Positive Mass Theorem

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1 Introduction

The positive mass theorem (or positive energy theorem) is a well-known result in general relativity. Roughly speaking, it states that the energy of an isolated system is nonnegative, and it is zero when there is no objects. Mathematically, we can formulate it as follows. Let (M, g, k) be a tuple of three data, where (M, g) is a n-dimensional Riemannian manifold with metric g and k is a symmetric two tensor, we will called such (M, g, k) an initial data set. We say that a Riemannian manifold (M, g) is asymptotically flat if ∂M has positive mean curvature with respect to the outer normal and there exists a compact subset $K \subset M$ such that any connected component of $M \backslash K$ is diffeomorphic to $\mathbb{R}^n \backslash B$ where B is a ball, and the a diffeomorphism $\Phi_i : N_i \to \mathbb{R}^n \backslash B$ can be seen as a chart, in this chart, we have

$$g_{ij}(x) = \delta_{ij} + O\left(|x|^{-p}\right) \quad \text{for some } p > 0,$$

where N_i is a connected component of $M \backslash K$. Intuitively, we may see the Riemannian manifold M "tends to" the Euclidean space (which is flat) as infinity. For a given initial data set (M, g, k), we define two quantity,

$$\mu = \frac{1}{2}\left(R_g + (\mathrm{Tr}_g\, k)^2 - |k|_g^2\right), \quad J = \mathrm{div}_g\, k - d(\mathrm{Tr}_g\, k)$$

where R_g is the scalar curvature of (M, g). We say an initial data set satisfies the dominant energy condition if $\mu \geqslant |J|$. Moreover, we define ADM energy E and linear momentum P by

$$E = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} \sum_i (g_{ij,i} - g_{ii,j})\nu^j \, dA,$$

$$P_i = \lim_{r\to\infty} \frac{1}{(n-1)\omega_{n-1}} \int_{S_r} (k_{ij} - (\mathrm{Tr}_g\, k) g_{ij})\nu^j \, dA$$

where ν is the outer unit normal of the sphere S_r and dA is its area element and ω_{n-1} is the volume of S_1. Here, the integrals are evaluated under the diffeomorphism $N_i \simeq \mathbb{R}^n \backslash B$. Then the general positive mass theorem is

Theorem 1.1. *Let (M, g, k) be a complete asymptotically flat initial data set satisfying the dominant energy condition $\mu \geqslant |J|$ with $3 \leqslant n < 8$. Then $E \geqslant |P|$.*

Another statement is when $n = 3$, we have

Theorem 1.2. *Let (M, g, k) be a complete asymptotically flat initial data set satisfying the dominant energy condition $\mu \geqslant |J|$. Suppose that $E = |P|$, then $E = |P| = 0$ and (M, g, k) arises as a spacelike slice of Minkowski spacetime $\mathbb{R}^{3,1}$. More precisely, M can be isometrically embedded into Minkowski space and g is the induced metric and k is the second fundamental form.*

2 Some Special Cases

2.1 Overview

In the report, we may not show Theorem 1.1, we can see some special cases instead. We will consider the case that $k \equiv 0$ be a zero tensor, it is said to be time-symmetric case, or Riemannian case. In this case, $P = 0$ by its definition, and the Theorem 1.1 becomes $E \geqslant 0$. Also, the dominant energy condition will become $R_g \geqslant 0$. Moreover, we assume that the $n = 3$ at first. Then this is actually the case that standard positive mass theorem proved by Richard Schoen and Shing-Tung Yau in two papers [1, 2], and we will only focus on this case. We now give a complete setting and the statement.

Let N be an asymptotically flat oriented three-dimensional manifold with $N \backslash K$ has k connected components $N_1, \ldots, N_k$. On each N_i, there is a coordinate x^1, x^2, x^3, and its metric tensor has following conditions

$$g_{ij} = \left(1 + \frac{M}{2r}\right)^4 \delta_{ij} + \ell_{ij}. \tag{2.1}$$

where ℓ_{ij} is the error term satisfies

$$|\ell_{ij}| \leqslant \frac{k_1}{1+r^2} = O(r^{-2}), \quad |\nabla \ell_{ij}| \leqslant \frac{k_2}{1+r^3} = O(r^{-3}), \quad |\nabla\nabla \ell_{ij}| \leqslant \frac{k_3}{1+r^4} = O(r^{-4}). \tag{2.2}$$

Here, ∇ means the gradient in Euclidean space. And on each N_i, we have a constant $M = M_i$ in the above which we call total mass of N_i. We now give a breif computation to derive the estimate of g, $\sqrt{g}$, $\frac{1}{g}$, $\frac{1}{\sqrt{g}}$, g^{ij}, Γ^k_{ij}, $R_{ijk\ell}$ where $g = \det(g_{ij})$ as usual. Observe that the main term of g is given by the product of diagonal of (g_{ij}). Hence we have

$$g = \left(1 + \frac{M}{2r}\right)^{12} + O(r^{-2})$$

and

$$\sqrt{g} = \left(1 + \frac{M}{2r}\right)^{6} + O(r^{-2}).$$

For $\frac{1}{g}$,

$$\frac{1}{g} = \frac{1}{(1+\frac{M}{2r})^{12} + O(r^{-2})} = \frac{1}{(1+\frac{M}{2r})^{12}} \frac{1}{1 + \frac{O(r^{-2})}{(1+\frac{M}{2r})^{12}}}.$$

Since $\lim_{r\to\infty}\left(1+\dfrac{M}{2r}\right)^{12}=1$, $(1+\frac{M}{2r})^{12}$ has a lower bound for r large. Observe that for $|z|\leqslant B$, we have

$$\frac{1}{1+B}\leqslant\frac{1}{1+z}\leqslant\frac{1}{1-B}.$$

Hence

$$\frac{1}{g}=\frac{1}{(1+\frac{M}{2r})^{12}}\left(1-\frac{\frac{O(r^{-2})}{(1+\frac{M}{2r})^{12}}}{1+\frac{O(r^{-2})}{(1+\frac{M}{2r})^{12}}}\right)=\frac{1}{(1+\frac{M}{2r})^{12}}\left(1-\frac{O(r^{-2})}{(1+\frac{M}{2r})^{12}}\right)=\frac{1}{(1+\frac{M}{2r})^{12}}+O(r^{-2}).$$

For the similar argument, we get that

$$\frac{1}{\sqrt{g}}=\frac{1}{(1+\frac{M}{2r})^{6}}+O(r^{-2}).$$

From the definition of g^{ij}, we can easy see that

$$g^{ij}=\frac{(-1)^{i+j}}{g}E_{ij}=\frac{\delta_{ij}}{(1+\frac{M}{2r})^{4}}+O(r^{-2})$$

where E is the adjugate matrix of (g_{ij}). Next,

$$\Gamma^k_{ij}=\frac{1}{2}g^{k\ell}(g_{i\ell,j}+g_{j\ell,i}-g_{ij,\ell}).$$

For $i\neq j$, $g_{ij,\ell}=O(r^{-3})$ from (2.2) directly. For $i=j$,

$$g_{ii,\ell}=-4\left(1+\frac{M}{2r}\right)^{3}\frac{x^j}{r^3}+O(r^{-3})=O(r^{-2}).$$

There result implies that

$$\Gamma^k_{ij}=O(r^{-2}).$$

Finally, using the local expression,

$$R_{ijk\ell}=(\Gamma^m_{jk,i}-\Gamma^m_{ik,j}+\Gamma^s_{jk}\Gamma^m_{is}-\Gamma^s_{ik}\Gamma^m_{js})g_{m\ell}$$

we can see that $R_{ijk\ell}=O(r^{-3})$. Now we give the statement of result.

Theorem 2.1. *Let $(N,\ g)$ as above. If the scalar curvature $R\geqslant 0$, then the total mass $M_i\geqslant 0$.*

The first key thing is to show the total mass M_i is equals to ADM energy so that Theorem 2.1 is the special case of Theorem 1.1. Since the dimension is three now,

$$E=\frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\sum_i(g_{ij,i}-g_{ii,j})\,\nu^j\,dA.$$

Note that we use the Einstein summation convention so the sum is summing over index j as well. By definition, $\nu^j = \frac{x^j}{r}$. Using (2.1) and noting that the term $i = j$ will cancel out, we get

$$\begin{aligned} E &= \frac{1}{16\pi} \lim_{r\to\infty} \sum_{i\neq j} \left(\ell_{ij,i} - \ell_{ii,j} - \frac{\partial}{\partial x^j} \left(1 + \frac{M}{2r}\right)^4 \right) \frac{x^j}{r} \, dA \\ &= \underbrace{\frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i\neq j} (\ell_{ij,i} - \ell_{ii,j}) \frac{x^j}{r} \, dA}_{\text{(I)}} + \underbrace{\frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} \sum_{i\neq j} 2M \left(1 + \frac{M}{2r}\right)^3 \frac{x^{j2}}{r^4} \, dA}_{\text{(II)}} \end{aligned}$$

For (I), by (2.2), we have

$$\left| \sum_{i\neq j} (\ell_{ij,i} - \ell_{ii,j}) \frac{x^j}{r} \right| = O(r^{-3}).$$

Hence,

$$\int_{S_r} \sum_{i\neq j} (\ell_{ij,i} - \ell_{ii,j}) \frac{x^j}{r} \, dA = O(r^{-1})$$

which implies that (I)$= 0$.
For (II), observe that

$$\sum_{i\neq j} x^{j2} = 2r^2.$$

Hence

$$\text{(II)} = \frac{M}{4\pi} \lim_{r\to\infty} \int_{S_r} \left(1 + \frac{M}{2r}\right)^3 \frac{1}{r^2} \, dA = \frac{M}{4\pi} \lim_{r\to\infty} 4\pi \left(1 + \frac{M}{2r}\right)^3 = M.$$

This shows that the ADM energy is equal to the total mass. Now we start to prove Theorem 2.1.

2.2 Proof of Theorem 2.1

For the proof, we will focus on a fix connected component N_i. By the assumption, N_i is diffeomorphic to $\mathbb{R}^3 \backslash B_{\sigma_0}(0)$ and write its coordinates x^1, x^2, x^3 with Euclidean length $r = |x|$ of $x = (x^1, x^2, x^3)$. We assume that $M < 0$ and $R \geqslant 0$. We will get a contradiction by following three steps. The first step is to show we may assume that $R > 0$ outside a compact set of $\mathbb{R}^3 \backslash B_{\sigma_0}(0)$. The second step is to show the existence of a complete area minimizing surface according to the assumption that $M < 0$. The third step is to show the surface in the second step cannot exist if $R \geqslant 0$.

Step 1

We want to show that there is another metric $\widetilde{g}_{ij}$ also satisfies (2.1), (2.2) which is conformal to the original metric g_{ij} such that its scalar curvature $\widetilde{R} \geqslant 0$ on N, $\widetilde{R} > 0$ outside a compact set of N_i and also has negative total mass for N_i.

Proof. Recall that the Laplacian of a function f is given by

$$\Delta f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}f\right).$$

From the estimate of $\frac{1}{\sqrt{g}}$, $\sqrt{g}$ and g^{ij}, we have

$$\begin{aligned}\Delta\frac{1}{r} &= \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\left(1+\frac{M}{2r}\right)^2\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\right) + O(r^{-5}) \\ &= \left(1+\frac{M}{2r}\right)\frac{M}{r^4} + O(r^{-5}) = \frac{M}{r^4} + O(r^{-5}).\end{aligned} \tag{2.3}$$

From this, there is a number $\sigma > \sigma_0$ so that

$$\Delta\frac{1}{r} < 0 \text{ for } r \geqslant \sigma.$$

Let $t_0 = -\dfrac{M}{8\sigma_0}$ and $\zeta(t)$ be a C^5 function from $(0, \infty)$ satisfies

$$\begin{aligned}&\zeta(t) = \begin{cases} t & \text{for } t < t_0 \\ \dfrac{3t_0}{2} & \text{for } t > 2t_0, \end{cases} \\ &\zeta'(t) \geqslant 0, \quad \zeta''(t) \leqslant 0 \text{ for } t \in (0, \infty).\end{aligned} \tag{2.4}$$

Define a C^5 function $\varphi : N \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} 1 + \dfrac{3t_0}{2}, & \text{if } x \in N\backslash N_i, \\ 1 + \zeta\left(-\dfrac{M}{4r}\right), & \text{if } x \in N_i = \mathbb{R}^3\backslash B_{\sigma_0}(0). \end{cases}$$

We compute $\Delta\varphi$ on N_i directly.

$$\nabla\varphi = \zeta'\left(-\frac{M}{4r}\right)\nabla\left(-\frac{M}{4r}\right).$$

Since $\operatorname{div}(fX) = \langle\nabla f, X\rangle + f\operatorname{div}X$. So,

$$\Delta\varphi = \operatorname{div}(\nabla\varphi) = \zeta''\left(-\frac{M}{4r}\right)\left\langle\nabla\left(-\frac{M}{4r}\right), \nabla\left(-\frac{M}{4r}\right)\right\rangle + \zeta'\left(-\frac{M}{4r}\right)\Delta\left(-\frac{M}{4r}\right)$$

Hence, we have

$$\Delta\varphi \leqslant 0 \text{ on } N, \text{ and } \Delta\varphi < 0 \text{ for } r > 2\sigma. \tag{2.5}$$

We now define the new metric

$$\widetilde{g}_{ij} = \varphi^4 g_{ij}.$$

It is not hard to see that $\tilde{g}_{ij}$ also (2.1), (2.2). For the connected component other than N_i, it just a constant scalar. For N_i, we have $r \geqslant \sigma_0$, whence $-\frac{M}{4r} \leqslant 2t_0$. Then the construction of φ yields

$$\begin{aligned}\tilde{g}_{ij} &= \left(1 - \frac{M}{4r}\right)^4 \left(1 + \frac{M}{2r}\right)^4 \delta_{ij} + O(r^{-2}) \\ &= \left(1 + \frac{M}{4r}\right)^4 \delta_{ij} + O(r^{-2})\end{aligned}$$

where the last equation follows from the coefficient of r^{-1} and constant are same. According to this expression, we find that the total mass of N_i under the metric $\tilde{g}_{ij}$ is $\widetilde{M} = \frac{M}{2} < 0$. By the formula of scalar curvature under conformal change (see Appendix A.2), we have (by taking $e^{2u} = \varphi^4$ in the formula)

$$\tilde{R} = \varphi^{-5} \left(-8\Delta\varphi + R\varphi\right).$$

Then from (2.5), we know $\tilde{R} \geqslant 0$ on N and $\tilde{R} > 0$ for $r > 2\sigma$ on N_i as desired. □

From now on, we will assume that the metric g_{ij} satisfies $R \geqslant 0$ on N, $R > 0$ outside a compact set of N_i and $M < 0$.

Step 2

We want to show that there exists a complete area minimizing surface S properly embedded in N so that $S \cap (N \backslash N_i)$ is compact, and $S \cap N_i$ lie between two parallel Euclidean 2-planes in 3-space defined by x^1, x^2, x^3.

Proof. Let $\sigma > 2\sigma_0$, and C_σ be the circle of radius σ centered at 0 in x^1x^2-plane. Let S_σ be the smooth embedded oriented minimal surface with boundary C_σ (We will assume the existence and this is the result using geometric measure theory, see [5]). We will find a sequence $\sigma_i \to \infty$ such that S_{σ_i} converges to the surface S we want. We need to check two things: $S_\sigma \cap (N \backslash N_i)$ is compact and $S \cap N_i$ lie between two planes.

(i) Since $N \backslash N_i$ is the union of connected components and a compact set, $S_\sigma \cap (N \backslash N_i)$ is a closed subset. To show that $S_\sigma \cap (N \backslash N_i)$ is compact, it suffices to show that there is a compact subset $K_0 \subset N$ such that

$$S_\sigma \cap (N \backslash N_i) \subset K_0 \text{ for all } \sigma > 2\sigma_0. \tag{2.6}$$

But we only need to show that $S_\sigma \cap N_j$ is bounded in N_j for all $j \neq i$ by Heine-Borel theorem. To see this, let y^1, y^2, y^3 be the coordinate associating N_j with $\mathbb{R}^3 \backslash B_{\tau_0}(0)$ satisfying (2.1), (2.2). Recall that the Hessian of a function f is defined by

$$\nabla^2_{ij} f = \frac{\partial^2 f}{\partial y^i \partial y^j} - \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} f.$$

We now compute the Hessian of $|y|^2$, it is clear that $\frac{\partial^2 |y|^2}{\partial y^i \partial y^j} = 2\delta_{ij}$. For the second term,

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} |y|^2 = 2 \sum_k \Gamma^k_{ij} y^k = O(|y|^{-1}).$$

We see that

$$\nabla^2_{ij} |y|^2 = 2\delta_{ij} + O(|y|^{-1}).$$

This shows that there exists $\tau_1 > \tau_0$ such that $|y|^2$ is a convex function for $|y| > \tau_1$. The boundary of S_σ is $C_\sigma \subset N_i$ by definition. We have $\partial(S_\sigma \cap N_j) \subset C_\sigma \cup \partial N_j$ with $\partial N_j = \partial B_{\tau_0}(0)$. Applying maximal principle, we conclude that

$$S_\sigma \cap N_j \subset B_{\tau_1}(0)$$

which shows $S_\sigma \cap N_j$ is bounded.

(ii) For any $h > 0$, we define

$$E_h = \{x \in \mathbb{R}^3 : |x^3| \leqslant h\}.$$

We will show that there exists $h > \sigma_0$ such that

$$N_i \cap S_\sigma \subset E_h \text{ for all } \sigma > 2\sigma_0. \tag{2.7}$$

First of all, we compute the Hessian of x^3 to get the estimate of it.

$$\nabla^2_{ij} x^3 = -\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right) x^3 = -\left(\Gamma^k_{ij} \frac{\partial}{\partial x^k}\right) x^3 = -\Gamma^3_{ij}.$$

Although we have already know the Christoffel symbols are $O(r^{-2})$, we need better asymptotic behavior here. Since $g_{ij} = (1 + \frac{M}{2r})^4 \delta_{ij} + \ell_{ij}$, we have

$$g_{ij,m} = -4\left(1 + \frac{M}{2r}\right)^3 \left(\frac{Mx^m}{2r^3}\delta_{ij}\right) + O(r^{-3}) = -\frac{2Mx^m}{r^3}\delta_{ij} + O(r^{-3}).$$

From this together with $g^{ij} = \frac{\delta_{ij}}{(1+\frac{M}{2r})^4} + O(r^{-2}) = \delta_{ij} + O(r^{-1})$,

$$\begin{aligned}
\nabla^2_{ij} x^3 = -\Gamma^3_{ij} &= -\frac{1}{2} g^{3m}(g_{im,j} + g_{jm,i} - g_{ij,m}) \\
&= -\frac{1}{2} g^{33}(g_{i3,j} + g_{j3,i} - g_{ij,3}) + O(r^{-3}) \\
&= \frac{Mx^j}{r^3}\delta_{i3} + \frac{Mx^i}{r^3}\delta_{j3} - \frac{Mx^3}{r^3}\delta_{ij} + O(r^{-3}).
\end{aligned} \tag{2.8}$$

Let $\overline{h}$ be the maximum for x^3 on $S_\sigma \cap N_i$, and this maximum attains at the point $x_0 \in S_\sigma$. If $\overline{h} \leqslant \sigma_0$, then $S_\sigma \cap N_i$ has an upper bound for x^3. We now suppose $\overline{h} > \sigma_0$, then

$$T_p S_\sigma = \left\langle \left.\frac{\partial}{\partial x^1}\right|_{x_0}, \left.\frac{\partial}{\partial x^2}\right|_{x_0} \right\rangle.$$

Let ν_1, ν_2 be the vector fields extend by $\frac{\partial}{\partial x^1}|_{x_0}$, $\frac{\partial}{\partial x^2}|_{x_0}$ near x_0. Let $(q_{ij})_{1\leqslant i,j\leqslant 2}$ be the induced metric of (g_{ij}) with respect to the basis ν_1, ν_2. The standard result says that the Riemannian connection on submanifold is projection of the original Riemannian connection. We denote the covariant derivative on S_σ by D and the unit normal vector field of S_σ by ν. We have

$$D_{\nu_i}\nu_j = \nabla_{\nu_i}\nu_j - \langle \nabla_{\nu_i}\nu_j, \nu\rangle \nu.$$

Evaluating at x_0 yields

$$D^2_{ij}x^3 = \nabla^2_{ij}x^3 + h_{ij}\nu(x^3)$$

where $h_{ij} = \langle \nabla_{\nu_i}\nu_j, \nu\rangle$ is the second fundamental form. From this we get that

$$q^{ij}D^2_{ij}x^3 = q^{ij}\nabla^2_{ij}x^3 + q^{ij}h_{ij}\nu(x^3).$$

Note that S_σ is a minimal surface, its mean curvature

$$H = q^{ij}h_{ij} = 0$$

which implies that the last term vanishes. Then we can use the estimate (2.8) to get that

$$q^{ij}D^2_{ij}x^3 = -\frac{2M\overline{h}}{r^3} + O(r^{-3}).$$

This shows that if $\overline{h}$ is sufficiently large, $q^{ij}D^2_{ij}x^3$ will greater than 0, and this contradicts that $\overline{h}$ is maximum at x_0. The minimum of x^3 is similar argument. Therefore, (2.7) is true.

Now, let $\rho > 2\sigma_0$ and define

$$A_\rho = (N\backslash N_i) \cup \{x : |x| \geqslant \sigma_0, (x^1)^2 + (x^2)^2 \leqslant \rho^2\}.$$

Then for any $\sigma > \rho$, by (2.6) and (2.7),

$$S_\sigma \cap A_\rho \subset (K_0 \cup E_h) \cap A_\rho \tag{2.9}$$

which is a compact set. We will assume the following "regularity estimate" which is developed by the geometric measure theory (for example [5]).

Regularity estimate.
Let $U_r(x)$ be the geodesic ball of radius r about $x \in N$. There exists $r_0 > 0$ so that for any $x_0 \in S_\sigma$ with $U_{r_0}(x_0) \cap C_\sigma = \varnothing$ and $S_\sigma \cap U_{r_0}(x_0)$ can be written as the graph of a C^3 function f_σ from $T_{x_0}(S_\sigma)$ to a normal coordinate on $U_{r_0}(x_0)$. Moreover, there is a constant c_1 depending only on (N, g) which bounds all derivatives of f_σ up to order three in $U_{r_0}(x_0)$.

Then by (2.9) and the regularity estimate, for fixed ρ, we can choose a sequence $\sigma_i^{(\rho)} \to \infty$ so that $S_{\sigma_i^{(\rho)}} \cap A_\rho$ converges in C^2 topology. This is true for any $\rho > 2\sigma_0$, so we can use the diagonal argument to find a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \to S$ where S is a embedded C^2-surface and the limit is uniformly in C^2-norm on compact subset of N. S is properly embedded by (2.9) and is area-minimizing on any compact subset of N by construction. Finally, (2.6) and (2.7) would imply that S also has the same properties which completes the proof of Step 2. $\square$

Step 3

In this step, we will show that the surface S constructed in **Step 2** cannot exist.

Proof. For any $\sigma \geqslant \sigma_0$, define

$$S_{(\sigma)} = [S \cap (N \backslash N_i)] \cup [S \cap B_\sigma(0)] .$$

The set $S_{(\sigma)}$ form as exhaustion of S, we now claim that

$$\operatorname{Area}(S_{(\sigma)}) \leqslant C_2 \sigma^2 \tag{2.10}$$

for a constant C_2 independent of σ. First note that if $S \pitchfork \partial B_\sigma(0)$, then the intersection is a 1-dimension manifold, i.e. a union of Jordan curves on $\partial B_\sigma(0)$. Let Ω be the region bounded by these curves on $\partial B_\sigma(0)$. Then it is clear that $\partial S_{(\sigma)} = \partial\Omega$. Since the area minimizing property of S gives

$$\operatorname{Area}(S_{(\sigma)}) \leqslant \operatorname{Area}(\Omega) \leqslant \operatorname{Area}(\partial B_\sigma(0)).$$

According to (2.1), (2.2), $g_{ij} = \delta_{ij} + o(1)$ as $r \to \infty$ which implies that g is uniformly equivalent to δ. Hence (2.10) is true for those $\sigma > \sigma_0$ with $S \pitchfork \partial B_\sigma(0)$. By transversality theorem (Appendix A.3), we have this is true for all $\sigma > \sigma_0$ except for a measure zero set, and (2.10) is true by approximation. For $a > 0$, we have

$$\begin{aligned}
\int_S \frac{1}{1+r^a} &= \int_{S_{(\sigma_0)}} \frac{1}{1+r^a} + \int_{S \backslash S_{(\sigma_0)}} \frac{1}{1+r^a} \\
&= \int_{S_{(\sigma_0)}} \frac{1}{1+r^a} + \int_{\sigma_0}^{\infty} \int_{\partial S_{(t)}} \frac{1}{|\nabla r|} \frac{1}{1+r^a}\, ds dt \\
&= \int_{S_{(\sigma_0)}} \frac{1}{1+r^a} + \int_{\sigma_0}^{\infty} \frac{1}{1+t^a} \int_{\partial S_{(t)}} \frac{1}{|\nabla r|}\, ds dt \\
&\leqslant \operatorname{Area}(S_{(\sigma_0)}) + \int_{\sigma_0}^{\infty} \frac{1}{1+t^a} \left(\frac{d}{dt} \operatorname{Area}(S_{(t)}) \right) dt
\end{aligned}$$

since

$$\frac{d}{dt} \operatorname{Area}(S_{(t)}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int_{\partial S_\ell} \frac{1}{|\nabla r|}\, ds d\ell = \int_{\partial S_{(t)}} \frac{1}{\nabla r}\, ds.$$

If $a > 2$, by integration by parts and using (2.10), we obtain

$$\begin{aligned}
\int_S \frac{1}{1+r^a} &\leqslant C_2 \sigma_0^2 + \left. \frac{\operatorname{Area}(S_{(t)})}{1+t^a} \right|_{\sigma_0}^{\infty} + a \int_{\sigma_0}^{\infty} \frac{t^{a-1}}{(1+t^a)^2} \operatorname{Area}(S_{(t)})\, dt \\
&\leqslant C_2 \sigma_0^2 + C_2 a \int_{\sigma_0}^{\infty} \frac{t^{a+1}}{(1+t^a)^2}\, dt < \infty.
\end{aligned} \tag{2.12}$$

Similarly, we have for $\sigma_0 < \sigma_1 < \sigma_2$,

$$\int_{S_{(\sigma_2)} \backslash S_{(\sigma_1)}} \frac{1}{r^2} \leqslant 2C_2 \log \frac{\sigma_2}{\sigma_1} + C_2. \tag{2.13}$$

Now, let $\{e_1, e_2, e_3\}$ be the orthonormal vector fields on N, and let K_{ij} be the sectional curvature of $\{e_i, e_j\}$, $K_{ii} = 0$ for convenience. Then

$$\mathrm{Ric}(e_i) = \sum_{j=1}^{3} K_{ij}$$

and

$$R = K_{12} + K_{13} + K_{23}.$$

Let ν be the unit normal vector field of S, and we may let $e_3 = \nu$. Denote II be the second fundamental form of S, i.e.

$$\mathrm{II} = (h_{ij}) = \begin{pmatrix} \langle \nabla_{e_1}\nu, e_1\rangle & \langle \nabla_{e_1}\nu, e_2\rangle \\ \langle \nabla_{e_2}\nu, e_1\rangle & \langle \nabla_{e_2}\nu, e_2\rangle \end{pmatrix}.$$

Denote $\|\mathrm{II}\|^2$ by

$$\|\mathrm{II}\|^2 = \sum_{i,j} h_{ij}^2.$$

Since S is a minimal surface, we have

$$\mathrm{Tr}\,\mathrm{II} = h_{11} + h_{22} = 0. \tag{2.14}$$

Recall that second variation formula for oriented codimension 1 (see [6]) is

$$\int_S f\left[\Delta f + f(\mathrm{Ric}(\nu) + \|\mathrm{II}\|^2)\right] \leqslant 0 \tag{2.15}$$

for all C^2 function f with compact support. Since $\mathrm{div}(f\nabla f) = \|\nabla f\|^2 + f\Delta f$, using divergence theorem and f has compact support, (2.15) yields

$$\int_S f^2\left(\mathrm{Ric}(\nu) + \|\mathrm{II}\|^2\right) \leqslant \int_S \|\nabla f\|^2 \tag{2.16}$$

holds for all C^2 function f with compact support. Since any Lipschitz function f with compact support can be approximated by C^2 function with compact support, (2.16) also holds for Lipschitz function. Let K be a Gaussian curvature of S, by Gauss equation,

$$K = K_{12} + h_{11}h_{22} - h_{12}^2. \tag{2.17}$$

According to (2.14), $h_{11} = -h_{22}$, whence (2.17) yields

$$\frac{1}{2}\|\mathrm{II}\|^2 = K_{12} - K.$$

Putting this equation into (2.16),

$$\int_S f^2\left(\mathrm{Ric}(\nu) + K_{12} - K + \frac{1}{2}\|\mathrm{II}\|^2\right) \leqslant \int_S \|\nabla f\|^2.$$

Observe that $\mathrm{Ric}(\nu) + K_{12} = R$, so

$$\int_S f^2 \left(R - K + \frac{1}{2}\|\mathrm{II}\|^2 \right) \leqslant \int_S \|\nabla f\|^2. \tag{2.18}$$

Now we will choose a function f in inequality. For $\sigma > \sigma_0$, define

$$\varphi(x) = \begin{cases} 1 & \text{on } S_{(\sigma)} \\ \dfrac{\log(\frac{\sigma^2}{r})}{\log \sigma} & \text{on } S_{(\sigma^2)} \backslash S_{(\sigma)} \\ 0 & \text{outside } S_{(\sigma^2)}. \end{cases}$$

Let ψ be a Lipschitz function on S with $|\psi| \leqslant 1$ and $\psi = 1$ outside a compact subset of S. Taking $f = \varphi\psi$ in (2.16) and using Cauchy-Schwarz inequality, AM-GM inequality gives

$$\begin{aligned} \int_S (\mathrm{Ric}(\nu) + \|\mathrm{II}\|^2)\varphi^2\psi^2 &\leqslant \int_S \|\psi\nabla\varphi + \varphi\nabla\psi\|^2 \\ &= \int_S \psi^2\|\nabla\varphi\|^2 + \varphi^2\|\nabla\psi\|^2 + 2\langle\psi\nabla\varphi,\, \varphi\nabla\psi\rangle \\ &\leqslant \int_S \psi^2\|\nabla\varphi\|^2 + \varphi^2\|\nabla\psi\|^2 + 2\|\psi\nabla\varphi\|\|\varphi\nabla\psi\| \\ &\leqslant 2\int_S \psi^2\|\nabla\varphi\|^2 + 2\int_S \varphi^2\|\nabla\psi\|^2. \end{aligned} \tag{2.19}$$

Observe that $\nabla\varphi = 0$ unless $x \in S_{(\sigma^2)} \backslash S_{(\sigma)}$. We compute $\nabla\varphi$ on $S_{(\sigma^2)} \backslash S_{(\sigma)}$ directly.

$$\nabla\varphi = \nabla\left(\frac{\log(\frac{\sigma^2}{r})}{\log\sigma}\right) = \nabla\left(\frac{\log(\sigma^2) - \log r}{\log\sigma}\right) = -\frac{1}{\log\sigma}\nabla(\log r).$$

We now compare $\nabla(\log r)$ and ∇r. We know

$$\left\langle \nabla(\log r),\, \frac{\partial}{\partial x^i} \right\rangle = \frac{\partial}{\partial x^i}\log r = \frac{2x^i}{r^2},$$

and

$$\left\langle \nabla r,\, \frac{\partial}{\partial x^i} \right\rangle = \frac{\partial}{\partial x^i} r = \frac{2x^i}{r}.$$

We see

$$\nabla(\log r) = \frac{\nabla r}{r}, \text{ and } \nabla\varphi = -\frac{1}{\log\sigma}\frac{\nabla r}{r} \text{ on } S_{(\sigma^2)} \backslash S_{(\sigma)}.$$

Then using $|\psi|^2 \leqslant 1$, (2.19) yields

$$\int_S (\mathrm{Ric}(\nu) + \|\mathrm{II}\|^2)\varphi^2\psi^2 \leqslant \frac{2}{(\log\sigma)^2}\int_{S_{(\sigma^2)} \backslash S_{(\sigma)}} \frac{\|\nabla r\|^2}{r^2} + 2\int_S \varphi^2\|\nabla\psi\|^2.$$

We know from (2.1), (2.2) that $\|\nabla r\| \leqslant C_3$ for some constant C_3. Then we have

$$\begin{aligned}\int_{S_{(\sigma)}} \|II\|^2\psi^2 = \int_{S_{(\sigma)}} \|II\|^2\varphi^2\psi^2 &\leqslant \int_S \|II\|^2\varphi^2\psi^2 \\ &\leqslant \frac{2}{(\log\sigma)^2}\int_{S_{(\sigma^2)}\setminus S_{(\sigma)}} \frac{\|\nabla r\|^2}{r^2} + 2\int_S \varphi^2\|\nabla\psi\|^2 - \int_S \mathrm{Ric}(\nu)\varphi^2\psi^2 \\ &\leqslant \frac{2C_3}{(\log\sigma)^2}\int_{S_{(\sigma^2)}\setminus S_{(\sigma)}} \frac{1}{r^2} + 2\int_S \|\nabla\psi\|^2 + \int_S |\mathrm{Ric}(\nu)|\,\psi^2.\end{aligned}$$

Then by (2.13), we obtain

$$\int_{S_{(\sigma)}} \|II\|^2\psi^2 \leqslant \frac{2C_2C_3}{\log\sigma} + \frac{2C_2C_3}{(\log\sigma)^2} + 2\int_S \|\nabla\psi\|^2 + \int_S |\mathrm{Ric}(\nu)|\,\psi^2.$$

Letting $\sigma \to \infty$, we conclude

$$\int_S \|II\|^2\psi^2 \leqslant 2\int_S \|\nabla\psi\|^2 + \int_S |\mathrm{Ric}(\nu)|\,\psi^2 \tag{2.20}$$

for any ψ with $|\psi| \leqslant 1$ and $\psi = 1$ outside a compact subset of S. We have already known that $R_{ijk\ell} = O(r^{-3})$, hence $\mathrm{Ric}(\nu) = O(r^{-3})$ as well. Let $\psi \equiv 1$, then (2.20) together with (2.12) and $a = 3$, we have

$$\int_S \|II\|^2 < \infty.$$

(Note that we will use the (2.20) for $\psi \not\equiv 1$ later.)
Again, using $R_{ijk\ell} = O(r^{-3})$, we have $K_{12} = O(r^{-3})$, and by (2.12), we have also $\int_S |K_{12}| < \infty$. Note that (2.17) implies that $|K| \leqslant |K_{12}| + \|II\|^2$, hence we have

$$\int_S |K| < \infty. \tag{2.21}$$

Now let $f = \varphi$ in (2.18) and taking $\sigma \to \infty$, we conclude

$$\int_S \left(R - K + \frac{1}{2}\|II\|^2\right) \leqslant 0. \tag{2.22}$$

Recall that we have $R \geqslant 0$ and $R > 0$ outside a compact subset of S. Therefore,

$$\int_S K \geqslant \int_S \left(R + \frac{1}{2}\|II\|^2\right) > 0. \tag{2.23}$$

Remark. Using Cohn-Vossen inequality $\int_S K \leqslant 2\pi\chi(S)$ and (2.23), we see that S is homeomorphic to $\mathbb{R}^2$.

To get a contradiction, we claim that $\int_S K \leqslant 0$. We will apply Gauss-Bonnet theorem with boundary and estimate the boundary terms.
Let $x = (x^1, x^2, x^3) \in \mathbb{R}^3$, denote $x' = (x^1, x^2, 0)$ and $r = |x'| = \sqrt{(x^1)^2 + (x^2)^2}$. Let

$$P_\sigma = \{x \in \mathbb{R}^3 | r' \leqslant \sigma\}$$

be the cylinder. For those $\sigma > \sigma_0$ with $\partial P_\sigma \cap S$ is transverse, according to the remark above, we know there is a circle $C_\sigma \subset \partial P_\sigma \cap S$ which is not homologous to zero in $\mathbb{R}^3 \backslash P_{\sigma_0}$. Let D_σ be the connected component of C_σ in $S \cap [(N \backslash N_i) \cup P_\sigma]$. We want to show that for large σ, D_σ is a disk. Combining the remark above and (2.21), we have S is conformally equivalent to the complex plane (we will assume this result from [7]). Hence, we have a conformal diffeomorphism $F : \mathbb{C} \to S$. If $F^{-1}(D_\sigma)$ is not simply connected, then there is a bounded domain $\mathcal{O}$ contained in $\mathbb{C} \backslash F^{-1}(D_\sigma)$. Thus on $\partial F(\mathcal{O})$ we have $r' = \sigma$, and inside $F(\mathcal{O})$ at some point we have $r' > \sigma$. Thus r' has a maximum at some point of $F(\mathcal{O})$. We claim that $(r')^2$ is a subharmonic function on S for large r'. By definition,

$$\Delta x^i = \sum_{j=1}^{2} e_j e_j x^i - D_{e_j} e_j x^i$$

where D is the covariant derivative on S. Now write

$$e_j = e_j^1 \frac{\partial}{\partial x^1} + e_j^2 \frac{\partial}{\partial x^2} + e_j^3 \frac{\partial}{\partial x^3}.$$

Then we have

$$e_j x^i = dx^i(e_j) = e_j^i.$$

On the other hand, using (2.1), (2.2),

$$\left\langle e_j, \frac{\partial}{\partial x^i} \right\rangle = e_j^i + O(r^{-1}).$$

From this, we conclude that

$$e_j x^i = \left\langle e_j, \frac{\partial}{\partial x^i} \right\rangle + O(r^{-1}).$$

Similar reason gives

$$D_{e_j} e_j x^i = \left\langle D_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle + O(r^{-2}).$$

Now,

$$\begin{aligned} e_j e_j x^i &= \left\langle \nabla_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle + \left\langle e_j, \nabla_{e_j} \frac{\partial}{\partial x^i} \right\rangle + O(r^{-2}) \\ &= \left\langle D_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle + h_{jj} \left\langle \nu, \frac{\partial}{\partial x^i} \right\rangle + \left\langle e_j, \nabla_{e_j} \frac{\partial}{\partial x^i} \right\rangle + O(r^{-2}). \end{aligned}$$

Also,

$$\left\langle e_j, \nabla_{e_j} \frac{\partial}{\partial x^i} \right\rangle = \sum_{k,\ell,m=1}^{3} \left\langle e_j^k \frac{\partial}{\partial x^k}, \Gamma_{\ell i}^m \frac{\partial}{\partial x^m} \right\rangle \leqslant |e_j^k||\Gamma_{\ell i}^m| \left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m} \right\rangle = O(r^{-2})$$

where the last equation is given by $e_k^j = O(1)$, $\Gamma_{\ell i}^m = O(r^{-2})$ and (2.1), (2.2). Hence, we get

$$e_j e_j x^i = \left\langle D_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle + h_{jj} \left\langle \nu, \frac{\partial}{\partial x^i} \right\rangle + O(r^{-2})$$

and

$$\Delta x^i = \sum_{j=1}^{2}\left(\left\langle D_{e_j}e_j, \frac{\partial}{\partial x^i}\right\rangle + h_{jj}\left\langle \nu, \frac{\partial}{\partial x^i}\right\rangle - \left\langle D_{e_j}e_j, \frac{\partial}{\partial x^i}\right\rangle\right) + O(r^{-2}) = O(r^{-2})$$

since $\sum h_{jj} = 0$. Now,

$$\Delta(r')^2 = \sum_{i=1}^{2}\Delta(x^i)^2 = 2\sum_{i=1}^{2}x^i\Delta x^i + \sum_{i=1}^{2}\langle \nabla x^i, \nabla x^i\rangle.$$

Note that

$$\nabla x^i = \sum_{j=1}^{2}(e_j x^i)e_j = \sum_{j=1}^{2}\left\langle e_j, \frac{\partial}{\partial x^i}\right\rangle e_j + O(r^{-1}).$$

Therefore,

$$\Delta(r')^2 = 2\sum_{i,j=1}^{2}\left\langle e_j, \frac{\partial}{\partial x^i}\right\rangle^2 + O(r^{-1}).$$

Since $\{e_1, e_2, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ are in $\mathbb{R}^3$, with $\{e_1, e_2\}$ and $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ are both linearly independent. Hence the main term above has an asymptotic lower bound $1 - O(r^{-1})$ by (2.1), (2.2), i.e.

$$\Delta(r')^2 = 2 - O(r^{-1})$$

which shows that $\Delta(r')^2 \geqslant 0$ for r sufficiently large. We may choose D_σ to be increasing in the sense that $D_{\overline{\sigma}} \supset D_\sigma$ for $\overline{\sigma} > \sigma$. By connectedness of S, D_σ forms an exhaustion of S. We apply Gauss-Bonnet theorem on D_σ,

$$\int_{D_\sigma} K = 2\pi - \int_{\partial D_\sigma} k$$

where k is the geodesic curvature of ∂D_σ with respect to the inner normal. Hence it suffices to show that there is a sequence $\sigma_i \to \infty$ so that

$$\int_{\partial D_{\sigma_i}} k \geqslant 2\pi - o(1). \tag{2.30}$$

We choose a frame $\{e_1, e_2, e_3\}$ where e_1 is positively oriented unit tangent vector of ∂D_σ, e_2 is the inner normal of D_σ, and $e_3 = \nu$ is the unit normal of S with respect to the metric g. Under this chosen frame,

$$k = \langle \nabla_{e_1}e_1, e_2\rangle$$

by definition. Since $r' = \sigma$ on ∂D_σ, $\langle e_1, \nabla r'\rangle = 0$ on ∂D_σ. Hence,

$$\langle \nabla_{e_1}e_1, \nabla r'\rangle + \langle e_1, \nabla_{e_1}\nabla r'\rangle = 0. \tag{2.31}$$

We now compute $\nabla r'$ and $\nabla_{e_1}\nabla r'$ directly. By definition,

$$\left\langle \nabla r', \frac{\partial}{\partial x^i}\right\rangle = \frac{\partial}{\partial x^i}r' = \begin{cases} \frac{x^i}{r'} = \frac{x^i}{\sigma} & \text{if } i = 1, 2, \\ 0 & \text{if } i = 3. \end{cases}$$

By (2.1), (2.2), we have

$$\nabla r' = \frac{x'}{\sigma} + O(\sigma^{-1}).$$

Here $x' = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$. For the second part, again by (2.1), (2.2)

$$\nabla_{e_1} \nabla r' = \frac{1}{\sigma} \nabla_{e_1} x' + O(\sigma^{-2}).$$

Write

$$e_1 = \sum_{j=1}^{3} e_1^j \frac{\partial}{\partial x^j}.$$

Then

$$\begin{aligned}\nabla_{e_1} x' &= e_1(x^1) \frac{\partial}{\partial x^1} + e_1(x^2) \frac{\partial}{\partial x^2} + \sum_{i=1}^{2} x^i \nabla_{e_1} \frac{\partial}{\partial x^i} \\ &= e_1(x^1) \frac{\partial}{\partial x^1} + e_1(x^2) \frac{\partial}{\partial x^2} + x^i e_1^j \Gamma_{ji}^k \frac{\partial}{\partial x^k}.\end{aligned}$$

We conclude

$$\nabla_{e_1} \nabla r' = \frac{e_1}{\sigma} - \frac{1}{\sigma} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle + O(\sigma^{-2}).$$

Putting these into (2.31) and moving terms around, we have

$$\left\langle \nabla_{e_1} e_1, \frac{x'}{\sigma} \right\rangle + \frac{1}{\sigma} - \frac{1}{\sigma} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 = O(\sigma^{-1}) \|\nabla_{e_1} e_1\| + O(\sigma^{-2}).$$

On the other hand,

$$\nabla_{e_1} e_1 = k e_2 - h_{11} \nu$$

by the choice of our $\{e_1, e_2, e_3\}$. Hence

$$k \left\langle e_2, \frac{x'}{\sigma} \right\rangle + \frac{1}{\sigma} - h_{11} \left\langle \nu, \frac{x'}{\sigma} \right\rangle - \frac{1}{\sigma} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 = O(\sigma^{-1}) \|\nabla_{e_1} e_1\| + O(\sigma^{-2}). \tag{2.32}$$

Suppose that $\sigma \in [\overline{\sigma}, 2\overline{\sigma}]$. We use the divergence theorem on the vector field $\frac{\partial}{\partial x^3}$ on the volume V enclosed by $D_{2\overline{\sigma}} \backslash D_{\overline{\sigma}}, \partial P_{2\overline{\sigma}}, \partial P_{\overline{\sigma}}$, and $\Omega_{\overline{\sigma}} = \{x | x^3 = -h, \overline{\sigma} \leqslant r' \leqslant 2\overline{\sigma}\}$ where h is the bound of $|x^3|$ for $x \in S \cap N_k$ (whose the existence is shown in **Step 2**). Since

$$\operatorname{div} \frac{\partial}{\partial x^3} = \sum_{i=1}^{3} \Gamma_{i3}^3 = O(\overline{\sigma}^{-2}),$$

divergence theorem gives

$$\int_{\partial V} \left\langle \frac{\partial}{\partial x^3}, \mathbf{n} \right\rangle = \int_V \operatorname{div} \frac{\partial}{\partial x^3} = O(1).$$

We now expand the left hand side,

$$\int_{\partial V}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle=\int_{D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}}\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle+\int_{\partial P_{2\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle+\int_{\partial P_{\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle-\int_{\Omega_{\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle.$$

Note that outer unit normal $\mathbf{n}$ of $\partial P_{2\overline{\sigma}}$ and $\partial P_{\overline{\sigma}}$ is the linear combination of $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$. Also, the outer unit normal of $\Omega_{\overline{\sigma}}$ is asymptotic to $\frac{\partial}{\partial x^3}$ by (2.1), (2.2). Hence (2.1), (2.2) implies that

$$\int_{\partial P_{2\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle+\int_{\partial P_{\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle=\int_{\partial P_{2\overline{\sigma}}}O(\sigma^{-1})+\int_{\partial P_{\overline{\sigma}}}O(\sigma^{-1})=O(1),$$

and

$$\int_{\Omega_{\overline{\sigma}}}\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle=\mathrm{Area}(\Omega_{\overline{\sigma}})+O(1).$$

So, we have

$$\int_{D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}}\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle-\mathrm{Area}(\Omega_{\overline{\sigma}})=O(1).$$

Now, let $\Omega'_{\overline{\sigma}}$ be the part of $\partial P_{2\overline{\sigma}}\cup\partial P_{\overline{\sigma}}$ between S and $\Omega_{\overline{\sigma}}$. Then $D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}$ has same boundary as $\Omega_{\overline{\sigma}}\cup\Omega'_{\overline{\sigma}}$. Using the area minimizing property of S on $D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}$, we get

$$\mathrm{Area}(D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}})=\mathrm{Area}(\Omega_{\overline{\sigma}})+O(\overline{\sigma}).\tag{2.33}$$

Therefore,

$$\int_{D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}}1-\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle=\mathrm{Area}(D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}})-\mathrm{Area}(\Omega(\overline{\sigma}))+O(1)=O(\overline{\sigma}).$$

On the other hand, coarea formula gives

$$\int_{\overline{\sigma}}^{2\overline{\sigma}}\int_{D_{2\overline{\sigma}}\cap\partial P_t}\left(1-\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle\right)\,ds dt=\int_{D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\setminus P_{\overline{\sigma}})}\|\nabla r'\|\left(1-\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle\right)$$

where ds is arclength of $D_{2\overline{\sigma}}\cap\partial P_t$. Since $D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\setminus P_{\overline{\sigma}})\subset D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}$, and

$$\|\nabla r'\|=\left\|\frac{x'}{r'}\right\|+O(\overline{\sigma}^{-1})=O(1),$$

we have

$$\int_{\overline{\sigma}}^{2\overline{\sigma}}\int_{D_{2\overline{\sigma}}\cap\partial P_t}\left(1-\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle\right)\,ds dt=O(\overline{\sigma}).\tag{2.34}$$

Again, using coarea formula, we have

$$\int_{\overline{\sigma}}^{2\overline{\sigma}}\int_{D_{2\overline{\sigma}}\cap\partial P_t}1\,ds dt=\int_{\overline{\sigma}}^{2\overline{\sigma}}\mathrm{Length}(D_{2\overline{\sigma}}\cap\partial P_t)\,dt=\int_{D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\setminus P_{\overline{\sigma}})}\|\nabla r'\|.$$

By (2.33),

$$\int_{\overline{\sigma}}^{2\overline{\sigma}}\mathrm{Length}(D_{2\overline{\sigma}}\cap\partial P_t)\,dt=O(\mathrm{Area}(D_{2\overline{\sigma}}\setminus D_{\overline{\sigma}}))=O(\overline{\sigma}^2).\tag{2.35}$$

Looking at (2.32), to get the estimate of geodesic curvature k, we also need to have the bound for second fundamental form on ∂D_σ. To do this, we apply the following function g to (2.20),

$$g(x) = \begin{cases} 0 & \text{for} x \in S \cap [(N \backslash N_i) \cup P_{\sqrt{\overline{\sigma}}}] \\ \frac{\log(r'/\sqrt{\overline{\sigma}})}{\log\sqrt{\overline{\sigma}}} & \text{for } x \in S \cap (P_{\overline{\sigma}} \backslash P_{\sqrt{\overline{\sigma}}}) \\ 1 & \text{for } x \notin S \cap [(N \backslash N_i) \cup P_{\overline{\sigma}}]. \end{cases}$$

We get

$$\int_{S\backslash[(N\backslash N_i)\cup P_{\overline{\sigma}}]} \|II\|^2 \leqslant \frac{2}{(\log\sqrt{\overline{\sigma}})^2} \int_{S\cap(P_{\overline{\sigma}}\backslash P_{\sqrt{\overline{\sigma}}})} \frac{\|\nabla r'\|^2}{(r')^2} + \int_{S\backslash[(N\backslash N_i)\cup P_{\sqrt{\overline{\sigma}}}]} |\operatorname{Ric}(\nu)|$$

since

$$\nabla \frac{\log(r'/\sqrt{\overline{\sigma}})}{\log\sqrt{\overline{\sigma}}} = \frac{1}{\log\sqrt{\overline{\sigma}}} \nabla \log\left(\frac{r'}{\sqrt{\overline{\sigma}}}\right) = \frac{1}{\log\sqrt{\overline{\sigma}}} \frac{\sqrt{\overline{\sigma}}}{r'} \nabla \frac{r'}{\sqrt{\overline{\sigma}}} = \frac{1}{\log\sqrt{\overline{\sigma}}} \frac{\nabla r'}{r'}.$$

Recall that we have already known by (2.1), (2.2) that

$$\|\nabla r'\| = O(1), \quad \operatorname{Ric}(\nu) = O(r^{-3}).$$

So,

$$\int_{D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\backslash P_{\overline{\sigma}})} \|II\|^2 \leqslant \frac{2C}{(\log\sqrt{\overline{\sigma}})} \int_{S\cap(P_{\overline{\sigma}}\backslash P_{\sqrt{\overline{\sigma}}})} \frac{1}{(r')^2} + \int_{S\backslash[(N\backslash N_i)\cup P_{\sqrt{\overline{\sigma}}}]} O(r^{-3}).$$

Similar reason as before, we have

$$\int_{S\cap(P_{\overline{\sigma}}\backslash P_{\sqrt{\overline{\sigma}}})} \frac{1}{(r')^2} = O(\log\sqrt{\overline{\sigma}}), \quad \int_{S\backslash[(N\backslash N_i)\cup P_{\sqrt{\overline{\sigma}}}]} O(r^{-3}) = O(\overline{\sigma}^{-1/2}).$$

It follows that

$$\int_{D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\backslash P_{\overline{\sigma}})} \|II\|^2 = o(1).$$

Again by coarea formula,

$$\int_{\overline{\sigma}}^{2\overline{\sigma}} \int_{D_{2\overline{\sigma}}\cap\partial P_t} \|II\|^2 \, dt = \int_{D_{2\overline{\sigma}}\cap(P_{2\overline{\sigma}}\backslash P_{\overline{\sigma}})} \|II\|^2 \|\nabla r'\|.$$

This implies that

$$\int_{\overline{\sigma}}^{2\overline{\sigma}} \int_{D_{2\overline{\sigma}}\cap\partial P_t} \|II\|^2 \, dt = o(1). \tag{2.36}$$

Now, (2.34), (2.35), and (2.36) imply that there exists $\sigma \in [\overline{\sigma}, 2\overline{\sigma}]$ satisfying

$$\int_{D_{2\overline{\sigma}}\cap\partial P_\sigma} \left(1 - \left\langle \nu, \frac{\partial}{\partial x^3} \right\rangle\right) = O(1), \quad \operatorname{Length}(D_{2\overline{\sigma}} \cap \partial P_\sigma) = O(\sigma), \quad \int_{D_{2\overline{\sigma}}\cap\partial P_\sigma} \|II\|^2 = o(\sigma^{-1}).$$

By Cauchy-Schwarz inequality, we have

$$\left(\int_{D_{2\overline{\sigma}}\cap\partial P_\sigma} \|II\|\right)^2 \leqslant \left(\int_{D_{2\overline{\sigma}}\cap\partial P_\sigma} 1\right) \int_{D_{2\overline{\sigma}}\cap\partial P_\sigma} \|II\|^2 = \operatorname{Length}(D_{2\overline{\sigma}} \cap \partial P_\sigma) o(\sigma^{-1}) = o(1).$$

Since ∂D_σ is a component of $D_{2\overline{\sigma}} \cap \partial P_\sigma$, the above result gives

$$\int_{\partial D_\sigma} 1 - \left\langle \nu, \frac{\partial}{\partial x^3} \right\rangle = O(1), \quad \mathrm{Length}(\partial D_\sigma) = O(\sigma), \quad \int_{\partial D_\sigma} \|\mathrm{II}\| = o(1). \tag{2.37}$$

We now will use (2.37) to estimate the terms in (2.32). Taking integral over ∂D_σ and using (2.37), we get

$$\int_{\partial D_\sigma} \left| k \left\langle e_2, \frac{x'}{\sigma} \right\rangle \right| = O(1) + O(\sigma^{-1}) \int_{\partial D_\sigma} \|\nabla_{e_1} e_1\| + \frac{1}{\sigma} \int_{\partial D_\sigma} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2. \tag{2.38}$$

We now want to bound the last term. By (2.1), (2.2), we have

$$\left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 + \left\langle e_2, \frac{\partial}{\partial x^3} \right\rangle^2 = 1 - \left\langle \nu, \frac{\partial}{\partial x^3} \right\rangle^2 + O(\sigma^{-1}).$$

Using (2.37) again, we have

$$\int_{\partial D_\sigma} \left\langle e_1, \frac{\partial}{\partial x^3} \right\rangle^2 + \left\langle e_2, \frac{\partial}{\partial x^3} \right\rangle^2 = O(1). \tag{2.39}$$

We now give a lower bound for $|\langle e_2, \frac{x'}{\sigma} \rangle|$. In fact, we will show that

$$\sup_{\partial D_\sigma} \left(1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle \right) = o(1). \tag{2.40}$$

By (2.1), and (2.2),

$$\begin{aligned} 1 - \left\langle \frac{x'}{\sigma}, e_2 \right\rangle^2 &= \left\langle \frac{x'}{\sigma}, \nu \right\rangle^2 + O(\sigma^{-1}) \\ &= \left\langle \frac{x'}{\sigma}, \nu - \frac{\partial}{\partial x^3} \right\rangle^2 + O(\sigma^{-1}) \\ &\leqslant \|\nu - \frac{\partial}{\partial x^3}\|^2 + O(\sigma^{-1}) \qquad \text{(Cauchy-Schwarz inequality)} \\ &= 2\left(1 - \left\langle \nu, \frac{\partial}{\partial x^3} \right\rangle \right) + O(\sigma^{-1}). \end{aligned}$$

Integral over ∂D_σ and use (2.37), we obtain

$$\int_{\partial D_\sigma} 1 - \left\langle \frac{x'}{\sigma}, e_2 \right\rangle^2 = O(1). \tag{2.41}$$

Remember that e_2 is the inner normal of D_σ and $\nabla r' = \frac{x'}{\sigma} + O(\sigma^{-1})$ is the outer normal, so

$$\left\langle \frac{x'}{\sigma}, e_2 \right\rangle \leqslant O(\sigma^{-1})$$

which implies that

$$1-\left\langle\frac{x'}{\sigma},e_2\right\rangle^2=\left(1-\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)\left(1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)\geqslant(1-O(\sigma^{-1}))\left(1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right).$$

Hence (2.41) implies that

$$\int_{\partial D_\sigma}1+\left\langle\frac{x'}{\sigma},e_2\right\rangle=O(1).\tag{2.42}$$

Since ∂D_σ is not homologous to zero in $\mathbb{R}^3\backslash P_{\sigma_0}$, its projection onto x^1x^2-plane is a circle of radius σ centered at zero. So, we have

$$\text{Length}(\partial D_\sigma)=2\pi\sigma+O(1).\tag{2.43}$$

Now, (2.42), (2.43) imply that there exists $x_0\in\partial D_\sigma$ so that

$$1+\left\langle\frac{x'}{\sigma},e_2\right\rangle(x_0)=O(\sigma^{-1}).\tag{2.44}$$

Let e_1 apply on $1+\langle\frac{x'}{\sigma},e_2\rangle$ along ∂D_σ,

$$\begin{aligned}e_1\left[1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right]&=\left\langle\nabla_{e_1}\frac{x'}{\sigma},e_2\right\rangle+\left\langle\frac{x'}{\sigma},\nabla_{e_1}e_2\right\rangle\\&=\frac{1}{\sigma}\langle e_1,e_2\rangle-\frac{1}{\sigma}\left\langle\frac{\partial}{\partial x^3},e_1\right\rangle\left\langle\frac{\partial}{\partial x^3},e_2\right\rangle-h_{12}\left\langle\frac{x'}{\sigma},\nu\right\rangle+O(\sigma^{-2})\\&\leqslant\frac{1}{2\sigma}\left(\left\langle\frac{\partial}{\partial x^3},e_1\right\rangle^2+\left\langle\frac{\partial}{\partial x^3},e_2\right\rangle^2\right)+\|II\|+O(\sigma^{-2}).\end{aligned}$$

Integral over ∂D_σ, and apply (2.37), (2.39), we have

$$\int_{\partial D_\sigma}\left|e_1\left[1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right]\right|=o(1).\tag{2.45}$$

Now, for any $x\in\partial D_\sigma$, write

$$1+\left\langle\frac{x'}{\sigma},e_2\right\rangle(x)=1+\left\langle\frac{x'}{\sigma},e_2\right\rangle(x_0)+\int_{x_0}^{x}e_1\left[1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right].$$

Then (2.44), (2.45) will imply (2.40). Finally, (2.38), (2.39), and (2.40) would imply

$$\int_{\partial D_\sigma}|k|=O(1)+O(\sigma^{-1})\int_{\partial D_\sigma}\|\nabla_{e_1}e_1\|.\tag{2.46}$$

Recall that

$$\nabla_{e_1}e_1=ke_2-h_11\nu,$$

we have

$$\|\nabla_{e_1}e_1\|\leqslant|k|+\|II\|.$$

Integral over ∂D_σ, and use (2.37),

$$\int_{\partial D_\sigma} \|\nabla_{e_1} e_1\| = \int_{\partial D_\sigma} |k| + o(1).$$

Then this combines with (2.46) will imply that

$$\int_{\partial D_\sigma} \|\nabla_{e_1} e_1\| = O(1), \quad \int_{\partial D_\sigma} |k| = O(1). \tag{2.47}$$

Integral (2.32) over ∂D_σ, and use the (2.37), (2.39), and (2.47), we obtain

$$\int_{\partial D_\sigma} k \left\langle e_2, \frac{x'}{\sigma} \right\rangle + \frac{1}{\sigma} \mathrm{Length}(\partial D_\sigma) \leqslant o(1).$$

It is equivalent to

$$\int_{\partial D_\sigma} k \geqslant \int_{\partial D_\sigma} k \left(1 + \left\langle e_2, \frac{x'}{\sigma} \right\rangle\right) + \frac{1}{\sigma} \mathrm{Length}(\partial D_\sigma) - o(1).$$

Using (2.40), (2.43), and (2.47), we conclude that

$$\int_{\partial D_\sigma} k \geqslant 2\pi - o(1).$$

Since this is true for arbitrary large σ in the interval $[\overline{\sigma}, 2\overline{\sigma}]$ for sufficiently large $\overline{\sigma}$, we can find a sequence $\sigma_i \to \infty$ so that (2.30) holds. This finishes our proof of claim. □

3 The Equality Case

3.1 Overview

In this section, we focus on the equality case. If we assume (2.1), (2.2) and furthermore that

$$|\nabla\nabla\nabla\ell_{ij}| + |\nabla\nabla\nabla\nabla\ell_{ij}| + |\nabla\nabla\nabla\nabla\nabla\ell_{ij}| \leqslant \frac{k_4}{1+r^5} = O(r^{-5}) \tag{3.1}$$

then we have the following theorem which is a special case for Theorem 1.2.

Theorem 3.1. *Let* (N, g) *as in* Theorem 2.1. *Suppose that on some connected component* N_i, (3.1) *is satisfied and its total mass is zero. If* $R \geqslant 0$ *on* N, *then* g *is flat. In fact,* (N, g) *is isometric to* $(\mathbb{R}^3, \delta)$.

3.2 Some lemmas

Lemma 3.2. *There is a constant* $c_1 > 0$ *depending on* N *and the constants* k_1, k_2, k_3 *in* (2.2) *so that for any function* ζ *with compact support on* N, *we have*

$$\left(\int_N \zeta^6\right)^{1/3} \leqslant c_1 \int_N \|\nabla\zeta\|^2.$$

Proof. We prove by contradiction. If the inequality is not true, then we can find a sequence of functions f_i with compact support such that

$$\int_N f_i^6 = 1, \quad \int_N \|\nabla f_i\|^2 \leqslant \frac{1}{i}. \tag{3.2}$$

Since N_k is indentified with $\mathbb{R}^2 \backslash B_{\sigma_0}(0)$ and g is uniformly equivalent to δ, using the Sobolev inequality on Euclidean space, we have

$$\left(\int_{N_k} f_i^6 \right)^{1/3} \leqslant (\text{const.}) \int_{N_k} \|\nabla f_i\|^2.$$

Combining with (3.2), we have $f_i \to 0$ in L^6-norm on N_k. Let g be a C^1 function defined on a precompact coordinate neighborhood $\mathcal{O} \subset N$. Again by Sobolev inequality in Euclidean space, we have

$$\inf_{\beta \in \mathbb{R}} \left(\int_{\mathcal{O}} (g - \beta)^6 \right)^{1/3} \leqslant (\text{const.}) \int_{\mathcal{O}} \|\nabla g\|^2.$$

Applying this inequality for $g = f_i|_{\mathcal{O}}$, and using (3.2), we obtain a sequence $\{\beta_i\}$ such that

$$\int_{\mathcal{O}} (f_i - \beta_i)^6 \to 0.$$

Since $\int_N f_i^6 = 1$, $\{\beta_i\}$ is bounded. We can find a convergent subsequence of $\{\beta_i\}$, we also write it $\{\beta_i\}$, and $\beta_i \to \beta$. Hence $f_i \to \beta$ in L^6-norm on $\mathcal{O}$. But, we have already known that $f_i \to 0$ in L^6-norm in N_k. Hence $\beta = 0$ on any coordinate neighborhood $\mathcal{O}$. Therefore, $f_i \to 0$ in L^6-norm on N_i which contradicts (3.2). □

We next study the equation of the form

$$\Delta v - f v = h \text{ on } N \tag{3.3}$$

where f, h are functions satisfy the following condition

$$|f| \leqslant \frac{k_7}{1+r^5}, \quad |h| \leqslant \frac{k_7}{1+r^5}, \quad |\nabla f| \frac{k_8}{1+r^5}, \quad |\nabla h| \leqslant \frac{k_8}{1+r^5} \tag{3.4}$$

on N_i. Let f_+, f_- be the positive and negative part of f as usual, i.e. $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Lemma 3.3. *Suppose that* (2.1), (2.2) *hold and* N_i *has zero total mass. There is a number* $\varepsilon_0 > 0$ *depending only on* N *and* k_1, k_2, k_3 *of* (2.2) *so that if*

$$\left(\int_N (f_-)^{3/2} \right)^{2/3} \leqslant \varepsilon_0,$$

then (3.3) *has a unique solution* v *defined on* N *satisfying* $v = O(r^{-1})$ *as* $r \to \infty$ *and* $\frac{\partial v}{\partial \mathbf{n}} = 0$ *on* ∂N, *where* $\mathbf{n}$ *is the outward unit normal vector to* ∂N. *Moreover,* v *satisfies*

$$v = \frac{A}{r} + \omega, \quad |\omega| \leqslant \frac{k_9}{1+r^2}, \quad |\nabla \omega| \leqslant \frac{k_1 0}{1+r^3}, \quad |\nabla\nabla\omega| \leqslant \frac{k_{11}}{1+r^4}$$

on N_i, *where*

$$A = -\frac{1}{4\pi}\int_N fv + h,$$

and the constants k_9, k_{10}, k_{11} *depend only on* k_1, k_2, k_3.

Proof. From now on, the constants c_n are only depending on k_1, k_2, k_3. We first solve the problem for $\sigma > \sigma_0$,

$$\begin{cases} \Delta v_\sigma - f v_\sigma = h & \text{on } N^\sigma = (N\backslash N_i) \cup (B_\sigma(0) \cap N_i) \\ v_\sigma = 0 & \text{on } \partial B_\sigma(0) \\ \dfrac{\partial v_\sigma}{\partial \mathbf{n}} = 0 & \text{on } \partial N \end{cases} \tag{3.5}$$

Suppose that v_σ satisfies (3.5), since $\operatorname{div}(v_\sigma \nabla v_\sigma) = \|\nabla v_\sigma\|^2 + v_\sigma \Delta v_\sigma$, using divergence theorem, we obtain

$$\begin{aligned} \int_{N^\sigma} \|\nabla v_\sigma\|^2 &= -\int_{N^\sigma} f v_\sigma^2 - \int_{N^\sigma} h v_\sigma \\ &\leqslant \int_{N^\sigma} (f_-) v_\sigma^2 + \int_{N^\sigma} |h||v_\sigma| \\ &\leqslant \left(\int_{N^\sigma} f_-^{3/2}\right)^{2/3} \left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \left(\int_{N^\sigma} |h|^{6/5}\right)^{5/6} \left(\int_{N^\sigma} v_\sigma^6\right)^{1/6}. \end{aligned}$$

By (3.4), we have

$$\left(\int_{N^\sigma} |h|^{6/5}\right)^{5/6} \leqslant c_2$$

where c_2 is independent of σ. Using this, Lemma 3.2, and the assumption, we have

$$\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} \leqslant c_1 \int_{N^\sigma} \|\nabla v_\sigma\|^2 \leqslant c_1 \varepsilon_0 \left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + c_1 c_2 \left(\int_{N^\sigma} v_\sigma^6\right)^{1/6}.$$

Choose $\varepsilon_0 = 1/(3c_1)$, and use the inequality $|ab| \leqslant \frac{1}{3}a^2 + \frac{3}{4}b^2$ with a be the integral term, $b = c_1 c_2$, we have

$$\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} \leqslant \frac{1}{3}\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \frac{1}{3}\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \frac{3}{4}(c_1 c_2)^2$$

which implies that

$$\int_{N^\sigma} v_\sigma^6 \leqslant \left(\frac{9}{4}(c_1 c_2)^2\right)^3 =: c_3.$$

Standard linear elliptic estimate (we assume this) imply that $\{v_\sigma : \sigma > \sigma_0\}$ is equicontinuous in C^2 topology on compact subsets of N. So, we can choose a sequence $\sigma_i \to \infty$ so that $v_{\sigma_i} \to v$ uniformly in C^2-norm on compact subsets of N. Hence, v is a solution of (3.3) defined on N satisfying

$$\frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \partial N, \quad \int_N v^6 \leqslant c_3, \quad \sup_N |v| \leqslant c_4 \tag{3.6}$$

where the last estimate follows from the L^6 estimate and standard linear theory (we assume this also). We now start to analyze the asymptotic behavior of v. For $x, y \in N_i$, let

$$\rho_x(y) = \left(g_{ij}(x)(y^i - x^i)(y^j - x^j)\right)^{1/2}.$$

By direct calculation, we can get

$$\Delta_y(\rho_x(y))^{-1} = -4\pi\delta_x(y) + \psi_x(y)$$

where $\delta_x(y)$ is a point mass at x, and use (2.1), (2.2),

$$|\psi_x(y)| \leqslant c_5|x-y|^{-2}|x|^{-3} \text{ for } B_1(x), \tag{3.7}$$

$$|\psi_x(y)| \leqslant c_6\left[\frac{1}{(1+|y|)^2|x-y|^3} + \frac{1}{|x|^2|x-y|^3} + \frac{1}{(1+|y|)^3|x-y|^2}\right] \text{ for } y \notin B_1(y). \tag{3.8}$$

Also, we have

$$c_7^{-1}|x-y| \leqslant \rho_x(y) \leqslant c_7|x-y|, \quad c_8^{-1} \leqslant |\nabla_y\rho_x(y)| \leqslant c_8, \quad \lim_{|x|\to\infty}|x|(\rho_x(y))^{-1} = 1. \tag{3.9}$$

Observe that $\operatorname{div}(f\nabla g) = \langle\nabla f, \nabla g\rangle + f\Delta g$, using divergence theorem, we have

$$\int_V\langle\nabla f, \nabla g\rangle + \int_V f\Delta g = \int_V \operatorname{div}(f\nabla g) = \int_{\partial V} f\frac{\partial g}{\partial \mathbf{n}}.$$

Similarly, we have

$$\int_V\langle\nabla f, \nabla g\rangle + \int_V g\Delta f = \int_{\partial V} g\frac{\partial f}{\partial \mathbf{n}}.$$

From these, we see that

$$\int_V f\Delta g - g\Delta f + \int_{\partial V} g\frac{\partial f}{\partial \mathbf{n}} - f\frac{\partial g}{\partial \mathbf{n}} = 0.$$

Apply this equation for $f = v$, $g = (\rho_x(y))^{-1}$, and $V = D_{\overline{\sigma}}(x) = \{y \in N_i : \rho_x(y) \leqslant \overline{\sigma}\}$ where $\overline{\sigma} \in (\sigma/2, \sigma)$, $\sigma \gg |x|$, and using (3.3), we get

$$\begin{aligned}
4\pi v(x) = &\int_{D_{\overline{\sigma}}(x)} \psi_x(y)v(y)\sqrt{g(y)}\,dy - \int_{D_{\overline{\sigma}}(x)} (fv+h)(y)(\rho_x(y))^{-1}\sqrt{g(y)}\,dy \\
&+ \frac{1}{\overline{\sigma}}\int_{\{\rho_x(y)=\overline{\sigma}\}} \frac{\partial v}{\partial \mathbf{n}}\,dA(y) - \int_{\{\rho_x(y)=\overline{\sigma}\}} v(y)\frac{\partial}{\partial \mathbf{n}}(\rho_x(y))^{-1}\,dA(y) \\
&- \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}(y)(\rho_x(y))^{-1}\,dA(y) + \int_{\partial B_{\sigma_0}(0)} v(y)(\rho_x(y))^{-1}\,dA(y)
\end{aligned} \tag{3.10}$$

where $\sqrt{g}dy$ is volume element and dA is the area element (both are relative to g). Next we want to let $\overline{\sigma}$ tends to ∞ to let the integral is over N_i, so we need to estimate the third and forth terms in (3.10). Again, using divergence theorem,

$$\int_{D_{\overline{\sigma}}(x)} \Delta v = \int_{\{\rho_x(y)=\overline{\sigma}\}} \frac{\partial v}{\partial \mathbf{n}} - \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}.$$

So, (3.3) implies that

$$\int_{\{\rho_x(y)=\overline{\sigma}\}} \frac{\partial v}{\partial \mathbf{n}} = \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}} + \int_{D_{\overline{\sigma}}(x)} \Delta v = \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}} + \int_{D_{\overline{\sigma}}(x)} (fv + h).$$

From (3.4), (3.6), we see that

$$\left| \int_{D_{\overline{\sigma}}(x)} (fv + h) \right| \leqslant c_9.$$

Therefore, we have

$$\left| \int_{\{\rho_x(y)=\overline{\sigma}\}} \frac{\partial v}{\partial \mathbf{n}} \right| \leqslant \left| \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial n} \right| + c_9 \leqslant c_{10}. \tag{3.11}$$

From (3.9), we see that Area$\{\rho_x(y) = \overline{\sigma}\} \leqslant c_{11}\overline{\sigma}^2$. Also,

$$\frac{\partial}{\partial \mathbf{n}} (\rho_x(y))^{-1} = \frac{1}{(\rho_x(y))^2} \frac{\partial}{\partial \mathbf{n}} \rho_x(y).$$

From (3.9) again, we see that

$$\begin{aligned} \left| \int_{\{\rho_x(y)=\overline{\sigma}\}} v \frac{\partial}{\partial \mathbf{n}} (\rho_x(y))^{-1} \right| &\leqslant \frac{c_{12}}{\overline{\sigma}^2} \int_{\{\rho_x(y)=\overline{\sigma}\}} |v| \\ &\leqslant \frac{c_{12}}{\overline{\sigma}^2} (c_{11}\overline{\sigma}^2)^{5/6} \left(\int_{\{\rho_x(y)=\overline{\sigma}\}} v^6 \right)^{1/6} \\ &\leqslant \frac{c_{13}}{\sigma^{1/3}} \left(\int_{\{\rho_x(y)=\overline{\sigma}\}} v^6 \right)^{1/6} \end{aligned}$$

where we use Hölder inequality. Using (3.6) and coarea formula, we may choose $\overline{\sigma} \in (\sigma/2, \sigma)$ so that

$$\int_{\rho_x(y)=\overline{\sigma}} v^6 \leqslant \frac{c_{14}}{\sigma}.$$

Thus, we obtain

$$\left| \int_{\{\rho_x(y)=\overline{\sigma}\}} v \frac{\partial}{\partial \mathbf{n}} (\rho_x(y))^{-1} \right| \leqslant c_{15}\sigma^{-1/2}. \tag{3.12}$$

Now let $\sigma \to \infty$ on (3.10), and using (3.11), (3.12), we get

$$\begin{aligned} 4\pi v(x) = &\int_{N_i} \psi_x(y) v(y) \sqrt{g(y)}\, dy - \int_{N_i} (fv + h)(y) (\rho_x(y))^{-1} \sqrt{g(y)}\, dy \\ &- \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}} (\rho_x(y))^{-1}\, dA(y) + \int_{\partial B_{\sigma_0}(0)} v \frac{\partial}{\partial \mathbf{n}} (\rho_x(y))^{-1}\, dA(y). \end{aligned} \tag{3.13}$$

Now, from (3.6), (3.7), (3.8), and Hölder inequality, we see

$$\left| \int_{N_i} \psi_x(y) v(y) \sqrt{g(y)}\, dy \right| \leqslant c_{16} \left[\int_{B_1(x)} |\psi_x(y)|\, dy + \int_{N_i \setminus B_1(x)} |\psi_x(y)||v(y)| \right] \tag{3.14}$$

$$\leqslant c_{17}|x|^{-3} + c_{18} \left(\int_{\mathbb{R}^3 \setminus B_1(x)} \left(\frac{1}{(1+|y|)^2|x-y|^3} + \frac{1}{(1+|y|)^3|x-y|^2} + \frac{1}{|x|^2|x-y|^3} \right)^{6/5} dy \right)^{5/6}.$$

By elementary calculus, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^3\setminus B_1(x)}\frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\leqslant&\left(\int_{B_{\frac{|x|}{2}}(x)\setminus B_1(x)}\frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\&+\left(\int_{B_{\frac{|x|}{2}}(x)}\frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\&+\left(\int_{\mathbb{R}^3\setminus(B_{\frac{|x|}{2}}(0)\cup B_{\frac{|x|}{2}}(x))}\frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\\leqslant& c_{19}(|x|^{-2}+|x|^{-5/2}+|x|^{-5/2})\leqslant c_{20}|x|^{-2}.\end{aligned}$$

Similarly, we have

$$\left(\int_{\mathbb{R}^3\setminus B_1(x)}\frac{dy}{(1+|y|)^{18/5}|x-y|^{12/5}}\right)^{5/6}\leqslant c_{21}|x|^{-2},\quad\int_{\mathbb{R}^3\setminus B_1(x)}\frac{dy}{|x-y|^{18/5}}\leqslant c_{22}.$$

Hence (3.14) yields

$$\left|\int_{N_i}\psi_x(y)\nu(y)\sqrt{g(y)}\,dy\right|\leqslant c_{23}|x|^{-2}.\tag{3.15}$$

Also, from (3.4), (3.6), and (3.9), we have

$$\left|\int_{B_{\frac{|x|}{2}}(x)}(f\nu+h)(y)(\rho_x(y))^{-1}\sqrt{g(y)}\,dy\right|\leqslant c_{24}|x|^{-3},\quad |x|(\rho_x(y))^{-1}\leqslant c_{25}\text{ for }y\notin B_{\frac{|x|}{2}}(x).$$

So we can apply dominated convergence theorem to the function

$$(f\nu+h)(y)|x|(\rho_x(y))^{-1}\sqrt{g(y)}\chi_{N_i\setminus B_{\frac{|x|}{2}}(x)}$$

where χ_E is the characteristic function of E. Then we have

$$\begin{aligned}&\lim_{|x|\to\infty}|x|\int_{N_i}(f\nu+h)(y)(\rho_x(y))^{-1}\sqrt{g(y)}\,dy\\=&\lim_{|x|\to\infty}\int_{N_i}(f\nu+h)(y)|x|(\rho_x(y))^{-1}\sqrt{g(y)}\chi_{N_i\setminus B_{\frac{|x|}{2}}(x)}\,dy\\&+\lim_{|x|\to\infty}\int_{N_i}(f\nu+h)(y)|x|(\rho_x(y))^{-1}\sqrt{g(y)}\chi_{B_{\frac{|x|}{2}}(x)}\,dy\\=&\int_{N_i}(f\nu+h)(y)\sqrt{g(y)}\chi_{N_i\setminus B_{\frac{|x|}{2}}(x)}\,dy+\int_{N_i}(f\nu+h)\sqrt{g(y)}\chi_{B_{\frac{|x|}{2}}(x)}\,dy\\=&\int_{N_i}(f\nu+h)(y)\sqrt{g(y)}\,dy.\end{aligned}\tag{3.16}$$

Now, using (3.9), (3.13), (3.15), and (3.16), we will see

$$A = \lim_{|x|\to\infty} 4\pi|x|\nu(x) = -\int_{N_i} (f\nu + h)(y)\sqrt{g}\, dy - \int_{\partial B_{\sigma_0}(0)} \frac{\partial \nu}{\partial \mathbf{n}}. \tag{3.17}$$

Using (3.13) and (3.17), we have

$$\nu = \frac{A}{r} + \omega$$

where

$$\begin{aligned} 4\pi\omega(x) = &\int_{N_i} \psi_x(y)\nu(y)\sqrt{g(y)}\, dy - \int_{N_i} (f\nu + h)(y) \left((\rho_x(y))^{-1} - |x|^{-1}\right) \sqrt{g}\, dy \\ &- \int_{\partial B_{\sigma_0}(0)} \frac{\partial \nu}{\partial \mathbf{n}} \left((\rho_x(y))^{-1} - |x|^{-1}\right)\, dA(y) + \int_{\partial B_{\sigma_0}(0)} \nu \frac{\partial}{\partial \mathbf{n}} (\rho_x(y))^{-1}\, dA(y). \end{aligned} \tag{3.18}$$

We see directly that

$$|(\rho_x(y))^{-1} - |x|^{-1}| \leqslant c_{26} \frac{|y|}{|x||x-y|},$$

combining with (3.4), (3.9), and (3.18), we have

$$|\omega(x)| \leqslant \frac{c_{27}}{1+r^2} \text{ for } x \in N_i. \tag{3.19}$$

Next, we estimate the derivatives of ω. We will assume the following result (Schauder estimate). Let L be an elliptic operator on the unit ball of $\mathbb{R}^3$ of the form

$$Lu = a_{ij}(\xi)\frac{\partial^2 u}{\partial \xi^i \partial \xi^j} + b_j(\xi)\frac{\partial u}{\partial \xi^j} + c(\xi)u(\xi)$$

where $\xi = (\xi^1, \xi^2, \xi^3)$ is the Cartesian coordinate in the ball. For any function $\varphi(\xi)$ defined on an open set Ω and real number λ with $0 < \lambda < 1$, define the following norms

$$\begin{aligned} |\varphi|_{0,\Omega} &= \sup_{\xi\in\Omega} |\varphi(\xi)| \\ |\varphi|_{0,\lambda,\Omega} &= \sup_{\xi\in\Omega} \frac{|\varphi(\xi) - \varphi(\bar{\xi})|}{|\xi - \bar{\xi}|^\lambda} \\ |\varphi|_{1,\lambda,\Omega} &= \sup_{\xi\in\Omega} |\nabla\varphi(\xi)| + |\nabla\varphi|_{0,\lambda,\Omega} \\ |\varphi|_{2,\lambda,\Omega} &= \sup_{\xi\in\Omega} |\nabla\varphi(\xi)| + \sup_{\xi\in\Omega} |\nabla\nabla\varphi(\xi)| + |\nabla\nabla\varphi|_{0,\lambda,\Omega} \end{aligned}$$

Let $B_r = \{\xi : |\xi| < r\}$. Suppose there is a positive number Λ so that

$$\sum_{i,j=1}^{3} |a_{ij}|_{0,\lambda,B_1} + \sum_{i=1}^{3} |b_i|_{0,\lambda,B_1} + |c|_{0,\lambda,B_1} \leqslant \Lambda, \quad \frac{|t|^2}{\Lambda} \leqslant a_{ij}(\xi)t^i t^j \;\forall t \in \mathbb{R}^3\backslash\{0\}, \quad \forall \xi \in B_1 \tag{3.20}$$

Then for any $C^{2,\lambda}$ function u on B_1, we have

$$|u|_{2,\lambda,B_{1/2}} \leqslant \overline{C}(|Lu|_{0,\lambda,B_1} + |u|_{0,B_1}) \tag{3.21}$$

where $\overline{C}$ depends only on λ, Λ.
We now fix a point $x_0 \in N_i$, and assume that $\sigma = \frac{1}{2}|x_0| > \sigma_0$, and let

$$\xi = \frac{1}{\sigma}(y - x_0)$$

where y is the asymptotic coordinate of N_i. Let $u(\xi) = \omega(y)$, $a_{ij}(\xi) = g^{ij}(y)$, $b_k(\xi) = \sigma g^{ij}(y)\Gamma^k_{ij}(y)$, and $c(\xi) = -\sigma^2 f(y)$, we have

$$Lu(\xi) = a_{ij}(\xi)\frac{\partial^2 u}{\partial \xi^i \partial \xi^j} + b_k(\xi)\frac{\partial u}{\partial \xi^k} + c(\xi)u(\xi) = \sigma^2(\Delta\omega(y) - f\omega(y)).$$

Using (3.3) and $\omega = \nu - \frac{A}{r}$, we see that

$$\Delta\omega - f\omega = Af|y|^{-1} + h - A\Delta|y|^{-1}.$$

Also, (2.1), (2.2) imply that

$$|\Delta|y|^{-1}| \leqslant c_{28}|y|^{-5}, \quad |\nabla(\Delta|y|^{-1})| \leqslant c_{29}|y|^{-6}$$

(remember that we now $M = 0$, and see (2.3)). Thus, together with (3.4), we have

$$|Lu|_{0,\lambda,B_1} \leqslant c_{30}\sigma^{-3+\lambda}.$$

Using (2.1), (2.2), we can easily see that (3.20) is satisfied for our operator Lu with a constant Λ independent of σ. Thus (3.21) implies that

$$|u|_{2,\Lambda,B_{1/2}} \leqslant c_{31}(\sigma^{-3+\lambda} + |u|_{0,B_1}).$$

By (3.19), we have for any $0 < \lambda < 1$,

$$|u|_{2,\lambda,B_{1/2}} \leqslant c_{32}\sigma^{-2}.$$

Rewrite to ω, we have

$$|\nabla\omega(x_0)| \leqslant c_{32}|x_0|^{-3}, \quad |\nabla\nabla\omega(x_0)| \leqslant c_{32}|x_0|^{-4}$$

as desired bound for derivatives of ω. The expression of A is just (3.17), and using divergence theorem on (3.3). Finally, we show the uniqueness. Suppose that $\bar{\nu}$ is another solution of (3.3) satisfying $\bar{\nu} = O(r^{-1})$ and $\frac{\partial\bar{\nu}}{\partial \mathbf{n}} = 0$ on ∂N. Then $u = \nu - \bar{\nu}$ satisfies

$$\Delta u - fu = 0, \quad u = O(r^{-1}), \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial N. \tag{3.22}$$

We will show that $u \equiv 0$. Let $\delta > 0$ be any number, and let

$$E_\delta = \{x \in N : u(x) \geqslant \delta\}.$$

Since $u = O(r^{-1})$, we know E_δ is compact. By (3.22), we have

$$\int_{E_\delta} u\Delta u = \int_{E_\delta} fu^2.$$

Since $\operatorname{div}(u\nabla u) = \|\nabla u\|^2 + u\Delta u$, using divergence theorem. we have

$$\int_{E_\delta} \operatorname{div}(u\nabla u) = \int_{\partial E_\delta} u\frac{\partial u}{\partial \mathbf{n}} = 0$$

by (3.22). Hence using Hölder inequality,

$$\begin{aligned}\int_{E_\delta} \|\nabla u\|^2 &= -\int_{E_\delta} fu^2 \\ &\leqslant \int_{E_\delta} (f_-)u^2 \leqslant \left(\int_{E_\delta} f_-^{3/2}\right)^{2/3} \left(\int_{E_\delta} u^6\right)^{1/3} \\ &\leqslant \varepsilon_0 \left(\int_{E_\delta} u^6\right)^{1/3}.\end{aligned}$$

Now applying $\zeta = u - \delta$ on E_δ and $\zeta \equiv 0$ on $N \backslash E_\delta$ to Lemma 3.2, we get

$$\left(\int_{E_\delta} (u-\delta)^6\right)^{1/3} \leqslant c_1 \int_{E_\delta} \|\nabla u\|^2.$$

Combine the above two inequalities and recall that $\varepsilon \leqslant 1/(3c_1)$, we have

$$\left(\int_{E_\delta} (u-\delta)^6\right)^{1/3} \leqslant c_1\varepsilon_0 \left(\int_{E_\delta} u^6\right)^{1/3} \leqslant \frac{1}{3}\left(\int_{E_\delta} u^6\right)^{1/3}.$$

Since $u = O(r^{-1})$, we see that

$$\int_N u^6 < \infty.$$

So let $\delta \to 0$, the above inequality implies that $u \leqslant 0$. Now, consider $-u$ which also satisfies (3.22), and the same argument will gives $u \geqslant 0$. Hence $u \equiv 0$, which completes the proof. □

The next lemma gives the property between conformal change.

Lemma 3.4. *Suppose* (N, g) *as* Theorem 2.1. *Let* R *be the scalar curvature of* (N, g) *and satisfy*

$$\frac{1}{8}\left(\int_N R_-^{3/2}\right)^{2/3} \leqslant \varepsilon_0$$

where ε_0 *is defined in* Lemma 3.3. *Then there is a unique positive function* φ *with* $\frac{\partial\varphi}{\partial\mathbf{n}} = 0$ *on* ∂N *so that* $\overline{g} = \varphi^4 g$ *is asymptotically flat, scalar flat, and total mass*

$$\overline{M} = -\frac{1}{32\pi}\int_N R\varphi.$$

Proof. In order for the metric $\overline{g}$ to be scalar flat, recall that the formula of scalar curvature between conformal change (see Appendix A.2)

$$\overline{R} = \varphi^{-5}[-8\Delta\varphi + R\varphi],$$

we must have

$$\Delta\varphi - \frac{1}{8}R\varphi = 0. \tag{3.23}$$

Then for $\nu = \varphi - 1$,

$$\Delta\nu - \frac{1}{8}R\nu = \frac{1}{8}R. \tag{3.24}$$

To make $\overline{g}$ is asymptotically flat, ν must satisfy the condition in Lemma 3.3. Then Lemma 3.3 gives a ν satisfying (3.24) with $\frac{\partial \nu}{\partial \mathbf{n}} = 0$ on ∂N. So, we have $\varphi = \nu + 1$ satisfies (3.23) and $\frac{\partial \varphi}{\partial \mathbf{n}} = 0$ on ∂N. Now, we show that φ is positive by contradiction. Suppose that

$$E = \{x \in N : \varphi(x) < 0\}$$

is not empty. Since φ is asymptotic to 1, E is precompact. Since $\operatorname{div}(f\nabla g) = \langle \nabla f, \nabla g\rangle + f\Delta g$, and by divergence theorem,

$$\int_E \operatorname{div}(\varphi\nabla\varphi) = \int_{\partial E} \langle \varphi\nabla\varphi, \mathbf{n}\rangle = 0$$

since $\frac{\partial \varphi}{\partial \mathbf{n}} = 0$ on ∂N. By Hölder inequality, we have

$$\int_E \|\nabla\varphi\|^2 = -\int_E \varphi\Delta\varphi = -\frac{1}{8}\int_E R\varphi^2 \leqslant \int_E (R_-)\varphi^2 \leqslant \left(\int_E R_-^{3/2}\right)^{2/3} \left(\int_E \varphi^6\right)^{1/3}.$$

Apply Lemma 3.2,

$$\left(\int_E \varphi^6\right)^{1/3} \leqslant c_1 \int_E \|\nabla\varphi\|^2 \leqslant c_1\varepsilon_0 \left(\int_E \varphi^6\right)^{1/3}$$

which is a contradiction since $\varepsilon \leqslant 1/(3\varepsilon_0)$ (we choose in the proof of Lemma 3.3). Hence $\varphi \geqslant 0$ on N. By maximum principle $\varphi > 0$ on N. Lastly, we show that ∂N has positive mean curvature relative to $\overline{g}$. Let H, $\overline{H}$ be the mean curvatures of ∂N with respect to g, $\overline{g}$ respectively. Then

$$\overline{H} = \varphi^{-2}\left(H + 4\varphi^{-1}\frac{\partial\varphi}{\partial \mathbf{n}}\right) = \varphi^{-2}H.$$

Hence $\overline{H} > 0$. The formula of $\overline{M}$ follows directly from the formula given in Lemma 3.3. □

Then we have the following special case of Lemma 3.4.

Corollary 3.5. *If $M = 0$, $R \geqslant 0$, and $R \not\equiv 0$, then there is a metric conformally equivalent to g which is asymptotically flat, scalar, so that N_i has negative total mass.*

Now, we are ready to prove Theorem 3.1.

3.3 Proof of Theorem 3.1

Theorem 2.1 and Corollary 3.5 implies that an asymptotically flat metric satisfying the hypothesis $M = 0$, $R \geqslant 0$, will have $R \equiv 0$ on N.n Now, we assume that g is a such metric and (3.1) is also satisfied. We define a one-parameter family of metric g_t on N by

$$(g_t)_{ij} = g_{ij} + t\,\mathrm{Ric}_{ij}$$

where Ric is the Ricci tensor of g. (3.1) would imply that g_t is asymptotically flat for small t, and ∂N has positive mean curvature with respect to $g = g_0$, so it has positive mean curvature with respect to g_t for small t by continuity. Let R_t be the scalar curvature of g_t, and keep in mind that $R_0 = R \equiv 0$. A formula in [8] says that

$$R_0' = \frac{d}{dt}\bigg|_{t=0} R_t = -\Delta R + \delta\delta\,\mathrm{Ric} - \|\,\mathrm{Ric}\,\|^2 \tag{3.25}$$

Since $R \equiv 0$, $\Delta R \equiv 0$, the corollary of second Bianchi identity gives

$$\delta\delta\,\mathrm{Ric} = 2\Delta R \equiv 0.$$

Then (3.25) becomes

$$R_0' = -\|\,\mathrm{Ric}\,\|^2. \tag{3.26}$$

On the other hand, $R_0 \equiv 0$ combined with (3.1) will gives that for small t,

$$\frac{1}{8}\left(\int_N R_{t-}^{3/2}\right)^{2/3} \leqslant \varepsilon_0$$

and ε_0 is independent of t when t is sufficiently small. Now, we can apply Lemma 3.4, which shows that there is a function φ_t so that $\varphi_t^4 g_t$ is asymptotically flat and scalar flat, and the mass

$$M(t) = -\frac{1}{32\pi}\int_N R_t \varphi_t \sqrt{g_t}\,dx. \tag{3.27}$$

Next, we claim that $M'(0)$ exists. For small h, define

$$\varphi^{(h)} = \frac{\varphi_h - \varphi_0}{h}.$$

Denote Δ_t be the Laplacian with respect to the metric g_t, and define the differential operator

$$\Delta^{(h)} v = \frac{\Delta_h v - \Delta_0 v}{h}.$$

Denote

$$R^{(h)} = \frac{R_h - R_0}{h}.$$

Under these notations, we found that $\varphi^{(h)}$ satisfies

$$\Delta_0 \varphi^{(h)} - \frac{1}{8} R_0 \varphi^{(h)} = -\Delta^{(h)} \varphi_h + \frac{1}{8} R^{(h)} \varphi_h. \tag{3.28}$$

Using (2.1), (2.2), and (3.1), we see that (3.28) satisfies the hypothesis of Lemma 3.3. Also, $\varphi^{(h)} = O(r^{-1})$, and the lemma says

$$|\varphi^{(h)}| \leqslant \frac{\gamma_1}{(1+r)^{-1}} \tag{3.29}$$

and γ_1 is independent of h. Standard linear theory applied to (3.28) shows that $\varphi^{(h)}$ has a local $C^{2,\alpha}$ bound depending on C^1 bound on R_0 and $-\Delta^{(h)}\varphi_h + \frac{1}{8}R^{(h)}\varphi_h$ (we assume this). Since these bounds are independent of h, we can find a sequence $h_i \to 0$ so that φ_{h_i} converges in $C^{2,\beta}$-norm for any $\beta < \alpha$ uniformly on compact subsets of N to a $C^{2,\alpha}$ function φ_0' satisfying

$$\Delta_0\varphi_0' - \frac{1}{8}R_0\varphi_0' = -\Delta_0'\varphi_0 + \frac{1}{8}R_0'\varphi_0$$

where

$$\Delta_0' = \frac{d}{dt}\bigg|_{t=0}\Delta_t, \quad R_0' = \frac{d}{dt}\bigg|_{t=0}R_t.$$

(3.29) gives that $\varphi_0' = O(r^{-1})$, then the uniqueness of Lemma 3.3 shows that the limit φ_0' is independent of the sequence $\{h_i\}$ we chosen. This means that

$$\frac{d}{dt}\bigg|_{t=0}\varphi_t$$

exists and equals to φ_0'. Also, (2.1), (2.2), and (3.1) implies that we have constants γ_2, γ_3 independent of h so that

$$|R^{(h)}| \leqslant \frac{\gamma_2}{1+r^{3+\alpha}}g^{(h)} = \frac{\gamma_3}{1+r^{1+\alpha}} \text{ on } N_k \tag{3.30}$$

where

$$g^{(h)} = \frac{g_h - g_0}{h}.$$

Now (3.29), (3.30) allows us to use dominated convergence theorem to conclude that

$$\begin{aligned}
M'(0) &= -\frac{1}{32\pi}\lim_{t\to 0}\int_N \frac{R_t\varphi_t\sqrt{g_t}}{t}\,dx \\
&= -\frac{1}{32\pi}\int_N \lim_{t\to 0}\frac{R_t\varphi_t\sqrt{g_t}}{t}\,dx \\
&= -\frac{1}{32\pi}\int_N R_0(\varphi_t\sqrt{g_t})'|_{t=0}\,dx - \frac{1}{32\pi}\int_N R_0'\varphi_0\sqrt{g_0}\,dx.
\end{aligned}$$

Since $R_0 \equiv 0$ and $\varphi_0 \equiv 1$, combining with (3.26) gives

$$M'(0) = \frac{1}{32\pi}\int_N \|\operatorname{Ric}\|^2. \tag{3.31}$$

Finally, if $\operatorname{Ric} \not\equiv 0$, we know that $M'(0) > 0$. In this case, we can find $t_0 < 0$ such that $M(t_0) < 0$. Therefore $\varphi_{t_0}^4 g_{t_0}$ is a metric with negative mass which contradicts Theorem 2.1. We get that $\operatorname{Ric} \equiv 0$. Since the dimension is 3, $\operatorname{Ric} \equiv 0$ would imply that g is flat. □

Appendix

A.1 Second Covariant Derivative

From the definition of covariant derivative of a tensor, we know if f is a function, then $\nabla^2 f$ is a $(0, 2)$-tensor defined as

$$(\nabla^2 f)(X, Y) = \nabla(\nabla f)(X, Y) = \nabla_X ((\nabla f)Y) - (\nabla f)(\nabla_X Y) = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f.$$

Hence this gives us a motivation to define the following notation.

Definition A.1. For vector fields X, Y, we define the **second covariant derivative** as

$$\nabla^2_{XY} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$

Under this notation we can see that there Riemann curvature tensor has the following formula

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = (\nabla_X \nabla_Y - \nabla_{\nabla_X Y}) - (\nabla_Y \nabla_X - \nabla_{\nabla_Y X}) = \nabla^2_{XY} - \nabla^2_{YX}.$$

Also let u be a smooth function, under this notation, we have

$$\begin{aligned} \langle \nabla_i \nabla u, e_j \rangle &= e_i \langle \nabla u, e_j \rangle - \langle \nabla u, \nabla_i e_j \rangle \\ &= \nabla_i \nabla_j u - \nabla_{\nabla_i e_j} u \\ &= \nabla^2_{ij} u. \end{aligned}$$

A.2 Conformal Change of Metric

We will derive some formulas to compare the tensors after conformal change. We will assume that $\tilde{g} = e^{2u} g$ where g is metric tensor.

Christoffel Symbols and Levi-Civita Connection

Recall that the formula of Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell}).$$

Then

$$\begin{aligned} \tilde{\Gamma}^k_{ij} &= \frac{1}{2} \tilde{g}^{k\ell}(\tilde{g}_{i\ell,j} + \tilde{g}_{j\ell,i} - \tilde{g}_{ij,\ell}) \\ &= \frac{1}{2} e^{-2u} g^{k\ell} \left((e^{2u} g_{i\ell})_j + (e^{2u} g_{j\ell})_i + (e^{2u} g_{ij})_\ell \right) \\ &= \frac{1}{2} e^{-2u} g^{k\ell} \left(e^{2u}(g_{i\ell,j} + g_{j\ell,i} - g_{ij,\ell}) + 2e^{2u}(g_{i\ell} u_j + g_{j\ell} u_i - g_{ij} u_\ell) \right) \\ &= \Gamma^k_{ij} + \delta^k_i u_j + \delta^k_j u_i - u_\ell g^{k\ell} g_{ij}. \end{aligned}$$

Now let $X = X^i e_i$, $Y = Y^j e_j$ be two vector fields. We have

$$\nabla_X Y = \nabla_{X^i e_i}(Y^j e_j) = X_i \nabla_i Y^j e_j = X^i Y^j_i e_j + X^i Y^j \Gamma^k_{ij} e_k.$$

Then

$$\begin{aligned}\widetilde{\nabla}_X Y &= X^i Y^j_i e_j + X^i Y^j \widetilde{\Gamma}^k_{ij} e_k \\ &= X^i Y^j_i e_j + X^i Y^j (\Gamma^k_{ij} + \delta^k_i u_j + \delta^k_j u_i - u_\ell g^{k\ell} g_{ij}) e_k \\ &= \nabla_X Y + X^i Y^j u_j e_i + X^i Y^j u_i e_j - g(X, Y) u_\ell g^{k\ell} e_k \\ &= \nabla_X Y + X(u) Y + Y(u) X - g(X, Y) \nabla u.\end{aligned}$$

Riemann Curvature Tensor

In the following $\langle X, Y\rangle$ means $g(X, Y)$. We know that

$$R_{ijk\ell} = \langle \nabla_i \nabla_j e_k - \nabla_j \nabla_i e_k, e_\ell \rangle.$$

Using the formula of $\widetilde{\nabla}_X Y$, we have

$$\begin{aligned}\widetilde{\nabla}_i \widetilde{\nabla}_j e_k =& \widetilde{\nabla}_i \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right) \\ =& \nabla_i \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right) + u_i \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right) \\ &+ \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right)(u) e_i - \langle e_i, \nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u \rangle \nabla u \\ =& \nabla_i \nabla_j e_k + u_{ik} e_j + u_k \nabla_i e_j + u_{ij} e_k + u_j \nabla_i e_k - g_{jk,i} \nabla u - g_{jk} \nabla_i \nabla u \\ &+ u_i \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right) + \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right)(u) e_i \\ &- \langle e_i, \nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u \rangle \nabla u.\end{aligned}$$

Simliarly, we have

$$\begin{aligned}\widetilde{\nabla}_j \widetilde{\nabla}_i e_k =& \nabla_j \nabla_i e_k + u_{jk} e_i + u_k \nabla_j e_i + u_{ij} e_k + u_i \nabla_j e_k - g_{ik,j} \nabla u - g_{ik} \nabla_j \nabla u \\ &+ u_j \left(\nabla_i e_k + u_k e_i + u_i e_k - g_{ik} \nabla u\right) + \left(\nabla_i e_k + u_k e_i + u_i e_k - g_{ik} \nabla u\right)(u) e_j \\ &- \langle e_j, \nabla_i e_k + u_k e_i + u_i e_k - g_{ik} \nabla u \rangle \nabla u.\end{aligned}$$

After cancelling the same terms, we obtain

$$\begin{aligned}\widetilde{\nabla}_i \widetilde{\nabla}_j e_k - \widetilde{\nabla}_j \widetilde{\nabla}_i e_k =& \nabla_i \nabla_j e_k - \nabla_j \nabla_i e_k + u_{ik} e_j - u_{jk} e_i - g_{jk,i} \nabla u + g_{ik,j} \nabla u \\ &- g_{jk} \nabla_i \nabla u + g_{ik} \nabla_j \nabla u + u_i u_k e_j - u_j u_k e_i - u_i g_{jk} \nabla u + u_j g_{ik} \nabla u \\ &+ \left(\nabla_j e_k + u_k e_j + u_j e_k - g_{jk} \nabla u\right)(u) e_i \\ &- \left(\nabla_i e_k + u_k e_i + u_i e_k - g_{ik} \nabla u\right)(u) e_j \\ &+ \Big[\langle e_j, \nabla_i e_k + u_i e_k - g_{ik} \nabla u \rangle - \langle e_i, \nabla_j e_k + u_j e_k - g_{jk} \nabla u \rangle\Big] \nabla u.\end{aligned}$$

By definition,

$$\widetilde{R}_{ijk\ell} = \widetilde{g}\left(\widetilde{\nabla}_i \widetilde{\nabla}_j e_k - \widetilde{\nabla}_j \widetilde{\nabla}_i e_k, e_\ell\right) = e^{2u} \langle \widetilde{\nabla}_i \widetilde{\nabla}_j e_k - \widetilde{\nabla}_j \widetilde{\nabla}_i e_k, e_\ell \rangle.$$

Combining the result computed above,

$$
\begin{aligned}
\widetilde{R}_{ijk\ell} = e^{2u}\Big[& R_{ijk\ell} + u_{ik}g_{j\ell} - u_{jk}g_{i\ell} - g_{jk,i}u_\ell + g_{ik,j}u_\ell - g_{jk}\nabla^2_{i\ell}u + g_{ik}\nabla^2_{j\ell}u \\
& + u_i u_k g_{j\ell} - u_j u_k g_{i\ell} - u_i u_\ell g_{jk} + u_j u_\ell g_{ik} \\
& + g_{i\ell}\left(\Gamma^s_{jk}u_s + 2u_j u_k - g_{jk}(\nabla u)(u)\right) \\
& - g_{j\ell}\left(\Gamma^s_{ik}u_s + 2u_i u_k - g_{ik}(\nabla u)(u)\right) \\
& + u_\ell\left(\Gamma^s_{ik}g_{sj} + u_i g_{jk} - u_j g_{ik} - \Gamma^s_{jk}g_{si} - u_j g_{ik} + u_i g_{jk}\right)\Big].
\end{aligned}
$$

By collecting the $g_{i\ell}$, $g_{j\ell}$, g_{ik}, g_{jk}, and u_ℓ terms together,

$$
\begin{aligned}
\widetilde{R}_{ijk\ell} = e^{2u}R_{ijk\ell} + e^{2u}\Big[& g_{i\ell}(-u_{jk} + u_j u_k + \Gamma^s_{jk}u_s - g_{jk}(\nabla u)(u)) \\
& + g_{j\ell}(u_{ik} - u_i u_k - \Gamma^s_{ik}u_s + g_{ik}(\nabla u)(u)) \\
& + g_{ik}(\nabla^2_{j\ell}u - u_j u_\ell) + g_{jk}(-\nabla^2_{i\ell}u + u_i u_\ell) \\
& + u_\ell\left(g_{ik,j} - g_{jk,i} + \Gamma^s_{ik}g_{sj} - \Gamma^s_{jk}g_{si}\right)\Big].
\end{aligned}
$$

Since

$$
g_{ik,j} = \Gamma^s_{ij}g_{sk} + \Gamma^s_{jk}g_{si}, \quad g_{jk,i} = \Gamma^s_{ij}g_{sk} + \Gamma^s_{ik}g_{si},
$$

we see that the last term vanishes. On the other hand,

$$
\nabla^2_{ij}u = u_{ij} - \Gamma^k_{ij}u_k.
$$

From this we see that

$$
\begin{aligned}
\widetilde{R}_{ijk\ell} = e^{2u}R_{ijk\ell} + e^{2u}\Big[& g_{i\ell}(-\nabla^2_{jk}u + \nabla_j u\nabla_k u) + g_{j\ell}(\nabla^2_{ik}u - \nabla_i u\nabla_k u) \\
& + g_{ik}(\nabla^2_{j\ell}u - \nabla_j u\nabla_k u) + g_{jk}(-\nabla^2_{i\ell}u + \nabla_i u\nabla_\ell u) \\
& + 2\left(-\frac{1}{2}g_{i\ell}g_{jk}(\nabla u)(u) + \frac{1}{2}g_{j\ell}g_{ik}(\nabla u)(u)\right)\Big]
\end{aligned}
$$

Since $\nabla u = g^{ij}u_i e_j$, we have

$$
\nabla u(u) = g^{ij}u_i u_j = |du|^2.
$$

Finally, we obtain

$$
\widetilde{R}_{ijkl} = e^{2u}R_{ijk\ell} + e^{2u}(g_{ik}T_{j\ell} + g_{j\ell}T_{ik} - g_{i\ell}T_{jk} - g_{jk}T_{i\ell}),
$$

where

$$
T_{ij} = \nabla^2_{ij}u - \nabla_i u\nabla_j u + \frac{1}{2}|du|^2 g_{ij}.
$$

Ricci Curvature and Scalar Curvature

By the definition,

$$\operatorname{Ric}_{jk} = g^{i\ell} R_{ijk\ell}.$$

Hence, from the formula of $\widetilde{R}_{ijk\ell}$, we have

$$\begin{aligned}
\widetilde{\operatorname{Ric}}_{jk} &= \tilde{g}^{i\ell}\widetilde{R}_{ijk\ell} = e^{-2u} g^{i\ell}\left(e^{2u}R_{ijk\ell} + e^{2u}(g_{ik}T_{j\ell} + g_{j\ell}T_{ik} - g_{i\ell}T_{jk} - g_{jk}T_{i\ell})\right) \\
&= \operatorname{Ric}_{jk} + \delta_k^\ell T_{j\ell} + \delta_j^i T_{ik} - nT_{jk} - g^{i\ell}g_{jk}T_{i\ell} \\
&= \operatorname{Ric}_{jk} - (n-2)T_{jk} - g^{i\ell}g_{jk}\left(\nabla^2_{i\ell}u - \nabla_i u\nabla_j u + \frac{1}{2}|du|^2 g_{i\ell}\right) \\
&= \operatorname{Ric}_{jk} - (n-2)\left(\nabla^2_{jk}u - \nabla_j u\nabla_k u + \frac{1}{2}|du|^2 g_{jk}\right) - g_{jk}\left(\Delta u - |du|^2 + \frac{n}{2}|du|^2\right) \\
&= \operatorname{Ric}_{jk} - (n-2)\left(\nabla^2_{jk}u - \nabla_j u\nabla_k u\right) - g_{jk}\left(\Delta u + (n-2)|du|^2\right).
\end{aligned}$$

For scalar curvature, we know

$$R = g^{jk}\operatorname{Ric}_{jk}.$$

Hence,

$$\begin{aligned}
\widetilde{R} &= \tilde{g}^{jk}\widetilde{\operatorname{Ric}}_{jk} \\
&= e^{-2u}g^{jk}\left(\operatorname{Ric}_{jk} - (n-2)(\nabla^2_{jk}u - \nabla_j u\nabla_k u) - g_{jk}(\Delta u + (n-2)|du|^2)\right) \\
&= e^{-2u}R - e^{-2u}\left((n-2)(\Delta u - |du|^2) + n(\Delta u + (n-2)|du|^2)\right) \\
&= e^{-2u}R - e^{-2u}\left(2(n-1)\Delta u + (n-1)(n-2)|du|^2\right).
\end{aligned}$$

For $n \neq 2$, we let $e^{2u} = v^{\frac{4}{n-2}}$. Then

$$u = \frac{2}{n-2}\log v.$$

By chain rule, we have

$$du = \frac{2}{n-2}\frac{dv}{v},$$

$$\Delta u = \frac{2}{n-2}\left(\frac{\Delta v}{v} - \frac{|dv|^2}{v^2}\right).$$

Putting these into the above formula, we obtain

$$\begin{aligned}
\widetilde{R} &= v^{-\frac{4}{n-2}}\left(R - 4\frac{n-1}{n-2}\left(\frac{\Delta v}{v} - \frac{|dv|^2}{v^2}\right) - 4\frac{n-1}{n-2}\frac{|dv|^2}{v^2}\right) \\
&= v^{-\frac{4}{n-2}}\left(R - 4\frac{n-1}{n-2}\frac{\nabla v}{v}\right) \\
&= e^{-2u}\left(R - 4\frac{n-1}{n-2}e^{-\frac{n-2}{2}u}\Delta(e^{\frac{n-2}{2}u})\right).
\end{aligned}$$

A.3 Transversality Theorem

Definition. Let M, N be two submanifolds of X. We say that M and N **intersect transversally** if for any $x \in M \cap N$,

$$T_xM + T_xN = T_xX,$$

and we denote it by $M \pitchfork N$.

Definition. Let $f : X \to Y$ be a smooth map between manifolds and Z is a submanifold of Y. We say that f is **transverse** to Z if for any $a \in f^{-1}(Z)$

$$df_a(T_aX) + T_{f(a)}Z = T_{f(a)}Y.$$

Also, we write $f \pitchfork Z$ in this case.

Recall that we know that if $f \pitchfork Z$, then $f^{-1}(Z)$ is a regular submanifold of X. Now, if $f : X \to Y$ is a smooth map with X has boundary, then we denote ∂f to be the map $f|_{\partial X}$. If $f \pitchfork Z$ and $\partial f \pitchfork Z$, then we see $f^{-1}(Z)$ is a manifold with boundary and

$$\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X.$$

Theorem (Thom's Transversality Theorem)**.** *Suppose that $F : X \times S \to Y$ is a smooth map between manifolds with only X has boundary, and Z is a submanifold (without boundary) of Y. If $F \pitchfork Z$ and $\partial F \pitchfork Z$, then $f_s \pitchfork Z$ and $f_s \pitchfork Z$ for almost all $s \in S$ where $f_s : X \to Y$ is defined as $f_s(x) = F(x, s)$. Here, almost all means that except for a measure zero set.*

Proof. Let $W = F^{-1}(Z)$. Let $\pi : X \times S \to S$ be the natural projection map. We will show that for any regular value s of $\pi|_W$, we have $f_s \pitchfork Z$, and for any regular value s of $\partial\pi|_W$. Then by Sard's theorem, we are done. Since $F \pitchfork Y$ clearly, W is a submanifold with boundary

$$\partial W = W \cap \partial(X \times S).$$

Let s be a regular value of $\pi|_W$. Choose any $x \in f_s^{-1}(Z)$ and let $f_s(x) = z \in Z$. By the assumption that $F \pitchfork Z$, we have

$$dF|_{(x,s)}(T_{(x,s)}(X \times S)) + T_zZ = T_zY.$$

That is, for any $a \in T_zY$, there exists $b \in T_{(x,s)}(X \times S)$ such that

$$dF|_{(x,s)}(b) - a \in T_zZ. \tag{A.1}$$

The thing we want to show is that there exist $v \in T_xX$ such that

$$df_s|_x(v) - a \in T_zZ. \tag{A.2}$$

Since

$$T_{(x,s)}(X \times S) = T_xX \times T_sS,$$

there is a vector $(w, e) \in T_xX \times T_sS$ corresponding to b we have chosen. If $e = 0$, then we choose $w = v$. Observe that

$$dF|_{(x,s)}(b) = df_s|_x(w)$$

in this case. Hence (A.1) implies (A.2).
Now, for $e \neq 0$, remember that s is the regular value of $\pi|_W$ and the map

$$d\pi|_{(x,s)} : T_xX \times T_sS \to T_sS$$

is just project to the second component. This means that there exists $(u, e) \in T_{(x,s)}W$, and hence

$$dF|_{(x,s)}(u, e) \in T_zZ. \tag{A.3}$$

Now let $v = w - u$, then (A.2) holds since

$$\begin{aligned} df_s|_x(v) - a &= df_s|_x(w - u) - a \\ &= dF_{(x,s)}((w, e) - (u, e)) - a \\ &= dF_{(x,s)}(b) - a - dF_{(x,s)}(u, e) \end{aligned}$$

is a vector in T_zZ which follows from (A.1) and (A.3). Therefore, (A.2) is true and $f_s \pitchfork Z$. For the $\partial f_s \pitchfork Z$ is exactly the same argument as $f_s \pitchfork Z$ by changing s to be the regular value of $\partial\pi|_W$. □

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