PROBLEM OVERVIEW

Abstract.

1. Optimization Problems

Suppose we have the following problem on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F}$ is the filtration generated by a Brownian motion $W = (W_t)_{t\geq 0}$:

$$J(x) := \inf_{a\in\mathcal{A}} \mathbb{E}\left[1/2\int_0^T a_s^2 ds + g(X_T^{0,x})\right].$$

This optimization is over $\mathcal{A} := \{a = (a_t)_{t\in[0,T]}$ such that a is adapted to $\mathbb{F}\}$. For simplicity we let

$$X_t = x + \int_0^t a_s ds + \sigma W_t.$$

and assume that g is continuously differentiable.

Our control is the "drift" of X. For a fixed control a we can consider another adapted process $\eta = (\eta_t)_{t\geq 0}$ and perturb a by $\epsilon > 0$ in the direction of η:

$$a + \epsilon\eta.$$

Naively, we can "differentiate" the objective function

$$F(a,x) = \mathbb{E}\left[\frac{1}{2}\int_0^T a_s^2 ds + g(X_T^{0,x})\right]$$

in an arbitrary perturbation direction η:

$$\delta_\eta F(a,x) := \lim_{\epsilon\downarrow 0} \frac{F(a+\epsilon\eta, x) - F(a,x)}{\epsilon}.$$

Note that:

$$\delta_\eta X_t = \int_0^t \eta_s ds$$

and under some integrability assumptions (so that we can naively pass limits under the integrals/expectations):

$$\begin{aligned}
\delta_\eta F(a,x) &= \mathbb{E}\left[\int_0^T a_s\eta_s ds + g'(X_T^{0,x})\int_0^T \eta_s ds\right] \\
&= \mathbb{E}\left[\int_0^T \left[a_s + g'(X_T^{0,x})\right]\eta_s ds\right].
\end{aligned}$$

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By Fubini's Theorem and Iterated Conditioning:

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left[a_s + g'(X_T^{0,x})\right] \eta_s ds\right] &= \int_0^T \mathbb{E}\left[(a_s + g'(X_T^{0,x}))\eta_s\right] ds \\
&= \int_0^T \mathbb{E}\left[\mathbb{E}\left[(a_s + g'(X_T^{0,x}))\eta_s | \mathcal{F}_s\right]\right] ds \\
&= \mathbb{E}\left[\int_0^T \mathbb{E}\left[(a_s + g'(X_T^{0,x}))\eta_s | \mathcal{F}_s\right] ds\right].
\end{aligned}$$

Taking out what is known:

$$\mathbb{E}\left[\int_0^T \mathbb{E}\left[(a_s + g'(X_T^{0,x}))\eta_s | \mathcal{F}_s\right] ds\right] = \mathbb{E}\left[\int_0^T \left[a_s + \mathbb{E}\left[g'(X_T^{0,x}) | \mathcal{F}_s\right]\right] \eta_s ds\right].$$

Taken together we have:

$$\delta_\eta F(a, x) = \mathbb{E}\left[\int_0^T \left[a_s + \mathbb{E}\left[g'(X_T^{0,x}) | \mathcal{F}_s\right]\right] \eta_s ds\right].$$

Like in calculus, to minimize F over adapted a, we solve for a satisfying the following first order condition **for all adapted** η:

$$\delta_\eta F(a, x) = \mathbb{E}\left[\int_0^T \left[a_s + \mathbb{E}\left[g'(X_T^{0,x}) | \mathcal{F}_s\right]\right] \eta_s ds\right] = 0.$$

It is possible to show that this holds if and only if

$$a_s = -\mathbb{E}\left[g'(X_T^{0,x}) | \mathcal{F}_s\right]$$

s

almost surely for almost every $t \in (0, T)$. If we apply the Martingale Representation Theorem we get that there exists an $a_0 = -\mathbb{E}[g'(X_T^{0,x})]$ and adapted $Z = (Z_t)_{t \geq 0}$ such that

$$a_t = a_0 + \int_0^t Z_s dW_s$$

and $a_T = -g'(X_T^{0,x})$. Putting this together with the dynamics of X (which depend on a) we arrive at the following Forward-Backward Stochastic Differential Equation (FBSDE):

$$\begin{cases} dX_t = a_t dt + \sigma dW_t, & X_0 = x \\ da_t = Z_t dW_t, & a_T = -g'(X_T^{0,x}). \end{cases}$$

This characterizes the first order condition. We can search for a solution of the optimization problem by searching over measurable initial conditions a_0 and adapted volatilities $Z = (Z_t)_{t \geq 0}$ so that we match the terminal condition of the FBSDE over all starting values x of X and paths of $W = (W_t)_{t \geq 0}$. To solve this numerically, we can discretize the FBSDE and parameterize a_0 and Z_t (on the discrete grid $0 = t_0 < t_1 < \cdots < t_N = T$) by neural networks that take as inputs X_0 and X_t, respectively. Using this approximation we can try to minimize the loss:

$$\boxed{\frac{1}{N} \sum_{i=1}^{N} (a_T(\omega_i) + g'(X_T^{0,x}(\omega_i)))^2}$$

over N samples $\omega_i \in \Omega$, $i = 1, \ldots, N$ that define a trajectory of the Brownian Motion W.