Algebraic Geometry

Orange

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1 Scheme

1.1 Motivation

Our main goal is to solve algebraic system

$$\begin{cases} f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ f_m(x_1, \ldots, x_n) = 0 \end{cases}$$

where f_i are all polynomials. We know that we can consider algebraic varieties over an infinite field k. But we want to deal with more general cases:

(i) when k is a finite field.

(ii) when k is still infinite, but consider the following cases:

  (a) $x^2 + 1$ has no solution in $\mathbb{R}$, but have 2 solutions in $\mathbb{C}$.

  (b) $x^2 + 1$, $x^3 + 2$ have no solution in $\mathbb{Q}$, but they are different polynomials clearly.

(iii) extend k to be a commutative ring with $1 (\neq 0)$.

(iv) given polynomials $f_i(x_1, \ldots, x_n)$ over $\mathbb{Z}$, we can solve this system over $\mathbb{Q}$, or over $\mathbb{Z}/p\mathbb{Z}$. We want to realize what the common relation between $\mathbb{Q}$ over $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}$.

1.2 Spectrum

From now on, if other specified, all rings are commutative with $1 (\neq 0)$.

Definition. Let A be a ring. The **spectrum** of A, denoted by $\operatorname{Spec} A$, is the set of all prime ideals of A.

Example. Let k be a field.

(i) $\operatorname{Spec} k = \{(0)\}$.

(ii) $\operatorname{Spec} k[t] = \{(0), (f) \mid f \text{ is irreducible.}\}$

(iii) $\mathbb{A}^n_k = \operatorname{Spec} k[t_1, \ldots, t_n]$.

(iv) $\operatorname{Spec} \mathbb{Z} = \{(0), (p) \mid p \text{ is a prime.}\}$

(v) $\operatorname{Spec} \left(k[t]/(t^2)\right) = \{(\bar{t})\}$.

Definition. Let $I \trianglelefteq A$ be an ideal, $V(I) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I\}$.

Proposition 1.1. Define $V(I)$ as a closed subset of $\operatorname{Spec} A$.

(i) This equips $\operatorname{Spec} A$ with a topological structure.

(ii) Let $f \in A$, define $D(f) = \{\mathfrak{p} \in \operatorname{Spec} A | f \notin \mathfrak{p}\}$, $D(f)$ forms a basis of open sets.

This topology is called the **Zariski topology** on $\operatorname{Spec} A$.

Proof.

(i) (1) $V(A) = \varnothing$, $V(0) = \operatorname{Spec} A$.

  (2) $I_i \trianglelefteq A$, $\bigcap_{i \in S} V(I_i) = V(\sum_{i \in S} I_i)$.

  (3) $V(I_1) \cup V(I_2) = V(I_1 I_2) = V(I_1 \cap I_2)$.

(ii) Let $U = \operatorname{Spec} A \backslash V(I)$ be an open set.

$$\begin{aligned} & \mathfrak{p} \in U \\ \Longrightarrow\ & I \nsubseteq \mathfrak{p} \\ \Longrightarrow\ & \exists f \in I,\ f \notin \mathfrak{p} \\ \Longrightarrow\ & \mathfrak{p} \in D(f). \end{aligned}$$

One can check that $D(f) \subset U$.

□

Definition. Let $\mathfrak{p} \in \operatorname{Spec} A$. If $\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\}$, then $\mathfrak{p}$ is called a **closed point**.

Remark. Let $\mathfrak{p} \in \operatorname{Spec} A$. Then $\mathfrak{p}$ is a closed point if and only if $\mathfrak{p}$ is a maximal ideal.

Example.

(i) $\mathbb{A}^1_k = \operatorname{Spec} k[t] = \begin{cases} (t-a),\ a \in k \\ (r(t)),\ r \text{ irreducible, } \deg r > 1 \\ (0) \end{cases}$

We see that $(t-a)$ corresponds to the points in $\mathbb{A}^1(k)$. So, the spectrum contains more "points" than the affine space. On the other hand, we call (0) the **generic point** since $V(0) = \operatorname{Spec} A$ or equivalent $\overline{\{(0)\}} = \operatorname{Spec} A$.

(ii) $\mathbb{A}^2_k = \operatorname{Spec} k[t_1, t_2] = \begin{cases} (t_1 - a, t_2 - b) \\ (f_i(t_1, t_2))_{i \in I}, \text{ e.g. } t_1 - t_2,\ t_1^2 + t_2^2 - 1, \ldots \\ (0) \end{cases}$

Similar to the above, $(t_1 - a, t_2 - b)$ corresponds to the points in $\mathbb{A}^2(k)$. But we can see that any irreducible variety is contained in the second case. So, the spectrum contains irreducible affine varieties.

Definition. Let $I \trianglelefteq A$ be an ideal. The **radical** of A is

$$\sqrt{I} := \{x \in A | x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

It is clear that $V(I) = V(\sqrt{I})$. On the other hand, if we use the following fact, we can see that $V(I) \subset V(J)$ if and only if $J \subset \sqrt{I}$.

Fact. $\sqrt{I} = \bigcap_{\substack{I \subset \mathfrak{p} \\ \mathfrak{p} \in \operatorname{Spec} A}} \mathfrak{p}$.

Proof. The proof is left as an exercise (you may want to prove the statement when $I = 0$ first). □

Proposition 1.2. Let $\varphi : A \to B$ be a ring homomorphism. For $\mathfrak{p} \in \operatorname{Spec} B$, we define $\psi : \operatorname{Spec} B \to \operatorname{Spec} A$ by $\psi(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$. Then ψ is continuous with respect to Zariski topology.

Proof. Let $I \trianglelefteq A$ be any ideal.

$$\begin{aligned} & \mathfrak{p} \in \psi^{-1}(V(I)) \\ \iff & \psi(\mathfrak{p}) \in V(I) \\ \iff & \varphi^{-1}(\mathfrak{p}) \supseteq I \\ \iff & \mathfrak{p} \supset \langle \varphi(I) \rangle \\ \iff & \mathfrak{p} \in V(\langle \varphi(I) \rangle). \end{aligned}$$

Hence, $\varphi^{-1}(V(I)) = V(\varphi(I))$ is closed. □

Corollary 1.3. Let $I \trianglelefteq A$ be an ideal. The quotient map $\pi : A \to A/I$ induces a homeomorphism from $\iota : \operatorname{Spec} A/I \to V(I) (\subset \operatorname{Spec} A)$ with subspace topology on $V(I)$.

Proof. By Proposition 1.2, $\iota : \operatorname{Spec} A/I \to \operatorname{Spec} A$ is continuous. Also, we know that $\mathfrak{p} \in \operatorname{Spec} A/I \iff \pi^{-1}(\mathfrak{p}) \in V(I)$. So, ι is injective and $\operatorname{Im}(\iota) = V(I)$. We consider its inverse

$$\iota^{-1} : V(I) \to \operatorname{Spec} A/I$$

by $Q \mapsto Q/I$. It remains to show that ι^{-1} is continuous.

$$\begin{aligned} & Q \in (\iota^{-1})^{-1}(V(\bar{J})) \\ \iff & \iota^{-1}(Q) \in V(\bar{J}) \\ \iff & Q/I \supset \bar{J} \\ \iff & Q \supset \pi^{-1}(\bar{J}) \end{aligned}$$

Namely, $Q \in (\iota^{-1})^{-1}(V(\bar{J})) \iff Q \in V(\pi^{-1}(\bar{J}))$. □

Corollary 1.4. Let $f \in A$ with $f^n \neq 0$ for all $n \in \mathbb{Z}_{>0}$. Let $S = \{1, f, f^2, \dots\}$ be a multiplicative subset of A. Denote A_f be the localization of A with respect to S. We have a canonical ring homomorphism $\varphi : A \to A_f$. Then $\psi : \operatorname{Spec} A_f \to \operatorname{Spec} A$ induces a homeomorphism from $\operatorname{Spec} A_f$ to $D(f) \subset \operatorname{Spec} A$.

Proposition 1.5. Spec A is quasi-compact[1], i.e. any open cover of Spec A admits a finite subcover.

Proof. Let $\{U_i\}_{i\in I}$ be a cover of Spec A. By Proposition 1.1 (ii), each U_i can be written as a union of $D(f_{i,j})$. Therefore, we can reduce to the case that $U_i = D(f_i)$.

$$\begin{aligned}
&\bigcup_{i\in I} D(g_i) = \operatorname{Spec} A \\
\Longleftrightarrow &\bigcap_{i\in I} V(g_i) = \varnothing \\
\Longleftrightarrow &V(\langle g_i\rangle_{i\in I}) = \varnothing \\
\Longleftrightarrow &1 \in \langle g_i\rangle_{i\in I} \\
\Longleftrightarrow &\exists I' \subset I \text{ with } |I'| < \infty \text{ such that } 1 = \sum_{i\in I'} a_i g_i
\end{aligned}$$

Therefore, $V(\langle g_i\rangle_{i\in I'}) = \varnothing$, i.e. $\bigcup_{i\in I'} D(g_i) = \operatorname{Spec} A$. □

1.3 Sheaves

Spectrum is not enough to distinguish different ring. For instance, Spec $k = \{(0)\}$ and Spec $k[t]/(t^2) = \{(\bar{t})\}$ are isomorphic as topological spaces. So we need more structures. Let X be an abstract topological space.

Definition. A **presheaf** $\mathscr{F}$ of abelian groups (or sets, modules, ...) on X consists of the following data:

(i) For any open set U of X, $\mathscr{F}(U)$ is an abelian group.

(ii) For a pair of open sets $V \subset U$, there is a group homomorphism $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

such that

(iii) $\mathscr{F}(\varnothing) = \{0\}$

(iv) $\rho_{UU} = \mathrm{id}_U$

(v) For open sets $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition. Let U be an open set of X. An element $s \in \mathscr{F}(U)$ is called a **section** of $\mathscr{F}$ over U.

Definition. A presheaf $\mathscr{F}$ on X is a **sheaf** if it also satisfies

(i) (Uniqueness) Let $\{U_i\}_{i\in I}$ be an open cover of $U \subset X$, and $s \in \mathscr{F}(U)$. If $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$.

[1] In algebraic geometry, compactness refers to Hausdorff and any open cover admits a finite subcover.

(ii) (Existence) Let $\{U_i\}_{i\in I}$ be an open cover of $U \subset X$, and $s_i \in \mathscr{F}(U_i)$ for all $i \in I$. If $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

In summary, a presheaf is a sheaf if the following is exact

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{F}(U) & \longrightarrow & \prod_i \mathscr{F}(U_i) & \longrightarrow & \prod_{i,j} \mathscr{F}(U_i \cap U_j) \\
 & & s & \longmapsto & s|_{U_i} & & \\
 & & & & (t_i) & \longmapsto & t_i|_{U_i\cap U_j} - t_j|_{U_i\cap U_j}
\end{array}
$$

Example.

(i) Let X be a topological space, $\mathscr{F}(U)$ consist of all continuous maps from U to $\mathbb{R}$. Then $\mathscr{F}$ is a sheaf.

(ii) Let $X = \mathbb{R}^n$, $\mathscr{F}(U)$ consist of all smooth maps from U to $\mathbb{R}$. Then $\mathscr{F}$ is a sheaf.

(iii) Fix an abelian group G. For any open subset $U \subset X$, define $\mathscr{F}(U)$ be the set of all locally constant map from U to G. Then $\mathscr{F}$ is a sheaf, called it **constant sheaf**.[2]

Definition. Let $\mathscr{F}$ be a sheaf on X, and $x \in X$. The **stalk** of $\mathscr{F}$ at x is defined as

$$\mathscr{F}_x := \varinjlim_{x\in U} \mathscr{F}(U)$$

where the partial order is given by $V \subset U \Leftrightarrow V \geqslant U$. Explicitly,

$$\mathscr{F}_x = \bigsqcup_{x\in U} \mathscr{F}(U)/\sim$$

with the equivalence relation $s \in \mathscr{F}(U)$, $t \in \mathscr{F}(V)$, $s \sim t$ if and only if there is an open subset $W \subset U \cap V$ such that $s|_W = t|_W$.
An element $s \in \mathscr{F}_x$ is called a **germ** at x.

Definition. Let X be a topological space. A **morphism** between sheaves $\mathscr{F}, \mathscr{G}$ on X is a family of maps $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ for any open subset $U \subset X$, such that for any pair of open sets $V \subset U$, the following diagram commutes

$$
\begin{array}{ccc}
\mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \\
{\scriptstyle \rho^{\mathscr{F}}_{UV}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathscr{G}}_{UV}} \\
\mathscr{F}(V) & \xrightarrow[\varphi_V]{} & \mathscr{G}(V)
\end{array}
$$

and we denote by $\varphi : \mathscr{F} \to \mathscr{G}$.

[2]If we define $\mathscr{F}(U) = G$ for all open subset $U \subset X$, then $\mathscr{F}$ is not a sheaf.

Remark. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ induces a morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ for all $x \in X$.

Definition. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. The kernel of φ is a sheaf defined by $\ker(\varphi)(U) := \ker \varphi_U$, called by **kernel sheaf**.

One can check that $\ker\varphi$ is a sheaf on X. How about the image and cokernel? We can define similarly,

$$\mathrm{Im}(\varphi)(U) := \mathrm{Im}\,\varphi_U,\ \mathrm{coker}(\varphi)(U) := \mathrm{coker}\,\varphi_U$$

They are only presheaves in general.

Example. Let $X = \mathbb{C}$, $\mathscr{O}_X^h$ be the sheaf of holomorphic functions, and $(\mathscr{O}_X^h)^*$ be the sheaf of invertible holomorphic functions. Consider the morphism

$$\begin{aligned} \exp : \mathscr{O}_X^h &\longrightarrow (\mathscr{O}_X^h)^* \\ f &\longmapsto e^{2\pi i f} \end{aligned}$$

Im(exp) is not a sheaf since log can only be defined locally.

Proposition 1.6 (Sheafification)**.** Let $\mathscr{F}$ be a presheaf on X. Define a presheaf $\mathscr{F}^\dagger$ on X as follows. For an open set $U \subset X$,

$$\mathscr{F}^\dagger(U) := \left\{ s : U \to \coprod_{x\in U} \mathscr{F}_x \,\middle|\, \begin{array}{c} 1.\ s(x)\in\mathscr{F}_x \text{ for all } x\in U \\ 2.\ \text{For any } x\in U, \text{ there exists } x\in V\subset U \text{ and } f\in\mathscr{F}(V) \\ \text{such that } s(y)=f_y \text{ for all } y\in V \end{array} \right\}.$$

Then

(i) $\mathscr{F}^\dagger$ is a sheaf, and there is a natural morphism of presheaves $\iota : \mathscr{F} \to \mathscr{F}^\dagger$.

(ii) $(\mathscr{F}^\dagger)_x = \mathscr{F}_x$ for all $x \in X$.

(iii) (Universal property) For any sheaf $\mathscr{G}$ on X and a morphism of presheaves $\psi : \mathscr{F} \to \mathscr{G}$, there is a unique morphism of sheaves $\psi^\dagger : \mathscr{F}^\dagger \to \mathscr{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F} \arrow[rr, "\psi"] \arrow[dr, "\iota"'] & & \mathscr{G} \\ & \mathscr{F}^\dagger \arrow[ur, "\psi^\dagger"'] & \end{tikzcd}$$

The sheaf $\mathscr{F}^\dagger$ is called the **sheafification** of $\mathscr{F}$ or **the sheaf associated to the presheaf** $\mathscr{F}$.

Proof. The proof is left as an exercise. □

Definition.

(i) For a morphism of sheaves on X, $\psi : \mathscr{F} \to \mathscr{G}$, $\operatorname{Im}(\psi)$, $\operatorname{coker}(\psi)$ are defined to be the sheafification of the presheaves $U \mapsto \operatorname{Im}\psi_U$, $U \mapsto \operatorname{coker}\psi_U$.

(ii) $\psi : \mathscr{F} \to \mathscr{G}$ is said to be **injective** if $\ker(\psi) = 0$.

(iii) $\psi : \mathscr{F} \to \mathscr{G}$ is said to be **surjective** if $\operatorname{Im}(\psi) = \mathscr{G}$.

(iv) $\psi : \mathscr{F} \to \mathscr{G}$ is said to be **isomorphism** if it is injective and surjective.

(v) We say that a sequence $\cdots \to \mathscr{F}_{i-1} \xrightarrow{\psi_{i-1}} \mathscr{F}_i \xrightarrow{\psi_i} \mathscr{F}_{i+1} \to \cdots$ is **exact** if $\operatorname{Im}(\psi_{i-1}) = \ker(\psi_i)$ at each stage.

One can check that a sequence is exact if and only if the corresponding sequence of stalks are exact.

Definition. Let $f : X \to Y$ be a continuous map between topological spaces, and $\mathscr{F}$ be a sheaf on X. For any open set $V \subset Y$,

$$f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V)).$$

$f_*\mathscr{F}$ is called the **direct image** and is a sheaf on Y.

Note that $(f_*\mathscr{F})_{f(x)} = \mathscr{F}_x$ is not true in general (Consider the constant map and constant sheaf).
Recall that we want to distinguish different rings, one idea is to consider all functions on the spectrum in the following viewpoint.
Consider the pair $(\operatorname{Spec} k[t_1, \dots, t_n], k[t_1, \dots, t_n])$, the second argument $k[t_1, \dots, t_n]$ consists of all polynomials function on $\mathbb{A}^n(k)$. Although $\mathbb{A}^n_k$ contains more points than $\mathbb{A}^n(k)$, we can view $k[t_1, \dots, t_n]$ as functions on $\mathbb{A}^n_k$.
Let $X = \operatorname{Spec} A$, our goal is to construct a sheaf $\mathscr{O}_X$ on $\operatorname{Spec} A$ such that $\mathscr{O}_X(X) = A$.
We know that $\{D(f)\}_{f\in A}$ forms an open basis of $\operatorname{Spec} A$, so we may expect that $\mathscr{O}_X(D(f)) \simeq A_f$ in the view of functions on $D(f)$ ($\mathfrak{p} \in D(f) \implies f \notin \mathfrak{p}$).
One may define $\mathscr{O}_X(D(f)) = A_f$, but what if $D(f) = D(g)$ for $f \neq g$, we don't want to define $\mathscr{O}_X(D(f))$ up to some isomorphism.

Definition. For any nonempty open set $U \subset X = \operatorname{Spec} A$,

$$\mathscr{O}_X(U) := \left\{ s : U \to \bigsqcup_{\mathfrak{p}\in U} A_\mathfrak{p} \,\middle|\, \begin{array}{c} \text{1. } s(\mathfrak{p}) \in A_\mathfrak{p} \text{ for all } \mathfrak{p}\in U \\ \text{2. For all } \mathfrak{p}\in U\text{, there exists } \mathfrak{p}\in V\subset U\text{, and } a, f\in A \\ \text{such that for any } \mathfrak{q}\in V, f\notin\mathfrak{q}, s(\mathfrak{q}) = \frac{a}{f} \end{array} \right\}.$$

One can check that $\mathscr{O}_X$ is a sheaf.

Theorem 1.7. There is an isomorphism $\psi_f : A_f \to \mathscr{O}_X(D(f))$ for all $f \in A$, and for any $D(g) \subset D(f)$, the following diagram commutes

$$\begin{tikzcd} A_f \arrow[r, "\psi_f"] \arrow[d, "\rho_{f,g}"'] & \mathscr{O}_X(D(f)) \arrow[d, "\text{res}"] \\ A_g \arrow[r, "\psi_g"'] & \mathscr{O}_X(D(g)) \end{tikzcd}$$

where $\rho_{f,g}$ is defined as follow. Since $D(g) \subset D(f)$, $V(g) \supset V(f)$, and $g \in \sqrt{(f)}$. So $g^n = af$ for some $n \in \mathbb{Z}_{>0}$, $a \in A$. Then $\rho_{f,g}(\frac{\alpha}{f^m}) = \frac{\alpha a^m}{g^{nm}}$.

Proof.

(i) Define ψ_f as follow

$$\psi_f\left(\frac{\alpha}{f^m}\right) : D(f) \to \bigsqcup_{\mathfrak{p}\in D(f)} A_{\mathfrak{p}}$$

by $\psi_f(\frac{\alpha}{f^m})(\mathfrak{p}) = \frac{\alpha}{f^m}$ for any $\mathfrak{p} \in D(f)$.

(ii) The diagram commutes:

$$\text{res} \circ \psi_f\left(\frac{\alpha}{f^m}\right) = \psi_f\left(\frac{\alpha}{f^m}\right)\Big|_{D(g)} : D(g) \to \bigsqcup_{\mathfrak{p}\in D(g)} A_{\mathfrak{p}}$$

sends $\mathfrak{p}$ to $\frac{\alpha}{f^m}$ for all $\mathfrak{p} \in D(g)$. On the other hand,

$$\psi_g\left(\rho_{f,g}\left(\frac{\alpha}{f^m}\right)\right) : D(g) \to \bigsqcup_{\mathfrak{p}\in D(g)} A_{\mathfrak{p}}$$

sends $\mathfrak{p}$ to $\rho_{f,g}(\frac{\alpha}{f^m}) = \frac{\alpha a^m}{g^{nm}}$. Since $\frac{\alpha}{f^m} = \frac{\alpha a^m}{g^{nm}}$, the diagram commutes.

(iii) ψ_f is injective:
Suppose that $\psi_f(\frac{\alpha}{f^m}) = 0$, then $\frac{\alpha}{f^m} = 0$ in $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in D(f)$. In other words, for any $\mathfrak{p} \in D(f)$, there exists $h \notin \mathfrak{p}$ such that $h\alpha = 0$ in A. Define

$$I := \text{Ann}(\alpha) := \{b \in A \mid b\alpha = 0\}$$

be an ideal in A. The above argument shows that $I \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in D(f)$, i.e. $V(I) \subset V(f)$. Therefore,

$$\begin{aligned} & f \in \sqrt{I} \\ \implies & f^r \in I \\ \implies & f^r\alpha = 0 \\ \implies & \frac{\alpha}{f^m} = 0 \text{ in } A_f. \end{aligned}$$

(iv) ψ_f is surjective:
Let $s \in \mathscr{O}_X(D(f))$. By construction of $\mathscr{O}_X$, there is an open cover $\{V_i\}_{i\in I}$ of $D(f)$ such that $s|_{V_i} = \frac{a_i}{g_i}$ for some a_i, $g_i \in A$ and $g_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in V_i$.

(1) Since $D(f)$ is homeomorphic to $\operatorname{Spec} A_f$ which is quasi-compact, we may assume $|I| < \infty$.

(2) Since $\{D(f)\}_{f\in A}$ is an open basis, there are $f_{i,j}$'s such that $\bigcup_j D(f_{i,j}) = V_i$ for all $i \in I$. Hence, we may assume $V_i = D(f_i)$ for some $f_i \in I$.

(3) Since $s|_{V_i} = \frac{a_i}{g_i}$ on $\mathfrak{p} \in V_i = D(f_i)$, $D(g_i) \supset D(f_i)$, $f_i^m = c_i g_i$. Also, $D(f_i^m) = D(f_i)$, we can assume that $V_i = D(f_i)$ and $s|_{V_i} = \frac{a_i}{f_i}$.

Now, on $V_i \cap V_j$, we have

$$\frac{a_i}{f_i} = \frac{a_j}{f_j} \text{ in } A_{f_i f_j},$$

i.e.

$$(a_i f_j - a_j f_i)(f_i f_j)^n = 0 \text{ in } A.$$

Replace f_i by f_i^{n+1}, we have

$$s|_{V_i} = \frac{a_i'}{f_i}, \text{ and } a_i' f_j - a_j' f_i = 0.$$

Since $D(f_i)$ covers $D(f)$,

$$\begin{aligned} 1 &= \sum_{i \in I} b_i f_i \text{ in } A_f \\ \implies f^r &= \sum_{i \in I} b_i' f_i \text{ in } A \text{ for some } r \in \mathbb{Z}_{>0}. \end{aligned}$$

Define $a := \sum_{i \in I} a_i' b_i'$. We claim that $\psi_f(\frac{a}{f^r}) = s$.

$$a f_j = \sum_{i \in I} a_i' b_i' f_j = \sum_{i \in I} a_j' b_i' f_i = a_j' f^r.$$

Hence $\frac{a_j'}{f_j} = \frac{a}{f^r}$, i.e. $\psi_f(\frac{a}{f^r})|_{V_i} = s|_{V_i}$ for all $i \in I$. By the uniqueness of sheaf, $\psi_f(\frac{a}{f^r}) = s$.

□

Corollary 1.8. $\mathcal{O}_X(\operatorname{Spec} A) = A$.

Proof. Apply Theorem 1.7 for $f = 1$. □

Proposition 1.9. Let $X = \operatorname{Spec} A$, $\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$.

1.4 Locally ringed spaces

Definition. Let X be a topological space, $\mathcal{O}_X$ be a sheaf of commutative ring on X, and $\mathcal{O}_{X,x}$ be local ring at each $x \in X$.

(i) $(X, \mathcal{O}_X)$ is called a **(locally) ringed space**.

(ii) $\mathcal{O}_X$ is called the **structure sheaf** on X.

(iii) $\mathfrak{m}_x$: the **maximal ideal** of $\mathcal{O}_{X,x}$. $\kappa_x := \mathcal{O}_{X,x}/\mathfrak{m}_x$ is called the **residue field** of X at x.

Example.

(i) Let $X = \mathbb{C}$, $\mathscr{O}_X^h$ be the sheaf of holomorphic function on X. Then $(X, \mathscr{O}_X^h)$ is a locally ringed space, $\mathfrak{m}_x$ consists of all holomorphic functions at x and zero at x, $\kappa_x = \mathbb{C}$.

(ii) Let $X = \operatorname{Spec}\mathbb{Z}$, $\mathscr{O}_X(D(f)) = \mathbb{Z}_f$. Then $\kappa_0 = \mathbb{Q}$, $\kappa_p = \mathbb{Z}_p/(p)\mathbb{Z}_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(iii) For general $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$. Let $\mathfrak{p} \in \operatorname{Spec} A$, $\kappa_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p})$.

Definition. A **morphism of (locally) ringed space** is a pair

$$(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

consisting of a continuous map $f : X \to Y$ and a morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ such that

$$f^\#_x : \mathscr{O}_{Y, f(x)} \to \mathscr{O}_{X, x}$$

is a local homomorphism (i.e. $f^\#_x(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$, or $(f^\#_x)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$) for all $x \in X$.

Remark. $(f, f^\#)$ is an isomorphism if f is a homeomorphism and $f^\#_x$ is an isomorphism for any $x \in X$.

Theorem 1.10. Every ring homomorphism $\varphi : A \to B$ gives a morphism of locally ringed space $(f, f^\#) : (\operatorname{Spec} B, B) \to (\operatorname{Spec} A, A)$ such that

$$\Phi : \operatorname{Hom}(A, B) \to \operatorname{Mor}(\operatorname{Spec} B, \operatorname{Spec} A)$$

is bijective.

Proof. Let $\varphi \in \operatorname{Hom}(A, B)$ define Φ as follow

$$\begin{aligned} \operatorname{Hom}(A, B) &\longrightarrow \operatorname{Mor}(\operatorname{Spec} B, \operatorname{Spec} A) \\ \varphi &\longmapsto (f, f^\#) \end{aligned}$$

$f : \operatorname{Spec} B \to \operatorname{Spec} A$ is defined by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. For $f^\#$, note that a sheaf is determined by an open basis, we can define $f^\#$ just for open basis for $\operatorname{Spec} A$.
Let $a \in A$. We know $f^{-1}(D(a)) = D(\varphi(a))$. Define $f^\#(D(a)) : \mathscr{O}_{\operatorname{Spec} A}(D(a)) \simeq A_a \to f_*\mathscr{O}_{\operatorname{Spec} B}(D(a)) = \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(a))) \simeq B_{\varphi(a)}$ be the map inducing by $\varphi : A \to B$.
We now check that $(f, f^\#)$ is a morphism of ringed space. f is continuous by Proposition 1.2. To check that $f^\#_\mathfrak{p}$ is local homomorphism, observe that the induced stalk map is given by $A_{f(\mathfrak{p})} = A_{\varphi^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. It is a local homomophism clearly.

(i) Φ is injective:
If $\Phi(\varphi_1) = \Phi(\varphi_2)$, then $f^\#_1 = f^\#_2$. Then

$$f^\#_1, f^\#_2 : A \simeq \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \mathscr{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \simeq B$$

are the same map, i.e., $\varphi_1 = \varphi_2$.

(ii) Φ is surjective: Let $(f, f^\#)$ be a morphism from $\operatorname{Spec} B$ to $\operatorname{Spec} A$. Define $\varphi : A \to B$ be the homomorphism $\varphi = f^\#(\operatorname{Spec} A)$. Since $f^\#$ is a morphism of sheaves, the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[f^\#_{\mathfrak{p}}]{} & B_{\mathfrak{p}} \end{array}$$

Since $f^\#_{\mathfrak{p}}$ is a local homomorphism, $(f^\#_{\mathfrak{p}})^{-1}(\mathfrak{p}B_{\mathfrak{p}}) = f(\mathfrak{p})A_{f(\mathfrak{p})}$. By the diagram, we have

$$\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})A_{f(\mathfrak{p})} \cap A = f(\mathfrak{p}),$$

i.e., $\Phi(\varphi) = (f, f^\#)$.

$\square$

Remark. The functor $A \to \operatorname{Spec} A$ is an anti-equivalence between the category of commutative rings and the category of the affine scheme.

1.5 Scheme

Definition.

(i) A **scheme** is a locally ringed space $(X, \mathscr{O}_X)$ admitting an open covering $(U_i)_{i \in I}$ such that $(U_i, \mathscr{O}_X|_{U_i})$ is isomorphic to $(\operatorname{Spec} A_i, \mathscr{O}_{\operatorname{Spec} A_i})$ as a ringed space for some ring A_i for any $i \in I$.

(ii) An element in $\mathscr{O}_X(U)$ is called a **regular function** on U.

(iii) An element in $\mathscr{O}_X(U)^*$ is called an **invertible regular function** on U.

(iv) A scheme which is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A is called an **affine scheme**.

Proposition 1.11. Let $(X, \mathscr{O}_X)$ be a scheme, and $U \subset X$ be an open subset of X. Then $(U, \mathscr{O}_X|_U)$ is a scheme as well.

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of X such that $(U_i, \mathscr{O}_X|_{U_i}) \simeq (\operatorname{Spec} A_i, \mathscr{O}_{\operatorname{Spec} A_i})$. Then $U = \bigcup_{i \in I}(U_i \cap U)$. Write $U_i \cap U = \bigcup_{j \in J(i)} V_{ij}$ with $V_{ij} = D(f_{ij}) \simeq \operatorname{Spec}(A_i)_{f_{ij}}$ for some $f_{ij} \in A_i$. Hence V_{ij} is a desired open cover for U. $\square$

Remark. Let X be an affine scheme, and $U \subset X$ be an open subset of X. Then $(U, \mathscr{O}_X|_U)$ is a scheme but NOT necessarily affine.

Example 1.12. Let k be a field. Let $U = \mathbb{A}^2_k \backslash \{(t_1, t_2)\}$ be an open subset of $\mathbb{A}^2_k$. Write $U = D(t_1) \cup D(t_2)$. The sheaf axiom gives the following exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{A}^2_k}(U) \xrightarrow{\varphi} \mathscr{O}_{\mathbb{A}^2_k}(D(t_1)) \times \mathscr{O}_{\mathbb{A}^2_k}(D(t_2)) \xrightarrow{\psi} \mathscr{O}_{\mathbb{A}^2_k}(D(t_1 t_2))$$

Let $(\frac{h_1}{t_1^{n_1}}, \frac{h_2}{t_2^{n_2}}) \in \mathscr{O}_{\mathbb{A}^2_k}(D(t_1)) \times \mathscr{O}_{\mathbb{A}^2_k}(D(t_2)) = k[t_1, t_2]_{t_1} \times k[t_1, t_2]_{t_2}$. If $(\frac{h_1}{t_1^{n_1}}, \frac{h_2}{t_2^{n_2}}) \in \ker\psi$, then $n_1 = n_2 = 0$ and $h_1 = h_2$. Namely, $\mathscr{O}_{\mathbb{A}^2_k}(U) = k[t_1, t_2]$. If U is affine, then $U = \operatorname{Spec} \mathscr{O}_{\mathbb{A}^2_k}|_U(U) = \mathbb{A}^2_k$, which is a contradiction.

Definition.

(i) A **morphism of schemes** $f : X \to Y$ is a morphism of locally ringed space from $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$.

(ii) Fix a scheme S. An S**-scheme** or **a scheme over** S is a scheme X with a morphism of schemes $\iota : X \to S$. ι is called the **structure morphism** over S. A **morphism between** S**-schemes** $f : X \to Y$ is a morphism of schemes such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota_X"'] & & Y \arrow[dl, "\iota_Y"] \\ & S & \end{tikzcd}$$

Example. Let $(X, \mathscr{O}_X)$ be a scheme, and $x \in X$. There is an affine open subset $U \subset X$ such that $x \in U$. Write $U = \operatorname{Spec} A$, then x can be viewed as a prime ideal $\mathfrak{p}$ in A. Then the stalk of $\mathscr{O}_X$ at x is $\mathscr{O}_{X,x} = (\mathscr{O}_X|_U)_x = A_{\mathfrak{p}} \twoheadrightarrow \kappa(\mathfrak{p})$. We may view a point x as a morphism $(x, \mathscr{O}_{\operatorname{Spec}\kappa(\mathfrak{p})}) \to (U, \mathscr{O}_X|_U) \hookrightarrow (X, \mathscr{O}_X)$

Definition. Let A be a ring, X be an A-scheme (Spec A-scheme), and B be an A-algebra. A B**-point of** X is an element in $\operatorname{Mor}_{\operatorname{Spec} A}(\operatorname{Spec} B, X) = X_A(B)$.

Example.

(i) Let L/k be a field extension, and X be a k-scheme. Then the L-point of X

$$X(L) = \{(p, \iota) \mid p \in X, \iota : \kappa(p) \to L\}.$$

(ii) Let $X = \mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[t]$, and $(t^2 + 1) \in X$. There are two points in $X(\mathbb{C})$ with image $(t^2 + 1)$, i.e. there are two ring homomorphisms from $\mathbb{R}[t]/(t^2 + 1)$ to $\mathbb{C}$.

(iii) Let k be a field, and $k[\varepsilon] := k[T]/(T^2)$ where $\varepsilon := T \mod T^2$. Let X be a k-scheme. Then

$$X(k[\varepsilon]) := \{(x, f) \mid x \in X, f : \mathscr{O}_{X,x} \to k[\varepsilon] \text{ is a local homomorphism}\}.$$

Note that f is a local homomorphism if and only if the following holds

$$\begin{cases} \kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x \xrightarrow{\bar{f}} k \\ \mathfrak{m}_x/\mathfrak{m}_x^2 \to k \text{ is a } k\text{-homomorphism.} \end{cases}$$

(iv) Let k be a field, $\mathbb{G}_m := \operatorname{Spec} k[T, T^{-1}]$, A be a k-algebra. The A-point of $\mathbb{G}_m$ is the invertible element in A, A^*.

1.6 Open immersions and closed immersions

Definition.

(i) Let X be a scheme. An **open subscheme** of X is an open subset $U \subset X$ with the structure sheaf $\mathscr{O}_X|_U$.

(ii) A morphism of schemes $f : X \to Y$ is an **open immersion** if there is an open subscheme $(V, \mathscr{O}_Y|_V)$ of Y such that $f : X \to V$ is an isomorphism as scheme.

Example.

(i) The canonical map $A \to A_f$ induces an open immersion of schemes $\operatorname{Spec} A_f \to \operatorname{Spec} A$.

(ii) Let p be a prime number, $\mathbb{F}_p$ be a finite field of characteristic p, and $X = \mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec}(\mathbb{F}_p[t])$. Define $F : \mathbb{F}_p[t] \to \mathbb{F}_p[t]$ by $t \mapsto t^p$. It induces $\operatorname{Spec} F : X \to X$ is identity on $\operatorname{Spec} \mathbb{F}_p[t]$ but is not an open immersion.

Definition. A morphism of schemes $f : X \to Y$ is a **closed immersion** if it satisfies the following

(i) f maps X homeomorphically onto a closed subset of Y.

(ii) The morphism $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ is surjective.

Namely, f is locally like $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$.

Remark. (ii) can be replaced by $f^\#_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective.

Definition. Let Y be a scheme. A scheme X is a **closed subscheme** of Y if there is a closed immersion $i : X \to Y$. Two closed subscheme X, X' are **identical** if there is an isomorphism $g : X \to X'$ with $i' \circ g = i$.

Example.

(i) $\operatorname{Spec} k[\varepsilon]$, $\operatorname{Spec} k[T]/(T)$ are two non-identical closed subscheme of $\mathbb{A}^1_k$.

(ii) Let $X = \operatorname{Spec} A$. For $V(I)$, there is a reduced structure on $V(I)$ which is given by $\operatorname{Spec}(A/\sqrt{I})$.

Theorem 1.13. Let $X = \operatorname{Spec} A$ and $j : Z \hookrightarrow X$ be a closed immersion. Then Z is affine and is isomorphic to $\operatorname{Spec} A/J$ for some $J \trianglelefteq A$.

Remark. The above ideal J is unique since if $\operatorname{Spec} A/J \simeq \operatorname{Spec} A/J'$, then $A/J \simeq \mathscr{O}_{\operatorname{Spec} A/J}(Z) \simeq \mathscr{O}_{\operatorname{Spec} A/J'}(Z) \simeq A/J'$. It follows that $J \simeq J'$.

Lemma 1.14. Let X be a scheme, $f \in \mathscr{O}_X(X)$. Define $X_f := \{x \in X \mid f_x \in \mathscr{O}^*_{X,x} \text{ i.e. } \overline{f_x} \neq 0 \text{ in } \kappa(x)\}$. Then

(i) X_f is an open subset of X.

(ii)
$$(*)\begin{cases} X = \bigcup_{i\in I} U_i \text{ is a finite affine open cover and} \\ U_i \cap U_j \text{ also admits a finite affine open cover for all } i, j \in I. \end{cases}$$

Suppose X satisfies $(*)$, then $\mathscr{O}_X(X) \to \mathscr{O}_X(X_f)$ induces an isomorphism from $\mathscr{O}_X(X)_f \to \mathscr{O}_X(X_f)$.

Remark. $(*)$ is true if X is affine.

Proof.

(i) Let $x \in X_f$. There is $g_x \in \mathscr{O}_{X,x}^*$ such that $f_x g_x = 1$. Then there is an open subset V_x and $\tilde{g} \in \mathscr{O}_X(V_x)$ such that $\tilde{g}_x = g_x$ and $f|_{V_x}\tilde{g} = 1$. It is easy to see that $V_x \subset X_f$ by $f|_{V_x}\tilde{g} = 1$, and hence $X_f = \bigcup_{x\in X_f} V_x$ is open.

(ii) We first consider the affine case. When $X = \operatorname{Spec} A$ for some ring A, we see that $X_f = D(f)$ and $\mathscr{O}_X(X)_f \simeq A_f \simeq \mathscr{O}_X(X_f)$. So the lemma is true for affine case.
For general case, let $y \in X_f$, there exists an open set V_y of y such that $f|_{V_y} g_y = 1$ in $\mathscr{O}_X(V_y)$ for some $g_y \in \mathscr{O}_X(V_y)$. On $V_{y_1} \cap V_{y_2}$,
$$g_{y_2}|_{V_{y_1}\cap V_{y_2}} = g_{y_2}|_{V_{y_1}\cap V_{y_2}} f|_{V_{y_1}\cap V_{y_2}} g_{y_1}|_{V_{y_1}\cap V_{y_2}} = g_{y_1}|_{V_{y_1}\cap V_{y_2}}.$$
Hence we can glue by sheaf axiom to get an inverse of $f|_{X_f}$ in $\mathscr{O}_X(X_f)$. This means the restriction map $\mathscr{O}_X(X) \to \mathscr{O}_X(X_f)$ can be extended to $\varepsilon : \mathscr{O}_X(X)_f \to \mathscr{O}_X(X_f)$. Now consider the exact sequence
$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \prod_{i\in I} \mathscr{O}_X(U_i) \longrightarrow \prod_{i,j\in I} \mathscr{O}_X(U_i \cap U_j).$$
By taking $\otimes_{\mathscr{O}_X(X)} \mathscr{O}_X(X_f)$, then
$$\begin{tikzcd} 0 \arrow[r] & \mathscr{O}_X(X)_f \arrow[r] \arrow[d, "\varepsilon"] & \prod_{i\in I} \mathscr{O}_X(U_i)_f \arrow[r] \arrow[d, "\psi"] & \prod_{i,j\in I} \mathscr{O}_X(U_i \cap U_j) \arrow[d, "\pi"] \\ 0 \arrow[r] & \mathscr{O}_X(X_f) \arrow[r] & \prod_{i\in I} \mathscr{O}_X(V_i) \arrow[r] & \prod_{i,j\in I} \mathscr{O}_X(V_i \cap V_j) \end{tikzcd}$$
where the first row is exact and $V_i = U_i \cap X_f = D(f|_{U_i})$. Since U_i is affine, $\mathscr{O}_X(U_i)_f \simeq \mathscr{O}_X(V_i)$ for any i, this shows ψ is an isomorphism. It is easy that ε is injective. Replace X by $U_i \cap U_j$, we can get π is injective. By diagram chasing, we can get that ε is surjective, and hence ε is an isomorphism.

□

Proposition 1.15. Let X be a scheme, $\mathscr{O}_X(X) = (f_1, \dots, f_r)$, and X_{f_i} are affine for all i. Then X is affine.

Proof. Since $\mathscr{O}_X(X) = (f_1, \dots, f_r)$, $X = \bigcup_{i=1}^r X_{f_i}$. $X_{f_i} \cap X_{f_j} = D(f_j|_{X_{f_i}}) \subset X_{f_i} = \operatorname{Spec} A_i$ is a principal open subset which is affine. Hence X satisfies $(*)$. By Lemma 1.14 (ii), $\mathscr{O}_X(X)_{f_i} \simeq \mathscr{O}_X(X_{f_i}) = A_i$. Let $A = \mathscr{O}_X(X)$. We have a natural map
$$A \longrightarrow A_{f_i} \xrightarrow[\iota_i]{\sim} A_i$$

From this, we get $u_i : X_{f_i} \to \operatorname{Spec} A$. We now claim that $u_i|_{X_{f_i}\cap X_{f_j}} = u_j|_{X_{f_i}\cap X_{f_j}}$. Note that the isomorphism $A_{f_i} \xrightarrow{\sim} A_i$ comes from the restriction map $\mathscr{O}_X(X) \xrightarrow{\text{res}} \mathscr{O}_X(X_f)$, this gives the following diagram commutes

$$\begin{tikzcd}
A \arrow[r] & A_{f_i} \arrow[r, "\sim"] \arrow[d] & A_i \arrow[d] \\
A \arrow[r] & A_{f_j} \arrow[r, "\sim"] & A_j
\end{tikzcd}$$

which proves the claim. We can define $u : X \to \operatorname{Spec} A$ by gluing $\{u_i : X_{f_i} \xrightarrow{\sim} D(f_i)\}$.
We now check that $u^{-1}(D(f_i)) = X_{f_i}$.

($\supset$) This is clear by construction.

($\subset$) Let $p \in u^{-1}(D(f_i))$, i.e. $u(p) \in D(f_i)$. Hence $u_j(p) = ((\iota_j)^{-1}(p)) \cap A \in D(f_i)$ which implies $p \in X_{f_i}$.

We now see that $\operatorname{Spec} A = \bigcup_{i=1}^r D(f_i)$, and $X = \bigcup_{i=1}^r X_{f_i}$ with $X_{f_i} \simeq D(f_i)$ for any i. Hence $X \simeq \operatorname{Spec} A$ is affine. □

Lemma 1.16. Let $\psi : C \to D$ be a ring homomorphism. If ψ is injective, then the image of $g = \operatorname{Spec} \psi : \operatorname{Spec} D \to \operatorname{Spec} C$ is dense in $\operatorname{Spec} C$. The converse is true if nilradical of C is zero.

Proof.

($\Rightarrow$) It suffices to show for any non-empty principal open subset $D(f)$ of $\operatorname{Spec} C$, we have $g(\operatorname{Spec} D) \cap D(f) \neq \varnothing$.
Recall that nilradical is the intersection of all prime ideals. Given a non-empty principal open subset $D(f) \subset \operatorname{Spec} C$, it follows that $f \notin \sqrt{0}$. By ψ is injective, $\psi(f)$ is not nilpotent in D. Hence, there is $\mathfrak{p} \in \operatorname{Spec} D$ such that $\psi(f) \notin \mathfrak{p}$. We get $f \notin \psi^{-1}(\mathfrak{p}) = g(\mathfrak{p})$.

($\Leftarrow$) Suppose ψ is not injective. By definition, we have $g(\operatorname{Spec} D) \subset V(\ker \psi) \subset \operatorname{Spec} C$. Since $g(\operatorname{Spec} D)$ is dense, we have $V(\ker \psi) = \operatorname{Spec} C$. This implies $\ker \psi \subset \sqrt{0}$ in C. But $\sqrt{0} = 0$ which contradicts that $\ker \psi \neq 0$.

□

Remark. Let $\psi : C \to D$ be an injective ring homomorphism, and $g = \operatorname{Spec} \psi : \operatorname{Spec} D \to \operatorname{Spec} C$. Then for any $c \in C$, $C_c \to D_{\psi(c)}$ is also injective. This will also imply $g^\# : \mathscr{O}_{\operatorname{Spec} C} \to g_*\mathscr{O}_{\operatorname{Spec} D}$ is injective.

Proof of Theorem 1.13. Since Z is a scheme, there is an affine open cover $\{V_i\}$ of Z. Since j maps homeomorphically to a closed subset $j(Z) \subset X$, there is a collection of open subset $\{U_i\}$ in X such that $V_i = j^{-1}(U_i)$. Write $U_i = \bigcup_\ell U_{i\ell}$ where $U_{i\ell} = D(g_{i\ell})$ for some $g_{i\ell} \in A$. Note that $j^\# : \mathscr{O}_X \to j_*\mathscr{O}_Z$, $j^\#(X) : A = \mathscr{O}_X(X) \to \mathscr{O}_Z(Z)$. Define $h_{i\ell} := j^\#(X)(g_{i\ell})$. Now,

$$Z_{h_{i\ell}} = j^{-1}(U_{i\ell}) \subset V_i$$

Since V_i is affine, $D(h_{i\ell}|_{V_i})$ is affine. We now have

$$Z = \bigcup_{i,\ell} D(h_{i\ell}|_{V_i})$$

and

$$D(h_{i_1\ell_1}|_{V_{i_1}}) \cap D(h_{i_2\ell_2}|_{V_{i_2}}) = D(h_{i_1\ell_1}|_{V_{i_1}\cap V_{i_2}} h_{i_2\ell_2}|_{V_{i_1}\cap V_{i_2}})$$

is affine as well. On the other hand, since Z is homeomorphic to a closed subset of $X = \operatorname{Spec} A$, this implies that Z is quasi-compact. Therefore, we may write

$$Z = \bigcup_{i=1}^{n} D(f_i) = \bigcup_{i=1}^{n} Z_{f_i}.$$

There exists $g_j \in A$ such that $j^{-1}(D(g_i)) = D(f_i)$ and $X = \bigcup_i D(g_i)$. Hence $1_A = \sum_{i\in I} a_i g_i$ for some $a_i \in A$. Then

$$1 = j^{\#}(X)\left(\sum a_i g_i\right) = \sum b_i f_i$$

for some $b_i \in \mathscr{O}_Z(Z)$. Then Proposition 1.15 implies that Z is affine, write $Z = \operatorname{Spec} B$. By Theorem 1.10, $j : Z \hookrightarrow X$ defined by a ring homomorphism $\psi : A \to B$. We can factor through the quotient,

$$\begin{tikzcd}
A \arrow[rr, "\psi"] \arrow[dr] & & B \\
& A/\ker\psi \arrow[ur, "\overline{\psi}"'] &
\end{tikzcd}$$

Since $\overline{\psi}$ is injective, $\overline{g}(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A/\ker\psi$.

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "\overline{g}"] \arrow[rr, bend right, "j"'] & \operatorname{Spec} A/\ker\psi \arrow[r, "i"] & \operatorname{Spec} A
\end{tikzcd}$$

Also, j is a closed immersion, we get $\overline{g}(\operatorname{Spec} B) \simeq \operatorname{Spec} A/\ker\psi$ as topological spaces. To finish the proof, we check that their sheaves are isomorphic.

$$\begin{tikzcd}
\mathscr{O}_X \arrow[rr, "j^{\#}"] \arrow[dr] & & j_*\mathscr{O}_Z \\
& i_*\mathscr{O}_{\operatorname{Spec} A/\ker\psi} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Since $j^{\#}$ is surjective, φ is surjective as well. Hence

$$(i_*\mathscr{O}_{\operatorname{Spec} A/\ker\psi})_{\psi^{-1}(\mathfrak{p})} \twoheadrightarrow (j_*\mathscr{O}_Z)_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \operatorname{Spec} B$, i.e.

$$(A/\ker\psi)_{\psi^{-1}(\mathfrak{p})} \twoheadrightarrow B_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \operatorname{Spec} B$. Hence $A/\ker\psi \twoheadrightarrow B$. $\square$

1.7 Proj construction

Lemma 1.17 (Gluing lemma)**.** Let S be a scheme, and $\{X_i\}$ a family of S-schemes. Given X_{ij} open subschemes of X_i and isomorphism of S-schemes f_{ij} such that

$$\begin{cases} f_{ij} : X_{ij} \simeq X_{ji} \\ f_{ii} = \mathrm{id}_{X_{ii}} \\ f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk} \end{cases}$$

and $f_{ik} = f_{jk} \circ f_{ij}$ for all i, j, k on $X_{ij} \cap X_{ik}$. Then there is a S-scheme X unique up to isomorphism with $g_i : X_i \hookrightarrow X$ an open immersions (of S-schemes) such that $g_i = g_j \circ f_{ij}$ on X_{ij} and $X = \bigcup g_i(X_i)$.

Proof. The proof is left as an exercise. □

Let $B = \bigoplus_{d\in\mathbb{Z}} B_d$ be a graded ring. Define $B_+ := \bigoplus_{d>0} B_d$ be an ideal.

Definition. Let B be a graded ring. Define

$$\operatorname{Proj} B = \{\mathfrak{p} \in \operatorname{Spec} B \mid \mathfrak{p}\text{: homogeneous}, \mathfrak{p} \not\supset B_+\}.$$

We define the topology on $\operatorname{Proj} B$ by subspace topology from $\operatorname{Spec} B$, i.e. the closed sets are

$$V_+(I) = \{\mathfrak{p} \in V(I) \mid \mathfrak{p}\text{: homogeneous}, \mathfrak{p} \not\supset B_+\}$$

for all $I \trianglelefteq B$ homogeneous ideal. Also, we define

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} B \mid f \notin \mathfrak{p}\}.$$

$\{D_+(f)\}$ forms an open basis on $\operatorname{Proj} B$.

Lemma 1.18.

(i) If $\mathfrak{p} \in \operatorname{Spec} B$, then $\mathfrak{p}^h := \bigoplus_{d\in\mathbb{Z}} I \cap B_d$ is a prime.

(ii) Let I, J be homogeneous ideal of B. Then $V_+(I) \subset V_+(J)$ if and only if $J \cap B_+ \subset \sqrt{I}$.

(iii) $\operatorname{Proj} B = \varnothing$ if and only if B_+ is nilponent.

Proof.

(i) Exercise.

(ii) ($\Leftarrow$) Let $\mathfrak{p} \in V_+(I)$. Then $\mathfrak{p} \supset \sqrt{I} \supset J \cap B_+ \supset JB_+$. But $\mathfrak{p} \not\supset B_+$, $\mathfrak{p} \supset J$. Hence, $\mathfrak{p} \in V_+(J)$.

($\Rightarrow$) Let $\mathfrak{p} \in V(I)$. Since I is homogeneous, $\mathfrak{p}^h \supset I$. By (i), $\mathfrak{p}^h$ is a prime. Consider the following two cases:

(1) $\mathfrak{p}^h \supset B_+$.

(2) $\mathfrak{p}^h \not\supset B_+$. Then $\mathfrak{p}^h \in V_+(I) \subset V_+(J)$, i.e. $J \subset \mathfrak{p}^h$.

The two cases implies that $J \cap B_+ \subset \mathfrak{p}^h \subset \mathfrak{p}$. Since this is true for all $\mathfrak{p}$, we have

$$J \cap B_+ \subset \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \sqrt{I}.$$

(iii) $\operatorname{Proj} B = V_+(0)$, $\varnothing = V_+(B_+)$. Apply (ii) by taking $I = (0)$ and $J = B_+$.

$\square$

Lemma 1.19. Let $B = \bigoplus_{d \in \mathbb{Z}} B_d$ be a graded ring, and $f \in B_d$ for some $d > 0$.

(i) There exists $u : D_+(f) \to \operatorname{Spec} B_{(f)}$ which is bijective.

(ii) If $g \in B_e$ for some $e > 0$, and $D_+(g) \subset D_+(f)$, then there exists a canonical homomorphism $B_{(f)} \to B_{(g)} \simeq (B_{(f)})_\alpha$ where $\alpha = g^{\deg f}/f^{\deg g}$. (Note that $f \in B_d$, B_f is also a graded ring with the grading $\deg(a/f^k) = \deg a - k \deg f$, and we define $B_{(f)} = (B_f)_0$.)

Proof.

(i) Since we have a natural inclusion $B_{(f)} \to B_f$, we have a canonical morphism $\operatorname{Spec} B_f \to \operatorname{Spec} B_{(f)}$ which sends $\mathfrak{q}$ to $\mathfrak{q} \cap B_{(f)}$. Consider the following diagram

$$\begin{tikzcd} D(f) \arrow[r, "\sim"] & \operatorname{Spec} B_f \arrow[d] \\ D_+(f) \arrow[u, "\text{inclusion}"] \arrow[r, dashed, "u"] & \operatorname{Spec} B_{(f)} \end{tikzcd}$$

We define $u(\mathfrak{p}) = \mathfrak{p}B_f \cap B_{(f)}$. We now check that u is bijective.

- u is surjective: Let $Q \in \operatorname{Spec} B_{(f)}$, then QB_f is a homogeneous ideal in B_f, and hence $\sqrt{QB_f}$ is a homogeneous ideal in B_f. We now claim that $\sqrt{QB_f}$ is a prime ideal in B_f. Indeed, if

$$\frac{a}{f^{e_1}} \cdot \frac{b}{f^{e_2}} \in \sqrt{QB_f}$$

with a, b are homogeneous elements in B. We can even modify $\frac{a}{f^{e_1}}, \frac{b}{f^{e_2}}$ so that $\frac{a}{f^{e_1}}, \frac{b}{f^{e_2}} \in B_{(f)}$. Then

$$\left(\frac{a}{f^{e_1}} \cdot \frac{b}{f^{e_2}}\right)^\ell \in QB_f \cap B_{(f)} = Q$$

$$\Longrightarrow \left(\frac{a}{f^{e_1}}\right)^\ell \text{ or } \left(\frac{b}{f^{e_2}}\right)^\ell \in Q$$

This shows our claim. Now, let $\mathfrak{p} = \sqrt{QB_f} \cap B$. We have $f \notin \mathfrak{p}$ and $\mathfrak{p}$ is homogeneous. Since $\deg f > 0$, we have $B_+ \not\subset \mathfrak{p}$. Then

$$u(\mathfrak{p}) = \left(\sqrt{QB_f} \cap B\right) B_f \cap B_{(f)} = \sqrt{QB_f} \cap B_{(f)} = Q.$$

- u is injective: Let $I \trianglelefteq B$ be a homogeneous ideal. We claim that if $IB_f \cap B_{(f)} \subset u(\mathfrak{p})$ for some $\mathfrak{p} \in D_+(f)$, then $I \subset \mathfrak{p}$ (Note that this claim would imply u is injective). Indeed, let $x \in I$, then

$$\begin{aligned} &\frac{x^{\deg f}}{f^{\deg x}} \in u(\mathfrak{p}) = \mathfrak{p}B_f \cap B_{(f)} \\ \Longrightarrow\ & x^{\deg f} \in \mathfrak{p}B_f \cap B = \mathfrak{p} \\ \Longrightarrow\ & x \in \mathfrak{p}. \end{aligned}$$

(ii) By (i), we know $D_+(f) \to \operatorname{Spec} B_{(f)}$ is bijective. Lemma 1.18 (ii) ($I = (f)$, $J = (g)$, $g \in J \cap B_+$) implies that $g^n = fb$ for some $n > 0$, $b \in B$. Then we have

$$\begin{aligned} B_{(f)} &\longrightarrow B_{(g)} \\ \tfrac{a}{f^r} &\longmapsto \tfrac{ab^r}{g^{nr}}. \end{aligned}$$

And the isomorphism

$$B_{(g)} \simeq (B_{(f)})_\alpha$$

is left as an exercise where $\alpha = g^{\deg f}/f^{\deg g}$.

$\square$

Theorem 1.20. Let $X = \operatorname{Proj} B$, $f \in B_d$ for some $d > 0$, and $u(f) : D_+(f) \to \operatorname{Spec} B_{(f)}$ be the map defined in Lemma 1.19.

(i) Let $g \in B_\ell$ for $\ell > 0$. If $D_+(g) \subset D_+(f)$, then the following diagram commutes

$$\begin{tikzcd} D_+(g) \arrow[r, "u(g)"] \arrow[d, hook] & \operatorname{Spec} B_{(g)} \arrow[d, "i_{f,g}"] \\ D_+(f) \arrow[r, "u(f)"] & \operatorname{Spec} B_{(f)} \end{tikzcd}$$

where $i_{f,g}$ is a homomorphism defined in Lemma 1.19. Hence $u(f)$ is a homeomorphism.

(ii) $\operatorname{Proj} B$ has a scheme structure.

(iii) The stalk of $\operatorname{Proj} B$ at $\mathfrak{p}$ $\mathscr{O}_{\operatorname{Proj} B, \mathfrak{p}} \simeq B_{(\mathfrak{p})} = (B_{\mathfrak{p}})_0$.

Proof.

(i) From the diagram

$$\begin{tikzcd} D(f) \arrow[r, "\sim"] & \operatorname{Spec} B_f \arrow[d] \\ D_+(f) \arrow[u, "\text{inclusion}"] \arrow[r, dashed, "u(f)"] & \operatorname{Spec} B_{(f)}, \end{tikzcd}$$

we see that $u(f)$ is continuous and bijective. The diagram

$$\begin{tikzcd} D_+(g) \arrow[r, "u(g)"] \arrow[d, hook] & \operatorname{Spec} B_{(g)} \arrow[d, "i_{f,g}"] \\ D_+(f) \arrow[r, "u(f)"] & \operatorname{Spec} B_{(f)} \end{tikzcd}$$

commutes comes from Lemma 1.19 (ii). It follows that $u(f)$ is an open map, and hence a homeomorphism.

(ii) Denote $X = \operatorname{Proj} B$. Define $\mathscr{O}_X(D_+(f)) = B_{(f)}$. Lemma 1.19 (ii) gives a canonical homomorphism $B_{(f)} \to B_{(g)}$. Hence $\mathscr{O}_X$ surely define a sheaf on X. Then $\operatorname{Proj} B = \bigcup_{\substack{f \in B_+ \\ \text{homogeneous}}} D_+(f)$ is a scheme.

(iii) Let $\mathfrak{p} \in \operatorname{Proj} B$, then there exists $f \in B_d$ for some $d > 0$ with $\mathfrak{p} \in D_+(f)$. Then

$$\mathscr{O}_{X,\mathfrak{p}} \simeq \mathscr{O}_{\operatorname{Spec} B_{(f)}, u(\mathfrak{p})} = B_{(\mathfrak{p})}.$$

□

Example. We know that $\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$. If $X = \operatorname{Proj} B$, then what is $\mathscr{O}_X(X)$?
For $B = k[x_0, \ldots, x_n]$, let $X = \bigcup_{i=0}^n D_+(x_i)$. Using the following exact sequence

$$0 \longrightarrow \mathscr{O}_X(X) \longrightarrow \prod_{i=0}^n \mathscr{O}_X(D_+(x_i)) \longrightarrow \prod_{\substack{i,j=0 \\ i \neq j}}^n \mathscr{O}_X(D_+(x_i x_j)),$$

we can show that $\mathscr{O}_X(X) = k$.

2 Noetherian schemes and morphisms of finite type

2.1 Noetherian topological spaces

Definition. A topological space X is **Noetherian** if all descending chains of closed subsets become stable (equivalently, all ascending chains of open subsets become stable).

Proposition 2.1. Let $X = \operatorname{Spec} A$. If A is a Noetherian ring, then X is Noetherian as topological space.

Proof. Let $\{V_i\}$ be a descending chain of closed subsets. Then write $V_i = V(I_i)$ where $I_i \trianglelefteq A$ with $I_i = \sqrt{I_i}$. Then $I_0 \subset I_1 \subset \cdots$. Since A is Noetherian, (I_i) is stable, and hence (V_i) is stable. □

Remark. The converse is not true. Let A be the ring of integer in $\overline{\mathbb{Q}_p}$. Then A is not Noetherian but $\operatorname{Spec} A = \{(0), (p)\}$ which is Noetherian.

Proposition 2.2. Let X be a topological space.

(i) Any open or closed subspace of a Noetherian space X is Noetherian.

(ii) X is Noetherian if and only if all open subspaces are quasi-compact.

(iii) Suppose X admits a finite open cover $\{X_i\}_{i\in I}$ with X_i Noetherian, then X is Noetherian.

Proof.

(i) Let U be an open subset of X. Given an ascending chain of open sets in U, $V_0 \subset V_1 \subset \cdots$ in U, then it is also an ascending chain of open sets in X. Since X is Noetherian, this chain would be stable, U is Noetherian.

(ii) $(\Rightarrow)$ Let $\{V_i\}$ be an open covering of X. Consider

$$S = \left\{ \bigcup_{i\in F} V_i \mid F \subset I, |F| < \infty \right\}.$$

Choose an ascending chain in S, since X is Noetherian, that chain would be stable. Hence $X = \bigcup_{i\in F} V_i$ for some $|F| < \infty$. Hence X is quasi-compact. By (i), any open subset of X is quasi-compact.

$(\Leftarrow)$ Let $\{U_i\}$ be an asceding chain of open sets in X. $U = \bigcup_{i=1}^{\infty} U_i$ is quasi-compact, hence $U = \bigcup_{i=1}^{n} U_i$ for some n. So, $U_j = U_n$ for all $j \geqslant n$.

(iii) Let $\{U_i\}$ be an ascending chain of open sets. Then $\{U_i \cap X_j\}$ is an ascending chain of open sets in X_j for all j. Since X_j are all Noetherian. $U_i \cap X_j = U_{n_j} \cap X_j$ for all $i \geqslant n_j$. Hence $U_i = U_n$ for all $i \geqslant n = \max\{n_i\}_{i\in I}$.

□

2.2 Noetherian schemes

Definition.

(i) A scheme X is **locally Noetherian** if there is an affine open covering $\{\operatorname{Spec} A_i\}$ of X such that A_i is Noetherian for all i.

(ii) A scheme X is **Noetherian** if it is locally Noetherian and quasi-compact.

Proposition 2.3. Let X be a scheme, and U be an open subset of X. If X is locally Noetherian (resp. Noetherian), then U is also locally Noetherian (resp. Noetherian).

Proof. Write $X = \bigcup_i U_i$ where $U_i = \operatorname{Spec} A_i$ and A_i is Noetherian. Since $U \cap U_i$ is an open set in U_i, $U \cap U_i = \bigcup_j \operatorname{Spec}(A_i)_{a_{ij}}$. Since A_i is Noetherian, $(A_i)_{a_{ij}}$ is Noetherian. Therefore, $U = \bigcup_i \bigcup_j \operatorname{Spec}(A_i)_{a_{ij}}$ is locally Noetherian.
For the Noetherian case, we know U is already locally Noetherian. Since X is quasi-compact, by Proposition 2.2 (ii), U is quasi-compact as well. Hence U is Noetherian. $\square$

Remark. A Noetherian scheme is a Noetherian topological space.

2.3 Morphisms of finite type

Definition. Let $f : X \to Y$ be a morphism of schemes. The property "P" is a **local property of morphisms** if the following two conditions are equivalent:

(i) f has property "P".

(ii) Y has an affine open cover $\{V_i\}$ such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ has property "P".

Definition. A morphism of schemes $f : X \to Y$ is **quasi-compact** if for all affine open subset $V \subset Y$, $f^{-1}(V)$ is quasi-compact.

Proposition 2.4. Quasi-compact is a local property of morphisms.

Proof. The implication (i)$\Rightarrow$(ii) is trivial. Conversely, let $V = \operatorname{Spec} A \subset Y$ be an affine open subset. Then $V \cap V_i$ is open in V_i, write $V \cap V_i = \bigcup_j V_{ij}$ with V_{ij} being principal open set in V_i, write $V_{ij} = \operatorname{Spec}(A_i)_{a_{ij}}$. Since V is quasi-compact, we can write

$$V = \bigcup_{i=1}^{r} \bigcup_{j=1}^{e_i} V_{ij}.$$

Since $f^{-1}(V_i)$ is quasi-compact, there are finitely many affine open $U_{i\ell} = \operatorname{Spec} B_{i,\ell}$ cover $f^{-1}(V_i)$. Define $U_{ij\ell} = f^{-1}(V_{ij}) \cap U_{i\ell}$, then we have $U_{ij\ell} = \operatorname{Spec}(B_{i,\ell})_{b_{ij\ell}}$ where $b_{ij\ell} = f^{\#}(a_{ij})$

$$B_{i,\ell} \xleftarrow{\quad f^{\#} \quad} A_i$$

$$\begin{tikzcd}
U_{i\ell} \arrow[r, "f|_{U_{i\ell}}"] & V_i = \operatorname{Spec} A_i \\
\operatorname{Spec}(B_{i,\ell})_{b_{ij\ell}} = U_{ij\ell} \arrow[u, "\text{inclusion}"] \arrow[r] & V_{ij} = \operatorname{Spec}(A_i)_{a_{ij}} \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

Now,

$$f^{-1}(V) = \bigcup_{i,j} f^{-1}(V_{ij}) = \bigcup_{i,j,\ell} U_{ij\ell},$$

with each $U_{ij\ell}$ is affine, and hence quasi-compact. Since the union is finite, $f^{-1}(V)$ is quasi-compact.

□

Definition. A morphism of schemes $f : X \to Y$ is said to be **locally of finite type** if for all affine open $V \subset Y$, $\mathscr{O}_X(U)$ is finitely generated $\mathscr{O}_Y(V)$-algebra, for all affine open $U \subset f^{-1}(V)$.

Remark. If $Y = \operatorname{Spec} k$ for a field k, then for any affine open subset $U \subset X$, $\mathscr{O}_X(U)$ is a k-algebra of finite type. That is, there exist an algebra homomorphism $k[x_1, \ldots, x_n] \twoheadrightarrow \mathscr{O}_X(U)$ (n may depends on U). Hence, $\mathscr{O}_X(U) = k[x_1, \ldots, x_n]/I$. We may view U as a affine variety in $\mathbb{A}^n(k)$.

Proposition 2.5. Locally of finite type is a local property.

Lemma 2.6. Let A be a ring, and B be an A-algebra (with a structure homomorphism $A \hookrightarrow B$).

(i) Suppose that $\operatorname{Spec} B = \bigcup_i D(b_i)$ for some $b_i \in B$, and B_{b_i} is finitely generated as A-algebra. Then B is finite generated as A-algebra.

(ii) Suppose that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an open immersion of schemes. Then B is a finitely generated A-algebra.

Proof.

(i) Since $\operatorname{Spec} B$ is quasi-compact, we may assume $\operatorname{Spec} B = \bigcup_{i=1}^n D(b_i)$. Then $B = (b_1, \ldots, b_n)$, i.e. there exists $\beta_i \in B$ such that

$$1 = \sum_{i=1}^n \beta_i b_i.$$

Since B_{b_i} is a finitely generated A-algebra, there are finitely many $a_{ij} \in B$ such that

$$B_{b_i} = A\left[\frac{a_{ij}}{b_i^{e_j}}\right]_j.$$

Define $C := A[a_{ij}, b_i]_{i,j} \subset B$. C is clearly a finite generated A-algebra with $C_{b_i} = B_{b_i}$. Define $D := C[\beta_i]_i \subset B$. We claim that $D = B$ (then B is finitely generated A-algebra clearly). Indeed, for $b \in B$, we have

$$\frac{b}{1} = \frac{c_i}{b_i^{\lambda_i}}$$

in $B_{b_i} = C_{b_i}$ for some $c_i \in C$. There exists M such that $bb_i^M = c_i b_i^M$ for all i since $\{b_i\}$ is finite. Since $1 = \sum_{i=1}^n \beta_i b_i$,

$$1 = \left(\sum_{i=1}^n \beta_i b_i\right)^{nM} = \sum_{i=1}^n \alpha_{i,M} b_i^M$$

for some $\alpha_{i,M} \in D$. Then

$$b = b\left(\sum_{i=1}^n \alpha_{i,M} b_i^M\right) = \sum_{i=1}^n c_i b_i^M \alpha_{i,M} \in D.$$

(ii) Let $i : A \to B$ be the structure homomorphism, then $\operatorname{Spec} i : \operatorname{Spec} B \to \operatorname{Spec} A$. There exists an affine open subset $U \subset \operatorname{Spec} A$ such that

$$(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B}) \xrightarrow[\sim]{\operatorname{Spec} i} (U, \mathscr{O}_{\operatorname{Spec} A}|_U)$$

is an isomorphism. There exists finitely many $a_i \in A$ such that $U = \bigcup_i D(a_i)$ since U is affine open. Then

$$\operatorname{Spec} i : (D(b_i), \mathscr{O}_{\operatorname{Spec} B}|_{D(b_i)}) \xrightarrow{\sim} (D(a_i), \mathscr{O}_{\operatorname{Spec} A}|_{D(a_i)})$$

is an isomorphism where $b_i = i(a_i)$. It follows that B_{b_i} is finitely generated A-algebra. By (i), B is a finitely generated A-algebra.

□

Lemma 2.7.

(i) Let X be an affine scheme, U be a principal open subset of X, and V be a principal open of U. Then V is a principal open of X.

(ii) Let X be a scheme, U, V be affine open subsets of X. Then for any $x \in U \cap V$, there is an open neighborhood $W \subset U \cap V$ such that W is a principal open in both U and V.

Proof.

(i) Write $X = \operatorname{Spec} A$, $U = \operatorname{Spec} A_f$, $V = \operatorname{Spec}(A_f)_{\frac{a}{f^r}} = \operatorname{Spec} A_{fa}$.

(ii) Let $x \in U \cap V$. Since $U \cap V$ is open in U, there is an principal open subset W_1 of U with $x \in W_1 \subset U \cap V$. On the other hand, W_1 is also open in V, there is a principal open subset W_2 of V such that $x \in W_2 \subset W_1$. We now claim that W_2 is a principal open subset in W_1. Write

$W_1 = \operatorname{Spec} A_1$, $V = \operatorname{Spec} B$. We have an inclusion $\iota : W_1 \hookrightarrow V$ which corresponds to $B \xrightarrow{\iota^\#} A_1$. Since W_2 is principal open in V, write $W_2 = \operatorname{Spec} B_b$. Then $\iota^{-1}(W_2) = W_2 = \operatorname{Spec}(A_1)_{\iota^\#(b)}$ which shows that W_2 is principal open in W_1. By (i), W_2 is also principal open in U.

□

Proof of Proposition 2.5. Recall that the statement (ii) is that there is an affine open covering $\{V_i\}$ of Y such that for any $U \subset f^{-1}(V_i)$, $\mathscr{O}_Y(V_i) \to \mathscr{O}_X(U)$ makes $\mathscr{O}_X(U)$ a finitely generated $\mathscr{O}_Y(V_i)$-algebra.
Now, take any affine open subset $V \subset Y$, write $V = \operatorname{Spec} A$. By Lemma 2.7 (ii), we can write $V \cap V_i = \bigcup_j V_{ij}$ with V_{ij} is principal open subset in both V and V_i. Now, write $f^{-1}(V_i) = \bigcup_\ell U_{i\ell}$ with $U_{i\ell}$ affine open. By assumption, $\mathscr{O}_X(U_{i\ell})$ is a finitely generated $\mathscr{O}_Y(V_i)$-algebra. Define

$$U_{ij\ell} = f^{-1}(V_{ij}) \cap U_{i\ell}.$$

Note that $U_{ij\ell}$ is principal open in $U_{i\ell}$, and hence $\mathscr{O}_X(U_{ij\ell})$ is a finitely generated $\mathscr{O}_X(U_{i\ell})$-algebra. Therefore,

$$\begin{aligned}
&\mathscr{O}_X(U_{ij\ell}) \text{ is finitely generated } \mathscr{O}_Y(V_i)\text{-algebra}\\
&\mathscr{O}_X(U_{ij\ell}) \text{ is finitely generated } \mathscr{O}_Y(V_{ij})\text{-algebra}\\
&\mathscr{O}_Y(V_{ij}) \text{ is finitely generated } \mathscr{O}_Y(V)\text{-algebra,}
\end{aligned}$$

where the last one follows from V_{ij} is principal open in V.

$$\begin{tikzcd}
U_{i\ell} \arrow[r] & V_i \\
U_{ij\ell} \arrow[u, hook] \arrow[r, hook] & V_{ij} \arrow[u, hook]
\end{tikzcd}$$

Now let $U = \operatorname{Spec} B \subset f^{-1}(V) = \bigcup_i f^{-1}(V_i) = \bigcup_{i,j,\ell} U_{ij\ell}$ be an affine open subset. For any $x \in U$, there exists $b(x) \in B$ such that $D(b(x))$ is principal open in both U and $U_{ij\ell}$. It follows that $\mathscr{O}_X(D(b(x)))$ is finitely generated $\mathscr{O}_Y(V)$-algebra since $\mathscr{O}_X(D(b(x)))$ is finitely generated $\mathscr{O}_X(U_{ij\ell})$-algebra.
For U is affine, U is quasi-compact. Hence, there exist finitely many x_i such that $U = \bigcup_{i=1}^r D(b(x_i))$. Then Lemma 2.6 (i) implies that $\mathscr{O}_X(U)$ is finitely generated $\mathscr{O}_Y(V)$-algebra. □

Definition. Let $f : X \to Y$ be a morphism of schemes. f is **of finite type** if f is locally of finite type and quasi-compact.

Corollary 2.8. A morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is of finite type if and only if B is finitely generated A-algebra.

Remark. Let $U \subset \operatorname{Spec} A$ be an open subset. $i : U \to \operatorname{Spec} A$ is locally of finite type but not of finite type in general. Consider $A = k[x_i]_{i \in \mathbb{Z}_{>0}}$ where k is a field. Let $\mathfrak{m} = (x_i | i \in \mathbb{Z}_{>0})$, and $U = \operatorname{Spec} A \backslash \mathfrak{m} = \bigcup_{i \in \mathbb{Z}_{>0}} D(x_i)$.

Proposition 2.9.

(i) If $f : X \to Y$, $g : Y \to Z$ are morphisms of finite type (resp. locally of finite type/quasi-compact), then $g \circ f$ is of finite type (resp. locally of finite type/quasi-compact).

(ii) If the composition of morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{g} Z$ is locally of finite type, then f is locally of finite type.

Proof.

(i) Let f, g be locally of finite type. Let $W \subset Z$ be affine open subset. Write

$$g^{-1}(W) = \bigcup_i V_i, \quad f^{-1}(V_i) = \bigcup_j U_{ij}$$

where V_i, U_{ij} are affine open. Then

$$U_{ij} \xrightarrow{f} V_i \to W$$

and

$$\begin{aligned} &\mathscr{O}_Y(V_i) \text{ is finitely generated } \mathscr{O}_Z(W)\text{-algebra} \\ &\mathscr{O}_X(U_{ij}) \text{ is finitely generated } \mathscr{O}_Y(V_i)\text{-algebra} \\ \Longrightarrow\ &\mathscr{O}_X(U_{ij}) \text{ is finitely generated } \mathscr{O}_Z(W)\text{-algebra.} \end{aligned}$$

By Proposition 2.5 and covering Z by such W will do the job.
For f, g are quasi-compact, let

$$\begin{aligned} &\{W_i\} \text{ be an affine open cover of } Z \\ &\{V_{ij}\} \text{ be an affine open cover of } g^{-1}(W_i) \\ &\{U_{ij\ell}\} \text{ be an affine open cover of } f^{-1}(V_{ij}) \end{aligned}$$

Since g is quasi-compact, it only needs finitely many i, j. Since f is quasi-compact, it only needs finitely many ℓ for each i, j. Then

$$(g \circ f)^{-1}(W_i) = \bigcup_{j,\ell} U_{ij\ell}$$

with the union is finite, and hence $g \circ f$ is quasi-compact.

(ii) Keep the notation as above. Write $U_{ij\ell} = \operatorname{Spec} C_{ij\ell}$, $V_{ij} = \operatorname{Spec} B_{ij}$, $W_i = \operatorname{Spec} A_i$.

$$U_{ij\ell} \longrightarrow V_{ij} \longrightarrow W_i$$

$$C_{ij\ell} \longleftarrow B_{ij} \longleftarrow A_i$$

Since $C_{ij\ell}$ is finitely generated A_i-algebra, $C_{ij\ell}$ is finitely generated B_{ij}-algebra.

□

Proposition 2.10. Let X be a Noetherian scheme. Then an open immersion $f : U \to X$ is of finite type.

Proof. Recall that if $g \in A$, A_g is an A-algebra of finite type. Therefore, f is locally of finite type. Since X is a Noetherian scheme, X is a Noetherian topological space, and U is as well by Proposition 2.2 (ii). Hence, U is quasi-compact, and f is quasi-compact. □

Proposition 2.11. Let S be a scheme, and X, Y be S-schemes. If X is a Noetherian scheme and is of finite type over S, then for any morphism of S-scheme $f : X \to Y$ is of finite type.

Proof. By Proposition 2.9 (ii), f is locally of finite type. Now, for any affine open subset $V \subset Y$, $f^{-1}(V) \subset X$ is open in X. Since X is Noetherian, $f^{-1}(V)$ is quasi-compact, and hence f is quasi-compact. □

Theorem 2.12. $X = \operatorname{Spec} A$ is a Noetherian scheme if and only if A is a Noetherian ring.

Proof. It suffices to prove ($\Rightarrow$). Write $\operatorname{Spec} A = X = \bigcup_{i=1}^{r} V_i$ with $V_i = \operatorname{Spec} A_i$, A_i is a Noetherian ring. Find $V_{ij} \subset V_i$ which are principal open both in X and V_i. Since X is quasi-compact, there exists finitely many V_{ij} such that

$$X = \bigcup_{i,j} V_{ij},\ V_{ij} = \operatorname{Spec}(A_i)_{g_{ij}} = \operatorname{Spec} A_{f_{ij}}$$

for some $g_{ij} \in A_i$, $f_{ij} \in A$. Since A_i is Noetherian, $(A_i)_{g_{ij}}$ is Noetherian, and $A_{f_{ij}}$ is Noetherian. Now, $X = \bigcup_{i,j} V_{ij}$ implies that $A = (f_{ij})_{i,j}$. Let $\varphi_{ij} : A \to A_{f_{ij}}$ be the natrual map. We claim that for any $I \trianglelefteq A$, we have

$$I = \bigcap_{i,j} \varphi^{-1}(\varphi_{ij}(I)A_{f_{ij}}).$$

($\subset$) This is clear.

($\supset$) Let $a \in \bigcap_{i,j} \varphi^{-1}(\varphi_{ij}(I)A_{f_{ij}})$. Then for any i, j,

$$\frac{a}{1} = \varphi_{ij}(a) = \frac{b_{ij}}{f_{ij}^{e_{ij}}}$$

for some $b_{ij} \in I$. Then $af_{ij}^{M} = b_{ij}f_{ij}^{M-e_{ij}}$. By using standard argument, we can see $a \in I$.

With this claim, we can conclude that A is Noetherian. □

Remark. Let $f : X \to Y$ be locally of finite type. This means there is an affine open cover $\{V_i\}$ of Y such that $f|_{f^{-1}(V_i)} : f^{-1}(V) \to V_i$ is locally of finite type, i.e. for all affine open subset $U \subset f^{-1}(V_i)$, $\mathscr{O}_X(U)$ is $\mathscr{O}_Y(V_i)$-algebra of finite type. Actually, this is equivalent to there exists an affine open cover $\{U_{ij}\}$ of $f^{-1}(V_i)$ such that $\mathscr{O}_X(U_{ij})$ is a $\mathscr{O}_Y(V_i)$-algebra of finite type.

Definition.

1. A morphism of schemes $f : X \to Y$ is said to be **affine** if for any affine open subset $V \subset Y$, $f^{-1}(V)$ is affine.

2. Let f be an affine morphism. f is said to be **finite** if $\mathscr{O}_X(f^{-1}(V))$ is a finitely generated $\mathscr{O}_Y(V)$-module for all affine open subset $V \subset Y$.

Example. Let L/k be a finite field extension. Then $\operatorname{Spec} L \to \operatorname{Spec} k$ is finite.

Proposition 2.13. Let $f : X \to Y$, $g : Y \to Z$ be morphisms of schemes. If f, g are affine (resp. finite), then $g \circ f$ is affine (resp. finite).

Proposition 2.14. Let $f : X \to Y$ be a morphism of schemes. Suppose Y has an affine open cover $\{V_i\}$ such that $f^{-1}(V_i)$ are affine for all i. Then

(i) f is affine (Affine is a local property).

(ii) Assume further that $\mathscr{O}_X(f^{-1}(V_i))$ is a finitely generated $\mathscr{O}_Y(V_i)$-module for all i, then f is finite (Finite is a local property).

Proof.

(i) Let $V = \operatorname{Spec} A \subset Y$ be an affine open subset. Then $V \cap V_i$ can be covered by V_{ij} which are all principal open in both V and V_i. Since V is quasi-compact, there exists finitely many i, j such that $V = \bigcup_{i,j} V_{ij}$. Write

$$V_{ij} = \operatorname{Spec}(A_i)_{a_{ij}} = \operatorname{Spec} A_{\alpha_{ij}}$$

for some $a_{ij} \in A_i$, $\alpha_{ij} \in A$. Then we have $f^{-1}(V) = \bigcup_{i,j} f^{-1}(V_{ij})$, and

$$\begin{tikzcd} f^{-1}(V_i) = \operatorname{Spec} B_i \arrow[r, "f|_{f^{-1}(V_i)}"] & V_i = \operatorname{Spec} A_i \\ f^{-1}(V_{ij}) = \operatorname{Spec}(B_i)_{b_{ij}} \arrow[u, hook] & V_{ij} = \operatorname{Spec}(A_i)_{a_{ij}} \arrow[u, hook] \end{tikzcd}$$

where $b_{ij} = f^{\#}(V_i)(a_{ij})$. Since $(\alpha_{ij})_{i,j} = A$, $(f^{\#}(V)(\alpha_{ij}))_{i,j} = \mathscr{O}_X(f^{-1}(V))$. Also $f^{-1}(V_{ij})$ is affine, then Proposition 1.15 would imply $f^{-1}(V)$ is affine.

(ii) Let $V = \operatorname{Spec} A \subset Y$ be an affine open subset. (i) implies that $f^{-1}(V) = U = \operatorname{Spec} B \subset X$ is affine. Since V_{ij} are principal in both V and V_i, $f^{-1}(V_{ij})$ are principal in both U and U_i. Write

$$\mathscr{O}_X(f^{-1}(V_{ij})) = B_{\beta_{ij}} = (B_i)_{b_{ij}}$$

for some $\beta_{ij} \in B$, $b_{ij} \in B_i$. By assumption, $(B_i)_{b_{ij}}$ is a finitely generated $(A_i)_{a_{ij}}$-module, write

$$(B_i)_{b_{ij}} = \sum_{\ell=1}^{\lambda(ij)} (A_i)_{a_{ij}} c_{ij\ell}.$$

Define C be an A-submodule of B generated by β_{ij}, $c_{ij\ell}$. Then for all $b \in B$, we have

$$\frac{b}{1} = \frac{c}{(b_{ij}^{e_{ij}})},$$

by standard argument, we will get $b \in C$. Hence $B = C$ is finite generated A-module.

□

2.4 Integral scheme

Definition. Let X be a scheme. X is said to be **reduced** if for any $x \in X$, the local ring $\mathscr{O}_{X,x}$ is a reduced ring, i.e. nilpotent radical is trivial.

Remark. It depends only on local.

Proposition 2.15. A scheme X is reduced if and only if for any open set $U \subset X$, $\mathscr{O}_X(U)$ is a reduced ring.

Proof.

($\Leftarrow$) For any $x \in X$, there exists an affine open neighborhood $U = \operatorname{Spec} A$. Then

$$\mathscr{O}_{X,x} = (\mathscr{O}_X|_U)_x = A_{\mathfrak{p}}$$

where x corresponding to a prime ideal $\mathfrak{p} \trianglelefteq A$. One can check that if A is reduced, then $A_{\mathfrak{p}}$ is reduced as well.

($\Rightarrow$) Let $f \in \mathscr{O}_X(U)$ with $f^n = 0$. Then $(f_x)^n = 0$ for all $x \in U$. By assumption, $f_x = 0$ for all $x \in U$, and hence $f = 0$ in $\mathscr{O}_X(U)$.

□

Example. A classical non-reduced ring is $k[t]/(t^2)$.

Remark.

(i) Let $X = \operatorname{Spec} A$, $N = \sqrt{0} \trianglelefteq A$. Then $\iota : \operatorname{Spec} A/N \hookrightarrow \operatorname{Spec} A$ is a closed immersion with $\operatorname{Spec}(A/N)$ reduced, denoted by A_{red}. Also, ι is an isomorphism as topological spaces.

(ii) In general, let X be a scheme. There exists X_{red} such that $\iota : X_{\text{red}} \to X$ is a closed immersion and an isomorphism as topological spaces.

Definition.

$$\mathscr{N}(U) := \{s \in \mathscr{O}_X(U) \mid s_x \text{ is nilpotent in } \mathscr{O}_{X,x} \text{ for all } x\}.$$

$\mathscr{O}_{X_{\text{red}}}$ is defined to be the quotient sheaf $\mathscr{O}_X/\mathscr{N}$. Then $(X_{\text{red}}, \mathscr{O}_{X_{\text{red}}})$ is the **reduced scheme** of X.

Definition.

(i) A topological space X is **irreducible** if whenever $X = X_1 \cup X_2$ with X_1, X_2 closed, then $X = X_1$ or $X = X_2$.

(ii) A maximal irreducible subspace of X is called an **irreducible component**.

Definition. A scheme X is **integral** if it is irreducible and reduced.

Proposition 2.16. Let $X = \operatorname{Spec} A$.

(i) A nonempty closed subset $Y = V(I) \subset X$ is irreducible if and only if $\sqrt{I}$ is a prime ideal. (X is irreducible if and only if $\sqrt{0}$ is prime.)

(ii) A is an integral domain if and only if X is integral.

(iii) The irreducible component of X are $V(\mathfrak{p}_i)$ for minimal prime $\mathfrak{p}_i \trianglelefteq A$.

Proof.

(i) ($\Leftarrow$) Since $\sqrt{I}$ is a prime, $V(I)$ is nonempty. Write $V(I) = V(J_1) \cup V(J_2) = V(J_1 J_2)$. Then $J_1 J_2 \subset \sqrt{I}$. Since $\sqrt{I}$ is a prime, we have $J_1 \subset \sqrt{I}$ or $J_2 \subset \sqrt{I}$. This means that $V(I) \subset V(J_1)$ or $V(I) \subset V(J_2)$.

($\Rightarrow$) $V(I)$ is nonempty, so $I \subsetneq A$. Recall that $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. Let $a, b \in A$ with $ab \in \sqrt{I}$. We have $V(I) \subset V(a) \cup V(b)$. Then

$$V(I) = (V(a) \cap V(I)) \cup (V(b) \cap V(I)).$$

Since $V(I)$ is irreducible, we have

$$V(a) \cap V(I) = V(I), \text{ or } V(b) \cap V(I) = V(I).$$

Therefore, $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

(ii) ($\Rightarrow$) If A is an integral domain, X is irreducible (by (i)) and reduced.

($\Leftarrow$) If X is integral, then X is reduced, and $\mathscr{O}_X(X) = A$ is reduced. Also, X is irreducible, (i) implies that $\sqrt{0}$ is a prime. Since A is reduced, $\sqrt{0} = 0$ is a prime, and hence A is an integral domain.

(iii) Let $V(I)$ be an irreducible component of X. By (i), $\sqrt{I}$ is a prime. We know $V(I) = V(\sqrt{I})$. If $V(I)$ is a maximal irreducible subset, then $\sqrt{I}$ is minimal.

$\square$

Remark. We see that $\operatorname{Spec} A$ is irredeucible if and only if the nilradical is the unique minimal prime ideal in A.

Proposition 2.17. Let X be an irreducible scheme. Then the topological space X has a unique point η such that $\overline{\{\eta\}} = X$. η is called the **generic point** of X. Assume further X is integral, then $\mathscr{O}_{X,\eta}$ is a field which is called the **function field** of X, and $\mathscr{O}_{X,\eta} = \mathrm{Frac}(\mathscr{O}_X(U))$ for any affine open $U \subset X$.

Proof. We first consider the case that $X = \operatorname{Spec} A$ is affine. Then Proposition 2.16 (i) shows that $\sqrt{0}$ is the unique minimal prime in A. Then $\overline{\sqrt{0}} = A$, i.e. $\sqrt{0}$ is the generic point.
For the general case, let U be an affine open subset of X. Since X is irreducible, U is irreducible as well. The affine case implies that there exists $\eta \in U$ such that $\overline{\{\eta\}} \cap U = U$. One can check that for any affine open subset V, we have $\eta \in V$. Hence $\overline{\{\eta\}} = X$.
For the last statement, for any affine open $V = \operatorname{Spec} A$. We have

$$\mathscr{O}_{X,\eta} = (\mathscr{O}_X|_V)_\eta.$$

Since X is reduced, $\mathscr{O}_X(V)$ is reduced. Since X is irreducible, V is irreducible. Hence $\mathscr{O}_X(V)$ is an integral domain by Proposition 2.16 (ii). Hence $\eta = (0) \trianglelefteq A$. Then

$$\mathscr{O}_{X,\eta} = (\mathscr{O}_X|_V)_\eta = \mathrm{Frac}\, A = \mathrm{Frac}(\mathscr{O}_X(V)).$$

$\square$

Example. Let $X = \operatorname{Spec} \mathbb{Z}$, then $\eta = (0)$, $\mathscr{O}_{X,\eta} = \mathbb{Q}$.

3 Dimension

3.1 Dimension of a ring and a scheme

Definition.

(i) Let A be a ring and $\mathfrak{p} \in \operatorname{Spec} A$. The **height** of $\mathfrak{p}$, denoted by $\operatorname{ht}(\mathfrak{p})$, is the supremum of the length of the strictly ascending chains of prime ideals contained in $\mathfrak{p}$

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}.$$

(ii) The **Krull dimension** of A is $\dim A = \sup_{\mathfrak{p} \in \operatorname{Spec} A} \operatorname{ht}(\mathfrak{p})$. For $A = \{0\}$, we define $\dim A = -\infty$.

Example.

(i) If k is a field, $\dim k = 0$.

(ii) Let $A = k[t]/(t^2)$, $\dim A = 0$.

(iii) $\dim \mathbb{Z} = 1$.

Do we have A is Noetherian if and only if $\dim A < \infty$?

The answer is no on both side. We can construct a non-Noetherian ring with finite Krull dimension. Let A be a ring of integer in $\overline{\mathbb{Q}_p}$ over $\mathbb{Z}_p$. Then A is not Noetherian, but $\operatorname{Spec} A = \{0, \mathfrak{p}\}$ where $\mathfrak{p} \cap \mathbb{Z}_p = p\mathbb{Z}_p$.

For an infinite Krull dimensional Noetherian ring, the famous example is Nagata's example. Let k be a field, and $A = k[x_i]_{i \in \mathbb{N}}$. Define

$$\mathfrak{p}_1 = (x_1),\ \mathfrak{p}_2 = (x_2, x_3), \ldots,\ \mathfrak{p}_m = (x_{\frac{m(m-1)}{2}+1}, \ldots, x_{\frac{m(m-1)}{2}+m}), \ldots$$

Let $\mathfrak{p} = \bigcup_{i=1}^{\infty} \mathfrak{p}_i$, and $S = A \backslash \mathfrak{p}$. One can see that $\dim S^{-1}A = \infty$. We now claim that $S^{-1}A$ is Noetherian. Indeed, let $I \trianglelefteq S^{-1}A$ be a proper ideal, and $i : A \to S^{-1}A$ be a natural map. $I \trianglelefteq S^{-1}A$ is proper if and only if $i^{-1}(I) \cap S = \varnothing$. Namely, $i^{-1}(I) \subset \bigcup_i \mathfrak{p}_i$. Then there exists a finite set F such that

$$i^{-1}(I) \subset \bigcup_{i \in F} \mathfrak{p}_i.$$

By prime avoidance, we have $i^{-1}(I) \subset \mathfrak{p}_i$ for some i. Hence $S^{-1}A$ is Noetherian.

Another question is do we have $\dim A = \dim(A/\mathfrak{p}) + \operatorname{ht}(\mathfrak{p})$? By definition, we have $\dim A \geqslant \dim A/\mathfrak{p} + \operatorname{ht}(\mathfrak{p})$.

Consider $A = R[x]$ where R is a discrete valuation ring, and $\mathfrak{p} = (px - 1)$. Then $A/\mathfrak{p}$ is a field and $\operatorname{ht}(\mathfrak{p}) = 1$ by Krull's principal ideal theorem, but $\dim A = 2$.

Fact.

I. (Going up) Let $f : A \to B$ be an injective ring homomorphism. Suppose B is integral over A, then

   (i) $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

   (ii) $\dim B = \dim A$.

II. (Krull's principal ideal theorem) Let A be a Noetherian ring, and $f \in A$ is not a unit. Then any prime ideal $\mathfrak{p}$ that is minimal among those containing f has at most height one.

III. (Normalization lemma) Let A be a finitely generated algebra over a field k.

   (i) $\dim k[x_1, \dots, x_n] = n$, and $\dim R[x_1, \dots, x_n] = \dim R + n$ for Noetherian ring R.

   (ii) There exists $y_1, \dots, y_r \in A$ which is algebraic independent over k such that A is integral over $k[y_1, \dots, y_r]$ (Then Fact I would imply $\dim A = r$).

   (iii) Assume further that A is an integral domain. Then $\dim A = \operatorname{trdeg} \operatorname{Frac} A$. For $\mathfrak{p} \in \operatorname{Spec} A$, $\dim A = \operatorname{ht}(\mathfrak{p}) + \dim A/\mathfrak{p}$.

   Remark. Let A be a finitely generated integral domain over k, and $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{p}) = 1$, then $\dim A/\mathfrak{p} = \dim A - 1$.

Definition. Let X be a topological space. The **dimension** of X is

$$\dim X := \sup\{n \mid Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n,\ Y_i \text{ are irreducible closed subset of } X\}.$$

Proposition 3.1. Let $X = \operatorname{Spec} A$. Then $\dim X = \dim A$.

Proof. The irreducible closed subsets of $\operatorname{Spec} A$ take the form $V(\mathfrak{p})$ with $\mathfrak{p}$ prime by Proposition 2.16. Also,

$$V(\mathfrak{p}_1) \subsetneq V(\mathfrak{p}_2) \Leftrightarrow \mathfrak{p}_1 \supsetneq \mathfrak{p}_2.$$

□

Proposition 3.2. Let X be a topological space. Then

(i) Let Y be a subspace of X with subspace topology, then $\dim Y \leqslant \dim X$.

(ii) Suppose that X is irreducible and of finite dimensional. Then for any closed subset $Y \subset X$ with $\dim Y = \dim X$, we have $Y = X$.

(iii) The dimension of X is equal to the maximal of the dimension of the irreducible components of X.

(iv) Let $\{U_i\}$ be an open cover of X. Then $\dim X = \sup_i\{\dim U_i\}$.

Proof.

(i) Let $Y_1 \subsetneq Y_2$ be irreducible closed subset in Y. Let X_i be the closure of Y_i in X. Since Y_i is irreducible in Y, we have X_i is irreducible in X. We claim that $X_1 \subsetneq X_2$ (then any chain in Y can be lifted to a chain in X). Indeed, $Y_i = Z_i \cap Y$ for some Z_i closed in X. Then we have

$$Y_i \subset \overline{Y_i} \cap Y \subset Z_i \cap Y = Y_i.$$

Hence $X_i \cap Y = Y_i$.

(ii) Let $Y_0 \subsetneq \cdots \subsetneq Y_r$ be a chain of irreducible closed subset of Y with $r = \dim Y$. Since Y is closed, it is also a chain in X. By using $\dim Y = \dim X$, one can see that $Y_r = X$. Then $X \subset Y$.

(iii) For any chain $Y_0 \subsetneq \cdots \subsetneq Y_r$, there exists an irreducible component containing Y_r.

(iv) (i) implies that $\dim U_i \leqslant \dim X$. So we have $\dim X \geqslant \sup\{\dim U_i\}$. Let $Y_0 \subsetneq \cdots \subsetneq Y_r$ be a chain of irreducible closed subset in X. For a fixed $x_0 \in Y_0$, choose i such that $x_0 \in U_i$. Then

$$\varnothing \neq Y_0 \cap U_i \subset \cdots \subset Y_r \cap U_i.$$

We now claim that each containing is strict, i.e. $Y_j \cap U_i \subsetneq Y_{j+1} \cap U_i$ for all j. Suppose not that we have $Y_j \cap U_i = Y_{j+1} \cap U_i$, then $Y_{j+1} \backslash Y_j \subset X \backslash U_i$. This implies that

$$Y_{j+1} = Y_j \cup \left[(X \backslash U_i) \cap Y_{j+1}\right].$$

But Y_{j+1} is irreducible and $Y_j \subsetneq Y_{j+1}$, we have $Y_{j+1} \subset X \backslash U_i$. However, $x_0 \in Y_{j+1} \cap U_i$ which is a contradiction.
On the other hand, $Y_j \cap U_i$ is clearly irreducible in U_i, we conclude that $\dim X = \sup \dim U_i$.

□

Example.

(i) If X is an irreducible scheme of dimension 0, then X consists of one point.

Proof. For any point $x \in X$, it is clear that $\{x\}$ is irreducible, and hence $\overline{\{x\}}$ is an irreducible closed subset of X. If $\overline{\{x\}} \neq X$, then we have a chain $\overline{\{x\}} \subsetneq X$, which shows that $\dim X \geqslant 1$. Therefore, we have x is a generic point for any $x \in X$. But the generic point is unique, X consists of one point. □

(ii) Let X be a Noetherian scheme of dimension 0. Write $X = \bigcup_{i=1}^{r} Y_i$ with Y_i irreducible component of X. By Proposition 3.2 (i), we have $\dim Y_i = 0$. Equip Y_i with reduced structure of closed subscheme. Then Y_i consists of a point. Therefore, $X = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ with each point $\mathfrak{p}_i$ is a closed point and open. Also, $X = \operatorname{Spec} \prod \mathscr{O}_{X,\mathfrak{p}_i}$.

(iii) Fact III implies that $\dim \mathbb{A}_k^n = n$.

(iv) $\dim X = \dim X_{\mathrm{red}}$ where X_{red} is the reduced structure of X. For affine case, it is just $\dim A/N = \dim A$ where N is the nilradical of A.

Definition. Let X be a topological space, Y be an irreducible closed subset of X. The **codimension** of Y in X is defined as

$$\operatorname{codim}(Y, X) := \sup\{n \mid Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n,\ Y_i \text{ irreducible in } X\}.$$

When $X = \operatorname{Spec} A$, and $Y = V(\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{Spec} A$. Then $\operatorname{codim}(Y, X) = \operatorname{ht}(\mathfrak{p})$. Also, from definition, we can see that $\dim Y + \operatorname{codim}(Y, X) \leqslant \dim X$. The equality does NOT hold in general even for affine case see Example.

3.2 Dimension of schemes of finite type over a field k

Theorem 3.3. Let X be an integral scheme of finite type over k, and K be the function field of X.

(i) X is of finite dimension and $\dim X = \operatorname{trdeg}_k K$.

(ii) All nonempty open $U \subset X$, we have $\dim U = \dim X$.

(iii) Let $x \in X$ be a closed point. Then $\dim X = \dim \mathscr{O}_{X,x}$.

Proof.

(i) Since X is integral, there is a unique generic point η such that $\mathscr{O}_{X,\eta} = K = \mathscr{O}_{U,\eta}$ for any nonempty open set U. Let $\{U_i\}$ be an affine open cover of X, from Proposition 3.2 (iv), $\dim X = \sup \dim U_i$. Therefore, it suffices to show (i) for any $U_i = \operatorname{Spec} B_i$ where B_i is a finitely generated k-algebra. From Fact III, we know $\dim U_i = \operatorname{trdeg} \mathscr{O}_{U_i,\eta}$. Hence $\dim X = \operatorname{trdeg} \mathscr{O}_{U,\eta} = \operatorname{trdeg} K$.

(ii) Note that (i) implies (ii) by $\mathscr{O}_{U,\eta} = K$ for any nonempty open subset U.

(iii) It suffices to prove for affine case $X = \operatorname{Spec} A$. From Fact III (iii), $\dim A = \dim A/\mathfrak{p} + \operatorname{ht}(\mathfrak{p}) = \dim \mathscr{O}_{X,\mathfrak{p}}$ for maximal $\mathfrak{p}$.

□

Corollary 3.4. Let X be a scheme of finite type over k. Then

(i) $\dim X < \infty$.

(ii) If X is irreducible, then $\dim X = \dim U$ for any nonempty open subset U.

Proof. Since X is of finite type over k, X is Noetherian. Write $X = \bigcup_{i=1}^{r} Y_i$ for Y_i irreducible component of X. We know $\dim X = \sup \dim Y_i$ from Proposition 3.2 (iii). Equip Y_i with reduced structure. Then Y_i is an integral scheme of finite type over k. Then Theorem 3.3 (i), $\dim Y_i < \infty$, and hence $\dim X < \infty$. For the second statement, replace X by X_{red}, we have X_{red} is an integral scheme of finite type over k. □

Definition. A Noetherian scheme X is **equidimensional** if all irreducible components of X have the same dimension.

Proposition 3.5. Let X be a scheme of finite type over k. Then

(i) For a nonempty open set $U \subset X$, $\dim U = \dim X$ if U is open dense or X is equidimensional.

(ii) If X is equidimensional, then for any irreducible closed subset $Y \subset X$, we have

$$\dim Y + \operatorname{codim}(Y, X) = \dim X.$$

Proof.

(i) Since X is finite type over k, X is Noetherian. Write $X = \bigcup_{i=1}^{r} Y_i$ with Y_i distinct irreducible component. Equip Y_i with the reduced structure.

- X is equidimensinoal: There exists an i such that $U \cap Y_i \neq \varnothing$ where Y_i is an integral scheme. From Proposition 3.2 (i) and the assumption that X is equidimensional, we have

$$\dim X = \dim Y_i = \dim U \cap Y_i \leqslant \dim U \leqslant \dim X.$$

  Hence, all inequalities are equal.
- U is open dense: We have $U \cap Y_i \neq \varnothing$ for all i, and $\dim U \cap Y_i = \dim Y_i$.

(ii) Let Y be an irreducible close subset of X. There exists a Y_i containing Y. The assumption gives $\dim Y_i = \dim X$. Let U be an affine open subset with $U \cap Y \neq \varnothing$. Then $U \cap Y_i \neq \varnothing$, and $\dim U \cap Y_i = \dim Y_i$. Let

$$Y = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r$$

be a chain of irreducible closed subset. Then

$$\varnothing \neq Y \cap U = Z_0 \cap U \subsetneq Z_1 \cap U \subsetneq \cdots \subsetneq Z_r \cap U$$

is a chain of irreducible closed in U. Also, let

$$F_0 = Y \cap U \subsetneq F_1 \subsetneq \cdots \subsetneq F_r$$

be a chain of irreducible closed subset in U, then

$$Y = \overline{Y \cap U} \subsetneq \overline{F_1} \subsetneq \cdots \subsetneq \overline{F_r}$$

is a chain in X where the closure is taken in X. We have the bijection

$$\begin{array}{c}\{\text{irreducible closed subsets in } X \text{ containing } Y\} \\ \big\downarrow{\scriptstyle u} \\ \{\text{irreducible closed subsets in } U \text{ containing } Y \cap U\}.\end{array}$$

We can reduce to the affine case, and replace X by X_{red}. Using Fact III, we have

$$\dim A = \dim A/\mathfrak{p} + \mathrm{ht}(\mathfrak{p}).$$

In other words,

$$\dim X = \dim Y + \operatorname{codim}(Y, X).$$

$\square$

Theorem 3.6. Let X be a scheme of finite type over k, and

$$E := \{x \in X \mid \{x\} \text{ is closed}\}$$

be a set consisting of closed points of X. Then E is dense in X.

Remark. This theorem is not true for algebraic variety over a finite field $\mathbb{F}_p$.

4 Fiber products and relative dimensions

Let S be a scheme, and X, Y be S-schemes, i.e. there exist structure morphisms $\iota_X : X \to S$, $\iota_Y : Y \to S$.

4.1 Fiber products and base change

We are looking for an object (called **fiber product**) W, which is an S-scheme, and morphisms of S-schemes $p : W \to X$, $q : W \to Y$ satisfying the following universal property:

For any S-scheme Z, and S-morphisms $f : Z \to X$, $g : Z \to Y$, there exists unique S-morphism $h : Z \to W$ such that $f = p \circ h$, $g = q \circ h$. In other words, any commutative diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& & Y \arrow[d, "\iota_Y"] \\
& X \arrow[r, "\iota_X"'] & S
\end{tikzcd}$$

can factor through W,

$$\begin{tikzcd}
Z \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] \arrow[dr, "h"] & & \\
& W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\iota_Y"] \\
& X \arrow[r, "\iota_X"'] & S
\end{tikzcd}$$

Theorem 4.1. The fiber product W (denoted by $X \times_S Y$) exists and is unique up to isomorphism. (Here isomorphism means if W and W' are both fiber product, then W is isomorphic to W' as X-scheme and Y-scheme.) If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$, then $X \times_S Y = \operatorname{Spec}(A \otimes_C B)$.

Proof.

(1) X, Y, S are affine.

Let Z be an S-scheme with S-morphisms f, g such that the following diagram commutes

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "\iota_Y"] \\
X \arrow[r, "\iota_X"'] & S
\end{tikzcd}$$

If Z is affine, then there exists unique h by the universal property of tensor product.

For general Z, let $\{Z_i\}$ be an affine open cover of Z. By the affine case, for any Z_i, there exists unique

$$h_i : Z_i \to \operatorname{Spec}(A \otimes_C B).$$

On $Z_i \cap Z_j$, let $\{V_{ij}\}$ be an affine open subset in $Z_i \cap Z_j$. There exists unique

$$u_{ij} : V_{ij} \to \operatorname{Spec}(A \otimes_C B).$$

But $h_i|_{V_{ij}}, h_j|_{V_{ij}}$ also satisfy the equations

$$f|_{V_{ij}} = p \circ (h_i|_{V_{ij}}), \quad g|_{V_{ij}} = q \circ (h_i|_{V_{ij}})$$
$$f|_{V_{ij}} = p \circ (h_j|_{V_{ij}}), \quad g|_{V_{ij}} = q \circ (h_j|_{V_{ij}}).$$

Hence, $h_i|_{V_{ij}}, h_j|_{V_{ij}}, u_{ij}$ are all equal by the uniqueness of u_{ij}.
Therefore, we can glue h_i to get

$$h : Z \to \operatorname{Spec}(A \otimes_C B)$$

and h is unique since h_i are all unique.

(2) Suppose that $X \times_S Y$ exists, then for any open subset $U \subset X$, $U \times_S Y$ exists (check that $p^{-1}(U)$ satisfies the universal property).

(3) Suppose that Y, S are affine.
Let $\{X_i\}$ be an affine open cover of X. (1) implies that $X_i \times_S Y$ exists for all i.

$$\begin{tikzcd} X_i \times_S Y \arrow[r, "q_i"] \arrow[d, "p_i"'] & Y \arrow[d] \\ X_i \arrow[r] & S \end{tikzcd}$$

By (2), we have $p_i^{-1}(X_i \cap X_j) \simeq (X_i \cap X_j) \times_S Y \simeq p_j^{-1}(X_i \cap X_j)$. By the universal property of fiber products, there exists unique

$$\alpha_{ij} : p_i^{-1}(X_i \cap X_j) \to p_j^{-1}(X_i \cap X_j).$$

On $X_i \cap X_j \cap X_\ell$, we have

$$\alpha_{i\ell} : p_i^{-1}(X_i \cap X_j \cap X_\ell) \xrightarrow{\sim} p_\ell^{-1}(X_i \cap X_j \cap X_\ell), \quad \alpha_{j\ell} \circ \alpha_{ij} = \alpha_{i\ell},$$

which are also follow from the uniqueness of $\alpha_{i\ell}$. Clearly, we have $\alpha_{ii} = \mathrm{id}_{p_i^{-1}(X_i)}$.
By gluing, we can get an object $X \times_S Y$ satisfying the universal property of fiber product.
Note that we can use similar argument to generalize Y to be a scheme (not necessarily affine).

(4) For S to be general scheme.
Let $\{S_i\}$ be an affine open cover of S. By (3), there exists $(X_i \times_{S_i} Y_i, p_i, q_i)$.

$$\begin{tikzcd} X_i \times_{S_i} Y_i \arrow[r] \arrow[d] & Y_i \arrow[d] \arrow[ddr, bend left] & \\ X_i \arrow[r] \arrow[drr, bend right] & S_i \arrow[dr, hook] & \\ & & S \end{tikzcd}$$

By the above diagram, one can see $X_i \times_{S_i} Y_i = X_i \times_S Y_i$, i.e. $X_i \times_{S_i} Y_i$ satisfies the universal property of X_i, Y_i over S.
By (2), $((X_i \cap X_j) \times_S Y_i, p_i, q_i)$ exists. Apply (2) again, $((X_i \cap X_j) \times_S (Y_i \cap Y_j), p_i, q_i)$ exists. Also, by universal property again, we have

$$((X_i \cap X_j) \times_S (Y_i \cap Y_j), p_i, q_i) \simeq ((X_i \cap X_j) \times_S (Y_i \cap Y_j), p_j, q_j).$$

So we can glue these local data to a scheme $X \times_S Y$.

□

Remark.

(i) If $S = \operatorname{Spec} C$, we will write $X \times_{\operatorname{Spec} C} Y = X \times_C Y$. Furthermore, if $Y = \operatorname{Spec} B$, we will write X_B.

(ii) The fiber product of schemes is not equal to the fiber product of sets, i.e.

$$X \times_S Y \neq \{(x, y) \mid x \in X, y \in Y, \iota_X(x) = \iota_Y(y)\}.$$

For instance, consider $X = Y = \mathbb{A}^1_k$. Then $X \times_k Y = \operatorname{Spec} k[x, y]$, and the prime ideal $(x^2 + y)$ do not lie in the RHS.

(iii) The universal property of fiber products can be formulated as

$$(X \times_S Y)(Z) = X_S(Z) \times_S Y_S(Z)$$

for all S-scheme Z.

(iv) In the above, we have always assumed the ring $0 \neq 1$. But Theorem 4.1 only makes sure that the existence of fiber product, it does not means that the fiber product of schemes is always nonempty. A clear example is that the spectrum of $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$.

Proposition 4.2. Let S be a scheme and X, Y be S-schemes.

(i) $X \times_S S = X$, and $X \times_S Y = Y \times_S X$.

(ii) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$ for an S-scheme Z.

(iii) $(X \times_S Y) \times_Y Z = X \times_S Z$ for a Y-scheme Z.

(iv) Let $U \subset X$ be an open subset. Then $p^{-1}(U)$ is the fiber product $U \times_S Y$ where $p : X \times_S Y \to X$ is the projection.

(v) Let $U \subset S$ be an open subset such that $\iota_X(X) \subset U$. Then $X \times_S Y = X \times_U Y_U$ where $Y_U = \iota_Y^{-1}(U)$.

Proof. All follow from the universal property of fiber product. For example, let us sketch the proof of (ii), consider the following commutative 3D diagram

$$\begin{tikzcd}
(X \times_S Y) \times_S Z \arrow[dd, red] \arrow[dr, blue] \arrow[drrr, red] & & Y \times_S Z \arrow[dd, dashed] \arrow[dr] & \\
& X \times_S (Y \times_S Z) \arrow[ur, red] \arrow[dd, red] & & Z \arrow[dd] \\
X \times_S Y \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dr] & \\
& X \arrow[rr] & & S
\end{tikzcd}$$

The red lines and black lines commute, so it will induce the blue line. Conversely, drawing similar picture would get the inverse of the blue line.
(Remark: This graph is not planar: by considering the two sets of vetices $\{X \times_S Y, X \times_S (Y \times_S Z), Z\}$ and $\{(X \times_S Y) \times_S Z, Y \times_S Z, S\}$, one can see that this graph contains a subdivision of a complete bipartite graph $K_{3,3}$.) □

Example.

(i) $X = \operatorname{Spec}(k[x_1, \dots, x_n]/(p_1, \dots, p_m))$ where k is a field. Let L/k be a field extension. Then

$$X_L = X \times_k L = \operatorname{Spec}(L[x_1, \dots, x_n]/(p_1, \dots, p_m))$$

where we view p_i as in $L[x_1, \dots, x_n]$.

(ii) Let X be a scheme. X can be regarded as a $\mathbb{Z}$-scheme. Then we have $X \times_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, and $X_{\mathbb{Q}} = X \times_{\mathbb{Z}} \mathbb{Q}$. $X_{\mathbb{Q}}$ is called a **generic fiber**.

Definition. Let $f : X \to Y$ be a morphism of schemes, $y \in Y$, and $\kappa(y)$ be its residue field. Define

$$X_y := X \times_Y \operatorname{Spec} \kappa(y).$$

This is called the **fiber of** f **over** y.

Proposition 4.3. The projection $p : X_y = X \times_Y \operatorname{Spec} \kappa(y) \to X$ induces a homeomorphism from X_y to $f^{-1}(y) \subset X$.

Proof. Choose an affine open neighborhood $V \subset Y$ of y. By Proposition 4.2 (iii),

$$X_y = (X \times_Y V) \times_V \operatorname{Spec} \kappa(\mathfrak{p}) = f^{-1}(V) \times_V \operatorname{Spec} \kappa(\mathfrak{p}).$$

We may assume Y is affine. For $U \subset X$ affine open subset, $p : p^{-1}(U) \to U$. We can also reduce to the case that X is affine.

$$f : \operatorname{Spec} B \to \operatorname{Spec} A, \quad X_y = \operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$$

We have the following commutative diagram

$$\begin{tikzcd}
B \arrow[rr, "p^{\#}(X)"] \arrow[dr, "\alpha_1"'] & & B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \\
& B \otimes_A A_{\mathfrak{p}} \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

where

$$\alpha_1 : \operatorname{Spec}(B \otimes_A A_{\mathfrak{p}}) = \operatorname{Spec}(S^{-1}B) \xrightarrow{\sim} \{\mathfrak{p} \in \operatorname{Spec} B \mid \mathfrak{p} \cap S = \varnothing\}$$

and

$$\alpha_2 : \operatorname{Spec}(B \otimes A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \xrightarrow{\sim} V(f(\mathfrak{p})S^{-1}B) \subset \operatorname{Spec}(S^{-1}B)$$

are homeomorphisms. Hence p is a homeomorphism from $\operatorname{Spec}(B \otimes A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ to

$$T := \{\mathfrak{q} \in \operatorname{Spec} B \mid \mathfrak{q} \cap S = \varnothing \text{ and } \mathfrak{q} \supset f^{\#}(Y)(\mathfrak{p})\}.$$

Note that

$$\begin{aligned}
&\mathfrak{q} \in T \\
\iff &(f^{\#}(Y))^{-1}(\mathfrak{q}) = f(\mathfrak{q}) = \mathfrak{p} \\
\iff &\mathfrak{q} \in f^{-1}(\mathfrak{p}).
\end{aligned}$$

□

Definition.

(i) Let S be a scheme, X, S' be S-schemes. We have the projection (the S'-structure)

$$P_{S'} : X_{S'} = X \times_S S' \to S'.$$

This process is called the **base change**. If $f : X \to Y$ an S-morphism,

$$f_{S'} = f \times \mathrm{id}_{S'} : X \times_S S' \to Y \times_S S'.$$

(ii) A morphism $f : X \to Y$ with property (P) is said to be **stable under base change** if

$$p_{Y'} : X \times_Y Y' \to Y'$$

has property (P) for all Y-scheme Y'.

$$\begin{tikzcd}
X \times_S S' \arrow[dr, "f_{S'}"] \arrow[dd] \arrow[drr, bend left] & & \\
& Y \times_S S' \arrow[r] \arrow[dd] & S' \arrow[dd] \\
X \arrow[dr, "f"'] \arrow[drr, dashed, bend left, "\iota_X"] & & \\
& Y \arrow[r, "\iota_Y"'] & S
\end{tikzcd}$$

Proposition 4.4. The following properties are stable under base change.

(i) Morphisms: locally of finite type, quasi-compact, affine, finite.

(ii) open immersion.

(iii) closed immersion.

Proof.

(1) Let Y' be a Y-scheme with the structure map $g : Y' \to Y$, $\{Y_i = \operatorname{Spec} C_i\}$ be an affine open cover of Y, $\{V_{ij} = \operatorname{Spec} A_{ij}\}$ be an affine open cover of $g^{-1}(Y_i)$, and $\{U_{ij} = \operatorname{Spec} B_{ij}\}$ be an affine open cover of $f^{-1}(Y_i)$.

$$\begin{array}{ccc} U_{i\ell} \times_{Y_i} V_{ij} & \xrightarrow{f'} & V_{ij} \\ \downarrow & & \downarrow g \\ U_{i\ell} & \xrightarrow[f]{} & Y_i \end{array}$$

Since f is locally of finite type, B_{ij} is a finitely generated C_i-algebra.
Also, $\{U_{i\ell} \times_{Y_i} V_{ij} = \operatorname{Spec}(B_{i\ell} \otimes_{C_i} A_{ij})\}$ covers $X \times_Y Y'$, and $B_{i\ell} \otimes_{C_i} A_{ij}$ is a finitely generated A_{ij}-algebra. Therefore, f' is locally of finite type.

(2) Suppose that f is quasi-compact. $f^{-1}(Y_i)$ is covered by finitely many $\{U_{ij}\}$. Hence, $(f')^{-1}(V_{i\ell})$ is covered by finitely many $\{U_{ij} \times_{Y_i} V_{i\ell}\}$, i.e. f' is quasi-compact.

(3) If f is affine, $f^{-1}(Y_i)$ is affine. Then

$$(f')^{-1}(V_{ij}) = f^{-1}(Y_i) \times_{Y_i} V_{ij}$$

is affine. Hence f' is affine.

(4) The case that f is finite is similar as locally of finite type.

(ii), (iii) are left as an exercise. □

Definition. Let k be a field, and X be a k-scheme. X is **geometrically integral (resp. reduced)** if $X \times_k \overline{k}$ is integral (resp. reduced).

Example. Let $X = \operatorname{Spec} \mathbb{C}$. We view X as an $\mathbb{R}$-scheme. Then X is not geometrically integral since $\operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ is not irreducible.

Remark. $X \times_k \overline{k}$ is never empty unless X is empty.

Proposition 4.5. Let k be a perfect field, X be an integral k-scheme, and K be the function field of X. Then the following are equivalent:

(i) X is geometrically integral.

(ii) $K \otimes_k \overline{k}$ is a field.

(iii) k is algebraic closed in K.

(iv) For all finite field extensions L/k, $X \times_k L$ is integral.

Proof. (ii) and (iii) are equivalent is an exercise (algebra). For the rest, use the following lemma (Lemma 4.6). □

Lemma 4.6. Let L/k be an algebraic extension, and X be an integral k-scheme. Then $X \times_k L$ is integral if and only if $K \otimes_k L$ is a field.

Proof.

($\Leftarrow$) Let $U = \operatorname{Spec} A$ be an affine open subset of X. Then $K = \mathscr{O}_{X,\eta} = \mathscr{O}_{U,\eta} = \operatorname{Frac} A$. We have the canonical map

$$A \otimes_k L \hookrightarrow (\operatorname{Frac} A) \otimes_k L,$$

then $A \otimes_k L$ is an integral domain (since $K \otimes L$ is a field). That is, U_L is an integral scheme. Now, X_L can be covered by $\{U_{i,L}\}$ with each $U_{i,L}$ is integral. This implies that X_L is reduced.
On the other hand, since X is integral, $U_i \cap U_j \neq \varnothing$. Hence, $U_{i,L} \cap U_{j,L} = (U_i \cap U_j)_L \subset X_L$ is nonempty. So, X_L is irreducible.

($\Rightarrow$) Since $X \times_k L$ is integral, $\operatorname{Spec}(A \otimes_k L) = U \times_k L \subset X \times_k L$ is integral. Hence $A \otimes_k L$ is an integral domain, $(\operatorname{Frac} A) \otimes_k L$ is an integral domain as well. Since $\operatorname{Frac} A \hookrightarrow (\operatorname{Frac} A) \otimes_k L$ is integral, $(\operatorname{Frac} A) \otimes_k L$ is a field.

□

4.2 Relative dimension

Let $f : X \to Y$ be a surjective morphism of scheme. Do we have $\dim X \geqslant \dim Y$?

Example. Let $Y = \operatorname{Spec} k[[t]] = \{(0), (t)\}$, and $X = \operatorname{Spec}(k((t)) \times k) = \{\mathfrak{p}_1 = (k((t)), 0), \mathfrak{p}_2 = (0, k)\}$. Consider the ring homomorphism

$$k[[t]] \xrightarrow{f^\#} k((t)) \times k$$

$$\varphi \longmapsto (\varphi, \varphi(0))$$

Then $f(\mathfrak{p}_1) = (f^\#)^{-1}(\mathfrak{p}_1) = (t)$, $f(\mathfrak{p}_2) = (0)$. We see that f is surjective, but $\dim X = 0$, $\dim Y = 1$.

Theorem 4.7. Let $f : X \to Y$ be a finite, surjective morphism of schemes. Then $\dim X = \dim Y$.

Proof. Let $\{Y_i\}$ be an affine open cover of Y with $Y_i = \operatorname{Spec} A_i$. Since f is finite, $f^{-1}(Y_i) = \operatorname{Spec} B_i$ is affine, and B_i is a finitely generated A_i-module. Therefore, B_i is integral over A_i with the structure map

$$A_i \xrightarrow{f_i^\#} B_i$$

where $f_i = f|_{f^{-1}(Y_i)}$. Since f is surjective, f_i is surjective. Now, we claim that $f_i^\#$ is injective. Suppsoe $N_i = \ker f_i^\# \neq 0$. Then

$$A_i \to A_i/N_i \to B_i \implies \operatorname{Spec} B_i \to V(N_i) \subset \operatorname{Spec} A_i.$$

f_i is surjective, hence $V(N_i) = \operatorname{Spec} A_i$, i.e. $N_i \subset \sqrt{0}$. We can reduce by a reduced structure. By going up, we have $\dim A_i = \dim B_i$. Then Proposition 3.2 (iv) and Proposition 3.1 gives

$$\dim X = \sup \dim B_i = \sup \dim A_i = \dim Y.$$

□

Definition. Let $f : X \to Y$ be a morphism of schemes. f is said to be **dominant** if $f(X)$ is dense in Y.

In particular, if X, Y are irreducible, then f is dominant if and only if $f(X)$ contains the generic point of Y.

Proof.

- ($\Leftarrow$) This is trivial.
- ($\Rightarrow$) We may assume X, Y are affine. Write $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Replace by a reduced structure, we have X, Y are integral. Let $N = \ker f^\#$ where $f^\# : A \to B$. Then

$$f(X) \subset V(N) \subset Y.$$

Since $f(X)$ is dense, we have $V(N) = Y$ which implies that $N = 0$.

□

Actually, the proof shows that f is dominant if and only if f sends the generic point of X to the generic point of Y.

Theorem 4.8. Let k be a field. Let $f : X \to Y$ be a dominant k-morphism where X, Y are integral scheme of finite type over k. Then

(i) The generic fiber X_η (η is the generic point of Y) is integral with function field K (K is the function field of X).

(ii) $\dim X_\eta = \dim X - \dim Y$.

Proof.

(i) Let $V = \operatorname{Spec} A \subset Y$, $U = \operatorname{Spec} B \subset f^{-1}(V)$ be affine open subsets. Note that A, B are integral domain. Since f is dominant, $f|_U : U \to V$ is dominant, and $f^\# : A \to B$ is injective.

$$\begin{tikzcd} X \times_Y \operatorname{Spec} \kappa(\eta) \arrow[r] \arrow[d, "p"'] & \operatorname{Spec} \kappa(\eta) \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that $p^{-1}(U) = \operatorname{Spec}(B \otimes_B \operatorname{Frac} A)$ which is homeomorphic to $(f|_U)^{-1}(\eta) = f^{-1}(\eta) \cap U$. To show X_η is integral, it suffices to show two things

(1) $p^{-1}(U)$ is reduced (since reduced is local property): $p^{-1}(U) = \operatorname{Spec}(B \otimes_A \operatorname{Frac} A)$. We claim that $B \otimes_A \operatorname{Frac} A$ is an integral domain. Indeed, $f^\# : A \to B$ is injective,

$$B \otimes_A \operatorname{Frac} A = \varinjlim_{\theta \in f^\#(A \setminus 0)} B\left[\frac{1}{\theta}\right]$$

is an integral domain. Hence $p^{-1}(U)$ is an integral scheme. It follows that X_η is reduced.

(2) $X_\eta \simeq f^{-1}(\eta)$ is irreducible: We claim that $p^{-1}(U)$ is dense in X_η (i.e. $f^{-1}(\eta) \cap U$ dense in $f^{-1}(\eta)$). But this follows from X is integral. Hence, X_η is irreducible.

Now,

$$B \subset \operatorname{Frac}(B \otimes_A \operatorname{Frac} A) \subset \operatorname{Frac} B,$$

we have $\operatorname{Frac} B = \operatorname{Frac}(B \otimes_A \operatorname{Frac} A)$.

(ii) X_η is a scheme over $\kappa(\eta) = \operatorname{Frac} A$. $X_\eta \to \operatorname{Spec} \kappa(\eta)$ is of finite type as X is of finite type over k. Then by Normalization lemma,

$$\begin{aligned} \dim X_\eta &= \operatorname{trdeg}_{\kappa(\eta)} K, \\ \dim X &= \operatorname{trdeg}_k K, \\ \dim Y &= \operatorname{trdeg}_k \kappa(\eta). \end{aligned}$$

Then it is clear that $\dim X_\eta = \dim X - \dim Y$.

$\square$

5 Global properties of morphisms

5.1 Separated morphisms

Let X be a topological space, $\Delta : X \to X \times X$ defined by $x \mapsto (x, x)$. Endow $X \times X$ with product topology. It is well-known that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

Definition.

(i) Let $f : X \to Y$ be a morphism of schemes. Define $\Delta_{X/Y} : X \to X \times_Y X$ by the induced map given by the universal property of fiber product

$$\begin{tikzcd} X \arrow[dr, "\Delta_{X/Y}"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Y X \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\Delta_{X/Y}$ is called **diagonal morphism**.

(ii) X is **separated over** Y is $\Delta_{X/Y}$ is a closed immersion.

(iii) X is **separated** if X is separated over $\operatorname{Spec} \mathbb{Z}$.

Remark. $x \in X \times_Y X$ and $p(x) = q(x)$, then it would not imply $x \in \operatorname{Im}(\Delta_{X/Y})$. For example, $Y = \operatorname{Spec} \mathbb{R}$, $X = \operatorname{Spec} \mathbb{C}$, $X \times_Y X$ has two points.

Proposition 5.1. Any morphism between affine schemes is separated.

Proof. Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $f : X \to Y$ be a morphism of schemes. Then

$$\begin{tikzcd} X \times_Y X & B \otimes_A B \arrow[d] & a \otimes b \arrow[d, mapsto] \\ X \arrow[u, "\Delta_{X/Y}"] & B & ab \end{tikzcd}$$

It is clear that $B \otimes_A B \to B$ is surjective, $\Delta_{X/Y}$ is a closed immersion. $\square$

Definition. Let k be a field. An **(algebraic) k-variety** is a separated scheme of finite type over k.

Proposition 5.1 tells that if $X = \operatorname{Spec} A$ with A being finitely generated k-algebra, then X is a k-variety.

Proposition 5.2. Let $f : X \to Y$ be a morphism of schemes. If $\operatorname{Im}(\Delta_{X/Y})$ is closed in $X \times_Y X$, then f is separated.

Proposition 5.3. To be separated is a local property of morphisms.

Proof. Let $\{X_i \times_Y X_i\}$ be an affine open cover of $X \times_Y X$. $f|_{X_i}$ is separated for any i, hence f is separated. $\square$

Proposition 5.4 (Separated Criterion)**.** Let $S = \operatorname{Spec} C$, X be an S-scheme. The following are equivalent

(i) X is separated over S.

(ii) For any affine open subsets U, V of X, $U \cap V$ is affine and the canonical homomorphism

$$\varphi_{UV} : \mathscr{O}_X(U) \otimes_C \mathscr{O}_X(V) \to \mathscr{O}_X(U \cap V)$$

$(s, t) \mapsto s|_{U\cap V} \cdot t|_{U\cap V}$ is surjective.

(iii) There is an affine open cover $\{U_i\}$ of X such that for any i, j, $U_i \cap U_j$ is affine and the homomorphism

$$\varphi_{ij} : \mathscr{O}_X(U_i) \otimes_C \mathscr{O}_X(U_j) \to \mathscr{O}_X(U_i \cap U_j)$$

is surjective.

Proof.

(ii)⇒(iii) This is trivial.

(i)⇒(ii) Write $\Delta = \Delta_{X/S}$.

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
U \cap V \arrow[dr, "h"] \arrow[drr, bend left, "\mathrm{id}_X|_{U\cap V}"] \arrow[ddr, bend right, "\mathrm{id}_X|_{U\cap V}"'] & & \\
& U \times_S V \arrow[r] \arrow[d] & V \arrow[d] \\
& U \arrow[r] & S
\end{tikzcd}$$

By the universal property, $\Delta|_{U\cap V} = h$. Since X is separated, Δ is a closed immersion, and $\Delta|_{U\cap V}$ is a closed immersion. Then

$$\Delta : U \cap V \to U \times_S V$$

is a closed subscheme of an affine scheme which is also affine, and

$$\Delta^{\#} : \mathscr{O}_X(U) \otimes_C \mathscr{O}_X(V) \twoheadrightarrow \mathscr{O}_X(U \cap V)$$

is surjective.

(iii)⇒(i) $\{U_i \times_S U_j\}$ is an affine open cover of $X \times_S X$. (iii) implies that

$$\Delta : U_i \cap U_j \to U_i \times_S U_j$$

is a closed immersion. Hence

$$\Delta : X \to X \times_S X$$

is a closed immersion.

□

Example.

(i) Let $\mathbb{P}^n_A := \operatorname{Proj} A[T_0, \dots, T_n]$, and $D_+(T_i)$ be the affine open subset. We have $D_+(T_i) \cap D_+(T_j) = D_+(T_iT_j)$. One can see that $\{D_+(T_i)\}$ satisfies Proposition 5.4 (iii), hence $\mathbb{P}^n_A$ is separated over A.

(ii) Let $X_1 = X_2 = \mathbb{A}^1_k$, and $U_1 = U_2 = \mathbb{A}^1_k \backslash \{\text{origin}\}$. Glue X_1, X_2 together along U_1, U_2 with the identity map, called this scheme Y. Then

$$X_1 \hookrightarrow Y, \quad X_2 \hookrightarrow Y$$

are open immersions. Then $X_1 \cap X_2$ in Y is equal $\mathbb{A}^1_k \backslash \{\text{origin}\}$. We have $X_1 \cap X_2 = \operatorname{Spec} k[t, 1/t]$, but

$$\mathscr{O}_Y(X_1) \otimes_k \mathscr{O}_Y(X_2) \to k[t, 1/t]$$

is not surjective. Hence Y is not separated over k.

Proposition 5.5.

(i) Closed immersions and open immersions are separated.

(ii) The composition of two separated morphisms is separated.

(iii) The separated morphisms are stable under base change.

(iv) Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of schemes. If $g \circ f$ is separated, then f is separated. In particular, any k-morphism between algebraic varieties are separated.

Proof.

(i) Let $U \hookrightarrow X$ be an open immersion. Then $U \times_X U = U$, and the diagonal morphism is actually $\mathrm{id}_U : U \to U \simeq U \times_X U$, which is clearly a closed immersion. For closed immersion, it is the same reason as open immersion.

(ii) Let $f : X \to Y$, $g : Y \to Z$ be two separated morphisms.

$$\begin{tikzcd}
X \arrow[dr, "\Delta_{X/Y}"] \arrow[rrrd, bend left, "\mathrm{id}_X"] \arrow[ddrr, phantom] \arrow[dddr, bend right, "\mathrm{id}_X"'] & & & & \\
& X \times_Y X \arrow[rr, "p_2"] \arrow[dd, "p_1"'] \arrow[dr, "h"] & & X \arrow[dd, "f"] \arrow[dddr, bend left] & \\
& & X \times_Z X \arrow[ur] \arrow[dl] & & \\
& X \arrow[rr, "f"'] \arrow[drrr, bend right] & & Y \arrow[dr, "g"] & \\
& & & & Z
\end{tikzcd}$$

From the above diagram, we see that $\Delta_{X/Z} = h \circ \Delta_{X/Y}$. Since f is separated, $\Delta_{X/Y}$ is a closed immersion. We now claim that h is a closed immersion. To see this, we consider

$$\mathrm{id}_X \times \Delta_{Y/Z} \times \mathrm{id}_X : X \times_Y X \times_Y Y \to X \times_Y (Y \times_Z Y) \times_Y X.$$

It suffices to show $h = \mathrm{id}_X \times \Delta_{Y/Z} \times \mathrm{id}_X$ (Since $\Delta_{Y/Z}$ is a closed immersion, and closed immersion is stable under base change, h is a closed immersion). Now, for any Y-scheme V, we have

$$\begin{tikzcd}
(X \times_Y X)_Y(V) \arrow[d, "\wr"] & \\
X_Y(V) \times X_Y(V) \arrow[d, "\wr"] & (\alpha, \beta) \arrow[d, mapsto] \\
(X_Y(V) \times Y_Y(V)) \times X_Y(V) \arrow[d, "\mathrm{id}_X \times \Delta_{Y/Z} \times \mathrm{id}_X"'] & (\alpha, \iota_V, \beta) \arrow[d, mapsto] \\
X_Y(V) \times Y_Z(V) \times Y_Z(V) \times X_Y(V) \arrow[d, "\wr"] & (\alpha, \iota_V, \iota_V, \beta) \arrow[d, mapsto] \\
X_Z(V) \times X_Z(V) & (\alpha, \beta)
\end{tikzcd}$$

Also, $h(\alpha, \beta) = (\alpha, \beta)$. By Yoneda lemma (only need to check any V-point), we have $h = \mathrm{id}_X \times \Delta_{Y/Z} \times \mathrm{id}_X$.

(iii) Since closed immersion is stable under base change, separated morphism is stable under base change.

(iv) $\Delta_{X/Z} = h \circ \Delta_{X/Y}$ as (ii). Since $g \circ f$ is separated, $\Delta_{X/Z}(X)$ is closed, and $h^{-1}(\Delta_{X/Z}(X))$ is closed. It is clear that $\Delta_{X/Y}(X) \subset h^{-1}(\Delta_{X/Z}(X))$. If we show the converse inclusion, then $\Delta_{X/Y}$ is a closed immersion. Indeed, let $s \in h^{-1}(\Delta_{X/Z}(X))$. Then

$$h(s) = \Delta_{X/Z}(a) = h(\Delta_{X/Y}(a))$$

for some $a \in X$. We want to show $s = \Delta_{X/Y}(a)$. Actually, we can reduce to affine case (since $h(s) = h(\Delta_{X/Y}(a))$, choose affine open neighborhoods for $p_1(s) = p_1(\Delta_{X/Y}(a))$ and $p_2(s) = p_2(\Delta_{X/Y}(a))$). Let $X = \operatorname{Spec} C$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} A$. Then

$$A \xrightarrow{g^\#} B \xrightarrow{f^\#} C.$$

We have

$$h^\# : C \otimes_A C \to C \otimes_B C.$$

It is clear that $h^\#$ is surjective in this affine neighborhood, and h is closed immersion in the affine neighborhood. Hence $s = \Delta_{X/Z}(a)$.

□

Lemma 5.6. Let Y be a separated Z-scheme. Then for any Y-schemes X_1, X_2, the canonical morphism $c : X_1 \times_Y X_2 \to X_1 \times_Z X_2$ is a closed immersion.

Proof. $\Delta_{Y/Z} : Y \to Y \times_Z Y$ is a closed immersion. Then consider

$$\mathrm{id}_{X_1} \times \Delta_{Y/Z} \times \mathrm{id}_{X_2} : X_1 \times_Y Y \times_Y X_2 \to X_1 \times_Y (Y \times_Z Y) \times_Y X_2.$$

Apply same argument as the proof of Proposition 5.5 (ii), one can see that

$$c = \mathrm{id}_{X_1} \times \Delta_{Y/Z} \times \mathrm{id}_{X_2}$$

is a closed immersion. □

Proposition 5.7. Let S be a scheme, X, Y be S-schemes. Suppose X is reduced and Y is separated over S. Let $f, g : X \to Y$ be S-morphisms such that there exists an open dense subset $U \subset X$ with $f|_U = g|_U$. Then $f = g$.

Proof.

$$\begin{tikzcd}
X \arrow[dr, "\varphi"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "\psi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

Also,

$$
\begin{tikzcd}
X \arrow[dr, "f"] \arrow[dddrr, bend right, "f"'] & & \\
& Y \arrow[dr, "\Delta_{Y/S}"] \arrow[ddr, bend right, "\mathrm{id}_Y"'] & \\
& & Y \times_S Y \arrow[d] \\
& & Y
\end{tikzcd}
$$

commutes. By the universal property of fiber product, $\Delta_{Y/S} \circ f = \psi$. Restrict this on U, we get

$$\psi|_U = \varphi|_U = \Delta_{Y/S} \circ f|_U.$$

Then $U \subset \varphi^{-1}(\Delta_{Y/S}(Y))$. Since U is dense, $X = \varphi^{-1}(\Delta_{Y/S}(Y))$. Similar for ψ. Hence, φ and ψ coincide on X as a morphism of topological spaces, and f, g coincide on X as a morphism of topological spaces. Now, we check the morphisms of sheaves are equal, i.e. $f^\# = g^\#$. To check the morphisms of sheaves are equal, we can check on an open cover, so we reduce to the affine case, and write $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. We have $f^\#, g^\# : A \to B$. Let $a \in A$, $b = f^\#(a) - g^\#(a)$. We know $b|_U = 0$. Therefore,

$$U \subset V(b) \subset \operatorname{Spec} B.$$

Since U is dense, $V(b) = \operatorname{Spec} B$. Since B is reduced, we have $b = 0$. □

5.2 Proper morphisms

We first recall that the proper map between topological spaces.

Definition. A continuous map between topological spaces $f : X \to Y$ is **proper** if the preimage of compact set is compact.

Remark. If X is Hausdorff, Y is locally compact Hausdorff, then a proper map is closed.

Definition.

(i) Let $f : X \to Y$ be a morphism of schemes. f is said to be **universally closed** if for all Y-scheme Y',

$$X \times_Y Y' \to Y'$$

is a closed map.

(ii) $f : X \to Y$ is called a **proper morphism** if f is of finite type, separated, universally closed.

Theorem 5.8. Let $f : X \to Y$ be a morphism of schemes. If f is finite, then f is proper.

Proof. Since f is finite, f is a closed map. By Proposition 4.4 (i), we know finite morphisms are stable under base change. Therefore, f is universally closed. On the other hand, since f is finite, f is affine. From Proposition 5.1, we conclude that f is separated. □

Example.

1. Closed immersions are proper.

2. Let k be a field. The projection map $\mathbb{A}^2_k = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$ is not proper.

Lemma 5.9. Let P be a property of morphisms. Suppose the following.

(i) Closed immersions satisfy P.

(ii) P stable under base change.

(iii) P stable under composition.

Then

(iv) If $X \to Z$, $Y \to Z$ verify P, then $X \times_Z Y \to Z$ verifies P.

(v) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a composition of morphisms of schemes. Suppose $g \circ f$ verifies P, and g is separated, then f verifies P.

Proof.

(iv)

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

By (ii), π_Y verifies P. By (iii), $X \times_Z Y \to Z$ verifies P.

(v)

$$\begin{tikzcd} X \arrow[dr, "h"] \arrow[drrr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & & \\ & X \times_Z Y \arrow[rr, "\pi_Y"] \arrow[d] & & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

By Lemma 5.6, h is closed. By (i), h verifies P. By (ii), π_Y verifies P. By (iii), f verifies P.

$\square$

Proposition 5.10.

(i) Composition of the proper morphisms is proper.

(ii) Closed immersion is proper.

(iii) Properness is stable under base change.

Corollary 5.11. Every k-morphism between proper k-varieties is proper.

Proof. Apply Proposition 5.10 and Lemma 5.9 (v). $\square$

5.3 Projective morphisms

Define $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ for any scheme Y and for any integer $n \geqslant 0$.

Definition. Let $f : X \to Y$ be a morphism of schemes. We say f is **projective** if there is a closed immersion $\iota : X \to \mathbb{P}^n_Y$ such that $\pi_Y \circ \iota = f$, i.e. the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr, "f"'] & & \mathbb{P}^n_Y \arrow[dl, "\pi_Y"] \\ & Y & \end{tikzcd}$$

Example. Let $Y = \operatorname{Proj} A[x_0, \ldots, x_n]$, $X = \operatorname{Proj}(A[x_0, \ldots, x_n]/I)$ for some homogeneous ideal I. Then $X \to Y$ is a projective morphism.

Proposition 5.12. Let A be a ring, B be a graded A-algebra ($A \subset B_0$). Let C be an A-algebra. Endow $B \otimes_A C$ a graded structure, $(B \otimes_A C)_d := B_d \otimes_A C$. Then $\operatorname{Proj}(B \otimes_A C) = (\operatorname{Proj} B) \times_A \operatorname{Spec} C$.

Proof. Let $\iota : B \hookrightarrow B \otimes_A C$ be a natural embedding homomorphism $b \mapsto b \otimes 1$. Now that this is grading preserving. Write $E = B \otimes_A C$. The above embedding gives a morphism

$$\operatorname{Proj} E \to \operatorname{Proj} B \to \operatorname{Spec} A.$$

Similarly, the embedding $C \hookrightarrow E$ gives

$$\operatorname{Proj} E \to \operatorname{Spec} C \to \operatorname{Spec} A.$$

Hence, they induce a morphism

$$\varphi : \operatorname{Proj} E \to (\operatorname{Proj} B) \times_A C.$$

Take $f \in B$ be a homogenous element. We have

$$D_+(\varphi^{\#}(\iota(f))) \to D_+(f) \times_A C.$$

Now since $B_{(f)} \hookrightarrow B_f$ is injective, $B_{(f)} \otimes C \hookrightarrow B_f \otimes C$ is also injective (degree 0 part). Since $B_f \otimes C \simeq E_{\varphi^{\#}(f)}$, this implies $B_{(f)} \otimes C \simeq E_{(\varphi^{\#}(f))}$. $\square$

Theorem 5.13. A projective morphism is proper.

Proof. Since Properness is stable under base change, we can assume $Y = \operatorname{Spec}\mathbb{Z}$. It suffices to show $\pi : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is proper since closed immersion is proper. It is clear that π is of finite type and separated. It remains to show that π is universally closed, i.e. for any scheme Z,

$$\mathbb{P}^n_Z \to Z$$

is closed. We may assume Z is affine.

Lemma 5.14. Let I be a homogeneous ideal in $A[x_0, \dots, x_n] = B$. Then

$$z \in Y \backslash \pi(V_+(I)) \iff \exists m > 0 \text{ with } (B/I)_m \otimes \kappa(z) = 0.$$

Proof. Since $V_+(I) \cap \pi^{-1}(y) = V_+(I \otimes \kappa(z))$,

$$\begin{aligned} z \notin \pi(V_+(I)) &\iff V_+(I \otimes_A \kappa(z)) = \varnothing \\ &\iff \sqrt{I \otimes_A \kappa(z)} \supset B_+ \otimes_A \kappa(z) \\ &\iff \exists m > 0 \text{ such that } B_m \otimes_A \kappa(z) \subset I \otimes_A \kappa(z). \end{aligned}$$

The last statement is equivalent to $(B/I)_m \otimes \kappa(y) = 0$. □

Note that $(B/I)_m$ is a finitely generated A-module. By Nakayama lemma,

$$(B/I)_m \otimes_A \mathscr{O}_{Z,z} = 0.$$

Hence, there exists $f \in A$ such that

$$f(B/I)_m = 0.$$

In other words,

$$(B/I)_m \otimes_A A_f = 0.$$

This shows that

$$D(f) \cap \pi(V_+(I)) = \varnothing.$$

Therefore, $\mathbb{P}^n_Z \to Z$ is a closed map. □

Lemma 5.15 (Segre embedding)**.** Let S be a scheme. Then there is a closed immersion

$$\mathbb{P}^n_S \times_S \mathbb{P}^m_S \to \mathbb{P}^{nm+n+m}_S$$

Proof. Reduce to the case that $S = \operatorname{Spec} Z$. Write

$$\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[T_0, \dots, T_n],$$

$$\mathbb{P}^m_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[S_0, \dots, S_m],$$

$$\mathbb{P}^{nm+n+m}_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[x_{ij}]_{\substack{0\leqslant i\leqslant n \\ 0\leqslant j\leqslant m}}.$$

Then define $f_{ij} : D_+(T_i) \times D_+(S_j) \to D_+(x_{ij})$ be the map induce by $x_{\alpha\beta}/x_{ij} \mapsto T_\alpha/T_i \otimes S_\beta/S_j$. Then f_{ij} can be glued. □

Proposition 5.16.

(i) Closed immersion is projective.

(ii) The composition of projective morphisms is projective.

(iii) Projective morphisms are stable under base change.

Proof.

(i) Let $n = 0$.

(ii)

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook] & & Y \arrow[rr] \arrow[dr, hook] & & Z \\ & \mathbb{P}^n_Y \arrow[ur, "\pi_Y"'] & & \mathbb{P}^m_Z \arrow[ur, "\pi_Z"'] & \end{tikzcd}$$

We have the following map

$$X \hookrightarrow \mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y \hookrightarrow \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}} \times Z \hookrightarrow \mathbb{P}^{nm+n+m}_{\mathbb{Z}} \times_{\mathbb{Z}} Z.$$

(iii) Closed immersion is stable under base change.

□

5.4 Discrete valuation rings and proper morphisms

Definition. Let K be a field. A **valuation** $\nu : K^\times \to \Gamma$ where $(\Gamma, +)$ is a totally ordered abelian group satisfying

(i) ν is a group homomorphism.

(ii) $\nu(\alpha + \beta) \geqslant \min\{\nu(\alpha), \nu(\beta)\}$.

Example. Let p be a prime in $\mathbb{Z}$. Define $\nu_p : \mathbb{Q}^\times \to \mathbb{Z}$ as following: let $\frac{x}{y} \in \mathbb{Q}^\times$ with $(x, y) = 1$, write

$$\frac{x}{y} = p^a \frac{a'}{b'}$$

with $(a', p) = (b', p) = 1$, then $\nu_p : \frac{x}{y} \mapsto a$.

Set $\nu(0) = \infty$. Define

$$\mathscr{O}_\nu := \{x \in K \mid \nu(x) \geqslant 0\}$$

which is called **valuation ring of** ν. We can see if $x, x^{-1} \in \mathscr{O}_\nu$, then $\nu(x) = 0$. Also,

$$\mathfrak{m}_\nu := \{x \in K \mid \nu(x) > 0\}$$

is the unique maximal ideal in $\mathscr{O}_\nu$.

Definition. Let $A \subset B$ be local rings. We say that B **dominates** A if the inclusion $A \hookrightarrow B$ is a local homomorphism.

Lemma 5.17. Let $\mathscr{O}_K$ be a valuation ring, and $K = \operatorname{Frac} \mathscr{O}_K$. For any local subring $A \subset K$ dominating $\mathscr{O}_K$, we have $A = \mathscr{O}_K$.

Proof. Suppose there exists $a \in A \backslash \mathscr{O}_K$, then $\nu(a) < 0$ where ν is the valuation associated to $\mathscr{O}_K$. Then $a^{-1} \in \mathfrak{m}_\nu \subset \mathscr{O}_K$. Since A dominates $\mathscr{O}_K$, we have $a^{-1} \in \mathfrak{m}_\nu \subset \mathfrak{m}_A$. Then $a \cdot a^{-1} = 1 \in \mathfrak{m}_A$ is a contradiction. $\square$

Theorem 5.18. Let X be a proper scheme over a valuation ring $\mathscr{O}_K$ and $K = \operatorname{Frac} \mathscr{O}_K$. Then the canonical map

$$X_{\mathscr{O}_K}(\mathscr{O}_K) \to X_K(K)$$

is bijective.

Proof.

- Injective: Since X is separated over $\mathscr{O}_K$, and $\operatorname{Spec} K$ is dense in $\operatorname{Spec} \mathscr{O}_K$. Use the same argument as Proposition 5.7, we have $X(\mathscr{O}_K) = X(K)$.

- Surjective: Let $\pi \in X(K)$, and it induces a morphism $\operatorname{Spec} K \to X \times_{\mathscr{O}_K} K$ still denoted it by π. Write $\operatorname{Spec} K = \{\eta\}$ Let $x = p(\pi(\eta))$, and $Z := \overline{\{x\}} \subset X$. Endow Z with reduced structure.

$$\begin{tikzcd}
\{\eta\} = \operatorname{Spec} K \arrow[dr, "\pi"] \arrow[drr, bend left, "\mathrm{id}"] & & \\
Z_K \arrow[r, hook] \arrow[d] & X \times_{\mathscr{O}_K} K \arrow[r] \arrow[d, "p"'] & \operatorname{Spec} K \arrow[d] \\
Z \arrow[r, hook] & X \arrow[r, "\iota_X"'] & \operatorname{Spec} \mathscr{O}_K
\end{tikzcd}$$

Note that $\pi(\eta)$ is a closed point in $X \times_{\mathscr{O}_K} K$. But $\pi(\eta)$ is dense in Z_K, we have $Z_K = \{x\} = \{\pi(\eta)\}$. On the other hand, ι_X is proper, we have $\iota_X(Z)$ is closed in $\operatorname{Spec} \mathscr{O}_K$. But $\eta \in \iota_X(Z)$, we have $\iota_X(Z) = \operatorname{Spec} \mathscr{O}_K$.
Let $s \in \operatorname{Spec} \mathscr{O}_K$ be the unique maximal ideal, and $t = \iota_Z^{-1}(s)$. We have

$$\mathscr{O}_K = \mathscr{O}_{K,s} \to \mathscr{O}_{Z,t}$$

is a local homomorphism.
Since Z is integral with generic point x, we have $\operatorname{Frac} \mathscr{O}_{Z,t} = \mathscr{O}_{Z,x} = K$. Then Lemma 5.17 implies $\mathscr{O}_{Z,t} = \mathscr{O}_K$. Then we have

$$\operatorname{Spec} \mathscr{O}_K = \operatorname{Spec} \mathscr{O}_{Z,t} \to Z \to X.$$

$\square$

Corollary 5.19. Let $f : X \to Y$ be a proper morphism. For any Y-scheme $\operatorname{Spec} \mathscr{O}_K$ where $\mathscr{O}_K$ is a valuation ring with $\operatorname{Frac} \mathscr{O}_K = K$, the canonical map $X(\mathscr{O}_K) \to X(\mathscr{O}_K)$ is bijective.

Proof. Let $Z = X \times_Y \operatorname{Spec} \mathscr{O}_K$. Then $Z(\mathscr{O}_K) = X(\mathscr{O}_K)$ and $Z_K(K) = X(K)$. Apply Theorem 5.18 for $Z \to \operatorname{Spec} \mathscr{O}_K$. $\square$

We have the following statement in general (we will not prove).

Theorem 5.20 (valuative criterion of properness)**.** Let $f : X \to Y$ be a morphism of finite type with Y locally Noetherian. Then the following are equivalent

1. f is separated (resp. proper).
2. For any $\nu : \operatorname{Spec} \mathscr{O}_K \to Y$ where $\mathscr{O}_K$ is a discrete valuation ring, then there is at most one lift $\tilde{\nu} : \operatorname{Spec} K \to X$ of ν (resp. a unique lift $\tilde{\nu}$).

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} \mathscr{O}_K \arrow[r, "\nu"] \arrow[ur, dashed, "\tilde{\nu}"] & Y
\end{tikzcd}$$

6 Local properties of morphisms

6.1 Normal schemes

Definition. A ring is **normal** if it is an integral domain and is integrally closed.

Example.

(i) $\mathbb{Z}$ is normal. More general, UFD is normal.

(ii) Ring of integer is normal.

(iii) Dedekind domain is normal.

Definition. Let X be a scheme. X is **normal at** x if $\mathscr{O}_{X,x}$ is normal. X is a **normal scheme** if X is irreducible and normal at all $x \in X$.

Proposition 6.1. Let X be an irreducible scheme. Then the following are equivalent:

(i) X is a normal scheme.

(ii) For any open subset $U \subset X$, $\mathscr{O}_X(U)$ is a normal integral domain.

(iii) Suppose further that X is quasi-compact, X is normal at its closed points.

Proof.

(i)⇒(ii) Let $V = \operatorname{Spec} A \subset X$ be an affine open subset. By definition, one can see that normal would imply integral. Hence A is an integral domain. Let $K = \operatorname{Frac} A$. We want to show that A is integrally closed.
Let $\alpha \in K$ be an integral element over A. Define

$$I_\alpha := \{a \in A \mid a\alpha \in A\}.$$

We have following two cases:

(1) $I_\alpha = A$: Then $\alpha \in A$.

(2) $I_\alpha \subsetneq A$: Then there exists a maximal ideal $\mathfrak{m} \trianglelefteq A$ with $I_\alpha \subset \mathfrak{m}$. Note that $\operatorname{Frac} A_\mathfrak{m} = K$, and $\alpha \in \operatorname{Frac} A_\mathfrak{m}$ and integral over $A_\mathfrak{m}$. Since X is normal at $\mathfrak{m}$, $\alpha \in A_\mathfrak{m}$. Namely, there exists $h \in A \backslash \mathfrak{m}$ such that $h\alpha \in A$. Then $h \in I_\alpha \subset \mathfrak{m}$ is a contradiction.

For general open subset $U \subset X$, cover U by affine open subsets, then use the above case.

(ii)⇒(iii) Let $x \in X$ be a closed point. Then there exists an affine open neighborhood $V = \operatorname{Spec} A \subset X$ containing x. Since A is a normal ring, $A_\mathfrak{m}$ is a normal ring.

(iii)⇒(i) Quasi-compactness is used to guarantee that X has a closed point (we will prove this later). Let $x \in X$. Since X is quasi-compact, then $\overline{\{x\}}$ is quasi-compact. Let $y \in \overline{\{x\}}$ be a closed point. Choose an affine open neighborhood $V = \operatorname{Spec} A$ of y. Then $A_{\mathfrak{m}} = \mathscr{O}_{X,y}$ is normal. Let $\mathfrak{p}$ be the prime ideal associated to the point x. Then $\mathscr{O}_{X,x} = A_{\mathfrak{p}}$ is a localization of $\mathscr{O}_{X,y}$ which is also normal.

□

Fact. A quasi-compact scheme X has a closed point.

Proof. Write $X = \bigcup_{i=1}^{n} U_i$ with $U_i = \operatorname{Spec} A_i$. Choose a closed point p_1 in U_1. Then $\overline{\{p_1\}}$ is irreducible (closure is taken in X). If $\overline{\{p_1\}} = \{p_1\}$, then we are done. Otherwise, $\{p_1\} \subsetneq \overline{\{p_1\}}$. Choose $p_2 \in \overline{\{p_1\}} \backslash \{p_1\}$, without loss of generality, say $p_2 \in U_2$ (Note that $p_2 \notin U_1$, and $p_1 \in U_2$). Furthermore, we can choose p_2 is closed in U_2. Now, we have $p_1 \notin \overline{\{p_2\}} \subsetneq \overline{\{p_1\}}$. If p_2 is not closed, then do similar way. We can conclude that X has a closed point by there is only finite U_i's. □

Remark. $X = \operatorname{Spec} A$ is normal if and only if A is normal.

Example.

(i) Since UFD is normal, $\mathbb{A}_k^n$ is normal, so is $\mathbb{P}_k^n$.

(ii) A classical non-normal scheme is $\operatorname{Spec}(k[x, y]/(y^2 - x^3))$.

Proposition 6.2. Let A be a Noetherian normal ring. Then

$$A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p}) \leqslant 1}} A_{\mathfrak{p}}.$$

Proof. Let $\alpha \in \operatorname{Frac} A \backslash A$. Then $I_\alpha \subsetneq A$. Consider

$$S = \{I_\alpha \mid \alpha \in \operatorname{Frac} A \backslash A\}.$$

Since A is Noetherian, there is an maximal element $I_\alpha \in S$. We claim that $I_\alpha \in \operatorname{Spec} A$. Indeed, if $ab \in I_\alpha$ with $a \notin I_\alpha$, i.e. $\alpha a \notin A$. Then $b \in I_{a\alpha}$. We have the natural inclusion $I_\alpha \subset I_{a\alpha}$. Since I_α is maximal, $b \in I_{a\alpha} = I_\alpha$. Write $\mathfrak{p} = I_\alpha$. We have two cases:

(1) $\alpha\mathfrak{p}A_{\mathfrak{p}} = A_{\mathfrak{p}}$: Then $\mathfrak{p}A_{\mathfrak{p}} = \alpha^{-1}A_{\mathfrak{p}}$. By Krull principal ideal theorem, $\operatorname{ht}(\mathfrak{p}) \leqslant 1$.

(2) $\alpha\mathfrak{p}A_{\mathfrak{p}} \subsetneq A_{\mathfrak{p}}$. Then $\alpha\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{p}A_{\mathfrak{p}}$. Since A is Noetherian, $A_{\mathfrak{p}}$ is Noetherian as well. Then $\mathfrak{p}A_{\mathfrak{p}}$ is a finitely generated $A_{\mathfrak{p}}$-module. By $\alpha\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{p}A_{\mathfrak{p}}$, we have

$$\alpha : \mathfrak{p}A_{\mathfrak{p}} \to \mathfrak{p}A_{\mathfrak{p}}$$

by multiplication. Then α is integral over $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}$ is normal, $\alpha \in A_{\mathfrak{p}}$, then there exists $h \in A \backslash \mathfrak{p}$ such that $h\alpha \in A$. $h \in I_\alpha = \mathfrak{p}$ is a contradiction.

In summary, the first case is the only case. In this case, $\alpha \notin A_{\mathfrak{p}}$ for some $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \leqslant 1$.

$\square$

Remark. Let A be an integral domain, then

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m}\text{: max.}} A_{\mathfrak{m}}$$

Corollary 6.3. Let X be a Noetherian normal scheme, and $F \subset X$ be a closed subset whose irreducible components are of codimension 2. Then

$$\mathscr{O}_X(X) \to \mathscr{O}_X(X - F)$$

is an isomorphism.

Proof. In our definition of normal scheme, X is integral. Then

$$\mathscr{O}_X(X) \hookrightarrow \mathscr{O}_X(X \backslash F) \hookrightarrow \mathscr{O}_{X,\eta}$$

is injective.

Since X is Noetherian, F is Noetherian, write $F = \bigcup_{i=1}^{m} F_i$ where F_i are irreducible component of F. Therefore, we reduce to the case that F is irreducible.

Let U be an affine open subset of X with $U \cap F \neq \varnothing$. Then

$$\operatorname{codim}(F \cap U, U) = \operatorname{codim}(F, X) = 2.$$

Therefore we reduce to the case that $X = \operatorname{Spec} A$ is affine.

Since F is a closed irreducible subset, $F = V(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} A$, and $\operatorname{codim}(F, X) = \operatorname{ht}(\mathfrak{p}) = 2$. By Proposition 6.2,

$$A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p}) \leqslant 1}} A_{\mathfrak{p}}.$$

We have for any $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \leqslant 1$, $\mathfrak{p} \in X \backslash F$. Hence

$$\mathscr{O}_X(X) \to \mathscr{O}_X(X \backslash F)$$

is surjective. $\square$

Definition. Let X be a scheme and $x \in X$.

(i) The **dimension of** X **at** x, denoted by $\dim_x X$, is the $\dim \mathscr{O}_{X,x}$.

(ii) $x \in X$ is **of codimension** r if $\overline{\{x\}}$ is of codimension r. If X is irreducible, codimension r if and only if $\dim_x X = r$.

Lemma 6.4. Let S be a scheme, and X, Y be S-schemes with Y of finite type over S. If one of the following holds:

(i) S is locally Noetherian.

(ii) X is integral.

Then all S-morphisms $\operatorname{Spec} \mathscr{O}_{X,x} \to Y$ extends to S-morphisms of an open subset $U \subset X$ containing x.

Theorem 6.5. Let S be a Noetherian scheme, X, Y be S-schemes of finite type. Suppose Y is proper over S and X is normal with fraction field K.
Let $U \subset X$ be a nonempty open subset, and $f : U \to Y$ is a morphism of S-schemes. Then there exists $V \supset U$ open and containing all codimension 1 points in X such that f extends uniquely on V. In particular, if $\dim X = 1$, f extends uniquely to X.

Proof. First of all, we show if there is an extension of f, then it must be unique.
Since Y is proper, it is separated. Since X is normal, X is reduced. By Proposition 5.7, the extension must be unique.
Now, let $x \in X$ be a point of codimension 1. Then $\mathscr{O}_{X,x}$ is a DVR since X is noetherian and normal.

$$\begin{tikzcd}
U \arrow[rr, "f"] & & Y \arrow[dr] & \\
\operatorname{Spec} \mathscr{O}_{X,\eta} \arrow[u, "\iota"] \arrow[urr, "f_\eta"'] \arrow[dr] & & & S \\
& \operatorname{Spec} \mathscr{O}_{X,x} \arrow[uur, dashed, "f_x"] \arrow[urr] & &
\end{tikzcd}$$

Since Y is proper, by the valuative criterion of properness, there exists $f_x : \operatorname{Spec} \mathscr{O}_{X,x} \to Y$. By Lemma 6.4, f_x extends to some open $U_x \ni x$,

$$\tilde{f}_x : U_x \to Y,$$

$\tilde{f}_x|_{U \cap U_x} = f|_{U \cap U_x}$ for any x with codimension 1. Repeat this process for codimension 1 point. ☐

Definition. Let k be a field, and X, Y be two integral k-varieties. We say that X, Y are **birational** if their function field are isomorphic.

Remark. If X, Y are birational, then Lemma 6.4 implies that there exists open subsets $U \subset X$, $V \subset Y$ such that $U \simeq V$.

Corollary 6.6. Let k be a field, X, Y be curves (k-varieties of dimension 1). If X, Y are proper over k and normal, then X, Y are isomorphic if and only if X, Y are birational.

Proof. Use the above remark and Theorem 6.5 ☐

Remark. Corollary 6.6 is false if $\dim X \geqslant 2$. For example, $\mathbb{P}^2_k$ is birational to $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

Example. Let k be an infinite field with characteristic different to 2. Let

$$X := V_+(a_0t_0^2 + a_1t_1^2 + a_2t_2^2) \subset \mathbb{P}^2_k.$$

Then X is birational to $\mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$.

($\Rightarrow$) By the above remark, there exists nonempty open subset U isomorphic to an open subscheme of $\mathbb{P}^1_k$. Then $U(k) \neq \varnothing$.

($\Leftarrow$) There exists $(\alpha_0, \alpha_1, \alpha_2) \in k^3$ not all zero. By linear algebra,

$$a_0t_0^2 + a_1t_1^2 + a_2t_2^2 \sim s_0s_1 + as_2^2.$$

Then

$$\left(\frac{s_1}{s_0}\right) + a\left(\frac{s_2}{s_0}\right)^2$$

is a $\mathbb{P}^1$-parametrization.

Definition. Let X be an integral scheme. A morphism $\pi : X' \to X$ is called **a normalization morphism** if X' is normal and if every dominant morphism $f : Y \to X$ with Y normal factors uniquely through π, i.e. the following diagram commutes

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & X \\ & X' \arrow[ur, "\pi"'] & \end{tikzcd}$$

Remark. If $\pi : X' \to X$ is a normalization of X, then for any nonempty U, $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a normalization of U.

Definition. A morphism $f : Y \to X$ is **integral** if f is affine and $\mathscr{O}_X(U) \to \mathscr{O}_Y(f^{-1}(U))$ is integral, i.e. for any $h \in \mathscr{O}_Y(f^{-1}(U))$, h is integral over $\mathscr{O}_X(U)$.

Lemma 6.7. Let $X = \operatorname{Spec} A$ with A integral, then the normalization morphism $\pi : X' \to X$ is given by $X' = \operatorname{Spec} A'$, and A' is the integral closure of A (in $\operatorname{Frac} A$).

Proposition 6.8. Let X be an integral scheme. Then there is a normalization morphism $\pi : X' \to X$ and is unique up to isomorphism (as X-scheme). Moreover, a morphism $f : Y \to X$ is the normalization morphism if and only if Y is normal and f is birational and integral.

Proof. Uniqueness follows from the universal property.
Let $\{U_i\}$ be an affine open cover of X. Then Lemma 6.7 implies there exists $\pi_i : U_i' \to U_i$ is the normalization. For $U_i \cap U_j$,

$$(U_i \cap U_j)' \simeq \pi_i^{-1}(U_i \cap U_j) \simeq \pi_j^{-1}(U_i \cap U_j),$$

we can glue U_i' to X'. □

Definition. Let X be an integral scheme, and $K(X)$ be its function field. Let L be an algebraic extension of $K(X)$. We define the **normalization of** X **in** L to be the integral morphism $\pi : X' \to X$ with X' normal and $K(X') = L$ such that π extends to the canonical morphism $\operatorname{Spec} L \to X$ ($\operatorname{Spec} L \to \operatorname{Spec} K(X) \hookrightarrow X$).

Remark. Let X be an integral k-variety where k is a field. Let $L/K(X)$ be a finite field extension. Let X' be the normalization of X in L, then $\pi : X' \to X$ is a finite morphism.

6.2 Regular schemes

Definition. Let X be a scheme and $x \in X$, $\mathfrak{m}_x \trianglelefteq \mathscr{O}_{X,x}$ be the unique maximal ideal, and $\kappa(x) := \mathscr{O}_{X,x}/\mathfrak{m}_x$. **Zariski tangent space** T_xX of X at x is defined by $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ (the dual space $\mathfrak{m}_x/\mathfrak{m}_x^2$ which is a vector space over $\kappa(x)$).

Lemma 6.9. Let A be a ring, $\mathfrak{m} \trianglelefteq A$ be a maximal ideal.

(i) $\mathfrak{m}/\mathfrak{m}^2 \simeq (A/\mathfrak{m}) \otimes_A \mathfrak{m}$ as A-module.

(ii) $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m}A_\mathfrak{m} \simeq \mathfrak{m}A_\mathfrak{m}/\mathfrak{m}^2A_\mathfrak{m}$ as A-module.

Example. Let k be a field, $X = \mathbb{A}_k^n = \operatorname{Spec} k[T_1, \dots, T_n], x \in X(k)$. Let $(a_1, \dots, a_n) \in k^n$, define $M_x := (T_i - a_i)_{i=1}^n$. By Lemma 6.9, we have

$$M_x/M_x^2 \simeq \mathfrak{m}_x/\mathfrak{m}_x^2.$$

We now claim that $T_xX \simeq k^n$.
Let $v = (v_1, \dots, v_n) \in k^n$, define

$$D_v(g) = \sum_{i=1}^n v_i \frac{\partial g}{\partial T_i}(a_i)$$

where $g \in M_x$. Then we have the induced map $D_v : M_x/M_x^2 \to k$. Hence $v \mapsto D_v$ is an isomorphism from k^n to $(M_x/M_x^2)^\vee$ (from Taylor expansion).

Proposition 6.10. Let $X = V(I) \subset Y = \mathbb{A}_k^n$, where $I \trianglelefteq k[T_1, \dots, T_n]$, $x \in X(k)$. Let $f : X \hookrightarrow Y$ be the closed immersion. By identifying $T_{f,x} : T_{X,x} \to T_{Y,f(x)} \simeq k^n$, $T_{X,x}$ can be identified with

$$\left\{ (t_1, \dots, t_n) \in k^n \;\middle|\; \sum_{i=1}^n t_i \frac{\partial P}{\partial T_i}(x) = 0, \forall P \in I \right\}.$$

Proof. f induces a map $X(k) \xhookrightarrow{f} Y(k)$. $f(x) \in Y(k)$ corresponds to a maximal ideal $\mathfrak{m}$, and $x \in X(k)$ corresponds to a maximal ideal $\mathfrak{n} = \mathfrak{m}/I \trianglelefteq (k[T_1, \dots, T_n]/I)$.

$$0 \to I/I \cap \mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0$$

is exact with each terms are vector space. By taking dual,

$$0 \to \left(\mathfrak{n}/\mathfrak{n}^2\right)^\vee \to \left(\mathfrak{m}/\mathfrak{m}^2\right)^\vee \to (I/I \cap \mathfrak{m})^\vee$$

is also exact. By using the identification,

$$\left(\mathfrak{n}/\mathfrak{n}^2\right)^\vee = \left\{(v_1, \ldots, v_n) \in k^n \;\middle|\; \sum_{i=1}^{n} v_i \frac{\partial g}{\partial T_i}(x) = 0,\ g \in I\right\}.$$

□

Definition. Let A be a Noetherian local ring and $\mathfrak{m}$ be its maximal ideal. A is said to be **regular** local ring if

$$\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim A$$

where $\kappa := A/\mathfrak{m}$.

Fact.

(i) For a Noetherian local ring A, $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim A$.

(ii) Let A be a regular local ring and $\mathfrak{p} \in \operatorname{Spec} A$. Then $A_\mathfrak{p}$ is a regular local ring.

(iii) All regular local ring are UFD.

Definition. Let X be a scheme, and $x \in X$. We say X is **regular at** x if $\mathscr{O}_{X,x}$ is regular. X is **regular** if X is regular at all $x \in X$. Non-regular points are called **singular points**.

Example. Let $X = \operatorname{Spec}(k[x, y]/(y^2 - x^3))$ for some field k with characteristic not 2, 3, and s_0 be the point corresponding to (x, y). Then

$$T_{X,s_0} := \{(v_1, v_2) \mid (v_1 \cdot 2y - v_2 \cdot 3x)|_{s_0} = 0\} = k^2.$$

Hence $\dim_k T_{X,s_0} = 2$. For any $s \neq (x, y)$, $\dim_k T_{X,s} = 1$. By Noether normalization, $\dim \mathscr{O}_{X,s_0} = 1 = \dim \mathscr{O}_{X,s}$. Therefore, s is regular for $s \neq s_0$.

Example.

(i) DVR is regular.

(ii) Let X be a normal Noetherian scheme, x is a codimension 1 point of X. Then $\mathscr{O}_{X,x}$ is a DVR. By (i), X is regular at x.

(iii) Regular, integral schemes are normal. But the converse is not true. For example, consider $X = \operatorname{Spec}\left(k[x, y, z]/(z^2 - xy)\right)$ (Check X is normal but not regular).

Proposition 6.11. If X is quasi-compact which is regular at all its closed points is regular.

Proof. Similar as the proof of Proposition 6.1 and use the Fact (ii). □

Theorem 6.12 (Jacobian criterion)**.** Let k be a field, X be an affine k-variety $\operatorname{Spec}(k[T_1, \dots, T_n]/(f_1, \dots, f_m))$. Let $x \in X(k)$. Then X is regular at x if and only if $J_x := (\frac{\partial f_i}{\partial T_j})_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}}$ is of rank $n - \dim_x X$ ($\dim_x X = \dim \mathscr{O}_{X,x}$).

Proof. Let $f : X \to \mathbb{A}^n_k$ be the closed immersion. By Proposition 6.10,

$$\dim T_{X,x} = n - \dim \left(I/I \cap \mathfrak{m}^2\right)^\vee = n - \operatorname{rk} J_x.$$

By the regularity of x, we will have $\dim \mathscr{O}_{X,x} = \dim T_{X,x}$. $\square$

6.3 Flat morphism

Definition. Let A be a ring and M be an A-module. M is **flat** if $\otimes_A M$ is left exact.

Example.

(i) Free A-modules are flat.

(ii) When A is a field.

(iii) $\mathbb{Z}/n\mathbb{Z}$ is not a flat $\mathbb{Z}$-module.

(iv) When A is a PID, M is an A-flat if and only if M is A-torsion free.

Fact.

(i) If M is finitely presented A-module, then M is A-flat if and only if M is a projective module of finite rank if and M is locally free, i.e. $\operatorname{Spec} A := \bigcup_i D(a_i)$, M_{a_i} is a free A_{a_i}-module.

(ii) Flatness is of local nature: M is flat if and only if $M_\mathfrak{p}$ is flat $A_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec} A$ if and only if $M_\mathfrak{m}$ is flat $A_\mathfrak{m}$-module for all maximal ideal $\mathfrak{m}$.

(iii) If $A \to B$ is flat and $B \to C$ are flat, then $A \to C$ is flat.

(iv) M is a flat A-module, then for any A-algebra B, $M \otimes_A B$ is B-flat.

Definition.

(i) Let $f : X \to Y$ be a morphism of scheme. f is **flat at** $x \in X$ of $f^\#_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is a flat morphism, i.e. $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{Y,y}$-module where $y = f(x)$.

(ii) f is flat if f is flat at all $x \in X$.

Proposition 6.13.

(i) Open immersions are flat.

(ii) The composition of flat morphisms are flat.

(iii) The flatness is stable under base change.

Lemma 6.14. Let $f : X \to Y$ be a flat morphism of schemes with Y irreducible. Then every nonempty open subset U of X dominates Y. If X has only finitely many irreducible components, then every one of them dominates Y.

Proof. Note that the second statement follows form the first. Indeed, write $X = \bigcup_{i=1}^{n} X_i$ with X_i distinct irreducible component. Consider

$$C_j := \bigcup_{i \neq j} X_i \subsetneq X$$

which is a closed subset in X. Then $X \backslash C_j = X_j \backslash C_j$ is a nonempty open subset.
It remains to show the first statement. It suffices to take $U = \operatorname{Spec} B$ nonempty affine open of X. Let $\eta \in Y$ be the generic point in Y. If f dominates Y, then $\eta \in \operatorname{Im}(f)$. Since η is contained in any nonempty open subset of Y, we may assume $Y = \operatorname{Spec} A$. Then the assumption gives

$$f^{\#} : A \to B$$

is flat. We now claim that U_η is nonempty. Recall that η corresponds to the nilradical $N \trianglelefteq A$. Then $\kappa(\eta) = \operatorname{Frac}(A/N)$ (or you may replace Y by Y_{red} and U by $U \times_Y Y_{\text{red}}$). We now have an exact sequence

$$0 \to A/N \to \operatorname{Frac}(A/N).$$

Since B is A-flat,

$$0 \to B \otimes_A (A/N) \to B \otimes_A \operatorname{Frac}(A/N).$$

Suppose that U_η is empty, i.e. $\operatorname{Spec}(B \otimes_A \operatorname{Frac}(A/N)) = \varnothing$. This means that $B \otimes_A \operatorname{Frac}(A/N)$ is nilpotent. The above exact sequence gives that $B \otimes_A (A/N) = B/f^{\#}(N)B$ is also nilpotent. Then B is nilpotnent which is a contradiction. Hence U_η is nonempty, i.e. $\eta \in \operatorname{Im}(f)$. Hence f dominates Y. $\square$

Lemma 6.15. Let A be a reduced ring and $a \in A$ is a zero divisor. Then there is a minimal prime $\mathfrak{p} \trianglelefteq A$ such that $a \in \mathfrak{p}$.

Proof. Consider a ring homomorphism $i_a : A \to A_a$. Since a is a zero divisor, i_a is not injective. Let $I = \ker(i_a)$. We claim that $V(I) \subsetneq \operatorname{Spec} A$. Suppose not, $I \subset \sqrt{0} \trianglelefteq A$. Since A is reduced, then $I = 0$ which is a contradiction. Then we have

$$D(a) \subset V(i) \subsetneq \operatorname{Spec} A.$$

Note that $\operatorname{Spec} A$ is reducible ($\operatorname{Spec} A = V(I) \cup (\operatorname{Spec} A \backslash D(a))$). Then there exists an irreducible component with the generic point $\eta \notin D(a)$. Namely, there is a minimal prime $\mathfrak{p}$ corresponding to the generic point η. And $\eta \notin D(a)$ implies that $a \in \mathfrak{p}$. $\square$

Proposition 6.16. Let Y be a Dedekind scheme (i.e. Noetherian, normal, and dimension 1), and $f : X \to Y$ be a morphism of schemes with X Noetherian and reduced. Then f is flat if and only if every irreducible component of X dominates Y.

Proof. The direction ($\Rightarrow$) is the statement of Lemma 6.14. Conversely, let $x \in X$ and $y = f(x) \in Y$. Since Y is Dedekind, there are only two cases:

- y is generic point: $\mathscr{O}_{Y,y}$ is a field, then f is flat at x.
- y is a closed point: $\mathscr{O}_{Y,y}$ is a DVR, and $\mathfrak{m}_y = (\pi)$ is the maximal ideal of $\mathscr{O}_{Y,y}$ where π is the uniformizer. To show that $f^\#_x$ is flat, it suffices to show $\mathscr{O}_{X,x}$ is a torsion-free $\mathscr{O}_{Y,y}$-module (see Example (iv)).
  It suffices to show $t := f^\#_x(\pi)$ is not a zero divisor. Suppose that t is a zero divisor, then by Lemma 6.15, there is a irreducible component with generic point η such that $\eta \notin D(t)$. There exists minimal $\mathfrak{p}$ with $t \in \mathfrak{p} \subset \mathfrak{m}_x \trianglelefteq \mathscr{O}_{X,x}$, then $(f^\#)^{-1}(\mathfrak{p}) \neq 0$ since $\pi \in (f^\#)^{-1}(\mathfrak{p})$. Hence $f(\eta) = y$ which is a closed point in this case. Then irreducible component does not dominate Y.

□

Theorem 6.17. Let $f : X \to Y$ be a flat morphism of locally Noetherian shcemes. Let $x \in X$ and $y = f(x)$. Then $\dim_x X_y = \dim \mathscr{O}_{X_y,x} = \dim_x X - \dim_y Y = \dim \mathscr{O}_{X,x} - \dim \mathscr{O}_{Y,y}$.

Lemma 6.18. Let $f : X \to Y$ be a morphism of schemes, and $x \in X$, $y = f(x)$. Then the local ring $\mathscr{O}_{X_y,x}$ is $\mathscr{O}_{X,x}/f^\#_x(\mathfrak{m}_y)\mathscr{O}_{X,x}$ where $\mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$.

Proof. Since the problem is local, we may assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. Then $f^\# : A \to B$, and $f^\# : A_\mathfrak{p} \to B_\mathfrak{q}$ where $\mathfrak{p} = (f^\#)^{-1}(\mathfrak{q})$.

$$(B \otimes_A \kappa(y))_{\mathfrak{q}\otimes\kappa(y)} = (B \otimes_A (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}))_{\mathfrak{q}\otimes(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})} = \left(S^{-1}B/\mathfrak{p}S^{-1}B\right)_{\mathfrak{q}S^{-1}B/\mathfrak{p}S^{-1}B} = B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$$

where $S = f^\#(A\backslash\mathfrak{p})$. □

Proof of Theorem 6.17. Base change by $\operatorname{Spec} \mathscr{O}_{Y,y} \to Y$

$$\begin{tikzcd} X \times_Y \operatorname{Spec} \mathscr{O}_{Y,y} \arrow[r, "f"] \arrow[d] & \operatorname{Spec} \mathscr{O}_{Y,y} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We can reduce to the case $Y = \operatorname{Spec} A$ for A is a Noetherian local ring and y is a closed point in Spec A. Write $\mathfrak{m}_y$ be the maximal ideal of A. After this reduction, we have dim Y is finite. We prove by induction on dim Y.

- dim Y=0: $\mathfrak{m}_y$ is also minimal in A. Then $\operatorname{Spec} Y = \{y\}$. X_y is homeomorphic to X. Hence $(X_y)_{red}$ is same as X_{red}.

- Suppose the statement holds for $\dim Y = d$ for some $d \geqslant 0$, now suppose $\dim Y = d+1$. Replace Y by Y_{red}.

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] & y \arrow[d, hook] \\
X' = X \times_Y Y_{\mathrm{red}} \arrow[r, "f"] \arrow[d] & Y_{\mathrm{red}} = Y' \arrow[d, hook, "\text{closed}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

From the upper-half diagram, we see that $X_y = X'_y$. Let $t \in \mathfrak{m}_y$ such that t is not a zero divisor. Since $Y' = \operatorname{Spec} A'$ is reduced, then Lemma 6.15, t is not contained in any minimal prime ideal and $t \in \mathfrak{m}_y$ (If $\mathfrak{m}_y \setminus \bigcup_{\mathfrak{p} \text{ minimal prime}} \mathfrak{p} = \varnothing$, then $\mathfrak{m}_y \subset \bigcup \mathfrak{p}$. By prime avoidance, we have $\dim Y = 0$ which is a contradiction). Then

$$\dim(A'/tA') = \dim A' - 1 = \dim A - 1.$$

$$f_x^{\#} : A' \to \mathscr{O}_{X',x}$$

Write $f_x^{\#}(t) = s$,

$$\iota : A' \xrightarrow{\times t} A'$$

is injective since t is not contained in any minimal prime and $t \in \mathfrak{m}_y$. Since $f_x^{\#}$ is flat,

$$\mathscr{O}_{X',x} \otimes_{A'} A' \xrightarrow{\times s} \mathscr{O}_{X',x} \otimes_{A'} A'$$

is injective as well. Since s is not a zero divisor, and $s \in \mathfrak{m}_x$ where $\mathfrak{m}_x$ is the maximal ideal in $\mathscr{O}_{X',x}$. Then

$$\dim\left(\mathscr{O}_{X',x}/s\mathscr{O}_{X',x}\right) = \dim \mathscr{O}_{X',x} - 1.$$

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] & y \arrow[d, hook] \\
X'' \arrow[r] \arrow[d] & \operatorname{Spec}(A'/tA') \arrow[d, hook] \\
X' \arrow[r] \arrow[d] & \operatorname{Spec} A' = Y' \arrow[d, hook] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

By induction hypothesis,

$$\dim X_y = \dim X''_y = \dim_x X'' - \dim_y Y'' = (\dim_x X - 1) - (\dim_y Y - 1).$$

$\square$

Theorem 6.19. Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k. Suppose Y is irreducible and X is equidimensional. Then for any $y \in Y$, X_y is either empty or pure with $\dim X_y = \dim X - \dim Y$.

Proof. Let Z be an irreducible component of X_y. Since X is of finite type over k, X_y is of finite type over $\kappa(y)$. Then Z contains a nonempty open subset (in X_y). By Theorem 3.6, there exists a closed point $x \in Z$ of X_y. By Noether normalization,

$$\dim Z = \dim_x Z.$$

Since X is equidimensional, $\dim X = \dim X_i$ for any irreducible component X_i of X. Choose an irreducible component X_i of X which contains Z.

$$\dim X = \dim X_i = \dim \mathscr{O}_{X_i,x} + \dim \overline{\{x\}}$$

by Noether normalization again. Since Y is irreducible,

$$\dim Y = \dim \mathscr{O}_{Y,y} + \dim \overline{\{y\}}.$$

It suffices to show $\dim \overline{\{y\}} = \dim \overline{\{x\}}$. Since $\overline{\{y\}}$ is irreducible and finite type over k,

$$\dim \overline{\{y\}} = \operatorname{trdeg}_k \kappa(y).$$

Similarly,

$$\dim \overline{\{x\}} = \operatorname{trdeg}_k \kappa(x).$$

Since $\kappa(y)$ is finite type over $\kappa(x)$, and $\kappa(y)$ is algebraic over $\kappa(x)$. Now, f is flat, by Theorem 6.17,

$$\dim Z = \dim X - \dim Y.$$

$\square$

Corollary 6.20. Blow-up, for example blow-up of $\mathbb{A}_k^2$ at origin is not flat.

$$\iota : \mathbb{A}_k^2 \to \mathbb{A}_k^2 \times \mathbb{P}_k^1 = \mathbb{P}_{k[x,y]}^1$$

$X = \overline{\iota(Y)} = V_+(\alpha y - \beta x)$ where α, β is the coordinate in $\mathbb{P}_k^1$.

6.4 Unramified morphisms

Definition. Let $f : X \to Y$ be a morphism of finite type between locally Noetherian schemes. Let $x \in X$ and $y = f(x)$.

(i) We say that f is **unramified at** x if $f^\#(\mathfrak{m}_y)$ generates $\mathfrak{m}_x$ and $\kappa(x)/\kappa(y)$ is a finite separable extension where $\mathfrak{m}_x$, $\mathfrak{m}_y$ are maximal ideals of $\mathscr{O}_{X,x}$ and $\mathscr{O}_{Y,y}$ respectively.

(ii) We say that f is **étale at** x if f is flat and unramified at x.

(iii) f is **unramified** (resp. **étale**) if f is unramified (resp. étale) at all points of x.

Remark. Lemma 6.18 implies that if f is unramified, then $\mathscr{O}_{X_y, x}$ is a field for any $x \in X_y$.

Proposition 6.21. Let $f : X \to Y$ be a morphism of finite type between locally Noetherian scheme. Then f is unramified if and only if for any $y \in Y$, X_y is finite over $\kappa(y)$, reduced and $\kappa(x)/\kappa(y)$ is separable for any $x \in X_y$. In fact, $X_y = \operatorname{Spec} \prod L_i$.

Example.

(i) If L/K is a separable field extension, then $\operatorname{Spec} L \to \operatorname{Spec} K$ is unramified and étale.

(ii) Let $L/\mathbb{Q}_p$ be a finite field extension. $\operatorname{Spec} \mathscr{O}_L \to \operatorname{Spec} \mathbb{Z}_p$ is unramified if and only if $p\mathbb{Z}_p$ is unramified in $\mathscr{O}_L$. Also, it would imply flatness.

(iii) Open immersion of locally Noetherian schemes is étale. Closed immersion of locally Noetherian schemes is unramified.

(iv) Let k be a field with characteristic not 2, $X = \operatorname{Spec}(k[x, y]/(y^2 - x))$ and $Y = \mathbb{A}^1_k = \operatorname{Spec} k[x]$.

- Let $s_1 = (x) \in Y$. Then $X_{s_1} = \operatorname{Spec} k[y]/(y^2)$ which is not reduced.
- Let $s_2 = (x - a)$ for $a \neq 0$. Then $X_s = \operatorname{Spec} k[y]/(y^2 - a)$, f is étale at each $x \in X_{s_2}$.

Proposition 6.22. Unramified (resp. étale) morphisms are stable under base change, composition and fibered products.

Proposition 6.23. Let $f : X \to Y$ be an étale morphism of finite type with Y locally Noetherian. Let $x \in X$ and $y = f(x)$. Then the following properties are true:

(i) $\dim \mathscr{O}_{X,x} = \dim \mathscr{O}_{Y,y}$ and f is quasi-finite.

(ii) The tangent map $T_{X,x} \to T_{Y,y} \otimes_{\kappa(y)} \kappa(x)$ is an isomorphism.

Proof.

(i) By Proposition 6.21 and Theorem 6.17.

(ii)
$$\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) = (\mathfrak{m}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(y)) \otimes_{\kappa(y)} \kappa(x) = \mathfrak{m}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(x)$$
$$\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathscr{O}_{X,x}} \kappa(y).$$

Since f is flat, $\mathfrak{m}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{X,x} = f^\#(\mathfrak{m}_y)\mathscr{O}_{X,x}$. Since f is unramified, $f^\#(\mathfrak{m}_y)\mathscr{O}_{X,x} = \mathfrak{m}_x$. Combining the equalities and tensor $\kappa(x)$ over $\mathscr{O}_{X,x}$,

$$\mathfrak{m}_y \otimes_{\mathscr{O}_{X,x}} \kappa(x) = \mathfrak{m}_x \otimes_{\mathscr{O}_{Y,y}} \kappa(x).$$

Hence

$$\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes_{\kappa(y)} \kappa(x) \simeq \mathfrak{m}_x/\mathfrak{m}_x^2.$$

□

Corollary 6.24. Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism of finite type and étale at $x \in X$. Then X is regular at x if and only if Y is regular at $f(x)$.

7 Coherent sheaves

7.1 $\mathscr{O}_X$-modules and coherent sheaves

Definition.

(i) Let $(X, \mathscr{O}_X)$ be a ringed topological space. A **sheaf of $\mathscr{O}_X$-module ($\mathscr{O}_X$-module)** $\mathscr{F}$ on X is a sheaf of abelian groups $\mathscr{F}$ on X such that $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module for all open subsets $U \subset X$ and compatible with restriction map i.e.

$$(a \cdot f)|_V = a|_V \cdot f|_V$$

for $V \subset U$, $a \in \mathscr{O}_X(U)$, and $f \in \mathscr{F}(U)$.

(ii) A **morphism** $f : \mathscr{F} \to \mathscr{G}$ **of $\mathscr{O}_X$-modules** is a morphism of sheaves such that $f|_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is a $\mathscr{O}_X(U)$-module homomorphism.

(iii) If $\mathscr{F}$, $\mathscr{G}$ are $\mathscr{O}_X$-module. Define $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is **the sheaf associated to the presheaf** $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.

(iv) An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be **locally free** if there is an open cover $\{U_i\}$ of X such that $\mathscr{F}|_{U_i} \simeq \mathscr{O}_X|_{U_i}^{(I_i)}$.
$\mathscr{F}$ is **free** if $\mathscr{F} \simeq \mathscr{O}_X^{(I)}$.
$\mathscr{F}$ is **locally free of rank** r if $|I_i| = r$.

(v) $\mathscr{F}$ is said to be **invertible** if it is locally free of rank 1. (Its inverse is given by $\mathscr{G} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$.)

Definition. Let $f : X \to Y$ be a morphism of schemes.

(i) Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $f_*\mathscr{F}$ is an $\mathscr{O}_Y$-module (called **direct image**) is defined by

$$f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$$

where V is an open subset of Y.

(ii) Let $\mathscr{G}$ be a sheaf on Y. Define $f^{-1}\mathscr{G}$ be the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{\substack{V \supseteq f(U) \\ V \subset Y \text{ open}}} \mathscr{G}(V).$$

(iii) $f^*\mathscr{G} := f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$ is called the **inverse image** of $\mathscr{G}$.

Definition. Let $X = \operatorname{Spec} A$, M is an A-module. Define

$$\widetilde{M}(U) := \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{c} 1.\ s(\mathfrak{p}) \in M_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U \\ 2.\ \text{For all } \mathfrak{p} \in U, \text{ there exists } \mathfrak{p} \in V \subset U, \text{ and } a \in M, f \in A \\ \text{such that for any } \mathfrak{q} \in V, f \notin \mathfrak{q}, s(\mathfrak{q}) = \frac{a}{f} \end{array} \right\}.$$

Proposition 7.1. Let $X = \operatorname{Spec} A$.

(i) Let $\{M_i\}_{i\in I}$ be a family of A-module. Then $(\bigoplus_{i\in I} M_i)^\sim \simeq \bigoplus_{i\in I} \widetilde{M_i}$.

(ii) For any A-modules M, N, $(M \otimes_A N)^\sim \simeq \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N}$.

(iii) Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of schemes. Then

  (1) $f_*(\widetilde{N}) \simeq ({}_A N)^\sim$ where N is an B-module.

  (2) $f^*(\widetilde{M}) \simeq (M \otimes_A B)^\sim$ where M is an A-module.

(iv) $L \to M \to N$ is an exact sequence of A-modules if and only if $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is an exact sequence of $\mathscr{O}_X$-module.

Proof.

(i) Locally, it commutes with direct sum.

(ii) Locally, it commutes with tensor product.

(iii) (1) Let $a \in A$, $f_*(\widetilde{N})(D(a)) = \widetilde{N}(f^{-1}(D(a))) = \widetilde{N}(D(b)) = N_b$ where $b = f^\#(a)$.
$({}_A N)^\sim(D(a)) = N_a = N \otimes_A A_a \simeq N_b$ by

$$\begin{array}{ccc} N_a & \longrightarrow & N_b \\ & & \\ \frac{n}{a^\ell} & \longmapsto & \frac{n}{f^\#(a)^\ell} \end{array}$$

  (2)

$$f^*(\widetilde{M})(D(b)) = \left\{ s : D(b) \to \bigsqcup_{q\in D(b)} M_{f(q)} \otimes_{A_{f(q)}} B_q \,\middle|\, \cdots\cdots \right\}.$$

One can construct a canonical map from $(M \otimes_A B)^\sim(D(b)) = M \otimes_A B_b \to f^*(\widetilde{M})(D(b))$ and show it is an isomorphism on stalk.

(iv) $\widetilde{L} \to \widetilde{M} \to \widetilde{N}$ is exact if and only if $L_x \to M_x \to N_x$ is exact for any $x \in X$ if and only if $P_x = 0$ for any $x \in X$ if and only if $P = 0$ if and only if $L \xrightarrow{i} M \xrightarrow{j} N$ is exact where $P = \ker j / \operatorname{Im} i$.

$\square$

Definition. Let X be a scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be **quasi-coherent** if there is an affine open cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{F}|_{U_i} \simeq \widetilde{M_i}$ where M_i is an A-module for any i.
If X is Noetherian, a quasi-coherent sheaf $\mathscr{F}$ is **coherent** if M_i is a finitely generated A_i-module.

Theorem 7.2. Let $X = \operatorname{Spec} A$, $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then there is an A-module M such that $\mathscr{F} \simeq \widetilde{M}$. (If A is Noetherian and $\mathscr{F}$ is coherent, then M is finitely generated A-module).

Proof. Let $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$ be an affine open cover of X such that $\mathscr{F}|_{U_i} \simeq \widetilde{M_i}$ where M_i is an A_i-module. Consider the exact sequence

$$0 \to \mathscr{F}(X) \to \prod_{i=1}^n \mathscr{F}(U_i) \to \prod_{i,j} \mathscr{F}(U_i \cap U_j).$$

Tensor A_f gives

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(X) \otimes A_f \arrow[r] \arrow[d, "\alpha"] & \prod_{i=1}^n \mathscr{F}(U_i) \otimes A_f \arrow[r] \arrow[d, "\beta"] & \prod_{i,j} \mathscr{F}(U_i \cap U_j) \otimes A_f \arrow[d, "\gamma"] \\
0 \arrow[r] & \mathscr{F}(X_f) \arrow[r] & \prod_{i=1}^n \mathscr{F}(U_i \cap X_f) \arrow[r] & \prod_{i,j} \mathscr{F}(X_f \cap U_i \cap U_j)
\end{tikzcd}$$

β is an isomorphism gives α is injective. Replace X by $U_i \cap U_j$ (since X is affine, X is separated, $U_i \cap U_j$ is affine), we get γ is injective, and α is surjective.

For Noetherian case, $\mathscr{F} = \widetilde{M}$. We have M_f is a finitely generated A_f-module, let $\{m_1, \dots, m_n\} \subset M$ be a generator of M_f over A_f. There exists finitely many f_i such that $X = \bigcup_{i \in I} D(f_i)$. Hence, there exists $\{m_{ij}\}_j$ generates M_{f_i} over A_{f_i}. Let

$$N = \langle m_{ij} \rangle_{i,j} \subset M,$$

one can see $N = M$ since $X = \bigcup_{i \in I} D(f_i)$. □

Corollary 7.3. Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then $\mathscr{F}$ is a quasi-coherent sheaf if and only if for any affine open subset $U \subset X$, we have $(\mathscr{F}(U))^\sim \simeq \mathscr{F}|_U$. (If X is Noetherian, $\mathscr{F}$ is coherent if $\mathscr{F}(U)$ is a finitely generated $\mathscr{O}_X(U)$-module.)

Corollary 7.4. Let X be a scheme.

(i) Let $f : \mathscr{G} \to \mathscr{H}$ be a morphism of quasi-coherent $\mathscr{O}_X$-module. Then $\ker(f)$, $\operatorname{Im}(f)$, $\operatorname{coker}(f)$ are also quasi-coherent.

(ii) Direct sum of quasi-coherent sheaves is quasi-coherent.

(iii) Tensor product of quasi-coherent sheaves is quasi-coherent.

Lemma 7.5. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module for $X = \operatorname{Spec} A$. Let $\{U_i\}_{i=1}^n$ be an affine open covering of X with $U_i = D(a_i)$. Suppose there is $f_{ij} \in \mathscr{F}(U_{ij})$, $U_{ij} := U_i \cap U_j$ such that

$$f_{ij}|_{U_{ijk}} + f_{jk}|_{U_{ijk}} - f_{ik}|_{U_{ijk}} = 0 \in \mathscr{F}(U_{ijk}) \qquad (\star\star)$$

where $U_{ijk} = U_i \cap U_j \cap U_k$. Then there exists $f_i \in \mathscr{F}(U_i)$ such that $f_{ij} = f_i|_{U_{ij}} - f_j|_{U_{ij}}$.

Corollary 7.6. Let X be an affine scheme. Let

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

be an exact sequence of $\mathscr{O}_X$-modules with $\mathscr{F}$ quasi-coherent. Then

$$0 \to \mathscr{F}(X) \to \mathscr{G}(X) \to \mathscr{H}(X) \xrightarrow{\pi} 0$$

is also exact.

Proof. It suffices to show the surjectivity of π. Let $c \in \mathscr{H}(X)$, there exists $\{U_i\}_{i \in I}$ affine open of X such that $c|_{U_i} = \pi(d_i)$ for some $d_i \in \mathscr{G}(U_i)$. Since X is affine, I can be a finite set. We have

$$\pi(d_i|_{U_{ij}} - d_j|_{U_{ij}}) = 0.$$

Then there exists $f_{ij} \in \mathscr{F}(U_{ij})$ such that

$$d_i|_{U_{ij}} - d_j|_{U_{ij}} = f_{ij}.$$

Hence f_{ij} satisfies $(\star\star)$ in Lemma 7.5. Therefore, by Lemma 7.5, there exists $f_i \in \mathscr{F}(U_i)$ such that $f_{ij} = f_i|_{U_{ij}} - f_j|_{U_{ij}}$. Let $d_i' = d_i - f_i$, then d_i' can be glued to $d \in \mathscr{G}(X)$. □

Proposition 7.7. Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then $\mathscr{F}$ is quasi-coherent if and only if for any $x \in X$, there is an open neighborhood $U \subset X$ of x such that there is an exact sequence

$$\mathscr{O}_X|_U^{(J)} \to \mathscr{O}_X|_U^{(I)} \to \mathscr{F} \to 0.$$

(If X is Noetherian, $\mathscr{F}$ is coherent if and only if $\mathscr{F}(U)$ is a finitely generated $\mathscr{O}_X(U)$-module.)

Proof. $(\Rightarrow)$ is easy. For the converse,

$$\mathscr{O}_X|_U^{(L)} \xrightarrow{\alpha} \mathscr{O}_X|_U^{(J)} \to \mathscr{O}_X|_U^{I} \to \mathscr{F}|_U \to 0$$

induces an exact sequence

$$0 \to \operatorname{coker}(\alpha) \to \mathscr{O}_X|_U^{(I)} \to \mathscr{F} \to 0.$$

Since Corollary 7.4, coker α is quasi-coherent.
By Corollary 7.6,

$$0 \to (\operatorname{coker}\alpha)(U) \to \mathscr{O}_X(U)^{(I)} \to \mathscr{F}(U) \to 0$$

is exact for U affine open subset of X.
Since $\mathscr{O}_X$, coker α are quasi-coherent, by Proposition 7.1,

$$\begin{tikzcd}
((\operatorname{coker}\alpha)(U))^\sim \arrow[r] \arrow[d, "f"] & (\mathscr{O}_X(U)^{(I)})^\sim \arrow[r] \arrow[d, "g"] & (\mathscr{F}(U))^\sim \arrow[r] \arrow[d, "h"] & 0 \\
\operatorname{coker}\alpha|_U \arrow[r] & \mathscr{O}_X|_U^{(I)} \arrow[r] & \mathscr{F}|_U \arrow[r] & 0
\end{tikzcd}$$

with f, g are isomorphisms. It follows that h is an isomorphism. □

Example. Let $j : U \to X$ be an open immersion of scheme, and $\mathscr{F}$ be a sheaf on U. Define $j!\mathscr{F}$ be a sheaf on X where $V \mapsto \mathscr{F}(V)$ if $V \subset U$, and $V \mapsto 0$ otherwise. Then $j!\mathscr{O}_X|_U$ on X is not quasi-coherent.

Proposition 7.8. Let $f : X \to Y$ be a morphism of schemes. If $\mathscr{G}$ is quasi-coherent (resp. coherent), then $f^*\mathscr{G}$ is quasi-coherent on X (resp. coherent)

Proof. We can reduce to $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. By Thoerem 7.2, $\mathscr{G} = \widetilde{M}$ where $M = \mathscr{G}(X)$. By Proposition 7.1, $f^*\mathscr{G} = (M \otimes_A B)^\sim$. □

Theorem 7.9.

(i) Let $f : X \to Y$ be a morphism of schemes with X Noetherian or X is quasi-compact and separated. Let $\mathscr{F}$ be quasi-coherent on X, then $f_*\mathscr{F}$ is quasi-coherent on Y.

(ii) If X, Y are Notherian, and f is finite. Then $f_*\mathscr{F}$ is coherent if $\mathscr{F}$ is coherent.

Proof.

(i) Let $\{U_i\}_{i=1}^r$ be an affine open cover of X with $\mathscr{F}|_{U_i} = \mathscr{F}(U_i)^\sim$ since $\mathscr{F}$ is quasi-coherent. Also, $U_i \cap U_j$ can be covered by finitely many affine open subsets since X is Noetherian or quasi-compact and separated. We reduce to the case that $Y = \operatorname{Spec} A$.

$$f^\# : A \to \mathscr{O}_X(X)$$

$$(f_*\mathscr{F})(D(g)) = \mathscr{F}(f^{-1}(D(g))) = \mathscr{F}(X_h)$$

where $h = f^\#(g)$. By the same argument as Lemma 1.14, $\mathscr{F}(X_h) = \mathscr{F}(X)_h = \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_X(X)_h$.

$$(f_*\mathscr{F})(D(g)) = \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_X(X)_h = \mathscr{F}(X)_g.$$

Hence $f_*\mathscr{F} = (\mathscr{F}(X))^\sim$.

□

Proposition 7.10. Let X be a scheme and Z be a closed subscheme of X. Let $\iota : Z \hookrightarrow X$ be the closed immersion. Then $\ker \iota^\#$ is quasi-coherent of ideals on X. The corresponding $Z \to \ker \iota^\#$ is a bijection between the closed subscheme of X and the quasi-coherent sheaves of ideals.

7.2 Quasi-coherent sheaves over projective spaces

Let $S = \bigoplus_{d \in \mathbb{Z}} S_d$. We assume S is generated by S_1 as an S_0-algebra. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded S=module. $\widetilde{M}$ on $\operatorname{Proj} S$ is defined as $\widetilde{M}(D_+(f)) = M_{(f)}$ which is quasi-coherent on $\operatorname{Proj} S$. Observe that if $M = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}_{\geqslant n_0}} M_n$, we have $\widetilde{M} = \widetilde{N}$.

Definition. Let $X = \operatorname{Proj} S$, for any $n \in \mathbb{Z}$, we set

$$\mathscr{O}_X(n) := (S(n))^{\sim}, \quad S(n)_d := S_{n+d}.$$

For $\mathscr{F}$ be an $\mathscr{O}_X$-module, define similarly $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Example. Let $X = \mathbb{P}^n_A$, $\mathscr{O}_X(-1)(X) = 0$, $\mathscr{O}_X(1)(X) = \langle T_0, \dots, T_n \rangle$ as A-module.

Proposition 7.11. The sheaf $\mathscr{O}_X(n)$ is invertible. For any S-module M, $(M(n))^{\sim} = \widetilde{M}(n) = \widetilde{M} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Proof. Since S is generated by S_1 over S_0, $\operatorname{Proj} S = \bigcup_{f \in S_1} D_+(f)$. Consider

$$i : S(n)_{(f)} \to S_{(f)}$$

by sending $\frac{m}{f^r}$ to $\frac{m}{f^{n+r}}$. One can construct its inverse. Hence i is an isomorphism as $\mathscr{O}_X(D_+(f)) = S_{(f)}$-module. For the second assertion,

$$\widetilde{M}(n)(D_+(f)) = M_{(f)} \otimes S(n)_{(f)} = (M \otimes S(n))_{(f)} = M(n)_{(f)} = (M(n))^{\sim}(D_+(f)).$$

$\square$

Lemma 7.12. Let X be either Noetherian or quasi-compact and separated. Let $\mathscr{F}$ be a quasi-coherent sheaf on X, $\mathscr{L}$ be an invertible sheaf on X. We fix a $s \in \mathscr{L}(X)$ and define

$$X_s := \{x \in X \mid s_x \mathscr{O}_{X,x} = \mathscr{L}_x\}$$

(i) Let $f \in \mathscr{F}(X)$. If $f|_{X_s} = 0$, then there exists $n > 0$ such that $f \otimes s^n = 0$ in $(\mathscr{F} \otimes \mathscr{L}^n)(X)$.

(ii) Let $g \in \mathscr{F}(X_s)$. Then there exists $n_0 > 0$ such that for any $n \geqslant n_0$, $g \otimes (s^n|_{X_s})$ can be lifted to a section in $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^n)(X)$.

Proof.

(i) Let $\{X_i\}_{i=1}^r$ be an affine open cover of X such that $\mathscr{L}|_{X_i} \simeq \mathscr{O}_X|_{X_i}$. Choose $e_i \in \mathscr{L}(X_i)$ such that $\mathscr{L}|_{X_i} = \mathscr{O}_X|_{X_i} \cdot e_i$. Then write $s|_{X_i} = h_i \otimes e_i$.

$$X_s \cap X_i = D(h_i) \subset X_i.$$

$$f|_{X_s} = 0 \implies f|_{D(h_i)} = 0 \implies \exists n \text{ such that } h_i^n \cdot f = 0 \text{ in } \mathscr{F}(X_i).$$

Then

$$(f \otimes s^n)|_{X_i} = (h_i^n f \otimes e_i^{\otimes n}) = 0.$$

$f \otimes s^n = 0$ in $(\mathscr{F} \otimes \mathscr{L}^n)(X)$.

(ii) Define $g_i := g|_{X_s \cap X_i} \in \mathscr{F}(D(h_i))$. Since $\mathscr{F}$ is quasi-coherent,

$$\mathscr{F}|_{X_i} = \mathscr{F}(X_i)^{\sim} = M_i^{\sim}.$$

Then $\mathscr{F}(D(h_i)) = (M_i)_{h_i}$. Hence there exists n such that $h_i^n g_i \in M_i$ for any i. Define

$$t_i := f_i \otimes e_i^n \in (\mathscr{F} \otimes \mathscr{L}^n)(X_i)$$

where

$$f_i := h_i^n g_i.$$

Now by construction,

$$t_i|_{X_i \cap X_j \cap X_s} - t_j|_{X_i \cap X_j \cap X_s} = (g_i \otimes s^n)|_{X_i \cap X_j \cap X_s} - (g_j \otimes s^n)|_{X_i \cap X_j \cap X_s}.$$

By (i), replace X by $X_i \cap X_j$, there exists r such that

$$(t_i - t_j)|_{X_i \cap X_j} \otimes s^r = 0 \text{ in } (\mathscr{F} \otimes \mathscr{L}^{n+r})(X_i \cap X_j).$$

Hence, we can glue $t_i \otimes s^r$ to a section $t \in (\mathscr{F} \otimes \mathscr{L}^{n+r})(X)$. So we can choose $n_0 \geqslant n + r$.

$\square$

Definition. Let X be a scheme.

(i) A quasi-coherent is said to be **finitely generated** if $\mathscr{F}(U)$ is a finitely generated A-module for $U = \operatorname{Spec} A$.

(ii) We say an $\mathscr{O}_X$-module, $\mathscr{F}$, is **generated by its global section at** x if $\mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_{X,x} \twoheadrightarrow \mathscr{F}_x$.

Definition. Let X be a projective scheme over $\operatorname{Spec} A$ ($i : X \hookrightarrow \mathbb{P}_A^n \to \operatorname{Spec} A$). Define $\mathscr{O}_X(n)$ to be $i^*(\mathscr{O}_{\mathbb{P}_A^n}(n))$. For an $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} i^*\mathscr{O}_{\mathbb{P}_A^n}(n)$.

Lemma 7.13. Let $f : X \to Y$ be an affine morphism, $\mathscr{F}$ be a quasi-coherent sheaf on X, $\mathscr{G}$ be a quasi-coherent sheaf on Y. Then

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) = f_*\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}.$$

Proof. Since f is affine, we may assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$. By Theorem 7.2, we have $\mathscr{F} = \widetilde{M}$, $\mathscr{G} = \tilde{N}$ where $M = \mathscr{F}(X)$, $N = \mathscr{G}(Y)$. Then $f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G})$ is quasi-coherent since f is affine. Then

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G})(Y) = (\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G})(X) = \mathscr{F}(X) \otimes_{\mathscr{O}_X(X)} f^*\mathscr{G}(X) = M \otimes_B (N \otimes_A B) = M \otimes_A N.$$

and $(f_*\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})(Y) = M \otimes_A N$. $\square$

Theorem 7.14 (Serre)**.** Let X be a projective scheme over A. Let $\mathscr{F}$ be a quasi-coherent finitely generated sheaf on X. Then there exists $n_0 > 0$ such that for any $n \geqslant n_0$, the sheaf $\mathscr{F}(n)$ is generated by a finite number of its global sections.

Proof. By definition, we have

$$i : X \hookrightarrow \mathbb{P}^n_A \to \operatorname{Spec} A$$

and X is quasi-compact and separated. By Thoerem 7.9, $i_*\mathscr{F}$ is quasi-coherent. Also, $(i_*\mathscr{F})(\mathbb{P}^n_A) = \mathscr{F}(X)$ and

$$(i_*\mathscr{F})_y = \begin{cases} \mathscr{F}_x, & y = i(x) \\ 0, & \text{otherwise.} \end{cases}$$

It suffices to prove for $X = \mathbb{P}^n_A = \bigcup_{i=0}^n D_+(T_i)$, write $B = A[T_0, \ldots, T_n]$ and $U_i = D_+(T_i)$. $\mathscr{F}|_{U_i} = (\mathscr{F}(U_i))^\sim$. $\mathscr{F}(U_i)$ is generated by $\{g_{ij}\}_{j=1}^{r_i}$ as $B_{(T_i)}$-module.
Let $\mathscr{L} = \mathscr{O}_X(1)$, $T_i \in \mathscr{O}_X(1)(U_i)$. By Lemma 7.12 (ii), there exists n_0 such that for any $n \geqslant n_0$, there exists $h_{ij} \in \mathscr{F}(n)(X)$ and $g_{ij} \otimes T_i^n = h_{ij}|_{U_i}$. Then $\mathscr{F}(n)(U_i)$ is generated by $\{g_{ij} \otimes T_i^n\}$. Hence $\mathscr{F}(n)$ is generated by h_{ij}. □

Corollary 7.15. Let X be a projective scheme over $\operatorname{Spec} A$. Then there is $m \in \mathbb{Z}$ and $r \geqslant 1$ such that $\mathscr{F}$ is a quotient sheaf of $\mathscr{O}_X(m)^r$.

Proof. By Theorem 7.14, there exists $n > 0$ such that $\mathscr{F}(n)$ is generated by global sections $s_1, \ldots, s_r \in \mathscr{F}(n)(X)$. Then we have a surjective morphism of sheaves

$$\mathscr{O}_x^r \twoheadrightarrow \mathscr{F}(n) \text{ by } (f_i)_i \mapsto \sum f_i s_i.$$

By tensoring,

$$\mathscr{O}_X(-n)^r \twoheadrightarrow \mathscr{F} \to 0.$$

□

Definition. Let $X = \operatorname{Proj} S$ (S is generated by S_1 as S_0-algebra). For any $\mathscr{O}_X$-module $\mathscr{F}$, define $\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$ as a graded S-module ($s \in S_d$, $t \in \mathscr{F}(n)(X)$, $s \otimes d \in \mathscr{O}_X(d) \otimes \mathscr{F}(n)(X) = \mathscr{F}(n+d)(X)$).

Proposition 7.16. Let A be a ring, $S = A[T_0, \ldots, T_n]$, $X = \operatorname{Proj} S$, then $\Gamma_*(\mathscr{O}_X) = S$.

Proof.

$$\mathscr{O}_X(d)(X) = \begin{cases} 0, & d < 0, \\ S_d, & d \geqslant 0. \end{cases}$$

□

Lemma 7.17. Let $X = \operatorname{Spec} A$, $\mathscr{F} = \widetilde{M}$ be a quasi-coherent sheaf on X and $\mathscr{G}$ be an $\mathscr{O}_X$-module. Then for any morphism of A-module $\varphi : M \to \mathscr{G}(X)$, there is a unique morphism of $\mathscr{O}_X$-module $\tilde{\varphi} : \mathscr{F} \to \mathscr{G}$ such that $\tilde{\varphi}(X) = \varphi$. If $\mathscr{G}$ is also quasi-coherent, then φ is an isomorphism if and only if $\tilde{\varphi}$ is an isomorphism.

Proof. Define

$$\tilde{\varphi}(D(f)) : M_f \to \mathscr{G}(D(f)) \text{ by } \frac{m}{f^n} \mapsto \varphi(m)|_{D(f)} \cdot \frac{1}{f^n}.$$

It is easy to see that this construction is compatible with restriction, and the rest are also easy. □

Theorem 7.18. Let $X = \operatorname{Proj} S$ where S is a graded ring generated by a finite number of elements of S_1 over S_0. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F} \simeq (\Gamma_*(\mathscr{F}))^\sim$ on X.

Proof. Write $X = \bigcup_{i=1}^{\ell} D_+(s_i)$ where $s_i \in S_1$, and $D_+(s_i) = U_i$ is affine.

$$\mathscr{F}|_{U_i} = (\mathscr{F}(U_i))^\sim.$$

Write $M = \Gamma_*(\mathscr{F})$

$$(\Gamma_*(\mathscr{F}))^\sim(U_i) = M_{(s_i)}.$$

Note that $M_{(s_i)}$ is generated by the elements of the form m/s_i^n with $m \in M_n$.

$$\mathscr{O}_X(n)(U_i) = s_i^n|_{U_i}\mathscr{O}_X(U_i).$$

$$\mathscr{F}(n)(U_i) = \mathscr{F}(U_i) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)(U_i)$$

since U_i is affine. Then for any $m \in M_n(X)$, we have a unique $f_i \in \mathscr{F}(U_i)$ such that $m|_{U_i} = f_i \otimes s_i^n$. Define $\varphi_{s_i}(m/s_i^m) = f_i$. Then we have a morphism

$$\varphi_{s_i} : (\Gamma_*(\mathscr{F}))^\sim(U_i) \to \mathscr{F}(U_i).$$

We want to show φ_{s_i} is an isomorphism.

- φ_{s_i} is injective: Apply Lemma 7.12 (i) to $s = s_i$, $U_i = X_{s_i}$, and $\mathscr{L} = \mathscr{O}_X(1)$.
- φ_{s_i} is surjective: Apply Lemma 7.12 (ii), there exists $n > 0$, $h \in \mathscr{F} \otimes (\mathscr{O}_X(1))^n(X) = \mathscr{F}(n)(X)$ such that for any $f_i \in \mathscr{F}(U_i)$, $h|_{U_i} = f_i \otimes S_i^n|_{U_i}$.

Again, since U_i is affine, by Lemma 7.17, we have

$$\widetilde{\varphi_{s_i}} : (\Gamma_*(\mathscr{F}))^\sim|_{U_i} \xrightarrow{\sim} \mathscr{F}|_{U_i}.$$

We can glue $\widetilde{\varphi_{s_i}}$ to get $\tilde{\varphi}$. $\square$

Theorem 7.19. Let A be a ring, $S = A[T_0, \dots, T_n]$, $X = \operatorname{Proj} S$. Then all closed subscheme Z of X is of the form $\operatorname{Proj} S/I$, where $I \trianglelefteq S$ is a homogeneous ideal. In particular, all projective scheme over A is of the form $\operatorname{Proj} T$ where T is a graded algebra of finite type over A.

Proof. Let $i : Z \hookrightarrow X$ be a closed immersion, $\mathscr{O}_X \to i_*\mathscr{O}_Z$ is surjective. Let $\mathscr{I}$ be the sheaf of ideals associated to Z which is quasi-coherent. Let $I = \Gamma_*(\mathscr{I})$. We have

$$0 \to \mathscr{I} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$$

after tensoring

$$0 \to \mathscr{I}\mathscr{O}_X \hookrightarrow \mathscr{O}_X(n).$$

Taking global sections, we have

$$0 \to \mathscr{I}\mathscr{O}_X(n)(X) \to \mathscr{O}_X(n)(X).$$

Since $I_d = \mathscr{I}\mathscr{O}_X(n)(X) \subset S_d$, I is a homogeneous ideal. By Theorem 7.18, $\tilde{I} \simeq \mathscr{I}$. By homework, we know $Z = \operatorname{Proj} S/I$. $\square$

Proposition 7.20. Let A be a ring and $Y = \mathbb{P}^n_A = \operatorname{Proj}(A[T_0, \dots, T_n])$. Let X be an A-scheme

(i) Let $f : X \to Y$ be a morphism of A-schemes. Then $f^*\mathscr{O}_Y(1)$ is an invertible sheaf on X and is generated by $d+1$ global sections.

(ii) Conversely, if $\mathscr{L}$ is an invertible sheaf on X generated by $s_0, \dots, s_d$, then there is a unique A-morphism $f : X \to Y$ such that $\iota : \mathscr{L} \xrightarrow{\sim} f^*\mathscr{O}_Y(1)$ and $f^*(T_i) - \iota(s_i)$.

Proof.

(i)
$$T_i \in \mathscr{O}_Y(1)(Y) \to \varinjlim_{V \supset f(X)} \mathscr{O}_Y(1)(V).$$

Let $\widetilde{T}_i$ be the image of T_i in $f^*\mathscr{O}_Y(1) = f^{-1}\mathscr{O}_Y(1) \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$. Call $\widetilde{T}_i \otimes 1$ by s_i. Then

$$f^*\mathscr{O}_Y(1)_x = \mathscr{O}_Y(1)_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{O}_{X,x},$$

$\{s_i|_x\}$ generates $f^{-1}\mathscr{O}_Y(1)_x$.

(ii) As $s_0, \dots, s_d$ generates $\mathscr{L}$ on X, $X = \bigcup_{i=0}^d X_{s_i}$. For any i, define

$$f_i : X_{s_i} \to D_+(T_i) = \operatorname{Spec} A[T_j/T_i]$$

by considering
$$f_i^\# : A[T_j/T_i] \to \mathscr{O}(X_{s_i}) \text{ by } T_j/T_i \mapsto a_{ji}$$
where $s_j = a_{ji}s_i$ (since s_i generates $\mathscr{L}$ on X_{s_i}). On the intersection $X_{s_i} \cap X_{s_\ell}$,

$$A[T_j/T_\ell]_{T_i/T_\ell} = A[T_j/T_i]_{T_\ell/T_i} \to \mathscr{O}(X_{s_i} \cap X_{s_\ell}).$$

Since $s_j = a_{ji}s_i$, $s_j = a_{j\ell}s_\ell$, $s_\ell = a_{\ell i}s_i$, we see that $a_{ji} = a_{j\ell}a_{\ell i}$. Hence $f_i^\#$, $f_\ell^\#$ induces same map on the intersection, f_i can be glued to get $f : X \to Y$.

□

Remark. f is a closed immersion if $\mathscr{O}_X(X_{s_i})$ is generated by a_{ji}.

Definition.

(i) An **immersion of schemes** is a morphism which is an open immersion followed by a closed immersion.

(ii) A morphism of schemes $f : X \to \operatorname{Spec} A$ is said to be **quasi-projective** if there is an immersion $i : X \to \mathbb{P}^d_A$ such that $f = \pi \circ i$ where $\pi : \mathbb{P}^d_A \to A$.

Lemma 7.21. Let $f : X \to Y$ be a morphism of schemes. Suppose that f is the composition of a closed immersion followed by an open immersion, and f is quasi-compact, then f is an immersion.

Definition.

(i) Let $f : X \to \operatorname{Spec} A$ be a morphism of schemes. We say that an invertible sheaf $\mathscr{L}$ on X is **very ample (relative to f)** if there is an immersion $i : X \to \mathbb{P}^d_A$ of A-schemes such that $\mathscr{L} \simeq \mathscr{O}_X(1) := i^*\mathscr{O}_{\mathbb{P}^d_A}(1)$.

(ii) If X is quasi-compact, $\mathscr{L}$ is an invertible sheaf on X. We say that $\mathscr{L}$ is **ample** if for all finitely generated quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there is $n_0 \geqslant 1$ such that for any $n \geqslant n_0$, $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections.

Remark. If X is a projective scheme, then $\mathscr{L}$ is very ample would imply ample.

Lemma 7.22. Let X be a Noetherian or quasi-compact and separable[3] scheme, $\mathscr{L}$ is an ample sheaf on X. Then for any $x \in X$, there exists $n > 0$ (depend on x) and a section $s \in \mathscr{L}^n(X)$ such that X_s is an affine open neighborhood of X.

Proof. Let $U \subsetneq X$ be an affine open neighborhood of x such that $\mathscr{L}|_U$ is free. Consider $C = X \backslash U$ with the reduced structure. Let $\mathscr{I}$ be the sheaf of ideal of $i : C \hookrightarrow X$. Note that if $x \in U$, $\mathscr{I}_x = \mathscr{O}_{X,x}$. We have

$$0 \to \mathscr{I} \to \mathscr{O}_X$$

after tensoring,

$$0 \to \mathscr{I} \otimes \mathscr{L}^n \to \mathscr{O}_X \otimes \mathscr{L}^n \simeq \mathscr{L}^n,$$

i.e. $\mathscr{I}\mathscr{L}^n \subset \mathscr{L}^n$. Since $\mathscr{L}$ is ample, there is n such that $(\mathscr{I}\mathscr{L}^n)$ is generated by global sections. There exists $s \in \mathscr{I}\mathscr{L}^n(X) \subset \mathscr{L}^n(X)$ such that $s_x\mathscr{O}_{X,x} = (\mathscr{I}\mathscr{L}^n)_x \simeq (\mathscr{L}^n)_x$. Define

$$X_s := \{x \in X \mid s_x\mathscr{O}_{X,x} = (\mathscr{L}^n)_x\}.$$

We now claim that $X_s \subset U$. Indeed, for any $y \in X_s$,

$$(\mathscr{L}^n)_y = s_y\mathscr{O}_{X,y} \subset (\mathscr{I}\mathscr{L}^n)_y.$$

Hence $\mathscr{I}_y = \mathscr{O}_{X,y}$, which implies that $y \in U$.
Now,

$$\mathscr{L}|_U = \mathscr{O}|_U \cdot e$$

for some $e \in \mathscr{L}(U)$. Write $s|_U = h \otimes e^n$. Then $X_s = X_s \cap U = D(h) \subset U$. $\square$

Lemma 7.23. Let X be a Noetherian or a quasi-compact and separated scheme. Let $\mathscr{L}$ be an invertible sheaf on X.

(i) Suppose there are $s_1, \ldots, s_r \in \mathscr{L}(X)$ such that X_{s_i} is affine for all i and $X = \bigcup_{i=1}^r X_{s_i}$. Then $\mathscr{L}$ is ample.

[3]For this case, see Appendix A.

(ii) Let U be an open quasi-compact subscheme of X. Then $\mathscr{L}$ is ample implies $\mathscr{L}|_U$.

Proof.

(i) Let $\mathscr{F}$ be a finitely generated quasi-coherent sheaf on X. Then $\mathscr{F}|_{X_{s_i}} = (\mathscr{F}(X_{s_i}))^\sim$. Since $\mathscr{F}$ is finitely generated, there exists $f_{i,j}$ generates $\mathscr{F}(X_{s_i})$. By Lemma 7.12 (ii), there exists $g_{ij} \in (\mathscr{F} \otimes \mathscr{L}^n)(X)$ such that

$$g_{ij}|_{X_{s_i}} = f_{ij} \otimes_i^n .$$

Then $\mathscr{F} \otimes \mathscr{L}^n$ is generated by g_{ij}.

(ii) Since $\mathscr{L}$ is ample, by Lemma 7.22, there exists $s_i \in \mathscr{L}^n(X)$ with X_{s_i} affine and $X = \bigcup_{i=1}^n X_{s_i}$ since X is quasi-compact. Wrtie $U = \bigcup_{\ell=1}^r V_\ell$ with V_ℓ affine open. Since X is Noetherian or seoarated, $V_\ell \cap X_{s_i}$ can be covered by finitely many $D(h_{ij}) \subset X_{s_i}$ where $h_{ij} \in \mathscr{O}(X_{s_i})$. Then

$$U \cap X_{s_i} = \bigcup_{j=1}^{n_i} D(h_{ij}).$$

Again by Lemma 7.12 (ii), there exists $t_{ij} \in \mathscr{O}_X \otimes (\mathscr{L}^n)^m(X)$ such that $t_{ij}|_{X_{s_i}} = h_{ij} \otimes s_i^m$. Then

$$X_{t_{ij}} = D(h_{ij}) = U_{s_{ij}}$$

where $s_{ij} = t_{ij}|_U$. By (i), $\mathscr{L}|_U$ is ample.

□

Theorem 7.24. Let $f : X \to \operatorname{Spec} A$ be a morphism of finite type. Suppose that X is Noetherian or f is separated. Let $\mathscr{L}$ be an ample sheaf on X. Then there is $m > 0$ such that $\mathscr{L}^m$ is very ample.

Proof. Since f is quasi-compact, X is quasi-compact. By Lemma 7.22 and X is quasi-compact, there exists n, $s_i \in \mathscr{L}^n(X)$ such that $X = \bigcup_{i=1}^r X_{s_i}$ with X_{s_i} being affine open. Since f is of finite type, $\mathscr{O}(X_{s_i})$ is a finitely generated A-algebra, write

$$\mathscr{O}(X_{s_i}) = A[f_{ij}], \quad f_{ij} \in \mathscr{O}(X_{s_i})$$

By Lemma 7.22, there exists $g_{ij} \in \mathscr{L}^{nm}(X)$ such that $g_{ij}|_{X_{s_i}} = f_{ij} \otimes s_i^m$. Note that $\{s_i^m, g_{ij}\} \subset \mathscr{L}^{mn}(X)$. Define

$$\pi X \to \operatorname{Proj} A[S_i, T_{ij}]_{i,j}$$

with $\pi^{-1}(D_+(S_i)) = X_{s_i}$. On the ring level

$$A[S_i, T_{ij}] \to \mathscr{O}_X(X_{s_i})$$

is surjective since $T_{ij}/S_i \mapsto f_{ij}$. Hence $\pi|_{\pi^{-1}(D_+(S_i))}$ is a closed immersion.

$$X \xrightarrow{\pi} \bigcup_i D_+(S_i) \hookrightarrow \operatorname{Proj} A[S_i, T_{ij}]$$

with first morphism is closed immersion and second morphism is open immersion. Also,

$$\pi^* \mathscr{O}_{\operatorname{Proj} A[S_i, T_{ij}]}(1) = \mathscr{L}^{nm}.$$

Hence $\mathscr{L}^{nm}$ is very ample. $\square$

Corollary 7.25. Let $X \to \operatorname{Spec} A$ be a morphism as in Theorem 7.24. Then $X \to \operatorname{Spec} A$ is quasi-projective if and only if there is an ample sheaf on X.

Proof.

- ($\Rightarrow$) There exists $\iota : X \to Y$ open immersion with Y projective. Then by Lemma 7.23 (ii), $\iota^* \mathscr{O}_Y(1)$ is ample.
- ($\Leftarrow$) By Theorem 7.24.

$\square$

8 Divisors

8.1 Meromorphic functions and associated primes

Definition. Let A be a ring, M be an A-module, $x \in M$.

$$\mathrm{Ann}(x) := \{a \in A \mid ax = 0\}.$$

Associated prime ideals of M are $\mathrm{Ass}(M) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} = \mathrm{Ann}(x) \text{ for some } x \in M\}$.

Fact. Let A be a Noetherian ring. Then

(i) $\mathrm{Ass}(M) \neq \emptyset$ for $M \neq 0$.

(ii) $\mathrm{Ass}_{S^{-1}A}(S^{-1}M) = \{S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}_A(M) \text{ and } \mathfrak{p} \cap S = \emptyset\}$.

(iii) The set of zero divisors of $A = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(A)} \mathfrak{p}$.

(iv) The minimal primes of A are in $\mathrm{Ass}\, A$.

Fact.

(i) If A is reduced, then the set of zero divisors is contained in the union of minimal primes.

(ii) The elements of minimal prime are zero divisors.

Fact. If A is a Noetherian ring, M is a finitely generated A-module, then $\mathrm{Ass}(M)$ is finite.

Fact. If $x \in M$ with $\mathrm{Ann}(x)$ is a prime $\mathfrak{p}$, then $\mathrm{Ass}(xA) = \{\mathfrak{p}\}$.

Note that if A is Noetherian and reduced, then $\mathrm{Ass}(A)$ are minimal primes.

Definition. Let X be a locally Noetherian scheme.

$$\mathrm{Ass}(\mathscr{O}_X) := \{x \in X \mid \mathfrak{m}_x \in \mathrm{Ass}_{\mathscr{O}_{X,x}}(\mathscr{O}_{X,x})\}$$

If $X = \operatorname{Spec} A$, $\mathrm{Ass}(\mathscr{O}_X) = \mathrm{Ass}(A)$. If $U \subset X$ is an open subset, then

$$\mathrm{Ass}(\mathscr{O}_X|_U) = \mathrm{Ass}(\mathscr{O}_X) \cap U.$$

Lemma 8.1. Let X be a locally Noetherian scheme, $U \subset X$ be a nonempty open subset, $\iota : U \hookrightarrow X$ canonical open immersion. Then $\mathscr{O}_X \to \iota_* \mathscr{O}_X|_U$ is injective if and only if $\mathrm{Ass}(\mathscr{O}_X) \subset U$.

Proof. We check on stalks, we may assume $X = \operatorname{Spec} A$.

- ($\Leftarrow$) It suffices to show that $\mathscr{O}_X(X) \to \mathscr{O}_X(U)$ is injective. Let $0 \neq a \in A = \mathscr{O}_X(X)$ with $a|_U = 0$. Then $\mathrm{Ass}(aA) \neq \emptyset$. Let $\mathfrak{p} \in \mathrm{Ass}(aA)$ with $\mathfrak{p} = \mathrm{Ann}(ab)$, then $\mathfrak{p} \in \mathrm{Ass}(A) \subset U$. But $a|_U = 0$ implies $a_{\mathfrak{p}} = 0$ in $A_{\mathfrak{p}}$. There exists $h \in A \backslash \mathfrak{p}$ such that $ha = 0$ which is a contradiction.

- ($\Rightarrow$) Let $\mathfrak{p} = \operatorname{Ann}(a) \in \operatorname{Ass}(A)$. If $\mathfrak{p} \notin U$, then $\mathfrak{p}\mathscr{O}_{X,x} = \mathscr{O}_{X,x}$ for any $x \in U$. Hence $a_x = 0$ for any $x \in U$, and $a|_U = 0$.

$\square$

Definition. Let X be a scheme. For any $U \subset X$, define

$$\mathscr{R}_X(U) := \{a \in \mathscr{O}_X(U) \mid a_x \text{ is not a zero divisor in } \mathscr{O}_{X,x}\ \forall x \in U\},$$

$$\mathscr{K}'_X(U) := \mathscr{R}_X(U)^{-1}\mathscr{O}_X(U),$$

and $\mathscr{K}_X$ be the sheafification of $\mathscr{K}'_X$.

Lemma 8.2.

(i) $\mathscr{R}_X(U) = R(\mathscr{O}_X(U))$ for affine open subset U.

(ii) $\mathscr{K}'_X(U) \to \prod_{x\in U} \mathscr{K}'_{X,x}$ is injective.

(iii) If X is locally Noetherian, then for any $x \in X$, $\mathscr{K}_{X,x} = \mathscr{K}'_{X,x} = \operatorname{Frac}(\mathscr{O}_{X,x})$.

Proposition 8.3. Let X be a locally Noetherian scheme, $U \subset X$ be an open subset containing $\operatorname{Ass}(\mathscr{O}_X)$. Let $\iota : U \to X$ be the canonical immersion. Then $\mathscr{K}_X \to \iota_*\mathscr{K}_U$ is an isomorphism.

Proof. By Lemma 8.1, $\mathscr{O}_X \to \iota_*\mathscr{O}_U$ is injective. Hence $\mathscr{K}'_X \to \iota_*\mathscr{K}'_U$ is injective, and hence $\mathscr{K}'_X \to \iota_*\mathscr{K}_U$ is injective.
For surjectivity, we first show the surjective of $\mathscr{K}'_X \to \iota_*\mathscr{K}'_U$. We can assume $X = \operatorname{Spec} A$. We check that $\mathscr{K}'_X(X) \to \mathscr{K}'_X(U)$ is surjective.
Let $I \trianglelefteq A$ such that $X\backslash U = V(I)$. By assumption $V(I) \cap \operatorname{Ass}(A) = \varnothing$. As A is Noetherian, $\operatorname{Ass}(A)$ is finite. By prime avoidance $I \not\subset \bigcup_{\mathfrak{p}_i\in\operatorname{Ass}(A)} \mathfrak{p}$. There exists $a \in I\backslash \bigcup_{\mathfrak{p}_i\in\operatorname{Ass}(A)} \mathfrak{p}_i$ (a is not a zero divisor). Then

$$U \supset D(a) \supset \operatorname{Ass}(A),$$

$$\mathscr{O}_X \hookrightarrow \iota_*\mathscr{O}_U \hookrightarrow \iota_*\mathscr{O}_{D(a)},$$

$$\mathscr{K}'_X \hookrightarrow \iota_*\mathscr{K}'_U \hookrightarrow \iota_*\mathscr{K}'_{D(a)}.$$

But $\mathscr{K}'_X(A) = \operatorname{Frac} A$ and $\mathscr{K}'_X(A_a) = \mathscr{K}'_X(A)$ since $D(a) \supset \operatorname{Ass}(A)$ implies $\operatorname{Ass}(A_a) = \operatorname{Ass}(A)$. Then $\mathscr{K}'_X(X) = \iota_*\mathscr{K}'_{D(a)}(X)$ implies $\mathscr{K}'_X(X) = \iota_*\mathscr{K}'_U(X) = \mathscr{K}'_X(U)$.
Now, let $s \in \mathscr{K}_X(U)$, there exists an affine open cover $\{U_i\}$ of U such that $s|_{U_i} = a_i/b_i \in \mathscr{K}'_X(U_i)$ where $a_i \in \mathscr{O}_X(U_i)$, $b_i \in R(\mathscr{O}_X(U_i))$. Then for any $\mathfrak{p} \in \operatorname{Ass}(\mathscr{O}_{U_i})$, $b_i \notin \mathfrak{p}$.

$$U_i \supset D(b_i) \supset \operatorname{Ass}(\mathscr{O}_{\mathscr{U}}).$$

Define $V = \bigcup_i D(b_i) \supset \operatorname{Ass}(\mathscr{O}_X)$. $s|_{V_i} \in \mathscr{O}_X(V_i)$. Then $s|_V \in \mathscr{O}_X(V) \subset \mathscr{K}'_X(V)$. By the surjectivity of prsheaves, we have $t \in \mathscr{K}'_X(X)$ such that $t|_V = s|_V$. Then $(t|_U)|_V = s|_V$. By injectivity of $\mathscr{K}_X \hookrightarrow \iota\mathscr{K}_U$, $t|_U = s$. $\square$

8.2 Cariter divisors

Definition. Let X be a scheme.

(i) $\mathrm{Div}(X) := \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$ is called **Cartier divisors**. $(\mathrm{Div}(X), +)$ is a group.

(ii) The image of $\Gamma(X, \mathscr{K}_X^*) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$ is called **principal Cartier divisor**.

(iii) $D_1 \sim D_2$ if and only if $D_1 - D_2$ is a principal divisor.

(iv) A Cartier divisor D is **effective** if the image of $\Gamma(X, \mathscr{O}_X \cap \mathscr{K}_X^*) \to \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*)$.

(v) $\mathrm{CaCl}\, X = \mathrm{Div}(X)/\sim$.

Let $X = \bigcup_i U_i$ with U_i open. Then a Cartier divisor can be written as $D = \{(U_i, f_i)\}_i$, with $f_i = a_i/b_i$ for $a_i \in \mathscr{O}_X(U_i)$ and $b_i \in \mathscr{R}_X(U_i)$ such that

$$f_i|_{U_i \cap U_j} \left(f_j|_{U_i \cap U_j}\right)^{-1} \in \mathscr{O}_X(U_i \cap U_j)^*.$$

Definition.

i Let X be a scheme. The **Picard group** is the isomorphism classes of invertible sheaves on X.

ii A Cartier divisor $D = \{(U_i, f_i)\}$ can associate to a sheaf $\mathscr{O}_X(D)|_{U_i} := f_i^{-1}\mathscr{O}_X|_{U_i} \subset \mathscr{K}_X|_U$

Proposition 8.4. Let X be a scheme.

(i) $\rho : D \mapsto \mathscr{O}_X(D)$ is additive, i.e. $\rho(D_1 + D_2) = \rho(D_1) \otimes_{\mathscr{O}_X} \rho(D_2)$.

(ii) ρ is injective $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$.

(iii) $\mathrm{Im}(\rho)$ is invertible sheaf contained in $\mathscr{K}_X$.

Proof.

(i) This is easy.

(ii) Principal divisors are in the kernel is clear. If $\rho(D) = 0$, i.e. $i : \mathscr{O}_X(D) \xrightarrow{\sim} \mathscr{O}_X$, let $f \in \mathscr{O}_X(D)$ such that $i(f) = 1$. Write $D = \{(U_i, f_i)\}$, then f and f_i^{-1} both generate $\mathscr{O}_X(D)|_{U_i}$. Hence $f \cdot f_i^{-1} \in \mathscr{O}_X^*(U_i)$, we can replace (U_i, f_i) by (U_i, f^{-1}). Hence D is principal.

(iii) Let $\mathscr{L}$ be an invertible sheaf contained in $\mathscr{K}_X$. Let $X = \bigcup_i U_i$, $\mathscr{L}|_{U_i} \simeq \mathscr{O}_X|_{U_i}$ as $\mathscr{O}_X$-module. Choose $f_i \in \mathscr{L}(U_i) \subset \mathscr{K}_X(U_i)$ such that $\mathscr{L}|_{U_i} = f_i\mathscr{O}_X|_{U_i}$, write $f_i = a_i/b_i$ as before. Then

$$\{(U_i, f_i^{-1})\} \in \Gamma(X, \mathscr{K}_X^*/\mathscr{O}_X^*).$$

Then

$$f_i^{-1} \cdot f_j|_{U_i \cap U_j} \in \mathscr{O}_X^*(U_i \cap U_j).$$

□

Corollary 8.5. Let X be a Noetherian scheme without embedded point (i.e. all associated primes are minimal (e.g. reduced)). Then $\rho : \mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism.

Proof. $\mathrm{Im}(\rho)$ is invertible sheaf is contained in $\mathscr{K}_X$. It suffices to show for any invertible sheaf $\mathscr{L}$. there exsits $i : \mathscr{L} \xrightarrow{\sim} \mathscr{L}' \subset \mathscr{K}_X$.
Since X is Noetherian, write $X = \bigcup_{i=1}^r X_i$ where X_i are irreducible components. Let ξ_i be the generic point of X_i, and $\{\xi_i\}_{i=1}^r = \mathrm{Ass}(\mathscr{O}_X)$. Let $U_i = X \backslash \bigcup_{i \neq j} X_j$, with $\xi_i \in U_i$. Shrink U_i with $\xi_i \in U_i$ such that $\mathscr{L}|_{U_i} \simeq \mathscr{O}_X|_{U_i}$.
Let $U = \bigcup_{i=1}^r U_i$, we have $\mathscr{L}|_U \simeq \mathscr{O}_X|_U$. By Lemma 8.1, we have $\mathscr{L} \hookrightarrow (\mathscr{L}|_U) \simeq \iota_* \mathscr{O}_X|_U \subset \iota_* \mathscr{K}_X|_U = \mathscr{K}_X$ where the last equality is by Proposition 8.3. □

Proposition 8.6. Let X be a quasi-projective over a Noetherian affine scheme. Then $\rho : \mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ is an isomorphism.

Proof. There is an affine open neighborhood $U \subset X$ of $\mathrm{Ass}(X)$. Then

$$f : X \xhookrightarrow{open} Y \xhookrightarrow{\text{closed}} \mathbb{P}^n_A$$

Then $Y \backslash X$ is closed in $\mathbb{P}^n_A$. There exists a homogeneous ideal I such that $V_+(I) = Y \backslash X$. Then $V_+(I) \cap \mathrm{Ass}(\mathscr{O}_X) = \varnothing$. Since A is Noetherian, there is $a \in I \backslash \bigcup_{\mathfrak{p} \in \mathrm{Ass}(\mathscr{O}_X)} \mathfrak{p}$.

$$V_+(a) \sup V_+(I) = Y \backslash X.$$

Then $D_+(a) \supset \mathrm{Ass}(\mathscr{O}_X)$. Denote $U = D_+(a)$ be an affine open set. Let $\mathscr{L}$ be an invertible sheaf on X with $\mathscr{L}|_U = (\mathscr{L}(U))^\sim$.
Let $B = R(\mathscr{O}_X(U))^{-1}\mathscr{O}_X(U) = \mathscr{K}_X'(U)$. Write $\mathcal{R} = R(\mathscr{O}_X(U))^{-1}$, and $\mathcal{N} := \mathcal{R}^{-1}\mathscr{L}(U)$. We want to show there is $\mathrm{Ass}(\mathscr{O}_X) \subset V \subset U$ such that $\mathscr{L}|_V$ is trivial. Since $\mathscr{O}_X(U) \to B$, $\widetilde{\mathcal{N}} = \iota^*(\mathscr{L}|_U)$ where $\iota : \mathrm{Spec}\, B \to U$. We claim that $\widetilde{\mathcal{N}}$ is trivial bundle on $\mathrm{Spec}\, B$.
Note that $\mathrm{Spec}\, B = \mathrm{Ass}(\mathscr{O}_X(U))$. Since A is Noetherian, we have $\mathrm{Spec}\, B$ is a fintie set. Then there are only finitely many maximal ideal $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$.

$$B/\cap \mathfrak{m}_i \simeq \bigoplus_{i=1}^r B/\mathfrak{m}_i$$

Then

$$\mathcal{N}/\mathcal{N} \cap_i \mathfrak{m}_i \simeq \bigoplus_{i=1}^r \mathcal{N}/\mathfrak{m}_i\mathcal{N} \simeq \bigoplus_{i=1}^r B/\mathfrak{m}_i = B/\cap \mathfrak{m}_i.$$

There is $\nu \in \mathcal{N}$, $\mathcal{N} = B\nu + I\mathcal{N}$ where $I = \cap_i \mathfrak{m}_i$. Then by Nakayama's lemma, we have $N = B\nu$. This shows the claim.

$$\iota^*(\mathscr{L}|_U) \simeq \mathscr{O}_{\mathrm{Spec}\, B}.$$

There exists $t \in R(\mathscr{O}_X(U))$ such that $V = D(t) \subset U$. Then $\mathscr{L}|_V = \mathscr{O}_V$ and $\mathrm{Ass}(\mathscr{O}_X) = \mathrm{Ass}(\mathscr{O}_U) \subset D(t)$ for $t \in R(\mathscr{O}_X(U))$. □

8.3 Weil divisor

Definition. Let X be a Noetherian scheme.

(i) A **prime cycle on** X is an irreducible closed subscheme of X.

(ii) A **cycle** is an element in $\mathbb{Z}^{(X)}$, i.e. $\sum_{x\in X} n_x\overline{\{x\}}$, almost all n_x is zero. n_x is called the **multiplicity at** x.

(iii) $\mathrm{Supp}(Z) := \bigcup_{n_x\neq 0}\overline{\{x\}}$, $Z = \sum_{x\in X} n_X\overline{\{x\}}$.

(iv) $Z \geqslant 0$ if and only if $n_x \geqslant 0$ for any $x \in X$. For any Z, we can write $Z = Z_0 - Z_\infty$ with $Z_0 \geqslant 0$ and $Z_\infty \geqslant 0$ where $Z_0 = \sum n_{x_i}\overline{\{x_i\}}$ and $Z_\infty = \sum n_{y_i}\overline{\{y_i\}}$ with $\{x_i\} \cap \{y_j\} = \varnothing$.

(v) A cycle Z is of **codimension 1**, denoted by $Z^1(X)$, if each component in $\mathrm{Supp}(Z)$ is of codimension 1.

Example. Let $X = \mathbb{P}^n_k$ where k is a field. Then $Z^1(X) = \bigoplus \mathbb{Z}[V_+(P)]$ where P is a homogeneous irreducible polynomial.

Definition. Let X be a Noetherian integral scheme. A cycle of codimension 1 is called a **Weil divisor**.

Lemma 8.7. Let X be a Noetherian integral scheme, and $f \in \mathscr{K}_X(X)^\times$. Then all but finitely many $x \in X$ of codimension 1, $f \in \mathscr{O}^*_{X,x}$.

Proof. Let $f = a/b$ for some open subset $U \subset X$, and $a, b \in \mathscr{O}_X(U)$. Let

$$S = \{V(a) \cup V(b) \cup X\backslash U\} = \bigcup_{i=1}^{r} C_i$$

where C_i are distinct irreducible component. There are only finitely many prime of codimension 1 in S. Hence $x \notin S$ would imply $f \in \mathscr{O}^*_{X,x}$. □

Let X be a normal Noetherian scheme, and $x \in X$ be codimension 1 point. We know $\mathscr{O}_{X,x}$ is a DVR. Then for any $f \in \mathscr{K}(X)^*$, we can define **multiplicity of** f **at** x by $\nu_x(f)$, and define

$$(f) := \sum_{\substack{x\in X\\ \dim \mathscr{O}_{X,x}}} \nu_x(f)\overline{\{x\}}$$

which is called a **principal divisor in** $Z^1(X)$. Define $(f) + (g) = (fg)$ and $Z_1, Z_2 \in Z^1(X)$, if $Z_1 \sim Z_2$ are **linearly equivalent** if and only if $Z_1 - Z_2 = (f)$ for some $f \in \mathscr{K}(X)^*$. We define

$$\mathrm{Cl}(X) = Z^1(X)/\sim.$$

Example. Let K be a number field, $\mathscr{O}_K$ be its ring of integers. Let $D \in Z^1(X)$ where $X = \operatorname{Spec} \mathscr{O}_K$. Then we can write

$$D = \sum_{i=1}^{r} n_i \mathfrak{p}_i$$

where $\mathfrak{p}_i$ is a prime in $\mathscr{O}_K$. Then D corresponds to the ideal $\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i^{n_i}$. In this sense, $\mathrm{Cl}(X)$ is the class group in the sense of algebraic number theory.

Proposition 8.8. Let A be a Noetherian domain. Then A is a UFD if and only if $X = \operatorname{Spec} A$ is normal and $\mathrm{Cl}(X) = 0$.

Lemma 8.9. Let A be a Noetherian domain. Then A is a UFD if and only if every prime of height 1 is principal.

Proof.

- $(\Rightarrow)$: This is easy.
- $(\Leftarrow)$: Choose $f \in A$ with f irreducible. Then minimal prime $\mathfrak{p} \supset (f)$ is height one. Then $\mathfrak{p} = (h)$. We have

  $$f = h \cdot f_2$$

  for some f_2. Since f is irreducible, f_2 is a unit, and $(f) = \mathfrak{p}$.

□

Proof of Proposition 8.8. For $(\Leftarrow)$.
Since A is a normal Noetherian ring, $A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p})=1}} A_{\mathfrak{p}}$. By definition, if $\mathfrak{p} \in \operatorname{Spec} A$ for $\operatorname{ht}(\mathfrak{p}) = 1$, then $\overline{\{\mathfrak{p}\}} \in Z^1(\operatorname{Spec} A)$. Then there is $f \in \operatorname{Frac}(A)$ such that $(f) = \overline{\{\mathfrak{p}\}}$. For any $\mathfrak{q} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{q}) = 1$,

$$\nu_{\mathfrak{q}}(f) = \begin{cases} 0 & \text{if } \mathfrak{q} \neq \mathfrak{p} \\ 1 & \text{if } \mathfrak{q} = \mathfrak{p}. \end{cases}$$

Hence $f \in \operatorname{Frac}(A_{\mathfrak{q}})$, and $f \in A$. We now claim that $\mathfrak{p} = (f)$.
Choose $g \in \mathfrak{p} \subset A$.

$$\nu_{\mathfrak{q}}(g/f) \geqslant 0$$

for all $\mathfrak{q} \neq \mathfrak{p}$.

$$\nu_{\mathfrak{p}}(g/f) \geqslant 0$$

since $g \in \mathfrak{p}$. Hence $g/f \in A$, and $g \in (f)$, then we can conclude by Lemma 8.9.
Conversely, UFD is normal is clear. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\operatorname{ht}(\mathfrak{p}) = 1$. Then $\mathfrak{p} = (h)$ for some irreducible $h \in A$. Let $D \in Z^1(\operatorname{Spec} A)$, write $D = \sum_{i=1}^{n} n_i \overline{\{\mathfrak{p}_i\}}$ with $\operatorname{ht}(\mathfrak{p}) = 1$ and $\mathfrak{p}_i = (h_i)$. Then $D = (f)$ where $f = \prod_i h_i^{n_i}$. □

Proposition 8.10. Let k be a field and $X = \mathbb{P}^n_k$. Define $\delta : \mathrm{Cl}(\mathbb{P}^n_k) \to \mathbb{Z}$ as the following. Let $x \in X$ be a point of codimension 1. Then $\overline{\{x\}} = V_+(P)$ for some P homogeneous irreducible in $k[T_0, \ldots, T_n]$, then define

$$\delta\left(\sum_{x:\text{ codim. } 1} n_x \overline{\{x\}}\right) = \sum n_x \deg(P_x) \in \mathbb{Z}.$$

Then δ is an isomorphism and any hyperplane in X generates $\mathrm{Cl}(X)$.

Proof. We first check that δ is well-defined. Let $f \in \mathscr{K}(X)$, we claim that $\delta((f)) = 0$. Write

$$f = \frac{\prod_{i=1}^r h_i^{\alpha_i}}{\prod_{j=1}^s g_j^{\beta_j}}$$

where $h_i, g_j \in k[T_0, \ldots, T_n]$ irreducible distinct homogeneous and $\deg(f) = 0$. Then

$$(f) = \sum \alpha_i D_+(h_i) - \sum \beta_j D_+(g_j).$$

Then $\delta(f) = \sum \alpha_i \deg h_i - \sum \beta_j \deg g_j = 0$. Hence $D \in \ker \delta$.
Surjectivity is clear. To show injectivity, let $D = \sum n_i V_+(P_i)$ with $\sum n_i \deg P_i = 0$. Let $f = \prod_i P_i^{n_i}$, then $\deg(f) = 0$, $f \in \mathscr{K}(X)$ with $(f) = D$. □

Proposition 8.11. Let X be a normal Noetherian scheme, $f \in \mathscr{K}(X)^*$, $U \subset X$ is an open subset. Then

(i) $f \in \mathscr{O}_X(U)$ if and only if $U \cap \mathrm{Supp}(f)_\infty = \varnothing$.

(ii) $x \in X$ is of codimension 1. Then $f \in \mathfrak{m}_x \mathscr{O}_{X,x}$ if and only if $x \in \mathrm{Supp}(f)_0$.

Example. $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.

From now on, let $(\star)$ be the property that Noetherian, integral, separated, regular in codimension 1.

Proposition 8.12. Let X be as $(\star)$. Let $Z \subsetneq X$ be a closed subscheme of X, and $p \in U := X \backslash Z$. Then

(i) There exists a surjective homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ defined by $D = \sum_i n_i Y_i \mapsto \sum n_i (Y_i \cap U)$.

(ii) If $\mathrm{codim}(Z, X) \geqslant 2$, then $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism.

(iii) If Z is irreducible of codimension 1, then we have the exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

where the first one is given by $1 \mapsto Z$.

Proof. Let $\overline{\{\eta\}} = Y \in Z^1(X)$ be a prime cycle. Then $Y \cap U$ is either empty set or is codimension 1 in U. Let $f \in \mathscr{K}(X)^* = \mathscr{O}^*_{U,\xi} = \mathscr{K}(U)^*$ where ξ is a generic point of X. Then $(f|_U) = \sum_i n_i (Y_i \cap U)$. Hence $\pi : \mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is well-defined. It is surjective since for any $Z_i \subset U$, $\overline{Z_i} \mapsto Z_i$ and $\overline{Z_i} \cap U = Z_i$. Since $\mathscr{O}_{X,\eta_i} = \mathscr{O}_{U,\eta_i}$ for $\eta_i \in U$, hence the second statement is true.
Foe the third one, if $D \in \ker \pi$, write $D = \sum n_i Y_i$ with $Y_i \cap U = \varnothing$. Then $Y_i \subset Z$ and $Y_i = Z$ since Y_i is of codimension 1. □

Example. Let Y be an irreducible curve of degree d in $\mathbb{P}^2_k$, i.e. $Y = V_+(P)$ for some P is irreducible of degree d. Then $V_+(\mathbb{P}^2_k) \simeq \mathbb{Z}/d\mathbb{Z}$. Since

$$\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^2_k) \to \mathrm{Cl}(\mathbb{P}^2_k \backslash Y) \to 0$$

and $\mathrm{Cl}(\mathbb{P}^2_k) \simeq \mathbb{Z}$ by $Y \mapsto d$.

Example. Let k be a field, $A = k[x, y, z]/(xy - z^2)$, and $X = \operatorname{Spec} A$. X satisfies $(\star)$. Then $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$. Consider $Y = (y, z) = \mathfrak{p}$ which is codimension 1. Then $\mathfrak{p}\mathscr{O}_{X,\mathfrak{p}} = (z)\mathscr{O}_{X,\mathfrak{p}}$. $\nu_{\mathfrak{p}}(y) = 2$ for $y \in A$. Note that $X\backslash Y = \operatorname{Spec} A_y$. Then for any $\mathfrak{q} \in \operatorname{Spec} A_y$, we have $\nu_{\mathfrak{q}}(y) = 0$. Hence $(y) = 2Y$. On the other hand, $A_y = k[y, z]_y$ is a UFD, hence $\mathrm{Cl}(X\backslash Y) = 0$. Therefore, Y is a generator of $\mathrm{Cl}(X)$. Finally, we claim that $\mathfrak{p} \neq (f)$ for some $f \in \mathrm{Frac}(A)$. But A is normal, $f \in A$. Consider $\mathfrak{m} = (x, y, z) \trianglelefteq A$, $\mathfrak{m}/\mathfrak{m}^2 = (\bar{x}, \bar{y}, \bar{z})$. Then $(\bar{y}, \bar{z})$ is contained in the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$.

Proposition 8.13. Suppose X satisfies $(\star)$. Then

(i) $X \times_{\mathbb{Z}} \mathbb{A}^1$ satisfies $(\star)$.

(ii) $\mathrm{Cl}(X) = \mathrm{Cl}(X \times \mathbb{A}^1)$.

Proof.

(i) Noetherian, integral, separated are clear. We check that $X \times_{\mathbb{Z}} \mathbb{A}^1$ is regular at codimension 1 point x. Let

$$\pi : X \times \mathbb{A} \to X$$

consider the two cases:

- $\pi(x)$ is of codimension 1: Choose an affine open neighborhood U contains $\pi(x) = y$ and y associated to the prime ideal $\mathfrak{p}_y$. Then $\mathscr{O}_{X\times\mathbb{A}^1,x} = (\mathscr{O}_X[t])_{\mathfrak{p}_y\mathscr{O}_X[t]}$ is a DVR since $\mathscr{O}_{X,\mathfrak{p}_y}$ is a DVR. Hence it is regular at x.
- $\pi(x)$ is the generic point of X: $\mathscr{O}_{X\times\mathbb{A}^1,x} = K[t]_{\mathfrak{p}_x}$ for some $\mathfrak{p}_x = (f)$ where $f \in K[t]$ and $K = \mathrm{Frac}(A)$. Then $\mathscr{O}_{X\times\mathbb{A}^1,x}$ is a DVR. Hence it is regular at x.

(ii) Set $\pi^* : \pi(X) \to \pi(X \times \mathbb{A}^1)$ defined by

$$\sum n_i Y_i \mapsto \sum n_i \pi^{-1}(Y_i).$$

- π^* is injective: If $\pi^*(D) = (f)$ for $f \in \mathrm{Frac}(X \times \mathbb{A}^1) = K(t)$. Image of π^* consists of points where $\pi(x)$ is codimension 1. Then $f \in K$, otherwise, $f = h/g$ with $h, g \in K[t]$, h, g coprime. At irreducible factor ℓ of h, g, $\nu_\ell(f) \neq 0$. Then (f) has support in the point with $\pi(x)$ is generic point in X. Hence $D = (f)$.

- π^* is surjective: Let $Z \in Z^1(X \times \mathbb{A}^1)$ be a prime cycle. Let z be the point with $\pi(z)$ is generic point in X. Then
$$\mathscr{O}_{X\times\mathbb{A}^1, z} = K[t]_{\mathfrak{p}}$$
where $\mathfrak{p} \neq 0$ is a prime in $K[t]$. Since $K[t]$ is PID, $\mathfrak{p} = (f)$. Write
$$f = \frac{h}{g}, \quad h \in A[t],\ g \in A$$
where $A = \mathscr{O}_X(U)$ for U is an affine open of X. Then $\nu_{\mathfrak{p}}(f) = 1$. For any $\mathfrak{q} \neq \mathfrak{p} \trianglelefteq K[t]$, $\nu_{\mathfrak{q}}(f) = 0$. Then
$$(f) = \mathfrak{p} + \sum n_i \mathfrak{q}_i$$
where $\pi(\mathfrak{q}_i)$ is codimension 1 and $\pi(\mathfrak{p})$ is generic point of X.

□

8.4 Pic(X), CaCl(X), Cl(X)

Let X be a Noetherian, integral, (separated), all local rings are UFD.

Let $D = \{(U_i, f_i)\}_{i=1}^n$, $f_i \in \mathscr{K}(X)^* \in \mathrm{CaCl}(X)$. Let $x \in U_i$ be of codimension 1. Then $\mathscr{O}_{X,x}$ is a DVR, $\mathrm{Frac}\, \mathscr{O}_{X,x} = K(X)$, there exists $\nu_x : K(X) \to \mathbb{Z}$.

Then define $[D] = \sum_{n_x} \overline{\{x\}}$ where the sum is over all codimension 1 point, $n_x := \nu(f_i)$ if $x \in U_i$. If $x \in U_i \cap U_j$, then $f_i/f_j \in \mathscr{O}_X(U_i \cap U_j)^*$. Hence $\nu_x(f_i/f_j)$. Hence we construct a morphism
$$i : \mathrm{CaCl}(X) \to \mathrm{Cl}(X).$$

Proposition 8.14. The morphism i defined above is an isomorphism.

Proof.

- Surjective: $Y = \overline{\{x\}}$, $x \in X$ is of codimension 1. Equip Y with the reduced structure, and let $\mathscr{I}_Y$ be a sheaf of ideals of Y. Let $x \in Y$, $\mathscr{O}_{X,x}$ is a UFD. $\mathscr{I}_{Y,x}$ is height 1 in $\mathscr{O}_{X,x}$, $\mathscr{I}_{X,x} = (h_x)$ for some $h_x \in \mathscr{O}_{X,x}$. Moreover, if $x \in U_i = \mathrm{Spec}\, A_i \subset X$, $\mathscr{I}_Y(U_i) = I_i$ for some prime of height 1. In this case, $I_{X,x} = I_i(A_i)_{I_i}$. There exists an affine open neighborhood U_x of x such that $I_i\mathscr{O}_X(U_x) = (h_x)$. If $U_i \cap Y = \varnothing$, then choose $f_i = 1$. Define $D = \{(U_x, h_x)\}$ to be the Cartier divisor since if $U_{x_1} \cap U_{x_2} \neq \varnothing$. Then
$$\mathscr{I}_Y|_{U_{x_1}\cap U_{x_2}} = h_{x_1}\mathscr{O}_X|_{U_{x_1}\cap U_{x_2}} = h_{x_2}\mathscr{O}_X|_{U_{x_1}\cap U_{x_2}},$$
which means that $h_{x_1}h_{x_2}^{-1} \in \mathscr{O}_X(U_{x_1} \cap U_{x_2})^*$.

  For $x \in X \backslash Y$, x is of codimension 1, then $\nu_x(1) = 0$. For $x \in Y$, by construction $\nu_x(h_x) = 1$ with x is of codimension 1. Then x is the generic point of Y. Hence $[D] = \overline{\{x\}}$.

- Injective: Let $D \in \mathrm{Div}(X)$ be a Cariter divisor. Write $D = \{(U_i, f_i)\}$ with $U_i = \operatorname{Spec} A_i$ affine open. Since local rings of X are UFD, A_i is normal. Then $[D] \geqslant 0$ implies that

$$f_i \in \bigcap_{\substack{\mathrm{ht}(\mathfrak{p})=1 \\ \mathfrak{p}\in \operatorname{Spec} A_i}} A_{i,\mathfrak{p}} = A_i.$$

Then $[D] = 0$ if and only if $[D] \geqslant 0$ and $[-D] \geqslant 0$. This shows that $f_i^{-1} \in A_i$. Hence $f_i \in A_i^{\times}$, and $D = 0$.

$\square$

Corollary 8.15. If X satisfies $(\star)$, then $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$.

Corollary 8.16. Let $X = \mathbb{P}^n_k$. Every invertible sheaf on X is isomorphic to $\mathscr{O}_X(\ell)$, for some $\ell \in \mathbb{Z}$.

Proof. Since $\mathrm{Cl}(X) \simeq \mathbb{Z}$, then Corollary 8.15 implies $\mathrm{Pic}(x) \simeq \mathrm{Cl}(X) \simeq \mathbb{Z}$. $\square$

Example. Let $Q := V_+(z_0z_3 - z_1z_2) \subset \mathbb{P}^3_k$. We know $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We first claim that $\mathrm{Cl}(Q) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection. Then

$$\begin{tikzcd} \pi_1^* : \mathrm{Cl}(\mathbb{P}^1) \arrow[r] & \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \\ x \arrow[r, mapsto] & \{x\} \times \mathbb{P}^1 \end{tikzcd}$$

$$\mathrm{Cl}(\mathbb{P}^1) \xrightarrow{\pi_2^*} \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{res} \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1)$$

where

$$x \mapsto \mathbb{P}^1 \times \{x\} \mapsto \mathbb{A}^1 \times \{x\}.$$

Since this is an isomorphism, π_2^* is injective. Similarly, π_1^* is injective as well.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r] & \mathrm{Cl}(\mathbb{A}^1 \times \mathbb{P}^1) \arrow[r] & 0 \\ & 1 \arrow[r, mapsto] & Y & \mathrm{Cl}(\mathbb{P}^1) \arrow[ul, "\pi_2^*"] \arrow[u, "\simeq"'] & \end{tikzcd}$$

Hence the claim is proved.

Let $f : Q \to \mathbb{P}^3$ be the natural closed immersion. We want to define the pullback $f^* : \mathrm{Cl}(\mathbb{P}^3) \to \mathrm{Cl}(Q) \simeq \mathbb{Z} \oplus \mathbb{Z}$ where $\mathrm{Cl}(\mathbb{P}^3) = \mathbb{Z}H$ for some hyperplane H. Let $Y = \overline{\{x\}}$ where $x \in \mathbb{P}^3$ of codimension 1. Then $x = V_+(P)$ for some P homogeneous irreducible in $k[z_0, \ldots, z_3]$. If $x \notin Q$, let $\overline{P}$ be the image of P in $k[z_0, \ldots, z_3]/z_0z_3 - z_1z_2$. Define $Y.Q = \sum_{\substack{x\in Q \\ x \text{ is codimension 1 in } Q}} n_x \overline{\{x\}}$ where $n_x := \nu_x(\overline{P})$. Extend by linearity. Let $D_1, D_2 \in Z^1(\mathbb{P}^3)$ with $\mathfrak{q} \notin \mathrm{Supp}(D)$ where $\mathfrak{q}$ is the generic point of Q with $D_1 \sim D_2$. Then $D_1 - d_2 = (f)$ with $\overline{f} \neq 0$ in $k[z_0, \ldots, z_3]/(z_0z_3 - z_1z_2)$. Then

$$D_1.Q - D_2.Q = (\overline{f}).$$

Hence $D_1.Q \sim D_2.Q$ in $Z^1(Q)$. Hence $\mathrm{Cl}(\mathbb{P}^2) \simeq \mathbb{Z}$.

There exists a homogeneous irreducible P_1 of degree 2 with $P_1 \neq z_0z_3 - z_1z_2$. If $Y = Q$, define $Y.Q := V_+(P_1).Q$. Suppose there is another P_2 satisfies the properties as P_1, then let $f = P_1/P_2$, we have $\mathfrak{q} \notin \mathrm{Supp}(f)$, and hence $V_+(P_1).Q \sim V_+(P_2).Q$. In this way,

$$f^* : \mathbb{Z} \simeq \mathrm{Cl}(\mathbb{P}^3) \to \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

by $H \mapsto (1, 1)$.

Let C be the curve define by $[t^3 : t^2u : tu^2 : u^3]$ which is clearly contained in Q. The question is that is there another hypersurface S such that the C is cut by S and Q.

Suppose there exists a closed irreducible of codimesion 1 $S \subset \mathbb{P}^3$ such that $S.Q = f^*(S) = C$, then C is type of (a, a). Note that

$$C \subset V_+(z_0z_3 - z_1z_2,\, z_1z_3 - z_2^2) = L \cup C$$

where $L = (z_2 = z_3 = 0)$. Let $Y := V_+(z_1z_3 - z_2^2)$, then

$$f^*(Y) = Y.Q = L + C$$

where $Y.Q = (2, 2)$ and $L = (1, 0)$, hence $C = (1, 2)$ is not of type (a, a). That is, there does not exist such S such that $S.Q = C$.

8.5 Divisor and Curves

Let k be a field, C be a curve over k.

Proposition 8.17. Let X be a proper regular curve over k, Y be a curve over k. $f : X \to Y$ be a morphism. Then either

(i) $f(X)$ is a point.

(ii) $f(X) = Y$ and $K(X)$ is a finite extension of $K(Y)$, and f is finite.

Proof. Since X is proper, Y is separated, f is proper, $f(X)$ is closed in Y. On the other hand, X is irreducible, $f(X)$ is irreducible. Hence $f(X)$ is a point or $f(X) = Y$. Since $K(Y) \hookrightarrow K(X)$ both are of transcendence degree 1, it is a finite extension. Now, let $V = \mathrm{Spec}\, A \subset Y$, we claim that $f^{-1}(V) = \mathrm{Spec}\, B$ where B is the integral closure of A in $K(X)$.

Note that a closed point $p \in X$ if and only if p is height one. Hence $\mathscr{O}_{X,p}$ is a DVR, any DVR contained in $K(X)$ is $\mathscr{O}_{X,p}$ since X is proper. Since B is finitely generated A-module and A, B are domain, we know $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective. Let $x \in f^{-1}(V)$, $p = f(x)$. Then

$$A \subset \mathscr{O}_{Y,p} \subset \mathscr{O}_{X,x} \subset K(X).$$

Since $\mathscr{O}_{X,x}$ is integrally closed, $B \subset \mathscr{O}_{X,x}$.

$$\begin{tikzcd} \mathrm{Spec}\, \mathscr{O}_{X,x} \arrow[r] \arrow[rr, bend left] & \mathrm{Spec}\, B \arrow[r, hook] & X \end{tikzcd}$$

Since X is proper, $x \in \mathrm{Spec}\, B$, hence $f^{-1}(V) = \mathrm{Spec}\, B$. $\square$

Definition. If $f : X \to Y$ is a finite morphism of regular curves over a field k. Define

$$f^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$$

by $\{(V_i, g_i)\} \mapsto \{(f^{-1}(V_i), f^\#(g_i))\}$ where $f^\# : \mathscr{O}_Y \hookrightarrow \mathscr{O}_X$ can be extended to $K(Y) \hookrightarrow K(X)$.

(In fact, for any integral k-variety X, Y and any finite dominant morphism f, we can define $\mathrm{CaCl}(Y) \xrightarrow{f^*} \mathrm{CaCl}(X)$.)

In Weil divisor, $\overline{\{x\}}$ where x is of codimension 1 in Y, let $U = \operatorname{Spec} A$ be an affine open neighborhood of x, then x corresponds to a height 1 prime $\mathfrak{p} \in \operatorname{Spec} A$. Since Y is regular, $\mathfrak{p}A_\mathfrak{p} = tA_\mathfrak{p}$ for some $t \in \mathfrak{p}$. Then there exists $\mathfrak{q} \in \operatorname{Spec} B$, $f(\mathfrak{q}) = \mathfrak{p}$.

$$f^*(\overline{\{x\}}) = \sum_{\substack{\mathfrak{q} \in \operatorname{Spec} B \\ f(\mathfrak{q}) = \mathfrak{p}}} \nu_\mathfrak{q}(f^\#(t)B_\mathfrak{q})\{\mathfrak{q}\}.$$

Then we have $\mathrm{Cl}(Y) \xrightarrow{f^*} \mathrm{Cl}(X)$.

Proposition 8.18. Let $f : X \to Y$ be a finite morphism of regular curves over a field k. Then for any $D \in Z^1(X)$, $\deg f^*D = \deg f \deg D$ where $\deg f := [K(X) : K(Y)]$ and $D = \sum_{i=1}^n n_i \overline{\{x_i\}} \in Z^1(X)$, $\deg D := \sum_i n_i [\kappa(x_i) : k]$.

Proof. Let $V = \operatorname{Spec} A \subset Y$ be an affine open subset, and $f^{-1}(V) = \operatorname{Spec} B$ is also affine where B is an integral closure of A in $K(X)$. Then B is a Dedekind domain. It suffices to check the statement of the form $D = \overline{\{y\}}$.

Let $\mathfrak{p} \in \operatorname{Spec} A$ be the corresponding prime to the point y, and let $t \in \mathfrak{p}$ such that $\mathfrak{p}A_\mathfrak{p} = (t)$. By definition

$$f^*(\overline{\{y\}}) = \sum_{f(\mathfrak{q} = \mathfrak{p})} \nu_\mathfrak{q}(f^\#(t))\overline{\{\mathfrak{q}\}}.$$

$$\deg f^*(\overline{\{y\}}) = \sum \nu_\mathfrak{q}(f^\#(t))[\kappa(\mathfrak{q}) : k].$$

Since B is Dedekind, $\mathfrak{p}B$ can be factorized

$$\mathfrak{p}B = \prod_i Q_i^{e_i}$$

with $Q_i \in \operatorname{Spec} B$ with $\mathrm{ht}(Q_i) = 1$. Then

$$B/\mathfrak{p}B = \prod_i B/Q_i^{e_i}B$$

is a vector space over $A/\mathfrak{p}$ of dimension $[K(X) : K(Y)]$.

On the other hand,

$$f^*(\{\mathfrak{p}\}) = \sum_i e_i \overline{\{Q_i\}},$$

$$\deg(f^*\{\mathfrak{p}\}) = \sum e_i[\kappa(Q_i) : k] = \sum e_i[\kappa(Q_i) : \kappa(\mathfrak{p})][\kappa(\mathfrak{p}) : k].$$

Comparing two sides of the equality, they are same. $\square$

A Remark on Lemma 7.22

Proposition A.1. Let X be quasi-compact separated, $U \subset X$ be a quasi-compact open subset of X, and $\mathscr{G}$ be quasi-coherent $\mathscr{O}_X$-module. For any finitely generated quasi-coherent sheaf $\mathscr{F}' \subset \mathscr{G}|_U$, there exists a quasi-coherent sheaf $\mathscr{F} \subset \mathscr{G}$ on X.

Proof. Define

$$\overline{\mathscr{F}} := \{s \in \mathscr{G}(V) \mid s|_{U \cap V} \in \mathscr{F}'(U \cap V)\}.$$

This is a sheaf: consider $\iota : U \hookrightarrow X$ be a canonical open immersion which is quasi-compact and separated. Then $\iota_* \mathscr{F} \subset \iota_*(\iota^* \mathscr{G})$ is quasi-coherent. Recall from homework that we have a canonical morphism $\mathscr{G} \to \iota_*(\iota^* \mathscr{G})$. Hence $\overline{\mathscr{F}}$ is the preimage of $\iota_* \mathscr{F}$ in $\mathscr{G}$. Hence $\overline{F}$ is quasi-coherent and $\overline{F}|_U = F'$. We consider the two cases

- $X = \operatorname{Spec} A$, $\overline{F} = (\overline{F}(X))^{\sim}$, write $N = \overline{F}(X)$. Then $N = \varinjlim_\lambda N_\lambda$ with N_λ are finitely generated A-module. Define $\overline{\mathscr{F}_\lambda} := \tilde{N_\lambda}$. There exists λ such that $\tilde{N_\lambda}|_U = \mathscr{F}'$. Then $\mathscr{F} = \tilde{N_\lambda}$.
- If $X = \operatorname{Spec} A_1 \cup \operatorname{Spec} A_2$. By case 1, there exists a finitely generated quasi-coherent $\mathscr{F}_1$ on $\operatorname{Spec} A_1 = U_1$ such that $\mathscr{F}|_{U_1 \cap U} = \mathscr{F}'|_{U_1 \cap U}$. $\mathscr{F}_1$ and $\mathscr{F}$ can be glued to a sheaf on $U_1 \cap U$, called $\mathscr{F}_1'$. On $U_2 = \operatorname{Spec} A_2$, consider $(U_1 \cap U) \cap U_2$. Again, by case 1, we have $\mathscr{F}_2$ on U_2.

  $$\mathscr{F}_2|_{(U_1 \cap U) \cap U_2} = \mathscr{F}_1'|_{(U_1 \cap U) \cap U_2}.$$

  Then $\mathscr{F}_2$, and $\mathscr{F}_1'$ can be glued to finitely generated quasi-coherent $\mathscr{F}_2' \subset \mathscr{G}$ on $U_1 \cup U_2 \cup U = X$.

□

Corollary A.2. Let X be quasi-compact separated shceme. Then every quasi-coherent $\mathscr{O}_X$-module is filtered inductive limit of its quasi-coherent subsheaf of finite type.

We now turn to the proof of Lemma 7.22, consider $\mathscr{I}\mathscr{L}^n \subset \mathscr{L}^n$. Let $s \in \mathscr{I}\mathscr{L}^n(X)$ such that $s|_x \mathscr{O}_{X,x} = \mathscr{L}_x^n$. $\mathscr{L}_x$ generated by s_x. There exists $t \in \mathscr{I}(U)$ with U being an affine open neighborhood of U such that $t\mathscr{O}|_U \subset \mathscr{L}|_U$. Then by Corollary A.2, there is an finitely generated quasi-coherent sheaf $\mathscr{I}'$ such that $\mathscr{I}'|_U = t\mathscr{O}|_U$.