Multi Period

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Objective:

population $k \in \{1,2\}$

$$J^k(g,\Gamma,a) = \mathbb{E}\left[\int_0^{T_2} \frac{\zeta^k}{2} g_t^2 + \frac{\gamma^k}{2}\Gamma_t^2 + \frac{\beta^k}{2} a_t^2 + S_t \Gamma_t \, dt\right.$$

$$\left. + (R - X_{T_1})_+ + \left(R - X_{T_2} + X_{T_1} - (X_{T_1} - R)_+\right)_+\right]$$

$X_{T_2} - X_{T_1} + (X_{T_1} - R)_+$ is the amt leftover after period 1 quota.

$S = (S_t)_{t \geq 0}$ market price of goods.

Equilibrium clearing condition:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \Gamma_t^i = 0$$

where here Γ^i denotes that we draw at random a population $k \in \{1,2\}$ according to their size and then a copy of Γ from that population.

Here: $dX_t = (h + g_t + \Gamma_t + C_t)dt + \sigma dW_t$, $X_0 \sim \xi$

$dC_t = a_t \, dt$, $C_0 = 0$

"Differentiating" X w.r.t. g and Γ in the direction η:

$$\partial_g X_t = \int_0^t \eta_s \, ds, \quad \partial_\Gamma X_t = \int_0^t \eta_s \, ds.$$

η adapted

If we differentiate w.r.t. a:

$$\partial_a C_t = \int_0^t \eta_s ds, \quad \partial_a X_t = \int_0^t \partial_a C_s \, ds = \int_0^t \int_0^s \eta_u du \, ds,$$

"Differentiating" J^k w.r.t. g in the direction η:

$$\partial_g J^k(g,\Gamma,a;\eta) = \mathbb{E}\left[\int_0^{T_2} \zeta^k g_t \eta_t \, dt - \mathbb{1}_{X_{T_1} < R}\, \partial_g X_{T_1}\right.$$

$$\left. - \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ < R}\, \partial_g X_{T_2} + \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ \leq R}\, \partial_g X_{T_1}\right.$$

$$- \mathbb{1}_{X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \, \mathbb{1}_{X_{T_1}>R} \, \partial_g X_{T_1} \Big]$$

$$= \mathbb{E}\Big[\int_0^{T_2} \zeta^k g_t \eta_t \, dt - \int_0^{T_1} \mathbb{1}_{X_{T_1}<R} \, \eta_t \, dt$$

$$- \int_{T_1}^{T_2} \mathbb{1}_{X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \, \eta_t \, dt$$

$$- \int_0^{T_1} \mathbb{1}_{X_{T_1}>R,\, X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \, \eta_t \, dt \Big]$$

$$= \mathbb{E}\Big[\int_0^{T_1} \Big[\zeta^k g_t - \mathbb{1}_{X_{T_1}<R} - \mathbb{1}_{X_{T_1}>R,\, X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \Big] \eta_t \, dt$$

$$+ \int_{T_1}^{T_2} \Big[\zeta^k g_t - \mathbb{1}_{X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \Big] \eta_t \, dt \Big]$$

Iterating expectations:

$$\partial_g J^k = \mathbb{E}\Big[\int_0^{T_1} \big[\zeta^k g_t - V_t - U_t \big] \eta_t \, dt + \int_{T_1}^{T_2} \big[\zeta^k g_t - \varphi_t \big] \eta_t \, dt \Big]$$

where:

$$V_t = \mathbb{E}\big[\mathbb{1}_{X_{T_1}<R} \,|\, \mathcal{F}_t \big] = \mathbb{P}\big(X_{T_1}<R \,|\, \mathcal{F}_t \big)$$ ← Prob. of non-compliance in period 1

$$U_t = \mathbb{E}\big[\mathbb{1}_{X_{T_1}>R,\, X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \,|\, \mathcal{F}_t \big]$$

$$= \mathbb{P}\big(X_{T_1}>R,\ X_{T_2}-X_{T_1}+(X_{T_1}-R)<R \,|\, \mathcal{F}_t \big)$$ ← Prob. of compliance in period 1 and non-compliance in period 2.

$$\varphi_t = \mathbb{E}\big[\mathbb{1}_{X_{T_2}-X_{T_1}+(X_{T_1}-R)_+<R} \,|\, \mathcal{F}_t \big]$$

$$= \mathbb{P}\big(X_{T_2}-X_{T_1}+(X_{T_1}-R)_+ <R \,|\, \mathcal{F}_t \big)$$ ← Prob. of non-compliance in period 2.

Similarly,

$$\text{"}\left(X_{T_2} - X_{T_1} + (X_{T_1} - K)_+ < K \mid \mathcal{F}_t \right)$$ below 2.

Similarly:

$$\partial_{p_2} J^k = \mathbb{E}\Bigg[\int_0^{T_1} \left[\gamma^k \Gamma_t + S_t - V_t - U_t\right] \eta_t \, dt + \int_{T_1}^{T_2} \left[\gamma^k \Gamma_t + S_t - Y_t\right] \eta_t \, dt \Bigg]$$

Lastly:

$$\partial_a J^k : \mathbb{E}\Bigg[\int_0^T \beta^k a_t \eta_t dt - \int_0^{T_1}\int_0^t \mathbb{1}_{X_{T_1} < R} \, \eta_u \, du \, dt$$

$$- \int_0^{T_2}\int_0^t \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ < R} \, \eta_u \, du \, dt$$

$$+ \int_0^{T_1}\int_0^t \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ < R} \, \eta_u \, du \, dt$$

$$- \int_0^{T_1}\int_0^t \mathbb{1}_{X_{T_1} > R,\, X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ < R} \, \eta_u \, du \, dt \Bigg]$$

$$= \mathbb{E}\Bigg[\int_0^T \beta^k a_t \eta_t dt - \int_0^{T_1}\int_0^t \left(\mathbb{1}_{X_{T_1} < R} + \mathbb{1}_{X_{T_1} > R, \dots}\right) \eta_u \, du \, dt$$

$$- \int_0^{T_2}\int_0^t \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R) < R} \, \eta_u \, du \, dt$$

$$+ \int_0^{T_1}\int_0^t \mathbb{1}_{X_{T_2} \dots < R} \, \eta_u \, du \, dt \Bigg]$$

Change order of integration

$$= \mathbb{E}\Bigg[\int_0^T \beta^k a_t \eta_t dt - \int_0^{T_1}\int_u^{T_1} \left(\mathbb{1}_{X_{T_1} < R} + \mathbb{1}_{X_{T_1} > R, \dots}\right) \eta_u \, dt \, du$$

$$- \int_0^{T_2}\int_u^{T_2} \mathbb{1}_{X_{T_2} \dots} \, \eta_u \, dt \, du$$

$$+ \int_0^{T_1}\int_u^{T_1} \mathbb{1}_{X_{T_2} \dots} \, \eta_u \, dt \, du \Bigg]$$

$$+\int_0^{T_1}\int_u^{T_1} \mathbb{1}_{X_{T_2}\dots}\, \eta_u \,dt\, du \Big]$$

$$= \mathbb{E}\Big[\int_0^T \beta^k a_t \eta_t dt - \int_0^{T_1} (T_1-u)\left(\mathbb{1}_{X_{T_1}<R} + \mathbb{1}_{\dots}\right)\eta_u du$$

$$- \int_0^{T_2} (T_2-u)\left[\mathbb{1}_{X_{T_2}\dots}\right]\eta_u du$$

$$+ \underbrace{\int_0^{T_1} (T_1-u)\,\mathbb{1}_{X_{T_2}\dots}\Big]\eta_u du}_{\int_0^{T_1} (T_2-u)\,\mathbb{1}_{X_{T_2}\dots} - (T_2-T_1)\,\mathbb{1}_{X_{T_2}\dots}\Big]\eta_u du} \Big]$$

$$= \mathbb{E}\Big[\int_0^{T_1} \left[\beta^k a_t - (T_1-t)(V_t+U_t) - (T_2-T_1)\,Y_t\right]\eta_t dt$$

$$+ \int_{T_1}^{T_2} \left[\beta^k a_t - (T_2-t)\,Y_t\right]\eta_t dt \Big]$$

Arguing that these should vanish for all η when (g, Γ, a) are optimal we get the first order conditions:

$$g_t^* = \frac{V_t+U_t}{\zeta^k}\mathbb{1}_{t\in[0,T_1]} + \frac{Y_t}{\zeta^k}\mathbb{1}_{t\in(T_1,T_2]}$$

$$\Gamma_t^* = \frac{V_t+U_t-S_t}{\gamma^k}\mathbb{1}_{t\in[0,T_1]} + \frac{Y_t-S_t}{\gamma^k}\mathbb{1}_{t\in[T_1,T_2]}$$

$$a_t^* = \frac{(T_1-t)(V_t+U_t) + (T_2-T_1)\,Y_t}{\beta^k}\mathbb{1}_{t\in[0,T_1]}$$

$$+ \frac{(T_2-t)\,Y_t}{\beta^k}\mathbb{1}_{t\in[T_1,T_2]}$$

$B^k \quad \forall t \in [T_1, T_2]$

Enforcing the equilibrium clearing condition at optimality:

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \nu^i = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{\gamma_1}\left((V_t^i + U_t^i - S_t)\mathbb{1}_{t\in[0,T_1]} + (\Psi_t^i - S_t^i)\mathbb{1}_{t\in[T_1,T_2]} \right)\mathbb{1}_{k=1} + \frac{1}{\gamma_2}(- - - - \cdot)\mathbb{1}_{k=2} \right\}$$

Assuming proportion of total pop. from k is π_k we have:

$$\left(\frac{\pi_1}{\gamma_1} + \frac{\pi_2}{\gamma_2}\right) S_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{\gamma_1}\left[(V_t^i + U_t^i)\mathbb{1}_{t\in[0,T_1]} + \Psi_t^i \mathbb{1}_{t\in[T_1,T_2]} \right]\mathbb{1}_{k=1} + \frac{1}{\gamma_2}\left[- - - - \right]\mathbb{1}_{k=2} \right\}$$

"Law of large numbers type of result"

Representative process for pop. 1

$$= \frac{\pi_1}{\gamma_1}\left[\mathbb{E}\left[V_t^1 + U_t^1\right]\mathbb{1}_{t\in[0,T_1]} + \mathbb{E}\left[\Psi_t^1\right]\mathbb{1}_{t\in[T_1,T_2]} \right]$$

representative process for pop. 2

$$+ \frac{\pi_2}{\gamma_2}\left[\mathbb{E}\left[V_t^2 + U_t^2\right]\mathbb{1}_{t\in[0,T_1]} + \mathbb{E}\left[\Psi_t^2\right]\mathbb{1}_{t\in[T_1,T_2]} \right],$$

We conclude:

$$S_t^* = \left(\omega_1 \mathbb{E}\left[V_t^1 + U_t^1\right] + \omega_2 \mathbb{E}\left[V_t^2 + U_t^2\right]\right)\mathbb{1}_{t\in[0,T_1]} + \left(\omega_1 \mathbb{E}\left[\Psi_t^1\right] + \omega_2 \mathbb{E}\left[\Psi_t^2\right]\right)\mathbb{1}_{t\in[T_1,T_2]}$$

Where $\omega_1 = \dfrac{\frac{\pi_1}{\gamma_1}}{\pi_1 \quad \pi_2}$, $\omega_2 = \dfrac{\frac{\pi_2}{\gamma_2}}{\pi_1}$

where $\quad \omega_1 = \dfrac{\gamma_1}{\dfrac{\pi_1}{\gamma_1} + \dfrac{\pi_2}{\gamma_2}} \quad , \quad \omega_2 = \dfrac{\gamma_2}{\dfrac{\pi_1}{\gamma_1} + \dfrac{\pi_2}{\gamma_2}}$

Using the Mtg representation them we have, for $k \in \{1,2\}$ the FBSDE system:

$$\begin{cases} dX_t^k = (h^k + g_t^{*,k} + \Gamma_t^{*,k} + C_t^{*,k})\,dt + \sigma^k dW_t \,, & X_0^k \sim \xi^k \\ dC_t^{*,k} = a^{*,k}\,dt \,, & C_0^{*,k} = 0 \\ dV_t^k = Z_t^{V,k}\,dW_t \,, & V_{T_1}^k = \mathbb{1}_{X_{T_1} < R} \\ dU_t^k = Z_t^{U,k}\,dW_t \,, & U_{T_1}^k = \mathbb{1}_{X_{T_1} > R}\, Y_{T_1} \\ dY_t^k = Z_t^{Y,k}\,dW_t \,, & Y_{T_2}^k = \mathbb{1}_{X_{T_2} - X_{T_1} + (X_{T_1} - R)_+ < R} \end{cases}$$

And the definitions of g^*, Γ^*, a^* are given above.

highlighted in red