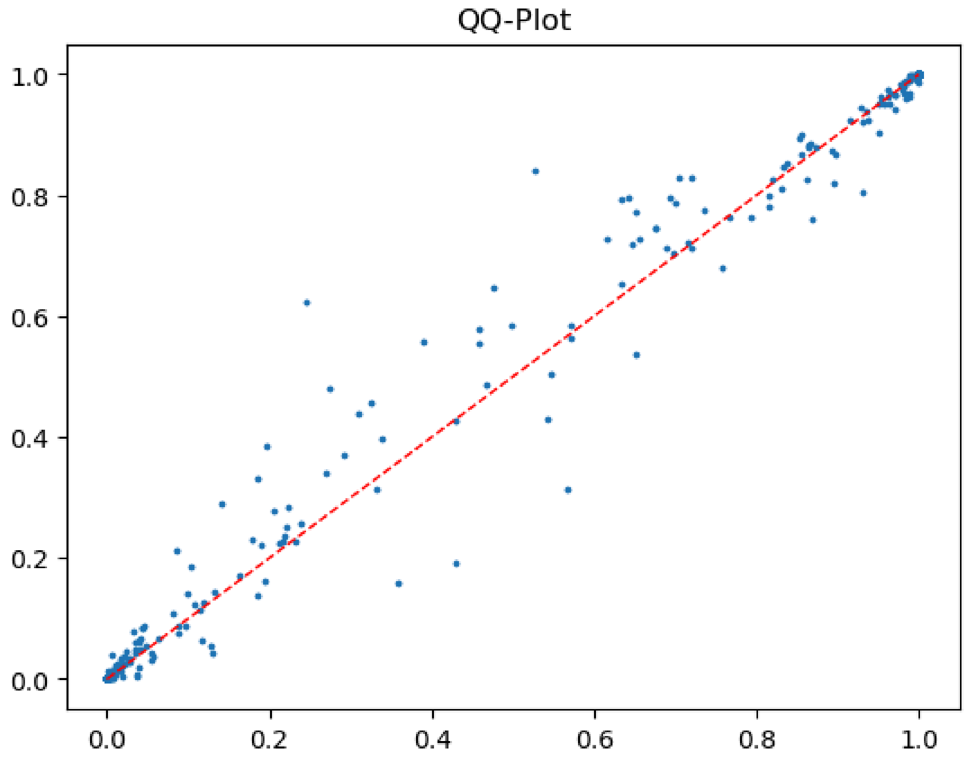


lr=0.002

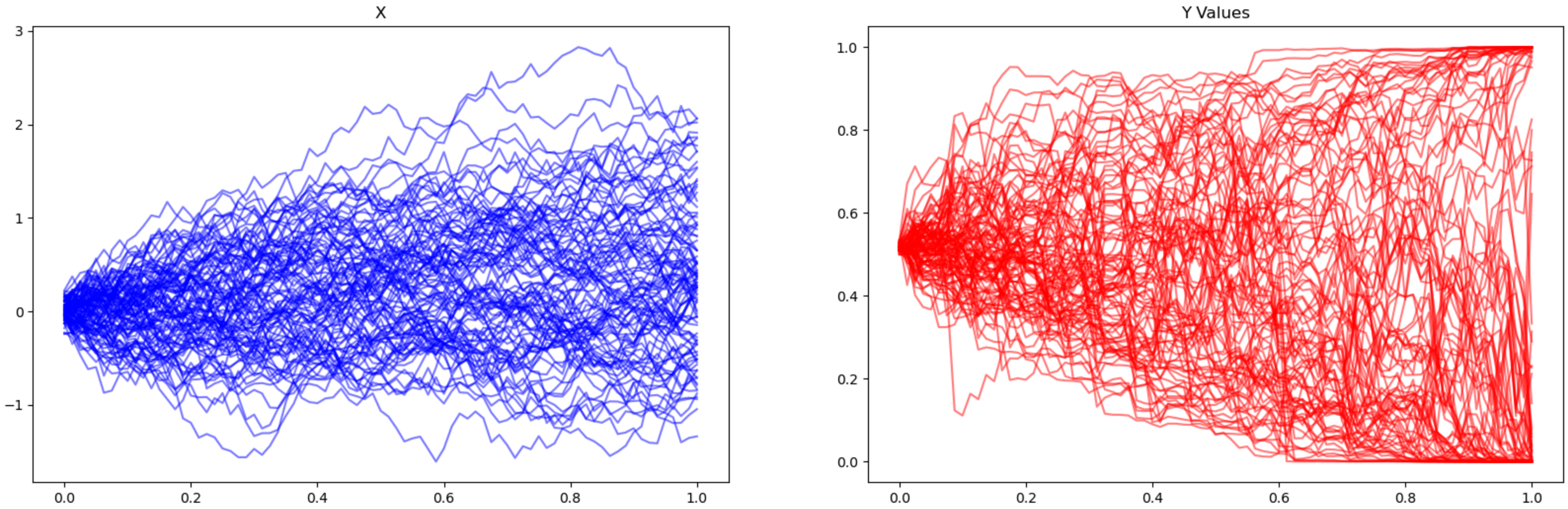
target: sigma=0.08

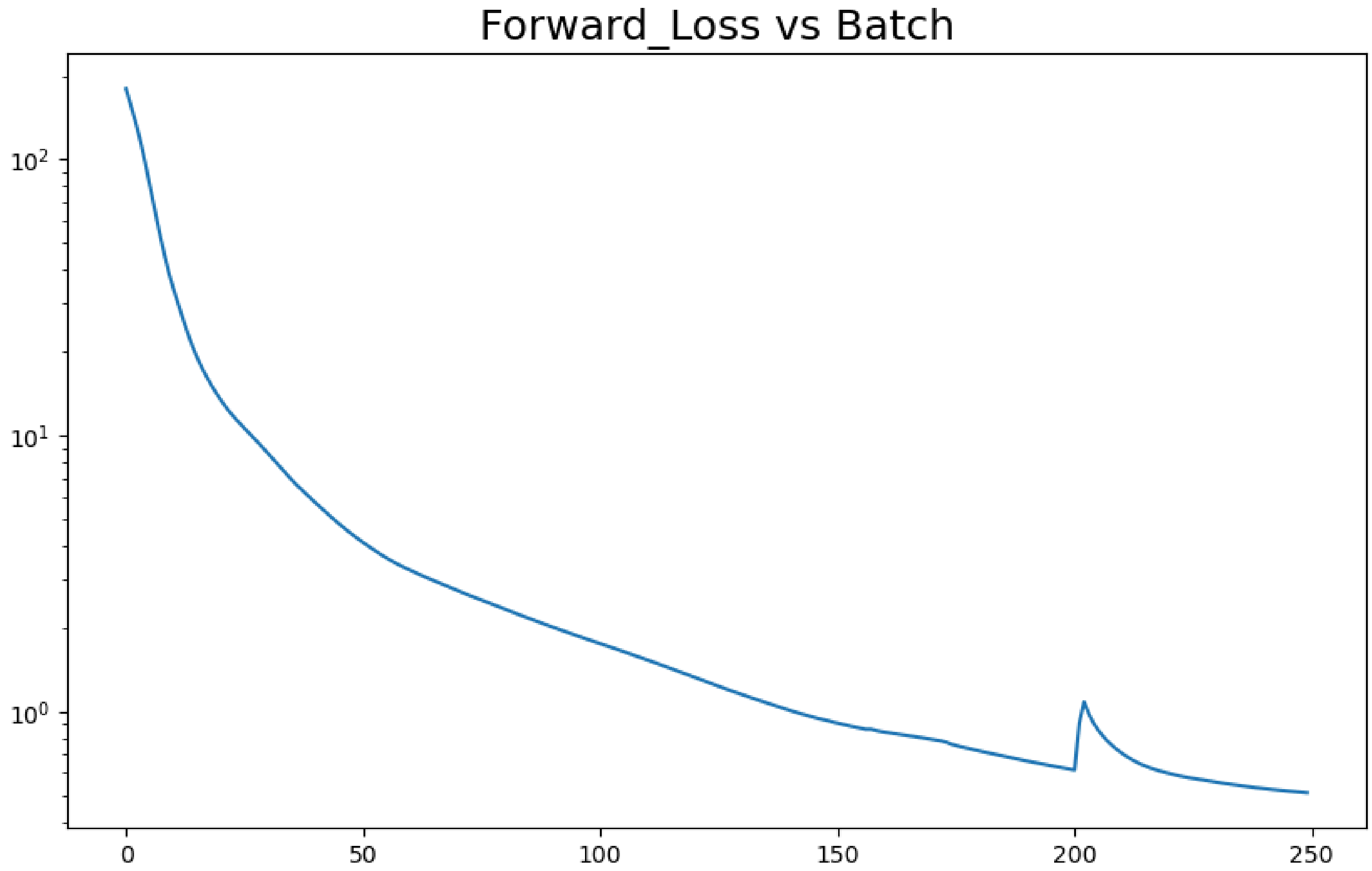
optimizer: Adamax()

scheduler: StepLR(step=100, gamma=0.999)

MaxBatch=250

OptimStep=20


X
3
2
1
0
−1
0.0
0.2
0.4
0.6
0.8
1.0
Y Values
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0



Forward_Loss vs Batch
10^2
10^1
10^0
0
50
100
150
200
250

In [ ]:
```python
import numpy as np
import torch as torch
import torch.nn as nn
import torch.nn.functional as F
import torch.optim as optim
import seaborn as sns
import pandas as pd
import matplotlib.pyplot as plt
import time
import random
from scipy.stats import norm
```

In [ ]:
```python
#Model and Params
#Numbers
NumTrain=500
NT=80
dt=1/NT
sigma=0.08
#Forward Loss
forward_losses = []

#Forward Loss
forward_losses = []
#Network Class for FBSDE
class Network(nn.Module):
    def __init__(self, lr, input_dims, fc1_dims, fc2_dims, n_outputs):
        '''
        lr: learning rate
        '''
        super(Network, self).__init__()

        #Pass input parameters
        self.input_dims = input_dims
        self.fc1_dims = fc1_dims
        self.fc2_dims = fc2_dims
        self.n_out = n_outputs

        #Construct network
        self.fc1 = nn.Linear(*self.input_dims, self.fc1_dims)
        nn.init.xavier_uniform_(self.fc1.weight)
        self.fc2 = nn.Linear(self.fc1_dims, self.fc2_dims)
        nn.init.xavier_uniform_(self.fc2.weight)
        self.fc3 = nn.Linear(self.fc2_dims, self.n_out)
        nn.init.xavier_uniform_(self.fc3.weight)

        self.optimizer = optim.Adam(self.parameters(), lr=lr)
        self.device = torch.device('cuda:0' if torch.cuda.is_available() else 'cpu')
        self.to(self.device)
```

```python
    def forward(self, input):
        x = F.relu(self.fc1(input))
        x= F.relu(self.fc2(x))
        output = self.fc3(x)
        return output

## Functions
def Sample_Init(N,mean=0,sd=0.1):
    '''
    Generate N samples of x0
    '''
    xi = np.random.normal(mean,sd,size=N)

    return torch.FloatTensor(xi).view(-1,1)

def SampleBMIncr(T, Npaths, Nsteps):
    # Returns Matrix of Dimension Npaths x Nsteps With Sample Increments of of BM
    # Here an increment is of the form dB
    dt = T / Nsteps
    dB = np.sqrt(dt) * np.random.randn(Npaths, Nsteps)
    return torch.FloatTensor(dB)

def target(x,sigma=sigma):
    x=x.detach().numpy()
    return torch.FloatTensor(-x/sigma)

# Forward Loss
def get_foward_loss_coupled(dB, init_x,NT, target,y0_model, z_models):
    x =  init_x
    # y = torch.rand_like(x)
    y_tilde=y0_model(x)
    y=torch.sigmoid(y_tilde)
    for j in range(1, NT+1):

        z = z_models[j-1](x)
        x =x+ y*dt+ dB[:,j].view(-1,1)
        y_tilde = (y_tilde +(z**2)*(1-2/(1+torch.exp(y_tilde)))/2*dt  + z * dB[:,j].view(-1,1))#.clamp(min=-1,max=1)
        y=torch.sigmoid(y_tilde)

    loss=torch.mean((y_tilde-target(x))**2)
    return loss


def get_target_path_coupled(dB, init_x,NumBM, NT,y0_model, z_models):
    x_path = torch.ones(NumBM,NT+1)
    y_path = torch.ones(NumBM,NT+1)
    x = init_x
    # y = torch.rand_like(x)
    y_tilde=y0_model(x)
    y=torch.sigmoid(y_tilde)
    x_path[:,0] = x.squeeze()
```

```
    y_path[:,0] = y.squeeze()
    for j in range(1, NT+1):
        z = z_models[j-1](x)
        x += y*dt+ dB[:,j].view(-1,1)
        y_tilde = (y_tilde +(z**2)*(1-2/(1+torch.exp(y_tilde)))/2 *dt + z * dB[:,j].view(-1,1))#.clamp(min=-1,max=1)
        y=torch.sigmoid(y_tilde)
        x_path[:,j] = x.squeeze()
        y_path[:,j] = y.squeeze()
    return x_path.detach(), y_path.detach()

class plot_results():
    def __init__(self,loss=forward_losses,sigma=sigma,Npaths=100,NumTrain=NumTrain,NT=NT):
        self.loss=loss
        self.x_path,self.y_path=get_target_path_coupled(dB, init_x, y0_model=y0_model_main, z_models=z_models_main, NumBM=NumTrain, NT=NT)
        self.number_of_paths=np.minimum(Npaths,NumTrain)
        self.sigma=sigma

    def FwdLoss(self,log=True):
        plt.figure(figsize=(10,6))
        plt.title("Forward_Loss vs Batch",fontsize=18)
        plt.plot(self.loss)

        if log==True:
            plt.yscale('log')

    def results(self,seed=0):
        random.seed(seed)
        idx_list = np.random.choice(NumTrain, self.number_of_paths, replace = False)
        x_plot = self.x_path.detach().numpy()[idx_list]
        y_plot = self.y_path.detach().numpy()[idx_list]
        t = np.array([i for i in range(NT+1)]) * 1/(NT)
        plt.figure(figsize=(20,6))
        plt.subplot(121)
        for i in range(self.number_of_paths):
                plt.plot(t,x_plot[i], color="blue", alpha=0.5)
        plt.title("X")

        plt.subplot(122)
        for i in range(self.number_of_paths):
                plt.plot(t,y_plot[i], color="red", alpha=0.5)
        plt.title("Y Values")

        ### Integrated Plots
        random.seed(seed)
        idx=random.randint(0,self.number_of_paths)
        plt.figure(figsize=(10,8))
        plt.subplot()
        plt.plot(t,x_plot[idx], color="blue", alpha=0.5,label='X')
        plt.plot(t,y_plot[idx], color="black", linestyle='--',alpha=0.5,label="Y Values")
        plt.hlines(y=[0,1],xmin=0,xmax=1,colors='firebrick',linestyles='-.')
        plt.title("Comparison of A Particular Path")
```

```python
        plt.legend()

    def qq_plot(self,sigma=sigma):
        plt.figure()
        plt.title("QQ-Plot")
        x_sigmoid=1/(1+np.exp(self.x_path[:,-1]/sigma))
        plt.scatter(x_sigmoid,self.y_path[:,-1],s=3)
        plt.plot(np.linspace(0,1,5),np.linspace(0,1,5),linestyle='--',linewidth=1,color='r')
```

In [ ]:

```python
## Train
torch.autograd.set_detect_anomaly(True)

dB = SampleBMIncr(1, Npaths=NumTrain, Nsteps=NT+1)
init_x =  Sample_Init(N=NumTrain)

#Forward Loss
forward_losses = []
#How many batches?
MaxBatch= 250

#How many optimization steps per batch
OptimSteps= 20

#Set Learning rate
learning_rate = 0.002

#Train on a single batch?
single_batch = True

#Set up main models for y0 and z (z will be list of models)
layer_dim = 10
y0_model_main = Network(lr=learning_rate, input_dims=[1], fc1_dims=layer_dim, fc2_dims=layer_dim,
                     n_outputs=1)
z_models_main = [Network(lr=learning_rate, input_dims=[1], fc1_dims=layer_dim, fc2_dims=layer_dim,
                     n_outputs=1) for i in range(NT)]



#Define optimization parameters
# params = list(y0_model_main.parameters())
params=[]
for i in range(NT):
    params += list(z_models_main[i].parameters())

#Set up optimizer and scheduler
optimizer = optim.Adamax(params, lr=learning_rate)
scheduler = torch.optim.lr_scheduler.StepLR(optimizer, step_size=100, gamma=0.999)


for k in range(0,MaxBatch):
```

```
    print("Batch Number: ", k+1)
    sloss=0
    #optimize main network wrt the foward loss
    for l in range(0,OptimSteps):
        optimizer.zero_grad()

        loss = get_foward_loss_coupled(dB, init_x,NT=NT,target=target, y0_model=y0_model_main, z_models=z_models_main)
        # print(loss)

        loss.backward()
        # print(params)
        # torch.nn.utils.clip_grad_norm_(parameters=params,max_norm=5,norm_type=1)
        optimizer.step()
        scheduler.step()
        nloss = loss.detach().numpy()
        sloss += nloss
        # print('OptimStep: '+ str(l+1))
        # print('forward_loss: ' + str(nloss))
    avgloss = sloss/OptimSteps
    print("Average Error Est: ", avgloss)
    forward_losses.append(avgloss)

    #Generate a new batch if using multiple batches
    if(not single_batch):
        dB = SampleBMIncr(1, Npaths=NumTrain, Nsteps=NT+1)
        init_x =  Sample_Init(N=NumTrain)


plot=plot_results(loss=forward_losses)
plot.FwdLoss()
plot.results()
plot.qq_plot()
```

```
Batch Number:  1
Average Error Est:  179.49754333496094
Batch Number:  2
Average Error Est:  156.0475814819336
Batch Number:  3
Average Error Est:  135.7221237182617
Batch Number:  4
Average Error Est:  116.17678680419922
Batch Number:  5
Average Error Est:  97.18975868225098
Batch Number:  6
Average Error Est:  79.9987850189209
Batch Number:  7
Average Error Est:  65.61859130859375
Batch Number:  8
Average Error Est:  54.43371505737305
Batch Number:  9
Average Error Est:  45.653940773010255
Batch Number:  10
Average Error Est:  38.753274536132814
Batch Number:  11
Average Error Est:  33.79064254760742
Batch Number:  12
Average Error Est:  29.773150062561037
Batch Number:  13
Average Error Est:  26.313890266418458
Batch Number:  14
Average Error Est:  23.369925880432127
Batch Number:  15
Average Error Est:  20.967220497131347
Batch Number:  16
Average Error Est:  19.060438442230225
Batch Number:  17
Average Error Est:  17.53457317352295
Batch Number:  18
Average Error Est:  16.259258937835693
Batch Number:  19
Average Error Est:  15.166566944122314
Batch Number:  20
Average Error Est:  14.225174236297608
Batch Number:  21
Average Error Est:  13.406475591659547
Batch Number:  22
Average Error Est:  12.674848747253417
Batch Number:  23
Average Error Est:  12.058139228820801
Batch Number:  24
Average Error Est:  11.521638917922974
Batch Number:  25
Average Error Est:  11.035692024230958
Batch Number:  26
```

```
Average Error Est:  0.5590780407190323
Batch Number:  231
Average Error Est:  0.5553162634372711
Batch Number:  232
Average Error Est:  0.551886796951294
Batch Number:  233
Average Error Est:  0.5488346487283706
Batch Number:  234
Average Error Est:  0.5458977192640304
Batch Number:  235
Average Error Est:  0.5429401367902755
Batch Number:  236
Average Error Est:  0.539850902557373
Batch Number:  237
Average Error Est:  0.5369081705808639
Batch Number:  238
Average Error Est:  0.534238263964653
Batch Number:  239
Average Error Est:  0.5312757402658462
Batch Number:  240
Average Error Est:  0.528806260228157
Batch Number:  241
Average Error Est:  0.5264797151088715
Batch Number:  242
Average Error Est:  0.5243109166622162
Batch Number:  243
Average Error Est:  0.5220732122659684
Batch Number:  244
Average Error Est:  0.5198907494544983
Batch Number:  245
Average Error Est:  0.5178260058164597
Batch Number:  246
Average Error Est:  0.5160125166177749
Batch Number:  247
Average Error Est:  0.5142452567815781
Batch Number:  248
Average Error Est:  0.5124104917049408
Batch Number:  249
Average Error Est:  0.5106629967689514
Batch Number:  250
Average Error Est:  0.5089240312576294
```