PROBABILITY THEORY

Random variables (discrete & continuous), probability density function, cumulative density function. Probability distributions- binomial & poison distributions; exponential & normal distributions.

Random variables

Introduction: In a random experiment, the outcomes (results) are governed by chance mechanism and the sample space S of such a random experiment consists of all outcomes of the experiment. When the outcomes of the sample space are non-numeric, they can be quantified by assigning a real number to every event of the sample space. This assignment rule, known as the random variable or stochastic variable. In other words a random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by X,Y,Z…….The set of all real number of a random variable X is called the range of X.

Example-1 While tossing a coin, suppose that the value 1 is associated for the outcome' head' and 0 for the outcome 'tail'. The sample space S=$\{H,T\}$and if X is the random variable then X (H) =1and X(T)=0, Range of X = $\{0,1\}$.

Example-2 A pair of fair dice is tossed. The sample space S consists of the 36 ordered pair (a,b)where a and b can be any integers between 1 and 6, that is S={(1,1),(1,2),………(6,6)}
Let X assign to each point (a,b) the maximum of its numbers, that is, X(a,b)=max(a,b). For example(1,1)=1,X(3,4)=4,X(5,2)=5.

Then x is a random variable where any number between 1 and 6 could occur, and no other number can occur. Thus the range space of X = {1, 2, 3, 4, 5, 6}.

Let Y assign to each point (a,b) the sum of its numbers, that is Y(a,b)=a+b. For example Y(1,1)=2, Y(3,4)=7, Y(6,3)=9. Then Y is a random variable where any number between 2 and 12 could occur, and no other number can occur. Thus the range space ={2,3,4,5,6,7,8,9,10,11,12}

Discrete random variables

Definition: a random variable is said to be discrete random variable if it's set of possible outcomes, the sample space S, is countable (finite or an unending sequence with a many elements as there are whole numbers).

Example:
1) Tossing a coin and observing the outcome.
2) Tossing a coin and observing the number of heads turning up.
3) Throwing a 'die' and observing the number of the face.

Continuous random variables

Definition: A random variable is said to be continuous random variables if sample space S contains infinite number of values.

Example:
1) Weight of articles.
2) Length of nails produced by a machine.
3) Observing the pointer on a speedometer/voltmeter.
4) Conducting a survey on the life of electric bulbs.

Generally counting problems corresponds to discrete random variables and measuring problems lead to continuous random variables.

PROBABILITY DISTRIBUTIONS
Probability distribution is the theoretical counterpart of frequency distribution, and plays an important role in the theoretical study of populations.

Discrete probability distribution:

Definition: If for each value x_i of a discrete random variable X, we assign number p(x_i) such that
i) p(x_i) ≥ 0, ii)$\sum_i \mathrm{p}(x_i) = 1$ then the function p(x) is called a probability function. If the probability that X takes the values x_i is p_i, then P(X=x_i) $= p_i$or p(x_i).The set of values [x_i, p(x_i)] is called a discrete probability distribution of the discrete random variable X. The function P(X) is called the probability density function (p.d.f) or the probability mass function (p.m.f)

Cumulative distribution function
The distribution function f(x) is defined by f(x) $= P(X \leq x) = \sum_{i=1}^{x} \mathrm{p}(x_i)$, x being an integer is called the cumulative distribution function(c.d.f)

The mean and variance of the discrete probability distribution
Mean (μ) $or\ expectation\ E(X) \ = \sum_i x_i\, p(x_i)$
Variance (V) = $\sum_i (x_i - \mu)^2\ p(x_i)$= $\sum_i x_i{}^2\, p(x_i) - [x_i p(x_i)]^2$=$\sum_i x_i{}^2\, p(x_i) - \mu^2$
Standard deviation (σ) $= \sqrt{V}$

1a: The random variable X has the following probability mass function

X	0	1	2	3	4	5
P(X)	K	3K	5K	7K	9K	11K

i) find K ii) find P(X<3) iii) find $P(3<X\leq 5)$ (iv) mean & variance

Solution: If X is a discrete random variable then $\sum_i P(x_i) = 1$

⇒ K+3K+5K+7K+9K+11K =1

⇒ 36K=1

⇒ K = 1/36

ii) P(X<3) = P(X=0)+P(X=1)+P(X=2)

= K+3K+5K = 9K = 9/36=1/4

iii) $P(3<X\leq 5)$ = P(X=4)+P(X=5) =9K+11K= 20K= 20/36=5/9

iv) Mean (μ) or $E(X) = \sum_i x_i\, p(x_i)$= 0(K)+1(3K)+2(5K)+3(7K)+4(9K)+5(11K)

=125K=125(1/36)= 3.4722

Variance=$\sum_i x_i{}^2\, p(x_i) - \mu^2$=$0^2$(K)+ 1^2 (3K)+ 2^2 (5K)+ 3^2 (7K)+ 4^2 (9K)+ 5^2 (11K) – $(3.4722)^2$

= 3K+20K+63K+144K+275K – $(3.4722)^2$

=505K– $(3.4722)^2$

=505(1/36) – $(3.4722)^2$

=1.9716

1.b A random variable X has the following probability function for various values of x

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2$+k

i) Find k ii) Evaluate P(x<6) (iii) P(x > 6) (iv) $P(3<x\leq 6)$. Also find the probability distribution and the cumulative distribution function of X.

Solution: (i) If X is a discrete random variable then $\sum_i P(x_i) = 1$and P(X)≥ 0

⇒ 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1

⇒10k^2+9k−1=0

⇒ k=1/10=0.01 and k=-1.

If k=-1, the second condition (P(X)≥ 0) fails and hence k$\neq -1$ ∴ k=1/10=0.1

(ii) P(x<6)=P(0)+P(1)+P(2)+P(3)+P(4)+P(5)

=0+0.1+0.2+0.2+0.3+0.01=0.81

(iii) P(x > 6)= 1- P(x < 6)= 1-0.81= 0.19

(iv) $P(3<x\leq 6)$=P(4)+P(5)+P(6)

=0.3+0.01+0.02=0.33

The probability distribution is as follows for k=0.1:

x	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

Cumulative distribution function of x is as follows:

x	0	1	2	3	4	5	6	7
f(x)	0	0+0.1 =0.1	0.1+0.2 =0.3	0.3+0.2 =0.5	0.5+0.3 =0.8	0.8+0.01 =0.81	0.81+0.02 =0.83	0.83+0.17 =1

2 a: Find 'k' such that the following distribution represents a finite probability distribution. Hence, find (i) k (ii) mean (iii) P(X≤1) (iv) P(X>1) (v) P(-1<X≤2)

X_i	-3	-2	-1	0	1	2	3
$P(X_i)$	k	2k	3k	4k	3k	2k	k

Solution: (i) If X is a discrete random variable then $\sum_i P(x_i) = 1$

⇒k+2k+3k+4k+3k+2k+k =1

⇒ 16k = 1

⇒ k=1/16

(ii) Mean $(\mu) = \sum_i x_i\, p(x_i)$ = -3xk+-2x2k+-1x3k+0x4k+1x3k+2x2k+3 x k

= -10k+10k = 0

(iii) P(X≤1) = P(X=-3)+P(X=-2)+P(X=-1)+ P(X=0)+P(X=1)

=k +2k+3k+4K+3k

= 13k = 13(1/16) = 13/16

(iv) P(X> 1) = P(X=2)+ P(X=3)

= 2k +k = 3k

= 3(1/16)= 3/16

(v) P(-1<X≤2) = P(X=0)+P(X=1)+ P(X=2)

= 4k + 3k +2k = 9k

= 9(1/16)= 9/16

2 b: The probability distribution of a finite random variable X is given by the following table:

X_i	-2	-1	0	1	2	3
p(X_i)	0.1	k	0.2	2k	0.3	k

i) Find the value of K and calculate the mean and variance. ii) Evaluate P(X<1) (iii) P(X >-1)

Solution: If X is a discrete random variable then $\sum_i P(x_i) = 1$

⇒0.1+k+0.2+2k+0.3+k =1

⇒ 0.6+4k = 1

⇒ 4k=1-0.6=0.4

⇒ k = 0.1

Mean $(\mu) = \sum_i x_i\, p(x_i)$ =-2x0.1+-1x0.1+0x0.2+1x0.2+2x0.3+3x0.1

=0.8

Variance (V) = $\sum_i x_i{}^2\, p(x_i) - \mu^2$

= 4x0.1+1x0.1+0x0.2+1x0.2+4x0.3+9x0.1$-(0.8)^2$

=2.16

ii)P(X<1) = P(X=-2)+P(X=-1)+P(X=0)

=0.1+0.1+0.2

=0.4

iii)P(X>-1) = P(X=0)+P(X=1)+P(X=2)+ P(X=3)

= 0.2 +2(.1)+0.3+0.1

= 0.8

In discrete probability distribution we are going to study .Geometric distribution & Poisson distribution.

GEOMETRIC DISTRIBUTION

Consider a sequence of trials, where each trial has only two possible outcomes (designated failure and success). The probability of success is assumed to be the same for each trial. In such a sequence of trials, the geometric distribution is useful to model the number of failures before the first success. The distribution gives the probability that there are zero failures before the first success, one failure before the first success, two failures before the first success, and so on.

Examples:

1.A newly-wed couple plans to have children, and will continue until the first girl. What is the probability that there are zero boys before the first girl, one boy before the first girl, two boys before the first girl, and so on?

2.A doctor is seeking an anti-depressant for a newly diagnosed patient. Suppose that, of the available anti-depressant drugs, the probability that any particular drug will be effective for a particular patient is p=0.6. What is the probability that the first drug found to be effective for this patient is the first drug tried, the second drug tried, and so on? What is the expected number of drugs that will be tried to find one that is effective?

3.A patient is waiting for a suitable matching kidney donor for a transplant. If the probability that a randomly selected donor is a suitable match is p=0.1, what is the expected number of donors who will be tested before a matching donor is found?

Definition: If p be the probability of success and x be the number of failures preceding the first success then this distribution is

$p(X = x) = q^x p$, x =0,1,2,3….., q = 1-p.

Properties: (i) $(x) > 0\ for\ 0 \leq p \leq 1$.

(ii) Total sum of probability, that is, $\sum_{x=0}^{\infty} P(x) = p \sum_{x=0}^{\infty} q^x = p.\frac{1}{1-q}\ 1..$

$\therefore \sum_{x=0}^{\infty} a\ r^x = a + ar + ar^2 + ar^3 + ar^4 + \cdots \dots \dots .. + \infty = \frac{a}{1-r}$.

Hence, P(x) is a probability function.

Mean and standard deviation of the Geometric distribution

Mean($\boldsymbol{\mu}$)=$\sum_{x=0}^{\infty} xP(x)$

$=\sum_{x=0}^{\infty} x\, q^x p$

$= p\sum_{x=1}^{\infty} x\, q^{x-1} q$

$= pq\sum_{x=1}^{\infty} \frac{d(q^x)}{dq}$

$=pq\,\frac{d}{dq}\sum_{x=1}^{\infty} q^x$

$=pq\,\frac{d}{dq}\left[\frac{q}{1-q}\right]$

$=pq\left[\frac{1}{(1-q)^2}\right]$

$\boldsymbol{\mu = \frac{q}{p}}$

Variance(V) $=\sum_{x=0}^{\infty} x^2 P(x) - \mu^2 = \frac{q}{p^2}$

$=\sum_{x=0}^{\infty} x^2\, q^x p - \left(\frac{q}{p}\right)^2$

$= p\sum_{x=0}^{\infty}[x^2 - x + x]\, q^x - \left(\frac{q}{p}\right)^2$

$= p\sum_{x=0}^{\infty} x[x-1]\, q^2 q^{x-2} + p\sum_{x=0}^{\infty} x\, q q^{x-1} - \left(\frac{q}{p}\right)^2$

$= pq^2\sum_{x=2}^{\infty} x[x-1]\, q^{x-2} + pq\sum_{x=1}^{\infty} x\, q^{x-1} - \left(\frac{q}{p}\right)^2$

$=pq^2\sum_{x=2}^{\infty}\frac{d^2(q^x)}{dq^2} + pq\sum_{x=1}^{\infty}\frac{d(q^x)}{dq} - \left(\frac{q}{p}\right)^2$

$=pq^2\frac{d^2}{dq^2}\sum_{x=2}^{\infty} q^x + pq\frac{d}{dq}\sum_{x=1}^{\infty} q^x - \left(\frac{q}{p}\right)^2$

$=pq^2\frac{d^2}{dq^2}\left[\frac{q^2}{1-q}\right] + pq\,\frac{d}{dq}\left[\frac{q}{1-q}\right] - \left(\frac{q}{p}\right)^2$

$=pq^2\left[\frac{2}{(1-q)^3}\right] + pq\left[\frac{1}{(1-q)^2}\right] - \left(\frac{q}{p}\right)^2$

$=pq^2\left[\frac{2}{p^3}\right] + pq\left[\frac{1}{p^2}\right] - \frac{q^2}{p^2}$

$= \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p}$

$\boldsymbol{V = \frac{q(q+p)}{p^2} = \frac{q}{p^2}}\ \therefore (q+p) = 1.$

3 (b) 3% of the product produced by a machine is found to be defective. Find the probability that first defective occurs in the (i) 5th item inspected (ii) first five inspected (iii) mean (iv) variance.

Solution: The probability of defective item is $\mathrm{p} = \frac{3}{100} = 0.03\ \&\ q = 1 - p = 0.97$. Let X denote the number of item produced by a machine until his first defective.

(i) $\mathrm{P}(\mathrm{X}=4)=q^4p=0.97^4\times 0.03=0.02656.$

(ii)$\mathrm{P}(0\leq \mathrm{X}\leq 4)=\mathrm{P}(\mathrm{X}=0)+\mathrm{P}(\mathrm{X}=1)+\mathrm{P}(\mathrm{X}=2)+\mathrm{P}(\mathrm{X}=3)+\mathrm{P}(\mathrm{X}=4)$

$$=q^0p+q^1p+q^2p+q^3p+q^4p$$

$$=(1+0.97^1+0.97^2+0.97^3+0.97^4)\times 0.03=0.14127$$

(iii) Mean = μ= $\frac{q}{p}=\frac{0.97}{0.03}=32.34$

(iv) Variance(V) $=\frac{q}{p^2}=\frac{0.97}{0.03^2}=1077.78$

4(a) What is the probability that the marketing representative must select (i) six people (ii) more than 6 people, before he finds one who attended the last home. The p.d.f of a Geometric Random Varible with 1-p=0.8.

Solution: Here, $\mathrm{q}=0.8$ & $p=1-q=0.2$.

(i) $\mathrm{P}(\mathrm{X}=6)=0.80^5\times 0.20=0.065536$

(ii) $\mathrm{P}(\mathrm{x}>6)=1-\mathrm{P}(\mathrm{X}\leq 6)$

= $1-\mathrm{P}(\mathrm{X}=0)-\mathrm{P}(\mathrm{X}=1)-\mathrm{P}(\mathrm{X}=2)-\mathrm{P}(\mathrm{X}=3)-\mathrm{P}(\mathrm{X}=4)-\mathrm{P}(\mathrm{X}=5)$

=1 $-[0.80^0+0.80^1+0.80^3+0.80^4+0.80^5]\times 0.20$

=0.262144

There are 26% chance.

4(b) If the probability that a target is destroyed on any one shot is 0.5. What is the probability that it would be destroyed on (i) 6th attempt (ii) more than 6 attempt (iii) mean (iv) variance?

Solution: Here, $\mathrm{q}=0.5$ & $p=1-q=0.5$.

(i) $\mathrm{P}(\mathrm{X}=6)=0.5^5\times 0.5=0.015625$

(ii) $\mathrm{P}(\mathrm{x}>6)=1-\mathrm{P}(\mathrm{X}\leq 6)$

= $1-\mathrm{P}(\mathrm{X}=0)-\mathrm{P}(\mathrm{X}=1)-\mathrm{P}(\mathrm{X}=2)-\mathrm{P}(\mathrm{X}=3)-\mathrm{P}(\mathrm{X}=4)-\mathrm{P}(\mathrm{X}=5)$

=1 $-[0.5^0+0.5^1+0.5^2+0.5^3+0.5^4+0.5^5]\times 0.5$

=0.015625

(iii) Mean = μ= $\frac{q}{p}=\frac{0.5}{0.5}=1$

(iv) Variance(V) $=\frac{q}{p^2}=\frac{0.5}{0.5^2}=2$

POISSON DISTRIBUTION

Poisson distribution is the discrete probability distribution of discrete random variable X, which has no upper bound. It is defined for non-negative values of x as follows: P(x)= $\frac{m^x e^{-m}}{x!}$

for x=0,1,2,3…… Here m>0 is called the parameter of the distribution. In binomial distribution the number successes out of total definite number of n trials is determined, whereas in Poisson distribution the number of successes at a random point time and space is determined.

Poisson distribution is suitable for 'rare' events for which the probability of occurrence p is very small and the of trials n is very large. Also binomial distribution can be approximated by Poisson distribution when n→ ∞ and p→ 0 such that m= np= constant.

Example of rare events:

i) Number of printing mistake per page.

ii) Number of accidents on a highway.

iii) Number of bad cheques at a bank.

iv) Number of defectives in a production center.

We have in case of binomial distribution, the probability of x successes out of n trials,

The distribution of probabilities for x=0,1,2,3….. is as follows.

X	0	1	2	3	…..
P(x)	e^{-m}	$\frac{me^{-m}}{1!}$	$\frac{m^2e^{-m}}{2!}$	$\frac{m^3e^{-m}}{3!}$	

Properties: (i) $(x) > 0\ for\ 0 \leq p \leq 1$.

(ii) Total sum of probability

$$\sum_{x=0}^{\infty} P(x) = e^{-m} + \frac{me^{-m}}{1!} + \frac{m^2e^{-m}}{2!} + \frac{m^3e^{-m}}{3!} + \cdots \ldots ..$$

$$= e^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= \quad e^{-m}e^{m} \qquad = 1$$

Hence, P(x) is a probability function.

Mean and standard deviation of the Poisson distribution

Mean($\boldsymbol{\mu}$)=$\sum_{x=0}^{\infty} xP(x)$

$$=\sum_{x=0}^{\infty} x\frac{m^x e^{-m}}{x!}$$

$$=\sum_{x=1}^{\infty} \frac{m^x e^{-m}}{(x-1)!}$$

$$=me^{-m}\sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= me^{-m}\left\{1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= me^{-m}e^{m}$$

Mean ($\boldsymbol{\mu}$) =m

Variance(V)=$\sum_{x=0}^{\infty} x^2P(x) - \mu^2$ ………(1)

Consider $\sum_{x=0}^{\infty} x^2P(x)$ = $\sum_{x=0}^{\infty}[x(x-1)+x]\frac{m^x e^{-m}}{x!}$

$$= \sum_{x=2}^{\infty}\frac{m^x e^{-m}}{(x-2)!}+\sum_{x=1}^{\infty}\frac{m^x e^{-m}}{(x-1)!}$$

$$=m^2e^{-m}\sum_{x=2}^{\infty}\frac{m^{x-2}}{(x-2)!} + m$$

$$=m^2e^{-m}\left\{1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\cdots\right\}+m$$
$$=m^2e^{-m}e^{m}+m$$

Thus, $\sum_{x=0}^{\infty}\ x^2P(x)$ $=m^2+m$

Substituting the value in equation (1) , variance is given by

Variance(V)=$\boldsymbol{m^2+m-m^2}$ = m

∴ Standard deviation($\boldsymbol{\sigma}$)=$\boldsymbol{\sqrt{m}}$

Mean and variance are equal in Poisson distribution.

5(b):2% of the fuses manufactured by a firm are found to be defective. Find the probability that a box containing 200 fuses contains i) no defective fuse ii)3 or more defective fuses (iii) at most one defective fuse

Solution: Let x is random variable denote the number of defective fuses. From question, the probability of defective fuse is =2/100=0.02.

Mean=m=np=200x0.02=4

Poisson distribution P(x)= $\frac{m^xe^{-m}}{x!}$= $\frac{4^xe^{-4}}{x!}$

i) P(x=0)= e^{-4}=0.0183

ii)P(x$\geq 3)=1-P(x<3)$

=1- [P(X=0)+P(x=1)+P(x=2)]

=1 $-e^{-4}$ [1+$\frac{4}{1!}+\frac{4^2}{2!}$]

=0.7621

(iii) P(x$\leq 1)=\mathrm{P}(\mathrm{x}=0)+\mathrm{P}(\mathrm{x}=1)$

=$e^{-4}+4e^{-4}=5e^{-4}$

= 5x0.0183=0.0915

6(a): The probability that individual suffers a bad reaction from an injection is 0.001. Find the probability that out of 2000 individuals (i) almost 2 ii) exactly 2 ii) more than 2 will get bad reaction.

Solution: As the probability of occurrence is very small, this follows Poisson distribution and we have

P(x)= $\frac{m^xe^{-m}}{x!}$

Mean=m=np=2000x0.001=2

i)P(x≤2)=$\mathrm{P}(\mathrm{x}=0)+\mathrm{P}(\mathrm{x}=1)+\mathrm{P}(\mathrm{x}=2)$

$=\frac{2^0e^{-2}}{0!}+\frac{2^1e^{-2}}{1!}+\frac{2^2e^{-2}}{2!}=e^{-2}+2e^{-2}+2e^{-2}=5e^{-2}$= 0.6767

(ii)P(x=2)=$\frac{2^2e^{-2}}{2!}$= $2e^{-2}=0.2707$

iii)P(x>2)=1-P(x≤ 2)

=1 – [P(x=0)+P(x=1)+P(x=2)]

= 1- $e^{-m}\left\{1+\frac{m}{1!}+\frac{m^2}{2!}\right\}$

=1 - $e^{-2}[1+2+2]$=0.3233

6(b): In a certain factory farming out razor blade, there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) two defective blades in a consignment of 10,000 packets.

Solution Let x is random variable denote the number of defective razor blades. From question, the probability of defective razor blade is =1/500=0.002.

Mean=m=np=10x0.002=0.02

Poisson distribution P(x)= $\frac{m^x e^{-m}}{x!}$

$=\frac{(0.02)^x e^{-0.02}}{x!}$

i) P(x=0)= $e^{-0.02}$=0.9802.

The number of packets containing no defective in a consignment of 10,000 packets is

= 10000 x 0.9802 = 9802 packets

ii) P(x=1)= $\frac{(0.02)^1 e^{-0.02}}{1!} = (0.02)e^{-0.02}$=$(0.02)$0.9802=0.0196

The number of packets containing no defective in a consignment of 10,000 packets is

= 10000 x 0.0196 = 196 packets

(iii) P(x=2)= $\frac{(0.02)^2 e^{-0.02}}{2!}$=$(0.02)(0.01)$0.9802=0.000196

The number of packets containing no defective in a consignment of 10,000 packets is

= 10000 x 0.000196 = 2 packets

7(a): A certain screw making machines produces on an average two defective out of 100 and pack them in boxes of 500. Find the probability that the box contains (i)three defectives (ii) At least one defective.

Solution: As the probability of occurrence is very small, this follows Poisson distribution. Let x is random variable denote the number of defective screw. From question, the probability of defective screw is p=2/100=0.02.

Mean=m=np=500x0.02=10 & P(x)= $\frac{m^x e^{-m}}{x!}$=$\frac{10^x e^{-10}}{x!}$.

i)P(x=3)= $\frac{10^3 e^{-10}}{3!} = 0.00757$

ii)P(x ≥ 1)=1 - P(x $= 0)$

= 1 - $\frac{10^0 e^{-10}}{0!}$ = 1 - e^{-10} =0.99995

7(b): The number of industrial injuries per working week in a particular factory is known to follow a Poisson distribution with mean 0.5. Find the probability that in a particular week there will be (i)less than 2 accidents (ii) more than 2 accidents .Also, find in a three week period there will be no accidents.

Solution Let x is random variable denote the number of industrial injuries per working week in a particular factory. From question, the mean is given that

Mean= m = 0.5

Poisson distribution P(x)= $\frac{m^x e^{-m}}{x!}$= $\frac{(0.5)^x e^{-0.5}}{x!}$

i) P(x<2) = P(0) + P(1) = $\frac{(0.5)^0 e^{-0.5}}{0!} + \frac{(0.5)^1 e^{-0.5}}{1!}$=$1.5e^{-0.5} = 0.9098.$

ii) P(x>2)= P(3) + P(4) + P(5)+--------------+ = 1- P(0) - P(1) -P(2)

=1- $\frac{(0.5)^0e^{-0.5}}{0!} - \frac{(0.5)^1e^{-0.5}}{1!} - \frac{(0.5)^2e^{-0.5}}{2!}$=$1-(1+0.5+\frac{0.25}{2})e^{-0.5} = 0.01439.$

(iii) Let x_1, x_2 and x_3 are the random variable denotes the number of industrial injuries in Ist ,IInd & IIIrd per working week in a particular factory with mean values m_1, m_2 and m_3 . Since Poisson distribution is additive property. Thus, the mean value is given by

m = m_1+ m_2 + m_3= 0.5+ 0.5+0.5 =1.5

Thus, the probability that there is no industrial injuries in in a three week period in a particular factory is given by with mean value 1.5

P(x=0) = P(0) = $\frac{(1.5)^0e^{-1.5}}{0!} = e^{-1.5} = 0.22313.$

Practice the problems 8 to 11(a) yourself, it is related to Poisson distribution and the answer of problems are as follows:

8. a) The number of misprints on a page of the *Daily Mercury* has a Poisson distribution with mean 1.2. Find the probability that the number of errors (i) on page four is 2 (ii) on page three is less than 3.

ANSWER: Here m = 1.2 (i) P(x=2) = P(2) = $\frac{(1.2)^2e^{-1.2}}{2!} = 0.2169$ (ii) P(x< 3) = P(0)+P(1)+P(2) = 0.8795

8.(b) A shop sells a particular make of video recorder. Assuming that the weekly demand for the video recorder is a Poisson variable with mean 3, find the probability that the shop sells (i) at least 3 in a week, (ii) at most 7 in a week, (iii) more than 20 in a month (4 weeks).

ANSWER: Here m = 3 (i) P(x≥ 3) =1- $8.5e^{-3} = 0.5768$ (ii) P(x≤ 7) = 19.8464 e^{-3}= 0.9881

(iii) More than 20 in a month = more than (20/4 = 5) in a week

P(x>5) = 1-P(0)-P(1)-P(2)-P(3)-P(4)-P(5)= 1 – 18.4 e^{-3} =0.08392

9(a) The number of runs scored by Ali in an innings of a cricket match is distributed according to a Poisson distribution with mean 4.5. Find the probability that he will score (i) exactly 4 in his next innings; (ii) at least three in his next innings; (iii) at least six in total in his next two innings.

ANSWER: Here m = 4.5 (i) P(x= 4) = 0.1898 (ii) P(x≥ 3) =1- $15.625e^{-4.5}$=0.8264

(iii) For next two innings, the mean value is given by

m = m_1+ m_2 = 4.5+ 4.5=9. Thus, the probability that at least six in total in his next two innings is given by P(x≥ 6) =1 - $\sum_{x=0}^{x=5} \frac{9^x e^{-9}}{x!}$ =1- $937.45e^{-9}$=0.8843

9(b) The number of parasites on fish hatched in the same season and living in the same pond follows a Poisson distribution with mean 3.6. Find, giving your answers to 3 decimal places, the probability that a fish selected at random will have (i) 4 or less parasites,(ii)exactly 2 parasites.

ANSWER: Here m = 3.6 (i) P(x≤ 4) = $\sum_{x=0}^{x=4} \frac{3.6^x e^{-3.6}}{x!} = 0.706$ (ii) P(x= 2) =0.1771

10.(a) The number of bacteria in one millilitre of a liquid is known to follow a Poisson distribution with mean 3. Find the probability that a 1 ml sample will contain no bacteria. If 100 samples are taken, find the probability that at most ten will contain no bacteria.

ANSWER: Here m = 3 (i) P(x= 0) =0.04979 (ii) P(x≤ 10) = $\sum_{x=0}^{x=10} \frac{3^x e^{-3}}{x!} \approx 1$

10.(b) A van hire firm has twelve vehicles available and has found that demand follows a Poisson distribution with mean 9.5. In a month of 25 working days, on how many days would you expect (i) demand to exceed supply;(ii) all vehicles to be idle;(iii) it to be possible to service 3 of the vans?

Solution Let x is random variable denote the number of vehicles available or demand of vehicles. From question, the mean is given that

Mean= m = 9.5

Poisson distribution P(x)= $\frac{m^x e^{-m}}{x!}$= $\frac{(9.5)^x e^{-9.5}}{x!}$

i) The demand will more than supply if x is greater than equal to 7. Thus,

P(x≤7) = 1- $\sum_{x=0}^{x=6} \frac{9.5^x e^{-9.5}}{x!} = 1 - 0.165 = 0.835.$

Thus, the number of days the demand will more than supply=$25 \times 0.835 = 20.875 \approx 21$

ii) P(x=0)= $\frac{(9.5)^0 e^{-9.5}}{0!} = e^{-9.5} = 0.000075$

Thus, the number of days the all vehicles to be idle =$25 \times 0.000075 = 0.00187 \approx 0$

(iii) P(x≤3) = $\sum_{x=0}^{x=3} \frac{9.5^x e^{-9.5}}{x!} = 0.015$.

Thus, the total number of days at least 3 of the vans are available =$25 \times 0.015 = 0.375 \approx 0 days$. It is not possible to service 3 of the vans.

11.(a) Wireless sets are manufactured with 25 soldered joints each. On the average 1 joint in 500 is defective. How many sets can be expected to be free from defective joints in a consignment of 10,000 sets.

ANSWER: Here m =np= 25(1/500)=0.05, P(x= 0) =$e^{-0.05}$, The total number of to be free from defective joints in a consignment of 10,000 sets is = 10,000× $e^{-0.05}$=9512 wireless sets

Continuous probability distribution:

The number of events are infinitely large the probability that a specific event will occur is practically zero for this reason continuous probability statement must be worded somewhat differently from discrete ones. Instead of finding the probability that x equals some value, we find the probability of x falling in a small interval. In this context we need a continuous probability function which is defined as follows.

Definition: If for every x belonging to the range of a continuous random variable X, we assign a real number P(x) satisfying the conditions

i) P(x) ≥ 0

ii) $\int_{-\infty}^{\infty} \mathrm{P}(x)\mathrm{dx} = 1$ then P(x) is called a Continuous probability function or probability density function(p.d.f).

If (a,b) is a subinterval of the range space of X then the probability that x lies in the (a,b) is defined to be the interval of P(x) between a and b. i.e., P(a≤x≤b) = $\int_{a}^{b} \mathrm{P}(x)\mathrm{dx}$.

Cumulative distribution function

If X is a continuous random variable with probability density function P(x) then the function f(x) is defined by f(x) $= P(X \leq x) = \int_{-\infty}^{x} P(x)\mathrm{dx}$ is called the cumulative distribution function(c.d.f)of X

The mean and variance of the continuous probability distribution

Mean (μ) or Expectation E(X) =$\int_{-\infty}^{\infty} x.p(x)dx$

Variance (V)= $\int_{-\infty}^{\infty}(x-\mu)^2.p(x)dx$=$\int_{-\infty}^{\infty}(x)^2.p(x)dx - \mu^2$

11(b): A random variable X has the density function P(x)=$\begin{cases} kx^2 & for \quad -3 \leq x \leq 3 \\ 0 & elsewhere \end{cases}$.

Find(i) k (ii) P(x$\leq$ 2) (iii) P(x>1) , (iv) P(1$\leq x \leq 2$).

Solution: If X is a continuous random variable then P(x) ≥ 0 & $\int_{-\infty}^{\infty} \mathrm{P}(x)\mathrm{dx} = 1$.

(i) That is, $\int_{-3}^{3} kx^2\mathrm{dx} = 1$

$\Rightarrow \left[\frac{kx^3}{3}\right]_{-3}^{3} = 1$

⇒ k=1/18

(ii) P(x$\leq$ 2)=$\int_{-3}^{2}\frac{x^2}{18}\mathrm{dx}$ = $\frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{2}$=$\frac{35}{54}$

(iii)P(x>1) =$\int_{1}^{3}\frac{x^2}{18}\mathrm{dx}$ = $\frac{1}{18}\left[\frac{x^3}{3}\right]_{1}^{3}$ =$\frac{26}{54}$= $\frac{13}{27}$.

(iv) P(1$\leq x \leq 2$) =$\int_{1}^{2}\frac{x^2}{18}\mathrm{dx}$ = $\frac{1}{18}\left[\frac{x^3}{3}\right]_{1}^{3}$ =$\frac{27}{54}$= $\frac{1}{2}$.

12(a). A random variable has the density function$f(x) = \frac{k}{1+x^2}, -\infty < x < \infty$. Determine k, $P(x > 0), P(0 < x < 1)$.

Solution: If X is a continuous random variable then i) f(x) ≥ 0 ii) $\int_{-\infty}^{\infty} \mathrm{f}(x)\mathrm{dx} = 1$.

That is, $\int_{-\infty}^{\infty}\frac{k}{1+x^2}\mathrm{dx} = 1$.

$\Rightarrow k[tan^{-1}(x)]_{-\infty}^{\infty} = 1$.

$\Rightarrow k[tan^{-1}(\infty) - tan^{-1}(-\infty)] = 1$.

⇒ k$\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1 \; or \; k = 1/\pi$.

Thus, $f(x) = \frac{1}{\pi(1+x^2)}$.

$P(x > 0)$=$\int_0^{\infty} f(x)dx$=$\int_0^{\infty} \frac{1}{\pi(1+x^2)} dx$

$= \frac{1}{\pi}[tan^{-1}(x)]_0^{\infty}$=$\frac{1}{\pi}[tan^{-1}(\infty) - tan^{-1}(0)] = \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$.

$P(0 < x < 1)$=$\int_0^1 f(x)dx$=$\int_0^1 \frac{1}{\pi(1+x^2)} dx$

$= \frac{1}{\pi}[tan^{-1}(x)]_0^1$=$\frac{1}{\pi}[tan^{-1}(1) - tan^{-1}(0)] = \frac{1}{\pi}\left[\frac{\pi}{4} - 0\right] = \frac{1}{4}$.

12(b). The probability density function of continuous random variable is given by $f(x) = ke^{-|x|}, -\infty < x < \infty$. Prove that $k = {}^{1}/_{2}$ and also find mean and variance.

Solution: If X is a continuous random variable then i) $\mathrm{f}(x) \geq 0$ ii) $\int_{-\infty}^{\infty} \mathrm{f}(x)\mathrm{dx} = 1$.

That is, $\int_{-\infty}^{\infty} ke^{-|x|}\mathrm{dx} = 1$.

$k\int_{-\infty}^{0} e^{-(-x)}\mathrm{dx} + k\int_0^{\infty} e^{-x}\mathrm{dx}$ =1.

$\Rightarrow k[e^x]_{-\infty}^{0} + k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$.

$\Rightarrow k[e^0 - e^{-\infty}] - k[e^{-\infty} - e^{-0}] = 1$

$\Rightarrow$ 2k=1 or k=1/2, $\therefore\ e^0 = e^{-0} = 1\ \&\ e^{-\infty} = 0$.

Mean (μ)=$\int_{-\infty}^{\infty} x.f(x)dx$=$\int_{-\infty}^{\infty} \frac{1}{2}\, xe^{-|x|}\mathrm{dx}$

$=\frac{1}{2}\int_{-\infty}^{0} xe^{-(-x)}\mathrm{dx} + \frac{1}{2}\int_0^{\infty} xe^{-x}\mathrm{dx}$

$=\frac{1}{2}[xe^x - e^x]_{-\infty}^{0} + \frac{1}{2}[x(-e^{-x}) - e^{-x}]_0^{\infty}$

$\mu = -\frac{1}{2}e^0 + \frac{1}{2}e^{-0} = 0$, $\therefore\ e^0 = e^{-0} = 1\ \&\ e^{-\infty} = 0$.

Variance(V) = $\int_{-\infty}^{\infty}(x)^2.f(x)dx - \mu^2 = \int_{-\infty}^{\infty} \frac{1}{2}\, x^2e^{-|x|}\mathrm{dx} - 0$

$=\frac{1}{2}\int_{-\infty}^{0} x^2e^{-(-x)}\mathrm{dx} + \frac{1}{2}\int_0^{\infty} x^2e^{-x}\mathrm{dx}$

$=\frac{1}{2}[x^2e^x - (2x)e^x + (2)e^x]_{-\infty}^{0} + \frac{1}{2}[x^2(-e^{-x}) - (2x)(e^{-x}) + (2)(-e^{-x})]_0^{\infty}$

$=\frac{1}{2}(2e^0) + \frac{1}{2}(2e^{-0})$= 2

13(a): The probability density function f(x) of continuous random variable is given by $P(x) = \begin{cases} kx(1-x), 0 \leq x \leq 1 \\ 0 \qquad\quad otherwise \end{cases}$. Determine k & $P(0 < x < 1/3)$.

Solution: If X is a continuous random variable then $\mathrm{P}(x) \geq 0\ \&\ \int_{-\infty}^{\infty} \mathrm{P}(x)\mathrm{dx} = 1$.

(i) That is, $\int_0^1 k(x - x^2)\mathrm{dx} = 1$

$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$ or $k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$

$\Rightarrow$ k=6 .

Thus, $(x) = \begin{cases} 6x(1-x), 0 \leq x \leq 1 \\ 0 \qquad\quad otherwise \end{cases}$.

(ii) P(0$\leq x \leq 1/3$) =$\int_0^{1/3} 6x(1-x)\mathrm{dx}$

$= 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1/3} = 6\left[\frac{1}{18}-\frac{1}{81}\right]$=0.2593

In continuous probability distribution we study Normal & Exponential distribution.

EXPONENTIAL DISTRIBUTION

Many scientific experiments involve the measurement of the duration of time X between an initial point of time and the occurrence of some phenomenon of interest. For Example X is the life time of a light bulb which is turned on left until it burns out.

The continuous random variable X having the probability density function

f(x) $=\begin{cases} \boldsymbol{\alpha} e^{-\alpha x} & for \quad x>0 \\ 0 & elsewhere \end{cases}$, where $\alpha>0$ is known as the exponential distribution. Here the only one parameter of the distribution is α.

Example of random variables modelled as exponential are

i) Duration of telephone calls
ii) Time require for repair of a component
iii) Service time at a server in a queue

Mean and standard deviation of the exponential distribution

Mean ($\boldsymbol{\mu}$) $= \int_{-\infty}^{\infty} x.f(x)dx = \int_0^{\infty} x.\boldsymbol{\alpha} e^{-\alpha x}dx = \boldsymbol{\alpha}\int_0^{\infty} x.e^{-\alpha x}dx$

$= \boldsymbol{\alpha}\left[x.\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}} - 1\frac{e^{-\alpha x}}{\boldsymbol{\alpha}^2}\right]_0^{\infty}$

$= \boldsymbol{\alpha}\left[0-\frac{1}{\boldsymbol{\alpha}^2}(0-1)\right] = \frac{1}{\boldsymbol{\alpha}}$

Mean ($\boldsymbol{\mu}$) $= \frac{\mathbf{1}}{\boldsymbol{\alpha}}$

Variance (V) = $\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

$= \boldsymbol{\alpha}\int_0^{\infty}(x-\mu)^2 e^{-\alpha x}dx$

$= \boldsymbol{\alpha}\left[(x-\mu)^2.\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}} - 2(x-\mu)\frac{e^{-\alpha x}}{\boldsymbol{\alpha}^2} + 2\frac{e^{-\alpha x}}{-\boldsymbol{\alpha}^3}\right]_0^{\infty}$

$=\boldsymbol{\alpha}\left[(0-\mu^2)\frac{1}{-\boldsymbol{\alpha}} - 2([0-(-\mu)]\frac{1}{\boldsymbol{\alpha}^2} - 2\frac{1}{\boldsymbol{\alpha}^3}(0-1)\right]$

$= \boldsymbol{\alpha}\left[\frac{\mu^2}{\boldsymbol{\alpha}} - \frac{2\mu}{\boldsymbol{\alpha}^2} + \frac{2}{\boldsymbol{\alpha}^3}\right]$

$= \boldsymbol{\alpha}\left[\frac{1}{\boldsymbol{\alpha}^3} - \frac{2}{\boldsymbol{\alpha}^3} + \frac{2}{\boldsymbol{\alpha}^3}\right] \qquad = \frac{1}{\boldsymbol{\alpha}^2}$

Standard deviation ($\boldsymbol{\sigma}$)=$\sqrt{\boldsymbol{V}}$= $\sqrt{\frac{\mathbf{1}}{\boldsymbol{\alpha}^2}} = \frac{\mathbf{1}}{\boldsymbol{\alpha}}$

Mean and Standard deviation is equal in exponential transformation.

14(a): In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for i) less than 10 minutes ii) 10 minutes or more iii) between 10minutes and 12 minutes .

Solution: The p.d.f of the exponential distribution is given by f(x)=$\alpha e^{-\alpha x}, for\ x > 0$ and mean = $1/\alpha$ .From question it is given that shower is exponentially distributed with mean 5 minutes. Thus, $1/\alpha = 5$ or $\alpha = 5$ and f(x) =$\frac{1}{5}e^{-\frac{x}{5}}$.

i) P(x<10)=$\int_0^{10}\frac{1}{5}e^{-\frac{x}{5}}dx = -\left[e^{-\frac{x}{5}}\right]_0^{10}$ =1- e^2=0.8647.

ii) P(x$\geq 10) = \int_{10}^{\infty}\frac{1}{5}e^{-\frac{x}{5}}dx$=$-\left[e^{-\frac{x}{5}}\right]_{10}^{\infty} = e^2 = 0.1353.$

iii) P(10<x<12)=$\int_{10}^{12}\frac{1}{5}e^{-\frac{x}{5}}dx$=$-\left[e^{-\frac{x}{5}}\right]_{10}^{12}$=-($e^{\frac{12}{5}} - e^{-2}$)=0.0446

14 (b) The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is (i) < 200 months, (ii) between 100 months and 25 years.

Solution: From question mean life of a compressor is 200 months.

That is, $1/\alpha = 200$ months or $\alpha = 0.005$. Hence, f(x)= $0.005e^{-0.005x}$, for $x > 0$.

i)P(x$< 200) = \int_0^{200} f(x)dx = \int_0^{10} 0.005e^{-0.005\,x}\,dx$.=0.005 $\left[\frac{e^{-0.005\,x}}{-0.005}\right]_0^{200}$ =1- e^{-1}=0.6321

ii) For 25 years= 25(12) months= 300 months. Thus the probability that the life of a compressor of that company is between 100 months and 300 months is

P(100$\leq x \leq 300) = \int_{100}^{300} f(x)dx = \int_{100}^{300} 0.005e^{-0.005\,x}dx. = 0.005\left[\frac{e^{-0.005\,x}}{-0.005}\right]_{100}^{300}$

=$-\left[e^{-1.5} - e^{-0.5}\right]$ =0.3834.

15 a) Studies of a single-machine-tool system showed that the time the machine operates before breaking down is exponentially distributed with a mean 10 hours. (i). Determine the failure rate and the reliability. (ii). Find the probability that the machine operates for at least 12 hours before breaking down. (iii). If the machine has already been operating 8 hours, what is the probability that it will last another 4 hours?

Solution : The p.d.f of the exponential distribution is given by f(x)=$\alpha e^{-\alpha x}, for\ x > 0$ and Mean(**μ**) = $1/\alpha$ =1/10=0.1 per hr.

(i) Failure rate=λ(t)=$1/\alpha$ =1/10=0.1 per hr, Reliability=R(t)= $e^{-\lambda t}$=$e^{-0.1t}$

(ii) P$(12 \leq x \leq \infty) = \int_{12}^{\infty} f(x)dx = \int_{100}^{300} 0.1e^{-0.1\,x}dx. = 0.1\left[\frac{e^{-0.1\,x}}{-0.1}\right]_{12}^{\infty}$ = $-\left[0 - e^{-0.1(12)}\right]$

=$e^{-1.2}$ =0.3012.

(iii) P$(8 \leq x \leq 12) = \int_8^{12} f(x)dx = \int_8^{12} 0.1e^{-0.1\,x}dx. = 0.1\left[\frac{e^{-0.1\,x}}{-0.1}\right]_8^{12}$ = $-\left[e^{-0.1(12)} - e^{-0.1(8)}\right]$ =$e^{-0.8} - e^{-1.2}$ =0.1481.

15(b): The sale per day in a shop is exponentially distributed with average sale amounting to Rs.100 and net profit is 8%. Find the probability that the net profit exceeds Rs.30 on a day.

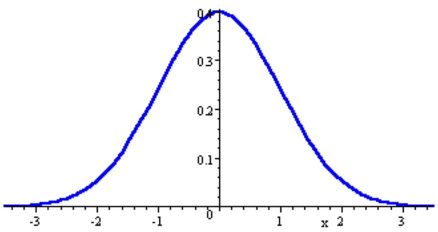
Solution: Let x be the random variable of the sale in the shop. Since x is an exponential variate the p.d.f f(x)= $\alpha e^{-\alpha x}, x > 0$ mean = 1/ α=100 or $\alpha = 0.01$.
Hence, f(x)= $0.01e^{-0.01x}$, $x > 0$
Let A be the amount the sale per day in a shop for which the profit is 8%. For net profit exceeds Rs.30 on a day, the sale of total amount must should be greater than
$\Rightarrow$ A. $\frac{8}{100} = 30 \therefore A = 375.$
Probability of profit exceeding Rs.30
= 1- Prob(profit$\leq Rs.30$)
= 1-Prob(sales $\leq Rs.375$)
$=1 - \int_0^{375} (0.01)e^{-0.01x}\ dx$
$=1+[e^{-0.01x}]_0^{375} = e^{-3.75}$=0.0235
The probability that the net profit exceeds Rs.30 on a day is $e^{-3.75} = 0.0235.$

NORMAL DISTRIBUTION

Normal distribution is the probability distribution of continuous random variable X, known as normal random variable or normal variate it is given by f(x)=$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2}/2\sigma^2$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$ & σ>0. Is known as normal distribution.
$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2}/2\sigma^2 dx$
μ & σ are the two parameters of the normal distribution is also known as Gaussian distribution.

PROPERTIES OF NORMAL DISTRIBUTION

(i)The graph of the normal distribution y=f(x) in the XY-plane is known as normal curve. Normal curve is symmetric about y axis, it is bell shaped the mean, median,& mode coincide & therefore normal curve is unimodal.
Normal curve is asymptotic to both positive & negative x-axis.

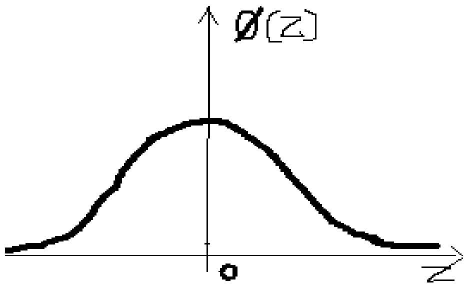


(ii)Area under the curve is unity, that is $\int_{-\infty}^{\infty} f(x)dx = 1.$
(iii)Probability that the continuous random variable lies between a & b is denoted by
P(a$\leq x \leq b$)& is given by $\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx$
(iv) Change of scale from x-axis to z-axis
P(a$\leq x \leq b$)=$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\ dx$. Taking $z = \frac{x-\mu}{\sigma}$ & $dz = \frac{dx}{\sigma}$, we obtain
P($z_1 \leq z \leq z_2$)=$\int_{z_1}^{z_2} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-z^2}{2}}\ \sigma dz$

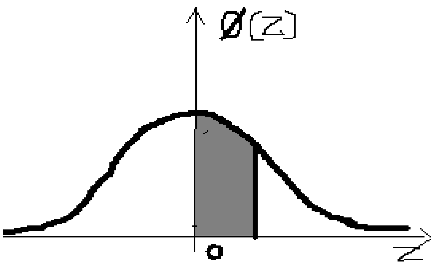
$$=\int_{z_1}^{z_2} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}\, dz \quad , \text{ Where } z_1=\frac{a-\mu}{\sigma},\ z_2=\frac{b-\mu}{\sigma}.$$

Normal distribution f(x) transformed by the standard variable Z is given by F(Z) $=\frac{1}{\sqrt{2\pi}} e^{-z^2}/2$ **with mean 0 & standard deviation 1 is known as standard normal distribution & its normal curve as standard normal curve.** The probability integral is tabulated for various values of Z varying from 0 to 3.9 & is known as normal table. Thus the entries in the normal table gives the area under the normal curve between the ordinates z=0 to z. Since normal curve is symmetric about y-axis the area from 0 to –z is same as the area from 0 to z. For this reason, normal table is tabulated only for positive values of z.

The integral in the RHS of (4) geometrically represents the area bounded by the standard normal curve F(Z) between z=z_1& z=z_2. Further in particular if $z_1 = 0$ *we have* $\emptyset(Z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2}/2$dz. This represents the area under the



standard normal curve z=0 to z.



Normal probability table

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549

0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3304	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999

Note:

1. $\int_{-\infty}^{\infty} \emptyset(Z)dz = 1$ ⇒P($-\infty < z \leq \infty) = 1$
2. $\int_{-\infty}^{0} \emptyset(Z)dz = \int_{0}^{\infty} \emptyset(Z)dz = 1/2$⇒P($-\infty \leq z \leq 0) = P(0 \leq z \leq \infty) = 1/2$
3. P($-\infty < z < z_1$)=P($-\infty < z \leq 0$)+P($0 \leq z < z_1$)=0.5+$\emptyset(z_1)$
4. P($z > z_2$)=0.5- $\emptyset(z_2)$

MEAN & STANDARD DEVIATION OF THE NORMAL DISTRIBUTION

Mean (μ) $= \int_{-\infty}^{\infty} \mathrm{x\ f(x)}\ dx$

$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x\ e^{-(x-\mu)^2}/2\sigma^2\ dx$

Putting t=$\frac{(x-\mu)}{\sigma\sqrt{2}}$ or x= $\mu +\ \sigma t\sqrt{2}$, we have dx= $\sigma\sqrt{2}$dt, t also varies from -∞ to ∞.

Mean=$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \mu +\ \sigma t\sqrt{2}e^{-t^2}\ \ \sigma\sqrt{2}$dt

$=\frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\ dt + \sigma\frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} te^{-t^2}\ dt$

$=\frac{2\ \mu}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t^2}\ dt +\ \sigma\frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} te^{-t^2}\ dt$

The second integral is 0 by standard property since $te^{-t^2}$is an odd function.

By gamma function $\int_{0}^{\infty} e^{-t^2}\ dt = \frac{\sqrt{\pi}}{2}$.

Hence, mean=$\frac{2\ \mu}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} + 0 =\ \mu$.

Hence the mean of the normal distribution is equal to mean of the given distribution.

Variance (V)= $\int_{-\infty}^{\infty}(x-\mu)^2\ f(x)dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-(x-\mu)^2}/2\sigma^2\ dx$

Substituting t=$\frac{x-\mu}{\sqrt{2\pi}}$, x= $\mu +\ \sigma t\sqrt{2}$, we have dx= $\sigma\sqrt{2}$dt, t also varies from -∞ to ∞.

Variance (V)=$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sigma^2 t^2 e^{-t^2}\ \ \sigma\sqrt{2}$dt

$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^2 e^{-t^2} dt$

$=\frac{2\sigma^2}{\sqrt{\pi}}2\int_{0}^{\infty} t^2 e^{-t^2} dt$

$=\frac{2\sigma^2}{\sqrt{\pi}}\int_{0}^{\infty} t\ (2te^{-t^2})dt$

Taking u=t, v=2te^{-t^2}

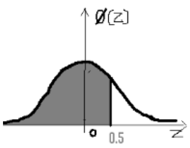
$\int uv$dt=u$\int v$dt $- \iint v$dt . $u'dt$

Variance (V)=$\frac{2\sigma^2}{\sqrt{\pi}}\{[te^{-t^2}]_0^{\infty} - \int_{0}^{\infty} -e^{-t^2}dt\}$

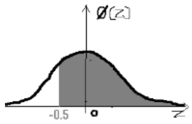
Variance (V)=$\frac{2\sigma^2}{\sqrt{\pi}}$[0+$\int_{0}^{\infty} e^{-t^2}dt$]=$\frac{2\sigma^2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}$=$\sigma^2$

The variance/standard deviation of the normal distribution is equals to the variance of the given distribution.

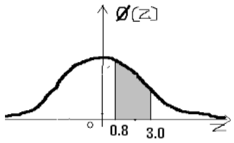
Area under standard normal curve to the left of z=1.5.



Area under standard normal curve to the right of z= -0.5.



Area under standard normal curve between z=0.8 to 3.



Area under the normal curve is distributed as follows:
95 % area lies in the Z-interval (-1.96, 1.96) or 5%(area of rejection) level of significance.
99 % area lies in the Z-interval (-2.58, 1.58) or 1%(area of rejection) level of significance.

Example: Find the following probabilities for the standard normal distribution with the help of normal probability table

a) P(-0.5$\leq z \leq 1.1$) b) P(z ≥ 0.60) c) P(z ≤ 0.75) d)P(0.2$\leq z \leq 1.4$)

Solution:

a) P(-0.5$\leq z \leq 1.1$) = P(-0.5$\leq z \leq 0) + \mathrm{P}(0 \leq z \leq 1.1)$
= P(0$\leq z \leq 0.5$)+ $\mathrm{P}(0 \leq z \leq 1.1)$
=$\emptyset(0.5) + \emptyset(1.1)$
=0.1915+0.3643
=0.5558

b) P(z ≥ 0.60) = P(z$\geq 0) - P(z \leq 0.60)$
= 0.5 $-\emptyset(0.60)$
=0.5 -0.2258 =0.2742

c) P(z ≤ 0.75) = P(z ≤ 0) + P(0$\leq$z ≤ 0.75)
=0.5 $+ \emptyset(0.75)$
=0.5 + 0.2422=0.7422

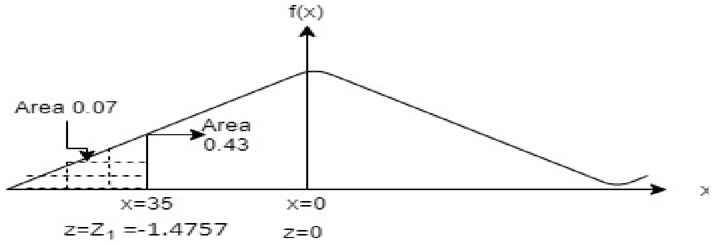
d)P(0.2$\leq z \leq 1.4$) = P(0$\leq z \leq 1.4) - \mathrm{d})\mathrm{P}(0 \leq z \leq 0.2)$
=$\emptyset(1.4) - \emptyset(0.2)$
=0.4192 -0.0793
=0.3399

16(b) In examination 7% of students score less than 35% marks and 89% of students score less than 60% marks, Find the mean and standard deviation, if the marks are normally distributed. It is given that if p (z) =$\frac{1}{\sqrt{2\pi}}\int_0^z e^{-z^2/2}\,dz$ then P (1.2263)=0.39 P(1.4757)=0.43.

Solution: Let the mean and standard deviation of distribution is μ & σ . Thus , z= $\frac{x-\mu}{\sigma}$.
It is given that in examination 7% of students score less than 35% marks. It implies that for

$P(x < 35) = P\left(z = z_1 < \left(\frac{35-\mu}{\sigma}\right)\right) = 0.5 - 0.07 = 0.43.$

Here , P(1.4757)=0.43,it implies that $\frac{35-\mu}{\sigma} = -1.4757$

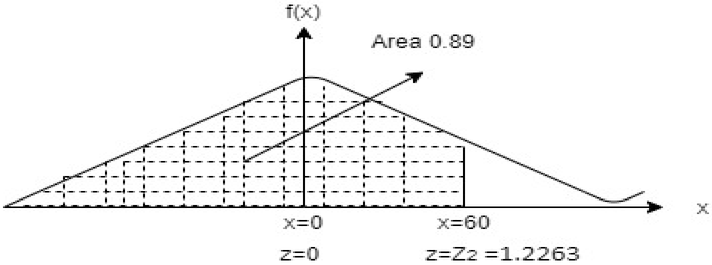


$\mu - 1.4757\sigma = 35$ ……………………………….(1)

Similarly, it is given that in examination 89% of students score less than 60% marks. It implies that for

$P(x < 60) = P\left(z = z_2 < \left(\frac{60-\mu}{\sigma}\right)\right) = 0.89 - 0.5 = 0.39.$

Here , P (1.2263)=0.39 , it implies that $\frac{60-\mu}{\sigma} = 1.2263$



$\mu + 1.2263\sigma = 60$ ……………………………….(2)

Solving the equation (1) & (2) we get

$\mu = 48.6537, \sigma = 9.2524.$

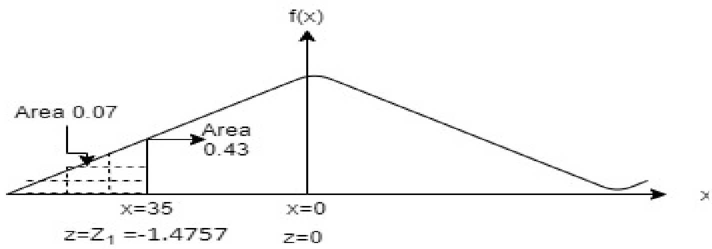
17a) In a normal distribution, 7% of the items are under 35 and 89% are under 63. Determine the mean and variance of the distribution. Given that P (1.2263)=0.39 P(1.4757)=0.43.

Solution: Let the mean and standard deviation of distribution is μ & σ . Thus , z= $\frac{x-\mu}{\sigma}$.

It is given that in examination 7% of items are under 35. It implies that for

$P(x < 35) = P\left(z = z_1 < \left(\frac{35-\mu}{\sigma}\right)\right) = 0.5 - 0.07 = 0.43.$

Here , P(1.4757)=0.43,it implies that $\frac{35-\mu}{\sigma} = -1.4757$

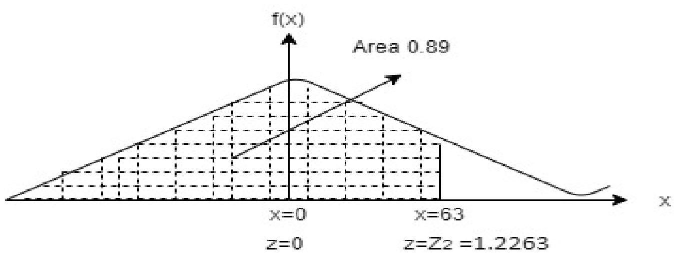


$\mu - 1.4757\sigma = 35$ ……………………………….(1)

Similarly, it is given that 89% are under 63. It implies that for

$P(x < 63) = P\left(z = z_2 < \left(\frac{63-\mu}{\sigma}\right)\right) = 0.89 - 0.5 = 0.39.$

Here , P (1.2263)=0.39 , it implies that $\frac{63-\mu}{\sigma} = 1.2263$



$\mu + 1.2263\sigma = 63$ ……………………………(2)

Solving the equation (1) & (2) we get

$\mu = 50.29, \sigma = 10.363.$

17.(b) Suppose the weights of 800 male students are normally distributed with mean 140 pounds and S.D 10 pounds. Find the number of students whose weight are (i) between 138 and 148 pounds (ii) more than 152 pounds.

Solution: Here, we have given $\mu = 140\ pounds$ & $\sigma = 10\ pounds$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-140}{10}$.

i) For $x = 138, \mathrm{z} = \frac{138-140}{10}$= - 0.2 & for $x = 148, \mathrm{z} = \frac{148-140}{10}$=0.8.

Thus the probability that students whose weight are in between 138 and 148 pounds

$P(138 < x < 148) = P(-0.2 < z < 0.8) = P(-0.2 < z < 0) + P(0 < z < 0.8)$

=$P(0 < z < 0.2) + P(0 < z < 0.8) = 0.0793 + 0.2881 = 0.3674.$

Thus number of students whose weight are in between 138 and 148 pounds are =800(0.3674)=293.92≈ 294.

ii) For $x = 152, \mathrm{z} = \frac{152-140}{10}$=1.2. Thus the probability that students whose weight are more than 152 pounds are

$P(x > 152) = P(z > 1.2) = 0.5 - P(0 < z < 1.2) = 0.5 - 0.3849 = 0.1151$

Thus number of students whose weight are more than 152 pounds are =800(0.1151)=92.08≈ 92.

18 a) A sales tax officer has reported that the average sales of the 500 business that he has to deal with during a year is Rs.36,000 with a standard deviation of 10,000. Assuming that the sales in these business are normally distributed, find (i) the number of business as the sales of which are Rs. 40,000. (ii) the percentage of business the sales of which are likely to range between Rs.30,000 and Rs. 40,000.

Solution: Here, we have given $\mu = Rs\ 36000$ & $\sigma = Rs\ 10000$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-36000}{10000}$.

i) For $x = 40000, \mathrm{z} = \frac{40000-36000}{10000}$=0.4. Thus the probability that number of business as the sales of which are Rs. 40,000

$P(0 < x < 40000) = P(0 < z < 0.4) = 0.1554$

Thus number of the number of business as the sales of which are Rs. 40,000 =500(0.1554)=77.7≈ 78.

(ii) For $x = 30000, \mathrm{z} = \frac{30000-36000}{10000}$=-0.6. Thus the probability that number of business as the sales of which are likely to range between Rs.30,000 and Rs. 40,000.

$P(30000 < x < 40000) = P(-0.6 < z < 0.4) = P(-0.6 < z < 0) + P(0 < z < 0.4)$

=$P(0 < z < 0.6) + P(0 < z < 0.4) = 0.2258 + 0.1554 = 0.3812.$

Thus number of the number of business as the sales of which are likely to range between Rs.30,000 and Rs. 40,000=500(0.3812)=190.6$\approx$ 191.

18 b) A manufacturer knows from experience that the resistance of resistors he produces is normal with mean 100 ohms and SD 2 ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms?

Solution: Here, we have given $\mu = 100\ ohms$ & $\sigma = 2\ ohms$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-100}{2}$.

For $x = 98$, z = $\frac{98-100}{2}$= -1 & for $x = 102$, z = $\frac{102-100}{2}$= 1.

Thus the probability that resistors will have resistance between 98 ohms and 102 ohms

$P(98 < x < 102) = P(-1 < z < 1) = P(-1 < z < 0) + P(0 < z < 1) = P(0 < z < 1) + P(0 < z < 1)$=$2P(0 < z < 1) = 2 \times 0.3413 = 0.6826$.

Thus the 68.26% resistors will have resistance between 98 ohms and 102 ohms.

19(a): A sample of 100 dry battery cells tested to find the length of life produced by a company and following results are recorded: mean life is 12 hrs, SD is 3 hrs. Assuming data to be normally distributed, find the expected life of a dry cell (i) have more than 15 hrs (ii) between 10 and 14 hrs.[P(0.667)=0.2486,P(1)=0.3413].

Solution: Here, we have given $\mu = 12\ hrs$ & $\sigma = 3hrs$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-12}{3}$.

i) For $x = 15$, z = $\frac{15-12}{3}$=1. Thus expected life of a dry cell have more than 15 hrs, that is

$P(x > 15) = P(z > 1) = 0.5 - P(0 < z < 1) = 0.5 - 0.3413 = 0.1587$

Thus number of dry cell have expected life more than 15 hrs=100(0.1587)=15.87$\approx$ 16.

ii) For $x = 10$, z = $\frac{10-12}{3}$=-0.667 & for $x = 14$, z = $\frac{14-12}{3}$=0.667. Thus expected life of a dry cell between 10 and 14 hrs, that is

$P(12 < x < 14) = P(-0.667 < z < 0.667) = P(-0.667 < z < 0) + P(0 < z < 0.667)$
= $2P(0 < z < 0.667) = 2(0.2486) = 0.4972$.

Thus number of dry cell have expected life between 10 and 14 hrs =100(0.4972)=49.72$\approx$ 50.

19(b): The mean weight of 1000 students during medical examination was found to be 70kg and S.D weight 6kg. Assume that the weight are normally distributed, find the number of students having weight (i) less than 65kg (ii) more than 75kg (iii) between 65kg to 75kg.[P(0.83)=0.2967].

Solution: Here, we have given $\mu = 70\ kg$ & $\sigma = 6kg$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-70}{6}$.

i) For $x = 65$, z = $\frac{65-70}{6} = -0.83$. Thus the probability that students having weight less than 65kg is given by:

$P(x < 65) = P(z < -0.83) = 0.5 - P(-0.83 < z < 0) = 0.5 - P(0 < z < 0.83)$
= $0.5-0.2967 = 0.2033$.

Thus number of students having weight less than 65kg =1000(0.2033)=203.3$\approx$ 203 students

ii) For $x = 75, z = \frac{75-70}{6}$ =0.83. Thus the probability that students having weight more than 75kg is given by:

$P(x > 75) = P(z > 0.83) = 0.5 - P(0 < z < 0.83) = 0.5 - P(0 < z < 0.83)$

= $0.5-0.2967 = 0.2033$.

Thus number of students having weight more than 75kg =1000(0.2033)=203.3$\approx$ 203 students.

(iii) The probability that students having weight between 65kg to 75kg is given by:

$P(65 < x < 75) = P(-0.83 < z < 0.83) = P(-0.83 < z < 0) + P(0 < z < 0.83)$

= $2P(0 < z < 0.83) = 2(0.2967) = 0.5934$.

Thus number of students having weight between 65kg to 75kg =1000(0.5934)=593.4$\approx$ 593 students.

20(a): Given that the mean height of students in a class is 158cms with SD of 20cms. Find how many students heights lie between 150cms and 170cms, if there are 100 students in the class.

Solution: Here, we have given $\mu = 158\ cms$ & $\sigma = 20\ cms$. Thus , z= $\frac{x-\mu}{\sigma}$ =$\frac{x-158}{20}$.

For $x = 150, z = \frac{150-158}{20}$=-0.4 & for $x = 170, z = \frac{170-158}{20}$=0.6.

Thus the probability that students heights lies between 150cms and 170cms, that is

$P(150 < x < 170) = P(-0.4 < z < 0.6) = P(-0.4 < z < 0) + P(0 < z < 0.6)$

= $P(0 < z < 0.4) + P(0 < z < 0.6) = 0.1916 + 0.2258 = 0.4174$.

Thus number of students whose heights lies in between 150cms and 170cms =100(0.4174)=41.74$\approx$ 42.

20(b): Suppose 2% of the people on the average are left handed. Find (i) the probability of finding 3 or more left handed (ii) the probability of finding ≤1 left handed.

Solution: As the probability of occurrence is very small, this follows Poisson distribution. Let x is random variable denote the number of people are left handed. From question, the mean value of people are left handed is

Mean=m=2/100=0.02. & P(x)= $\frac{m^x e^{-m}}{x!}$=$\frac{(0.02)^x e^{-0.02}}{x!}$.

i)P(x ≥ 3)= $1 - \sum_{x=0}^{x=2} \frac{(0.02)^x e^{-0.02}}{x!} = 1 - 1.0202e^{-0.02}$

ii)P(x ≤ 1)= $\sum_{x=0}^{x=1} \frac{(0.02)^x e^{-0.02}}{x!} = 1.02e^{-0.02}$