## Relations

Let A and B be two sets. Then the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$, is called the cartesian Product, or Cross Product or Product set of A and B. and is denoted by $A \times B$.

Thus, $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

**Note** : $A \times B \neq B \times A$.

$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}$.

and $(a,b) \neq (b,a)$

$A \times A = \{(a,b) \mid a \in A,\ b \in A\}$.

The product $A \times A = A^2$.

If a set A has m elements and a set B has n elements, then 'a' can be chosen from A in m ways and with every one of these choices (of a), b can be chosen from B in n ways.

Accordingly, (a, b) can be chosen in $m \times n$ ways. This means that $A \times B$ has exactly mn elements.

**Result** : If A and B are finite sets with $|A| = m$ and $|B| = n$, then $A \times B$ is a finite set with $|A \times B| = mn$.

If A and B are finite sets, then
$|A \times B| = |A||B|$.

From this result, we have,
$|B \times A| = |B||A| = |A||B| = |A \times B|$.

and $|A \times A| = |A|^2$.

Ex: If A has 5 elements and B has 8 elements, then $A \times B$ and $B \times A$ will have $5 \times 8 = 8 \times 5 = 40$ elements each, $A \times A$ will have $5 \times 5 = 25$ elements, $B \times B$ will have $8 \times 8 = 64$ elements

Since $|A \times B| = mn$ then $|A| = m$, & $|B| = n$, it follows that $|P(A \times B)| = 2^{mn}$.

This means that, when $|A| = m$ and $|B| = n$, $A \times B$ has $2^{mn}$ number of subsets.

Ex: (1) Let $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{4, 6\}$. Find

a) $(A \times B$ (b) $B \times A$ (c) $B \times C$ (d) $A \times C$

(e) $(A \cup B) \times C$ (f) $A \cup (B \times C)$ (g) $(A \times B) \cup C$.

(h) $A \cap (B \times C)$ (i) $(A \times B) \cup (B \times C)$.

(j) $(A \times B) \cap (B \times A)$ (k) $(A \times B) \cap (B \times C)$.

Sol: (a) $A \times B = \{(1,2), (1,3), (3,2)\ (3,3)\ (5,2), (5,3)\}$

(b) $B \times A = \{(2,1)\ (2,3), (2,5), (3,1)\ (3,3)\ (3,5)\}$.

(c) $B \times C = \{(2,4)\ (2,6), (3,4), (3,6)\}$.

(d) $A \times C = \{(1,4)\ (1,6)\ (3,4)\ (3,6), (5,4), (5,6)\}$.

(e) $(A \cup B) \times C = \{1, 2, 3, 5\} \times \{4, 6\}$.

$= \{(1,4)\ (1,6), (2,4)\ (2,6)\ (3,4)\ (3,6), (5,4)\ (5,6)\}$

(f) $A \cup (B \times C) = \{1, 3, 5\} \cup \{(2,4), (2,6), (3,4), (3,6)\}$

$= \{1, 3, 5, (2,4), (2,6), (3,4), (3,6)\}$.

$(A \times B) \cup C = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3), (4,6)\}$.

$A \cap (B \times C) = \phi$

$(A \times B) \cup (B \times C) = \{(1,2), (1,3), (3,2), (3,3), (5,2)\ (5,3), (2,4), (2,6), (3,4), (3,6)\}$.

$(A \times B) \cap (B \times A) = \{(3,3)\}$, $(A \times B) \cap (B \times C) = \phi$.

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## Zero-one Matrices and Directed graphs

Consider the sets $A=\{a_1, a_2, \ldots, a_n\}$ and $B=\{b_1, b_2, \ldots, b_n\}$ of orders $m$ and $n$ respectively. Then $A \times B$ consists of all ordered pairs of the form $(a_i, b_j)$, $1 \le i \le m$, $1 \le j \le n$, which are $mn$ in number.

Let $R$ be a relation from $A$ to $B$ so that $R$ is a subset of $A \times B$.

Now, let us put $m_{ij} = (a_i, b_j)$ and assign the values 1 or 0 to $m_{ij}$ according to the following rule:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R. \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$

The $m \times n$ matrix formed by these $m_{ij}$'s is called the relation matrix for $R$, and is denoted by $M_R$ or $M(R)$. Since $M(R)$ contains only 0 and 1 as its elements, $M(R)$ is called the zero-one matrix for $R$.

It is to be noted that the rows of $M_R$ corresponds to the elements of $A$ and the columns to those of $B$.

When $B = A$, the matrix $M_R$ becomes an $n \times n$ matrix whose elements are $m_{ij} = (a_i, a_j)$ with

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R. \end{cases}$$

For example: Consider the sets $A = \{0, 1, 2\}$ and $B = \{p, q\}$, and the relation $R$ from $A$ to $B$ defined by $R = \{(0, p), (1, q), (2, p)\}$

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Here, $A = \{a_1, a_2, a_3\} = \{0, 1, 2\}$ and $B = \{b_1, b_2\} = \{p, q\}$.
We note that

$m_{11} = (a_1, b_1) = (0, p) = 1$ because $(0,p) \in R$.

$m_{12} = (a_1; b_2) = (0, q) = 0$, because $(0,q) \notin R$.

$m_{21} = (a_2, b_1) = (1, p) = 0$

$m_{22} = (a_2, b_2) = (1, q) = 1$

$m_{31} = (a_3, b_1) = (2, p) = 1$

$m_{32} = (a_3, b_2) = (2, q) = 0$

Accordingly, the matrix of the relation R is

$$M(R) \equiv M_R = [m_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Ex ② Consider the set $A = \{1, 2, 3, 4\}$ and a relation R defined on A by $R = \{(1,2), (1,3), (2,4), (3,2)\}$.
Thus, here, $A = \{a_1, a_2, a_3, a_4\} = B$ where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$.

Accordingly, $m_{ij} = (a_i, a_j) = (i, j)$,
$i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$ and we find that

$m_{11} = (1,1) = 0$ $\because (1,1) \notin R$

$m_{12} = (1,2) = 1$, $\because$ ~~(1,2) ∉ R~~ $\because (1,2) \in R$.

$m_{13} = (1, 3) = 1$ $\because (1,3) \in R$.

$m_{1,4} = (1,4) = 0$.

$m_{21} = (2,1) = 0$, $m_{22} = (2,2) = 0$, $m_{23} = (2,3) = 0$, $m_{2,4} = (2,4) = 1$

$m_{31} = (3,1) = 0$, $m_{3,2} = (3,2) = 1$, $m_{33} = (3,3) = 0$, $m_{34} = (3,4) = 0$.

$m_{41} = (4,1) = 0$ $m_{42} = (4,2) = 0$ $m_{43} = (4,3) = 0$, $m_{44} = (4,4) = 0$.

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Thus, the matrix of R is

$$M_R = [m_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Remarks** : For any relation R from a finite set A to a finite set B, the following results are obvious:

(1) $M_R$ is the zero matrix iff $R = \phi$.

(2) Every element of $M_R$ is 1 iff $R = A \times B$.

## Digraph of a Relation :

Let R be a relation on a finite set A. Then R can be represented pictorially as described below:

1) Draw a small circle for each element of A & label the circle with the corresponding element of A.

2) These circles are called vertices / nodes.

3) Draw an arrow, called an edge, from a vertex x to a vertex y iff $(x, y) \in R$.

4) The resulting pictorial representation of R is called a directed graph or digraph of R.

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the origin or the source for that edge, and a vertex where an edge ends is called the terminus for that edge.

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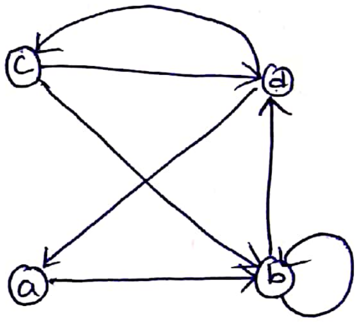
A vertex which is neither a source nor a terminus of any edge is called an isolated vertex.

An edge for which the source and terminus are ~~not~~ one and the same vertex is called a loop.

The number of edges (arrows) terminating at a vertex is called the in-degree of that vertex. and the number of edges (arrows) leaving a vertex is called the out-degree of that vertex.

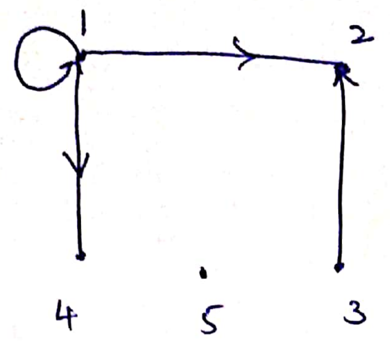
For example: consider the set $A=\{a,b,c,d\}$ and the relation $R=\{(a,b),(b,b),(b,d),(c,b),(c,d),(d,a),(d,c)\}$ defined on $A$.

The digraph of this relation is as shown below:




Observe that, since $(b,b)\in R$, there is a loop at the vertex $b$. we also see that in-degrees of the vertices $a, b, c, d$ are $1, 3, 1, 2$ respectively. Further, the out-degrees of $a,b,c,d$ are $1, 2, 2, 2$ respectively.

Ex ② : Consider the set $A=\{1,2,3,4,5\}$ and the relation $R=\{(1,1),(1,2),(1,4),(3,2)\}$ defined on $A$. The digraph of this relation is as shown below:




we observe that the above digraph has

* a loop at the vertex 1.
* the vertex 5 is an isolated vertex; no edge leaves this vertex & no edge terminates at this vertex.

Also, in-degrees & the out-degrees of the vertices 1, 2, 3, 4 are 1, 2, 0, 1 & 3, 0, 1, 0 respectively.

## Problems

① Let $A = \{1,2\}$ and $B = \{p,q,r,s\}$ and let the relation $R$ from $A$ to $B$ be defined by
$R = \{(1,q), (1,r), (2,p), (2,q), (2,s)\}$
write down the matrix of $R$.

Sol. $A \times B = \{(1,p), (1,q), (1,r), (1,s), (2,p), (2,q), (2,r), (2,s)\}$
$R = \{(1,q), (1,r), (2,p), (2,q), (2,s)\}$.

$(1,p) \notin R \to 0$, $(1,q) \in R \to 1$, $(1,r) \in R \to 1$, $(1,s) \notin R$

$(2,p) \in R$, $(2,q) \in R$, $(2,r) \notin R$, $(2,s) \in R$.

$$\therefore M_R = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

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(a)

(2) Let $A = \{1,2,3,4\}$ and let $R$ be the relation on $A$. defined by $xRy$ iff "x divides y", written $x/y$.

(a) write down $R$ as a set of ordered pairs.

(b) Draw the digraph of $R$.

(c) Determine the in-degrees and out-degrees of the vertices in the digraph.

25

$A \times A = m \times n$
$= 16$.

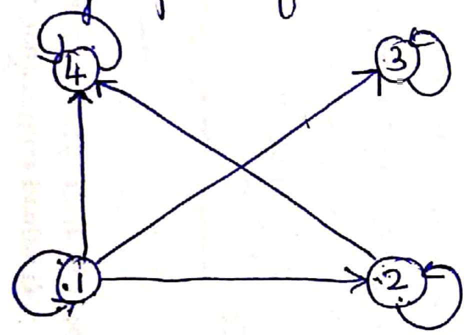
(a) we observe that,

$1/1$, $1/2$, $1/3$, $1/4$,    $R = \{$

$2/2$, $2/4$, $3/3$, $4/4$.

Hence, $R = \{(1,1), (1,2) (1,3) (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(b) The digraph of $R$ is as shown below.




$R = \{1,1), (1,2), (1,4), (1,3),$
$(2,2) (2,4), (3,3), (4,4),\}$

(c) In-degrees of 1, 2, 3, 4 are : 1, 2, 2, 3 respectively.
Out-degrees of 1, 2, 3, 4 are : 4, 2, 1, 1 "

(3) Let $A = \{1,2,3,4,6\}$ and $R$ be a relation on $A$ defined by $aRb$ iff a is a multiple of b. Represent the relation $R$ as a matrix and draw its digraph.

Sol: $R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4),$
$(6,1), (6,2), (6,3), (6,6)\}$.

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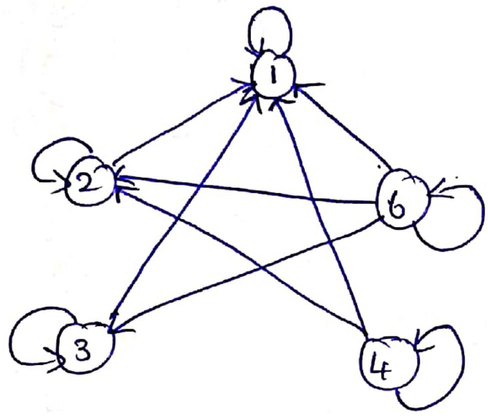
Matrix of R is $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

The digraph of R is




11 (a)

4.) Draw the relation R from a set A to a set B a described by the following matrix:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$ ~~3×4~~ 4×3

$A = \{a, b, c, d\}$

$B = \{x, y, z\}$.

$R = \{(a,x), (a,z), (b,x), (b,y), (c,z), (d,x)\}$

Soln The given ~~mad~~ $M_R$ is $4 \times 3$ matrix.

$\therefore |A| = 4$ , $|B| = 3$.

If $|A| = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3\}$, then by observing the elements of $M_R$, we find that,

$(a_1, b_1) \in R$ $\quad (a_1, b_2) \notin R$ $\quad (a_1, b_3) \in R$

$(a_2, b_1) \in R$ $\quad (a_2, b_2) \in R$ $\quad (a_2, b_3) \notin R$

$(a_3, b_1) \notin R$ $\quad (a_3, b_2) \notin R$ $\quad (a_3, b_3) \in R$

$(a_4, b_1) \in R$ $\quad (a_4, b_2) \notin R$ $\quad (a_4, b_3) \notin R$

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Thus $R = \{(a_1,b_1), (a_1,b_3), (a_2,b_1), (a_2,b_2), (a_3,b_3), (a_4,b_1)\}$

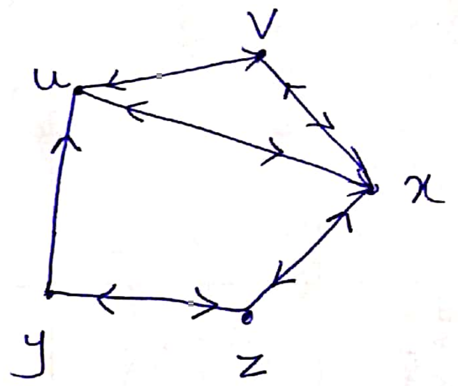
(5). Let $A = \{u, v, x, y, z\}$ and $R$ be a relation on $A$ whose matrix is as given below. Determine $R$ and also draw the associated digraph.

$$M(R) = \begin{array}{c} \\ u \\ v \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} u & v & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Sol.

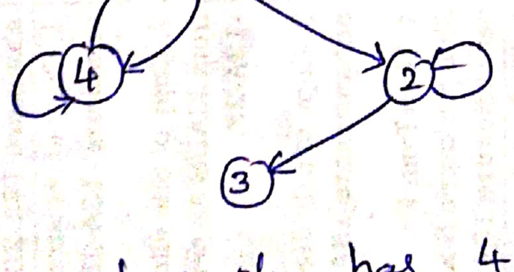
$R = \{(u,v), (u,x), (v,u), (v,x), (x,u)\ (x,v), (x,z), (y,u), (y,z), (z,x)\ (z,y)\}$.

Digraph of R :




11 (b)

(6) Find the relation represented by the digraph given below. Also write down its matrix:




The above digraph has 4 vertices. The relation R represented by it is defined on the set $A = \{1, 2, 3, 4\}$ and is given by

$R = \{(1,2), (1,4)\ (2,2)\ (2,3), (4,1)\ (4,4)\}$

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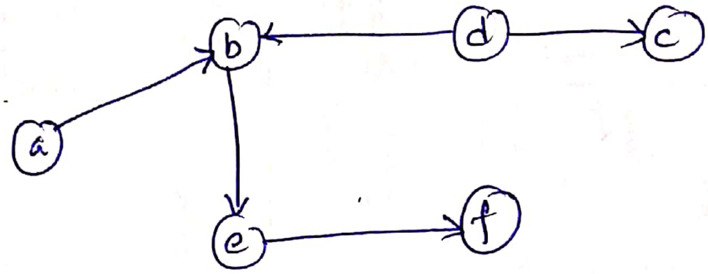
The matrix of R is,
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(7) For $A = \{a, b, c, d, e, f\}$, the digraph in figure below represents a relation R on A. Determine R as well as its associated relation matrix.



Sol. $R = \{(a,b), (b,e)\ (b,b)\ (d,c), (e,f)\}$.

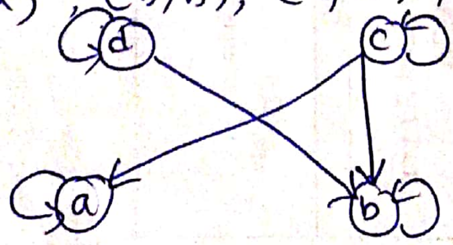
matrix of R is
$$M(R) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

13 (a)

(8) Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix
$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

construct the digraph of R & list the in-degrees and out-degrees of all vertices.

$R = \{(a,a), (b,b), (c,a), (c,b), (c,c), (d,b) (d,d)\}$



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In-degrees of {a, b, c, d} are : 2, 3, 1, 1
out-degrees of {a, b, c, d} are : 1, 1, 3, 2.

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# Operations on Relations

II

Since a relation is a subset of the cartesian product of two sets, the set-theoretic operations may be used to construct new ~~to~~ relations from given ~~m~~ relations.

## Union and Intersection of Relations :

Union : Given the relations $R_1$ and $R_2$ from a set $A$ to a set $B$, the union of $R_1$ and $R_2$, denoted by $R_1 \cup R_2$, is defined as a relation from $A$ to $B$ with the property that $(a,b) \in R_1 \cup R_2$ if and only if $(a,b) \in R_1$ or $(a,b) \in R_2$.

Intersection : The intersection of $R_1$ and $R_2$, denoted by $R_1 \cap R_2$, is defined as a relation from $A$ to $B$ with the property that $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.

Evidently, $R_1 \cup R_2$ is the union of the sets $R_1$ & $R_2$ and $R_1 \cap R_2$ is the intersection of the sets $R_1$ & $R_2$ in the universal set $A \times B$.

Complement of a Relation : Given a relation $R$ from a set $A$ to a set $B$, the complement of $R$, denoted by $\overline{R}$, is denoted as a relation from $A$ to $B$ with the property that $(a,b) \in \overline{R}$ iff $(a,b) \notin R$.

In other words, $\overline{R}$ is the complement of the set $R$ in the universal set $A \times B$.

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**Converse of a Relation**: Given a relation R from a set A to a set B, the converse of R, denoted by $R^c$, is defined as a relation from B to A with the property that $(a,b) \in R^c$ iff $(b,a) \in R$.

**Results**: (i) If $M_R$ is the matrix of R, then $(M_R)^T$, the transpose of $M_R$, is the matrix of $R^c$, &

(ii) $(R^c)^c = R$.

10 (b)

Ex. (1). Consider the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. and the relations $R = \{(a,1), (b,1), (c,2), (c,3)\}$ and $S = \{(a,1), (a,2), (b,1), (b,2)\}$ from A to B. Determine $\bar{R}$, $\bar{S}$, $R \cup S$, $R \cap S$, $R^c$ and $S^c$.

Soln: We have,

$A \times B = \{(a,1), (a,2), (a,3)\ (b,1)\ (b,2), (b,3), (c,1), (c,2), (c,3)\}$

$\therefore \bar{R}$ = complement of R in $A \times B = (A \times B) - R$.

$= \{(a,2), (a,3), (b,2)\ (b,3), (c,1)\}$.

$\bar{S}$ = complement of S in $A \times B = (A \times B) - S$.

$= \{(a,3), (b,3), (c,1)\ (c,2)\ (c,3)\}$.

$R \cup S = \{(a,1), (a,2), (b,1), (b,2), (c,2), (c,3)\}$.

$R \cap S = \{(a,1), (b,1)\}$.

$R^c = \{(1,a), (1,b), (2,c), (3,c)\}$. Converse of $R = R^c$, $(a,b) \in R$, $(b,a)$

$S^c = \{(1,a), (2,a), (1,b), (2,b)\}$.

(2) Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R$ and $S$ from $A$ to $B$ are represented by the following matrices. Determine the relations $\overline{R}$, $R \cup S$, $R \cap S$ and $S^c$ and their matrix representations.

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 0 \end{array}, \quad M_S = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \end{array}$$

$3 \times 4$

Soln: From $M_R$ & $M_S$,

$R = \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\}$.

$S = \{(1,1), (1,2), (1,3) (1,4), (2,4), (3,2), (3,4)\}$.

$\therefore \overline{R} = \{(1,2), (1,4), (2,1), (2,2), (2,3), (3,4)\}$.

$R \cup S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,1) (3,2), (3,3), (3,4)\}$

$R \cap S = \{(1,1), (1,3), (2,4), (3,2)\}$

$S^c = \{(1,1), (2,1), (3,1), (4,1), (4,2), (2,3), (4,3)\}$.

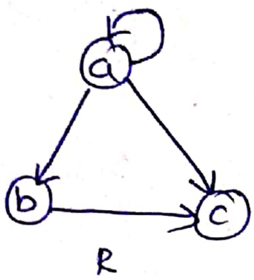
$$M(\overline{R}) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}; \quad M((R \cup S)) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \end{array}$$

$$M((R \cap S)) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \end{array}, \quad M(S^c) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 \end{array}$$

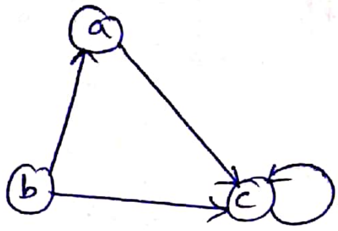
$3 \times 4$

(3) The digraphs of two relations on the set
13(b) $A = \{a, b, c\}$ are given below. Draw the digraphs of $\bar{R}$, $R \cup S$, $R \cap S$ & $R^c$.

Sol..



R



S.

Soln: $R = \{(a,a)\ (a,b), (a,c)\ (b,c)\}$.

$S = \{(a,c), (b,a)\ (b,c), (c,c)\}$.

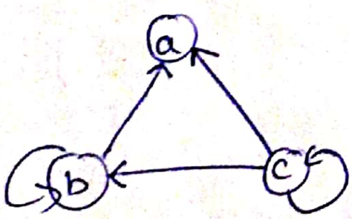
$\therefore\ \bar{R} = \{(b,a), (b,b), (c,a)\ (c,b), (c,c)\}$.

$R \cup S = \{(a,a), (a,b), (a,c), (b,a), (b,c), (c,c)\}$.

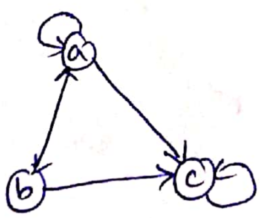
$R \cap S = \{(a,c), (b,c)\}$.

$R^c = \{(a,a), (b,a), (c,a)\ (c,b)\}$.

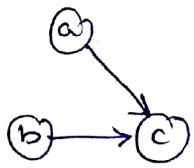
Digraphs of $\bar{R}$, $R \cup S$, $R \cap S$ & $R^c$ are as follows:



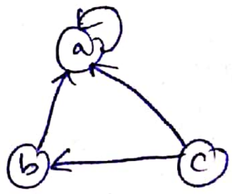
$\bar{R}$



$R \cup S$



$R \cap S$



$R^c$.

=

## Composition of Relations :

Consider a relation R from a set A to a set B and a relation S from the set B to a set C. With these relations in hand, we can define a new relation, called the product or the composition of R and S, from the set A to the set C. This new relation, denoted by $R \circ S$, is defined as follows:

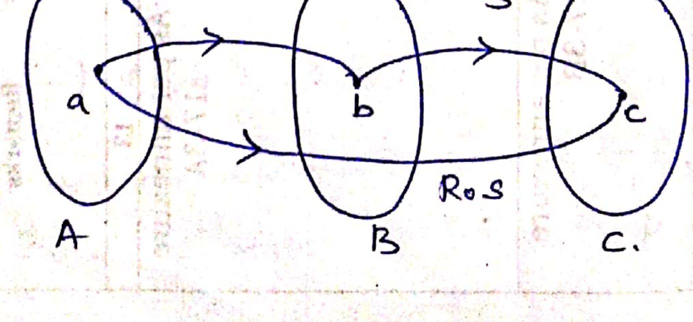
If a is in A, and 'c' is in C, then $(a, c) \in R \circ S$ iff there is some b in B such that $(a, b) \in R$ and $(b, c) \in S$. That,

$$R \circ S = \{(a, c) \mid a \in A,\ c \in C \text{ and there exists } b \in B \text{ with } (a, b) \in A \text{ and } (b, c) \in B\}.$$

In other words, a is related to c by $R \circ S$ in two stages: if we can get c from a in B by relation R
* first to some b in B by relation R and then from b to c by relation S.

The relation $R \circ S$ is interpreted as "R followed by S".

$R \subseteq (A \times B)$, $S \subseteq (B \times C)$, and $R \circ S \subseteq (A \times C)$.



Note (1) $R \circ S \neq S \circ R$

(2) If $R$ is a relation on $A$, then $R \circ R$ is again a relation on $A$.

$R \circ R = R^2$, $(R \circ R) \circ R = R^3, \ldots$

In general, $R^n$ is a relation on $A$ defined recursively by (i) $R^1 = R$, (ii) $R^n = R \circ R^{n-1}$ for $n \geq 2$.

**Theorem**: Let $R$ be a relation from a set $A = \{a_1, a_2, \ldots, a_m\}$ to a set $B = \{b_1, b_2, \ldots, b_n\}$ & $S$ be a relation from the set $B$ to a set $C = \{c_1, c_2, \ldots, c_p\}$. Then the matrices of $R$, $S$ & $R \circ S$ satisfy the identity.

$$M(R) \times M(S) = M(R \circ S).$$

Ex. (1) Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$.
Also, let $R_1$ be a relation from $A$ to $B$, defined by

$R_1 = \{(1, x), (2, x), (3, y)\ (3, z)\}$,

$R_2$ & $R_3$ be relations from $B$ to $C$, defined by

$R_2 = \{(w, 5), (x, 6)\}$, $R_3 = \{(w, 5), (w, 6)\}$.

Find $R_1 \circ R_2$ and $R_1 \circ R_3$.

Soln: $(1, x) \in R_1$ and $(x, 6) \in R_2$ ; $\therefore (1, 6) \in R_1 \circ R_2$
$(2, x) \in R_1$ and $(x, 6) \in R_2$ ; $\therefore (2, 6) \in R_1 \circ R_2$

$\therefore R_1 \circ R_2 = \{(1, 6), (2, 6)\}$.

There is no element $(a, b) \in R_1$ such that $(b, c) \in R_3$. Hence $R_1 \circ R_3 = \phi$.

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~~② For the relations $R_1 = \{1,2,3,4$~~ ~~$A = \{1,2,3,4$~~

~~$R_1 = \{(1,$~~

② For the relations $R_1 = \{(1,x),(2,x),(3,y),(3,z)\}$

17 ⓐ & $R_2 = \{(w,5),(x,6)\}$ of the sets $A = \{1,2,3,4\}$

& $B = \{w,x,y,z\}$, $C = \{5,6,7\}$, find $M(R_1)$, $M(R_2)$ &

$M(R_1 \circ R_2)$. Verify that $M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$.

& 17.

$$M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_1 \circ R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_1) \cdot M(R_2) = \underset{4\times4}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \underset{4\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$= \underset{4\times3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} = M(R_1 \circ R_2).$$

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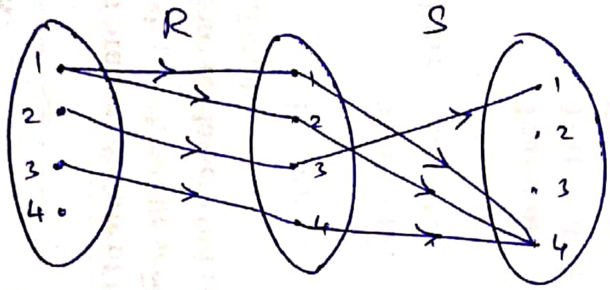
(3) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,3), (3,4)\}$, $S = \{(3,1), (4,4), (2,4), (1,4)\}$. be relations on A. Determine the relations $R \circ S$, $S \circ R$, $R^2$ and $S^2$.

Soln : $R \circ S = \{(1,4), (2,1), (3,4)\}$.




$S = \{(3,1), (4,4), (2,4), (1,4)\}$

$R = \{(1,1), (1,2), (2,3), (3,4)\}$.

$R \circ S$.

$S \circ R = \{(3,1), (3,2)\}$.

$R \circ R = R^2 = \{(1,1), (1,2), (1,3), (2,4)\}$.

$S \circ S = S^2 = \{(3,4), (4,4), (2,4), (1,4)\}$.

(4) If $A = \{1, 2, 3, 4\}$ and $R, S$ are relations on A defined by $R = \{(1,2), (1,3), (2,4), (4,4)\}$, $S = \{(1,1), (1,2), (1,3), (1,4), (2,3), (2,4)\}$. find $R \circ S$, $S \circ R$, $R^2$, $S^2$, write down their matrics.

Sol. $R \circ S = \{(1,3), (1,4)\}$., $S \circ R = \{(1,2), (1,3), (1,4), (2,4)\}$

$R^2 = R \circ R = \{(1,4), (2,4)\}$

$S^2 = S \circ S = \{(1,2), (1,3), (1,4)\}$

$$M(R \circ S) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad M(S \circ R) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$M(R^2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(S^2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(5) Let $A = \{a, b, c\}$ and $R$ and $S$ be relations on $A$ whose matrices are as given below:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the composite relations $RoS$, $SoR$, $RoR$, $SoS$ and their matrices.

Soln. $R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b)\}$.

$S = \{(a,a), (b,b), (b,c), (c,a), (c,c)\}$.

$RoS = \{(a,a), (a,c), (b,a)\ (b,b)\ (b,c), (a,c), (c,b)\ (c,c)\}$.
$(b,c), (c,a),$

$SoR = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,c), (c,b)\}$

$RoR = R^2 = \{(a,a), (a,b), (b,a), (b,b), (a,c), (b,c), (c,a)$
$(c,b), (c,c)\}$.

$SoS = S^2 = \{(a,a), (b,c)\ (b,a), (c,a), (b,b), (c,c)\}$.

$$M(RoS) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad M(SoR) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$M(RoR) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M(SoS) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

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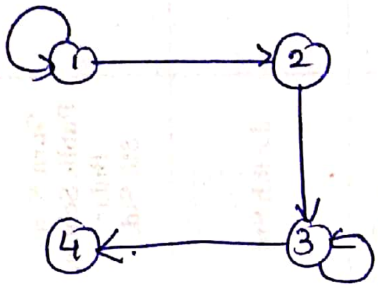
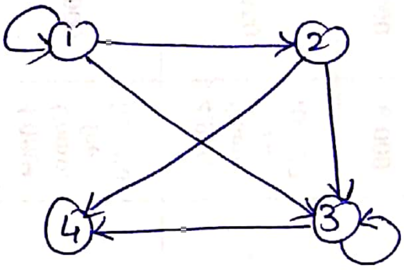
16(b) 6) Let $R = \{(1,1), (1,2), (2,3), (3,3), (3,4)\}$ be a relation on $A = \{1, 2, 3, 4\}$. Draw the digraph of R. Obtain $R^2$ and draw the digraph of $R^2$.

Soln. Digraph of R.




$R \circ R = R^2 = \{(1,1), (1,2), (1,3), (2,3), (3,4) (2,4), (3,3)\}$.

Digraph of $R^2$




7) Let $R = \{(1,2), (1,3), (2,4), (3,2)\}$ be a relation on $A = \{1, 2, 3, 4\}$. Write down the relation matrix $M(R)$ of R. Compute $[M(R)]^2$ and hence obtain $R^2$.

Soln.

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[M(R)]^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $[M(R)]^2 = M(R^2)$,

$R^2 = \{(1,2), (1,4), (3,4)\}$.

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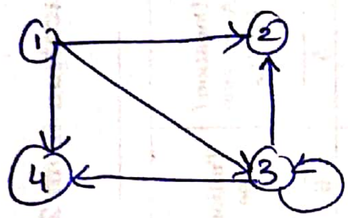
(8) If $A = \{1, 2, 3, 4\}$ and $R$ is a relation on $A$ defined by $R = \{(1,2), (1,3), (2,4), (3,2), (3,3), (3,4)\}$. find $R^2$ and $R^3$. write down their digraphs.

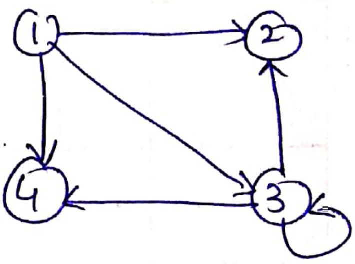
Sol : $R^2 = R \circ R = \{(1,2), (1,3)\ (1,4), (3,2), (3,3), (3,4)\}$.

$R^3 = R^2 \circ R = \{(1,2), (1,3), (1,4), (3,2), (3,3), (3,4)\}$.

Digraphs of $R^2$ & $R^3$ are :



$R^2$



$R^3$

15(a)

(9) For the relation $R = \{(1,2), (1,3), (2,4), (3,2), (3,3), (3,4)\}$, on the set $A = \{1,2,3,4\}$, find $M(R)$, $M(R^2)$, $M(R^3)$. verify that $M(R^2) = [M(R)]^2$ & $M(R^3) = [M(R)]^3$.

Soln.

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R \circ R = \{(1,2), (1,3)\ (1,4), (3,2), (3,3), (3,4)\}$.

$$M(R^2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M(R^3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

According to the theorem, $M(R) \times M(S) = M(R \circ S)$ we have, $M(R) \times M(R) = M(R^2)$ & $M(R^2) \times M(R) = M(R^3)$.

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$$\therefore [M(R)]^2 = M(R)\cdot M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^2).$$

Also,

$$[M(R)]^3 = M(R)\cdot [M(R)]^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^3).$$

19) Let $R = \{(1,2),(3,4),(2,2)\}$ and $S = \{(4,2),(2,5),(3,1),(1,3)\}$

17(b) be relations on the set $A = \{1,2,3,4,5\}$. Find the following: $R\circ(R\circ S)$, $R\circ(S\circ R)$, $S\circ(R\circ S)$, $S\circ(S\circ R)$.

Ans.

$R\circ S = \{(1,5),(3,2),(2,5)\}$, & ~~S~~

$S\circ R = \{(4,2),(3,2),(1,4)\}$

$\therefore R\circ(R\circ S) = \{(1,5),(2,5)\}$, $R\circ(S\circ R) = \{(3,2)\}$.

$S\circ(R\circ S) = \{(4,5),(3,5),(1,2)\}$.

$S\circ(S\circ R) = \{(3,4),(1,2)\}$

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# Properties of Relations

## Some important properties of relations defined on a set :

**Reflexive Relation :** A relation R on a set A is said to be reflexive if $(a,a) \in R$, for all $a \in A$.

In other words, a relation R on a set A is reflexive whenever every element a of A is related to itself by R. (i.e $aRa$, $\forall a \in A$).

**For example :** ① The relation " is less than or equal to " is a reflexive relation on the set of all real numbers. Because, $a = a$ for every real number a.

② If $A = \{1, 2, 3, 4\}$, then the relation $R = \{(1,1), (2,3), (3,3)\}$. is not reflexive. Because $4 \in A$ but $(4,4) \notin R$.

* The matrix of a reflexive relation must have 1's on its main diagonal.

**Symmetric Relation :** A relation R on a set A is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$.

It follows that R is not symmetric if there exist $a, b \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$.

A relation which is not symmetric is called an asymmetric relation.

Ex: If $A = \{1,2,3\}$ and $R_1 = \{(1,1), (1,2), (2,1)\}$ & $R_2 = \{(1,2), (2,1), (1,3)\}$ are relations on A, then, $R_1$ is symmetric but $R_2$ is asymmetric ! because, $(1,3) \in R_2$ but $(3,1) \notin R_2$.

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(2) It is evident that for the matrix $M_R = [m_{ij}]$ of a symmetric relation the following property holds:

If $m_{ij} = 1$ then $m_{ji} = 1$ and if $m_{ij} = 0$ then $m_{ji} = 0$.

This means that the matrix $M_R$ of a symmetric relation R is such that the $(i, j)^{th}$ element of $M_R$ is equal to the $(j, i)^{th}$ element of $M_R$.

In other words, the matrix of a symmetric relation is a symmetric matrix.

(3) In the digraph of a symmetric relation, if there is an edge from vertex a to vertex b, then there is an edge from b to a, this means that if two vertices are connected by an edge, they must always be connected in both directions.

<u>Irreflexive Relation</u> : A relation on a set A is said to be irreflexive if $(a, a) \notin R$ for any $a \in A$.

That is, a relation R is irreflexive if no element of A is related to itself by R.

<u>Ex</u> : The relation "is less than" and "is greater than" are irreflexive on the set of all real numbers.

<u>Note</u> : (1) The matrix of an irreflexive relation must have 0's on its main diagonal.

(2) The digraph of an irreflexive relation has no cycle of length 1 at any vertex.

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**Antisymmetric Relation**: A relation $R$ on a set $A$ is said to be antisymmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a=b$.

It follows that, $R$ is not antisymmetric if there exists $a, b \in A \ni (a,b) \in R$ & $(b,a) \in R$ but $a \neq b$.

Ex: The relation "is less than or equal to" on the set of all real numbers is an antisymmetric relation because if $a \leq b$ and $b \leq a$ then $a=b$.

**Results**: (1) If $M_R = [m_{ij}]$ is the matrix of an antisymmetric relation, then, for $i \neq j$, we have either $m_{ij}=0$ or $m_{ji}=0$.

(2) In the digraph of an antisymmetric relation for two different vertices $a$ & $b$, there cannot be a ~~d~~ bidirectional edge between $a$ & $b$.

**Transitive Relation**:

A relation $R$ on a set $A$ is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, for all $a, b, c \in A$.

It follows that $R$ is not transitive if there exist $a, b, c \in A$ such that $(a,b) \in R$ & $(b,c) \in R$ but $(a,c) \notin R$.

Ex: The relation "is less than or equal to"

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and "is greater than or equal to" are transitive relations on the set of all real numbers. Because, if $a \le b$ and $b \le c$, then $a \le c$ and if $a \geq b$ and $b \geq c$ then $a \geq c$, for all real numbers $a, b, c$.

② If we consider the set $A = \{1, 2, 3\}$ and the relations $R_1 = \{(1,1), (1,2), (2,3), (1,3), (3,1), (3,2)\}$ and $R_2 = \{(1,2), (2,3), (1,3), (3,1)\}$ on $A$, then $R_1$ is transitive but $R_2$ is not transitive.

<u>Results</u> ① A relation $R$ on a set $A$ is transitive iff its matrix $M_R = [m_{ij}]$ has the following property:

If $m_{ik} = 1$ and $m_{kj} = 1$, then $m_{ij} = 1$.

② A relation $R$ on set $A$ is transitive iff it satisfies the following property.

If there is a path of length greater than 1 from vertex 'a' to vertex 'b', then there is a path of length 1 from a to b.

In other words: $R$ is transitive iff $R^n \subseteq R$ for all $n \geq 1$.

—

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Problems :

① Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on $A$ :

(i) $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$.

(ii) $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$.

(iii) $R_3 = \{(1,1), (2,2), (3,3)\}$.

(iv) $R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$.

(v) $R_5 = \{(1,1), (2,3), (3,3)\}$.

(vi) $R_6 = \{(2,3), (3,4), (2,4)\}$.

(vii) $R_7 = \{(1,3), (3,2)\}$.

Soln : (i) $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$.
This relation is symmetric and irreflexive, but neither reflexive nor transitive.

(ii) $R_2 = \{(1,1)\ (2,2)\ (3,3), (2,3)\}$.
$R_2$ is reflexive and Transitive but not symmetric.

(iii) & (iv) $R_3 = \{(1,1), (2,2), (3,3)\}$. & $R_4 = \{(1,1), (2,2)\ (3,3), (2,3), (3,2)\}$
$R_3$ and $R_4$ are both reflexive and symmetric.

(v) $R_5$ is neither reflexive nor symmetric

(vi) $R_6$ is Transitive and irreflexive, but not symmetric.

(vii) $R_7$ is irreflexive, but neither transitive nor symmetric.

10 ② Let $A = \{1, 2, 3, 4\}$. Determine the nature of the following relations on $A$.

(1) $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}$

(2) $R_2 = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

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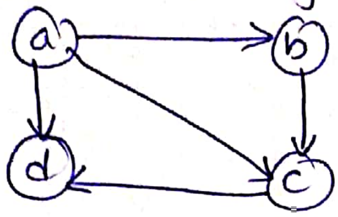
$R_3$ represented by the following digraph.




Soln. (1) $R_1$ is reflexive, ~~&~~ symmetric and transitive.

(2) $R_2$ is transitive

(3) By examining the edges in the digraph, we find that the ~~co~~ relation $R_3$ is both asymmetric and antisymmetric.

5(b) (3) Find the nature of the relations represented by the following matrices.

(a) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Soln. (a) Here, the given matrix is symmetric.
That is, $a_{ji} = a_{ij}$ for $i, j = 1, 2, 3$;
Therefore, the corresponding relation is symmetric.

(b) Here, the given matrix has 1's on its main diagonal & is symmetric.
Therefore, the corresponding relation is reflexive & symmetric.

(c) Here, the given matrix is not symmetric.
Therefore, the corresponding relation is not symmetric.
Further, the presence of 1 in the $(1,4)^{th}$ & $(4,1)^{th}$ positions of the matrix indicates that the relation is not antisymmetric.

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④ Show that the relation R represented

(a) by the matrix $M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is transitive.

Sol: Let $A = \{a, b, c\}$ be the set on which R is defined.

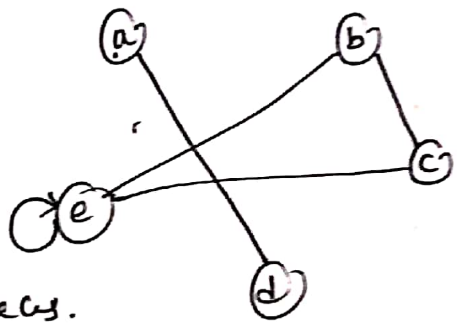
Then, by examining the given $M_R$, we find that

$R = \{(a,a), (a,b), (a,c), (b,c), (c,c)\}$.

By examining the elements of R, we find that R is transitive.

⑤ Let $A = \{a, b, c, d, e\}$, and $R = \{(a,d), (d,a), (c,b), (b,c), (c,e), (e,c), (b,e), (e,b), (e,e)\}$ be a symmetric relation on A. Draw the graph of R. Is R connected?

Sol: Graph of R:




The graph shows that R is not connected, since graph has two pieces.

⑥ On the set $Z^+$, a relation R is defined by aRb iff. a divides b (exactly). Prove that R is reflexive, transitive and antisymmetric, but not symmetric.

Soln.. For any $a \in Z^+$, the statement "a divides b" is true.

Thus, aRa for all $a \in Z^+$. Hence R is reflexive

Next, we note that, for any $a, b \in Z^+$,
"a divides b" need not imply that "b divides a".
For example, 3 divides 6 but 6 does not divide 3".
Thus, aRb does not always imply bRa.
Hence R is <u>not symmetric</u>.

Further, "a divides b" and "b divides a" imply that " $a = b$.
Thus aRb & bRa imply, $a=b$.
Therefore, R is <u>antisymmetric</u>.

Lastly, we note that for any $a, b, c \in Z^+$,
"a divides b" and "b divides c" imply that
"a divides c".
Thus, aRb and bRc imply aRc.
Hence, <u>R is transitive</u>.

(7) Let S be a universal set. On P(S), define a relation R by $(A, B) \in R$ iff. $A \subseteq B$.
Prove that R is reflexive, antisymmetric and transitive, but not symmetric.

Soln: For any subset A of S, (i.e. for any $A \in P(S)$)
we have, $A \subseteq A$, $\Rightarrow (A, A) \in R$.
Hence R is <u>reflexive</u> on P(S).

For any $A, B \in P(S)$, $A \subseteq B$ & $B \subseteq A \Rightarrow A = B$.
That is, $(A, B) \in R$ & $(B, A) \in R$, $\Rightarrow A = B$.
Hence R is <u>antisymmetric</u> on P(S).

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For any $A, B, C \in P(S)$, if $A \subseteq B$ and $B \subseteq C$, we have $A \subseteq C$.

That is, if $(A, B) \in R$ & $(B, C) \in R$, then $(A, C) \in R$.

Hence $R$ is transitive on $P(S)$.

For any $A, B \in P(S)$, $A \subseteq B$ does not ~~necessily~~ necessarily imply that $B \subseteq A$.

That is, $(A, B) \in R$ does not always imply that $(B, A) \in R$.

Therefore, $R$ is not symmetric on $P(S)$.

3(b)

8) Let $R$ be a relation on a set $A$. Prove ~~that~~ the following:

(1) $R$ is reflexive iff $\overline{R}$ is irreflexive,

(2) ~~If~~ $R$ is reflexive, so is $R^c$.

(3) If $R$ is symmetric, so are $R^c$ and $\overline{R}$.

(4) If $R$ is transitive, so is $R^c$.

Sol: (1) Suppose $R$ is reflexive. ~~The~~

Then $(a, a) \in R$ for every $a \in A$.

Consequently, $(a, a) \notin \overline{R}$ for any $a \in A$.

This means that $\overline{R}$ is irreflexive.

By reversing the steps, we find that if $\overline{R}$ is irreflexive then $R$ is reflexive.

(2) Suppose $R$ is reflexive.

Then $(a, a) \in R$ for all $a \in R$.

Consequently, $(a, a) \in R^c$ as well. $\therefore$ $R^c$ is reflexive.

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(3) Take any $(a,b) \in R^c$.
Then $(b,a) \in R$. 
Consequently, $(a,b) \in R$, because $R$ is symmetric
This implies $(b,a) \in R^c$
Thus, $R^c$ is also symmetric.

Next take any $(a,b) \in \overline{R}$.
Then $(a,b) \notin R$.
Consequently, $(b,a) \notin R$, because $R$ is symmetric
This imply that, $(b,a) \in \overline{R}$.
Thus, $\overline{R}$ is also symmetric.

(4) Take any $(a,b), (b,c) \in R^c$.
Then $(b,a), (c,b) \in R$.
This implies that $(c,a) \in R$, because $R$ is transitive
Therefore, $(a,c) \in R^c$.
Thus, $R^c$ is transitive.

2 (a)
(9) Let $R$ and $S$ be relations on a set $A$.
Prove the following:
(1) If $R$ and $S$ are reflexive, so are $R \cap S$ & $R \cup S$.
(2) If $R$ and $S$ are symmetric, so are $R \cap S$ & $R \cup S$.
(3) If $R$ and $S$ are antisymmetric, so is $R \cap S$.
(4) If $R$ and $S$ are transitive, so is $R \cap S$.

Soln : (1) Suppose $R$ and $S$ are reflexive.
Then $(a,a) \in R$ and $(a,a) \in S$ $\forall a \in A$.
Consequently, $(a,a) \in R \cap S$ & $(a,a) \in R \cup S$.
$\therefore$ $R \cap S$ & ~~$R \cup$~~ $R \cup S$ are reflexive.

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(2) Suppose $R$ and $S$ are symmetric.

Take any $(a,b) \in R \cap S$.

Then $(a,b) \in R$ and $(a,b) \in S$.

Therefore, $(b,a) \in R$ and $(b,a) \in S$.

Consequently, $(b,a) \in R \cap S$.

Hence $R \cap S$ is symmetric.

Take any $(x,y) \in R \cup S$.

Then $(x,y) \in R$ or $(x,y) \in S$.

Therefore, $(y,x) \in R$ or $(y,x) \in S$.

Consequently, $(y,x) \in R \cup S$.

Hence $R \cup S$ is symmetric.

(3) Suppose $R$ and $S$ are antisymmetric.

Take any $(a,b), (b,a) \in R \cap S$.

Then $(a,b), (b,a) \in R$ & $(a,b), (b,a) \in S$.

By the antisymmetry of $R$ (or $S$), it follows that $b = a$.

Thus, $R \cap S$ is antisymmetric.

(4) Suppose $R$ and $S$ are transitive.

Take $(a,b), (b,c) \in R \cap S$.

Then $(a,b) \in R$, $(a,b) \in S$.

$(b,c) \in R$, $(b,c) \in S$.

This imply, $(a,c) \in R$ & $(a,c) \in S$, so that $(a,c) \in R \cap S$.

$\therefore$ $R \cap S$ is transitive.

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# Equivalence Relations & Hasse Diagram

A relation $R$ on a set $A$ is said to be an equivalence relation on $A$ if
(i) $R$ is reflexive,
(ii) $R$ is symmetric and
(iii) $R$ is transitive, on $A$.

A trivial example of an equivalence relation is the relation "is equal to" on the set of all real numbers, $\mathbb{R}$.

(1) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$. be a relation on $A$. verify that $R$ is an equivalence relation.

Soln: We have to show that $R$ is reflexive, symmetric and transitive.

* We note that all of $(1,1), (2,2), (3,3), (4,4)$ belong to $R$. That is, $(a,a) \in R$ $\forall a \in A$.
$\therefore$ $R$ is reflexive relation.

* We note that, $(1,2), (2,1) \in R$ & $(3,4), (4,3) \in R$
That is, whenever, $(a,b) \in R$ then $(b,a) \in R$ for $a, b \in A$.
$\therefore$ $R$ is a symmetric relation.

* We note that, $(1,2), (2,1), (1,1) \in R$, $(2,1), (1,2), (2,2) \in R$
$(4,3), (3,4), (4,4) \in R$.
That is, whenever, $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$, for all $a, b, c \in A$.
$\therefore$ $R$ is Transitive relation.
Hence $R$ is an equivalence relation.

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② Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (1,3), (4,1), (4,4)\}$.

be a relation on A. Is R an equivalence relation ?

Soln : We note that,

(i) $(a,a) \in R$ for every $a \in R$.

$\therefore$ R is reflexive,

(ii) $(4,1) \in R$, but $(1,4) \notin R$.

$\therefore$ R is not symmetric.

Since R is not symmetric, R is not an equivalence relation.

③ A relation R on a set $A = \{a, b, c\}$ is represented by the following matrix :

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determine whether R is an equivalence relation.

Soln : From $M_R$ we have,

$R = \{(a,a), (a,c), (b,b), (c,c)\}$.

We note that, $(a,c) \in R$ but $(c,a) \notin R$.

$\therefore$ R is not symmetric.

$\therefore$ R is not an equivalence relation.

Accordingly, R is not symmetric matrix, R is not [Since $M_R$ is not symmetric relation ].
symmetric

⑤ The digraph of a relation R on the set $A = \{1, 2, 3\}$ is as given below. Determine whether R is an equivalence relation.

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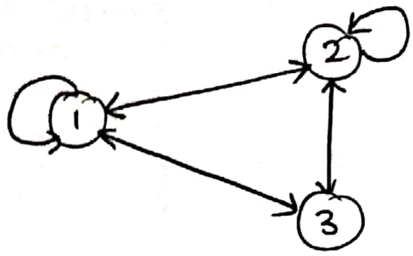

From the digraph we have,

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}$

We note that the given relation is symmetric and transitive. but not reflexive, because, $(3,3) \notin R$.

Therefore, R is not an equivalence relation.

7(b)

(7) Let S be the set of all non-zero integers, and $A = S \times S$. On A, define the relation R by $(a,b) R (c,d)$ if and only if $ad = bc$. Show that R is an equivalence relation.

Soln. * We note that $(a,a) R (a,a)$, because, $aa = aa$ for any $a \in S$.

Therefore, R is reflexive on A.

* Suppose $(a,b) R (c,d)$. Then $ad = bc$. and therefore, $cb = da$.

Hence, $(c,d) R (a,b)$.

$\therefore$ R is symmetric on A.

* Suppose that $(a,b) R (c,d)$ and $(c,d) R (e,f)$.

Then ~~cb = d~~ $ad = bc$ and $cf = de$, $\Rightarrow af = be$

Hence, $(a,b) R (e,f)$.

$\therefore$ R is transitive on A

Hence R is an equivalence relation on A.

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# Partial Orders

A relation $R$ on a set $A$ is said to be a partial ordering relation or a partial order on $A$ if
(i) $R$ is reflexive (ii) $R$ is antisymmetric and
(iii) $R$ is transitive on $A$.

Partially ordered set / Poset : A set $A$ with a partial order $R$ defined on it is called a partially ordered set / an ordered set / poset, and is denoted by the pair $(A, R)$.

① The most familiar partial order is the relation "less than or equal to", denoted by $\leq$, on the set $Z$ of all integers.
[Because, this relation is reflexive, antisymmetric and transitive].
Thus, $(Z, \leq)$ is a poset.

② The relation "is greater than or equal to" denoted by $\geq$, is also a partial order on $Z$; that is $(Z, \geq)$ is also a poset.

③ The "divisibility relation" on the set $Z^+$ defined by a divides b (denoted by $a/b$) for all $a, b \in Z^+$ is a partial order on $Z^+$.

④ The "subset relation" $\subseteq$ defined on the power set of a set $S$ is a partial order on $S$.
Thus, for any set $S$, $(P(S), \subseteq)$ is a poset.

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The relation "is less than" and "is greater than" are not partial orders on Z; because they are not reflexive.

The relation "congruent modulo n" defined on the set of all integers Z is also not a partial order, because this relation is not antisymmetric.

## Digraph of a partial order:

Since a partial order is a relation on a set, we can think of the digraph of a partial order if the set is finite.

## Fundamental property of digraphs of partial orders.

**Property:** The digraph of a partial order has no cycle of length greater than 1.

## Hasse Diagrams:

Since a partial order is reflexive, at every vertex in the digraph of a partial order there would be a cycle of length 1. While drawing the digraph of a partial order, we need not exhibit ~~of~~ such cycles explicitly; they will be automatically understood.

If, in the digraph of a partial order, there is an edge from a vertex 'a' to a vertex 'b' and there is an edge from the vertex 'b' to a vertex 'c', then there should be an edge from 'a' to 'c'. (because of transitivity)

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As such, we need not exhibit an edge from a to c explicitly ; it will be automatically understood. (by convention).

To simply the format of the digraph of a partial order, we represent the vertices by (dots / bullets) & draw the digraph in such a way that all edges point upward.

Hasse Diagram: The digraph of a partial order drawn by adopting the conventions ~~indicated is the~~ ~~above~~ is called a poset diagram / the Hasse diagram for the partial order.

Problems :

① Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3) (3,3), (3,4), (1,4), (4,4)\}$.

verify that R is a partial order on A. Also write down the Hasse diagram for R.

Soln : We observe that the given relation R is reflexive and transitive.

Further, R does not contain ordered pairs of the form $(a,b)$ and $(b,a)$ with $b \neq a$.

Therefore, R is antisymmetric.

$\therefore$ R is a partial order on A.

The Hasse ~~diagram~~ diagram for R must exhibit the relationships between the elements of A as defined by R ; if $(a,b) \in R$, there must be an upward edge from a to b.

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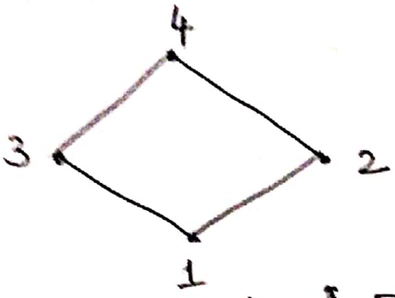
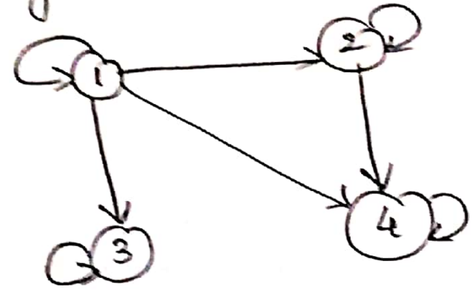
By the ordered pairs contained in R, we find that the Hasse diagram of R is as shown below:

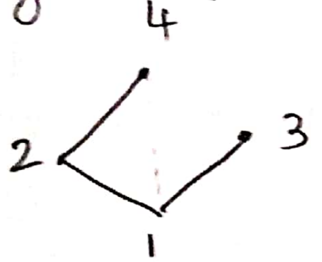



(2) A partial order R on the set $A = \{1, 2, 3, 4\}$ is represented by the following graph. Draw the Hasse diagram for R.

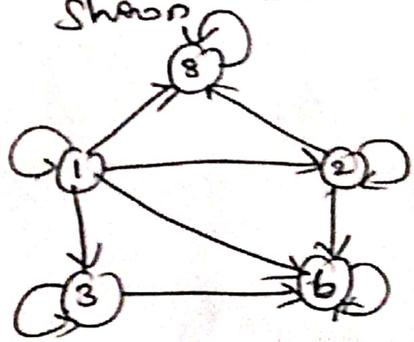



Sol: By observing the given digraph, we note that $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4)\}$

The Hasse diagram for this R is as shown below




(3) The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below:

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verify that (A, R) is a poset & write ~~for~~ down its Hasse diagram.

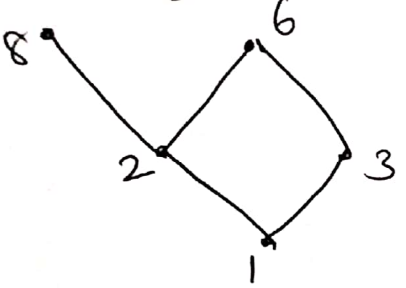
Sol $R = \{(1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6)$
$(2,8)\ (3,3), (3,6), (6,6), (8,8)\}$.

We check that R is reflexive, transitive and antisymmetric.

$\therefore$ R is a partial ordered on A.

That is (A, R) is a poset.

The Hasse diagram for R is as shown:



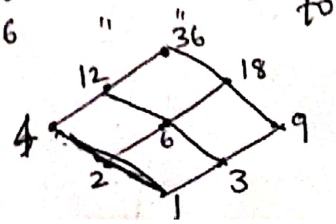

④ If Draw the Hasse diagram representing the positive divisors of 36.

Sp. The set of all positive divisors of 36 is
$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

The relation R of divisibility (that is aRb iff a divides b) is a partial order on this set. The Hasse diagram for this partial order is ~~xeqn~~ as shown in figure.

We note that, under R,

1 is related to all elements of $D_{36}$.
2 " " " 2, 4, 6, 12, 18, 36.
3 " " " 3, 6, 9, 12, 18, 36.
4 " " " 4, 12, 36
6 " " " 6, 12, 18, 36
9 is related to 9, 18, 36
12 is " " 12 & 36.
18 " " " 18 & 36
36 " " to 36.

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1,2,6. 46, 45

(14) (a) Let $A = B = \{a, b, c, d\}$ and $R = \{(a,a), (a,c), (b,c), (c,a), (d,b), (d,d)\}$ and $S = \{(a,b), (b,c), (c,a), (c,b), (d,c)\}$. IV

Compute $M_{(R\cap S)}$, $M(R^c)$ and $M(\bar{S})$.

Sol:

$$M(R\cap S) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 0 \\ c & 1 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 \end{array}$$

$R\cap S = \{(b,c), (c,a)\}$.

$$M(R^c) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

$R^c = \{(a,a), (c,a), (c,b), (a,c), (b,d), (d,d)\}$.

$$M(\bar{S}) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 1 \\ b & 1 & 1 & 0 & 1 \\ c & 0 & 0 & 1 & 1 \\ d & 1 & 1 & 0 & 1 \end{array}$$

$\bar{S} = (A \times B) - S$, $S = \{(a,b), (b,c), (c,a), (c,b), (d,c)\}$

$A \times B = \{(a,a), (a,b), (a,c), (a,d), (b,a), (b,b), (b,c), (b,d), (c,a), (c,b), (c,c), (c,d), (d,a), (d,b), (d,c), (d,d)\}$

$\bar{S} = \{(a,a), (a,c), (a,d), (b,a), (b,b), (b,d), (c,c), (c,d), (d,a), (d,b), (d,d)\}$.

$\therefore$ ~~$M(\bar{S})$~~

14) (b) Let $A = B = \{1, 2, 3\}$, and $R$ & $S$ be relations on $A$ whose matrices are as given below. Find the matrices of $\overline{R}$, $R^c$, $R \cap S$ and $R \cup S$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Sol:

$R = \{(1,1), (1,3), (2,2), (2,3)\}$.

$S = \{(1,2), (1,3), (2,1), (2,2), (3,2)\}$

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$\overline{R} = (A \times B) - R = \{(1,2), (2,1), (3,1), (3,2), (3,3)\}$.

$$M_{\overline{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$R^c = \{(1,1), (3,1), (2,2), (3,2)\}$.

$$M(R^c) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$R \cap S = \{(1,3), (2,2),\}$

$$M(R \cap S) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$R \cup S = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,2)\}$.

$$M(R \cup S) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \quad //$$

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15) (b) For the relation $R = \{(1,2),(1,3),(2,4),(3,2),(3,3),(3,4)\}$ on the set $A = \{1,2,3,4\}$, find $R^2$ & $R^3$ and draw ~~write~~ their digraphs.

Soln : Let $A = \{1,2,3,4\}$

$R = \{(1,2),(1,3),(2,4),(3,2),(3,3),(3,4)\}$

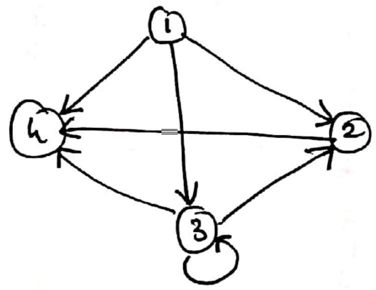
$R \circ R = R^2 = \{(1,4)\ (1,2)\ (3,4),\ (1,3),\ (2,4),\ (3,2),\ (3,3)\}$

$[R \circ S = \{(a,c) \mid a \in A,\ c \in C,\ \&\ \exists\ b \in B \text{ with } (a,b) \in R\ \&\ (b,c) \in S\}]$

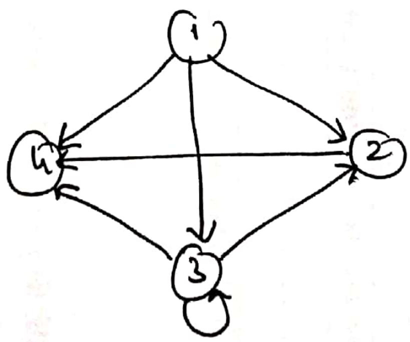
~~$(R \circ R) \circ R = (R \circ R^2) = \{(1,4),\ (1,2),$~~

$R \circ (R \circ R) = R \circ R^2 = \{(1,4),\ (1,2)\ (3,4),\ (3,2),\ (1,3),\ (2,4),$
$(3,3),\ (\cancel{3,4})\}.$

Digraph of $R^2$ :



Digraph of $R^3$ :



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