In [ ]:
```
import numpy as np
import torch as torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
import seaborn as sns
import pandas as pd
import matplotlib.pyplot as plt
from datetime import datetime
import random
from scipy.stats import norm

import os
import pathlib

from Model import *
from utils import *

torch.autograd.set_detect_anomaly(True)
```

Out[ ]:
```
<torch.autograd.anomaly_mode.set_detect_anomaly at 0x3110fb010>
```

Problem & Objective

$$F(g,\ \Gamma,\ a,\ X) := \mathbb{E}\left[\int_0^{T_2} \frac{1}{2}\left(\left(\xi {g_t}^2 dt + \gamma {\Gamma_t}^2 + \beta {a_t}^2\right) + S_t\Gamma_t\right) dt + w(K - X_{T_1})_+ + w(K - X_{T_2} + X_{T_1} - (X_{T_1} - K)_+)_+\right] \tag{1}$$

$$\mathcal{J}(g,\ \Gamma,\ a,\ X) := \inf_{g,\,\Gamma,\,a} F(g,\ \Gamma,\ a,\ X) \tag{2}$$

, where:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \Gamma_t^{(i)} \& = 0 \tag{3}$$

$$dX_t = (h + g_t + \Gamma_t + C_t)\, dt + \sigma dB_t, \quad X_0 \sim \xi = \mathcal{N}(v, \eta) \tag{4}$$

$$dC_t = a_t dt, \quad C_0 = 0 \tag{5}$$

And by differentiating X w.r.t. g and Γ in an arbitrary direction η:

$$\partial_g X_t = \int_0^t \eta_s ds \tag{6}$$

$$\partial_\Gamma X_t = \int_0^t \eta_s ds \tag{7}$$

Differentiate X and C w.r.t. a in an arbitrary direction η:

$$\partial_a C_t = \int_0^t \eta_s ds \tag{8}$$

$$\partial_a X_t = \int_0^t \partial_a C_s ds = \int_0^t \int_0^s \eta_u du\, ds \tag{9}$$

Thus, differentiate $\mathcal{J}$ w.r.t. g in an arbitrary direction η and by iterating expectations:

$$\begin{aligned}
\partial_g \mathcal{J}(g,\ \Gamma,\ a,\ \eta) &= \mathbb{E}\left[\int_0^{T_2} \xi g_t \eta_t\, dt - w \mathbb{I}_{X_{T_1}<K} \partial_g X_{T_1} - w \mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K} \left(\partial_g X_{T_2} - \partial_g X_{T_1} + \mathbb{I}_{X_{T_1}>K} \partial_g X_{T_1}\right)\right] && (10)\\
&= \mathbb{E}\Bigg[\int_0^{T_1} \left(\xi g_t - w \mathbb{I}_{X_{T_1}<K} - w \mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K}\right) \eta_t dt && (11)\\
&\quad + \int_{T_1}^{T_2} \left(\xi g_t - w \mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K}\right) \eta_t dt\Bigg] && (12)\\
&= \mathbb{E}\Bigg[\int_0^{T_1} \left(\xi g_t - w \mathbb{E}\left[\mathbb{I}_{X_{T_1}<K} | \mathcal{F}_t\right] - w \mathbb{E}\left[\mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K} | \mathcal{F}_t\right]\right) \eta_t dt && (13)\\
&\quad + \int_{T_1}^{T_2} \left(\xi g_t - w \mathbb{E}\left[\mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K} | \mathcal{F}_t\right]\right) \eta_t dt\Bigg] && (14)\\
&= \mathbb{E}\Bigg[\int_0^{T_1} \left(\xi g_t - w \mathbb{P}\left(X_{T_1} < K | \mathcal{F}_t\right) - w \mathbb{P}\left(X_{T_2} - X_{T_1} + (X_{T_1} - K)_+ < K | \mathcal{F}_t\right)\right) \eta_t dt && (15)\\
&\quad + \int_{T_1}^{T_2} \left(\xi g_t - w \mathbb{P}\left(X_{T_2} - X_{T_1} + (X_{T_1} - K)_+ < K | \mathcal{F}_t\right)\right) \eta_t dt\Bigg] && (16)\\
&= 0 && (17)
\end{aligned}$$

for all η when g is optimal. So we get the first order condition of g_t:

$$g_t = \frac{w\left[\mathbb{P}\left(X_{T_1} < K|\mathcal{F}_t\right) + \mathbb{P}\left(X_{T_2} - X_{T_1} + \left(X_{T_1} - K\right)_+ < K|\mathcal{F}_t\right)\right]}{\xi}\,\mathbb{I}_{t\in[0,T_1)} \tag{18}$$

$$+ \quad \frac{w\mathbb{P}\left(X_{T_2} - X_{T_1} + \left(X_{T_1} - K\right)_+ < K|\mathcal{F}_t\right)}{\xi}\,\mathbb{I}_{t\in[T_1,T_2]} \tag{19}$$

Here we have 2 options: 1) either choose the target to be indicator functions themselves 2) or weighted/scaled indicator functions (i.e., $w\,\mathbb{I}(\cdot),\ 0 < w < 1$). Since when scaled, the loss function will see smaller target values, leading to smaller error gradients and more stable updates, avoiding large oscillations. Thus by scaling/weighting the target indicator functions, we can achieve a smoother convergence process and greater convergence rate.

Hence, g can be written as:

$$g_t = \frac{V_t + U_t}{\xi}\,\mathbb{I}_{t\in[0,T_1)} + \frac{Y_t}{\xi}\,\mathbb{I}_{t\in[T_1,T_2]} \tag{20}$$

, where

$$V_t = \mathbb{E}\left[w\mathbb{I}_{X_{T_1}<K}|\mathcal{F}_t\right] \qquad = w\mathbb{P}\left(X_{T_1} < K|\mathcal{F}_t\right) \tag{21}$$

$$U_t = \mathbb{E}\left[w\mathbb{I}_{X_{T_1}>K,\, X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K}|\mathcal{F}_t\right] \qquad = w\mathbb{P}\left(X_{T_1} > K,\ X_{T_2} < 2K|\mathcal{F}_t\right) \tag{22}$$

$$Y_t = \mathbb{E}\left[w\mathbb{I}_{X_{T_2}-X_{T_1}+(X_{T_1}-K)_+<K}|\mathcal{F}_t\right] \qquad = w\mathbb{P}\left(X_{T_2} - X_{T_1} + \left(X_{T_1} - K\right)_+ < K|\mathcal{F}_t\right) \tag{23}$$

Similarly, we differentitate $\mathcal{J}$ w.r.t. Γ and a in an arbitrary direction η and apply iterating expectations, arguing that all η terms should be 0 for any η. Thus we get the following first order conditions:

$$\Gamma_t = \frac{V_t + U_t - S_t}{\gamma}\,\mathbb{I}_{t\in[0,T_1)} + \frac{Y_t - S_t}{\gamma}\,\mathbb{I}_{t\in[T_1,T_2]} \tag{24}$$

$$a_t = \frac{\left(T_1 - t\right)\left(V_t + U_t\right) + \left(T_2 - T_1\right)Y_t}{\beta}\,\mathbb{I}_{t\in[0,T_1)} + \frac{\left(T_2 - t\right)Y_t}{\beta}\,\mathbb{I}_{t\in[T_1,T_2]} \tag{25}$$

Enforcing the Equilibrium Market Clearing Condition ar optimality:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\Gamma^{(i)} \tag{26}$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma}\left[\left(V_t^{(i)} + U_t^{(i)} - S_t\right)\,\mathbb{I}_{t\in[0,T_1)} + \left(Y_t^{(i)} - S_t\right)\,\mathbb{I}_{t\in[T_1,T_2]}\right] \tag{27}$$

$$= 0 \tag{28}$$

Thus, we get the market clearing price as follows:

$$S_t = \mathbb{E}\left[V_t + U_t\right] \ \mathbb{I}_{t\in[0,T_1)} + \mathbb{E}\left[Y_t\right] \ \mathbb{I}_{t\in[T_1,T_2]} \tag{29}$$

So putting all these together and extending to the 2-population scenario ($k \in \{1,2\}$):

$$g_t^{(k)} = \frac{V_t^{(k)} + U_t^{(k)}}{\zeta^{(k)}} \ \mathbb{I}_{t\in[0,T_1)} + \frac{Y_t^{(k)}}{\zeta^{(k)}} \ \mathbb{I}_{t\in[T_1,T_2]} \tag{30}$$

$$\Gamma_t^{(k)} = \frac{V_t^{(k)} + U_t^{(k)} - S_t}{\gamma^{(k)}} \ \mathbb{I}_{t\in[0,T_1)} + \frac{Y_t^{(k)} - S^t}{\gamma^{(k)}} \ \mathbb{I}_{t\in[T_1,T_2]} \tag{31}$$

$$a_t^{(k)} = \frac{(T_1 - t)(V_t^{(k)} + U_t^{(k)}) + (T_2 - T_1)Y_t}{\beta^{(k)}} \ \mathbb{I}_{t\in[0,T_1)} + \frac{(T_2 - t)Y_t^{(k)}}{\beta^{(k)}} \ \mathbb{I}_{t\in[T_1,T_2]} \tag{32}$$

$$S_t = \left(\frac{\frac{\pi_1}{\gamma_1}}{\frac{\pi_1}{\gamma_1} + \frac{\pi_2}{\gamma_2}} \mathbb{E}[V_t^{(1)} + U_t^{(1)}] + \frac{\frac{\pi_2}{\gamma_2}}{\frac{\pi_1}{\gamma_1} + \frac{\pi_2}{\gamma_2}} \mathbb{E}[V_t^{(2)} + U_t^{(2)}] \right) \mathbb{I}_{t\in[0,T_1)} \tag{33}$$

$$+ \left(\frac{\frac{\pi_1}{\gamma_1}}{\frac{\pi_1}{\gamma_1} + \frac{\pi_2}{\gamma_2}} \mathbb{E}[Y_t^{(1)}] + \frac{\frac{\pi_2}{\gamma_2}}{\frac{\pi_1}{\gamma_1} + \frac{\pi_2}{\gamma_2}} \mathbb{E}[Y_t^{(2)}] \right) \mathbb{I}_{t\in[T_1,T_2]} \tag{34}$$

$$\begin{cases} dX_t^{(K)} &= (h^{(k)} + g_t^{(k)} + \Gamma_t^{(k)} + C_t^{(k)})dt + \sigma^{(k)} dW_t^{(k)} & , \quad X_0^{(k)} \sim \xi^{(k)} \\ dC_t^{(k)} &= a_t^{(k)} dt & , \quad C_0^{(k)} = 0 \\ dV_t^{(k)} &= Z_t^{V,(k)} dW_t^{(k)} & , \quad V_{T_1}^{(k)} = w * \mathbb{I}_{X_{T_1} < K} \\ dU_t^{(k)} &= Z_t^{U,(k)} dW_t^{(k)} & , \quad U_{T_1}^{(k)} = 1 * Y_{T_1} \mathbb{I}_{X_{T_1} > K} \\ dY_t^{(k)} &= Z_t^{Y,(k)} dW_t^{(k)} & , \quad Y_{T_2}^{(k)} = w * \mathbb{I}_{X_{T_2} - X_{T_1} + (X_{T_1} - K)_+ < K} \end{cases}$$

In [ ]:
```
from IPython.display import display, Math, Latex
```

In [ ]:
```
display(Math(r'F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx'))
```

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k} dx$$

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