$

$u = \int(-6xy + 6y)dx + g(y)$

$u = -3x^2y + 3y^3 + 6xy + g(x)$

$u = -3x^2y + 6xy + g(y)$

Choose $g(y) = \cancel{\frac{3x^2}{y}} + y^3$ & $g\cancel{(x) = -3xy^2}\ g(x) = 0.$

$$\Rightarrow \boxed{v = 6xy - 3x^2y + y^3\ \cancel{3xy^2}}$$

$f(z) = u + iv.$

$f(z) =$

→ S.T $\cancel{v=}$ $v = \left(r - \frac{1}{r}\right)\sin\theta$ is Harmonic & find its Harmonic conjugate.

$v = \left(r - \frac{1}{r}\right)\sin\theta.$

L.E $= \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}.$

$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta \qquad \frac{\partial^2 v}{\partial r^2} = \frac{-2}{r^3}\sin\theta.$

$\frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta \qquad \frac{\partial^2 v}{\partial \theta^2} = -\left(r - \frac{1}{r}\right)\sin\theta.$

L.E → $-\frac{2}{r^3}\sin\theta + \frac{\sin\theta}{r} + \frac{\sin\theta}{r^3} + \frac{-1}{r}\sin\theta + \frac{1}{r^3}\sin\theta = 0.$

∴ v is Harmonic.

C.R :- $u_r = \frac{1}{r}v_\theta \quad ; \quad u_\theta = -r v_r.$

$\frac{\partial u}{\partial r} = \frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta.$

$\int$ w.r.t. r.

$u = \int\left(1 - \frac{1}{r^2}\right)\cos\theta\, dr + f(\theta)$

$u = \left(r + \frac{1}{r}\right)\cos\theta + f(\theta) = r\cos\theta + \frac{1}{r}\cos\theta + f(\theta).$

$\frac{\partial u}{\partial \theta} = -r\left(1 + \frac{1}{r^2}\right)\sin\theta = \left(-r - \frac{1}{r}\right)\sin\theta.$

$\int$ w.r.t θ.

$u = \int -\left(r + \frac{1}{r}\right)\sin\theta\, d\theta + g(r).$

$u = \left(r + \frac{1}{r}\right)\cos\theta + g(r).$

$u = r\cos\theta + \frac{1}{r}\cos\theta + g(r).$

$\cancel{\text{Choose } f(\theta) = \frac{1}{r}\cos\theta}$ & $g(r) = \frac{2}{r^3}\cancel{\cos\theta}.$

Choose $f(\theta) = g(r) = 0.$

$$\boxed{u = r\cos\theta + \frac{1}{r}\cos\theta}$$

→ S.T $u = \frac{\cos\theta}{r}$ is harmonic. Find its Harmonic conjugate & also find the Corresponding analytic function.

L.E :- $\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{1}{r^2}\frac{d^2u}{d\theta^2} = 0.$

$\frac{du}{dr} = \frac{-\cos\theta}{r^2}$ $\qquad \frac{d^2u}{dr^2} = \frac{2\cos\theta}{r^3}$

$\frac{du}{d\theta} = \frac{-\sin\theta}{r}$ $\qquad \frac{d^2u}{d\theta^2} = \frac{-\cos\theta}{r}$

$\frac{2\cos\theta}{r^3} - \frac{\cos\theta}{r^3} - \frac{\cos\theta}{r^3} = 0.$

⇒ u is Harmonic.

C.R :- $\cancel{u_r = \frac{1}{r}v_\theta}$; $\cancel{u_r}$

$v_r = -\frac{1}{r}u_\theta$; $\qquad v_\theta = r u_r.$

$\cancel{u_\theta := \frac{\partial u}{\partial\theta} = \frac{-\sin\theta}{r}}$

$\cancel{\int w.r.t\ \theta.}$

$\cancel{u_i = \frac{-1}{r}\int -\sin\theta d\theta}$

$\frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{-\sin\theta}{r}.$

$\int$ w.r.t r.

$v = \sin\theta \int \frac{1}{r^2} dr. + f(\theta)$

$v = -\frac{\sin\theta}{r} + f(\theta).$

$\frac{\partial v}{\partial \theta} = r \cdot \frac{-r\cos\theta}{r^2}.$

$\int$ w.r.t θ.

$v = \frac{1}{r}\int -\cos\theta\, d\theta + g(r)$

$v = \frac{1}{r} - \sin\theta + g(r).$

Let $f(\theta) = -\frac{\sin\theta}{r}$ 0. & $g(r) = -\frac{\cos\theta}{r}$ 0

$v = -\cancel{\frac{\cos\theta}{r}} \; \frac{-\sin\theta}{r}$

$f(z) = u + iv.$

$f(z) = \frac{\cos\theta}{r} + i\left(\cancel{\frac{-\cos\theta}{r}} - \frac{\sin\theta}{r}\right)$ $\quad \theta \to 0,\ r \to z.$

$f(z) = \frac{1}{z} \cancel{-i\left(\frac{1}{z}\right)} = \frac{1}{z}\cancel{(1-i)}$ //

Miscellenoous

9→ Find the analytic function $f(z)$ as a function of z given $u+v = x^3-y^3+3xy(x-y)$ (Sum of its real & imaginary part)

$u+v = x^3-y^3+3x^2y-3xy^2$

Diff. partially w.r.t. x

$u_x+v_x = 3x^2+6xy-3y^2 \rightarrow (1)$

Diff. partially w.r.t y.

$u_y+v_y = -3y^2+3x^2-6xy \rightarrow (2)$

Using C.R eq^n $u_y = -v_x$ & $v_y = u_x$

$-v_x+u_x = -3y^2+3x^2-6xy. \rightarrow (2)$

Add (1) & (2). $2u_x = 3x^2+\cancel{6xy}-3y^2-3y^2+3x^2-\cancel{6xy}$

$2u_x = 6x^2-6y^2$

$\boxed{u_x = 3x^2-3y^2}$

Subtract:- (1) - (2).

$2v_x = \cancel{3x^2}+6xy-3y^2+\cancel{3y^2}-\cancel{3x^2}+6xy$

$2v_x = 12xy$

$\boxed{v_x = 6xy}$

$f'(z) = u_x+iv_x$

$f'(z) = (3x^2-3y^2)+i(6xy).$

$x \rightarrow z$
$y \rightarrow 0.$

$f'(z) = 3z^2$

$\int$ w.r.t z.

$\boxed{f(z) = z^3+C}$

→ Find the analytic function $f(z) = u+iv$ if $u-v = e^x(\cos y - \sin y)$.

Diff. w.r.t x.

$$u_x - v_x = e^x(\cos y - \sin y) \rightarrow (1)$$

Diff. w.r.t y.

$$u_y - v_y = e^x[-\sin y - \cos y]$$

Using C.R eqn.

$u_y = -v_x \quad v_y = u_x$.

$$-v_x - u_x = e^x[-\sin y - \cos y]$$

$$u_x + v_x = e^x[\sin y + \cos y] \rightarrow (2)$$

add (1) & (2) $\Rightarrow$ $2u_x = e^x\cos y - \cancel{e^x\sin y} + \cancel{e^x\sin y} + e^x\cos y$

$$\boxed{u_x = e^x\cos y}$$

(1) − (2) $\Rightarrow$ $-2v_x = \cancel{e^x\cos y} - e^x\sin y - e^x\sin y - \cancel{e^x\cos y}$

$$\boxed{v_x = e^x\sin y}$$

$$f'(z) = u_x + iv_x.$$

$$f'(z) = e^x\cos y + i(e^x\sin y)$$

$x \to z \quad y \to 0.$

$$f'(z) = e^z$$

$\int$ w.r.t z.

$$\boxed{f(z) = e^z + c}$$

→ $u-v = (x-y)(x^2+4xy+y^2)$

$$u-v = x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3$$

$$u-v = x^3 - y^3 + 3x^2y - 3xy^2$$

Diff. w.r.t x.

$$u_x - v_x = 3x^2 + 6xy - 3y^2 \rightarrow (1)$$

Diff. w.r.t y.

$$u_y - v_y = -3y^2 + 3x^2 - 6xy$$

10 FRIDAY WEEK 02 • 010-

C.R eq^n :- $\cancel{u_y = -v_y}$ $v_x = -u_y$ $u_x = v_y$.

$-v_x - u_x = -3y^2 + 3x^2 - 6xy \rightarrow ②$

$v_x + u_x = 3y^2 - 3x^2 + 6xy \rightarrow ②$

① + ② ⇒ $2u_x = 3x^2 + 6xy - 3y^2 + 3y^2 - 3x^2 + 6xy$

$\boxed{u_x = 6xy}$

① - ② ⇒ $-2v_x = 3x^2 + 6xy - 3y^2 - 3y^2 + 3x^2 - 6xy$

$-2v_x = 6x^2 - 6y^2$

11 SATURDAY WEEK 02 • 011-

$\boxed{v_x = 3y^2 - 3x^2}$

$f'(z) = u_x + i v_x$.

$f'(z) = 6xy + i(3y^2 - 3x^2)$

$f'(z) = \cancel{z} - 3z^2 i$

$\int$ w.r.t z.

$\boxed{f(z) = -iz^3 + c}$

12 SUNDAY

27 28 29 30 31

13 MONDAY WEEK 03 • 013-353

$\rightarrow u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$.

Diff. w.r.t r.

$u_r + v_r = \frac{-2}{r^3}(\cos 2\theta - \sin 2\theta) \rightarrow ①$

Diff. w.r.t θ.

$u_\theta + v_\theta = \frac{1 \cancel{-2}}{r^2}(\cancel{+}\sin 2\theta + \cos 2\theta)$

$u_r = \frac{1}{r} v_\theta$ $\quad v_r = \frac{-1}{r} u_\theta$.

$v_\theta = r u_r$ $\quad u_\theta = -r v_r$.

14 TUESDAY WEEK 03 • 014-352

$\cancel{\frac{1}{r}(v_\theta - u_\theta) = \frac{2}{r^2}(\sin 2\theta + \cos 2\theta)}$

$\cancel{(v_\theta - u_\theta) = -2}$

$-r v_r + r u_r = \frac{-2}{r^2}(\sin 2\theta + \cos 2\theta)$.

$-v_r + u_r = \frac{-2}{r^3}(\sin 2\theta + \cos\theta) \rightarrow ②$.

① + ② ⇒ $2u_r = -\cancel{\frac{2}{r^3}} \frac{-2}{r^3}\cos 2\theta + \frac{2}{r^3}\sin 2\theta - \frac{2}{r^3}\sin 2\theta - \frac{2}{r^3}\cos 2\theta$

$\boxed{u_r = \frac{-2\cos 2\theta}{r^3}}$

ow

tion

is

the

ing

the

Scanned by CamScanner

$(1)-(2) \Rightarrow \cancel{2}v_r = \frac{-\cancel{2}}{r^3}\,(2\sin 2\theta)$

$\boxed{v_r = \frac{2\sin 2\theta}{r^3}}$

$f'(z) = e^{-i\theta}\,(u_r + i v_r)$

$f'(z) = e^{-i\theta}\left[\frac{-2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right]$

$\theta \to 0 \quad r \to z$

$f'(z) = \frac{-2}{z^3}$ $\qquad -3+1$

$\int w.r.t\ z.$

$\boxed{f(z) = \frac{1}{z^2} + c}$

$\rightarrow u+v = r(\cos\theta + \sin\theta) + \frac{1}{r}(\cos\theta - \sin\theta)$

$u_r + v_r = \cos\theta + \sin\theta - \frac{1}{r^2}(\cos\theta - \sin\theta) \rightarrow (1)$

$u_\theta + v_\theta = r(-\sin\theta + \cos\theta) + \frac{1}{r}(-\sin\theta - \cos\theta).$

$v_\theta = r u_r \qquad u_\theta = -r v_r$

$-r v_r + r u_r = r(-\sin\theta + \cos\theta) + \frac{1}{r}(-\sin\theta + \cos\theta)$

$-v_r + u_r = (-\sin\theta + \cos\theta) - \frac{1}{r^2}(\sin\theta + \cos\theta) \rightarrow (2)$

$(1)+(2) \Rightarrow 2u_r = 2\cos\theta - \frac{1}{r^2}\,2\cos\theta.$

$u_r = \cos\theta - \frac{1}{r^2}\cos\theta$

$\boxed{u_r = \cos\theta\left(1 - \frac{1}{r^2}\right)}$

$(1)-(2) \Rightarrow 2v_r = 2\sin\theta - \frac{1}{r^2}(\cos\theta - \cancel{\cos\theta}\sin\theta) + \frac{1}{r^2}(\sin\theta + \cos\theta)$

$\boxed{v_r = \sin\theta\left(1 + \frac{1}{r^2}\right)}$

$f'(z) = e^{-i\theta}\,(u_r + i v_r)$

$f'(z) = e^{-i\theta}\left[\cos\theta\left(1 - \frac{1}{r^2}\right) + i\sin\theta\left(1 + \frac{1}{r^2}\right)\right]$

$\theta \to 0 \quad r \to z$

$f'(z) = \left(1 - \frac{1}{z^2}\right)$ $\qquad z^{-2+1}$

$\boxed{f(z) = z + \frac{1}{z} + c}$

Application to flow problems:-

The real & imaginary part of an analytic function play important role in flow of 2-Dimensional fluid & heat flows.

Eg:- In a 2-D irrotational flow of an incompressable fluid, Velocity vector happens to be a gradient of scalar harmonic function. This scalar function is called velocity potentianal (ϕ). The

Harmonic conjugate of this function is called Stream function for flow (ψ).

The analytic function for which ϕ is a real part & ψ is a imaginary part is called the complex potential for the flow & is denoted by $w(z)$

$w(z) = \phi + i\psi$

Since $w(z) = \phi + i\psi$ is an analytic function, the C.R eq^n applied to this function yields the following relation b/w ϕ & ψ

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ; \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

Consequently the curves ϕ = constants & ψ = constants are orthogonal.
The curves ϕ = constants are called equipotential curves & the curves ψ = constants are called streamlines.

→ An electrostatic field in xy plane is given by $\phi = 3x^2y - y^3$. Find the stream function → ψ

$$\Rightarrow \frac{\partial \phi}{\partial x} = 6xy \qquad \frac{\partial \phi}{\partial y} = 3x^2 - 3y^2$$

By using CR eq^n for complex potential we have,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ; \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \psi}{\partial y} = 6xy \qquad \frac{\partial \psi}{\partial x} = 3y^2 - 3x^2$$

$\int$ w.r.t y. $\qquad \int$ w.r.t x.

$\psi = 3xy^2 + f(x) \qquad \psi = 3xy^2 - x^3 + g(y)$

By choosing $f(x) = -x^3$, $g(y) = 0$

$$\boxed{\psi = 3xy^2 - x^3}$$

→ P.T $\phi = x^2 - y^2$, $\psi = \frac{y^3}{x^2+y^2}$ is Harmonic & hence find the ~~analytic function~~ complex potential.

LHS:- $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$

$\frac{\partial \phi}{\partial x} = 2x \qquad \frac{\partial^2 \phi}{\partial x^2} = 2 \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0$

$\frac{\partial \phi}{\partial y} = -2y \qquad \frac{\partial^2 \phi}{\partial y^2} = -2 \qquad \therefore \phi$ is Harmonic.

$\frac{\partial \psi}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^2} \qquad \frac{\partial^2 \psi}{\partial x^2} = \frac{-2y \cdot (x^2+y^2)^2 - 8x (x^2+y^2)}{(x^2+y^2)^4}$

$\frac{\partial \psi}{\partial y} = \frac{1(x^2+y^2) - 2y}{(x^2+y^2)^2} = \frac{2x(x^2+y^2)^2 - (x^2+y^2-2y)\,2x(x^2+y^2)^2}{(x^2+y^2)^4}$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{-2y(x^2+y^2)\,(x^2+y^2)\,(-2yx^2 - 2y^3 - 4xy^2)}{(x^2+y^2)^4}$

$\frac{\partial^2 \psi}{\partial y^2} = (x^2+y^2)^2\,(2x -$

$\frac{\partial \psi}{\partial x} = \frac{-y(2x)}{(x^2+y^2)^2}$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{-2y(x^2+y^2)^2 - [-y\,2xy\,2(x^2+y^2)\cdot 2x]}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)\,[-2y(x^2+y^2) + 8x^2y]}{(x^2+y^2)^4}$

$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$

$\frac{\partial \psi}{\partial y} = \frac{1(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 \psi}{\partial y^2} = \frac{-2y(x^2+y^2)^2 - (x^2-y^2)[2(x^2+y^2)\cdot 2y]}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)\,[-2y(x^2+y^2) - (x^2-y^2)\,4y]}{(x^2+y^2)^4}$

$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$ //

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

$$w(z) = \phi + iy.$$

$$= (x^2 - y^2) + i\left(\frac{y}{x^2+y^2}\right)$$

$x \to z \quad y \to 0$

$$\boxed{w(z) = z^2}$$

→ 2 concentric circular cylinders of radii r_1, r_2 $(r_1 < r_2)$ are kept in potentials ϕ_1 & ϕ_2 resp. Using complex function $w = a \log z + c$. P.T capacitance per unit length of the capacitor formed by them is $\frac{2\pi\lambda}{\log(r_2/r_1)}$, where λ is a dielectric constant of the medium.

$z = re^{i\theta}$.

$$\phi + i\psi = a \log(re^{i\theta}) + c$$

$$= a[\log r + \log e^{i\theta}] + c.$$

$$= a \log r + a i\theta + c$$

$$\phi + i\psi = (a \log r + c) + ia\theta.$$

$\phi = a \log r + c \qquad \psi = a\theta$

Let $\phi_1 = a \log r_1 + c$ & $\phi_2 = a \log r_2 + c$

Thus the potential difference $= \phi_2 - \phi_1$

$$\phi_2 - \phi_1 = a[\log r_2 - \log r_1]$$

$$\phi_2 - \phi_1 = a \log\left(\frac{r_2}{r_1}\right)$$

Also the total charge (flux) is given by

$$\int_0^{2\pi} d\psi = \int_0^{2\pi} a\, d\theta = \underline{\underline{2\pi a}}. \to \text{Total charge.}$$

The capacitance without di-electric $= \frac{\text{Charge}}{\text{Potential diff}}$

$$= \frac{2\pi a}{a \log\left(\frac{r_2}{r_1}\right)}$$

NOTES

$$C = \frac{2\pi}{\log(r_2/r_1)}$$

A medium of di-electric constant λ increases the potential difference to λ times in ~~vaccum~~ vacuum for ~~the~~ same charge.

$\therefore$ Capacitance with di-electric $= \dfrac{2\pi\lambda}{\log(r_2/r_1)}$

Application to flow problems.

The real and imaginary part of analytic functn play important role in the study 2-dimensional fluid & heat . In a 2-dimensional

the velocity vector happens to be gradient of a scalar harmonic functn. This scalar functn is called the velocity potential & is denoted by ϕ. The harmonic conjugate of this functn is called the stream function for the flow and is usually denoted by ψ. The analytic function for which ψ is a real part and ψ is a imaginary part is called the complex potential for the flow & is denoted by $w(z)$

$$w(z) = \phi + i\psi.$$

Since $w(z) = \phi + i\psi$ is an analytic function, the CR eqn applied to this functn yield the following relation between ϕ & ψ.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ; \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}.$$

Consequently the curves ϕ = constant, ψ = constant are orthogonal. The curves ϕ = constant are called the equipotential lines and the curves ψ = constant are called stream lines.

7) An electrostatic field in the x-y plane is given by the potential functions $\phi = 3x^2y - y^3$, find the stream functn.

Solⁿ: $\psi = ?$

$\Rightarrow \frac{\partial \phi}{\partial x} = 6xy$.

$\frac{\partial \phi}{\partial y} = 3x^2 - 3y^2$.

By using the CR eqⁿ for a complex potential we have,

$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ $\qquad$ $\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$.

$\frac{\partial \psi}{\partial y} = 6xy$ $\qquad$ $\frac{\partial \psi}{\partial x} = -3x^2 + 3y^2$.

$\int$ w.r.t y. $\qquad$ $\int$ w.r.t x

$\psi = 6x\frac{y^2}{2} + f(x)$ $\qquad$ $\psi = -x^3 + 3xy^2 + g(y)$. — (2)

$= 3xy^2 + f(x)$ — (1) $\qquad$ choose $f(x) = -x^3$ $\quad$ $g(y) = 0$.

$$\boxed{\psi = 3xy^2 - x^3}$$

2) P.T $\phi = x^2 - y^2$ and $\psi = \frac{y}{x^2+y^2}$ is harmonic & hence find the analytic functn.

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ $\qquad$ $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$

$\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y)$ $\qquad$ $\frac{\partial}{\partial x} y \frac{-1}{(x^2+y^2)^2} \cdot 2x + \frac{\partial}{\partial y} y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y + \frac{1}{x^2+y^2}$

$2 - 2 = 0$. $\qquad$ $-2\left[xy\left(\frac{+2}{(x^2+y^2)^3}\right) \cdot 2x + \frac{1}{x^2+y^2} \cdot 2x\right] + -\left[2y^2\left(\frac{-2}{(x^2+y^2)^3} \cdot 2y\right) + \frac{1}{(x^2+y^2)^2} \cdot 4y\right] + \frac{-1}{(x^2+y^2)^2} \cdot 2y$.

$\Rightarrow \frac{-4xy}{(x^2+y^2)^3} + \frac{-4x}{x^2+y^2}$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$.

$\frac{\partial^2 \psi}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) - (-2xy)(2)(x^2+y^2)(2x)}{(x^2+y^2)^4}$

$= \frac{(x^2+y^2)\left[-2y(x^2+y^2) + 8x^2y\right]}{(x^2+y^2)^4}$.

$= \frac{-2yx^2 - 2y^3 + 8x^2y}{(x^2+y^2)^3} =$

27. Two concentric cylinders of a radii R_1, R_2 $(r_1 < r_2)$ are kept at potentials ϕ_1 & ϕ_2 respectively. Using complex functions $w = a\log z + c$. P.T the capacitance per unit length of the capacitor form by them is $\frac{2\pi\lambda}{\log r_2/r_1}$ where $\lambda \to$ dielectric constant of the medium.

Soln

$$\phi + i\psi = a\log(re^{i\theta}) + c$$

$$= a\left[\log r + \log e^{i\theta}\right] + c$$

$$= a\log r + a\log e^{i\theta} + c \quad = a\log r + a\,i\theta + c$$

$$= (a\log r + c) + ia\theta$$

Let $\phi = a\log r + c$ $\qquad \psi = a\theta \Rightarrow d\psi = a\,d\theta$

Let $\phi_1 = a\log r_1 + c$ $\qquad \phi_2 = a\log r_2 + c_0$

Thus potential diff $= \phi_2 - \phi_1$

$$= a\left[\log r_2 - \log r_1\right]$$

$$= a\left[\log \frac{r_2}{r_1}\right]$$

Also the total charge (flux) is given by,

$$\int_0^{2\pi} d\psi = \int_0^{2\pi} a\,d\theta = a\,\theta\Big]_0^{2\pi} = 2\pi a$$

The capacitance without dielectric $= \frac{\text{charge}}{\text{potential diff}}$

$$= \frac{2\pi \cancel{a}}{\cancel{a}\log\left(\frac{r_2}{r_1}\right)} = \frac{2\pi}{\log\left(\frac{r_2}{r_1}\right)}$$

A medium of dielectric constant λ increases the p.d to λ times that in vaccum for the same charge

$\therefore$ The capacitance with dielectric $= \frac{2\pi\lambda}{\log r_2/r_1}$