

1. 设 n 为自然数. 证明: $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$.

$$\begin{aligned} \text{令 } \theta = \frac{\pi}{2} - \varphi, \text{ 则 } \frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} &= \frac{1+\cos\varphi+i\sin\varphi}{1+\cos\varphi-i\sin\varphi} = \frac{2\cos^2\frac{\varphi}{2}+2i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\cos^2\frac{\varphi}{2}-2i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}} \\ &= \frac{\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}-i\sin\frac{\varphi}{2}} = (\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2})^2 = \cos\varphi + i\sin\varphi. \end{aligned}$$

$$\therefore \left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\varphi + i\sin n\varphi = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right).$$

2. 设复数 z_1, z_2, z_3 满足 $\frac{z_2-z_1}{z_3-z_1} = \frac{z_1-z_3}{z_2-z_3}$, 证明 $|z_2-z_1| = |z_3-z_1| = |z_2-z_3|$.

$$\text{由 } \frac{z_2-z_1}{z_3-z_1} = \frac{z_1-z_3}{z_2-z_3}, \text{ 且 } z_1^2+z_2^2+z_3^2 = z_1z_2+z_2z_3+z_3z_1 \Rightarrow \frac{1}{2}(z_1-z_2)^2 + \frac{1}{2}(z_1-z_3)^2 + \frac{1}{2}(z_2-z_3)^2 = 0.$$

$$\Rightarrow z_1 = z_2 = z_3$$

$$\therefore \frac{z_2-z_1}{z_3-z_1} = \frac{z_1-z_3}{z_2-z_3} = \frac{z_2-z_3+z_3-z_1}{z_3-z_1} = \frac{z_2-z_3}{z_3-z_1} + 1, \text{ 令 } a = \frac{z_2-z_3}{z_3-z_1}. \text{ 由 } |a| = -\frac{1}{a} \Rightarrow a^2+a+1=0.$$

$$\Rightarrow a = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \Rightarrow |a| = \left| \frac{z_2-z_3}{z_3-z_1} \right| = 1 \Rightarrow |z_2-z_3| = |z_3-z_1|, \text{ 由理可证其他.}$$

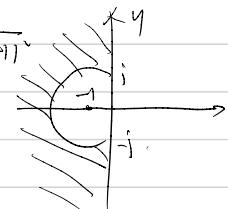
(等边三角形).

△ 画出复变函数图像并判断是否为区域.

3. 试判别满足条件 $0 < \arg \frac{z-1}{z+1} < \frac{\pi}{4}$ 的点组成的点集是否为区域?

$$\text{设 } z = x+iy, \text{ 令 } \theta = \arg \frac{z-1}{z+1}. \text{ 由 } \frac{z-1}{z+1} = \frac{x+(y-1)i}{x+(y+1)i} = \frac{[x+(y-1)i][x-(y+1)i]}{[(x+(y+1)i)[x-(y+1)i]} = \frac{x^2+y^2-1}{x^2+y^2+2} - i \frac{2y}{x^2+y^2+2},$$

$$\theta = \arctan \frac{-2y}{x^2+y^2-1}, \text{ 由 } 0 < \arg \frac{z-1}{z+1} < \frac{\pi}{4} \text{ 得 } 0 < \frac{-2y}{x^2+y^2-1} < 1. \Rightarrow \begin{cases} x > 0 \\ (x+1)^2+y^2 > 2. \end{cases} \Rightarrow$$



$$0 < \left| \frac{z-2}{z-3} \right| \leq 1 \text{ 的图象.} \rightarrow \text{充分考虑几何意义} \Rightarrow |z-2| \leq |z-3| \text{ 且 } z \neq 3$$

$$-\frac{\pi}{4} < \arg \frac{z-1}{z+1} < \frac{\pi}{4} \text{ 的图象} \Rightarrow \frac{x+1-y}{x-y} > 0 \text{ 且 } \frac{x+y}{x-y} < 0$$

△ 求 z 平面上的曲线 C 经过映射 $w=f(z)$ 到 w 平面上的像曲线 T .

4. 求函数 $w=\frac{1}{z}$ 把 z 平面上以 $(1,0)$ 为圆心, 1 为半径的圆周 C 映射成 w 平面上的像曲线 T .

$$\text{① } C = |z-1|=1. \text{ 令 } z = x+iy, w = u(x,y) + i v(x,y), \text{ 由 } w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$$

$$\text{由 } u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2} \text{ 得 } x = \frac{u}{u^2+v^2}, y = -\frac{v}{u^2+v^2}. \text{ 代入 } C \text{ 中得: } u = \frac{1}{2}.$$

$$\text{② } C = |z-1|=1. \text{ 即 } |z-1|^2 = (z-1)(\bar{z}-1)=1. \text{ 由 } w = \frac{1}{z} \text{ 得 } z = \frac{1}{w}, \bar{z} = \frac{1}{\bar{w}} \text{ 代入 } C \text{ 中得:}$$

$$(\frac{1}{w}-1)(\frac{1}{\bar{w}}-1)=1 \Rightarrow \frac{1}{w\bar{w}} - (\frac{1}{w} + \frac{1}{\bar{w}}) = 0 \Rightarrow w + \bar{w} = 1, \text{ 又因为 } \operatorname{Re} w = \frac{1}{2}(w + \bar{w}). \text{ 由 } \operatorname{Re} w = \frac{1}{2}$$

△ 讨论复变函数 $w=f(z)$ 的可导性 (讨论极限).

5. 讨论函数 $w=f(z)=|z|^2$ 的可导性

$$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z}+\bar{\Delta z}) - z\bar{z}}{\Delta z} = \bar{z} + \Delta z + 2\frac{\bar{\Delta z}}{\Delta z}.$$

$$\text{当 } z=0 \text{ 时, } \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \bar{\Delta z} = 0.$$

$$\text{当 } z \neq 0 \text{ 时, 若取 } \Delta z = \Delta x \rightarrow 0 \text{ 得: } \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \bar{z} + z. \quad] \Rightarrow z \neq 0 \text{ 时, } w=f(z) \text{ 导数不存在.}$$

$$\text{若取 } \Delta z = i\Delta y \rightarrow 0 \text{ 得: } \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \bar{z} - z.$$

○ 求映射 $w = z + \frac{1}{z}$ 下圆周 $|z|=2$ 的像.

$$\Rightarrow \text{设 } w = u + iv, z = x + iy. \text{ 则 } |z| = 2 \Rightarrow x^2 + y^2 = 4$$

$$w = z + \frac{1}{z} = x + iy + \frac{1}{x+iy} = x + iy + \frac{x-iy}{x^2+y^2} = \left(x + \frac{x}{x^2+y^2}\right) + i\left(y - \frac{y}{x^2+y^2}\right) = u + iv.$$

$$\Rightarrow u = x + \frac{x}{x^2+y^2} = \frac{5}{4}x, v = y - \frac{y}{x^2+y^2} = \frac{3}{4}y. \Rightarrow \frac{16}{25}u^2 + \frac{16}{9}v^2 = 16 \Rightarrow \frac{u^2}{\frac{25}{16}} + \frac{v^2}{\frac{9}{16}} = 1. \text{(椭圆)}$$

○ 函数 $w = z^2$ 将平面上直线段 $\operatorname{Re} z = 1, -1 \leq \operatorname{Im} z \leq 1$ 变成 w 平面上什么曲线?

$$\Rightarrow \begin{cases} x=1 \\ y \in [-1, 1] \end{cases} w = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy \Rightarrow \begin{cases} u = 1 - y^2 \\ v = 2y \end{cases} \text{ 请参考参数方程 } u + \frac{v^2}{4} = 1$$

即 $u = 1 - \frac{v^2}{4}, v \in [-2, 2]$ 为一抛物线.

○ 设 $|z_0| < 1$. 试证明: 若 $|z|=1$, 则 $\left| \frac{z-z_0}{1-\bar{z}_0 z} \right| = 1$.

$$\text{若 } |z| < 1, \text{ 则 } \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| < 1$$

$$\Rightarrow \text{若 } |z|=1, \text{ 则 } |z| = 1, \text{ 则 } \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| = \frac{|z-z_0|}{|z||1-\bar{z}_0 z|} = \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|} = \frac{|z-z_0|}{|\bar{z}-\bar{z}_0|} = 1.$$

$$\text{若 } |z| < 1, \text{ 则 } |z-z_0|^2 = |z|^2 + |\bar{z}_0|^2 - 2\operatorname{Re}(z\bar{z}_0), |1-\bar{z}_0 z|^2 = |1+\bar{z}_0|^2 |z|^2 - 2\operatorname{Re}(z\bar{z}_0).$$

$$\text{得 } \frac{|z-z_0|^2}{|1-\bar{z}_0 z|^2} = \frac{|z|^2 + |\bar{z}_0|^2 - 2\operatorname{Re}(z\bar{z}_0)}{|1+|\bar{z}_0|^2|z|^2 - 2\operatorname{Re}(z\bar{z}_0)} < 1. \text{ 故 } \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| < 1.$$

○ 求下列方程所表示的曲线. $\left| \frac{z-a}{1-\bar{a}z} \right| = 1$ ($|a| < 1$)

$$\Rightarrow \left| \frac{z-a}{1-\bar{a}z} \right| = \frac{|z-a|}{|1-\bar{a}z|} = \frac{|\bar{z}||z-a|}{|\bar{z}||1-\bar{a}z|} = \frac{|\bar{z}||z-a|}{|\bar{z}-\bar{a}|} = |\bar{z}| = 1. \text{ 即 } x^2 + y^2 = 1 \text{ (单位圆周).}$$

○ 试利用 $1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}$, 记 $\alpha =$

$$\left\{ 1 + \cos\theta + 2\cos 2\theta + \dots + n\cos n\theta = \frac{\sin(n+\frac{1}{2})\theta + \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}} \right.$$

$$\left. \sin\theta + \sin 2\theta + \dots + \sin n\theta = \frac{\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \right.$$

6. 讨论函数 $w = f(z) = \operatorname{Im} z$ 的可导性.

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Im} \Delta z}{\Delta z} = \frac{\Delta y}{\Delta z} = \frac{\Delta y}{\Delta x + i\Delta y}.$$

$$\textcircled{1} \quad \Delta z = \Delta x \rightarrow 0, \quad \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta y}{\Delta x + i\Delta y} = 0. \quad] \quad w = f(z) \text{ 不可导.}$$

$$\textcircled{2} \quad \Delta z = i\Delta y \rightarrow 0, \quad \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta y}{\Delta x + i\Delta y} = \frac{1}{i}. \quad] \quad w = f(z) \text{ 不可导.}$$

7. 证明函数 $f(z) = \sqrt{|xy|}$ 在 $z=0$ 满足 CR 条件, 但在 $z=0$ 处不可导.

由题可知 $w = f(z) = u(x,y) + i v(x,y)$, $\Rightarrow u(x,y) = \sqrt{|xy|}, v(x,y) = 0$.

$$\text{且 } \frac{\partial u}{\partial x} \Big|_{x=y=0} = 0, \quad \frac{\partial v}{\partial y} \Big|_{x=0, y=0} = 0, \quad \frac{\partial u}{\partial y} \Big|_{x=y=0} = 0, \quad \frac{\partial v}{\partial x} \Big|_{x=y=0} = 0. \Rightarrow \text{满足 CR 条件}$$

$$\text{又 } \frac{\Delta w}{\Delta z} \Big|_{z=0} = \frac{\sqrt{|(x+\Delta x)(y+\Delta y)|}}{\Delta z} = \frac{\sqrt{|\Delta x \Delta y|}}{\Delta x + i \Delta y}, \quad \Delta y = k \Delta x \text{ 且 } \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{\sqrt{|k|(1)}}{\Delta x + i k \Delta x} = \frac{\sqrt{|k|}}{1+i k} \Rightarrow \text{不可导.}$$

△ 解析函数

8. 设函数 $f(z) = u(x,y) + i v(x,y)$ 在区域 D 内解析, 且 $v = u^2$, 证明 $f(z)$ 在 D 内为常数.

$$\text{由 } v = u^2 \text{ 得 } \frac{\partial v}{\partial x} = 2u \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y} = 2u \frac{\partial u}{\partial y}. \quad \text{由柯西黎曼条件 } \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2u \frac{\partial u}{\partial y} = -2u \frac{\partial u}{\partial x} = -4u^2 \frac{\partial u}{\partial x} \Rightarrow (1+4u^2) \frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0.$$

△ 调和函数的概念

9. 设 $f(z) = u(x,y) + i v(x,y)$ 为解析函数, 且满足: $u(x,y) + v(x,y) = x^2 - y^2 + 2(x+y) + 2xy$. $f(0) = 0$. 试用调和函数求解.

由 $f(z) = u(x,y) + i v(x,y)$, 有 $i f(z) = -v(x,y) + i u(x,y)$ 仍为解析函数. 所以 $u(x,y)$ 是 $-v(x,y)$ 的共轭调和函数. 从而 $u(x,y) + v(x,y)$ 是 $u(x,y) - v(x,y)$ 的共轭调和函数. 由 CR 条件 $(u-v)_x = (u+v)_y = 2x - 2y + 2$. 对上式两端关于 x 积分:

$$u - v = \int (2x - 2y + 2) dx = x^2 - 2xy + 2x + \varphi(y) \quad (\text{偏积分}).$$

$$\Rightarrow \text{求 } (u-v)_y = -2x + \varphi'(y) = -(u+v)_x = -2x + 2y - 2. \quad \text{因此 } \varphi'(y) = 2y - 2 \Rightarrow \varphi(y) = y^2 - 2y + C.$$

$$\Rightarrow u - v = x^2 - y^2 + 2x - 2y - 2 + y^2 - 2y + C. \quad \text{联立题给条件 } u + v = x^2 - y^2 + 2(x+y) + 2xy \text{ 可解出 } u, v.$$

△ 三角函数和双曲函数.

10. 证明 $w = \cos z$ 将直线 $x=C_1$ 和 $y=C_2$ 分别变成双曲线和椭圆.

$$\Rightarrow \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh iy - i \sin x \sinhy.$$

$$\text{得 } u = \cos x \cosh hy, \quad v = -\sin x \sinhy.$$

$$\text{由 } \cosh^2 hy - \sinh^2 hy = 1 \text{ 得 } \frac{u^2}{\cos^2 x} - \frac{v^2}{\sinh^2 hy} = 1. \Rightarrow \text{双曲线.}$$

$$\text{由 } \sinh^2 hy + \cosh^2 hy = 1 \text{ 得 } \frac{u^2}{\cosh^2 hy} + \frac{v^2}{\sinh^2 hy} = 1 \Rightarrow \text{椭圆.}$$

11. 判断极限是否存在、有无意义.

$$(1) \lim_{z \rightarrow i} \frac{z-i}{z(1+z^2)}$$

$$(2) \underbrace{z^2}_{} = \underbrace{z^2 - i^2}_{=} = (z+i)(z-i), \text{ 但 } \frac{z-i}{z(1+z^2)} = \frac{1}{z(z+i)} \Rightarrow \lim_{z \rightarrow i} = \frac{1}{i \times 2i} = -\frac{1}{2}$$

12. 复变函数:

$$f(z) = \begin{cases} \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}, & z \neq 0 \\ 0, & z=0. \end{cases}$$

说明: (1) $f(z)$ 在 $z=0$ 处连续. (2) $f(z)$ 在 $z=0$ 处满足 C-R 方程. (3) $f(z)$ 在 $z=0$ 不可导.

$$(1) z \neq 0 \text{ 时} = f(z) = u(x,y) + i v(x,y) = \frac{x^3-y^3}{x^2+y^2} + i \frac{x^3+y^3}{x^2+y^2}.$$

$$\Rightarrow u(x,y) = \frac{x^3-y^3}{x^2+y^2} = (x-y) \cdot \frac{x^2+xy+y^2}{x^2+y^2} = r(\cos\theta - \sin\theta)(1+\sin\theta\cos\theta) \Rightarrow \lim_{r \rightarrow 0} u(r\cos\theta, r\sin\theta) = 0.$$

$$\text{同理可得: } \lim_{r \rightarrow 0} v(x,y) = \lim_{r \rightarrow 0} \frac{x^3+y^3}{x^2+y^2} = \lim_{r \rightarrow 0} r(\cos\theta+\sin\theta)1+\sin\theta\cos\theta = 0.$$

则 $\lim_{z \rightarrow 0} f(z) = 0 = f(0)$. 因此 $f(z)$ 在 $z=0$ 连续.

$$(2) u_x(0,0) = \frac{\partial u}{\partial x}|_{(0,0)} = \lim_{\Delta x \rightarrow 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(\Delta x^3 - 0) - 0}{\Delta x} = 1.$$

$$v_y(0,0) = \frac{\partial v}{\partial y}|_{(0,0)} = \lim_{\Delta y \rightarrow 0} \frac{v(0, \Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{(0 - \Delta y^3) - 0}{\Delta y} = -1.$$

同理可得: $v_x(0,0) = 1$, $u_y(0,0) = 1$.

则 $f(z)$ 在 $z=0$ 有: $u_x = v_y$, $u_y = -v_x \Rightarrow$ 满足 C-R 条件.

$$(3) \frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z}, \text{ 当 } z \text{ 沿曲线 } y=kx \text{ 趋近于 } 0 \text{ 时有}$$

$$\lim_{\substack{x \rightarrow 0 \\ y=kx}} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \rightarrow 0 \\ y=kx}} \frac{\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} - 0}{x+iy} = \lim_{\substack{x \rightarrow 0 \\ y=kx}} \frac{x^3(1+k^3)+ix^3(1+k^3)}{x^2(1+k^2) \cdot x(1+ik)} = \lim_{\substack{x \rightarrow 0 \\ y=kx}} \frac{1+k^3+i(1+k^3)}{(1+k^2)(1+ik)} = \frac{1+k^3+i(1+k^3)}{(1+k^2)(1+ik)}$$

则 $f(z)$ 在 $z=0$ 不可导.

△判断复变函数在何处可导、何处解析

① 通过复变函数解析的充要条件判断.

13. 判断 $(1) f(z) = x^3 - y^3 + 2x^2y^2i$ 在何处解析

$$\Rightarrow u(x,y) = x^3 - y^3, v(x,y) = 2x^2y^2 \Rightarrow u_x = 3x^2, u_y = -3y^2, v_x = 4xy^2, v_y = 4x^2y.$$

由 $u_x = v_y$ 解得: $x=0$ 或 $y=\frac{3}{4}$; 由 $u_y = -v_x$ 解得: $y=0$ 或 $x=\frac{3}{4}$.

则 $f(z)$ 只在 $(0,0), (\frac{3}{4}, \frac{3}{4})$ 两点解析.

(2) $f(z) = \bar{z}$.

$$\text{令 } z = x+iy, f(z) = x-iy \Rightarrow u=x, v=-y \Rightarrow u_x=1, u_y=0, v_x=0, v_y=-1 \Rightarrow u_x \neq v_y, u_y \neq -v_x$$

则 $f(z)$ 在复平面内处处不解析.

② 解析函数经四则运算、复合、反函数仍为解析函数(初等解析函数)

对多项式 $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n (a_0 \neq 0)$ 在全平面上解析.

对有理分式 $= \frac{P(z)}{Q(z)}$ 在除使 $Q(z)=0$ 的点之外处处解析.

A CR 方程的极坐标形式

14. 证明 CR 的极坐标形式为: $\frac{\partial u}{\partial r} = -\frac{1}{r} \frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.

\Rightarrow 在极坐标下 $f(z) = u(r, \theta) + iv(r, \theta)$.

1°. 当 Δz 沿经向 $\Delta z = \Delta r e^{i\theta}$ 邻近于 0, ($\Delta\theta = 0$) 时:

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta r, \theta \rightarrow 0} \frac{u(r+\Delta r, \theta) + iv(r+\Delta r, \theta) - u(r, \theta) - iv(r, \theta)}{\Delta r e^{i\theta}} \\ &= \lim_{\Delta r \rightarrow 0} \frac{u(r+\Delta r, \theta) - u(r, \theta)}{\Delta r e^{i\theta}} + \lim_{\Delta r \rightarrow 0} \frac{iv(r+\Delta r, \theta) - iv(r, \theta)}{\Delta r e^{i\theta}} = \left[\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right] e^{-i\theta} \end{aligned}$$

2°. 当 Δz 沿切向 $\Delta z = ire^{i\theta}$ 邻近于 0, ($\Delta r = 0$) 时:

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{ire^{i\theta}, \theta \rightarrow 0} \frac{u(r, \theta+Δθ) + iv(r, \theta+Δθ) - u(r, \theta) - iv(r, \theta)}{ire^{i\theta} Δθ} \\ &= \lim_{Δθ \rightarrow 0} \frac{u(r, \theta+Δθ) - u(r, \theta)}{ire^{i\theta} Δθ} + \lim_{Δθ \rightarrow 0} \frac{iv(r, \theta+Δθ) - iv(r, \theta)}{ire^{i\theta} Δθ} = \left[\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right] (-i) \frac{1}{r} e^{-i\theta} \end{aligned}$$

对比两式, 有: $\frac{\partial u}{\partial r} = -\frac{1}{r} \frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

△ 给定复变函数的实部或虚部, 确定其虚部或实部 (求亚反曲线簇)

若给定复变函数 $f(z)$ 的实部 $u(x, y)$. ① 根据 CR 条件可以得到 $U(x, y)$ 的两个偏导数
 $\Rightarrow \begin{cases} U_x = -U_y \\ U_y = U_x \end{cases}$. ② 根据 $U(x, y)$ 对 x, y 的偏微分求原函数. 三种方法 =

a. 偏积分 = 首先对 U_x 积分, 注意常数项为 y 的函数. $\Rightarrow \int U_x dx = \int -U_y dy = U(x, y) + \underline{\varphi(y)}$.

b. 曲线积分 = $\Rightarrow U(x, y) = \int U_x dx + U_y dy = \int (U_y) dx + U_x dy$. (注意 dy 积分下限有时会产生负项!)

c. 全微分 = 法.

15. 已知在区域 D 内, $U(x, y)$ 和 $V(x, y)$ 为共轭调和函数. 求 $aU(x, y) + bV(x, y)$ 的共轭调和函数.

\Rightarrow 运用曲线积分法 = 令 $f(x, y) = aU(x, y) + bV(x, y)$, 其共轭调和函数为 $h(x, y)$

则有 $\frac{\partial f}{\partial x} = \frac{\partial h}{\partial y} = aU_x + bV_x$, $\frac{\partial f}{\partial y} = -\frac{\partial h}{\partial x} = aU_y + bV_y$.

$$\begin{aligned} \therefore h(x, y) &= \int -(aU_y + bV_y) dx + (aU_x + bV_x) dy = a \int -U_y dx + U_x dy + b \int -V_y dx + V_x dy \\ &= a \int U_x dx + V_y dy + b \int -U_x dx - V_y dy = a \int dU - b \int dV = aU(x, y) - bV(x, y). \end{aligned}$$

16. 下列等式是否正确? $\rightarrow \ln z^2 = \ln z + \ln z \neq 2 \ln z$.

(1) $\ln z^2 = 2 \ln z$ = 不正确. $\ln z^2 = 2 \ln r + i(2\theta + 2k\pi)$, $2 \ln z = 2 \ln r + i(2\theta + 4k\pi)$.

(2) $e^{\ln z} = z$ = 正确. 幂函数的定义

(3) $\ln \sqrt{z} = \frac{1}{2} \ln z$ = 不正确. 因为 $\frac{1}{2} \ln z = \frac{1}{2} (\ln r + i(\theta + 2k\pi)) = \frac{1}{2} \ln r + i(\frac{1}{2}\theta + k\pi)$ 而 $\sqrt{z} = \sqrt{r} e^{\frac{i(\theta+2k\pi)}{2}} = \sqrt{r} e^{\frac{\theta}{2}i} = \sqrt{r} e^{\frac{\theta}{2}i}, -\sqrt{r} e^{\frac{\theta}{2}i}$.

$\Rightarrow \ln \sqrt{z} = \begin{cases} \ln \sqrt{r} e^{\frac{\theta}{2}i} = \frac{1}{2} \ln r + i(\frac{\theta}{2} + 2(k+1)\pi) \end{cases}$.

(4) $\ln e^z = z$ = 不正确, 分析见下页

△ 圆的参数方程: $(z-z_0) = re^{i\theta}$.
表示一个圆周: $|z-z_0|=r$

令 $w = e^z = u+iv$. 由 $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ 得 $u = e^x \cos y, v = e^x \sin y$.
 $\Rightarrow \ln e^z = \ln w = \ln \sqrt{u^2+v^2} + i(\arctan \frac{v}{u} + 2k\pi) = \ln \sqrt{e^{2x} \cos^2 y + e^{2x} \sin^2 y} + i(\arctan(\tan y) + 2k\pi) = x + iy + 2k\pi \neq z$
由此可见 $\ln e^z = z$, 且 $w = e^z$ 和 $z = \ln w$ 不能相互推导.

A 题型: 解复方程

17. 求解下列方程.

(1) $e^{2z} - 1 - \sqrt{3}i = 0$

\Rightarrow 运用 $\ln e^z = z$ 進行求解: $e^{2z} = 1 - \sqrt{3}i \Rightarrow z = \frac{1}{2} \ln(1 - \sqrt{3}i) = \frac{1}{2} \ln 2 + \frac{1}{2}i(-\frac{\pi}{3} + 2k\pi) = \frac{1}{2} \ln 2 + i(k - \frac{1}{3})\pi$.

(2) $\ln z = \frac{\pi}{2}i \Rightarrow e^{\ln z} = z$ ($\ln z = \frac{\pi}{2}i$ → 错误说法: 则寻求的都是单值式)

(3) $\cos z = 2 \Rightarrow z = \text{Arcos } 2 = -i \ln(2 \pm \sqrt{3}) = 2k\pi - i \ln(2 \pm \sqrt{3}), k=0,1,2\dots$

(4) $\sin z - \cos z = 2 \Rightarrow \sqrt{2} \sin(z - \frac{\pi}{4}) = 2 \Rightarrow z - \frac{\pi}{4} = \text{Arcsin } \sqrt{2} \Rightarrow z = (2k + \frac{1}{4})\pi - i \ln(\sqrt{2} \pm 1)$.

(5) $\cos z = i \sinh 5$

$\Rightarrow \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y = i \sinh 5$

对比可得: $\begin{cases} \cos x \cosh y = 0 \\ -\sin x \sinh y = \sinh 5 \end{cases}$ 因为 $\cosh y = \frac{e^y + e^{-y}}{2} \neq 0$, 故 $\cos x = 0 \Rightarrow x = (k + \frac{1}{2})\pi$.
 $\Rightarrow \sin x = (-1)^k \Rightarrow (-1)^{k+1} \sinh y = \sinh 5 \Rightarrow y = (-1)^{k+1} 5$.

$\therefore z = (k + \frac{1}{2})\pi + i(-1)^{k+1} 5$.

(6) $\sin iz = i$.

$\Rightarrow i z = \text{Arcsin } i = -i \ln(-1 \pm \sqrt{2}) = \begin{cases} -i(\ln(-1 + \sqrt{2}) + 2k\pi i) = 2k\pi - i \ln(-1 + \sqrt{2}) \\ -i(\ln(-1 - \sqrt{2}) + (T\pi + 2k\pi)i) = (2k+1)\pi - i \ln(-1 - \sqrt{2}) \end{cases}$

$\Rightarrow z = \begin{cases} -\ln(-1 + \sqrt{2}) - 2k\pi i \\ -\ln(-1 - \sqrt{2}) - (2k+1)\pi i \end{cases}$ 注意此处 $\pm \sqrt{2}$ 的 arg 值不相同.

(7) $\sin z + i \cos z = 4i \Rightarrow \cos z - i \sin z = 4 \Rightarrow e^{-iz} = 4$

(8) $|tanh z| = 1$

$\Rightarrow \tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z} - 1}{e^{2z} + 1} \Rightarrow |e^{2z} - 1| = |e^{2z} + 1|$.

$\because e^{2z} \neq 0$. 由 $e^{2z} = 1$, 则 $e^{2x}(\cos 2y + i \sin 2y) = 1 \Rightarrow \begin{cases} e^{2x} = 1 \\ \cos 2y = 0 \end{cases}$

$\Rightarrow x=0, y = \frac{\pi}{4} + \frac{k\pi}{2} \Rightarrow z = i(\frac{1}{4} + \frac{k}{2})\pi$.

18. 设 $f(z) = u(x,y) + iv(x,y)$ 为 $z = x+iy$ 的解析函数. 试

$$w(z, \bar{z}) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) \text{ 证明 } \frac{\partial w}{\partial \bar{z}} = 0.$$

$$\Rightarrow \text{直接求导} = \frac{\partial w}{\partial \bar{z}} = U_x \cdot \frac{1}{2} + U_y \cdot (-\frac{1}{2i}) + iV_x \cdot -\frac{1}{2} + iV_y \cdot (\frac{1}{2i}) \\ = \frac{1}{2}U_x + \frac{1}{2}U_y + \frac{1}{2}V_x - \frac{1}{2}V_y = 0$$

$\Delta f(z)$ 可导 / 解析 $\Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0$, 证明:

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \left(\frac{1}{-i} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0$$

19. 设 $f(z)$ 为解析函数. 则有:

$$(1) \left(\frac{\partial}{\partial x} |f(z)| \right)^2 + \left(\frac{\partial}{\partial y} |f(z)| \right)^2 = |f'(z)|^2 \quad (2) \frac{\partial^2}{\partial x^2} |f(z)|^2 + \frac{\partial^2}{\partial y^2} |f(z)|^2 = 4|f'(z)|^2$$

20. 设 D 为关于实轴对称的区域. 证明: 函数 $f(z)$ 与 $\bar{f}(\bar{z})$ 在 D 内同时解析.

证一: 设 $f(z) = u(x,y) + iv(x,y)$. 则 $\bar{f}(\bar{z}) = u(x,-y) - iv(x,-y)$.

由 $f(z)$ 解析 $\Leftrightarrow u_x = v_y, u_y = -v_x$. 而 $\varphi(x,y) = u(x,y), \psi(x,y) = -v(x,y)$

$$\Rightarrow \varphi_x = u_x, \varphi_y = -u_y, \psi_x = -v_x, \psi_y = v_y$$

则在 φ, ψ 可微且, $\varphi_x = \psi_y, \varphi_y = -\psi_x$. 即 $\bar{f}(\bar{z})$ 解析.

证二: $\lim_{z \rightarrow z_0} g(z) = \overline{f(\bar{z})}$. 由 \exists :

$$\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{\overline{f(\bar{z})} - \overline{f(\bar{z}_0)}}{z - z_0} = \lim_{z \rightarrow z_0} \frac{(f(\bar{z}) - f(\bar{z}_0))}{\bar{z} - \bar{z}_0}$$

若 $f(z)$ 在 z_0 可导. 则由上式可知 $g(z) = \overline{f(\bar{z})}$ 在 \bar{z}_0 可导且 $g'(\bar{z}_0) = \overline{f'(\bar{z}_0)}$

反之. 若 $g(z)$ 在 z_0 可导. 则 $f(z) = \overline{g(\bar{z})}$ 在 \bar{z}_0 可导. 由解析定义知 $f(z)$ 和 $\overline{f(\bar{z})}$ 同时解析.

21. 判断下列函数的多值性.

$$(1) \sin \sqrt{z}$$

$$(2) \cos \sqrt{z}$$

$$(3) \frac{\sin \sqrt{z}}{\sqrt{z}}$$

$$(4) \ln \sin z$$

$$\Rightarrow (1) \sin \sqrt{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{z})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n \sqrt{z}}{(2n+1)!} \Rightarrow \text{多值函数 (22值)}$$

$$(2) \cos \sqrt{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{z})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!} \Rightarrow \text{单值函数.}$$

$$(3) \frac{\sin \sqrt{z}}{z} = \frac{1}{\sqrt{z}} \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{z})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!} \Rightarrow \text{单值函数.}$$

(4) $\ln \sin z = \ln \xi$. $\xi = 0$ 为支点. 即 $z = k\pi$ ($k = 0, \pm 1, \pm 2, \dots$) 为支点 \Rightarrow 多值函数 2 无限值.

Δ 初等解析函数的级数表达式 (泰勒展开).

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

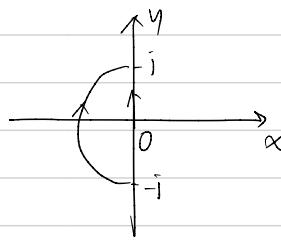
△复变函数积分基本操作

1. 计算从 $A = -i$ 到 $B = i$ 的积分 $\int_C |z| dz$ 的值. 其中 C 为:

(1) 线段 \overline{AB} . (2) 左半单位圆.

(1) 令 $z = x + iy$. 则 $|z| = \sqrt{x^2 + y^2}$. $dz = dx + idy$.

$$\Rightarrow \int_C |z| dz = \int_{x=0}^1 \sqrt{x^2 + y^2} (dx + idy) = i \left[\int_{-1}^0 -y dy + \int_0^1 y dy \right] = i \left[-\frac{1}{2}y^2 \Big|_{-1}^0 + \frac{1}{2}y^2 \Big|_0^1 \right] = i.$$



或者利用参数方程: 写出 z 的参数方程 $z(t) = x(t) + iy(t)$. $x(t) = 0$. $y(t) = t$. ($t \in [-1, 1]$)

$$\text{即 } z(t) = iy(t). |z| = |y| = |t|. dz = dy = dt \Rightarrow \int_C |z| dz = \int_{-1}^1 |t| dt = i \int_{-1}^0 t dt + i \int_0^1 t dt = i.$$

(2). 直接用参数方程较方便: $z = e^{it}$, $t: \frac{\pi}{2}\pi \rightarrow \frac{1}{2}\pi$. $\Rightarrow |z| = |e^{it}| = 1$, $dz = ie^{it} dt$.

$$\Rightarrow \int_C |z| dz = \int_{\frac{\pi}{2}}^{\frac{1}{2}\pi} ie^{it} dt = i \int_{\frac{\pi}{2}}^{\frac{1}{2}\pi} (\cos t + i \sin t) dt = i \left[\int_{\frac{\pi}{2}}^{\frac{1}{2}\pi} \cos t dt + i \int_{\frac{\pi}{2}}^{\frac{1}{2}\pi} \sin t dt \right] = 2i.$$

○计算积分 $\int_{|z|=1} \frac{dz}{z}$, $\int_{|z|=1} \frac{dz}{|z|}$, $\int_{|z|=1} \frac{|dz|}{z}$, $\int_{|z|=1} \left| \frac{dz}{z} \right|$.

(1) $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$. $\Rightarrow \oint_{|z|=1} \frac{dz}{z} = \oint_{|z|=1} ie^{i\theta} d\theta = 2\pi i$

(2) $\oint_{|z|=1} \frac{dz}{|z|} = \oint_{|z|=1} dz = 0$

(3) $z = e^{i\theta}$, $|dz| = |ie^{i\theta} d\theta| = d\theta \Rightarrow \int_{|z|=1} \frac{|dz|}{z} = \oint_{|z|=1} e^{-i\theta} d\theta = i \oint_{|z|=1} de^{-i\theta} = i \cdot e^{-i\theta} \Big|_0^{\pi} = 0$

(4) $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta \Rightarrow |z| = 1$. $|dz| = |ie^{i\theta} d\theta| = d\theta \Rightarrow \oint_{|z|=1} \left| \frac{dz}{z} \right| = \oint_{|z|=1} d\theta = 2\pi$

2. 设 C_r 为圆周 $|z|=r$ 在第 I 象限中的部分. 方向为正. 函数 $f(z)$ 在 C_r 上连续. 且

$\lim_{z \rightarrow 0} zf(z) = 0$. 证明: $\lim_{r \rightarrow 0} \int_{C_r} f(z) dz = 0$. 并计算极限 $\lim_{r \rightarrow 0} \int_{C_r} \frac{P(z)}{z} dz$. 其中 $P(z) = C_0 + C_1 z + C_2 z^2 + \dots + C_n z^n$, $C_0 \neq 0$.

(1) 由 $\lim_{z \rightarrow 0} zf(z) = 0$ 得: $\forall \varepsilon > 0$. $\exists \delta > 0$. 当 $|z| < \delta$ 时有 $|zf(z)| < \varepsilon$.

C_r 的参数方程为 $z = re^{i\theta}$, $\theta = 0 \rightarrow \frac{\pi}{2}$. 则 $|\int_{C_r} f(z) dz| = |\int_{C_r} f(re^{i\theta}) rie^{i\theta} d\theta| \leq \int_0^{\frac{\pi}{2}} |f(re^{i\theta})| r |d\theta| \leq \frac{\pi}{2} r$

$$\text{则 } \lim_{r \rightarrow 0} \int_{C_r} f(z) dz = 0.$$

(2)

△积分的估值不等式的应用: $|\int_C f(z) dz| \leq \int_C |f(z)| |dz| \leq ML$.

○设 r 为半径为 1 的上半正向半圆周. 求证: $|\int_r \frac{e^z}{z} dz| \leq \pi e$.

$$\Rightarrow \left| \int_r \frac{e^z}{z} dz \right| \leq \int_r \left| \frac{e^z}{z} \right| |dz| = \int_r \frac{|e^z|}{|z|} |dz| = \int_r |e^z| |dz| = \int_r e^x |dz| \leq e \int_r |dz| = \pi e$$

○设 C 为正向圆周 $|z|=2$ 在第 I 象限中的部分. 求证: $|\int_C \frac{1}{1+z^2} dz| \leq \frac{\pi}{3}$.

$$\Rightarrow \left| \int_C \frac{1}{1+z^2} dz \right| \leq \int_C \left| \frac{1}{1+z^2} \right| |dz| \leq \int_C \frac{1}{|1+z^2|} |dz| \leq \int_C \frac{1}{|1|z|^2 - 1} |dz| = \frac{1}{3} \int_C |dz| = \frac{\pi}{3}.$$

⇒ 搭配使用: $|z_1 z_2| = |z_1||z_2|$. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1||z_2|}{|z_2|}$, $|z_1 + z_2| \leq |z_1| + |z_2|$, $|z_1 - z_2| \geq ||z_1| - |z_2||$.

0 试证 $\left| \int_{|z|=r} \frac{dz}{(z+a)(z-a)} \right| < \frac{2\pi r}{|r^2 - |a|^2|}$ ($r > 0, |a| \neq r$). (熟读运用不等式)

$$\text{原式} \leq \int_{|z|=r} \frac{|dz|}{|z^2 - a^2|} < \int_{|z|=r} \frac{|dz|}{|r^2 - |a|^2|} = \frac{2\pi r}{|r^2 - |a|^2|}$$

△ 积分基本定理的应用 (注意被积函数的奇点) 及其推论

3. 计算积分 $\int_C (z^2 + 2e^{-z} + \sin z) dz$. 其中 C 为正向圆周 $|z|=4$.

由 $(z^2 + 2e^{-z} + \sin z)$ 解析, 得原式 = 0 (判断复变函数解析性的第二条)

4. 计算积分 $\int_C \frac{e^z}{z(z-1)} dz$. 其中 C 为正向圆周 $|z+1| = \frac{1}{2}$.

在积分区域内 $f(z)$ 无奇点且解析, 因此原式为 0.

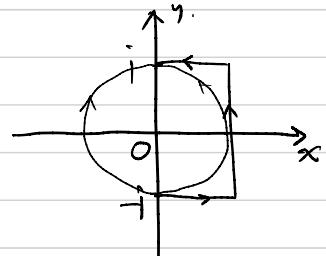
0. 计算积分 $\int_{-i}^i \frac{dz}{z}$. 其路径如右图 [推论 3.1]

→ 被积函数 $f(z) = \frac{1}{z}$ 的解析区域为: 从原点沿负实轴剪开的平面.

由积分基本定理推论, 该积分与路径无关. 设定路径为右半单位圆周.

$$\text{即 } z = e^{i\theta}, dz = ie^{i\theta} d\theta \Rightarrow \int_{-i}^i \frac{dz}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta} \cdot ie^{i\theta} d\theta = i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi i$$

$$\begin{aligned} \text{若按照原路径计算: } \int_{-i}^i \frac{dz}{z} &= \int_0^1 \frac{dx}{x-i} + \int_{-1}^{-i} \frac{idy}{y+i} + \int_1^0 \frac{dx}{x+i} \\ &= \ln(i) - \ln(-i) + \ln(-i) - \ln(i) \\ &= \ln(i) - \ln(-i) = i(\frac{\pi}{2} + 2k\pi) - i(-\frac{\pi}{2} + 2k\pi) = \pi i \end{aligned}$$



但是若设定路径为左半单位圆周, 积分为: $\int_{-i}^i \frac{dz}{z} = - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{-i\theta} \cdot ie^{i\theta} d\theta = -\pi i$

(由此可以理解为: 此题中 $z=0$ 为被积函数的奇点, 则 $f(z)$ 的解析区域因为是单连通区域, 因此从原点出发剪开一个平面, 认为沿负实轴剪开的平面是解析区域, 则左半圆周上 $f(z)$ 并不解析, 因此结果不正确)

同样地若设定路径为从 -i 到 i 的直线段, 则 $\int_{-i}^i \frac{dz}{z} = \int_{-1}^1 \frac{idy}{y} \Rightarrow \text{错误}$

0. 证明: $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$. (由 $\int_C \frac{dz}{z+2} = 0$, C 为正向圆周 $|z|=2$ 证明)

$$\Rightarrow z = \cos\theta + i\sin\theta, dz = -\sin\theta + i\cos\theta \Rightarrow \int \frac{dz}{z+2} = \int \frac{-\sin\theta + i\cos\theta}{\cos\theta + 2 + i\sin\theta} d\theta = \int \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$$

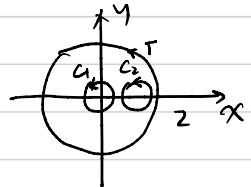
△柯西定理推广(复合闭路定理)的应用

→ 运用复合闭路定理：对于积分 $\oint_C f(z) dz$, 被积函数在 D 内有奇点。设奇点为 z_1, z_2, \dots, z_n ，
则作 n 个包含奇点的正向圆周 C_1, C_2, \dots, C_n 。原积分可化为 $(\oint_{C_1} + \oint_{C_2} + \dots + \oint_{C_n}) f(z) dz$ ，其中前者
由积分基本定理可知为 0。后者运用结论计算：

(遇此类积分可直接将积分曲线变化为包含奇点的正向曲线 → 推论)

$$\text{即 } \oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz \quad (\text{其中 } C_k \text{ 为包含 } f(z) \text{ 奇点的正向闭曲线})$$

0 熟练运用结论： $I = \oint_C \frac{1}{(z-z_0)^m} dz = \begin{cases} 2\pi i, & m=1 \\ 0, & m \neq 1 \end{cases}$ 其中 C 为包含 z_0 的任意正向闭区域 $\left(\oint_C \frac{1}{(z-z_0)^m} dz \right)$



△柯西积分公式的应用 ($\oint_C \frac{f(z)}{z-z_0} dz$)

→ 运用柯西积分公式计算闭线积分：对于积分 $\oint_C \frac{f(z)}{z-z_0} dz$ 来说， $z=z_0$ 为被积函数的奇点。

若积分曲线 C 围成的区域 D 不包含该奇点（即被积函数处处解析），则原积分为 0。

若 D 内包含奇点且只有唯一奇点 $z=z_0$ ，则原积分可用 $f(z)$ 在该点函数值表示： $2\pi i f(z_0) = \oint_C \frac{f(z)}{z-z_0} dz$

0 求积分 $\oint_{|z|=1} \frac{e^z}{z} dz$ 的值并证明定积分 $\int_0^{\pi} e^{i\theta} \cos(\sin\theta) d\theta = \pi$

$$\Rightarrow \oint_{|z|=1} \frac{e^z}{z} dz = 2\pi i, \text{ 令 } z=e^{i\theta}, \text{ 则有 } \int_0^{2\pi} e^{i\theta} [\cos(\sin\theta) + i\sin(\sin\theta)] d\theta = \int_0^{2\pi} \frac{e^z}{z} dz = 2\pi i$$

$$\text{又 } \int_0^{2\pi} e^{i\theta} \cos(\sin\theta) d\theta = \int_0^{2\pi} \operatorname{Re}[e^{i\theta} (\cos(\sin\theta) + i\sin(\sin\theta))] d\theta = \operatorname{Re}\left[\int_0^{2\pi} \frac{e^z}{z} dz\right] = 2\pi i$$

$$\text{又 } \int_0^{\pi} e^{i\theta} \cos(\sin\theta) d\theta = \left(\int_0^{\pi} + \int_{\pi}^{2\pi}\right) e^{i\theta} \cos(\sin\theta) d\theta, \text{ 其中 } \int_{\pi}^{2\pi} e^{i\theta} \cos(\sin\theta) d\theta = \int_{-\pi}^0 e^{i(\theta+2\pi)} \cos(\sin(\theta+2\pi)) d\theta$$

$$= \int_0^{\pi} e^{i(\theta+2\pi)} \cos(\sin(\theta+2\pi)) d\theta, \text{ 故而 } \int_0^{\pi} e^{i\theta} \cos(\sin\theta) d\theta = \pi$$

$$\text{或者也由 } \int_{\pi}^{2\pi} e^{i\theta} \cos(\sin\theta) d\theta = \int_{-\pi}^0 e^{i(\theta+2\pi)} \cos(\sin(\theta+2\pi)) d\theta = \int_0^{-\pi} e^{i(\theta+2\pi)} \cos(\sin(\theta+2\pi)) d\theta = \int_0^{\pi} e^{i\theta} \cos(\sin\theta) d\theta$$

$$0 \text{ 计算积分 } \oint_{|z|=2} \frac{z^2 - z + 1}{z-1} dz. \Rightarrow \text{原式} = 2\pi i (2z^2 - 2z + 1) \Big|_{z=1} = 4\pi i$$

△ 不能把有奇点的因子约掉。

△解析函数高阶导数公式的应用 ($\oint_C \frac{f(z)}{(z-z_0)^n} dz$)

○ 计算积分 $I = \int_0^{2\pi} e^{\cos\theta} (\cos(n\theta) - \sin(n\theta)) d\theta$

→ 尽量将此积分代入为关于 z 的积分式，即可运用定理求解。

$e^{\cos\theta} (\cos(n\theta) - \sin(n\theta))$ 和 $z = re^{i\theta}$ 比较相似，因此考虑将其转化为 $f(z)$ 的积分。

$$\text{由题有 } e^{\cos\theta} [\cos(n\theta) - \sin(n\theta)] = e^{\cos\theta + i\cos\theta - i\sin\theta} = e^{e^{i\theta} + i\cos\theta} = e^{-i\theta} \cdot e^{i\cos\theta}.$$

则令 $z = e^{-i\theta}$ 。上式可化为 $= e^z \cdot z^{-n} = \frac{e^z}{z^n}$ 且有 $dz = -ie^{-i\theta} d\theta = -iz d\theta$ 。即 $d\theta = \frac{1}{iz} dz$

$$\Rightarrow I = \int_0^{2\pi} e^{\cos\theta} [\cos(n\theta) - \sin(n\theta)] d\theta = \oint_{|z|=1} \frac{e^z}{z^{n+1}} dz \quad (\text{注意积分区域的变化})$$

$$\text{由高阶导数公式} = \oint_{|z|=1} \frac{e^z}{z^{n+1}} dz = \frac{1}{i} \cdot \frac{2\pi i}{n!} \cdot (e^z)^{(n)} \Big|_{z=0} = \frac{2\pi}{n!}.$$

○ 设函数 $f(z)$ 在 $|z| \leq 2$ 上解析，且在 $|z|=2$ 上有 $|f(z)-z| \leq |z|$ ，证明 $|f'(1)| \leq 8$

建立积分 $I = \oint_{|z|=2} \frac{f(z)}{(z-1)^2} dz$ ，则显然 $I = 2\pi i f'(1)$ 。

另一方面 $\left| \oint_{|z|=2} \frac{f(z)}{(z-1)^2} dz \right| \leq \oint_{|z|=2} \frac{|f(z)|}{|(z-1)|^2} |dz|$ ，其中由 $|f(z)-z| \leq |f(z)|-|z| \leq |z| \leq 2$ 得 $|f(z)| \leq 4$

对 $|z|=2$ 有 $|(z-1)^2| \leq 1$ 。则上式 $\leq 16 \oint_{|z|=2} |dz| = 16\pi$ ，即 $|f'(1)| \leq \left| \frac{I}{2\pi i} \right| = \frac{|I|}{2\pi} \leq 8$

或者建立对 $|z|=2$ 的积分。

○ 设函数 $f(z)$ 在 $|z| < 1$ 内解析，且 $f(0)=1$, $f'(0)=2$ 。计算积分 $I_1 = \oint_{|z|=1} \frac{(z+1)^2}{z^2} f(z) dz$ 。

1) 计算积分 $I_1 = \int_0^{2\pi} \cos^2 \frac{\theta}{2} f(e^{i\theta}) d\theta$ 。

(1) 由柯西导数公式容易得 $I_1 = 2\pi i [(z+1)^2 f(z)]' \Big|_{z=0} = 8\pi i$

(2) 将(1)中令 $z = e^{i\theta}$ ，则 $I_1 = 4i \int_0^{2\pi} \cos^2 \frac{\theta}{2} f(e^{i\theta}) d\theta$ ，即 $I_1 = \frac{1}{4} I = 2\pi$ 。

○ 计算积分 $\oint_{|z|=1} (z+\frac{1}{z})^{2n} \frac{dz}{z}$ ，并证明 $\int_0^{2\pi} \cos^{2n} \theta d\theta = \frac{2\pi (2n)!}{2^{2n} (n!)^2} = 2\pi \cdot \frac{(2n)!!}{(2n)!!}$

→ 基本上都是令 $z = e^{i\theta}$ 。

● 设 D 为单连通区域， $z_0 \in D$ ， $f(z)$ 在 D 内除 z_0 外均解析，且 $|f(z)|$ 在 z_0 的邻域内有界。证明：
 $\oint_C f(z) dz = 0$ ，其中 C 为 D 内任一包含 z_0 的闭曲线。

→ 用积分基本定理的推论知： $\oint_C f(z) dz = \oint_{C'} f(z) dz$ ，其中 C' 为 $z=z_0$ 邻域内包含 z_0 在内的闭曲线

则由 $|f(z)|$ 在 z_0 邻域内有界得： $\exists M > 0$ ，使得 $|f(z)| \leq M$

同时有 $|\oint_C f(z) dz| \leq \oint_C |f(z)| |dz| \leq M \oint_C |dz|$ ，由 C' 为任意闭曲线，则当 C' 趋于无限小时 $\oint_{C'} |dz| \rightarrow 0$

即 $0 < |\oint_C f(z) dz| \leq M \oint_C |dz| \rightarrow 0$ 。由 ϵ 的任意性可知 $|\oint_C f(z) dz| = 0$ ，即 $\oint_C f(z) dz = 0$ 。

设 $f(z) = \oint_{|\zeta|=2} \frac{\zeta^3 + 2\zeta + 1}{(\zeta - z)^2} d\zeta$, 求 $f'(i)$. [例 3.18]

$$\Rightarrow f(z) = \oint_{|\zeta|=2} \frac{\zeta^3 + 2\zeta + 1}{(\zeta - z)^2} d\zeta = 2\pi i (\zeta^2 + 2\zeta + 1) \Big|_{\zeta=z} = 2\pi i (iz^2 + 2z) \Rightarrow f'(i) = -12\pi,$$

求积分 $\oint_{|z|=1} \frac{e^z}{z} dz$. 计算 $\int_0^\pi e^{i\theta} (\sin \theta) d\theta = \pi$.

$$\Rightarrow \int_0^\pi e^{i\theta} [\underbrace{\cos(\sin \theta)}_{\text{实部}} + i \underbrace{\sin(\sin \theta)}_{\text{虚部}}] d\theta = 2 \int_0^\pi e^{i\theta} \cos(i\theta) d\theta.$$

△ 复数项级数

○ 题型 判别级数收敛性

$$(1) \sum_{n=1}^{+\infty} \frac{(1+i)^n}{2^n \cos n}, \quad (2) \sum_{n=1}^{+\infty} \frac{i^n}{n}, \text{ 和 } \sum_{n=1}^{+\infty} \frac{i^n}{n^2} (n>0), \quad (3) \sum_{n=1}^{+\infty} \frac{1}{(1+i)^n}, \quad (4) \sum_{n=1}^{+\infty} (1+\frac{1}{n}) e^{\frac{i\pi}{n}} \text{ 和 } \sum_{n=1}^{+\infty} \frac{e^{\frac{i\pi}{n}}}{n}, \quad (5) \sum_{n=1}^{+\infty} \frac{1+(1-i)^{2n}}{n}$$

$$(6) \sum_{n=1}^{\infty} \sin(n\pi + \frac{1}{n}), \quad (7) \sum_{n=1}^{\infty} \frac{i^n}{n^n}, \quad (8) \sum_{n=1}^{\infty} \frac{\cos ni}{2^n}, \quad (9) \sum_{n=1}^{\infty} \frac{1}{1+(\frac{1}{2})^n}$$

→ (1) 首先判断函数项是否收敛: $\left| \frac{(1+i)^n}{2^n \cos n} \right| \leq \left| \left(\frac{1+i}{2} \right)^n \right| = \left| \frac{1+i}{2} \right|^n = 1 \Rightarrow \text{无法判断.}$

考虑比较审敛法: $\left| \frac{(1+i)^n}{2^n \cos n} \right| = \left| \frac{(1+i)^n}{2^n} \cdot \frac{1}{\cos n} \right| = \frac{2}{e^{n\pi} + e^{-n\pi}} < \frac{2}{e^n} < 1$. 由于被数 $\sum_{n=1}^{\infty} \frac{2}{e^n}$ 收敛, 则原级数收敛.

(2) 首先有 $\sum_{n=1}^{\infty} \left| \frac{i^n}{n^n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^n} \rightarrow \text{发散.}$

再有: $\sum_{n=1}^{\infty} \frac{i^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} (\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}) \Rightarrow \operatorname{Re} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2}$, 由交错级数相关性质知其收敛.

同理 $\operatorname{Im} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \cdot \sin \frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \rightarrow \text{收敛}, \text{ 所以原级数是收敛的.}$

$$(3) \sum_{n=1}^{+\infty} \left| \frac{1}{(1+i)^n} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{1+i} \right|^n = \sum_{n=1}^{\infty} \left(\frac{1}{1+i} \right)^n = \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow \text{收敛} \Rightarrow \text{绝对收敛}$$

(4)

$$(5) \sum_{n=1}^{\infty} \frac{1+(-i)^{2n+1}}{n} = \sum_{n=1}^{\infty} \frac{1+(-1)^n i}{n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n} (-1)^n i \right) \rightarrow \text{显然发散.}$$

$$(6) \sum_{n=1}^{\infty} \sin(n\pi + \frac{1}{n}) = \sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n} \rightarrow \text{交错级数收敛}$$

$$(7) \sum_{n=1}^{\infty} \left| \frac{i^n}{n^n} \right| = \sum_{n=1}^{\infty} \frac{1}{n^n} > \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow \text{发散.}$$

$$\text{又 } \sum_{n=1}^{\infty} \frac{i^n}{n^n} = \sum_{n=1}^{\infty} \frac{1}{n^n} (\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}) = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{(2n)^n} + \frac{(-1)^{n-1}}{(2n-1)^n} \right] \rightarrow \text{收敛.} \quad \left. \begin{array}{l} \\ \end{array} \right\} \rightarrow \text{条件收敛}$$

$$(8) \sum_{n=1}^{\infty} \left| \frac{\cos ni}{2^n} \right| < \sum_{n=1}^{\infty} \frac{1}{2^n} \rightarrow \text{收敛} \Rightarrow \text{绝对收敛.}$$

$$(9) \sum_{n=1}^{\infty} a_n = \frac{1}{1+(1-i)^n} + \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{1+\frac{1}{2^n}(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2})} = \neq 0 \rightarrow \text{发散.}$$

○ 题型 求幂级数的收敛半径

$$(1) \sum_{n=0}^{\infty} (-i)^n \cdot \frac{2n+1}{2^n} \cdot z^{2n+1}, \quad (2) \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^n \cdot (z-i)^{n(n+1)}$$

$$(1) \Rightarrow \sum f_n(z) = (-i)^n \cdot \frac{2n+1}{2^n} \cdot z^{2n+1}, \text{ 由 } \lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-i)^{n+1} \cdot \frac{2n+3}{2^{n+1}} \cdot z^{2n+3}}{(-i)^n \cdot \frac{2n+1}{2^n} \cdot z^{2n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-i)^{2n+2}}{2} \cdot z^2 \right| = \frac{1}{2} |z|^2.$$

当 $\frac{1}{2} |z|^2 < 1$ 时, 幂级数绝对收敛; 当 $\frac{1}{2} |z|^2 > 1$ 时, 幂级数发散, 故收敛半径 $R = \sqrt{2}$.

$$(2) \sum f_n(z) = \left(\frac{1}{n} \right)^n \cdot (z-i)^{n(n+1)} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| =$$

→ 对于缺项幂级数, 不能用系数模比值/根值法确定收敛半径, 可使用数项级数的比值/根值判别法, $\lim_{n \rightarrow \infty} \frac{C_{n+1}}{C_n}$, $\lim_{n \rightarrow \infty} \sqrt[n]{C_n}$.

○ 题型 讨论级数 $\sum_{n=0}^{\infty} (z^{n+1} - z^n)$ 的收敛性

$$\Rightarrow \sum_{n=0}^{\infty} (z^{n+1} - z^n) = z^{n+1} - 1. \text{ 则当 } z=1 \text{ 时, 级数收敛于 } 0, \text{ 当 } |z| < 1 \text{ 时, 级数收敛于 } -1.$$

当 $z=-1$ 时, 由 $(-1)^{n+1} = -1$ 或知级数发散. 当 $z = e^{i\theta}$ 时, $\operatorname{Im} \sum n z^n = \operatorname{Im} \sum n e^{in\theta} = \operatorname{Im} \sum n \cos n\theta + i \sum n \sin n\theta$ 为无极限, 因而发散.

当 $|z| > 1$ 时, 显然发散.

【题型】求下列级数的和函数

$$(1) \sum_{n=1}^{\infty} (-1)^{n-1} n z^n.$$

$$(2) \frac{1}{1x2} + \frac{z}{2x3} + \frac{z^2}{3x4} + \frac{z^3}{4x5} + \dots$$

$$\Rightarrow (1) S(z) = \sum_{n=1}^{\infty} (-1)^{n-1} n z^n = z - 2z^2 + 3z^3 - 4z^4 + \dots$$

$$\Rightarrow zS(z) = z^2 - 2z^3 + 3z^4 - 4z^5 + \dots \text{ 两式相加: } (2+1)S(z) = z - z^2 + z^3 - z^4 + \dots = -(-z + z^2 - z^3 + z^4 + \dots)$$

$$= -\frac{-z(1+z)}{1+z} = \frac{z}{1+z}, |z| < 1$$

$$\text{即 } S(z) = \frac{z}{(1+z)^2}, |z| < 1$$

$$(2) S(z) = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n(n+1)} \Rightarrow z^2 S(z) = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n(n+1)}. \text{ 求导有 } [z^2 S(z)]' = \sum_{n=1}^{\infty} z^n = \lim_{n \rightarrow \infty} \frac{1-z^n}{1-z} = \frac{1}{1-z}, |z| < 1.$$

$$\Rightarrow [z^2 S(z)]' = \int \frac{1}{1-z} dz = -\ln|1-z|. \Rightarrow z^2 S(z) = \int (-\ln|1-z|) dz = z - (z-1)\ln|1-z| \Rightarrow S(z) = \frac{1}{2} - \frac{z-1}{2z} \ln|1-z|. |z| < 1.$$

【题型】将 $f(z)$ 展开成形如 $\sum_{n=0}^{\infty} C_n (z-z_0)^n$ 的幂级数

1) 将 $f(z) = \frac{1}{3z-2}$ 展开成形如 $\sum_{n=0}^{\infty} C_n (z-2)^n$ 的幂级数

$$\Rightarrow f(z) = \frac{1}{3z-2} = \frac{1}{3(z-2)+4} = \frac{1}{4} \cdot \frac{1}{1-\frac{3}{4}(z-2)} = \frac{1}{4} \sum_{n=0}^{\infty} \left[-\frac{3}{4}(z-2) \right]^n = \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{4^{n+1}} (z-2)^n.$$

(思路: 形如 $f(z) = \frac{1}{az+bz}$ 的函数通过等比数列的求和公式逆向进行转换)

其收敛区域由几何级数知 $\left| -\frac{3}{4}(z-2) \right| = \frac{3}{4}|z-2| < 1$ 即 $|z-2| < \frac{4}{3}$, 收敛半径为 $\frac{4}{3}$.

△泰勒级数

设 $0 < r < 1$, 利用函数 $f(z) = \frac{1}{1-z}$ 的幂级数证明:

$$\sum_{n=0}^{+\infty} r^n \cos n\theta = \frac{r \cos \theta}{1 - 2r \cos \theta + r^2}; \quad \sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}$$

\Rightarrow 思路 = 观察等式左侧发现比较相似, 等式右侧分子可化为 $(r \cos \theta)^2 + r^2 \sin^2 \theta$ 形式
考虑将两个等式相加, 用实部和虚部的方法.

$$\Rightarrow \sum_{n=0}^{\infty} r^n (\cos n\theta + i \sin n\theta) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \text{ 其中 } z = r(\cos \theta + i \sin \theta) \text{ 且 } \frac{1}{1-z} = \frac{1}{1-r \cos \theta - i r \sin \theta}$$

$$= \frac{1}{1-(r \cos \theta) - i r \sin \theta} = \frac{r \cos \theta + i r \sin \theta}{(r \cos \theta)^2 + r^2 \sin^2 \theta} = \frac{r \cos \theta}{1 - 2r \cos \theta + r^2} + i \frac{r \sin \theta}{1 - 2r \cos \theta + r^2} \text{ 由实部、虚部对应相等即证}$$

设函数 $f(z) = \frac{1}{1-z-z^2}$ 在 $z=0$ 处的泰勒级数为 $\sum_{n=0}^{\infty} a_n z^n$. (1) 求级数的收敛半径.

(2) 导出 a_n 满足的递推关系式. (3) 证明: $n=0, 1, 2, \dots$

$$\frac{1}{2\pi i} \oint_{|\xi|=R} \frac{1+\xi^2 f(\xi)}{(\xi-z)^{n+1}} d\xi = \frac{f^{(n)}(z)}{n!}, |z| < r < R. \quad \therefore a_n = \frac{f^{(n)}(z_0)}{n!}$$

$\Rightarrow (1)$ $R = \frac{\sqrt{5}-1}{2}$ (2) 此表达式 $f(z) = \frac{1}{1-z-z^2}$ 不便于使用间接法. 于是使用直接法 将问题转化为求 $f(z)$ 的 n 阶导数. $\Rightarrow (1-z-z^2)f(z)=1$, 即有 $C_0(1-z-z^2)f(z) + C_1(1+z)f'(z) + C_2(-2)f''(z)=0$

代入 $z=0$ 得: $f^{(n)}(0) - n f^{(n+1)}(0) - n(n+1) f^{(n+2)}(0) = 0$, 又有 $f(0)=1$, $f'(0)=1$, $f''(0)=4$.

$$\text{即 } f^{(3)}(0) = 3f^{(2)}(0) + 3 \times 2 f^{(1)}(0) \Rightarrow \frac{f^{(3)}(0)}{3!} = \frac{f^{(2)}(0)}{2!} + \frac{f^{(1)}(0)}{1!} \Rightarrow f^{(n)}(0) = \frac{f^{(n+1)}(0)}{(n+1)!} + \frac{f^{(n+2)}(0)}{(n+2)!} \Rightarrow a_n = a_{n+1} + a_{n+2}.$$

$$f^{(4)}(0) = 4f^{(3)}(0) + 4 \times 3 f^{(2)}(0) \Rightarrow \frac{f^{(4)}(0)}{4!} = \frac{f^{(3)}(0)}{3!} + \frac{f^{(2)}(0)}{2!} \quad \text{其中 } a_0 = a_1 = 1.$$

$$(3) \frac{1}{2\pi i} \oint_{|\xi|=R} \frac{1+\xi^2 f(\xi)}{(\xi-z)^{n+1}} d\xi = \frac{1}{2\pi i} \oint_{|\xi|=R} \frac{1-\xi}{(\xi-z)^{n+1}} d\xi, \text{ 其中 } \frac{1+\xi^2 f(\xi)}{1-\xi} = f(\xi), \text{ 证毕.}$$

[题型: 求函数的泰勒展开式] $\left\{ \begin{array}{l} \text{在某点 } z=z_0 \text{ 处展开} \Rightarrow \text{找圆域} \Rightarrow \text{找最近奇点} \\ \text{比较系数法} \\ \text{在某圆域 } |z-z_0| < R \text{ 内展开} \end{array} \right.$

○ 将 $f(z) = \sec z$ 在 $z=0$ 处泰勒展开.

$\Rightarrow f(z) = \sec z = \frac{1}{\cos z}$. 其收敛半径易求得 $R = \frac{\pi}{2}$, 故当 $|z| < \frac{\pi}{2}$ 时, $f(z)$ 在 $z=0$ 处的泰勒级数可设为:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$$

由幂级数在收敛圆内绝对收敛的性质得:

$$f(z) = \sec z \cdot \cos z = (c_0 + c_1 z + c_2 z^2 + \dots) \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right)$$

$$= c_0 + \left(-\frac{c_0}{2!} + c_2 \right) z^2 + \left(\frac{c_0}{4!} - \frac{c_1}{2!} + c_4 \right) z^4 + \dots$$

经比较系数后可得: $c_0 = 1$, $c_2 - \frac{c_0}{2!} = 0$, $c_4 - \frac{c_1}{2!} + \frac{c_0}{4!} = 0, \dots$, $c_{2n} - \frac{c_{2n-2}}{2!} + \frac{c_{2n-4}}{4!} + \dots + (-1)^n \frac{c_0}{(2n)!} = 0$.

即 $c_0 = 1$, $c_2 = \frac{1}{2!}$, $c_4 = \frac{5}{4!}, \dots$. 故 $\sec z = 1 + \frac{1}{2!} z^2 + \frac{5}{4!} z^4 + \dots$ ($|z| < \frac{\pi}{2}$)

(直接法) \Rightarrow 直接由 $c_n = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{f(\xi)}{(\xi-z)^{n+1}} d\xi = \frac{f^{(n)}(z)}{n!}$ 公式求出泰勒系数

(间接法) \Rightarrow ① 求导; ② 积分; ③ 借助已知函数的泰勒展开.

○ 将 $f(z) = \frac{1}{z^3}$ 在 $z=1$ 处泰勒展开.

$$\Rightarrow f(z) = \frac{1}{z} \cdot \frac{d^2}{dz^2} \left(\frac{1}{z} \right) = \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{1-z} \right) = \frac{1}{2} \frac{d}{dz} \left[\sum_{n=0}^{\infty} (1-z)^{-n} (z-1)^n \right] = \sum_{n=0}^{\infty} \frac{(-1)^n n(n-1)}{2} \cdot (z-1)^{n-2}, |z-1| < 1.$$

奇点为 $z=0$, 故收敛半径 $R=1=|z-1| < 1$

\Rightarrow 对 $f(z) = \frac{1}{z^n}$ ($n>1$) 在 $z=0$ 处泰勒展开用求导 $f(z) = \frac{1}{a+bz}$ 直接凑

○ 将 $f(z) = \frac{1}{(z-a)(z-b)}$ ($ab \neq 0$) 在 $z=0$ 处展成泰勒级数并指明收敛域.

对于 $\sum_{n=0}^{\infty} (-1)^n$ 类型的级数其收敛域为 $|z| < 1 \Rightarrow |z-1| < 1$.

△ 罗朗级数.

↑

已知 $f(z) = \sum_{n=0}^{\infty} (-1)^n (z-2)^n + \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n$ 的收敛域.

\Rightarrow 对正幂项级数 $\sum_{n=0}^{\infty} (-1)^n (z-2)^n = \sum_{n=0}^{\infty} (-\frac{1}{2})^n (2-z)^n \Rightarrow$ 收敛半径 $R = \frac{1}{\limsup_{n \rightarrow \infty} |(-\frac{1}{2})^n|} = 2$, 即 $|z-2| < 2$.

对负幂项级数 $\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n \Rightarrow$ 公比 $\zeta = \frac{1}{2}$, 则原式 $= \sum_{n=0}^{\infty} (-1)^n \zeta^n = \frac{1}{1-\zeta} = \frac{2}{2-1}$, 奇点 $z=1$, 则收敛半径 $R=1$, 即 $|z| < 1 \Rightarrow |z-2| > 1$

将 $|z-2| < 2$ 及 $|z-2| > 1$ 代入级数, 均发散, 综上: 收敛域为环域 $1 < |z-2| < 2$

0 将函数 $f(z) = \frac{\sin z}{z}$ 在圆环域 $0 < |z| < +\infty$ 内展开为罗朗级数 \Rightarrow 级数中心为 $z=0$

{ 将函数 $f(z) = \frac{1}{z}$ 在点 $z=1$ 处展开为泰勒级数. \Rightarrow 级数中心为 $z=1$

\Rightarrow 注意泰勒级数和罗朗级数的区别!

在某圆环域内展开

[题型: 求函数的罗朗展开式] { 在以某点为中心的收敛域内展开

0 将函数 $f(z) = \frac{1}{z^2 - 6z + 5}$ 在以 $z=2$ 为中心的邻域内级数展开

$\Rightarrow f(z) = \frac{1}{z-5} - \frac{1}{z-1}$, 其零点为 $z=1, z=5$. 则其收敛域被分为3个部分: $|z-2| < 1, 1 < |z-2| < 3, |z-2| > 3$

在 $|z-2| < 1$ 内: $\frac{1}{z-5} = \frac{1}{(z-2)-3} = -\frac{1}{3} \cdot \frac{1}{1-(\frac{z-2}{3})} = -\frac{1}{3} \sum_{n=0}^{\infty} (\frac{z-2}{3})^n$; $\frac{1}{z-1} = \frac{1}{1+(z-2)} = \sum_{n=0}^{\infty} (-1)^n (z-2)^n \Rightarrow f(z) = \sum_{n=0}^{\infty} \left[-\frac{1}{3} \frac{1}{3^n} + (-1)^{n+1} \right] (z-2)^n$

$|z-2| > 3$: $\frac{1}{z-5} = -\frac{1}{3} \cdot \frac{1}{1-(\frac{z-2}{3})} = -\frac{1}{3} \sum_{n=0}^{\infty} (\frac{z-2}{3})^n$; $\frac{1}{z-1} = \frac{1}{z-2} \cdot \frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2} \sum_{n=0}^{\infty} \left(\frac{1}{z-2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{(z-2)^{n+1}}$

$\Rightarrow f(z) = \sum_{n=0}^{\infty} \left[-\frac{1}{3} \frac{1}{3^n} (z-2)^n + (-1)^{n+1} \frac{1}{(z-2)^{n+1}} \right]$

$|z-2| > 3$: $\frac{1}{z-5} = -\frac{1}{3} \cdot \frac{1}{1-(\frac{z-2}{3})} = -\frac{1}{1-(z-2)} \cdot \frac{1}{1-(\frac{z-2}{z-2})} = -\frac{1}{z-2} \sum_{n=0}^{\infty} \frac{z^n}{(z-2)^n} = -\sum_{n=0}^{\infty} \frac{z^n}{(z-2)^{n+1}}$

$\frac{1}{z-2} = \sum_{n=0}^{\infty} (-1)^n (\frac{1}{z-2})^{n+1} \Rightarrow f(z) = \sum_{n=0}^{\infty} \left[-z^n + (-1)^{n+1} \right] \frac{1}{(z-2)^{n+1}}$.

\Rightarrow 级数 $f(z) = \frac{1}{z-2}$ 能直接展开的条件为 $|z| < 1$, 否则要创造此条件, 答案中必定含有因子 $(z-2)$.
以题给中心, 首先由奇点确定多个收敛域

0 将 $f(z) = \frac{1}{z^2 - 2z + 1}$ 在以 i 为中心的邻域内级数展开.

0 将 $f(z) = e^{\frac{1}{z^2}}$ 在 $|z| > 1$ 内展开.

0 将 $f(z) = \frac{2}{(z-1)(z-3)}$ 在指定点展开为罗朗级数. (1) $z=0$. (2) $z=1$ 或 $z=3$.

○ 求 $f(z) = e^{z+\frac{1}{z}}$ 在 $0 < |z| < +\infty$ 内的罗朗展开式

$$\Rightarrow e^{z+\frac{1}{z}} = \sum_{n=-\infty}^{\infty} C_n z^n = C_0 + \sum_{n=1}^{\infty} C_n (z^n + z^{-n}) \text{ 其中 } C_n = \frac{1}{2\pi i} \int_{\Gamma} e^{2\cos\theta} \cos n\theta d\theta, n=0,1,2,\dots.$$

$$(C_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{z+\frac{1}{z}}}{z^{n+1}} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{2\cos\varphi}}{e^{in\varphi}} \cdot ie^{i\varphi} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} e^{2\cos\varphi} \cos n\varphi d\varphi)$$

○ 试证：在 $0 < |z| < +\infty$ 内下列展开式成立 $\cosh(z + \frac{1}{z}) = C_0 + \sum_{k=1}^{\infty} C_k (z^k + z^{-k})$

$$\text{其中 } C_k = \frac{1}{2\pi i} \int_{\Gamma} \cosh(z) \cosh(2\cos\varphi) d\varphi.$$

$\Rightarrow W = z + \frac{1}{z}$ 在复平面上只有一个奇点 $z=0$ ，又 $\cosh w$ 在 W 的平面上解析，因此 $\cosh(z + \frac{1}{z})$ 在复平面上也只有 $z=0$ 一个奇点。即在 $0 < |z| < +\infty$ 环域内解析。由罗朗定理得 $\cosh(z + \frac{1}{z})$ 可罗朗展开。

$$\cosh(z + \frac{1}{z}) = \sum_{k=-\infty}^{\infty} C_k z^k = \sum_{k=0}^{\infty} C_k z^k + \sum_{k=0}^{-\infty} C_k z^k = C_0 + \sum_{k=1}^{\infty} C_k z^k + \sum_{k=1}^{\infty} C_k z^{-k}$$

其中 $C_k = \frac{1}{2\pi i} \int_{T_p} \frac{\cosh(z+z^{-1})}{z^{k+1}} dz$. T_p 表示任意圆周 $|z|=p$ ($p>0$). 若取 $p=1$. 则沿圆周 $T_p: z=e^{i\varphi}, 0 \leq \varphi \leq 2\pi$ 有：

$$C_k = \frac{1}{2\pi} \int_0^{2\pi} \cosh(e^{i\varphi} + e^{-i\varphi}) e^{-ni\varphi} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\varphi) \cos n\varphi d\varphi - \frac{i}{2\pi} \int_0^{2\pi} \cosh(2\cos\varphi) \sin n\varphi d\varphi$$

令 $\varphi = 2\pi - \theta$. 则 $\int_0^{2\pi} \cosh(2\cos\varphi) \sin n\varphi d\varphi = - \int_0^{2\pi} \cosh(2\cos\theta) \sin n\theta d\theta$. 则 $\int_0^{2\pi} \cosh(2\cos\varphi) \sin n\varphi d\varphi = 0$

$$\text{且 } C_k = \frac{1}{2\pi} \int_0^{2\pi} \cosh(2\cos\varphi) \cos n\varphi d\varphi. \text{ 且有 } C_{-k} = C_k. \Rightarrow \cosh(z + \frac{1}{z}) = C_0 + \sum_{k=1}^{\infty} C_k (z^k + z^{-k}).$$

○ (与上题类似) 试证 $\sin[t(z + \frac{1}{z})] = C_0 + \sum_{k=1}^{\infty} C_k (z^k + z^{-k})$. $0 < |z| < +\infty$. 其中 t 为与 z 无关的实参数。

$$C_k = \frac{1}{2\pi} \int_0^{2\pi} \sin(t\cos\varphi) \cos k\varphi d\varphi (k=0,1,2,\dots).$$

$$\Rightarrow \sin[t(z + \frac{1}{z})] = \sum_{k=1}^{\infty} C_k z^k.$$

$$\text{其中 } C_k = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{\sin[t(\xi + \xi^{-1})]}{\xi^{k+1}} d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\sin[t(e^{i\varphi} + e^{-i\varphi})]}{e^{ik\varphi}} ie^{i\varphi} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(2t\cos\varphi)}{e^{ik\varphi}} d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin(2t\cos\varphi) (\cos k\varphi - i \sin k\varphi) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \sin(2t\cos\varphi) \cos k\varphi d\varphi.$$

△ 孤立奇点

[题型：确定函数的奇点及其类型，极点及其阶数。]

0 在复平面内找出函数 $f(z) = \frac{z^2(z-1)}{(\sin\pi z)^2}$ 的孤立奇点，并确定类型及可能的阶数

\Rightarrow 分母 $(\sin\pi z)^2$ 的零点为 $z=n$ ($n=0, \pm 1, \pm 2, \dots$)，则 $f(z)$ 具有奇点 $z=n$ ($n=0, \pm 1, \pm 2, \dots$)。

① 又 $\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \frac{z^2(z-1)}{(\sin\pi z)^2} = -\frac{1}{\pi^2}$ 即 $z=0$ 为可去奇点；(也可由 $z=0$ 为分母分子的二阶零点得到)

② $(\sin\pi z)'|_{z=n} = \pi \cos\pi z|_{z=n} = (-1)^n \pi \neq 0$ 可知 $z=n$ 为 $\sin\pi z$ 的一阶零点，则为 $(\sin\pi z)^2$ 的二阶零点。

(也可由 $(\sin\pi z)^2|_{z=n}=0$, $(\sin\pi z)'|_{z=n}=0$, $(\sin\pi z)''|_{z=n} \neq 0$ 得到)

而 $f(z)$ 的分子有一阶零点 $= z=\pm 1$ ，因为 $z^2=(z+1)(z-1)$ ，所以 $z=\pm 1$ 为 $f(z)$ 的一阶极点。

($\lim_{z \rightarrow \pm 1} f(z) = \lim_{z \rightarrow \pm 1} \frac{z^2(z-1)}{(\sin\pi z)^2} = \lim_{z \rightarrow \pm 1} \frac{z^2}{\sin\pi z} \Rightarrow -1$ 极点)

③ 对于 $z=\pm 2, \pm 3, \dots$ 其为 $f(z)$ 的二阶极点。

0 确定函数 $f(z) = \frac{z}{e^{z-1}} e^{\frac{1}{z-1}}$ 的孤立奇点并确定其类型。

\Rightarrow 由 $e^{z-1}=0$ 得 $z=\ln 1=2k\pi i$ ，则有奇点 $z=1, 2k\pi i$ ($k=0, \pm 1, \dots$)

① $\lim_{z \rightarrow 1} f(z) = \lim_{z \rightarrow 1} \frac{z}{e^{z-1}} e^{\frac{1}{z-1}} = \frac{1}{e^0} e^\infty \Rightarrow$ 不存在，因此 $z=1$ 为本性奇点。

② 对分母有： $(e^{z-1})'|_{z=2k\pi i} = 0$, $(e^{z-1})''|_{z=2k\pi i} = 1 \neq 0 \Rightarrow z=2k\pi i$ 为一阶极点 ($k \neq 0$)

③ $k=0$ 时， $\lim_{z \rightarrow 0} f(z) = e^1$ ，即 $z=0$ 为可去奇点。

△ 讨论函数无穷远点性质

$$\begin{aligned}|z| < 1 &: f(z) = \frac{z}{2} \left(\frac{1}{z-1} - \frac{1}{z-3} \right) = \frac{z}{2} \left(\frac{1}{3-z} - \frac{1}{z} \right) \\&= \frac{z}{2} \left(\frac{1}{3} - \frac{1}{1-\frac{z}{3}} - \frac{1}{z} \right) \\&= \frac{z}{2} \cdot \left(\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} - \sum_{n=0}^{\infty} z^n \right) = \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{1}{3^{n+1}} - 1 \right) z^n.\end{aligned}$$

$$1 < |z| < 3: f(z) = \frac{z}{2} \left(\frac{1}{3} + \frac{1}{z-3} + \frac{1}{z-1} \right) = \frac{z}{2} \cdot \left(\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \right) = \sum_{n=0}^{\infty} \left(\frac{z^{n+1}}{2 \cdot 3^{n+1}} + \frac{1}{2 z^{n+1}} \right)$$

$$3 < |z| < \infty: f(z) = \frac{1}{2} \left(\frac{1}{z-3} - \frac{1}{z-1} \right) = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{3^n}{z^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{1}{2} \left(-3^n \right) \frac{1}{z^n}.$$

$$0 < |z-1| < 2: f(z) = \frac{z-1}{(z-1)(z-3)} + \frac{1}{(z-1)(z-3)} = \frac{1}{z-3} + \frac{1}{z-1} - \frac{1}{z-3}$$

$$\boxed{-\frac{1}{2} \cdot \frac{1}{z-1} + \frac{3}{2} \cdot \frac{1}{z-3}} \Rightarrow -\sum_{n=0}^{\infty} \frac{(z-1)^{n+1}}{2^{n+1}}.$$

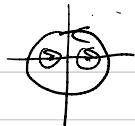
$$2 < |z-1| < \infty: \frac{1}{z-3} = \frac{1}{(z-1)-2} = \frac{1}{z-1} \cdot \frac{1}{1-\frac{2}{z-1}} = \frac{1}{z-1} \sum_{n=0}^{\infty} \frac{2^n}{(z-1)^{n+1}} = \sum_{n=0}^{\infty} \frac{2^n}{(z-1)^{n+1}}$$

$$\frac{A}{z-1} + \frac{B}{z-3}$$

$$\begin{aligned}A(z-1)+B(z-3) \\ A+B=1 \\ 2A+2B=0 \\ 2A=-1 \\ A=-\frac{1}{2}\end{aligned}$$

$$f(z) = \frac{z-3}{(z-1)(z-3)} + 3 \cdot \frac{1}{(z-1)(z-3)} = \frac{1}{z-1} + 3 \cdot \frac{1}{z-3} - \frac{1}{z-1} = \frac{1}{z-1} - 3 \sum_{n=0}^{\infty} \frac{(z-1)^{n+1}}{2^{n+1}}$$

$$\text{若 } |a| > 2, \oint_{|z|=2} \frac{e^{iz} \sin z}{z^2 - a^2} dz = \oint_{|z|=2} \frac{e^{iz} \sin z}{(z+a)(z-a)} dz$$



若 $|a| > 2$, 则 $|z|=2$ 内无极点. $\oint = 0$

若 $|a| < 2$, 则 两奇点 $z=a, z=-a$.

$$\begin{aligned} \oint_{|z|=2} &= \oint_{C_1} \frac{e^{iz} \sin z}{z-a} dz + \oint_{C_2} \frac{e^{iz} \sin z}{z-a} dz \\ &= 2\pi i \left(\frac{e^{iz} \sin z}{z-a} \right) \Big|_{z=a} + 2\pi i \left(\frac{e^{iz} \sin z}{z+a} \right) \Big|_{z=-a}. \end{aligned}$$

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{1 + a \sin \theta} d\theta \\ \text{令 } z = e^{i\theta}, \sin \theta = &\frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz} \\ dz = &\frac{1}{iz} dz \end{aligned}$$

$$\begin{aligned} &= \int_{|z|=1} \frac{1}{1 + a \cdot \frac{z^2 - 1}{2iz}} \cdot \frac{1}{iz} dz \\ &= \int_{|z|=1} \frac{\frac{2iz}{z^2 + a(z^2 - 1)}}{2iz + a(z^2 - 1)} dz \\ &= \int_{|z|=1} \frac{2}{2iz + a(z^2 - 1)} dz \end{aligned}$$

$$\begin{aligned} &a z^2 + 2iz - a \\ &= a(z^2 + \frac{2i}{a} z - 1) \\ &= a(z^2 + \frac{2i}{a} z - \frac{1}{a^2} + \frac{1}{a^2} - 1) \\ &= a(z + \frac{i}{a})^2 + \frac{1}{a} - a. \\ (z + \frac{i}{a})^2 &= 1 - \frac{1}{a^2} = \frac{a^2 - 1}{a^2} \\ z + \frac{i}{a} &= i \sqrt{\frac{a^2 - 1}{a^2}} \\ z &= \frac{\sqrt{a^2 - 1}}{a} i - \frac{1}{a} = \frac{i}{a} (\sqrt{a^2 - 1} - 1) \text{ 极点.} \end{aligned}$$

$$\begin{aligned} \frac{2}{2i + 2az} &= \frac{1}{az + i} \\ &= \frac{1}{i\sqrt{a^2 - 1} + i} = \frac{2\pi}{\sqrt{a^2 - 1}} \end{aligned}$$

$$\sin^2 \theta = \frac{(z^2 - 1)^2}{-4z^2}$$

$$z = e^{i\theta} \Rightarrow \sin \theta = \frac{z - \bar{z}}{2i} = \frac{z^2 - 1}{2iz}, d\theta = \frac{1}{iz} dz$$

$$\int_0^{\pi} \operatorname{Im} \sin^2 \theta dz = \frac{1}{4} \int_{|z|=1} \frac{1}{1 - (z^2 - 1)^2} \cdot \frac{1}{iz} dz$$

$$= \frac{1}{4i} \int_{|z|=1} \frac{1}{z - \frac{(z^2 - 1)^2}{4z}} dz = \frac{1}{4i} \int_{|z|=1} \frac{4z}{4z^2 - (z^2 - 1)^2} dz$$

$$= \frac{1}{4i} \int_{|z|=1} \frac{4z}{(z^2 + 2z + 1)(-z^2 + 2z + 1)} dz.$$

$$z = \frac{-2i\sqrt{8}}{2} = -i\sqrt{2} = \underline{-i\sqrt{2}}$$

$$z = \frac{-2\pm\sqrt{8}}{-2} = 1 \mp i\sqrt{2} = \underline{1-i\sqrt{2}} \quad \underline{\text{和} \quad \frac{1+i\sqrt{2}}{1-i\sqrt{2}}}$$

$$-i \cdot \frac{z}{8z - 4z(z^2 - 1)}$$

$$= -i \cdot \frac{z^2 - 1}{8(i\sqrt{2}) - 4(i\sqrt{2} - 1) \cdot 2(1 - i\sqrt{2})} = -i \cdot \frac{z^2 - 1}{8(2 - i\sqrt{2})}$$

$$-i \cdot \frac{1 - i\sqrt{2}}{8(1 - i\sqrt{2}) - 4(1 - i\sqrt{2}) \cdot 2(1 - i\sqrt{2})} = -i \cdot \frac{1 - i\sqrt{2}}{8(1 - i\sqrt{2})^2}$$

$$-i \cdot \frac{1 - i\sqrt{2}}{16(2 - i\sqrt{2})} \quad 2\sqrt{2} + 2 \rightarrow -i\sqrt{2}$$

$$= -i \cdot \frac{(i\sqrt{2} - 1)(i\sqrt{2} + 1)}{16(2 - i\sqrt{2})(i\sqrt{2} + 1)}$$

$$= -i \cdot \frac{1}{16\sqrt{2}}$$

$$= -\frac{i}{16\sqrt{2}}$$

$$-\frac{1}{16\sqrt{2}} \times 2\pi i = \frac{2}{16\sqrt{2}}\pi i = \frac{1}{8\sqrt{2}}\pi i$$

设 z_0 为函数 $f(z)$ 的m阶极点, 函数 $g(z)$ 在 z_0 处解析. 证明: $\text{Res}[g(z) \cdot \frac{f'(z)}{f(z)}, z_0] = -mg(z_0)$.

\Rightarrow 由 z_0 为 $f(z)$ 的m阶极点可设 $f(z) = \frac{\psi(z)}{(z-z_0)^m}$. 其中 $\psi(z)$ 在 z_0 处非零解析.

那 $f'(z) = \frac{\psi'(z) - m\psi(z)}{(z-z_0)^{m+1}}$, $z=z_0$ 为其 $m+1$ 阶极点. 那么会 $T(z) = g(z) \frac{f'(z)}{f(z)}$, 则 $z=z_0$ 为 $T(z)$ 的一阶极点.

$$\text{则 } \text{Res}[T(z), z_0] = \lim_{z \rightarrow z_0} (z-z_0) \cdot g(z) \frac{f'(z)}{f(z)} = g(z_0) \lim_{z \rightarrow z_0} \frac{(z-z_0) \cdot \frac{\psi'(z) - m\psi(z)}{(z-z_0)^{m+1}}}{\frac{\psi(z)}{(z-z_0)^m}} = g(z_0) \lim_{z \rightarrow z_0} \frac{(\psi'(z) - m\psi(z))}{\psi(z)} = -mg(z_0).$$

设 $z=z_0$ 是函数 $f(z)$ 的m阶极点. 证明: 对任意给定的正整数n, 有

$$\text{Res}[f(z), z_0] = \frac{1}{(m+n-1)!} \lim_{z \rightarrow z_0} \left\{ \frac{d^{m+n}}{dz^{m+n}} [(z-z_0)^{m+n} f(z)] \right\}.$$

\Rightarrow 由题可设 $f(z) = \frac{\psi(z)}{(z-z_0)^m}$, 其中 $\psi(z)$ 在 z_0 处解析且非零, 那么 $(z-z_0)^m f(z) = \psi(z)$.

$(z-z_0)^{m+n} f(z) = (z-z_0)^n \psi(z) \Rightarrow z=z_0$ 为其n阶零点.

$$\text{那么有 } \frac{d^{m+n}}{dz^{m+n}} [(z-z_0)^{m+n} f(z)] = \frac{d^{m+n}}{dz^{m+n}} [(z-z_0)^n \psi(z)] = \frac{d^{m-1}}{dz^{m-1}} \left[\frac{d^n}{dz^n} ((z-z_0)^n \psi(z)) \right]$$

$$\text{其中 } \frac{d^n}{dz^n} ((z-z_0)^n \psi(z)) = C_n (z-z_0)^n \psi^{(m)}(z) + \dots + C_n ((z-z_0)^n)^{(m)} \psi(z). \text{ 说明 } g(z) + n! \psi(z) \text{ 显然 } g(z_0) = 0$$

$$\text{则 } \frac{d^{m-1}}{dz^{m-1}} \left[\frac{d^n}{dz^n} ((z-z_0)^n \psi(z)) \right] = \frac{d^{m-1}}{dz^{m-1}} [g(z) + n! \psi(z)] = \frac{d^{m-1}}{dz^{m-1}} g(z) + \frac{d^{m-1}}{dz^{m-1}} [n! \psi(z)].$$

$$\Rightarrow \text{Res}[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \psi(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [n! \psi(z)]$$

$$\text{而 } \lim_{z \rightarrow z_0} \left[\frac{d^{m-1}}{dz^{m-1}} g(z) \right] = 0. \text{ 故 } \lim_{z \rightarrow z_0} \left[\frac{d^{m-1}}{dz^{m-1}} g(z) + \frac{d^{m-1}}{dz^{m-1}} (n! \psi(z)) \right] = \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [n! \psi(z)]$$

$$\text{那 } \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \psi(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^n \psi(z)] \\ = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m+n-1}}{dz^{m+n-1}} [(z-z_0)^{m+n} f(z)].$$

设C为正向圆周 $|z|=2$. 计算积分 (1) $I = \oint_C z \left[e^{\frac{1}{z-1}} + \frac{1}{(z-1)(z+1)} \right] dz$. (2) $I = \oint_C \frac{1}{z \sin(\frac{1}{z})} dz$

(1) 该题目要求在 $\oint_C z e^{\frac{1}{z-1}} dz$. 其只有一个无穷奇点 $z=1$, 且

$$I = \oint_C z e^{\frac{1}{z-1}} dz = 2\pi i \text{Res}[ze^{\frac{1}{z-1}}, 1] = -2\pi i \text{Res}[ze^{\frac{1}{z-1}}, \infty] = 2\pi i \text{Res}[\frac{1}{z} e^{\frac{1}{z-1}}, \infty]$$

$$(2) I = \oint_C \frac{1}{z \sin(\frac{1}{z})} dz = 2\pi i \text{Res}[\frac{1}{z \sin(\frac{1}{z})}, 0] = -2\pi i \text{Res}[\frac{1}{z \sin(\frac{1}{z})}, \infty] = 2\pi i \text{Res}[\frac{1}{z \sin(\frac{1}{z})}, 0] = \frac{2\pi i}{\sin 1}$$

留数定理yyds!

① 计算积分

$$(1) \int_{-\infty}^{+\infty} \frac{x \sin bx}{x^4 + a^4} dx \quad (a > 0, b > 0) \quad (\int_{-\infty}^{+\infty} \frac{\cos x}{x^4 + 1} dx).$$

$$\Rightarrow \text{原式} = \operatorname{Im} \int_{-\infty}^{+\infty} \frac{z}{z^4 + a^4} e^{ibz} dz, \text{ 由 } z^4 + a^4 = 0 \text{ 得 } 4 \text{ 个根 } z = ae^{\frac{(2k+1)\pi i}{4}}, \text{ 在上半平面的两个: } z = ae^{\frac{\pi i}{2}}, ae^{\frac{3\pi i}{2}}.$$
$$\int_{-\infty}^{+\infty} \frac{z}{z^4 + a^4} e^{ibz} dz = 2\pi i [\operatorname{Res}(\frac{ze^{ibz}}{z^4 + a^4}, ae^{\frac{\pi i}{2}}) + \operatorname{Res}(\frac{ze^{ibz}}{z^4 + a^4}, ae^{\frac{3\pi i}{2}})] = \left(\frac{e^{\frac{ab}{2}}}{4a^2 e^{\frac{\pi i}{2}}} + \frac{e^{\frac{3ab}{2}}}{4a^2 e^{\frac{3\pi i}{2}}} \right) 2\pi i$$
$$= \frac{\pi i}{2a^2} \left(\frac{e^{\frac{ab}{2}}}{1} + \frac{e^{\frac{3ab}{2}}}{-1} \right) = \frac{\pi}{2a^2} (e^{\frac{ab}{2}} - e^{\frac{3ab}{2}}) = \frac{\pi}{2a^2} e^{-\frac{ab}{2}} (e^{\frac{ab}{2}} - e^{-\frac{ab}{2}}) = \frac{\pi}{a^2} e^{-\frac{ab}{2}} \sin \frac{ab}{2}$$

$$(2) I = \int_0^{2\pi} \frac{\cos \theta}{1 - 2a \cos \theta + a^2} d\theta$$

$$\text{令 } z = e^{i\theta}, \text{ 则 } dz = \frac{1}{iz} dz, \cos \theta = \frac{z^2 + z^{-2}}{2}, \cos \theta = \frac{z + z^{-1}}{2}.$$

$$\Rightarrow I = \oint_{|z|=1} \frac{\frac{z^2 + z^{-2}}{2}}{1 - 2\frac{z + z^{-1}}{2} + a^2} \cdot \frac{1}{iz} dz = \frac{1}{i} \oint_{|z|=1} \frac{z^2 + 1}{z^3(z^2 - az^2 + a^2)} dz = \frac{1}{2i} \oint_{|z|=1} \frac{z^6 + 1}{z^3(a^2 - 1)(a - z)} dz. \Rightarrow \text{唯一极点} z = \frac{1}{a}, a. \Rightarrow I = 0$$

只有 $z = 0$, a 在积分区域内, 则有 $I = \pi \operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, 0] + \pi \operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, a]$

$$= -\pi \operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, \infty] - \pi \operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, a]$$

$$\text{其中 } -\operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, \infty] = \operatorname{Res}[\frac{1 + z^6}{z^3(a - z)(a^2 - 1)}, 0] = -\frac{1}{a}; -\operatorname{Res}[\frac{z^6 + 1}{z^3(a^2 - 1)(a - z)}, a] =$$

[题型：求函数 $f(t)$ 的傅立叶积分].

→ 用到的公式为：

$$\begin{aligned} f(t) &= \frac{1}{\pi} \int_0^{+\infty} A(w) \cos wt dw + \frac{1}{\pi} \int_0^{+\infty} B(w) \sin wt dw \\ &= \frac{1}{\pi} \int_0^{+\infty} \left[\int_{-\infty}^{+\infty} f(\tau) \cos w\tau d\tau \right] \cos wt dw + \frac{1}{\pi} \int_0^{+\infty} \left[\int_{-\infty}^{+\infty} f(\tau) \sin w\tau d\tau \right] \sin wt dw. \end{aligned}$$

先判断 $f(t)$ 的奇偶性，再计算出 $A(w)$ 和 $B(w)$ 表达式，将 $f(t)$ 表示上式第一项即可。

○ 求下列函数的傅立叶积分.

$$(1) f(t) = \begin{cases} \sin t, & |t| \leq \pi \\ 0, & |t| > \pi. \end{cases}$$

[题型：证明积分式]

① 运用函数的傅立叶积分形式进行证明.

$$\begin{aligned} 2 \int_0^{\pi} \sin t \sin wt dt &= \int_0^{\pi} (\cos(1+wt)t - \cos(1-w)t) dt \\ &= \int \cos(1-w)t dt - \int \cos(1+w)t dt \\ &\quad \text{(-w = -ss)} \\ -\frac{1}{2} (\cos A - \cos B) &= \sin \frac{A+B}{2} \sin \frac{A-B}{2} \\ &= \frac{1}{\pi w} \sin((tw)\pi) \left[\int_0^{\pi} -\frac{1}{\pi w} \sin((tw)t) dt \right] \\ &= \frac{\sin((tw)\pi)}{\pi w} - \frac{\sin((tw)\pi)}{\pi w} \\ \frac{A+B}{2} &= t \quad A+B = 2wt \\ \frac{A-B}{2} &= wt \quad A-B = 2w^2t \\ 2A &= 2t + 2w^2t \\ A &= (wt+1)t \\ B &= (1-w)t \\ &= \frac{(tw)\sin((tw)\pi) - (tw)\sin((tw)\pi)}{1-w^2} \\ &= \frac{[\sin((tw)\pi) - \sin((tw)\pi)] + w[\sin((tw)\pi) + \sin((tw)\pi)]}{1-w^2} \\ &= \end{aligned}$$

[题型：求函数 $f(t)$ 的傅立叶变换]

$$\frac{2w \cos \pi - 2 \sin \pi \cos \pi}{1-w^2}.$$

$$= \left(\frac{2 \sin \pi}{\pi w} \right)$$

① 设 $F(w) = \int f(it) dt$. 证明 1) $\bar{F}(-w) = \int f(-t) dt$; 2) $|F(w)| = |\bar{f}(-w)|$.

1) \Rightarrow 由 $f(t)$ 为实值函数. 由 $F(w) = \int_{-\infty}^{+\infty} f(t) e^{-int} dt$ 有 $\bar{F}(w) = \int_{-\infty}^{+\infty} f(t) \overline{e^{int}} dt = \int_{-\infty}^{+\infty} f(t) e^{int} dt = \bar{f}(-w)$.
即 $|F(w)| = |\bar{f}(-w)| = |\bar{F}(-w)|$.

② 设 $F(w) = \int f(it) dt$. 记 $f(it)$ 为复值函数的充分必要条件为 $\bar{F}(w) = F(-w)$.
关键是在傅里叶变换中, 其积与积分号可以交换. 那么积分再取共轭 \Leftrightarrow 取共轭再积分.

③ 求函数 $f(w) = e^{-wt}$ 的傅里叶正弦和余弦变换, 其中 $w > 0$.

I 题型：傅里叶变换性质的应用

○ 求各函数的傅里叶变换。

$$(1) f(t) = \delta(t-1)(t-2)^2 \sin t. \quad (\delta(t) \text{ 函数})$$

\Rightarrow 利用 $\delta(t)$ 函数独特的筛选性质有 $f(t) = \delta(t-1)(t-1)^2 \sin t = \delta(t-1) \sin t$.

$$\text{且 } F[f(t)] = F[\delta(t-1) \sin t] = \sin 1 \cdot e^{-jw} F[\delta(t)] = \sin 1 \cdot e^{-jw}.$$

$$(2) f(t) = u(t)e^{-t} \cos t. \quad (u(t) \text{ 函数})$$

$$\Rightarrow F[f(t)] = \int_{-\infty}^{+\infty} f(t) e^{-jw t} dt = \int_0^{+\infty} e^{-t} \cos t \cdot e^{-jw t} dt = \int_0^{+\infty} e^{-(jw+1)t} \cos t dt = \frac{jw+1}{(jw+1)^2 + 1}$$

$$(3) f(t) = \frac{\sin t}{t}.$$

$$(4) f(t) = \frac{1}{a^2+t^2}.$$

$$\Rightarrow \text{由 } F[e^{-at}] = \frac{2a}{a^2+w^2} \Leftrightarrow \frac{1}{2a} F[e^{-at}] = \frac{1}{a^2+w^2} = f(w) = F[\frac{1}{a} e^{-|at|}].$$

$$\text{所以 } F[f(t)] = 2\pi \cdot \frac{1}{2a} e^{-|aw|} = \frac{\pi}{a} e^{-|aw|}.$$

$$(5) f(t) = t e^{-it} \sin t.$$

$$\Rightarrow F[f(t)] = F[t(\cos t \sin t)] - i F[\sin^2 t].$$

$$\text{其中 } F(\cos t \sin t) = \frac{\pi i}{2} (\delta(w+2) - \delta(w-2)). \quad F[\sin^2 t] = \pi f(t) - \frac{\pi}{2} (\delta(w+2) + \delta(w-2))$$

○ 求各函数的傅里叶逆变换 (注意把逆变换和对称性区分开)

$$(1) F(w) = \cos 2w.$$

令 $f(t) = \cos 2w$, 则有 $F[f(t)] = \pi(\delta(w+2) + \delta(w-2))$. 那么由对称性有:

$$F[\pi(\delta(t+2) + \delta(t-2))] = 2\pi f(-w) = 2\pi \cos 2w. \quad \text{所以 } F^{-1}(\cos 2w) = \frac{1}{2} [\delta(t+2) - \delta(t-2)].$$

A~ 和求逆变换的方法 (像函数看起来很像原函数) 直接对其进行傅里叶变换, 再利用对称性.

$$(2) F(w) = -2\pi \delta''(w).$$

$$\text{① } \Rightarrow F(w) = j \cdot (2\pi j \delta'(w))'. \quad \text{设 } F[g(t)] = 2\pi j \delta'(w), \text{ 且 } F^{-1}[F(w)] = t g(t)$$

$$\text{对 } F[g(t)] = j(2\pi j \delta'(w)) = j F[1], \text{ 且 } g(t) = t. \quad \text{那么 } F^{-1}[F(w)] = t^2.$$

$$\text{② } F[-2\pi \delta''(t)] = -2\pi F[\delta''(t)] = -2\pi(jw)^2 F[\delta(t)] = 2\pi w^2. \quad \text{所以 } F[2\pi t^2] = 2\pi F[t^2] = 2\pi [-2\pi \delta''(-w)]$$

$$\text{即 } F[t^2] = -2\pi \delta''(w). \Rightarrow$$

$$(3) F(w) = \frac{1}{2+4w^2}. \quad (\text{单边指数型})$$

$$(4) F(w) = \frac{-2iw}{(w^{\frac{3}{4}})(w^{\frac{5}{4}})}. \quad (\text{双边指数型})$$

$$\Rightarrow F(w) = \frac{-2iw}{(w^{\frac{3}{4}})(w^{\frac{5}{4}})} = \frac{1}{3} \left(\frac{2iw}{w^{\frac{5}{4}}} - \frac{2iw}{w^{\frac{3}{4}}} \right), \text{ 其中 } F^{-1}\left[\frac{2iw}{w^{\frac{5}{4}}}\right] = e^{2t} u(t) - e^{-2t} u(t). \quad F^{-1}\left[\frac{-2iw}{w^{\frac{3}{4}}}\right] = e^{4t} u(t) - e^{4t} u(-t)$$

$$\text{所以 } F^{-1}[F(w)] = \frac{1}{3} [(e^{2t} - e^t) u(t) + (e^{-t} - e^{-2t}) u(t)]$$

$$\text{即 } F^{-1}[F(w)] = \begin{cases} \frac{1}{3}(e^{2t} - e^t), & t < 0 \\ 0, & t = 0 \\ \frac{1}{3}(e^{-t} - e^{-2t}), & t > 0. \end{cases}$$

$$15) F(w) = \frac{2\sin w}{w}.$$

$$\Rightarrow \text{令 } g(t) = u(t+1) - u(t-1). \text{ 则 } F[g(t)] = \int_{-\infty}^{+\infty} g(t) e^{iwt} dt = \int_{-1}^1 e^{iwt} dt = \frac{e^{iwt} - e^{-iwt}}{iw} = \frac{2\sin w}{w}$$

$$\text{且 } F^{-1}[F(w)] = [u(t+1) - u(t-1)]. \Rightarrow F^{-1}[F(w)] = \begin{cases} 1, & |t| \leq 1 \\ \frac{1}{2}, & |t|=1 \Rightarrow \text{注意间断点} \\ 0, & |t| > 1 \end{cases}$$

[题型：卷积及卷积定理].

0. 知道 $f(t) = \begin{cases} e^{-t}, t \geq 0 \\ 0, t < 0 \end{cases}$, $g(t) = \begin{cases} \sin t, 0 \leq t \leq \frac{\pi}{2} \\ 0, \text{otherwise.} \end{cases}$, 计算 $f(t) * g(t)$.

\Rightarrow 直接由卷积定义：

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(s) g(t-s) ds = \begin{cases} 0, & t \leq 0 \\ \int_0^t f(s) g(t-s) ds, & 0 \leq t \leq \frac{\pi}{2} \\ \int_0^{\frac{\pi}{2}} f(s) g(t-s) ds, & t > \frac{\pi}{2}. \end{cases}$$

$$\text{其中 } \int_0^t f(s) g(t-s) ds = \int_0^t e^{-s} \sin(t-s) ds = \frac{1}{2} (\sin t - \cos t + e^{-t}).$$

$$\int_0^{\frac{\pi}{2}} f(s) g(t-s) ds = \int_0^{\frac{\pi}{2}} e^{-s} \sin(t-s) ds = \frac{1}{2} e^{-t} (1 + e^{\frac{\pi}{2}}).$$

0 计算 $f(t) = \frac{t^2}{(1+t^2)^2}$ 的傅里叶变换. [适用留数定理计算傅里叶变换]

$$\Rightarrow F[f(t)] = \int_{-\infty}^{+\infty} \frac{t^2}{(1+t^2)^2} e^{-iwt} dt$$

$$\text{①当 } w > 0 \text{ 时. 上式} = \int_{+\infty}^{-\infty} \frac{(-t)^2}{(1+(-t)^2)^2} e^{-i w (-t)} dt = \int_{-\infty}^{+\infty} \frac{t^2}{(1+t^2)^2} e^{iwt} dt = 2\pi i \text{Res} \left[\frac{t^2}{(1+t^2)^2}, i \right]$$

$$= 2\pi i \lim_{t \rightarrow i} \left[\frac{t^2}{(t+i)^2} e^{iwt} \right]' = -\frac{\pi}{2} e^{-w} (w+i) = \frac{\pi}{2} e^{-w} (hw)$$

△仔细计算!!!

$$\text{②当 } w < 0 \text{ 时. 上式} = \int_{-\infty}^{+\infty} \frac{-t^2}{(1+t^2)^2} e^{iwt} dt = 2\pi i \text{Res} \left[\frac{-t^2}{(1+t^2)^2} e^{iwt}, i \right] = \frac{\pi}{2} e^w (hw)$$

$$\text{综上: } F[f(t)] = \frac{\pi}{2} e^{-|w|} (|hw|).$$

0 求 $f(t) = e^{-bt} u(t) \sin bt$ 的傅里叶变换 [适用卷积定理求傅里叶变换].

$$F[e^{-bt} u(t) \sin bt].$$

$$\begin{aligned} &\stackrel{(1)}{=} F[e^{-bt} u(t)] * F[\sin bt] \\ &= \int_0^{+\infty} e^{-bt} \cdot e^{-iwt} dt \quad \downarrow \\ &\quad \int_0^{+\infty} e^{-(b+iw)t} dt \\ &\quad - \frac{1}{b+iw} e^{-(b+iw)t} \Big|_0^{+\infty} \\ &\quad \xrightarrow{\frac{1}{b+iw}} \\ &= \left(\frac{1}{b+iw} \right) * \left[i\pi (\delta(w+b) - \delta(w-b)) \right]. \quad \begin{matrix} -s \pi w^{-b} \\ s \pi w^{+b} \\ s \pi (w^0) \end{matrix} \\ &= i\pi \int_{-\infty}^{+\infty} \frac{1}{b+iw+s} \cdot (\delta(w-s+b) - \delta(w-s-b)) ds \\ &= i\pi \left[\frac{-1}{b+iw+b} + \frac{1}{b+iw-b} \right] = i\pi \left[\frac{1}{b+iw+b} - \frac{1}{b+iw-b} \right] \\ &= i\pi \cdot \frac{2ib}{(b+iw)^2 + b^2} = \frac{-2\pi b}{(b+iw)^2 + b^2}. \end{aligned}$$

[題型=解微分方程]

$$0 x'(t) - 9 \int_{-\infty}^t x(s) ds = e^{-9t}, \text{ 其中 } \int_{-\infty}^{+\infty} x(t) dt = 0. \quad \Delta \text{ 指数信号变换公式的运用(1)}$$

$$\Rightarrow \mathcal{F}[x'(t)] - 9 \mathcal{F}\left[\int_{-\infty}^t x(s) ds\right] = \mathcal{F}[e^{-9t}] = i\omega X(\omega) - \frac{9}{i\omega} X(\omega) = \frac{2}{(\omega + 9)^2}$$

$$\text{且 } X(\omega) = \frac{-2i\omega}{(\omega^2 + 9)(\omega^2 + 1)} = \frac{i\omega}{4(\omega^2 + 9)} - \frac{i\omega}{4(\omega^2 + 1)}, \text{ 由于 } e^{-\alpha t} u(t) - e^{-\beta t} u(t-t) = \frac{-i\omega}{\beta^2 - \alpha^2} \text{ 且}$$

$$X(\omega) = \frac{1}{8} [(e^{3t} u(t-t) - e^{-3t} u(t)) - (e^{t} u(t-t) - e^{-t} u(t-t))]$$

$$= \begin{cases} \frac{1}{8}(e^{-t} - e^{-3t}), & t > 0 \\ 0, & t = 0 \text{ (注意间断点).} \\ \frac{1}{8}(e^{3t} - e^t), & t < 0. \end{cases}$$

$$0 x''(t) - x(t) = \delta(t).$$

$$\Rightarrow \mathcal{F}[x''(t)] - \mathcal{F}[x(t)] = \mathcal{F}[\delta(t)] = (i\omega)^2 X(\omega) - X(\omega) = 1 \Rightarrow X(\omega) = -\frac{1}{\omega^2}$$

$$\text{且 } X(\omega) = \frac{1}{2}(e^t u(t-t) - e^{-t} u(t)) \Rightarrow X(\omega) = \begin{cases} -\frac{1}{2}e^{-t}, & t \geq 0 \\ \frac{1}{2}e^t, & t < 0 \end{cases} \quad (\text{当 } t=0 \text{ 时 } X(\omega) = \frac{1}{2}).$$

$$0 \int_{-\infty}^{+\infty} \frac{y(\xi)}{(t-\xi)^2 + a^2} d\xi = \frac{1}{t^2 + b^2}, \quad 0 < a < b. \quad \Delta \text{ 卷积公式的运用(1)}$$

$$0 \int_{-\infty}^{+\infty} e^{-|t-\xi|} x(\xi) d\xi = f(t).$$

$$0 \int_{-\infty}^{+\infty} y(\xi) \sin \omega \xi d\xi = \begin{cases} \sin \omega, & 0 \leq \omega \leq \pi \\ 0, & \omega > \pi. \end{cases} \quad \Delta \text{ 正余弦傅里叶(逆)变换公式的运用.}$$

$$0 \int_0^{+\infty} x(\omega) \sin \omega t d\omega = e^{-t} (t > 0).$$

Δ 常见傅里叶变换对(逐一证明)

①

$$O e^{-at} u(t) \sim \frac{1}{a+iw}; \quad F[e^{at} u(t)] \sim \frac{1}{a-iw}. \quad (\text{单边指数})$$

$$\text{证明: } F[e^{-at} u(t)] = \int_0^{+\infty} e^{-(a+iw)t} dt = -\frac{1}{a+iw} e^{-(a+iw)t} \Big|_0^{+\infty} = \frac{1}{a+iw}.$$

$$F[e^{at} u(t)] = \int_{-\infty}^0 e^{(a-iw)t} dt = \frac{1}{a-iw} e^{(a-iw)t} \Big|_{-\infty}^0 = \frac{1}{a-iw}.$$

$$O e^{-at} u(t) + e^{at} u(t) \sim e^{-at} \sim \frac{2a}{a^2+w^2}; \quad e^{at} u(t) - e^{-at} u(-t) \sim \frac{-2iw}{a^2+w^2}. \quad (\text{双边指数})$$

$$\text{证明: } F[e^{-at} u(t) + e^{at} u(t)] = \frac{1}{a+iw} + \frac{1}{a-iw} = \frac{2a}{a^2+w^2}. \quad (\text{实际上 } e^{-at} u(t) + e^{at} u(t) = e^{-at})$$

$$F[e^{-at} u(t) - e^{at} u(-t)] = \frac{1}{a+iw} - \frac{1}{a-iw} = \frac{-2iw}{a^2+w^2}.$$

拉普拉斯变换

△基本性质: $\mathcal{L}[f(t)] = F(p)$.

(1) 线性性质: $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$.

(2) 相似性: $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{p}{a}\right)$. $\mathcal{L}^{-1}[F(ap)] = \frac{1}{a} f\left(\frac{t}{a}\right)$.

(3) 延迟性: (时域) $\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-pt_0} F(p)$. (注意是 $f(t-t_0)u(t-t_0)$ 而不是 $f(t-t_0)$). \Rightarrow 时移

(4) 平移性: (频域). $F(p-p_0) = \mathcal{L}[e^{p_0 t} f(t)]$. \Rightarrow 频移

(5) 微分性质: (时域微分) $= \mathcal{L}[f'(t)] = pF(p) \Rightarrow F(p) = \left(\frac{1}{p} \mathcal{L}[f'(t)] = \mathcal{L}[f(t)] \right)$
 (频域微分) $= F'(p) = -\mathcal{L}[tf'(t)] \Rightarrow f'(t) = \left(\frac{\mathcal{L}^{-1}[F'(p)]}{-t} = \mathcal{L}^{-1}[F'(p)] \right)$

(6) 积分性质: (时域积分) $= \mathcal{L}\left[\int_0^t f(s)ds\right] = \frac{F(p)}{p}$ $\Rightarrow p\mathcal{L}\left[\int_0^t f(s)ds\right] = \mathcal{L}[f(t)]$
 (频域积分) $= \int_p^{+\infty} F(s)ds = \mathcal{L}\left[\frac{f(t)}{t}\right] \Rightarrow \int_p^{+\infty} \mathcal{L}[f(t)]dt = \mathcal{L}\left[\frac{f(t)}{t}\right]$.

7) 周期性: $\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-pt}dt}{1-e^{-pT}}$

[题型: 求拉普拉斯变换]

O (1) $\sin(t-2)$, (2) $\sin t \cdot u(t-2)$. (3) $\sin(t-2) \cdot u(t-2)$. (4) $e^{at} \cdot u(t-2)$.

$$O(1) \frac{1-\cos t}{t^2}$$

$$O(2) \int_0^t te^{-st} \sin nt dt$$

$$O(3) |\cos t|.$$

[题型: 求拉普拉斯逆变换].

$$O(1) \frac{e^{-sp+1}}{p}$$

$$(2) \frac{4p}{(p^2+4)^2}$$

$$(3) \frac{2p+5}{p^2+4p+13}$$

$$(4) \ln \frac{p^2+1}{p}$$

[题型：用拉普拉斯变换求微积分方程]

【题型：用拉普拉斯变换求广义积分】

数理方程

[题型：建立定解问题]

① 长为 l 的均匀杆，侧面绝热。一端温度为 T_0 ，另一端有恒定热流 q 进入（单位时间通过单位面积流入的热量）。杆初始温度分布为 $\frac{1}{2}x(l-x)$ 。写出定解问题。

② 一均匀杆原长为 l ，一端固定，另一端沿杆轴线方向被拉长 ϵ 而静止。突然放手任其振动。建立振动方程与定解条件。

[题型: 二维拉普拉斯方程]

① 求解矩形域 $0 \leq x \leq a, 0 \leq y \leq b$ 内的拉普拉斯方程:

$$U_{xx} + U_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b.$$

② 满足边界条件: $\begin{cases} U|_{x=0} = Ay(b-y), \\ U|_{y=0} = 0, \quad U|_{y=b} = 0, \end{cases}$

求解: 用分离变量法, 设 $U(x,y) = X(x)Y(y)$. 由 $X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$

得到 $\begin{cases} X''(x) - \lambda X(x) = 0 \\ Y''(y) + \lambda Y(y) = 0 \end{cases}$ 进行求解:

由 $U|_{y=0} = 0, U|_{y=b} = 0$ 得 $Y(0) = Y(b) = 0$ 得固有问题是 $\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(b) = 0. \end{cases}$

解得 $\lambda = (\frac{n\pi}{b})^2, n=1, 2, \dots$, $Y(y) = C_2 \sin \frac{n\pi}{b} y$. ($U|_{y=0} = 0, U|_{y=b} = 0$ 第一类边界条件)

再由 $X''(x) - \lambda X(x) = 0$ 得 $X(x) = C_1 e^{\frac{n\pi}{b} x} + C_2 e^{-\frac{n\pi}{b} x}$

$$\Rightarrow U(x,y) = \sum_{n=1}^{\infty} (a_n e^{\frac{n\pi}{b} x} + b_n e^{-\frac{n\pi}{b} x}) \sin \frac{n\pi}{b} y = \sum_{n=1}^{\infty} (c_n \cosh \frac{n\pi}{b} x + d_n \sinh \frac{n\pi}{b} x) \sin \frac{n\pi}{b} y.$$

由边界条件 $U|_{x=0} = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{b} y = Ay(b-y)$.

$$\begin{cases} U|_{x=a} = \sum_{n=1}^{\infty} (c_n \cosh \frac{n\pi}{b} a + d_n \sinh \frac{n\pi}{b} a) \sin \frac{n\pi}{b} y = 0 \end{cases}$$

即得方程组 $\begin{cases} d_n = \frac{2}{b} \int_0^b Ay(b-y) \sin \frac{n\pi}{b} y dy \quad (\text{正交}) \\ c_n \cosh \frac{n\pi}{b} a + d_n \sinh \frac{n\pi}{b} a = 0. \end{cases}$

$$\text{其中 } d_n = -\frac{2}{b} \int_0^b Ay(b-y) \sin \frac{n\pi}{b} y dy = 2A \int_0^b y \sin \frac{2\pi}{b} y dy - \frac{2A}{b} \int_0^b y^2 \sin \frac{n\pi}{b} y dy = \frac{4Ab^2 [(-1)^n - 1]}{n^2 \pi^3}.$$

$$c_n = -\frac{\sinh \frac{n\pi}{b} a}{\cosh \frac{n\pi}{b} a}, \quad d_n = \frac{4Ab^2 [(-1)^n - 1]}{n^2 \pi^3} \sinh \frac{n\pi}{b} a.$$

$$\text{通解: } U(x,y) = \sum_{n=1}^{\infty} (c_n \cosh \frac{n\pi}{b} x + d_n \sinh \frac{n\pi}{b} x) \sin \frac{n\pi}{b} y.$$

转换后计算简便

不是所有时候都适合转换

1. 分组计算

(2) 满足边界条件 $\begin{cases} U|_{x=0} = 0, \quad U|_{x=a} = Ay, \quad 0 < y < b, \\ U|_{y=0} = 0, \quad U|_{y=b} = 0, \quad 0 < x < a. \end{cases}$

设 $U(x,y) = X(x)Y(y)$, $\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$

得固有值问题 $\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y'(0) = Y'(b) = 0 \end{cases}$ 易解得 $Y(y) = C_n \cos \frac{n\pi}{b} y, \lambda = \left(\frac{n\pi}{b}\right)^2, n=0,1,2,\dots$

可解得: $X(x) = \begin{cases} A_0 x + A_1, & n=0, \\ C_1 e^{-\frac{n\pi}{b}x} + C_2 e^{\frac{n\pi}{b}x} = C_1 \cosh \frac{n\pi}{b}x + C_2 \sinh \frac{n\pi}{b}x, & n \neq 0. \end{cases}$

可得解具有形式: $U(x,t) = a_0 + a_1 x + \sum_{n=1}^{+\infty} [a_n \cosh \frac{n\pi}{b}x + b_n \sinh \frac{n\pi}{b}x] \cos \frac{n\pi}{b}y.$

$\Rightarrow U|_{x=0} = a_0 + \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi}{b}y = 0. \Rightarrow a_0 = a_n = 0.$

$U|_{x=b} = a_1 a + \sum_{n=1}^{+\infty} b_n \sinh \frac{n\pi a}{b} \cos \frac{n\pi}{b}y = Ay. \text{ 从而有:}$

$$b_n = \frac{1}{\sinh \frac{n\pi a}{b}} \cdot \frac{2}{b} \int_0^b A y \cos \frac{n\pi}{b} y dy = \frac{2(-1)^{n-1} J A b}{n^2 \pi^2 \sinh \frac{n\pi a}{b}}$$

$\Delta a_1 a = \frac{1}{2} \left(\frac{2}{b} \int_0^b A y dy \right) = \frac{Ab}{2} \Rightarrow a_1 = \frac{Ab}{2a}$

从而解为: $U(x,t) = \frac{Ab}{2a} + \frac{2Ab}{\pi^2} \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2 \sinh \frac{n\pi a}{b}} \sinh \frac{n\pi}{b} x \cos \frac{n\pi}{b} y.$

$$U_{xx} + U_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi.$$

$$(3). \begin{cases} U_x(0, y) = y - \frac{\pi}{2}, \\ U_y(x, 0) = 0. \end{cases} \quad \begin{cases} U_x(\pi, y) = 0, \\ U_y(x, \pi) = 0. \end{cases} \quad 0 \leq y \leq \pi$$

$$\Rightarrow \Delta U(x, y) = X(x)Y(y) \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda.$$

可解得 $Y(y) = \cos ny, \lambda = n^2, n = \underline{0, 1, 2, \dots}$

$$\text{由 } X''(x) - \lambda X(x) = 0 \text{ 可解得: } \begin{cases} X_0(x) = a_0 + a_1 x, n=0 \\ X_n(x) = C_n e^{nx} + D_n e^{-nx}, n=1, 2, \dots \end{cases}$$

设 $U(x, y) = a_0 + a_1 x + \sum_{n=1}^{+\infty} (C_n e^{nx} + D_n e^{-nx}) \cos ny$. 代入边界条件:

$$\begin{cases} U_x(0, y) = y - \frac{\pi}{2} = a_1 + \sum_{n=1}^{+\infty} n(C_n - D_n) \cos ny \end{cases}$$

$$U_x(\pi, y) = 0 = a_1 + \sum_{n=1}^{+\infty} n(C_n e^{n\pi} - D_n e^{-n\pi}) \cos ny.$$

$$\text{易得: } a_1 = 0, \quad \begin{cases} C_n e^{nx} - D_n e^{-nx} = 0. \end{cases}$$

$$n(C_n - D_n) = \frac{2}{\pi} \int_0^\pi (y - \frac{\pi}{2}) \cos ny dy = \frac{2}{n^2 \pi} [(-1)^n - 1].$$

$$\Rightarrow C_n = \frac{e^{-n\pi} [1 - (-1)^n]}{n^2 \pi \sinh n\pi}, \quad D_n = \frac{e^{n\pi} [1 - (-1)^n]}{n^2 \pi \sinh n\pi}.$$

解为 $U(x, y) = a_0 + \sum_{n=1}^{+\infty} \frac{2[1 - (-1)^n]}{n^2 \pi} \frac{\cosh n(x - \pi)}{\sinh n\pi} \cos ny$. a_0 为任意常数.

I 题型：求解定解问题（非齐次）】. 特解法和分离变量法.

$$(1) \begin{cases} u_{tt} - u_{xx} = \sin 2x, & 0 < x < \pi, t > 0, \\ u|_{x=0} = 0, \quad u|_{x=\pi} = 0, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0. \end{cases}$$

$$(2) \begin{cases} \underline{u_t} - a^2 u_{xx} = Ax, & 0 < x < l, t > 0, \\ u|_{x=0} = 0, \quad u|_{x=l} = 0 \\ u|_{t=0} = 0. \end{cases}$$

[固有函数法]

$$0 \begin{cases} u_{tt} = a^2 u_{xx} + A, & 0 < x < l, t > 0. \\ u|_{x=0} = 0, \quad u_x|_{x=l} = 0, & t > 0. \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0, & 0 < x < l. \end{cases}$$

二阶 Laplace 方程

→ 满足该齐次边界条件的固有函数为 $\sin \frac{(2n-1)\pi}{2l} x$, $n=1, 2, \dots$. 故设解具有以下形式

$$u(x,t) = \sum_{n=1}^{+\infty} T_n(t) \sin \frac{(2n-1)\pi}{2l} x.$$

将自由项按固有函数展开: $A = \sum_{n=1}^{+\infty} A_n \sin \frac{(2n-1)\pi}{2l} x$. 其中 A_n 为:

$$A_n = \frac{2}{l} \int_0^l A \sin \frac{(2n-1)\pi}{2l} x dx = \frac{2A}{l} \cdot \frac{-2l}{(2n-1)\pi} \cos \frac{(2n-1)\pi}{2l} x \Big|_0^l = \frac{4A}{(2n-1)\pi}$$

把解和自由项代入原方程:

$$\sum_{n=1}^{+\infty} T_n''(t) \sin \frac{(2n-1)\pi}{2l} x = -\frac{(2n-1)^2 \pi^2}{4l^2} a^2 \sum_{n=1}^{+\infty} T_n(t) \sin \frac{(2n-1)\pi}{2l} x + \sum_{n=1}^{+\infty} \frac{4A}{(2n-1)\pi} \sin \frac{(2n-1)\pi}{2l} x$$

$$\left\{ \begin{array}{l} T_n''(t) = -\frac{(2n-1)^2 \pi^2 a^2}{4l^2} T_n(t) + \frac{4A}{(2n-1)\pi} \\ T_n(0) = 0, \quad T_n'(0) = 0. \end{array} \right.$$

得到关于 $T_n(t)$ 的常系数微分方程.

$$\text{解此方程得: } T_n(t) = C_1 \cos \frac{(2n-1)\pi a}{2l} t + C_2 \sin \frac{(2n-1)\pi a}{2l} t + \frac{16A l^2}{(2n-1)^3 \pi^3 a^3}$$

$$\text{由初始条件得: } C_1 = -\frac{16A l^2}{(2n-1)^3 \pi^3 a^3}, \quad C_2 = 0. \quad \text{即} \quad \hookrightarrow \text{常数法解法}$$

$$T_n(t) = -\frac{16A l^2}{(2n-1)^3 \pi^3 a^3} \cos \frac{(2n-1)\pi a}{2l} t + \frac{16A l^2}{(2n-1)^3 \pi^3 a^3}.$$

$$\text{则解为: } u(x,t) = \sum_{n=1}^{+\infty} \frac{16A l^2}{(2n-1)^3 \pi^3 a^3} \left(1 - \cos \frac{(2n-1)\pi a}{2l} t \right) \sin \frac{(2n-1)\pi}{2l} x.$$

△ 此题只能用固有函数法, 用待解法错误.

※

$$0 \begin{cases} u_t = a^2 u_{xx} - bu, & 0 < x < \pi, t > 0 \\ u_x|_{x=0} = 0, \quad u_x|_{x=\pi} = 0, & t > 0 \\ u|_{t=0} = x, & 0 < x < \pi. \end{cases}$$

~物理热传导方程

该定解问题对应的固有值问题为: $\begin{cases} X'(x) + \lambda X(x) = 0 \\ X(0) = X'(\pi) = 0. \end{cases}$ 解得 $\lambda = n^2$, $X_n = \cos nx, n=0, 1, 2, \dots$

则该定解问题的解的形式为: $u(x,t) = \sum_{n=0}^{+\infty} T_n(t) \cos nx.$

该问题没有自由项, 将形式代入方程有:

$$\sum_{n=0}^{+\infty} T_n'(t) \cos nx + a^2 n^2 \sum_{n=0}^{+\infty} T_n(t) \cos nx + b \sum_{n=0}^{+\infty} T_n(t) \cos nx = 0$$

可得关于 $T_n(t)$ 的常系数方程:

$$\begin{cases} T_n'(t) + n^2 a^2 T_n(t) + b T_n(t) = 0 \\ T_n(0) = x, \text{ } \cancel{X} \end{cases}$$

解此方程得: $T_n(t) = C_n e^{-(n^2 a^2 + b)t}$, 那 $u(x,t) = \sum_{n=0}^{+\infty} C_n e^{-(n^2 a^2 + b)t} \cos nx.$

由初始条件 $u|_{t=0} = x$ 得: $u|_{t=0} = \sum_{n=0}^{+\infty} C_n \cos nx = x.$

$$\text{那么 } C_n = \frac{2}{\pi} \int_0^\pi x \cos nx dx = \frac{2}{n\pi} \int_0^\pi x d \sin nx$$

$$= \frac{2}{n\pi} x \sin nx \Big|_0^\pi - \frac{2}{n\pi} \int_0^\pi \sin nx dx = \frac{2}{n^2 \pi} \cos nx \Big|_0^\pi = \frac{2[(-1)^n - 1]}{n^2 \pi}$$

$$C_0 = \frac{1}{\pi} \int_0^\pi x dx = \frac{1}{2\pi} x^2 \Big|_0^\pi = \frac{\pi}{2} \quad (\cancel{\frac{1}{\pi}}) \text{ 利用偶函数对称性}$$

但要注意 $n=0$ 的情况

$$\text{即有 } C_n = \begin{cases} \frac{\pi}{2}, & n=0 \\ \frac{4}{n^2 \pi}, & n=2k-1 \\ 0, & n=2k. \end{cases} \Rightarrow \frac{2[(-1)^n - 1]}{n^2 \pi}.$$

$$\text{综上: } u(x,t) = \frac{\pi}{2} e^{-bt} - \frac{4}{\pi} e^{-bt} \sum_{k=1}^{+\infty} \frac{1}{(2k-1)^2} e^{-(2k-1)^2 a^2 t} \cos((2k-1)x)$$

$$(u(x,t) = \frac{\pi}{2} e^{-bt} - \frac{2}{\pi} e^{-bt} \sum_{n=1}^{+\infty} \frac{[(-1)^n - 1]}{n^2} e^{-n^2 a^2 t} \cos nx)$$

[题型：非齐次边界条件的处理]

○ 求解定解问题 $\begin{cases} u_{tt} = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u|_{x=0} = kt, \quad u|_{x=l} = bt, & t > 0 \end{cases}$

$$u|_{t=0} = E(t - \frac{x}{l}), \quad u|_{t=0} = k(t - \frac{x}{l}), \quad 0 < x < l.$$

先将非齐次边界条件化为齐次，设 $W(x,t) = A(x)t + B(x)$

$$\text{由 } W|_{x=0} = B(0) = kt, \quad W|_{x=l} = A(l)t + B(l) = bt \text{ 得 } W(x,t) = \frac{k-b}{l}tx + kt$$

由 $U(x,t) = V(x,t) + W(x,t)$ ，其中 $V(x,t)$ 满足：

$$\begin{cases} v_{tt} = a^2 v_{xx}, & 0 < x < l, t > 0 \\ v|_{x=0} = 0, \quad v|_{x=l} = 0, & t > 0 \\ v|_{t=0} = E(t - \frac{x}{l}), \quad v|_{t=0} = k(t - \frac{x}{l}) - \frac{k-b}{l}x \end{cases}$$

设 $V(x,t) = X(x)T(t)$ ，得：

$$\begin{cases} X''(x) + \lambda^2 X(x) = 0 \\ X(0) = X(l) = 0 \end{cases} \text{ 解出 } \lambda = (\frac{n\pi}{l})^2, \quad X(x) = C \sin \frac{n\pi}{l} x, \quad n=1, 2, \dots$$

$$\text{得 } T(t) = C_1 \cos \frac{n\pi a}{l} t + C_2 \sin \frac{n\pi a}{l} t, \quad \Rightarrow \quad V(x,t) = \sum_{n=1}^{+\infty} [C_1 \cos \frac{n\pi a}{l} t + C_2 \sin \frac{n\pi a}{l} t] \sin \frac{n\pi}{l} x$$

由 $v|_{t=0} = E(t - \frac{x}{l})$ 及 $v|_{t=0} = k(t - \frac{x}{l}) - \frac{k-b}{l}x$ 得：

$$a_n = \frac{2}{l} \int_0^l E(t - \frac{x}{l}) \sin \frac{n\pi}{l} x dx = \frac{2E}{n\pi}.$$

$$b_n = \frac{2}{n\pi a} \int_0^l [k(t - \frac{x}{l}) - \frac{k-b}{l}x] \sin \frac{n\pi}{l} x dx = (-1)^n \frac{2bl}{n^2 \pi^2 a}.$$

$$\text{所以 } U(x,t) = \sum_{n=0}^{+\infty} \left[\frac{2E}{n\pi} \cos \frac{n\pi a}{l} t + (-1)^n \frac{2bl}{n^2 \pi^2 a} \sin \frac{n\pi a}{l} t \right] \sin \frac{n\pi}{l} x + \frac{k-b}{l}tx + kt$$

[題型: Fourier 積分變換]

(波动方程)

$$0 \left\{ \begin{array}{l} U_{tt} - U_{xx} = t \sin x, \quad -\infty < x < +\infty, \quad t > 0 \\ U|_{t=0} = 0, \quad U_t|_{t=0} = \sin x. \end{array} \right.$$

$$\Rightarrow \text{設 } \tilde{U}(w, t) = F[U(x, t)], \text{ 原方程化為: } \left\{ \begin{array}{l} \frac{\partial^2 \tilde{U}}{\partial t^2} + w^2 \tilde{U} = i\pi [\delta(w+1) - \delta(w-1)]t = f(w)t \\ \tilde{U}|_{t=0} = 0, \quad \tilde{U}_t|_{t=0} = f(w) \end{array} \right.$$

①解常微分方程:

由 $t^2 + w^2 = 0$ 得齊次通解: $\tilde{U} = C_1 \cos wt + C_2 \sin wt$.

設非齊次通解為 $\tilde{U} = C_1(t) \cos wt + C_2(t) \sin wt$. 得到:

[常數變易法]

$$\left\{ \begin{array}{l} \cos wt C_1'(t) + \sin wt C_2'(t) = 0 \\ -wsinwt C_1'(t) + wcoswt C_2'(t) = f(w)t. \end{array} \right. \Rightarrow \left\{ \begin{array}{l} C_1'(t) = \frac{f(w)}{w} \cos wt \cdot t \\ C_2'(t) = -\frac{f(w)}{w} \sin wt \cdot t. \end{array} \right.$$

$$\text{積分: } C_1(t) = -\frac{f(w)}{w} \int t \sin wt dt = \frac{f(w)}{w^2} [t \cos wt - \frac{\sin wt}{w} + C_3]$$

$$C_2(t) = \frac{f(w)}{w} \int t \cos wt dt = \frac{f(w)}{w^2} [t \sin wt + \frac{\cos wt}{w} + C_4]$$

$$\text{故 } \tilde{U}(w, t) = \frac{f(w)}{w^2} t + \frac{f(w)}{w^2} [C_3 \cos wt + C_4 \sin wt].$$

由 $\tilde{U}|_{t=0} = 0$ 及 $\tilde{U}_t|_{t=0} = f(w)$ 得: $C_3 = 0$. $C_4 = \frac{1}{w} (1 - \frac{1}{w^2})$.

$$\text{故 } \tilde{U}(w, t) = \frac{i\pi}{w^2} [\delta(w+1) - \delta(w-1)]t + \frac{i\pi}{w} (1 - \frac{1}{w^2}) [\delta(w+1) - \delta(w-1)] \sin wt.$$

②求逆變換:

$$U(x, t) = \mathcal{F}^{-1}[\tilde{U}(w, t)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{U}(w, t) e^{inx} dw$$

$$= \frac{it}{2} \int_{-\infty}^{+\infty} \frac{1}{w^2} [\delta(w+1) - \delta(w-1)] e^{inx} dw + \frac{i}{2} \int_{-\infty}^{+\infty} \frac{1}{w} (1 - \frac{1}{w^2}) [\delta(w+1) - \delta(w-1)] \sin wt e^{inx} dw$$

$$= \frac{it}{2} [e^{-ix} - e^{ix}] = \frac{t}{2i} [e^{ix} - e^{-ix}] = ts \sin x.$$

(利用 $\delta(w)$ 的篩選性質).

$$0 \quad \begin{cases} u_t + au_x = f(x, t), & -\infty < x < +\infty, t > 0 \\ u|_{t=0} = \varphi(x), & -\infty < x < +\infty. \end{cases}$$

$$\text{记 } \tilde{u}(w, t) = \mathcal{F}[u(x, t)], \tilde{f}(w, t) = \mathcal{F}[f(x, t)], \tilde{\varphi}(w) = \mathcal{F}[\varphi(x)].$$

$$\text{由 } \begin{cases} \frac{\partial \tilde{u}}{\partial t} + aiw\tilde{u} = \tilde{f}(w, t). \\ \tilde{u}|_{t=0} = \tilde{\varphi}(w) \end{cases} \text{ 由-阶常数变易法:}$$

$$\begin{aligned} \tilde{u}(w, t) &= e^{-aiwtdt} \left(\int_0^t \tilde{f}(w, \tau) e^{aiw\tau dt} d\tau + C \right) = e^{-aiwt} \left(\int_0^t \tilde{f}(w, \tau) e^{aiw\tau dt} d\tau + C \right) \\ &= \int_0^t \tilde{f}(w, \tau) e^{-ai(t-\tau)w} d\tau + Ce^{-aiwt}. \end{aligned}$$

$$\text{由 } \tilde{u}|_{t=0} = \tilde{\varphi}(w) \text{ 得: } C = \tilde{\varphi}(w). \text{ 因此 } \tilde{u}(w, t) = \int_0^t \tilde{f}(w, \tau) e^{-ai(t-\tau)w} d\tau + \tilde{\varphi}(w) e^{-aiwt}.$$

$$\text{由于 } \mathcal{F}[\delta(x-at)] = e^{-aiwt}, \mathcal{F}[\delta(x-a(t-\tau))] = e^{-ai(t-\tau)w}. \text{ 因此}$$

$$\tilde{u}(w, t) = \tilde{\varphi}(w) \tilde{g}(x-at) + \int_0^t \tilde{f}(w, \tau) \tilde{g}(x-a(t-\tau)) d\tau$$

从逆变换:

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}[\tilde{u}(w, t)] = \varphi(x) * \delta(x-at) + \int_0^t f(x, \tau) * \delta(x-a(t-\tau)) d\tau. \quad \text{※逆变换} \\ &= \int_{-\infty}^{+\infty} \varphi(x-\xi) \delta(\xi-at) d\xi + \int_0^t \left[\int_{-\infty}^{+\infty} f(x-\xi, \tau) \delta(\xi-a(t-\tau)) d\xi \right] d\tau \\ &= \varphi(x-at) + \int_0^t f(x-a(t-\tau), \tau) d\tau. \end{aligned}$$

$$0 \quad \begin{cases} u_t - a^2 u_{xx} - bu_x - cu = 0, & -\infty < x < +\infty, t > 0. \\ u(x, 0) = \varphi(x), & -\infty < x < +\infty. \end{cases}$$

$$\text{记 } \tilde{u}(w, t) = \mathcal{F}[u(x, t)], \tilde{\varphi}(w) = \mathcal{F}[\varphi(x, t)].$$

$$\Rightarrow \begin{cases} \frac{\partial \tilde{u}}{\partial t} + a^2 i w \tilde{u} - b \tilde{u} - c \tilde{u} = 0. \\ \tilde{u}|_{t=0} = \tilde{\varphi}(w). \end{cases} \text{ 解得: } \tilde{u} = \tilde{\varphi}(w) e^{-(a^2 w^2 - biw - ct)}$$

$$\text{通过配方: } \tilde{u}(w, t) = \tilde{\varphi}(w) e^{-\frac{b^2 - 4a^2 c}{4a^2} t} e^{-a^2 t (w - \frac{bi}{2a^2})^2}, \text{ 又因为 } \mathcal{F}\left[\frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}}\right] = e^{-\frac{a^2 t w^2}{4}}$$

$$\text{因此 } \mathcal{F}^{-1}\left[e^{-\frac{b^2 - 4a^2 c}{4a^2} t} e^{-a^2 t (w - \frac{bi}{2a^2})^2}\right] = e^{-\frac{b^2 - 4a^2 c}{4a^2} \cdot \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2 t}} \cdot e^{-\frac{bx}{2a^2}}} = \frac{e^{ct}}{2a\sqrt{\pi t}} e^{\frac{(x+bt)^2}{4a^2 t}}$$

$$\text{因此 } u(x, t) = \mathcal{F}^{-1}[\tilde{\varphi}(w) e^{-(a^2 w^2 - biw - ct)}] = \mathcal{F}^{-1}[\tilde{\varphi}(w)] * \mathcal{F}^{-1}[e^{-(a^2 w^2 - biw - ct)}]$$

$$= \varphi(x) * \frac{e^{ct}}{2a\sqrt{\pi t}} e^{\frac{(x+bt)^2}{4a^2 t}} = \frac{e^{ct}}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \varphi(x-\xi) e^{\frac{(\xi+bt)^2}{4a^2 t}} d\xi$$

[题型: Laplace 稳定变换].

$$0 \left\{ \begin{array}{l} u_{xy}=1, \quad x>0, \quad y>0 \\ u|_{x=0}=y+1, \quad u|_{y=0}=1. \end{array} \right.$$

(这里的“1”就是 $u(x)$ 单位阶跃).

对 $u(x,y)$ 关于 x 取 L 变换. 记 $\tilde{u}(py) = \mathcal{L}[u(x,y)]$.

$$\text{由 } \mathcal{L}\left[\frac{\partial u(x,y)}{\partial x}\right] = \frac{\partial}{\partial y}\left[\mathcal{L}\left[\frac{\partial u(x,y)}{\partial x}\right]\right] = \frac{\partial}{\partial y}[p\tilde{u}(py)-y-1] = p\frac{\partial \tilde{u}}{\partial y}-1$$
$$\mathcal{L}[1] = \frac{1}{p}.$$

$$\text{原方程化为 } \left\{ \begin{array}{l} p\frac{\partial \tilde{u}}{\partial y}-1 = \frac{1}{p} \\ \tilde{u}|_{y=0} = \frac{1}{p} \end{array} \right. \Rightarrow \tilde{u} = (\frac{1}{p^2} + \frac{1}{p})y + \frac{1}{p}.$$

$$\text{取逆变换: } u(x,y) = \mathcal{L}^{-1}[\tilde{u}(py)] = \mathcal{L}[(\frac{1}{p^2} + \frac{1}{p})y + \frac{1}{p}] = xy + y + 1.$$

$$0 \left\{ \begin{array}{l} u_{xx} + u_{xt} = 0, \quad x>0, t>0 \\ u|_{x=0} = \psi(t), \quad u_x|_{x=0} = 0. \end{array} \right.$$

$$\left. \begin{array}{l} u|_{t=0} = \psi(x), \quad \psi(0) = \psi(0). \end{array} \right.$$

$$\text{记 } \tilde{u}(pt) = \mathcal{L}[u(x,t)], \quad \tilde{\psi}(p) = \mathcal{L}[\psi(x)]$$

$$\text{由 } \mathcal{L}\left[\frac{\partial^2 u}{\partial x^2}\right] = p^2\tilde{u} - p\tilde{u}|_{x=0} - \tilde{u}_{xx}|_{x=0} = p^2\tilde{u} - p\psi, \quad \mathcal{L}\left[\frac{\partial u}{\partial x}\right] = p\tilde{u} - \tilde{u}|_{x=0} = p\tilde{u} - \psi$$

$$\text{原方程化为 } \left\{ \begin{array}{l} \frac{\partial \tilde{u}}{\partial t} + p\tilde{u} = \psi + \frac{\psi'}{p} \\ \tilde{u}|_{t=0} = \tilde{\psi}(p). \end{array} \right.$$

$$\text{由常数变易法: } \tilde{u} = e^{-\int p dt} \left(\int_0^t (\psi + \frac{\psi'}{p}) e^{p\tau} d\tau + C \right) = e^{-pt} \left(\int_0^t (\psi + \frac{\psi'}{p}) e^{p\tau} d\tau + \tilde{\psi}(p) \right)$$

$$\text{其中 } \int_0^t (\psi + \frac{\psi'}{p}) e^{p\tau} d\tau = \int_0^t \psi e^{p\tau} d\tau + \int \frac{\psi'}{p} e^{p\tau} d\tau = \int_0^t \frac{\psi}{p} de^{p\tau} + \int \frac{\psi'}{p} e^{p\tau} d\tau = \frac{\psi}{p} e^{pt}$$

$$\text{所以 } \tilde{u} = e^{-pt} \left(\frac{\psi}{p} e^{pt} + \tilde{\psi}(p) \right) = \frac{\psi}{p} + \tilde{\psi}(p) e^{-pt}.$$

$$\text{取逆变换: } u(x,t) = \mathcal{L}^{-1}[\tilde{u}(pt)] = \psi(t) + \psi(x-t) \tilde{u}(x-t)$$

$u(x-t)$ 为单位阶跃函数. 即: $u(x-t) = \begin{cases} \psi(t), & x-t < 0 \\ \psi(t) + \psi(x-t), & x-t > 0. \end{cases}$

1. 正确个数()

(1) 设 $z=\infty$ 为 $f(z)$ 可去奇点, $\lim_{z \rightarrow \infty} \operatorname{Res}[f(z), \infty] = 0$.

(2) 设 $f(z)$ 在 $|z| > R > 0$ 内解析, 且 $\lim_{z \rightarrow \infty} z \cdot f(z) = 0$. $\lim_{z \rightarrow \infty} \operatorname{Res}[f(z), \infty] = 0$

(3) 设 $f(z)$ 的罗朗展开: $f(z) = \sum_{n=-\infty}^{-1} \frac{a_n}{(z-1)^n} + \sum_{n=0}^{+\infty} \frac{b_n}{(z-1)^n}$. $\lim_{z \rightarrow 1} \operatorname{Res}[f(z), 1] = 0$.

- (A) 1. 0. (B) 2. (C) 3. (D) 4.

2. 给定复平面上点集:

(1) $|z-2+i| \leq 1$. (2) $0 \leq \arg z \leq \frac{\pi}{4}$. (3) $|z-4| \geq |z|$. (4) $|2z+3| > 4$.

其中构成区域的有()

- (A) 1. (B) 2. (C) 3. (D) 4.

3. 设 $f(z) = 1 + 2z + 3z^2 + \dots + 2012z^{2011}$. $\lim_{|z|=1} \oint \frac{(z)^{2011} f(z)}{z} dz = ()$.

4. $f(z) = \frac{z+2}{z(z+1)}$ 在 $|z| > 1$ 内的罗朗级数为 _____.

5. 用 Laplace 变换计算积分. $\int_0^{+\infty} \frac{x \sin x}{1+x^2} dx$.

6. $\lim_{z \rightarrow z_0} \frac{\operatorname{Im}(z) - \operatorname{Im}(z_0)}{z - z_0} = ()$. A. i B. -i C. 0 D. 不存在.

↳ 该式为 $f(z) = \operatorname{Im}(z)$ 导数定义式, $f(z) = \operatorname{Im} z$ 不满足 CR 条件, 因此不可导.

7. 不正确的是().

(A) 积分 $\oint_{|z|=r} \frac{1}{z-a} dz$ 值与 r 大小无关. (B) $|\int_C (x^2 + iy^2) dz| < 1$, 其中 C 为 i 到 j 的残段.

(C) 若 $f(z)$ 在 $0 < |z| < 1$ 内解析, 且沿任圆周 $C: |z| = r$ ($0 < r < 1$) 积分为 0. $\lim_{z \rightarrow 0} f(z)$ 在 $z=0$ 处解析.

(D) 设 $g(z)$ 在区域 D 内有定义, 且 $f'(z) = g(z)$. $\lim_{z \rightarrow 0} g'(z)$ 存在且解析.

↳ (C) 存在可去奇点. (D) 由 $f'(z) = g(z)$ 知 $f(z)$ 解析. $\lim_{z \rightarrow 0} f'(z)$ 也解析. $\lim_{z \rightarrow 0} g'(z)$ 也解析.

8. $\operatorname{Res}[f(z), 0] = 0$ 的是().

(A) $f(z) = \frac{1}{z^2-1} - \frac{1}{z}$. (B) $f(z) = \frac{\sin z}{z} - \frac{1}{z}$. (C) $f(z) = \frac{\sin z + \cos z}{z}$ (D) $f(z) = \frac{e^z - 1}{z^2}$.

9. $z=0$ 有函数 $\frac{1}{z^2 - \sin z^2}$ 的 m 阶极点. $\lim m =$ _____.

(D) 调和函数 $u = \frac{x}{x^2+y^2}$ 的共轭调和函数为 _____.

11. 设幂级数 $\sum_{n=0}^{+\infty} C_n (z-2)^n$ 在 $z=4$ 收敛而在 $z=2+2i$ 发散. $\lim_{z \rightarrow 4^-} f(z)$ 为 _____.

12. 将 $f(z) = \frac{1}{z^2(z^2+1)}$ 在 $0 < |z-1| < 1$ 内展开为罗朗级数

13. [证明题] 试证: 复数函数 $f(z) = \cos(z + \frac{1}{z})$ 的罗朗级数展开式 $\sum_{n=-\infty}^{+\infty} C_n z^n$ 的系数为:

$$C_n = \frac{1}{2\pi i} \int_0^{2\pi} \cos(\theta) \cos(2\cos\theta) d\theta.$$

$$\Rightarrow \forall z = e^{i\theta}, \text{ 有 } f(z) = \frac{1}{2} [e^{i(z+\frac{1}{z})} + e^{-i(z+\frac{1}{z})}] = \frac{1}{2} [e^{2i\cos\theta} + e^{-2i\cos\theta}] = \cos(2\cos\theta).$$

$$\underline{C_n = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\cos(2\cos\theta)}{e^{in+i\theta}} ie^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\cos\theta) [\cos n\theta - i\sin n\theta] d\theta}$$

令 $\theta = \pi - \theta$ 可得 $\int_0^{2\pi} \sin n\theta \cos(2\cos\theta) d\theta = 0$. 原式得证.

14. 不正确的是()

- (A) ∞ 是 $\sin \frac{1}{z}$ 的本性奇点.
 (C) ∞ 是 $\frac{1}{\sin z}$ 的孤立奇点.

- (B) ∞ 是 $\sin z$ 的本性奇点.
 (D) ∞ 是 $\frac{1}{\sin \frac{1}{z}}$ 的孤立奇点.

15. 函数 $f(x) = u(x)e^{-x} \sin x$ 的傅立叶变换为().

- (A) $\frac{1}{(1+iw)^2 + 1}$. (B) $\frac{1}{(1-iw)^2 + 1}$. (C) $\frac{1}{(1+iw)^2 - 1}$. (D) $\frac{1}{(1-iw)^2 - 1}$

$$\Rightarrow \int_0^{+\infty} \sin x \cdot e^{-(1+iw)x} dx = \frac{1}{p^2 + 1} \Big|_{p=1+iw} = \frac{1}{(1+iw)^2 + 1} \quad (\text{这里用了 Laplace 变换})$$

$$e^{-at} u(t) \Rightarrow \frac{1}{a+iw}.$$

16. 级分 $\oint_{|z|=2} \sin \frac{1}{z-1} = \underline{\quad}$.

$$\oint_{|z|=2} \sin \frac{1}{z-1} = 2\pi i \operatorname{Res}[\sin \frac{1}{z-1}, 1]. \text{ 由 } \sin \frac{1}{z-1} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)! (z-1)^{2n+1}} \text{ 得 } C_{-1} = 1 \text{ 且 } \lim_{n \rightarrow \infty} \frac{(-1)^n}{(2n+1)!} = 0.$$

Δ 用罗朗级数展开求留数.

$$\operatorname{Res}[f(z), 1] = -\operatorname{Res}[f(z), \infty] = \operatorname{Res}[\frac{1}{z-1} f(\frac{1}{z}), 0].$$

17. $f(z) = \tan z + \frac{2}{1+z}$ 在 $z=2$ 处泰勒级数收敛半径为 $\underline{\quad}$.

18. 若 $z=z_0$ 分别为 $f_1(z), f_2(z)$ 的 m 阶和 n 阶极点. 则 $z=z_0$ 是 $f_1(z)f_2(z)$ 的 $\underline{\quad}$ 阶极点.

19. 设函数 $f(z)$ 在 $z=0$ 的邻域内罗朗级数为 $f(z) = \sum_{n=-\infty}^{+\infty} C_n z^n$. 求 C_n . 其中 $n=-\infty, \dots, 0, 1$.

20. 用 Laplace 变换求微分方程 $\begin{cases} ty''(t) + 2y'(t) + ty(t) = 2\sin t \\ y(0) = -1, \quad y'(0) = 0. \end{cases}$

21. 设 $f(z) = u(x, y) + iv(x, y)$ 在区域 D 内解析.

(1) 若在 D 内 $f'(z) = 0$, 则 $f(z)$ 为常数.

(2) 若 $u(x, y)$ 在 D 内为常数, 则 $f(z)$ 为常数.

(3) 若在 D 内 $u(x, y) = v(x, y)$, 则 $f(z)$ 为常数. 正确的有 (). D.

(A) 0 个. (B) 1 个. (C) 2 个. (D) 3 个.

22. 已知 $|e^{i\theta}-1|=2$. 则 $\theta=$ ____.

\Rightarrow 2k 解法: $\theta_0 = \pi$. $e^{2\pi i}$ 周期为 $2\pi \Rightarrow \theta = (2k+1)\pi$, $k=0, \pm 1, \pm 2, \dots$

23. 将函数 $f(z) = \frac{1}{z+1}$ 在区域 D 内展开为形如 $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$ 的级数. 且区域 D 为 $|z|<1$ 和 $|z|>1$. 中心为 $z=0$.

24. 对一维热传导方程混合初边值问题 $\begin{cases} u_t = a^2 u_{xx}, \quad 0 < x < l, \quad t > 0 \\ u_x(0, t) = A, \quad u_x(l, t) = -A, \quad t > 0, \quad \text{其中 } A \neq 0. \quad \text{进} \\ u(x, 0) = B, \quad 0 \leq x \leq l. \end{cases}$

将边界条件代入 $u(x, t) = v(x, t) + w(x)$, 得 $v(x, t)$ 满足齐次边值条件. 且 $w(x) = -\frac{A}{l}x^2 + Ax$.

25. $f(t) = g(t-1)(t-2)^2 \sin t$ 用 Fourier 变换 = $\sin x$

26. 计算积分 $\int_0^{+\infty} \frac{x^2}{(1+x^2)(a^2+x^2)} dx$. \Rightarrow 原式 = $\frac{1}{2} \int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)(a^2+x^2)} dx$

27. 用 Laplace 变换求解常微分方程边值问题. $\begin{cases} y''(t) + y(t) = \cos nt, \quad 0 < t < \frac{\pi}{2} \\ y(0) = 0, \quad y(\frac{\pi}{2}) = 1. \end{cases}$

$\Rightarrow p^2 Y(p) - py(0) - y'(0) + Y(p) = \frac{20}{p^2+4}$. $Y(p) = \frac{20}{(1+p^2)(p^2+4)} + \frac{y'(0)}{p^2+1} = \frac{y'(0)+\frac{20}{3}}{p^2+1} - \frac{10}{3} \cdot \frac{2}{p^2+4}$

$y(t) = \mathcal{L}^{-1}[Y(p)] = \frac{20}{3} \int_0^t \sin nt dt - \frac{10}{3} \sin 2t$. 由 $y(\frac{\pi}{2}) = 1$ 得 $y'(0) + \frac{20}{3} = 1$. $\Rightarrow y(t) = \sin t - \frac{10}{3} \sin 2t$.

注意 $\mathcal{L}[y''(t)] = p^2 Y(p) - py(0) - y'(0)$!!!

28. 函数 $f(z) = \frac{z^3}{(z^2-1)(z+1)}$ 在复平面上所有有限点处留数的和是 ().

(A) 2.

(B) -2.

(C) 1.

(D) -1.

29. 对于 $f(z) = \frac{1}{z} e^{\frac{z}{z+1}}$ 在区域 $|z| < +\infty$ 内展开成洛朗级数，只要求含有 z^{-2} 到 z^0 的项。

① 间接法：

$$\Rightarrow e^{\frac{z}{z+1}} = e^{(1-\frac{1}{z+1})} = e \cdot e^{-\frac{1}{z+1}}, \text{ 其中 } e^{-\frac{1}{z+1}} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{(z+1)^n} = 1 - \frac{1}{z+1} + \frac{1}{2!} \left(\frac{1}{z+1}\right)^2 - \frac{1}{3!} \left(\frac{1}{z+1}\right)^3 + \dots$$

$$\text{其中 } \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} = \frac{1}{z} \cdot \sum_{n=0}^{+\infty} \frac{(-1)^n}{z^n} = \frac{1}{z} \left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots$$

$$\left(\frac{1}{1+z}\right)^2 = -\frac{d}{dz} \frac{1}{1+z} = -\frac{d}{dz} \left(\frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z^2} - \frac{2}{z^3} + \frac{3}{z^4} - \frac{4}{z^5} + \dots$$

$$\left(\frac{1}{1+z}\right)^3 = \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}}\right) = \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \dots\right) = -\frac{1}{z^3} + \frac{3}{z^4} - \frac{6}{z^5} + \frac{10}{z^6} + \dots$$

$$\text{代入得: } e^{-\frac{1}{z+1}} = 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} + \dots + \frac{1}{2z^2} - \frac{1}{z^3} + \frac{3}{2z^4} - \frac{2}{z^5} + \dots + \frac{1}{6z^3} - \frac{1}{2z^4} + \frac{1}{z^5} - \frac{5}{3z^6} + \dots \\ = 1 - \frac{1}{z} - \frac{1}{2z^2} + \dots$$

$$\text{则 } f(z) = \frac{1}{z} e^{\frac{z}{1+z}} = \frac{1}{z} - \frac{1}{2z^2} + \dots$$

② 直接法： 设 $w = \frac{1}{z}$. $g(w) = f(\frac{1}{w}) = we^{\frac{1}{w+1}}$. w 为 $g(w)$ 的极点。 $g(w) = \sum_{n=0}^{+\infty} c_n w^n$. $c_n = \frac{g^{(n)}(0)}{n!}$.

$$\Rightarrow c_0 = g(0) = 0, c_1 = g'(0) = e, c_2 = \frac{g''(0)}{2!} = -e. \text{ 则 } g(w) = ew - ew^2 + \dots$$

得 $f(z) = \frac{e}{z} - \frac{e}{z^2} + \dots$ 罗朗级数不能用比公式。

或者 $f(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$, 其中 $c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$.

$$c_{-2} = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z \cdot z^{-2+1}} dz = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z^2} dz = \text{Res}[\frac{1}{z^2} e^{\frac{1}{z+1}}, 0] = -e.$$

$$c_{-1} = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z \cdot z^{-1+1}} dz = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z} dz = \text{Res}[\frac{1}{z} e^{\frac{1}{z+1}}, 0] = e.$$

$$c_0 = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z \cdot z^{0+1}} dz = \frac{1}{2\pi i} \oint_C \frac{e^{\frac{z}{1+z}}}{z} dz = \text{Res}[e^{\frac{1}{z+1}}, 0] = 0.$$

$$\text{则 } f(z) = \frac{e}{z} - \frac{e}{z^2} + \dots$$

30. 设曲线 C 为正向圆周 $|z|=2$. 计算积分 $I = \oint_C \frac{1}{z \sin(\frac{1}{z})} dz$.

$$I = -2\pi i \text{Res}[f(z), \infty] = 2\pi i \text{Res}[\frac{1}{z^2} f(\frac{1}{z}), 0].$$

31. 设 $\alpha > 0$. 计算积分 $I = \int_0^\alpha \frac{1}{(5-3\sin\frac{2\pi\theta}{\alpha})} d\theta$.

32. 用 Laplace 變換求解 $\begin{cases} y'(t) + 3y(t) + 2 \int_0^t y(s) ds = e^{-t} + \sin ht \\ y(0) = 0. \end{cases}$

$$\text{記 } Y(p) = \mathcal{L}[y(t)]. \Rightarrow pY(p) + 3Y(p) + \frac{2Y(p)}{p} = \frac{1}{p+1} + \frac{1}{p^2-1}$$

$$\Rightarrow Y(p) = \frac{1}{(p+1)^2(p+2)} + \frac{1}{(p^2-1)(p+1)(p+2)} = \frac{1}{(p+1)^2(p+2)} + \frac{1}{(p+1)^2(p-1)(p+2)}$$

$$y(t) = \mathcal{L}^{-1}[Y(p)] = \text{Res}[Y(p)e^{pt}, (\pm 1, \mp 2)] = \left[\frac{e^{pt}}{p+2} + \frac{e^{pt}}{(p-1)(p+2)} \right] \Big|_{p=-1} + \left[\frac{(p+1)e^{pt}}{(p+1)^2(p+2)} + \frac{e^{pt}}{(p+1)^2(p+2)} \right] \Big|_{p=1} + \left[\frac{e^{pt}}{(p+1)^2} + \frac{e^{pt}}{(p+1)^2(p-1)} \right] \Big|_{p=-2} = \frac{1}{12}e^t + \frac{5}{4}e^{-t} - \frac{t}{2}e^{-t} - \frac{4}{3}e^{-2t}. \quad [\text{複反演公式}]$$

$$\text{或者: } Y(p) = \frac{p^2}{(p+1)(p+2)(p+1)^2} = \frac{A}{p+1} + \frac{B}{p+2} + \frac{Cp+D}{(p+1)^2}.$$

$$\Rightarrow A = \lim_{p \rightarrow 1} Y(p)(p+1) = \lim_{p \rightarrow 1} \frac{p^2}{(p+2)(p+1)^2} = \frac{1}{12}.$$

$$B = \lim_{p \rightarrow -2} Y(p)(p+2) = \lim_{p \rightarrow -2} \frac{p^2}{(p+1)^2(p+1)} = -\frac{1}{3}$$

$$D = \lim_{p \rightarrow 0} Y(p)(p+1)^2 - \lim_{p \rightarrow 0} \left[\frac{A(p+1)}{p+1} + \frac{B(p+1)^2}{p+2} \right] = \frac{3}{4} \quad [\text{求待定系數}]$$

$$-C+D = \lim_{p \rightarrow 1} Y(p)(p+1)^2 = -\frac{1}{2} \Rightarrow C = \frac{5}{4}$$

$$\therefore Y(p) = \frac{1}{12} \cdot \frac{1}{p+1} - \frac{4}{3} \cdot \frac{1}{p+2} + \frac{5}{4} \cdot \frac{1}{(p+1)} - \frac{1}{2} \cdot \frac{1}{(p+1)^2}$$

问题时对n取值作划分

△自由端为三角形形状，但在边界条件得到 $U_{tt}(t)$ 的富解

$$33. \text{求解定解问题} \quad \begin{cases} U_{tt} - a^2 U_{xx} = \sin \frac{2\pi}{l} x \sin \frac{2\alpha t}{l}, & 0 < x < l, t > 0 \\ U(0, t) = 0, \quad U(l, t) = 0, & t > 0 \\ U(x, 0) = 0, \quad U_t(x, 0) = 0, & 0 < x < l \end{cases}$$

[固有频率法]

该齐次边界条件下应的固有值 $\lambda = (\frac{n\pi}{l})^2$, $n=1, 2, \dots$ 固有频率为 $\sin \frac{n\pi}{l} x$, $n=1, 2, \dots$

初设解具有下形式 $U(x, t) = \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi}{l} x$
自由项 $\sin \frac{2\pi}{l} x \sin \frac{2\alpha t}{l}$

将 $U(x, t)$ 代入方程得: $\sum_{n=1}^{+\infty} T_n''(t) \sin \frac{n\pi}{l} x + \frac{\ell^2 a^2}{n^2 \pi^2} \sum_{n=1}^{+\infty} T_n(t) \sin \frac{n\pi}{l} x = \sin \frac{2\pi}{l} x \sin \frac{2\alpha t}{l}$.

比较系数得: $T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = \begin{cases} \sin \frac{2\alpha t}{l}, & n=2 \\ 0, & n \neq 2. \end{cases}$

由初值条件 $U(x, 0) = 0, U_t(x, 0) = 0$ 得 $T_n(0) = 0, T_n'(0) = 0$.

得 $T_n(t)$ 的常微分方程定解问题,

(1) $n \neq 2$ 时: $\begin{cases} T_n''(t) + \left(\frac{n\pi a}{l}\right)^2 T_n(t) = 0, & \text{通解为 } T_n(t) = A_n \cos \frac{n\pi a}{l} t + B_n \sin \frac{n\pi a}{l} t \\ T_n(0) = T_n'(0) = 0. \end{cases}$

由初值条件得 $A_n = B_n = 0$. 即 $T_n(t) = 0, n \neq 2$.

(2) $n=2$ 时. $\begin{cases} T_2''(t) + \left(\frac{n\pi a}{l}\right)^2 T_2(t) = \sin \frac{2\pi a}{l} t \\ T_2(0) = T_2'(0) = 0. \end{cases}$

$\pm \frac{2\pi a}{l}$ 为特征根. 即通解为 $T_2(t) = C_1 \cos \frac{2\pi a}{l} t + C_2 \sin \frac{2\pi a}{l} t$.

设特解为 $T_2^*(t) = t(A \cos \frac{2\pi a}{l} t + B \sin \frac{2\pi a}{l} t)$. 代入方程得 $T_2^*(t) = -\frac{l}{4\pi a} t \cos \frac{2\pi a}{l} t$.

$$\Rightarrow T_2(t) = C_1 \cos \frac{2\pi a}{l} t + C_2 \sin \frac{2\pi a}{l} t - \frac{l}{4\pi a} t \cos \frac{2\pi a}{l} t.$$

由 $T_2(0) = T_2'(0) = 0$ 得 $C_1 = 0, C_2 = \frac{l^2}{8\pi^2 a^2}$.

$$\text{综上 } T_2(t) = \frac{l^2}{8\pi^2 a^2} \sin \frac{2\pi a}{l} t - \frac{l}{4\pi a} t \cos \frac{2\pi a}{l} t.$$

$$\text{综上 } U(x, t) = \frac{l^2}{8\pi^2 a^2} \sin \frac{2\pi a}{l} t \sin \frac{2\pi}{l} x - \frac{l}{4\pi a} t \cos \frac{2\pi a}{l} t \sin \frac{2\pi}{l} x.$$

△初值条件为三角形形状. 在最后用正交性求待定系数

只有 1 个: 由 $U_t - a^2 U_{xx} = 0, 0 < x < l, t > 0$.

$$\begin{cases} U_{x=0} = 0, \quad U_{x=l} = 0, \quad t > 0. \end{cases}$$

$$U_{t=0} = \sin \frac{\pi}{l} x + 3 \sin \frac{3\pi}{l} x, 0 \leq x \leq l.$$

34. 不正确的是 (A).

(A) 设 $f(z) = \frac{g(z)}{(z-z_0)^n}$, n 为正整数. $g(z)$ 在 z_0 点解析. 则 z_0 是 $f(z)$ 的 n 阶极点.

(B) 若 $f(z)$ 在 D 内任一点 z_0 的全部域内可展成泰勒级数. 则 $f(z)$ 在 D 内解析.

(C) 若 $f(z)$ 在单连域 D 内连续, 且沿 D 内任一条简单闭曲线的积分值为 0. 则 $f(z)$ 在 D 内解析.

(D) 若 $f(z)$ 在单连域 D 解析. 则积分 $\int_{z_0}^{z_1} f(z) dz$ 与路径无关.

35.

5. 下列命题中正确的是

(C)

(A) 如果 $f(z)$ 在 $|z| \leq 1$ 内可导, 则 $f(z)$ 在 $|z| \leq 1$ 内解析; X

(B) 如果 $f(z) = u(x, y) + iv(x, y)$ 在 $|z| < 1$ 内解析, 则 $u(x, y)$ 是 $v(x, y)$ 的共轭调和函数;

(C) 如果 $f(z)$ 在 $z = 0$ 点可展成 Taylor 级数, 则 $f(z)$ 在 $z = 0$ 点的某去心邻域内解析;

(D) 如果 $f(z)$ 在 $0 < |z| < 1$ 内解析, 且沿此区域内的任一闭曲线积分等于 0, 则 $z = 0$ 为

$f(z)$ 的解析点. X 有奇点.

36. 不正确的是 (B)

(A) $\overline{e^z} = e^{\bar{z}}$ (B) $\ln z^2 = 2 \ln z$. (C) $\overline{\ln z} = \ln \bar{z}$. (D) $\overline{\sin z} = \sin \bar{z}$.

37. 设 $x^2 + x + 1 = 0$. 则 $x^6 + x^7 + x^8 + \dots$ = ().

(A) 2. (B) 1 (C) 0 (D) -1

$$\begin{aligned} \Rightarrow x^2 + x + 1 = 0 \Rightarrow x + \frac{1}{x} = -1. \quad x^6 + x^7 + x^8 + \dots \\ = x^6 \left[\left(x + \frac{1}{x} \right)^2 - 1 \right] = 0. \end{aligned}$$

38. 设 $f(t)$ 的 Fourier 变换为 $F(w) = \pi \delta(w+2)$, 则 $f(t) + f(-t)$ 的 Laplace 变换为 (A).

(A) $\frac{P}{P^2 + 4}$ (B) $\frac{2}{P^2 + 4}$ (C) $\frac{P}{P^2 - 4}$ (D) $\frac{2}{P^2 - 4}$.

$$\Delta \text{计算积分} \int_0^{+\infty} \frac{x \sin x}{1+x^2} dx$$

$$\text{令 } f(t) = \int_0^{+\infty} \frac{x \sin t x}{1+x^2} dx. \text{ 则 } \mathcal{L}[f(t)] = F(p) = \int_0^{+\infty} \left[\int_0^{+\infty} \frac{x}{1+x^2} \sin tx dx \right] e^{-pt} dt$$

$$\Rightarrow F(p) = \int_0^{+\infty} \frac{x}{1+x^2} dx \int_0^{+\infty} \sin tx e^{-pt} dt = \int_0^{+\infty} \frac{x^2}{(1+x^2)(p^2+x^2)} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)(p^2+x^2)} dx$$

$$\begin{aligned} \text{积分} \int_{-\infty}^{+\infty} \frac{x^2}{(1+x^2)(p^2+x^2)} dx &= 2\pi i \operatorname{Res}[f(p), i] + 2\pi i \operatorname{Res}[f(p), -i] = 2\pi i \left[\frac{-1}{2i(p^2+1)} + \frac{-t^2}{2t(p^2+1)} \right] \\ &= \frac{\pi i}{2(p^2+1)} \quad \text{且 } f(t) = \mathcal{L}[F(p)] = \frac{\pi i}{2} e^{-t}. \text{ 令 } t=1 \Rightarrow \text{原积分} = \frac{\pi i}{2e} \end{aligned}$$

$$39. \text{ 计算积分} \oint_{|z|=2} \frac{z^3}{1+z} e^{\frac{1}{z}} dz.$$

$$40. \text{ 设 } f(t) = \begin{cases} \sin t, |t| \leq \pi \\ 0, |t| > \pi. \end{cases}$$

(1) 求 $f(t)$ 的 Fourier 变换 $F(w)$.

(2) 计算广义积分 $\int_0^{+\infty} \frac{\sin wt \sin wt}{1-w^2} dw$. 其中 w 为参数.