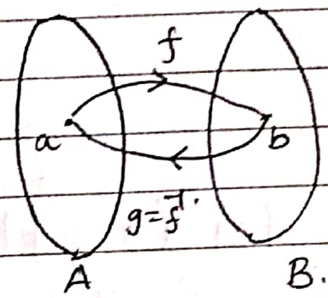
Invertible functions

A function $f : A \to B$ is said to be invertible if there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, where I_A is the identity function on A and I_B is the identity function on B.

Then, g is called an <u>inverse</u> of f and we write $g = f^{-1}$.



Ex: (1) Let $A = \{1,2,3,4\}$ and f & g be functions from A to A given by $f = \{(1,4), (2,1), (3,2), (4,3)\}$ and $g = \{(1,2), (2,3), (3,4), (4,1)\}$. Prove that f and g are inverses of each other.

Soln:

$(g \circ f)(1) = g[f(1)] = g(4) = 1 = I_A(1)$.
$(g \circ f)(2) = g[f(2)] = g(1) = 2 = I_A(2)$
$(g \circ f)(3) = g[f(3)] = g(2) = 3 = I_A(3)$.
$(g \circ f)(4) = g[f(4)] = g(3) = 4 = I_A(4)$.

$(f \circ g)(1) = f[g(1)] = f[2] = \cancel{3}\ 1 = I_A(1)$
$(f \circ g)(2) = f(g(2)) = f(3) = 2 = I_A(2)$.
$(f \circ g)(3) = f[g(3)] = f(4) = 3 = I_A(3)$
$(f \circ g)(4) = f[g(4)] = f(1) = 4 = I_A(4)$.

Thus for all $x \in A$, we have $(g \circ f)(x) = I_A(x)$, and $(f \circ g)(x) = I_A(x)$.

Therefore, g is an inverse of f & f is ~~th~~ an inverse of g.

2) Consider the function $f: R \to R$ defined by $f(x) = 2x+5$. Let a function $g: R \to R$ be defined by $g(x) = \frac{1}{2}(x-5)$. Prove that g is an inverse of f.

Sol: Then, we check that, for any $x \in R$,

$$(g \circ f)(x) = g[f(x)] = g(2x+5)$$
$$= \frac{1}{2}[2x+5-5]$$
$$= x = I_R(x)$$

$$(f \circ g)(x) = f[g(x)] = f\{\tfrac{1}{2}(x-5)\}$$
$$= 2\{\tfrac{1}{2}(x-5)\} + 5$$
$$= x-5+5 = x = I_R(x)$$

Hence this shows that g is ~~the~~ an inverse of f and also f is an inverse of g.

5(a) Theorem 1: 3

If a function $f: A \to B$ is invertible then it has a unique inverse. Further, if $f(a) = b$, then $f^{-1}(b) = a$.

Proof: Suppose $f: A \to B$ is invertible and it has g and h as inverses.

Then g and h are functions from $B \to A$ such that

$$(g \circ f) = I_A, \quad h \circ f = I_A$$

$$(f \circ g) = I_B, \quad h \circ g = I_B,$$

Then we find that,

$$h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$$

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This proves that h and g are not different. Thus, f has a unique inverse (when it is invertible).

Now, suppose that $f(a) = b$.

Then, if g is the inverse of f, we have,

$$a = I_A(a) = (g \circ f)(a) = g[f(a)] = g(b).$$

$$a = I_A(a) = (g \circ f)(a) = g[f(a)] = g(b)$$

Since $g = f^{-1}$, this proves that $f^{-1}(b) = a$.

Remarks (1) If f is invertible, the statements $f(a)=b$ and $a = f^{-1}(b)$ are equivalent.

(2) If $f = \{(a,b) \mid a \in A, b \in B\}$ is invertible, then $f^{-1} = \{(b,a) \mid b \in B, a \in A\}$ and conversely.

(3) If f is invertible then f^{-1} is invertible, and $(f^{-1})^{-1} = f$.

(4) If f^{-1} exists, then $f^{-1} = f^c$, where f^c is the converse of f. (Since f is a relation - being a function from A to B, the converse f^c of f exists) ~~as~~ ~~to~~

(5) A relation R from a set A to a set B is an invertible function from A to B iff R^c is an invertible function from B to A.

Theorem 2: A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof: First suppose that f is invertible.

Then there exists a unique function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

Take any $a_1, a_2 \in A$. Then

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$f(a_1) = f(a_2)$

$\Rightarrow g\{f(a_1)\} = g\{f(a_2)\}$

$\Rightarrow I_A(a_1) = I_A(a_2)$.

$\Rightarrow a_1 = a_2$

This proves that f is one-to-one.

Next, take any $b \in B$, Then $g(b) \in A$ and $b = I_B(b) = (f \circ g)(b) = f\{g(b)\}$.

Thus, b is the image of an element $g(b) \in A$ under f.

Therefore, f is onto.

Conversely, suppose that f is one-to-one & onto. Then for each $b \in B$ there is a unique $a \in A$ such that $b = f(a)$.

Now consider the function $g : B \to A$ defined by $g(b) = a$. Then

$(g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a)$, &

$(f \circ g)(b) = f\{g(b)\} = f(a) = b = I_B(b)$.

These show that f is invertible with g as the inverse.

Thm 3: Let A and B be finite sets with $|A| = |B|$ and f be a function from A to B. Then the following statements are equivalent.

(1) f is one-to-one.

(2) f is onto

(3) f is invertible.

Proof: Suppose $f : A \to B$ is one-to-one. Since A and B are finite sets with

Proof: $|A| = |B|$, it follows that f is onto.
Consequently, f is invertible.
Conversely, suppose f is invertible, then f is one-to-one & onto.
$\therefore$ The three statements are equivalent.

Thm 4: If $f : A \to B$ and $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: Since f and g are invertible functions, they are both one-to-one and onto.
Consequently, $g \circ f$ is both one-to-one & onto.
Therefore, $g \circ f$ is invertible.
Now, the inverse f^{-1} of f is a function from B to A and inverse g^{-1} of g is a function from C to B.
Therefore, if $h = f^{-1} \circ g^{-1}$ then h is a function from C to A.
We find that,

$$(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ I_B \circ g^{-1}.$$
$$= g \circ g^{-1} = I_C.$$

and

$$h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)$$
$$= f^{-1} \circ [g^{-1} \circ g] \circ f$$
$$= f^{-1} \circ I_B \circ f$$
$$= f^{-1} \circ f$$
$$= I_A$$

The above expressions show that h is the inverse of $g \circ f$; that is $h = (g \circ f)^{-1}$.
Thus, $(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}$.

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Ex (3) Consider the function $f: R \to R$ defined by $f(x) = x^2$, $\forall x \in R$. Is f invertible?

Soln: For any $a \in R$, we have $f(a) = a^2$ and $f(-a) = (-a)^2 = a^2$.

Thus, both a and $-a$ have the same image a^2 under f.

Therefore, f is not one-to-one.

Consequently, f is not invertible.

Ex (4) Let $A = \{x \mid x$ is real and $x \geq -1\}$ and $B = \{x / x$ is real and $x \geq 0\}$.

Consider the function $f: A \to B$ defined by $f(a) = \sqrt{a+1}$, for all $a \in A$. Show that f is invertible and determine f^{-1}.

Soln First we have to check the f is one-to-one and onto.

Take any $a_1, a_2 \in A$.

Then $f(a_1) = \sqrt{a_1+1}$ and $f(a_2) = \sqrt{a_2+1}$, and

$f(a_1) = f(a_2) \Rightarrow \sqrt{a_1+1} = \sqrt{a_2+1}$

$\Rightarrow a_1 + 1 = a_2 + 1$, so that $a_1 = a_2$

Hence f is one-to-one

Take any $b \in B$. Then $b = f(a)$ holds if $b = \sqrt{a+1}$ or $b^2 = a+1$, or $a = b^2 - 1$.

Since $b \geq 0$, we note that $b^2 - 1 \geq -1$.

Thus, every $b \in B$ has $a = b^2 - 1$ as a preimage in A under f.

Hence f is onto as well.

This proves that f is invertible.

The inverse of f is given by

$f^{-1}(b) = b^2 - 1 \quad \forall\ b \in B$

//

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Ex 5. Find the inverse of the function $f(x) = e^x$ defined from $R \to R^+$.

Sol: For any $x_1, x_2 \in R$,

$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$.

Therefore, f is one-to-one.

Taken any $y \in R^+$, and put $x = \log_e y$.

Then, $x \in R$ & $e^x = y$ that is $f(x) = y$.

Thus, every $y \in R^+$ has $x = \log_e y$ as its pre image in R under f.

Therefore, f is onto.

Accordingly, f is an invertible function.

The inverse of f is given by

$f^{-1}(y) = \log_e y \quad \forall\ y \in R^+$.

(6) Let $A = B = R$, the set of all real numbers, and the functions $f: A \to B$ and $g: B \to A$ defined by,

$f(x) = 2x^3 - 1, \quad \forall\ x \in A;$

$g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}, \quad \forall\ y \in B.$

Show that each of f & g is the inverse of the other.

Sol: For any $x \in A$

$(g \circ f)(x) = g(f(x)) = g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}$, where $y = f(x)$

$= \left\{\frac{1}{2}(2x^3 - 1) + 1\right\}^{1/3}, \quad \because y = f(x) = 2x^3 - 1.$

$= x$.

Thus $g \circ f = I_A$.

Next, for any $y \in B$,

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$$(f\circ g)(y) = f(g(y)) = f\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]$$

$$= 2\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]^3 - 1$$

$$= 2\left[\frac{1}{2}(y+1)\right] - 1 = y.$$

Thus, $f\circ g = I_B$.

~~∴ each f & g~~

∴ each of f & g are invertible functions, and are inverse of each other.

(7) Let $A = B = C = R$, and $f : A \to B$ and $g : B \to C$ be defined by
$f(a) = 2a+1$, $g(b) = \frac{1}{3}b$, $\forall a \in A$, $b \in B$.
Compute $g\circ f$ and show that $g\circ f$ is invertible, what is $(g\circ f)^{-1}$?

Sol:- We have, $(g\circ f)(a) = g(f(a))$

$$= g(2a+1)$$

$$= \frac{1}{3}(2a+1)$$

~~We check that f is 1-1~~

Thus $g\circ f : A \to C$ is defined by $(g\circ f)(a) = \frac{1}{3}(2a+1)$

We check that f is invertible with $f^{-1}(b) = \frac{1}{2}(b-1)$ and g is invertible with $g^{-1}(c) = 3c$.

∴ $g\circ f$ is invertible, and its inverse is given by, $(g\circ f)^{-1}(c) = (f^{-1}\circ g^{-1})(c)$

$$= f^{-1}\left[g^{-1}(c)\right]$$

$$= f^{-1}(3c)$$

$$= \frac{1}{2}(3c-1)$$

13 (a)

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(8) Let I_A and I_B denote the identity functions on sets A and B resply. For any function $f : A \to B$ then prove that, $f \circ I_A = f = I_B \circ f$.

Soln Suppose $f : A \to B$ be a function and let I_A & I_B be identity functions on A & B resply

Let $a \in A$. Then

$(f \circ I_A)(a) = f(I_A(a)) = f(a)$.

$(I_B \circ f)(a) = I_B(f(a)) = f(a)$.

Hence, for all $a \in A$, $(f \circ I_A)(a) = f(a)$ and $(I_B \circ f)(a) = f(a)$;

Thus, $f \circ I_A = f$ and $I_B \circ f = f$.

$\therefore$ $\boxed{f \circ I_A = f = I_B \circ f}$

(9) Let f and g be functions from R to R defined by $f(x) = ax + b$ and $g(x) = cx + d$. What is relationship must be satisfied by a, b, c and if $f \circ g = g \circ f$?

Soln. Given: $f(x) = ax + b$ and $g(x) = cx + d$.

$(f \circ g)(x) = f(g(x)) = f(cx + d) = a(cx + d) + b$

$= acx + ad + b$

$(g \circ f)(x) = g(f(x)) = g(ax + b) = c(ax + b) + d$

$= acx + bc + d$.

The two compositions have to be equal $(f \circ g) = (g \circ f)$

$acx + ad + b = acx + bc + d$.

Subtract acx from each side of the above equation

$\boxed{ad + b = bc + d}$

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Put $fog(x) = gof(x)$ so,

$a(cx + d) + b = c(ax + b) + d$

$a\cancel{cx} + ad + b = a\cancel{cx} + bc + d.$

$ad + b = bc + d$

$\Rightarrow \boxed{a = c}$

~~So, for fog = gof, ad + b = bc + d~~

13 (b) Let $A = \{1, 2, 3\}$ and f, g, h be functions as follows : $f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,3)\}$

$h = \{(1,1), (2,2), (3,1)\}.$

Find fog, gof, fohog.

Soln)

$fog = f[g(1)] = f[2] = 3$

$fog = f[g(2)] = f(3) = 1$

$fog = f[g(3)] = f(3) = 3$

$gof = g[f(1)] = g(2) = 1$

$gof = g[f(2)] = g(3) = 3.$

$gof = g[f(3)] = g(1) = 2$

$fohog = f[hog(1)] = f[h(g(1))] = f[h(2)] = f(2) = 3$

$fohog = f[hog(2)] = f[h(g(2))] = f[h(1)] = g(1) = 2$

$fohog = f[hog(3)] = f[h(g(3))] = f[h(3)] = f(1) = 2.$

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Stirling Numbers of the second kind:

Let A and B be finite sets with $|A| = m$, and $|B| = n$, where $m \geq n$. Then the number of onto functions from A to B is given by the formula:

$$p(m,n) = \sum_{k=0}^{n} (-1)^k \, ({}^nC_{n-k}) \, (n-k)^m$$

With $p(m,n)$ given by the above formula, the number $\{p(m,n)/n!\}$ is called the stirling number of the second kind and is denoted by $S(m,n)$.

Thus, by definition,

$$\boxed{S(m,n) = \frac{p(m,n)}{n!} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, ({}^nC_{n-k}) (n-k)^m}$$

for $m \geq n$.

This number represents the number of ways in which it is possible to assign m distinct objects into n identical places (containers) with no place (container) left empty.

Note: It is easy to check that $S(m,1) = 1$ for all $m \geq 1$ and $S(m,m) = 1$

It can be shown that the number of possible ways to assign m distinct objects to n identical places with empty places allowed is given by the formula, $\boxed{p(m) = \sum_{i=1}^{n} S(m,i)}$, for $m \geq n$

Ex: (1) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{w, x, y, z\}$. Find the number of onto functions from A to B.

Soln: Here $m = |A| = 7$ and $n = |B| = 4$.

Therefore, the number of onto functions from A to B is

$$P(7,4) = \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^7$$

${}^nC_n = 1$, ${}^nC_{n-1} = n$. ${}^nC_1 = 1$

$$= {}^4C_4 \times 4^7 - {}^4C_3 (4-1)^7 + {}^4C_2 (4-2)^7 - {}^4C_1 (4-3)^7 + 0.$$

$$= 4^7 - 4 \times 3^7 + 6 \times 2^7 - 4$$

$$= 8400.$$

(2) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

(a) Find how many functions are there from A to B. How many of these are one-to-one? How many are onto?

(b) Find how many functions are there from B to A. How many of these are onto? How many are one-to-one?

Soln: Here, $|A| = 4$ and $|B| = n = 6$.

$\therefore$ (i) The number of functions possible from A to B is $n^m = 6^4 = 1296$.

(ii) The number of functions possible from B to A is $m^n = 4^6 = 4096$.

(iii) The number of one-to-one functions possible from A to B is $\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360$.

(iv) There is no one-to-one functions from B to A.

(v) There is no onto function from A to B.

(vi) The number of onto functions from B to A is

$$p(6,4) = \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^6$$

$$= 4^6 - 4\times 3^6 + 6\times 2^6 - 4 = 1560.$$

Ex. (3) Evaluate $S(5,4)$ and $S(8,6)$.

Soln: By definition, we have,

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m$$

Therefore,

$$S(5,4) = \frac{1}{4!}\sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^5$$

$$= \frac{1}{4!}\left\{4^5 - \left({}^4C_3\right)\times 3^5 + \left({}^4C_2\right)\times 2^5 - \left({}^4C_1\right)\times 1^5\right\}$$

$$= \frac{1}{4!}\left\{4^5 - 4\times 3^5 + 6\times 2^5 - 4\right\}$$

$$= \frac{240}{4!} = 10 \,//$$

and $$S(8,6) = \frac{1}{6!}\sum_{k=0}^{6} (-1)^k \, {}^6C_{6-k} \, (6-k)^8$$

$$= \frac{1}{6!}\left\{6^8 - \left({}^6C_5\right)\times 5^8 + \left({}^6C_4\right)\times 4^8 - \left({}^6C_3\right)\times 3^8 + \left({}^6C_2\right)\times 2^8 - \left({}^6C_1\right)\times 1^8\right\}$$

$$= \frac{1}{6!}\left\{6^8 - (6\times 5^8) + (15\times 4^8) - (20\times 3^8) + (15\times 2^8) - 6\right\}.$$

$$= 266 \,//$$

(4)

i) There are six programmers who can assist eight executives. In how many ways can the executives be assisted so that programmer assists atleast one executive ?

Soln: Let A denote the set of executives and B denote the set of programmers.

Then the required number is equal to the number of onto fuctions from A to B.

This number is $p(8,6) = (6!) \times S(8,6)$.

$$\text{W.K.T, } S(8,6) = \frac{1}{6!}\sum_{k=0}^{6} ({}^6C_{6-k})(6-k)^8 = 266.$$

$$\therefore p(8,6) = (6!) \times 266 = 720 \times 266 = 191520$$

This is the required number.

17 (a) Ex (5). Find the number of ways of distributing four distinct objects among three identical containers, with some container (s) possibly empty.

Soln Here, the number of objects is $m=4$ and the number of containers is $n=3$.

Therefore, the required number is

$$p(4) = \sum_{i=1}^{3} S(4,i) = S(4,1) + S(4,2) + S(4,3)$$

Now, $$S(4,1) = \frac{1}{1!}\sum_{k=0}^{1}(-1)^k({}^1C_{1-k})(1-k)^4 = 1$$

$$S(4,2) = \frac{1}{2!}\sum_{k=0}^{2}(-1)^k({}^2C_{2-k})(2-k)^4 = \frac{1}{2}\{2^4 - 2\times 1^4\} = 7$$

$$S(4,3) = \frac{1}{3!}\sum_{k=0}^{3}(-1)^k({}^3C_{3-k})(3-k)^4$$

$$S(4,3) = \frac{1}{6}\{3^4 - 3\times 2^4 + 3\times 1^4\} = 6.$$

Thus the required number is

$$p(4) = 1+7+6 = 14.$$

Ex 6 If m and n are positive integers with $1 \le n \le m$, prove that

$$S(m+1, n) = S(m, n-1) + n\,S(m,n).$$

Sol. Let $A = \{a_1, a_2, \ldots, a_m, a_{m+1}\}$.

We consider the distribution of m+1 elements of A among n identical containers with no container left empty.

We note that, the elements $a_1, a_2, \ldots, a_m$ of A can be distributed among n-1 identical containers (with none left empty) in $S(m, n-1)$ ways.

If a_{m+1} is placed in the remaining container, $S(m, n-1)$ represents the number of ways of distributing all the m+1 elements of A among n containers, with m elements distributed among n-1 containers and the remaining one element placed in the remaining container.

Next, we note that the elements $a_1, a_2, \ldots, a_m$ can be distributed among n identical containers (with none left empty) in $S(m,n)$ ways.

If a_{m+1} is placed in any one of these ~~con~~ n containers, then for each choice of the container into which a_{m+1} is placed, there are $S(m,n)$ ways of distributing all the $m+1$ elements of A among n containers.

Since there are n such possible choices, there are in all $nS(m,n)$ ways of distributing all the $m+1$ elements of A among n containers with m elements distributed among n ~~to~~ containers and the $(m+1)^{st}$ element also put into any one of these n containers.

Thus:

Total no. of ways of distributing the $m+1$ elements of A among n identical containers

$= S(m, n-1) + nS(m,n)$.

$\therefore$ $\boxed{S(m+1, n) = S(m, n-1) + nS(m,n)}$

Q 7 Evaluate $S(8,7)$, given that $S(7,6) = 21$.

Soln: $S(8,7) = S(7,6) + 7S(7,7)$

[using the formula: $S(m+1,n) = S(m,n-1) + nS(m,n)$]

$S(8,7) = 21 + 7(1)$ [$\because S(7,7) = 1$

$= 21 + 7 = 28$ //.

15 (a)

Q 8 Given that $S(8,4) = 1701$, $S(8,5) = 1050$ and $S(8,6) = 266$, evaluate $S(10,6)$.

Sln: We have, $S(m+1,n) = S(m,n-1) + nS(m,n)$

$\therefore S(10,6) = S(9,5) + 6S(9,6)$.

$$= \{S(8,4) + 5S(8,5)\} + 6 \times \{S(8,5) + 6S(8,6)\}$$ (7)

$$= S(8,4) + 11S(8,5) + 36S(8,6).$$

using the given values, we have,

$$S(10,6) = 1701 + (11 \times 1050) + (36 \times 266)$$

$$= 22,827.$$

15 (b) Evaluate $S(9,4)$, $S(7,4)$, $S(8,4)$ & $S(9,5)$

2

Pigeonhole Principle

16(a) If m pigeons occupy n pigeon holes and if $m>n$, then two or more pigeons occupy the same pigeohole.

This is restated as follows:

If m pigeons occupy n pigeonholes, where $m>n$, then at least one pigeonhole must contain two or more pigeons in it.

This is known as the Pigeonhole Principle.

Illustration: If 6 pigeons occupy 4 pigeonholes, then at least one pigeonhole must contain two or more pigeons in it.

As a simple application of the principle, if 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalization of The pigeonhole Principle

If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain $(p+1)$ or more pigeons, where $p=\lfloor (m-1)/n \rfloor$

16(b)

Ex (1) How many persons must be chosen in order that at least five of them will have birth days in the same calender month?

Soln: Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays, in the Same month is, by the generalized pigeonhole principle, equal to $\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1$.

This number is 5. if

$$\left\lfloor \frac{n-1}{12} \right\rfloor + 1 = 5 \quad \text{or} \quad \boxed{n = 49}$$

Thus, the no. of persons is 49.

$p+1$

$(100+1)$

19 ⓐ Prove that if 101 integers are selected from the set $S = \{1, 2, 3, \ldots, 200\}$, then at least two of these are such that one divides the other.

Soln. Let $X = \{1, 3, 5, \ldots, 199\}$.

Then every integer n between 1 and 200 (inclusive) is of the form $n = 2^k x$, where k is an integer ≥ 0 and $x \in X$.

Thus, every element of S corresponds to some $x \in X$.

The set X has 100 distinct elements and therefore, if 101 elements of S are selected, then at least two of them say a & b, $a \neq b$ must correspond to the same $x \in X$.

Thus $a = 2^m x$, $b = 2^n x$, for some integers $m, n \geq 0$.

Evidently, a divides b if $m \leq n$ and ~~b divides~~ b divides a if $n < m$.

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(3) Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any 3 of these students are chosen to be a debating leams from the class, the sum of their shirts ~~no~~ numbers is used as the code numbers of the teams. Show that if any 8 of the 20 are selected, then from these 8 we may form at least two different teams having the same code number.

Sln: From the 8 of the 20 students selected, we can form $C(8,3) = 56$ different teams.

According to the way in which the code number of a team is determined, we note that the smallest possible code number is $1+2+3=6$ and the largest " " " is $18+19+20=57$.

Thus, the code numbers vary from 6 to 57. (inclusive), and there are 52 in number.

As such, only 52 code numbers (pigeon holes) are available for the 56 possible teams (pigeons). Consequently, by the pigeonhole principle, at least two different teams will have the same code number.