Chapter 1: Discrete-time signals and systems.

1. Continuous-time signal / Analog signal $\xrightarrow[x[n] = x_c(nT)]{\text{Sampling}}$ Discrete-time signal / Digital signal

2. Classification of sequences.

Right-sided $x[n]=0$, for $n<N_1$ Two-sided. not right or left

Left-sided. $x[n]=0$. for $n>N_2$.   Finite-length $x[n]=0$. for $n<N_1$ and $n>N_2$. $(N_2>N_1)$

Causal $x[n]=0$. for $n<0$.

Anticasual $x[n]=0$. for $n>0$.

3. Basic sequences.

(1) Unit sample sequence. $\delta[n] = \begin{cases} 1 & n=0 \\ 0, & n\neq 0. \end{cases}$

$\Rightarrow x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$. $\Rightarrow x[n] * \delta[n-n_0] = x[n-n_0]$.

(2) Unit step sequence. $u[n] = \begin{cases} 1, & n\geq 0 \\ 0, & n<0. \end{cases}$

$u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{k=-\infty}^{n} \delta[k]$. $\delta[n] = u[n] - u[n-1]$.

(3) Rectangular sequence.   (4) Exponential sequence. $x[n] = a^n$.

$R_N[n] = \begin{cases} 1, & 0\leq n\leq N-1 \\ 0. & \text{other}. \end{cases}$

(5) Sinusoidal sequence.

alias: $x[n] = A\sin(\omega n+\Phi) = A\sin((\omega-2\pi)n+\Phi))$.

When $\omega: \pi\to 2\pi$. $x[n]$ oscillates like $\omega = -\pi\sim 0$.

When $\omega: 2\pi\to 3\pi$. $x[n]$ oscillates like $\omega = 0\sim\pi$.

only need to consider ω in $[-\pi, \pi)$

① sinusoidal continuous-time signal: $x(t) = A\sin(\Omega t+\Phi)$. $T = \frac{2\pi}{\Omega}$.

② sinusoidal sequence: $x[n] = A\sin(\omega n+\Phi)$.

$$\frac{2\pi}{\omega} = \begin{cases} \text{integer } N: & T=N \\ \text{ration } \frac{P}{Q}: & T=P \\ \text{irration} = & T=\infty \end{cases}$$

4. Symmetry of sequences.

For a real sequence: $\begin{cases} x[n] = x[-n] : \text{ even symmetric} \\ x[n] = -x[-n] : \text{ odd symmetric} \end{cases}$

Any real sequence : $x[n] = x_e[n] + x_o[n] = \dfrac{x[n]+x[-n]}{2} + \dfrac{x[n]-x[-n]}{2}$.

For a complex sequence: $\begin{cases} x[n] = x^*[-n] & \text{Conjugate-symmetric} \\ x[n] = -x^*[-n]. & \text{Conjugate-antisymmetric}. \end{cases}$

Any complex sequence: $x[n] = x_e[n] + x_o[n] = \dfrac{x[n]+x^*[-n]}{2} + \dfrac{x[n]-x^*[-n]}{2}$.

5. Energy of sequences.: $E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n]$.

6. Covolution: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]. = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$.

(1) $x[n] * \delta[n] = x[n]$, $x[n] * \delta[n-n_0] = x[n-n_0]$

(2) If $x[n] \neq 0$. for $N_0 \le n \le N_1$. length $= L_1$.

$h[n] \neq 0$. for $N_2 \le n \le N_3$. length $= L_2$.

then $y[n] = x[n] * h[n]$. for $N_0 + N_2 \le n \le N_1 + N_3$. length $= L_1 + L_2 - 1$.

7. Discrete systems

(1) define: $h[n] = T\{\delta[n]\}$. $s[n] = T\{u[n]\}$.

(2) some systems:

① Amplification: $y[n] = 2x[n]$.

② Mid-value filter: $y[n] = \text{med}\{x[n-1], x[n], x[n+1]\}$.

③ Echo system: $y[n] = x[n] + ax[n-n_d]$.

④ Delay: $y[n] = x[n-5]$.

⑤ Accumulator: $y[n] = \sum_{k=-\infty}^{0} x[n-k]$.

⑥ Moving average: $y[n] = \dfrac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x[n-k]$.

(3) Classification of discrete-time system.

① Memoryless (static) system.
output $y[n]$ depends only on the $x[n]$ at the same $\underline{n}$.

② Linear system: $T\{ax_1[n]+bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$.

③ Time-invariant system.
if $T\{x[n]\} = y[n]$. $T\{x[n-n_0]\} = y[n-n_0]$.

④ Causal system. the output $n=n_0$ only depend on $n \le n_0$.

⑤ stable system. (BIBO).

8. LTI system: $y[n] = T\{x[n]\} = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$
(可线性叠加. 输出不随时间变化).

An LTI system can be completely charaterized by $h[n] \rightarrow T\{\delta[n]\}$.

(1) Classification of LTI system.
① FIR (finite impulse response) $\Rightarrow$ $h[n]$'s length is finite.
② IIR (infinite impulse response).

(2) Causal: $h[n]=0$ for $n<0$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$
$$= \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k].$$

(3) Stable: If $|x[n]| < \infty$. $|T\{x[n]\}| = |y[n]| < \infty$
$\Rightarrow$ Absolutely summable: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]. \le \left|\sum_{k=-\infty}^{\infty} x[n-k]h[k]\right|$$
$$\le \sum_{k=-\infty}^{\infty} |x[n-k]h[k]| \le \max(|x|) \sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

$\rightarrow$ FIR system is always stable.

9. Linear constant-coefficient difference equation.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$N=0$: non-recursive DE: $h[n] = \frac{b_n}{a_0}$. (FIR)

$N>0$: recursive DE: (FIR/IIR)

Example: Accumulator system (IIR).

$$y[n] = \sum_{k=0}^{\infty} x[n-k] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \Rightarrow h[n]=u[n].$$

$\Downarrow$ IIR.

$$y[n-1] = \sum_{k=0}^{\infty} x[n-1-k]. \Rightarrow \text{DE}: y[n]-y[n-1]=x[n].$$

⇒ IIR的差分方程一定有递归.

Example: Moving average system (FIR).

[1]: $y[n] = \frac{1}{M_2+1}\sum_{k=0}^{M_2} x[n-k] = x[n]*h[n]. \Rightarrow h[n] = \begin{cases} \frac{1}{M_2+1}, & 0\le n\le M_2 \\ 0, & \text{other} \end{cases}$ ⇒ FIR

[2]: $y[n]-y[n-1] = \frac{1}{M_2+1}\sum_{k=0}^{M_2} x[n-k] - \frac{1}{M_2+1}\sum_{k=0}^{M_2} x[n-1-k] = \frac{1}{M_2+1}(x[n]-x[n-M_2-1])$.

⇒ FIR的差分方程有递归也可无递归.

Recursive Computation. 以n=0为分界点计算. 若给$\delta[n-1]$. 则以n=1为分界点.

0. $y[n]-ay[n-1]=x[n]$. $y[-1]=1$. $x[n]=\delta[n]$. calc $y[n]$.

$y[0] = ay[-1]+x[0] = a+1$. $y[-2] = \frac{1}{a}(y[-1]-x[-1]) = a^{-1}$.

$y[1] = a(a+1)+0$ $y[-3] = a^{-2}$.

$y[2] = a\cdot a(a+1)+0$. $y[-4] = a^{-3}$.

$\vdots$ $\cdots$

$y[n] = a^n(a+1)$, $n\ge 0$. $y[n] = a^{n+1}$, $n<0$.

0 A continous-time signal $x_c(t) = \sin\frac{5\pi}{4}t$. sampling frequency is 1KHz. from $t=0$ (corresponding to $x[0]$).

(a) Discrete-time sequence $x[n]$.
(b) The period of $x[n]$ for $n\ge 0$.
(c) Calculate convolution with $h[n]=1$ for $-2<n<3$. and $h[n]=0$ otherwise.

⇒ (a) $T = \frac{1}{f} = \frac{1}{1000}$. $x[n] = x_c[nT] = \sin\frac{\pi n}{800}$, $n\ge 0$.

(b). $T_d = \frac{2\pi}{\omega} = 1600$

(c) $x[n]*h[n] = \sum_{k=-2}^{3} h[k]x[n-k] = \sum_{k=-2}^{3} x[n-k] = \sin\frac{n-3}{800} + \sin\frac{n-2}{800} + \sin\frac{n-1}{800} + \sin\frac{n}{800} + \sin\frac{n+1}{800} + \sin\frac{n+2}{800}$.

Chapter 2. Discrete-time Fourier Transform.

1. Discrete-time Fourier transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

IDTFT: $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$. (DTFT has period of 2π).

Compare:
$$\begin{cases} X_c(\Omega) = \int_{-\infty}^{+\infty} x_c(t)e^{-j\Omega t}dt \\ x_c(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_c(\Omega)e^{j\Omega t}d\Omega \end{cases}$$

2. Magnitude and Phase.

The $\mathcal{F}$ Transform is a complex-valued function of ω.

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$$

Magnitude: $|X(e^{j\omega})|$ Phase: $\angle X(e^{j\omega})$

principle phase: $-\pi < ARG(X(e^{j\omega})) < \pi$.

continuous phase: $arg(X(e^{j\omega}))$.

$$X(e^{j(\omega+2\pi)}) = \sum_{-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n} = X(e^{j\omega}) \Rightarrow \text{has period of } 2\pi.$$

3. Convergence: If $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$. then $|X(e^{j\omega})| < \infty$. $\Rightarrow$ absolutely summable.

$\hookrightarrow$ conditions of existence of $\mathcal{F}$ transform.

Some sequences that are not absolutely summable but square summable have $\mathcal{F}$ transforms.

e.g: $x[n] = \frac{\sin(\omega_c n)}{\pi n}$, $-\infty < n < \infty$ $\Rightarrow$ $X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c. \end{cases}$

$\because IFT[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin(\omega_c n)}{\pi n}$.

By introducing $\delta(\omega)$. $u[n]$ and period sequences $\Rightarrow$ have $\mathcal{F}$ transform

neither absolutely summable nor square summable.

e.g: $x[n]=1$. $\Rightarrow$ $X(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} 2\pi\delta(\omega+2\pi r)$. $\cancel{= 2\pi\delta(\omega)}$

$\Downarrow$

$X(e^{j\omega})$以2π为周期.

4. Relationship between DTFT and Z transformation.

$$X(e^{j\omega}) = X(z)\big|_{|z|=1}, \text{ that is } z=e^{j\omega}$$

$r \leq 1$

$$\Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] (re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n} = FT[x[n] r^{-n}].$$

5. Properties of DTFT.

(1) $ax[n] + by[n] \longleftrightarrow aX(e^{j\omega}) + bY(e^{j\omega})$.

(2) $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$ (time shifting)

(3) $e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$ (frequency shifting)

(4) $x[-n] \longleftrightarrow X(e^{-j\omega})$, $x[n] \longleftrightarrow X^*(e^{j\omega})$. (time reversal for real sequency).

(5) $x[n] * y[n] \longleftrightarrow X(e^{j\omega}) Y(e^{j\omega})$. (convolution).

(6) $x[n] y[n] \longleftrightarrow \frac{1}{2\pi} X(e^{j\omega}) * Y(e^{j\omega})$ (windowing)

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta \text{ (periodic convolution).}$$

(7) $\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$

$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$ ($X(e^{j\omega}) = Y(e^{j\omega})$). (Parseval theorem).

$|X(e^{j\omega})|^2$: Energy spectral Density. 能量谱密度.

$\Downarrow$

unit: $J \cdot s = J/Hz$

$|x_c(t)|^2 \Rightarrow$ unit: J/s.

(8)* Symmetry.

$$\begin{cases} x^*[n] \longleftrightarrow X^*(e^{-j\omega}). \\ x[-n] \longleftrightarrow X(e^{-j\omega}). \\ x^*[-n] \longleftrightarrow X^*(e^{j\omega}). \end{cases}$$

Proof:
$$\begin{cases} X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[-n]e^{j\omega(-n)} = \sum_{n=-\infty}^{\infty} x[-n]e^{-j(-\omega)(-n)} \\ \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} = [\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\omega)n}]^* = X^*(e^{-j\omega}). \end{cases}$$

$$\begin{cases} Re(x[n]) \longleftrightarrow X_e(e^{j\omega}) = \dfrac{X(e^{j\omega})+X^*(e^{-j\omega})}{2}. \\ jIm(x[n]) \longleftrightarrow X_o(e^{j\omega}) = \dfrac{X(e^{j\omega})-X^*(e^{-j\omega})}{2}. \\ x_e[n] \longleftrightarrow Re[X(e^{j\omega})] = \dfrac{X(e^{j\omega})+X^*(e^{j\omega})}{2}. \\ x_o[n] \longleftrightarrow jIm[X(e^{j\omega})] = \dfrac{X(e^{j\omega})-X^*(e^{j\omega})}{2}. \end{cases}$$

$$\boxed{\begin{array}{c} x[n] = x_e[n] + x_o[n] \\ \Downarrow \\ X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}). \end{array}}$$

(9) Symmetry for real sequences.

$$X(e^{j\omega}) = X^*(e^{-j\omega}) = X^*(e^{-j(\omega-2\pi)}) = X^*(e^{j(2\pi-\omega)}).$$

实序列: $Re[X(e^{j\omega})] = Re[X(e^{-j\omega})]$ $\quad |X(e^{j\omega})| = |X(e^{-j\omega})|$ (偶对称)

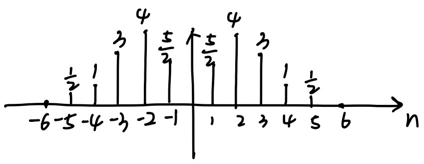
$Im[X(e^{j\omega})] = -Im[X(e^{-j\omega})].$ $\quad \angle X(e^{j\omega}) = -\angle X(e^{-j\omega}).$ (奇对称)

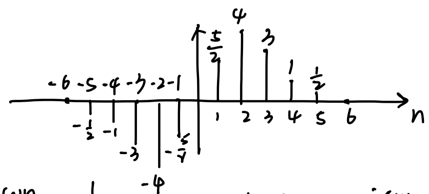
Example:

(1) $x[n] = x_e[n] + x_o[n]$.

(2) Calc the DTFT of $x_e[n]$ and $x_o[n]$.

$\Rightarrow$ (1) $x_e[n] = \dfrac{x[n]+x[-n]}{2}$ $\qquad x_o[n] = \dfrac{x[n]-x[-n]}{2}$





(2). $DTFT[x_e[n]] = \sum_{n=-6}^{6} x_e[n]e^{-j\omega n} = \frac{1}{2}[e^{j5\omega}+e^{-j5\omega}] + [e^{j4\omega}+e^{-j4\omega}] + 3[e^{j3\omega}+e^{-j3\omega}) + 4[e^{j2\omega}+e^{-j2\omega}) + \frac{5}{2}(e^{j\omega}+e^{-j\omega}]$

$DTFT[(x_o[n])] = \sum_{n=-6}^{6} x_o[n]e^{j\omega n} = \frac{1}{2}[e^{j5\omega}-e^{-j5\omega}] + [e^{j4\omega}-e^{-j4\omega}] + 3[e^{j3\omega}-e^{-j3\omega}) + 4[e^{j2\omega}-e^{-j2\omega}) + \frac{5}{2}(e^{j\omega}-e^{-j\omega}]$

(3) $X(e^{j\omega}) = e^{-j5\omega} + 2e^{-j4\omega} + 6e^{-j3\omega} + 8e^{-j2\omega} + 5e^{-j\omega}$.

则有: $DTFT(x_e[n]) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{j\omega})]. = X_R(e^{j\omega})$ Real even symmetric

$DTFT(x_o[n]) = \frac{1}{2}(X(e^{j\omega}) - X^*(e^{j\omega})]. = jX_I(e^{j\omega}).$ Pure imaginary. odd symmetric

6. Discrete Fourier Series (DFS)

(1) Discretize

(2) DFS (离散傅里叶级数).

For periodic sequence $\tilde{x}[n]$. $\Rightarrow$ $\tilde{x}[n] = \tilde{x}[n+rN]$. has a period of N.

Fourier Series:

$$\begin{cases} \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}, \ k=-\infty,\dots\infty. & (W_N^{kn} = e^{-j2\pi kn/N}) \\ \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}, \ n=-\infty,\dots\infty. \end{cases}$$

连续周期信号: $x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t}$, $\omega_s = \frac{2\pi}{T}$, $F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\omega_s t} dt$

$e^{jn\omega_s t} \longrightarrow 2\pi\delta(\omega - n\omega_s)$ 则 $x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_s t} \longrightarrow 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_s)$

$\tilde{X}[k]$ has a period of N: $\tilde{X}[k+rN] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(k+rN)n} = \tilde{X}[k]$.

(7) Fourier transform of periodic signals.

From $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi k n}{N}}$, $n=-\infty,\dots\infty$. to: $\tilde{X}(e^{j\omega}) = \sum_{k=0}^{N-1} \frac{2\pi}{N} \tilde{X}[k] \delta(\omega - \frac{2\pi k}{N})$. $\omega \in [0, 2\pi)$.

Fourier Series

$\tilde{X}(e^{j\omega}) = \sum_{k=0}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta(\omega - \frac{2\pi k}{N})$. $\omega \in (-\infty, \infty)$. 频谱离散

$\Rightarrow$ DFS can be used to calculate the spectrum of periodic signals.

① Fourier Series $\tilde{X}[k]$. ② Period $= N$.

(8) Properties of DFS.

① Linearity: $a\tilde{x}_1[n] + b\tilde{x}_2[n] \xleftrightarrow{DFS} a\tilde{X}_1[k] + b\tilde{X}_2[k]$.

合成序列的周期为分周期的最小公倍数. $N_1=4$. $N_2=6$. 则 $N=12$.

② Time-shifting: $\tilde{x}[n-m] \longleftrightarrow W_N^{km}\tilde{X}[k]$.

③ Frequency-shifting: $W_N^{-nl}\tilde{x}[n] \longleftrightarrow \tilde{X}[k-l]$.

④ Duality: $\tilde{x}[n] \longleftrightarrow N\tilde{x}[-k]$.

⑤ Symmetry:

$\tilde{x}^*[n] \longleftrightarrow \tilde{X}^*[-k]$, $\tilde{x}^*[-n] \longleftrightarrow \tilde{X}^*[k]$.

$$\begin{cases} Re(\tilde{x}[n]) = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[n]) \longleftrightarrow \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k]) = \tilde{X}_e[k]. \\ jIm(\tilde{x}[n]) = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[n]) \longleftrightarrow \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k]) = \tilde{X}_o[k]. \end{cases}$$

实部 ⇌ 偶分量
虚部 ⇌ 奇分量

$$\begin{cases} \tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n]) \longleftrightarrow \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[k]) = Re(\tilde{X}[k]). \\ \tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n]) \longleftrightarrow \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[k]) = jIm(\tilde{X}[k]). \end{cases}$$

⑥ Symmetry for real sequences.

$\tilde{x}[n]=\tilde{x}^*[n] \Rightarrow \tilde{X}[k]=\tilde{X}^*[-k]=\tilde{X}^*[N-k]$.

$\Rightarrow Re(\tilde{X}[k])=Re(\tilde{X}[-k]) \Rightarrow |\tilde{X}[k]|=|\tilde{X}[-k]|$ ⇒ $\tilde{X}[k]$相对$k=0$, $k=N/2$共轭对称.

$Im(\tilde{X}[k])=-Im(\tilde{X}[k])$ $\qquad \angle\tilde{X}[k]=-\angle\tilde{X}[-k]$.

7. Periodic convolution.

Define: $\tilde{x}_3[n]=\tilde{x}_1[n]\circledast\tilde{x}_2[n]=\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$

$\tilde{x}_1[n], \tilde{x}_2[n], \tilde{x}_3[n]$ all have period of N. (p36).

properties: $\tilde{x}_3[n]=\tilde{x}_1[n]\circledast\tilde{x}_2[n] \Rightarrow \tilde{X}_3[k]=\tilde{X}_1[k]\tilde{X}_2[k]$.

$\tilde{x}_3[n]=\tilde{x}_1[n]\tilde{x}_2[n] \Rightarrow \tilde{X}_3[k]=\frac{1}{N}\tilde{X}_1[k]\circledast\tilde{X}_2[k]$.

8. Different types of FT.

(1) Continuous Time - Continuous Freq.

(2) Discrete Time - Continuous Freq.

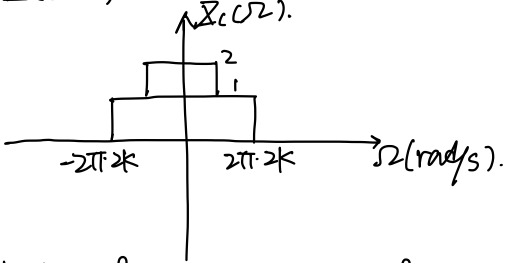
(3) Discrete Time - Discrete Freq.

0 (p39). A continuous-time signal has the following spectrum $X_c(\Omega)$. in the freq. domain.

1. Calc the $x_c(t)$.

2. Sampling frequency is 4k Hz. calc $x[n]$ and $X(e^{jw})$

3. Compare $X_c(\Omega)$. $X(e^{jw})$



$\Rightarrow$ 1. $X_c(\Omega)=\begin{cases}1, & |\Omega|\le 2\pi f_0.\\ 0, & \text{otherwise}.\end{cases}+\begin{cases}1. & |\Omega|\le\pi f_0.\\ 0, & \text{otherwise}.\end{cases}$ where $f_0=2k(Hz)$.

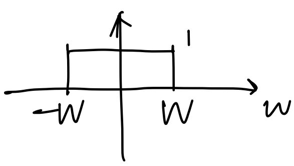
or $X_c(\Omega)=\begin{cases}2, & |\Omega|\le\pi f_0\\ 1, & \pi f_0<|\Omega|\le 2\pi f_0.\\ 0, & \text{otherwise}.\end{cases}$ where $f_0=2k$ Hz.

then $x_c(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X_c(\Omega)e^{j\Omega t}d\Omega.=\frac{1}{2\pi}\int_{-\pi f_0}^{\pi f_0}e^{j\Omega t}d\Omega+\frac{1}{2\pi}\int_{-2\pi f_0}^{2\pi f_0}e^{j\Omega t}d\Omega$

$=\frac{e^{j\pi f_0t}-e^{-j\pi f_0t}}{2j\pi t}+\frac{e^{2j\pi f_0t}-e^{-2j\pi f_0t}}{2j\pi t}=\frac{\sin(2\pi f_0t)}{\pi t}+\frac{\sin(\pi f_0t)}{\pi t}$

2. $T=\frac{1}{f}=\frac{1}{4000}$. $x[n]=x_c[nT]=\frac{\sin(\pi n)}{\pi nT}+\frac{\sin(\frac{\pi n}{2})}{\pi nT}$

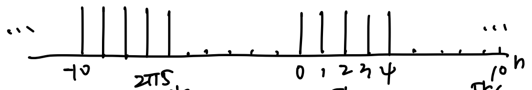
A $\frac{\sin Wt}{\pi t}\xrightarrow{\mathcal{F}}$



3. $X(e^{j\omega}) = \begin{cases} \frac{1}{T}, & |\omega| \le \pi \\ 0, & \text{otherwise.} \end{cases} + \begin{cases} \frac{1}{T}, & |\omega| \le \frac{\pi}{2} \\ 0, & \text{otherwise.} \end{cases}$

4. $X(e^{j\omega}) = \frac{1}{T} X_c(\Omega)\Big|_{\Omega=\frac{\omega}{T}}, \ |\omega| < \pi.$

O (P28) Given periodic sequence:



$\Rightarrow N=10$

$$\Rightarrow \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} = \sum_{n=0}^{4} W_{10}^{kn} = \frac{1-e^{-j\frac{2\pi}{10}5k}}{1-e^{-j\frac{2\pi}{10}k}} = \frac{1-W_{10}^{5k}}{1-W_{10}^{k}} = \frac{W_{10}^{5k/2}(W_{10}^{-5k/2}-W_{10}^{5k/2})}{W_{10}^{k/2}(W_{10}^{-k/2}-W_{10}^{k/2})} \quad \text{(DFS)}$$

$$= \frac{W_{10}^{k5/2}\cdot 2j\sin(2\pi k5/20)}{W_{10}^{k/2}\cdot 2j\sin(2\pi k/20)} = \underbrace{e^{-j4\pi k/10}}_{\text{广义线性相位}} \cdot \underbrace{\frac{\sin(\pi k/2)}{\sin(\pi k/10)}}_{\text{广义幅度}}$$

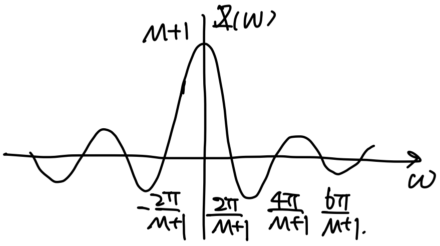
In general. given a rectangular sequence with length of $(M+1)$: $x[n] = \begin{cases} 1, & 0 \le n \le M. \\ 0. & \text{other.} \end{cases}$

FT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = \frac{\sin(\omega\cdot\frac{M+1}{2})}{\sin(\frac{\omega}{2})} e^{-j\omega\frac{M}{2}}$ (DTFT)

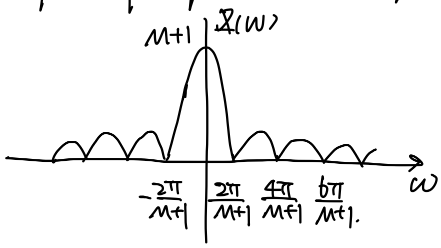
广义幅度: $X(\omega) = \frac{\sin(\omega(M+1)/2)}{\sin \omega/2}$. 幅度谱: $|X(\omega)|$. 广义相位: $e^{-j\omega\frac{M}{2}}$ (线性相位)

给定M值，使$W(\omega)=0$的最小频率值为 $\omega = \pm\frac{2\pi}{M+1}$. 则幅度谱的主瓣宽是 $\frac{4\pi}{M+1}$.

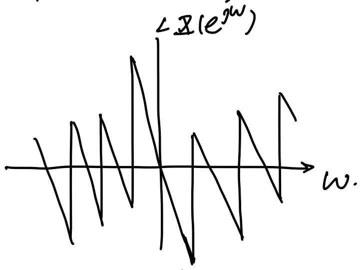
代入$\omega=0$可得主瓣的峰值幅度为 $X(\omega)|_{\omega=0} = M+1$.



广义幅度



幅度



相位.

$\Rightarrow$ 幅度为负的频段 相位$\pm\pi$.

Chapter 03 Z Transform.

1. The z-transform of a sequence $x[n]$ is defined as:

$$X(z)=\sum_{n=-\infty}^{\infty} x[n] z^{-n},\ z=re^{j\omega}\ \text{(complex variable)}$$

$$x[n]=\frac{1}{2\pi j}\int_C X(z) z^{n-1} dz \qquad C: 围线积分\ e^{j\omega}\ \omega=-\pi\to\pi$$

⇒ r 为收敛域半径. 当 $r=1$ 时. $X(z)=X(e^{j\omega})$. 相当于傅立叶变换 (单位圆).

2. ROC is bounded with the poles.

3. z-transforms:

(1) $x[n]=a^n u[n] \longrightarrow X(z)=\frac{1}{1-az^{-1}},\ |z|>|a|$. (Right-sided)

在ROC内 $X(z)$ 的闭式表达式才有意义. ROC内 $X(z)$ 一定不发散. ROC外 $X(z)$ 一定发散. 其闭式表达式不一定发散. 发散只发生在极点处.

(2) $x[n]=-a^n u[-n-1] \longrightarrow X(z)=\frac{1}{1-az^{-1}}.\ |z|<|a|$. (Left-sided).

4. properties:

(1) $\begin{cases} \text{Causal} \Rightarrow \text{ROC contains } z=\infty. \\ \text{Anti-causal} \Rightarrow \text{ROC contains } z=0. \end{cases}$

(2) ROC of finite-length sequence is the whole plane.
Stable / FT exists ⇒ ROC contains $|z|=1$.

(3) $nx[n] \longrightarrow -z\frac{dX(z)}{dz}$.

o Example: $u[n]\to\frac{1}{1-z^{-1}}.\ |z|>1$. then $nu[n]\longrightarrow\frac{z^{-1}}{(1-z^{-1})^2}.\ |z|>1$.

(4) $x^*[n]\longrightarrow X^*(z^*)$. $\begin{cases} Re(x[n])\longrightarrow \frac{X(z)+X^*(z^*)}{2} \\ jIm(x[n])\longrightarrow \frac{X(z)-X^*(z^*)}{2} \end{cases}$.

实数序列 z 变换的零极点共轭对称.

(5) $x[-n]\longrightarrow X(\frac{1}{z})$. $x^*[-n]\longrightarrow X^*(\frac{1}{z^*})$.

实偶序列的零极点共轭对称且倒数 (反演) 4个一组.

(6) ① $x[n]$ is causal: $x[0]=\lim_{z\to\infty}X(z)$. $\lim_{n\to\infty}x[n]=\lim_{z\to1}(z-1)X(z)$. (终值/零值定理)

② $x[n]$ is anti-causal: if $x[0]=0,\ n>0$. then $x[0]=\lim_{z\to0}X(z)$.

5. Inverse z-transform.

(1) Partial fraction expansion: $X(z) = \frac{N(z)}{D(z)} = \sum_{m=1}^{N} \frac{A_m}{1-z_m z^{-1}}$, $A_m = (1-z_m z^{-1}) X(z) \big|_{z=z_m}$

(2) Long division

(3) Power series.(幂指数序列展开).

Chapter 4 Transform Analysis of Linear Time-Invariant System.

1. System function 系统函数：

(1) $H(z)=\sum_{n=-\infty}^{\infty}h[n]z^{-n}$. For LTI system: $Y[n]=H[n]X[n]$.

(2) Difference equation.

For a linear constant coefficient difference equation of an LTI system:

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k]$$

causal sequences.

The $H(z)$ has the form: $H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M}b_k z^{-k}}{\sum_{k=0}^{N}a_k z^{-k}}$.

(3) Initial rest condition 初始松弛条件.

if $x[n]=0$ for $n<n_0$. then $y[n]=0$ for $n<n_0$.

⇒ 该条件体现了LTI系统的因果性. 因果LTI系统一定满足该条件.

Difference equation + initial rest condition ⇒ causal LTI system.

2. Properties of zeros. poles. ROC.

(1) 若实系数或 $h[n]$ 为实序列. 则复数零点(极点)两两共轭. $H(z)=H^*(z^*)$

(2) FIR: $h[n]$ 有限长.

对有限长序列(该序列作为 $h[n]$ 时对应FIR系统). 其ROC为整个平面($z=0.\infty$ 可能例外.
因此该系统一定稳定, 除 $z=0.\infty$ 外没有其他极点, 则 $H(z)$ 为整式. 对应差分方程可以无递归.
(全零点型)

$\Rightarrow H(z)=\sum_{k=0}^{M}h[k]z^{-k}$. $y[n]=\sum_{k=0}^{M}h[k]x[n-k]=\sum_{k=0}^{M}b_k x[n-k]$ (差分方程)

(3) IIR: $h[n]$ 无限长.

此时ROC不是整个平面. 因此存在除 $z=0.\infty$ 以外的其他极点. $H(z)$ 为有理分式
差分方程有递归. 若除 $z=0$ 外没有零点. 为全零点型

3. Frequency Response (频响)

(1) Define: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$. $\Rightarrow$ DTFT$\{h[n]\}$.

For LTI system: $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$.

$\Rightarrow$ 只有当系统稳定时才存在频响 ($z=e^{j\omega}$).

$\Downarrow$

4. Magnitude response.

$$\boxed{H(e^{j\omega}) = |H(e^{j\omega})|e^{\angle H(e^{j\omega})}}$$

(1) Define: $|H(e^{j\omega})|$.

Magnitude square function: $|H(e^{j\omega})|^2 = H(e^{j\omega})^* H(e^{j\omega})$.

(2) Gain (dB): $10\log_{10}|H(e^{j\omega})|^2 = 20\lg|H(e^{j\omega})|$.

5. Phase response.

(1) Define: $\angle H(e^{j\omega})$.

(2) Principle value phase: $-\pi \underline{\underline{<}} ARG(H(e^{j\omega})) \underline{\underline{\le}} \pi$

Continuous phase: $\arg(H(e^{j\omega}))$.

(3) Group delay (群延迟) 单位: 采样点, $= grd[H(e^{j\omega})] = -\dfrac{d\arg(H(e^{j\omega}))}{d\omega}$

$\Rightarrow |Y(e^{j\omega})| = |H(e^{j\omega})|\cdot|X(e^{j\omega})|$. $\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$

$grd\{Y(e^{j\omega})\} = grd\{H(e^{j\omega})\} + grd\{X(e^{j\omega})\}$.

- 幅度响应 $|H(e^{j\omega})|$ 对信号的幅度进行调整.
- 相位响应 $\angle H(e^{j\omega})$ 对信号的相位进行调整.
- 群延迟 $grd\{H(e^{j\omega})\}$ 对信号作时域上的移位. 单位是采样点,

6. Complex exponential input.

(1) $x[n] = e^{j\omega_0 n}$, $-\infty < n < \infty$ Eigenvector

$y[n] = e^{j\omega_0 n}H(e^{j\omega_0})$. Eigenvalue $\Rightarrow$ 改变幅值和相位.

(2) $x[n] = \sum_k a_k e^{jk\omega_0 n}$ non Eigenvector.

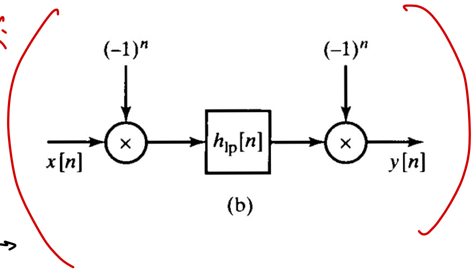
$y[n] = \sum_k a_k e^{jk\omega_0 n}H(e^{jk\omega_0})$.

(3) $x[n] = e^{j\omega_0 n}u[n]$ non Eigenvector

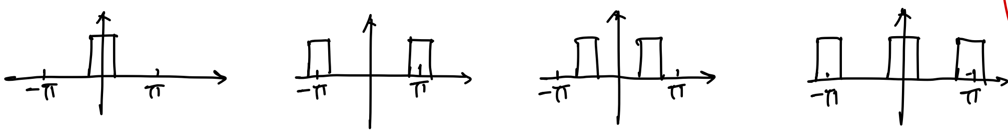
(此时需用定义计算 $y[n]$)

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_0(n-k)}u[n-k] = \underbrace{H(e^{j\omega_0})e^{j\omega_0 n}}_{\text{稳态响应}} - \underbrace{\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega_0 k}\right)e^{j\omega_0 n}}_{\text{瞬态响应}}.$$

若系统稳定，瞬态响应最终会消失.

※ 

7. Ideal frequency-selective filter.



8. Frequence response for rational system function.

$$\Rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = B z^{N-M}\frac{\prod_{k=1}^{M}(z-c_k)}{\prod_{k=1}^{N}(z-d_k)}$$

c_k：零点，d_k：极点.

Example: PPT p31

9. All-pass system.

(1) Define: the frequency-response magnitude is constant.

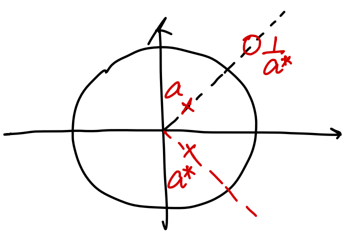
$$|H_{ap}(e^{j\omega})| = \text{constant} \quad (\text{not } H(e^{j\omega}) = \text{constant}).$$

◦ Example: $H(z) = \frac{z^{-1}-a^*}{1-az^{-1}} = \frac{1-a^*z}{z-a}$, $\Rightarrow \begin{cases} \text{zero}: a^{*-1} = r^{-1}e^{j\theta} \\ \text{pole}: a = re^{j\theta} \end{cases}$

方向相同，幅度相乘等于1

let $a = \frac{1}{2}e^{j\frac{\pi}{4}}$.

then zeros and poles:



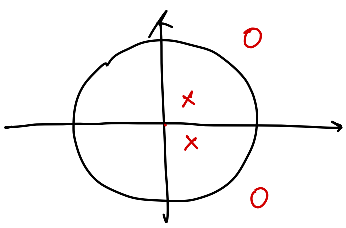
$\Rightarrow a^* = \frac{1}{2}e^{-j\frac{\pi}{4}}$. $\frac{1}{a^*} = 2e^{j\frac{\pi}{4}}$.

⇒ 全通系统零极点一定分布在单位圆两侧，(共轭反演)

$H(z) = z$ 时，零点为原点，极点为无穷远.

$H(z) = z^{-1}$ 时 ……

特别地，若为实系数或实序列：零点对互为共轭，极点对互为共轭.



10. Minimum-phase system 最小相位系统.

(1) Inverse system: $H(z)H_i(z)=1$ / $H_i(z)=H^{-1}(z)$.

ROC of $H(z)\cdot H_i(z)$ must overlap.

一个因果稳定系统,其逆系统不一定因果稳定.(零点在单位圆外时).

零极点全在单位圆内,其逆系统也因果稳定.此时该系统为最小相位系统.

全通系统不能保证其逆系统因果稳定.

(2) 最小相位系统

⇒ 零极点全在单位圆内(不能有 $z=\infty$ 这个零极点,零点不能在单位圆之外).

↳ 例如 $H(z)=z$ 或 z^{-1}.

其单位冲激响应 $h[n]$ 称为最小相位序列.

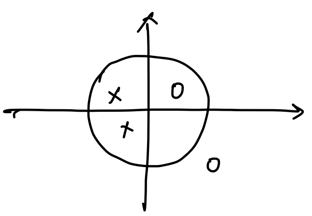
最大相位系统: 所有零极点在单位圆外.

(3) An rational system can be divided:

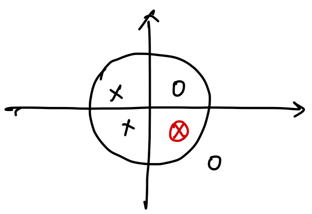
$$H(z)=H_{min}(z)H_{ap}(z).$$ (添加系数使 $|H_{ap}(z)|=1$).

⇒ $|H(e^{j\omega})|=|H_{min}(e^{j\omega})|$, 经过全通系统,幅度不变,相位产生失真.

o Example:

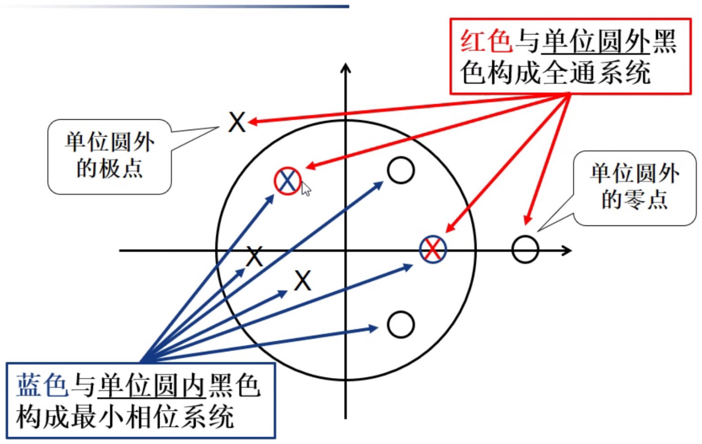


⇒



⇒ 一个全通系统和最小相位系统的叠加.

Minimum-phase and all-pass decomposition



黑色:原来的零/极点

红色蓝色:单位圆外零/极点的共轭反演位置的零极点

0 (1) $H(z)=\dfrac{1+4z^{-1}}{1-0.2z^{-1}}$ (2) $H(z)=\dfrac{(1+2jz^{-1})(1-2jz^{-1})}{1-0.5z^{-1}}$.

$$\Rightarrow (2)\ H(z)=\frac{(z+2j)(z-2j)}{z(z-\frac{1}{2})} = \frac{(z+2j)(z-2j)}{z(z-\frac{1}{2})}\cdot\frac{z+\frac{1}{2}j}{z+\frac{1}{2}j}\cdot\frac{z-\frac{1}{2}j}{z-\frac{1}{2}j}$$

$$=\underbrace{4\frac{(z+\frac{1}{2}j)(z-\frac{1}{2}j)}{z(z-\frac{1}{2})}}_{\text{最小相位系统}}\cdot\underbrace{\frac{(z+2j)(z-2j)}{(z+\frac{1}{2}j)(z-\frac{1}{2}j)}\cdot\frac{1}{4}}_{\text{全通系统}}$$

⇓ 幅值为1 代入 $z=j$.

11. Amplitude-response compensation.

(1) Distorting system: $H_d(z)=H_{min}(z)H_{ap}(z)$.

Let $H_c(z)=\dfrac{1}{H_{min}(z)}$. then $G(z)=H_d(z)H_c(z)=H_{ap}(z)$.

(补偿系统) 使得对幅值不产生影响, 相位会失真.

(2) 最小相位系统具有最小的相位延迟和群延迟.

① phase delay: $-\arg[H_{min}(e^{j\omega})] < -\arg[H(e^{j\omega})]$.

② group delay: $grd[H_{min}(e^{j\omega})] < grd[H(e^{j\omega})]$.

在最小相位系统后边级联任意多个全通系统可组成多个具有相同幅度响应的系统.
即全通系统的群延迟一定大于0: $grd[H_{ap}(e^{j\omega})]>0$

12. Linear-phase system (线性相位系统).

(1) $\Rightarrow$ $H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\text{nonegative}}\underbrace{e^{-j\omega\alpha}}_{\text{线性相位}}$, $|\omega|<\pi$.

$\Rightarrow$ $\arg[H(e^{j\omega})]=-\omega\alpha$.

$grd[H(e^{j\omega})]=\alpha$ (real).

0 Example: Ideal delay system.

$h_{id}[n]=\delta[n-m] \Rightarrow H_{id}(e^{j\omega})=e^{-j\omega m}$, $|\omega|<\pi$ ⇒ 作用是延迟m个采样点.

positive or negative

(2) $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha+j\beta}$, $|\omega|<\pi$. ⇒ Generalized linear phase system.

⇒ $A(e^{j\omega})$ is real function (广义幅度).

$\Phi(\omega)=-\omega\alpha+\beta$ (广义相位).

$grd[H(e^{j\omega})]=\alpha$ (real). 常数群延迟.

○ Example: Hilbert transform.

$$H(e^{j\omega})=\begin{cases}-j=e^{-j\frac{\pi}{2}}, & 0<\omega<\pi\\ j=e^{j\frac{\pi}{2}}, & -\pi\le\omega<0.\end{cases} \Rightarrow \begin{cases}A(e^{j\omega})=\begin{cases}-1, & 0<\omega<\pi\\ 1, & -\pi\le\omega<0\end{cases}\\ \phi(\omega)=\frac{\pi}{2}\end{cases} \text{ 或 } \begin{cases}A(e^{j\omega})=1\\ \phi(\omega)=\begin{cases}-\frac{\pi}{2}, & 0<\omega<\pi\\ \frac{\pi}{2}, & -\pi\le\omega<0.\end{cases}\end{cases}$$

○ Example: Differenitator (微分器).

$$H(e^{j\omega})=j\omega=\omega e^{j\frac{\pi}{2}}. \Rightarrow \begin{cases}A(e^{j\omega})=\omega\\ \phi(\omega)=\frac{\pi}{2}\ (\alpha=0, \beta=\frac{\pi}{2})\end{cases} \text{ 或. } \begin{cases}A(e^{j\omega})=|\omega|\\ \phi(\omega)=\begin{cases}\frac{\pi}{2}, & 0<\omega<\pi\\ -\frac{\pi}{2}, & -\pi\le\omega<0\end{cases}\end{cases}$$

(3) Conditions of linear phase system. (ppt-P49).

① 必要条件: $\sum_{n=-\infty}^{+\infty} h[n]\sin(\omega(n-\alpha)+\beta)=0.$

② 充分条件:

if $h[M-n]=h[n]$. M is an integer. then $\begin{cases}\beta=0 \text{ or } \pi\\ \alpha=\frac{M}{2}.\end{cases}$ (关于$\frac{M}{2}$偶对称)

if $h[M-n]=-h[n]$. M is an integer. then $\begin{cases}\beta=\frac{\pi}{2} \text{ or } \frac{3\pi}{2}\\ \alpha=\frac{M}{2}.\end{cases}$ (关于$\frac{M}{2}$奇对称).

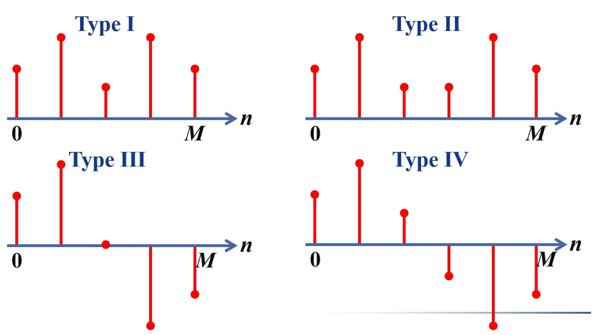
(4) Causal generalized linear phase (FIR) system.

对因果系统. 其为线性相位系统的条件为(充分条件):

并且该系统一定是有限长的. 其长度为 $M+1$. 对称中心为 $\frac{M}{2}$.

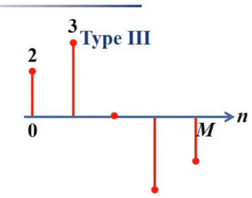
$h[n]=\pm h[M-n]$. $0\le n\le M$. $\text{len}\{h[n]\}=M+1$.

奇偶对称 序列长度的奇偶. ⇒ 4种组合



Causal generalized linear phase (FIR) system

Exercise



$h[n]=2\delta[n]+3\delta[n-1]-3\delta[n-3]-2\delta[n-4]$

$H(e^{j\omega})=2+3e^{-j\omega}-3e^{-3j\omega}-2e^{-4j\omega}$

$=e^{-2j\omega}(2e^{2j\omega}+3e^{j\omega}-3e^{-j\omega}-2e^{-2j\omega})$

$=e^{-2j\omega}(4j\sin 2\omega+6j\sin\omega)$ ⇒ 幅度表达式都是以sin连加.

$\omega=0,\pi$ → 固定零点

对称中心.

△ 群延迟为2 (对称中心).

相位为$(-2\omega+\frac{\pi}{2})$. ⇒ $je^{-2j\omega}$

幅度为 $4\sin 2\omega$. $6\sin\omega$.

结论：

Amplitude response

$$TypeI: A(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cos\left[\left(\frac{M}{2}-n\right)\omega\right] = \sum_{k=0}^{M/2} a[k]\cos(k\omega)$$

$$where: a[0]=h[M/2], a[k]=2h[M/2-k], k=1,2...M/2$$

$$TypeII: A(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cos\left[\left(\frac{M}{2}-n\right)\omega\right] = \sum_{k=1}^{(M+1)/2} b[k]\cos(\omega(k-1/2))$$

$$where: b[k]=2h[(M+1)/2-k], k=1,2...(M+1)/2$$

$$TypeIII: A(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cos\left[\left(\frac{M}{2}-n\right)\omega-\frac{\pi}{2}\right] = \sum_{k=1}^{M/2} c[k]\sin(k\omega)$$

$$where: c[k]=2h[M/2-k], k=1,2...M/2$$

$$TypeIV: A(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cos\left[\left(\frac{M}{2}-n\right)\omega-\frac{\pi}{2}\right] = \sum_{k=1}^{(M+1)/2} d[k]\sin(\omega(k-1/2))$$

$$where: d[k]=2h[(M+1)/2-k], k=1,2...(M+1)/2$$

$$A(e^{j\omega}) = \sum_{n=0}^{M} h[n]\cos\left[\left(\frac{M}{2}-n\right)\omega-\frac{\pi}{2}\right] \qquad \Phi(\omega) = -\frac{M}{2}\omega+\frac{\pi}{2}$$

由于幅度值均由cos, sin的迭加构成，所以频谱有固定零点。

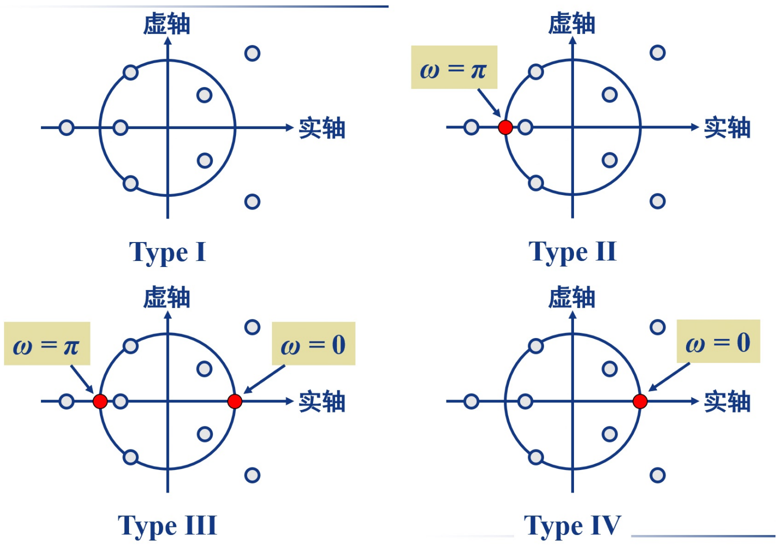
Type I：没有固定零点。　　Type II：有一个固定零点 $w=\pi$.

Type III：有两个固定零点：$w=0,\pi$　　Type IV：有一个固定零点，$w=0$.

⇒ Type II 不能做高通滤波器。Type III 只能做带通滤波器。

固定零点：

Fixed zeros



Properties of zeros of causal generalized linear phase system.

When $h[n]$ is even symmetric (Type I & II): $H(z) = z^{-M}H(z^{-1})$.

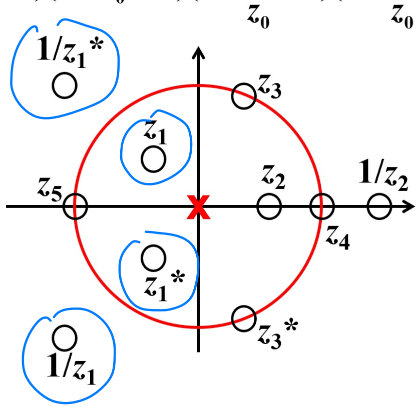
odd symmetric (Type III & VI): $H(z) = -z^{-M}H(z^{-1})$.

代入 $z=1,-1$ 可得到不同情况下的固定零点.

If $h[n]$ is real, 在 z_1, $\frac{1}{z_1}$ 是零点的基础上, 其共轭 z_1^*, $\frac{1}{z_1^*}$ 也是零点.

△广义线性相位系统一定是FIR系统, 只有零点.

$$(1 - z_0 z^{-1})(1 - z_0^* z^{-1})(1 - \frac{1}{z_0} z^{-1})(1 - \frac{1}{z_0^*} z^{-1})$$



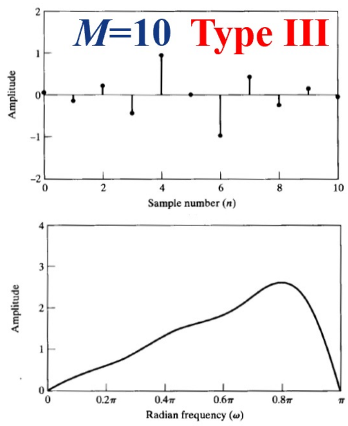
⇒存在一些简并情况.

一般情况是4个一对出现.

○ Example: choice of $h[n]$ length (PPT P64).

Limitation on filter application

Type I	
Type II	$H(\pi) = 0$
Type III	$H(0) = 0$, $H(\pi) = 0$
Type IV	$H(0) = 0$



	M是偶数				M是奇数			
	低通	高通	带通	带阻	低通	高通	带通	带阻
h[n]偶对称	I Y	Y	Y	Y	II Y	N	Y	N
h[n]奇对称	III N	N	Y	N	IV N	Y	Y	N

Chapter 5. Sampling of continuous-time signals.

1. Ideal Sampling: $x[n] = x_c(nT)$

Mathmatical model: $x_c(t) \longrightarrow \boxed{C/D} \longrightarrow x(n) = x_c(nT)$. (with T into C/D)

$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$

$\Rightarrow x_c(t) \longrightarrow \otimes \longrightarrow x_s(t) \longrightarrow$ 冲击函数序列转换为离散序列 $\longrightarrow x[n]$.

step1: $x_s(t) = x_c(t)\cdot s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$

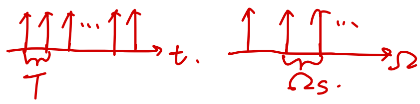
相乘.不是相卷　　　　冲激函数序列.

step2:

sampling period: T (second).
sampling rate: $f_s=\frac{1}{T}$ (samples/second)
sampling rate: $\Omega_s=\frac{2\pi}{T}$. (rad/second)

$\Rightarrow s(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT) \xrightarrow{\mathcal{F}} S(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(\Omega-n\Omega_s)$ $\left(S(\omega)=\sum_{n=-\infty}^{\infty}\omega_s\delta(\omega-n\omega_s)\right)$.

↳ 时域上的周期为T(s). 频域上的周期为$\Omega_s=\frac{2\pi}{T}$ (rad/s).



时域: $x_s(t) = x_c(t)\cdot s(t)$. $\Rightarrow$ 频域: $X_s(\Omega) = X_c(\Omega) * S(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\Omega-n\Omega_s)$

$\Rightarrow$时域归一化: $n=\frac{t}{T}$. $x_s(t) = x_c(nT)\,\delta(t-nT) = x[n]$.

频域归一化: $\omega=\Omega\cdot T$. $X_s(\Omega) = X(e^{j\omega}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(\frac{\omega}{T}-n\frac{2\pi}{T})$.

(转换前. $x_s(nT)$的横坐标为t. 冲激函数序列分布在采样周期T的整数倍 $t=nT$ (s)上. 转换后, 横坐标乘以$\frac{1}{T}$. 用$n=1,2,\cdots$来描述. 实质上就是指周期T的n倍时间.

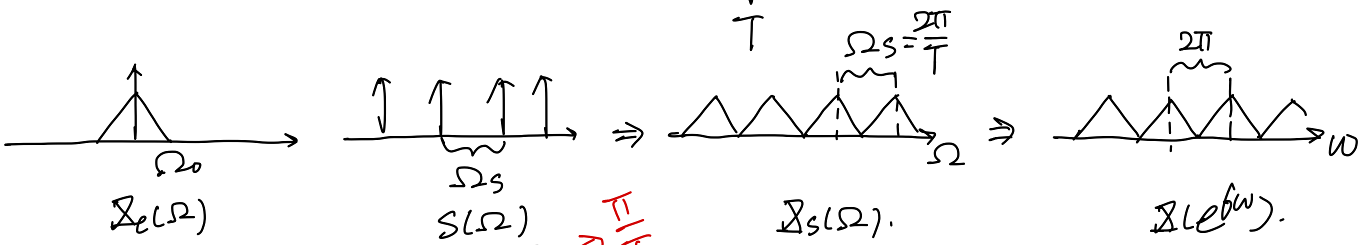
转换前. $X_s(\Omega)$的横坐标为Ω. 单位为rad/s. 转换后. 横坐标乘以T. 用ω. (rad)来描述). 转换之前$\Omega=\frac{\omega}{T}$. 周期为T. 转换后ω的周期为2π.)

2. Nyquist sampling: $\Omega_s \geq 2\Omega_0$. $\Omega_s=\frac{2\pi}{T}$.

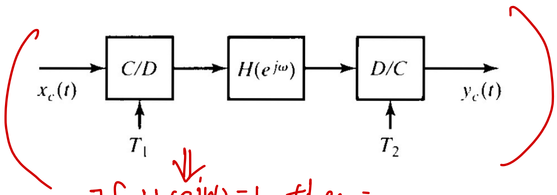
临界采样 (critical sampling): $\Omega_s = 2\Omega_0$.

3. Ideal Reconstruction.

Mathmatical model: $x[n] \longrightarrow \underset{\uparrow T}{\boxed{D/C}} \longrightarrow x_r(t)$.



⇒ 使用 $H_r(\Omega) = \begin{cases} T, & |\Omega| \le \frac{\Omega_s}{2} \Rightarrow \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\Omega_s}{2}. \end{cases}$ 进行滤波.



If $H(e^{j\omega})=1$. then:
$Y(e^{j\omega}) = \frac{T_2}{T_1} X(e^{j\omega})$.

Step1: convert from sequence to impulse train.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT).$$

Step2: Ideal reconstruction filter $\Rightarrow h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T}$.

$$x_r(t) = x_s(t) * h_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

($\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. $\mathrm{sinc}(n) = \begin{cases} 1, & n=0 \\ 0, & \text{others} \end{cases}$).

4. Reconstruction aliasing.

5. Discrete-time processing of continuous-time signals

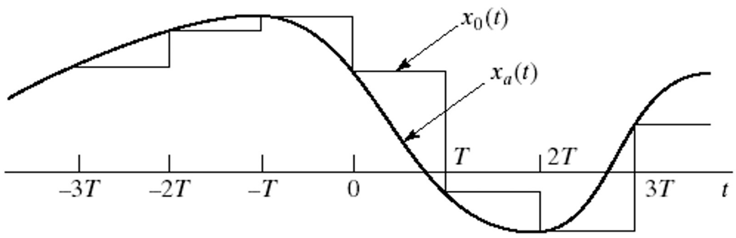
6. Digital process of analog signals.

$\xrightarrow{x_c(t)}$ (1) Anti-aliasing filter $\xrightarrow{x_a(t)}$ (2) Sample and hold ($\uparrow T$) $\xrightarrow{x_0(t)}$ (3) A/D converter ($\uparrow T$) $\xrightarrow{x[n]}$ (4) Discrete-time system $\xrightarrow{y[n]}$ D/A converter ($\uparrow T$) $\longrightarrow$ (5) Reconstruction filter $\to y_r(t)$

(1) 抗混叠滤波　(3) 模数转换.　(5) 重构滤波.

(2) 采样保持.　(4) 数模转换

(1) Sampling and hold.



hold　sampled

$$x_0(t) = \underline{x_s(t)} * h_0(t) =$$

$$\left[\sum_n x_c(nT)\delta(t-nT)\right] * h_0(t)$$

$$= \left[\sum_n x_c(nT) h_0(t-nT)\right]$$

$$h_0(t) = \begin{cases} 1, & 0<t<T \\ 0, & \text{others.} \end{cases}$$

(2) Analog-to-digital converter.

Sample – Quantize – Encode.

(3) Compensated reconstruction filter.

7. Sampling rate

(1) reduction (down-sampling. decimation).

$$x_d[n] = x[nM]$$

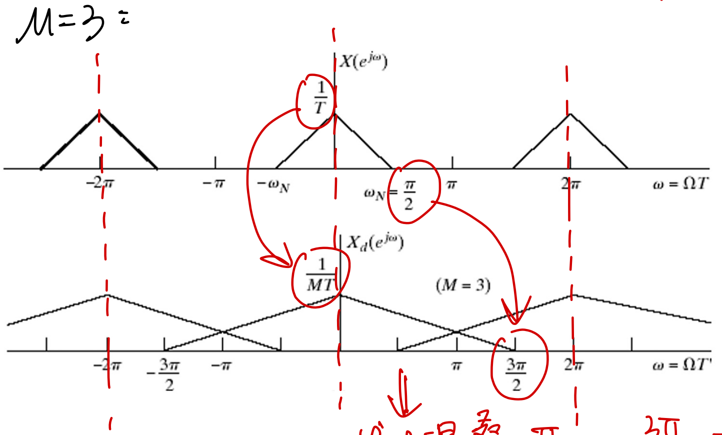
Math Model: $x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x[nM]$.

$x[n]$ ⇓ Sampling period: T $\qquad$ $x_d[n]$ ⇓ Sampling period: $T' = MT$

$$\Rightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j\frac{\omega-2\pi i}{M}}).$$

→ 作用是为了使降采样后的频谱满足2π为周期.

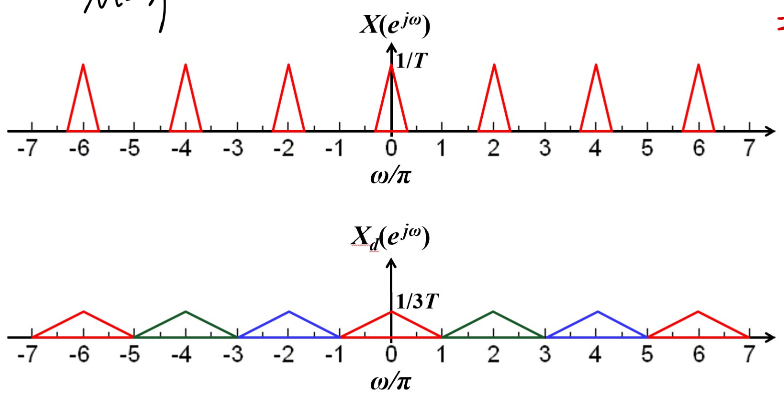
$M=3$:



⇒ 横坐标扩大为 M 倍.
纵坐标缩小 $\frac{1}{M}$

⇓ 发生混叠. $\frac{\pi}{2}\times 3 = \frac{3\pi}{2} > \pi$.

- Example:
  $M=3$.



$$X_d(e^{j\omega}) = 1/3\cdot X(e^{j\omega/3}) + 1/3\cdot X(e^{j(\omega-2\pi)/3}) + 1/3\cdot X(e^{j(\omega-4\pi)/3})$$

⇒ $X_c(j\Omega)$: $\Omega_N = \frac{\pi}{3}$, $T=1$

$X(e^{j\omega})$: $\omega_N = \Omega_N T = \frac{\pi}{3}$

$X_d(e^{j\omega})$: $\omega_N = \Omega_N \cdot 3T = \pi$

Down-sampling system: (Decimator)

the conditions for no aliasing ⇒ $\omega_N \le \frac{\pi}{M}$ or $\omega_N M \le \pi$

downsampling by a factor of M = low pass filtering + compression

$x[n] \longrightarrow$ Lowpass Filter (Gain = 1, Cutoff = $\frac{\pi}{M}$) $\xrightarrow{\tilde{x}[n]} \boxed{\downarrow M} \longrightarrow \tilde{x}_d[n] = \tilde{x}[nM]$

Cutoff = $\frac{\pi}{M}$ → 防止发生混叠.

(2) increase (up-sampling. interpolation).

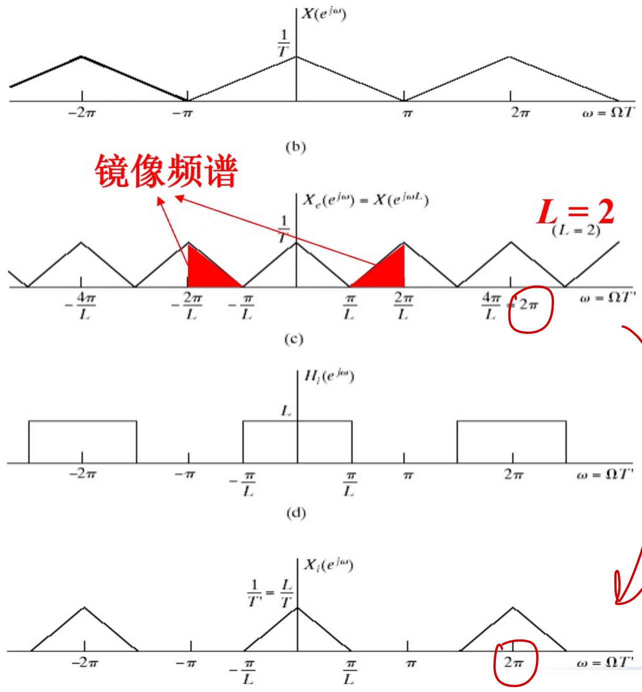
$$x_e[n] = \begin{cases} x[\frac{n}{L}], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{others.} \end{cases} \quad \text{or} \quad x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL].$$

Math Model: $x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n]$

$\Rightarrow X_e(e^{j\omega}) = X(e^{j\omega L})$.

横坐标(ω)收缩 $\frac{1}{L}$.
~~纵坐标扩大L倍~~. 纵坐标不变

L=2:



$X(e^{j\omega})$

$X_e(e^{j\omega}) = X(e^{j\omega L})$

$H_i(e^{j\omega})$

低通滤波：只取需要的频段.
幅值补偿：幅值扩大L倍.

$X_i(e^{j\omega})$

up-sampling system: (interpolator)

$\Rightarrow$ expansion + anti-mirror filtering.

$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_e[n]$ Lowpass Filter $\longrightarrow x_i[n]$.

Gain = L $\Rightarrow$ 幅值补偿

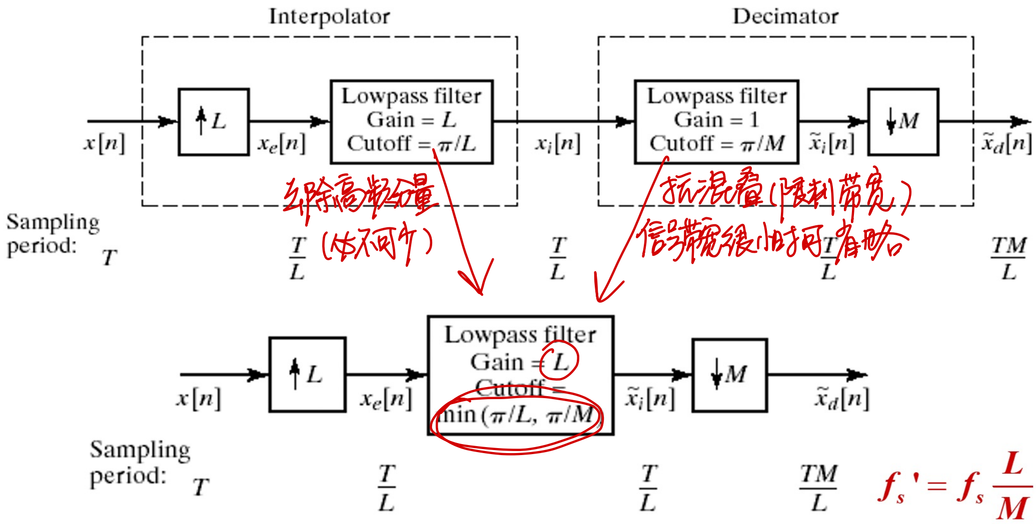
Cutoff = $\frac{\pi}{L}$

填充0值，再通过低通滤波器：将高频跳变消除，时域上0值填为采样值
填0(不改变幅度)

频率：先将频谱压缩，再通过低通滤波器得到真正需要的部分.

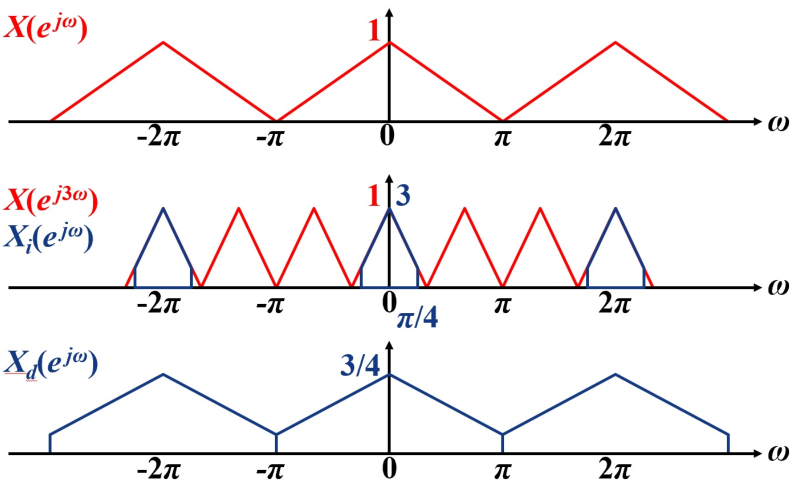
(3) Changing the sampling rate by a non-integer factor

By combining decimation and interpolation, it is possible to change the sampling rate by a non-integer factor.



$$f_s' = f_s \frac{L}{M}$$

o Example:



上例只损失1/4的高频
如果先抽选则损失3/4的高频
L/M>1时无高频损失

先内插后抽选的好处
1. 合并抗混迭和反镜像滤波器
2. 尽可能减少高频信息损失

Chapter 06 Discrete Fourier Transform

1. system function: $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$.

For LTI systems: $Y(z) = X(z)H(z)$.

zero-state response: $y[n] = \mathcal{Z}^{-1}\{X(z)H(z)\} = x[n] * h[n]$.

2. Discrete Fourier Transform.

For each sequence $x[n]$ with finite length N:

Periodic sequence: $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN] = x[((n))_N]$.

$\Rightarrow x[n] = \tilde{x}[n] R_N[n]$.

DFT and DFS:

$$\text{DFS}: \begin{cases} \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N}, \ k=-\infty,\cdots\infty \\ \tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N}, \ n=-\infty\cdots\infty \end{cases}$$

$\Rightarrow$ 对于每一个k，均有 $\frac{n}{N} = \frac{0}{N}, \frac{1}{N}, \cdots \frac{N-1}{N}$.

$$\text{DFT}: \begin{cases} X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \ k=0,\cdots N-1 \\ x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, \ n=0,\cdots N-1. \end{cases}$$

$$\Rightarrow \begin{cases} X[0] = \sum_{n=0}^{N-1} x[n] \\ x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]. \end{cases}$$

隐含周期性.

DFT and DTFT:

$$\begin{cases} \text{DFT} = X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \ k=0,\cdots N-1. \\ \text{DTFT} = X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}. \end{cases}$$

$\Rightarrow \omega = 2\pi \cdot \frac{k}{N}, \ k=0,\cdots N-1$.

$X[k] = X(e^{j2\pi k/N}) = X(z)\big|_{z=e^{j2\pi k/N}}$

$\Rightarrow$ 单位圆上等间隔采样. 采样点数为 $\underline{N}$.

3. Extending sequence length by zero padding.

Sampling points less than the length of sequence:

M-point samples of the DTFT:

$$X[k] = X(e^{jw})\Big|_{w=\frac{2\pi}{M}k} = \sum_{n=-\infty}^{\infty} x[n] W_M^{kn},\ k=0,\cdots M-1.$$

The IDTFT according to these samples:

$$x'[n] = \left(\sum_{r=-\infty}^{\infty} x[n-rM]\right) R_M[n].$$

4. Sampling theory in frequency domain.

When sampling points in frequency domain $M \geq$ sequence length N.
then can reconstruct the signal in time domain.

5. Properties of DFT.

calculation:

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & \cdots & W^{N-1} \\ W^0 & W^2 & W^3 & \cdots & W^{N-2} \\ \vdots & & & & \\ W^0 & W^{N-1} & W^{N-2} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}.$$

$\Rightarrow N^2$ times

(1) Circular shift.

$$\begin{cases} x[((n-m))_N] R_N[n] \xleftrightarrow{DFT} W_N^{km} X[k] \\ W_N^{-nl} x[n] \xleftrightarrow{DFT} X[((k-l))_N] R_N[k]. \end{cases}$$

$\Rightarrow$ 类似于傅里叶变换中的时移.

(2) Duality.

$$X[n] \xleftrightarrow{DFT} N x[N-k],\ k=0,\cdots N-1.$$

$\Rightarrow X[k] = DFT\{x[n]\}.\ Y[k] = DFT\{X[n]\} = N x[N-k].$

(3) symmetry.

$$x^*[n] \stackrel{DFT}{\leftrightarrow} X^*[((-k))_N]R_N[k] = X^*[N-k]$$

$$x^*[((-n))_N]R_N[n] = x^*[N-n] \stackrel{DFT}{\leftrightarrow} X^*[k]$$

$$\mathrm{Re}\{x[n]\} = \frac{1}{2}(x[n]+x^*[n]) \stackrel{DFT}{\leftrightarrow} \frac{1}{2}(X[k]+X^*[N-k]) = X_{ep}[k]$$

ep = 对称分量 . op = 反对称分量

$$j\,\mathrm{Im}\{x[n]\} = \frac{1}{2}(x[n]-x^*[n]) \stackrel{DFT}{\leftrightarrow} \frac{1}{2}(X[k]-X^*[N-k]) = X_{op}[k]$$

$$x_{ep}[n] = \frac{1}{2}(x[n]+x^*[N-n]) \stackrel{DFT}{\leftrightarrow} \frac{1}{2}(X[k]+X^*[k]) = \mathrm{Re}\{X[k]\}$$

$$x_{op}[n] = \frac{1}{2}(x[n]-x^*[N-n]) \stackrel{DFT}{\leftrightarrow} \frac{1}{2}(X[k]-X^*[k]) = j\,\mathrm{Im}\{X[k]\}$$

$\left\{\begin{array}{l}\text{实部} \rightleftharpoons \text{对称分量} \\ \text{虚部} \rightleftharpoons \text{反对称分量}\end{array}\right.$

$\Rightarrow$ $X_{ep}[k]$ = 圆周(周期)共轭对称分量. (关于 $\frac{N}{2}$ 共轭对称).

$X_{op}[k]$ = 圆周(周期)共轭反对称分量.

Any finite-length sequence can be decomposed as:

$X[k] = X_{ep}[k] + X_{op}[k]$. $\left\{\begin{array}{l} X_{ep}[k] = \frac{1}{2}(X[k]+X^*[N-k]) = X_{ep}^*[N-k] \\ X_{op}[k] = \frac{1}{2}(X[k]-X^*[N-k]) = -X_{op}^*[N-k]. \end{array}\right.$

(14) Real sequence: $X[k] = X^*[((-k))_N]R_N[k]$.

$\Rightarrow$ $\mathrm{Re}(X[k]) = \mathrm{Re}(X[N-k])$. $\mathrm{Im}(X[k]) = -\mathrm{Im}(X[N-k])$.

(15) Parseval's theorem.

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k].$$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

(16)* Circular convolution.

$$y[n] = \underbrace{x_1[n]\,(N)\,x_2[n]}_{N\text{点卷积}} = (x_1[((n))_N] \circledast x_2[((n))_N])R_N[n].$$

$$= \left(\sum_{m=0}^{N-1} x_1[((m))_N]\,x_2[((n-m))_N]\right)R_N[n].$$

$$= \left(\sum_{m=0}^{N-1} x_1[m]\,x_2[((n-m))_N]\right)R_N[n].$$

$\Rightarrow$ Length of $x_1[n]$. $x_2[n]$ and $y[n]$ are all N.

$\Rightarrow$ $y[n] = x[n]\,(N)\,\delta[n-1] = x[((n-1))_N]R_N[n]$.

↳ 与 $\delta[n]$ 互数循环卷积相当于循环移位

Using DFT to calculate circular convolution:

$$\begin{cases} x_1[n] (N) x_2[n] \xleftrightarrow{DFT} X_1[k] X_2[k]. \\ x_1[n] x_2[n] \xleftrightarrow{DFT} \frac{1}{N} X_1[k] (N) X_2[k]. \end{cases}$$

$$\Rightarrow y[n] = x_1[n] (N) x_2[n] = \left(\sum_{m=0}^{N-1} x_1[m] x_2[((n-m))_N] \right) R_N[n].$$

Relation between linear convolution and circular convolution.

$$y[n] = x[n] * h[n] \Rightarrow x[n] (N) h[n] = \sum_{r=-\infty}^{\infty} y[n+rN] R_N[n].$$

Δ If the length of two sequence are L_1. L_2.

the point of circular convolution $N \geq L_1 + L_2 - 1$.

then $x[n] * h[n] = x[n] (N) h[n]$.

Summary of DFT and convolution

(1) Calculate N-point circular convolution by linear convolution

(a) $y[n] = x[n] * h[n]$

(b) $x[n] (N) h[n] = \sum_{r=-\infty}^{\infty} y[n+rN] R_N[n]$

(2) Calculate linear convolution by circular convolution

(a) zero-padding $x[n]$ **and** $h[n]$ **to length** $N \geq L_1+L_2-1$

(b) $x[n] * h[n] = x[n]\ (N)\ h[n]$

(3) Calculate linear convolution by DFT

(a) zero-padding $x[n]$ **and** $h[n]$ **to length** $N \geq L_1+L_2-1$

(b) N-point DFT of $x[n]$ **and** $h[n]$

(c) $x[n] (N) h[n] = IDFT\{X[k] H[k]\}$

$x[n] * h[n] = x[n] (N) h[n]$

Chapter 07. FFT: Fast Fourier Transform.

1. DFT:

$$\begin{cases} X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, & k=0,\dots,N-1. \\ x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}, & n=0,\dots,N-1. \end{cases}$$

$\begin{cases} n\text{: 求和变量} \\ k\text{: 自变量} \end{cases}$

$$\Rightarrow \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & \cdots & W^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ W^0 & W^{N-1} & W^{N-2} & \cdots & W^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

复数乘法 N^2 次.

FFT: $N\log_2 N$

2. Decimation in frequency (频域抽选)

$W = e^{-j2\pi/N}$. $e^{-j2\pi kn/N} = W^{kn}$.

o Example: $N=8$:

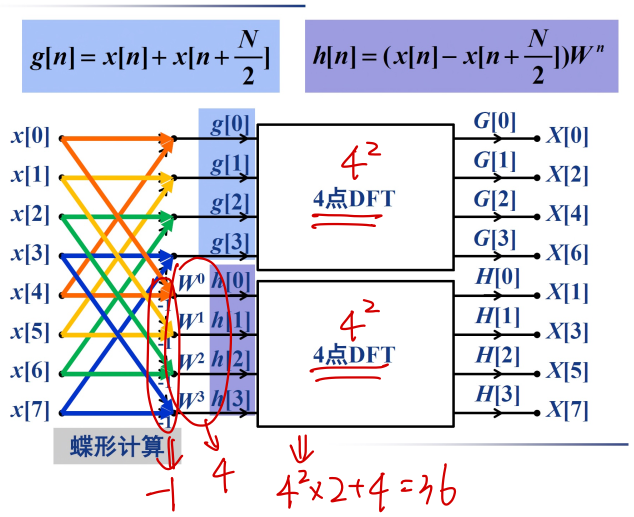
	n= 0	1	2	3	4	5	6	7
k = 0	W^0	W^0	W^0	W^0	W^0	W^0	W^0	W^0
1	W^0	W^1	W^2	W^3	W^4	W^5	W^6	W^7
2	W^0	W^2	W^4	W^6	W^8	W^{10}	W^{12}	W^{14}
3	W^0	W^3	W^6	W^9	W^{12}	W^{15}	W^{18}	W^{21}
⋮								
7								

$\Rightarrow \begin{cases} G(r) = X[2r],\ r=0,1,2,3. & \Rightarrow G[r] = X[0], X[2], X[4], X[6] \\ H(r) = X[2r+1],\ r=0,1,2,3 & \Rightarrow H[r] = X[1], X[3], X[5], X[7]. \end{cases}$

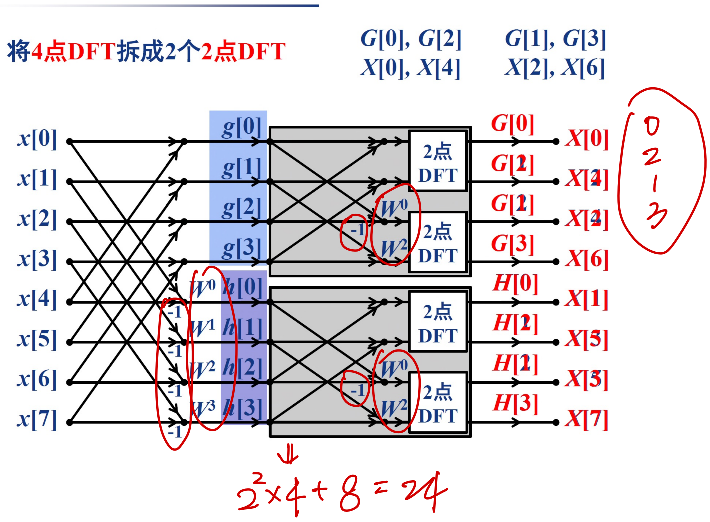
$\Rightarrow \begin{cases} g[n] = x[n] + x[n+\frac{N}{2}], & n=0,1,2,3 \\ h[n] = (x[n] - x[n+\frac{N}{2}])W^n, & n=0,1,2,3. \end{cases}$

⇒ 由 $x[n]$ 得到 $g[n]$, $h[n]$. 再计算4点DFT $= 4^2\times 2+4 = 36$. 8点DFT: $8^2 = 64$.

First decomposition



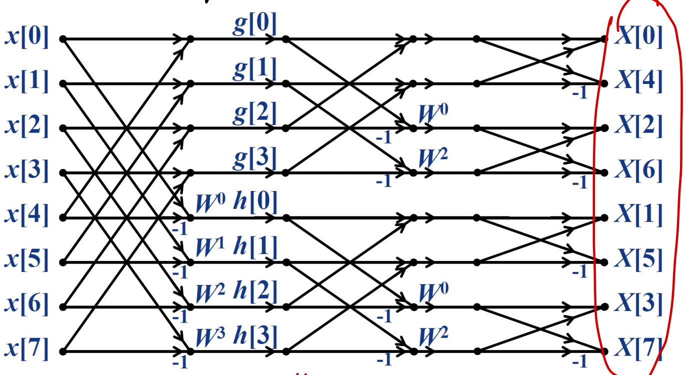
Second decomposition

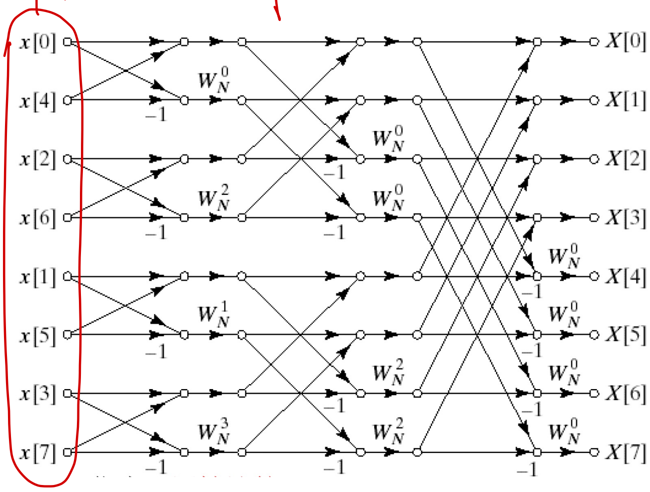


$$\boxed{\text{2 point DFT: } k=0,1. \quad \begin{cases} X[0] = x[0]+x[1] \\ X[1] = x[0]-x[1]. \end{cases}}$$

8 point FFT: $N\log_2 N = 8\times 3 = 24$.

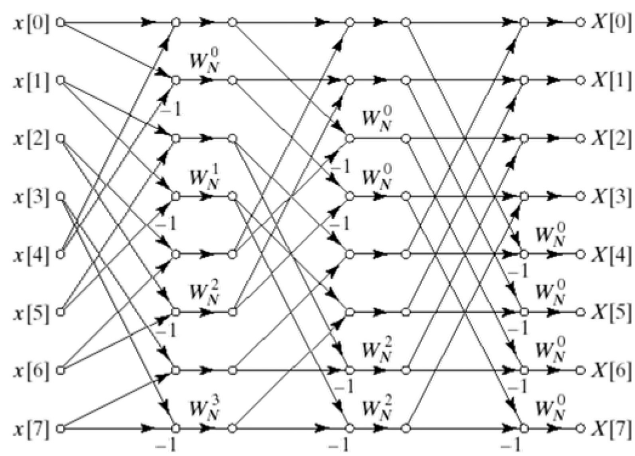
→ 0426. 1537.





同址计算. 非顺序读取数据

非同址计算
顺序读取数据 ⇐



{ 权重 W_N^r 一定在蝶形网络后面
每一级所取的权重不变.

△ N 必须为 2 的指数幂. 不足补 0

○ Example:

Question: An FFT analyzer requires a number of sampling points $N=2^m$ (m is a positive integer). If an input signal $x_c(t)$ has a maximum frequency of 5 kHz (1 kHz = 1000 Hz) and we want to achieve a frequency resolution smaller than 10Hz. $\Delta f = 10Hz$

(1) What is the minimum number of N?

(2) What is the maximum sampling period of T?

(3) What is the minimum sampling time window?

$\Rightarrow W = 5kHz$, $f_s \geq 2W = 10kHz$, then $T = \frac{1}{f_s} \leq 0.1ms$

$\boxed{\Delta f \Delta t = \frac{1}{N}}$. $\Delta t = T \leq 0.1ms$ then $N = \frac{1}{\Delta f \Delta t} \geq \frac{1}{10\times 0.1\times 10^{-3}} = 10^4$. → $\Delta f \leq 10Hz$, $\Delta t \leq 0.1ms$

Minimum number of N is $1024 = 2^{10}$. $m=10$.

△ 首先确定采样频率 f_s. T.

3. Decimation in time (时域抽选)

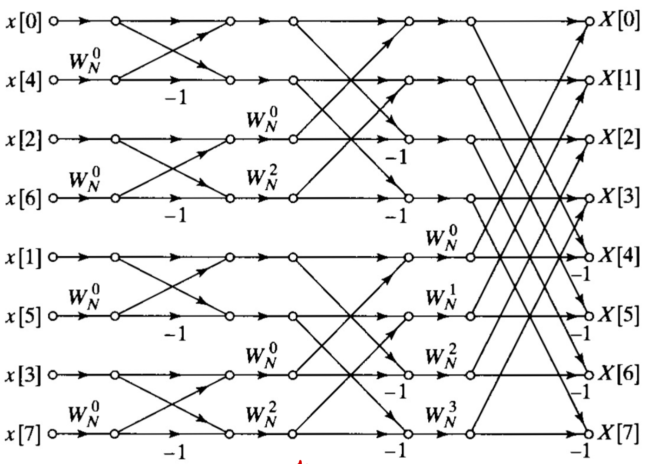
DFT: $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$, $k=0, \cdots, N-1$.

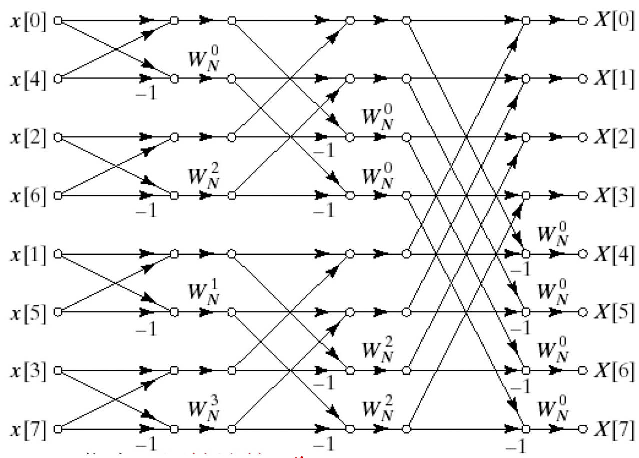
let: $\begin{cases} g[n] = x[2n], & n=0, \cdots, \frac{N}{2}-1 \\ h[n] = x[2n+1], & n=0, \cdots, \frac{N}{2}-1 \end{cases}$ $\begin{cases} G[k] = \sum_{n=0}^{\frac{N}{2}-1} g[n] W_{N/2}^{kn}, & k=0, \cdots, \frac{N}{2}-1 \\ H[k] = \sum_{n=0}^{\frac{N}{2}-1} h[n] W_{N/2}^{kn}, & k=0, \cdots, \frac{N}{2}-1 \end{cases}$

then: $\begin{cases} X[k] = G[k] + H[k] W_N^k, & k=0, \cdots, \frac{N}{2}-1 \\ X[k+\frac{N}{2}] = G[k] - H[k] W_N^k, & k=0, \cdots, \frac{N}{2}-1 \end{cases}$

let: $\begin{cases} g_1[n] = g[2n], & n=0 \cdots \frac{N}{4}-1 \\ g_2[n] = g[2n+1], & n=0 \cdots \frac{N}{4}-1 \end{cases}$ $\begin{cases} h_1[n] = h[2n], & n=0 \cdots \frac{N}{4}-1 \\ h_2[n] = h[2n+1], & n=0 \cdots \frac{N}{4}-1 \end{cases}$

then $\begin{cases} G[k] = G_1[k] + G_2[k] W_{N/2}^k, & k=0 \cdots \frac{N}{4}-1 \\ G[k+\frac{N}{4}] = G_1[k] - G_2[k] W_{N/2}^k, & k=0 \cdots \frac{N}{4}-1 \end{cases}$ $\begin{cases} H[k] = H_1[k] + H_2[k] W_{N/2}^k, & k=0 \cdots \frac{N}{4}-1 \\ H[k+\frac{N}{4}] = H_1[k] - H_2[k] W_{N/2}^k, & k=0 \cdots \frac{N}{4}-1 \end{cases}$





$\Downarrow$ DIT.　　　　$\Downarrow$ DIF

⇒ 拓扑结构完全一致. 但权重位置不同.

W_{128}^{16} 一共出现在 5 级.

W_{128}^{16} 11月22日

$W_{128}^0 \sim W_{128}^{127}$

① $W_{128}^0 \sim W_{128}^{63}$ W_{128}^{16}

② $W_{64}^0 \sim W_{64}^{31}$ W_{64}^{8}

③ $W_{32}^0 \sim W_{32}^{15}$ W_{32}^{4}

④ $W_{16}^0 \sim W_{16}^{7}$ W_{16}^{2}

⑤ $W_8^0 \sim W_8^3$ W_8^1

⑥ $W_4^0 \sim W_4^1$

⑦ W_2^0.

4. Bit-reversed order

5. Coefficients: W_N^r for $r = 0, 1, \cdots \frac{N}{2}-1$.

$\Rightarrow$ 计算N点FFT.总共需要$\frac{N}{2}$个W_N系数.

6. IFFT.

(1) $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$, $k = 0 \cdots N-1$. $\Rightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$, $n = 0 \cdots N-1$.

(2) $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} = \frac{1}{N} \left[\sum_{k=0}^{N-1} X^*[k] W_N^{kn} \right]^* = \frac{1}{N} [FFT(X^*[k])]^*$

(3) $FFT[X[k]] = \sum_{k=0}^{N-1} X[k] W_N^{kn} = N x[N-n]$.

$\Rightarrow x[n] = \frac{1}{N} FFT\{X[k]\} \Big|_{n=N-n}$.

Chapter 08 Filter Design Techniques

How to design a discrete-time filter?

Method 1: Use continuous-time (analog) filter design techniques
+ frequency transform / mapping.

$\Rightarrow$ Impluse invariance, bilinear.

Method 2: Window method.

1. use analog filter design techniques

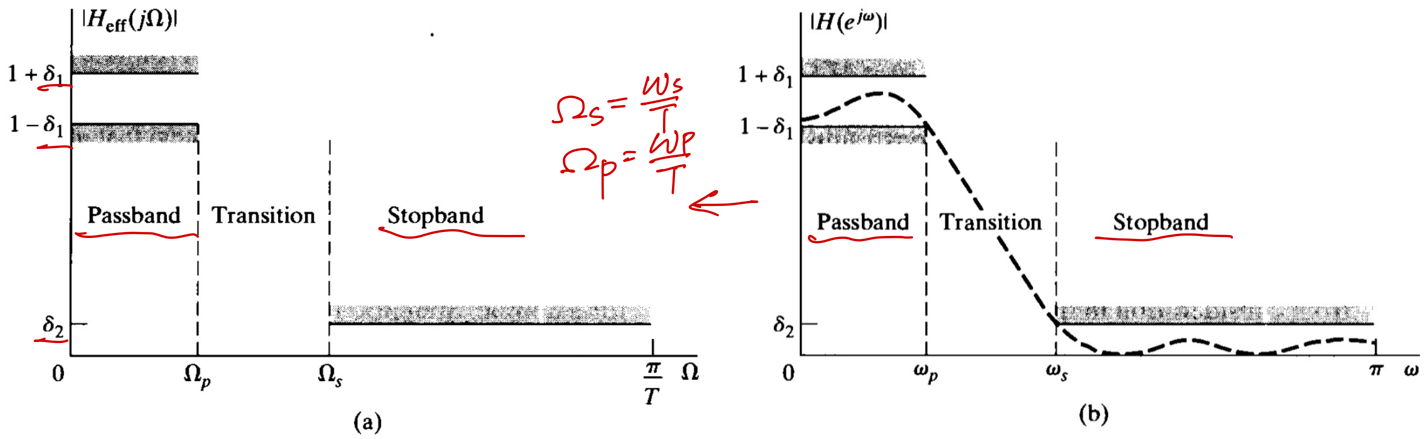
$$\xrightarrow{x_a(t)} (C/D) \xrightarrow{x[n]} (H(e^{j\omega})) \xrightarrow{y[n]} (D/C) \longrightarrow y_a(t) \quad \Rightarrow \text{CDC system}$$

$\uparrow T$ (C/D), $\uparrow T$ (D/C)

Frequency transform / mapping: $\omega = \Omega T$

$$H_{eff}(\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T}. \quad \omega = \pi \\ 0 & , |\Omega| > \frac{\pi}{T}. \end{cases}$$

Magnitude response 幅频图:



Decimal unit: $\delta_1 = \delta_p \quad \delta_2 = \delta_s$

$$\Rightarrow \begin{cases} \alpha_p = 20 \cdot \lg(1-\delta_p) < 0. & \text{通带最大衰减} \\ \alpha_s = 20 \cdot \lg \delta_s < 0. & \text{阻带最大衰减} \end{cases}$$

o Example: $\delta_p = 0.05 \Rightarrow \alpha_p = -0.45\,dB \quad \delta_s = 0.1. \quad \alpha_s = -20\,dB.$

3dB 截止频率 ω_c. 功率降到 $\frac{1}{2}$. $|H(e^{j\omega_c})| = \frac{1}{\sqrt{2}}$. $20\lg|H(e^{j\omega_c})| = -3dB$.

5%　　10% 透过率

2. Butterworth filter design.

power response: $|H_c(\Omega)|^2 = \dfrac{1}{1+(\Omega/\Omega_c)^{2N}}$. Ω_c 3dB截止频率.

poles (极点): $1+(\Omega/\Omega_c)^{2N}=0 \Rightarrow \Omega_k = \Omega_c j\exp(\frac{\pi}{2N}+k\frac{2\pi}{2N})$. $k=0\sim 2N-1$.

3. Transform from $H_c(s)$ to $H(z)$

$H_c(s)$ ↓ continuous time; $H(z)$ ↳ discrete time.

- 1 因果稳定的系统经过映射后仍因果稳定

$H_c(s)$在S平面左半平面内极点，能映射到$H(z)$在Z平面单位圆内.

1) Imaginary axis of s plane $\Rightarrow$ unit circle of z plane.

2) a causal stable $H_c(s)$ $\Rightarrow$ a causal stable $H(z)$.

4. Impulse invariance method 冲激不变法.

The impulse response of discrete-time filter is chosen proportional to equally spaced samples of the impulse response of continuous-time filter

$\Rightarrow h[n] = T_d h_c(t)|_{t=nT_d} = T_d h_c(nT_d)$.

$$H_c(s) \longrightarrow h_c(t) \xrightarrow[h[n]=T_d h_c(nT_d)]{} h[n] \longrightarrow H(z).$$

- Example: Exponential decay. 记

$h_c(t) = Ae^{s_0 t}u(t)$. $\Rightarrow h[n] = T_d h_c(nT_d) = T_d A e^{s_0 n T_d} u[n]$.

$H_s = \dfrac{A}{s-s_0}$, $Re(s)<0$. $\longrightarrow$ $H(z) = \dfrac{T_d A}{1-e^{s_0 T_d}z^{-1}}$. $|e^{s_0 T_d}|<1$ or $Re(s_0 T_d)<0$.

△ 首先对H_s进行部分分式分解.

e.g. $H_c(s) = \dfrac{s+a}{(s+a)^2+b^2} = \dfrac{1/2}{(s+a)+jb} + \dfrac{1/2}{(s+a)-jb}$.

Relation between $H_c(\Omega)$ and $H(e^{j\omega})$.

$\Rightarrow H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c(\Omega + k\Omega_s)\Big|_{\Omega=\frac{\omega}{T_d}}$. $\Omega_s = \frac{2\pi}{T_d}$.

if $H_c(\Omega)$ is bandlimited, $H_c(\Omega) = 0$, $|\Omega| \geq \frac{\pi}{T_d}$.

then: $H(e^{j\omega}) = H_c(\frac{\omega}{T_d})$, $|\omega| \leq \pi$.

Design steps:

$\Omega_p = \omega_p / T_d$

$\Omega_s = \omega_s / T_d$

1) Discrete-time spec $\Rightarrow$ continuous-time spec

2) Design $H(s)$ Prototype.

3) Transform $H_c(s)$ to $H(z)$. = $H(z) = H_c(s)\Big|_{\frac{1}{s-s_k} = \frac{T_d}{1-e^{s_k T_d} z^{-1}}} = \sum_{k=0}^{N-1} \frac{T_d A_k}{1-e^{s_k T_d} z^{-1}}$.

Pros and cons:

$\Rightarrow$ pros: Linear mapping of frequency. ($\omega = \Omega T_d$.) → 线性.

cons: Aliasing in frequency response.

restriction $\Rightarrow$ can not be appied to high pass and band stop filters which are not bandlimited.

5. Bilinear

$\Rightarrow$ Map one Ω to only ω to solve the problem of frequency aliasing.

One Ω is mapped to many ω before: $s = j\Omega$. $z = e^{j\omega} = e^{j\Omega T_d}$.

($\omega = 2\arctan(\frac{\Omega T_d}{2})$) bilinear transform $\Rightarrow$ $\boxed{\Omega = \frac{2}{T_d}\tan\frac{\omega}{2}}$ → 非线性.

↳ 1) $\Omega = -\infty \sim \infty$ is mapped to $\omega = -\pi \sim \pi$.

2) when $\Omega \to 0$, $\omega = \Omega T$.

impulse invariance: $\omega = \Omega T_d$.

Pro: avoiding the problem of aliasing.

cons: The transformation between s and z is nonlinear.

Mapping the entire imaginary-axis in the s-plane to one revolution of the unit circle in the z-plane.

Restriction: only situations which the corresponding warping of the frequency axis is acceptable.

↳ 模拟微分器(幅度线性响应)不能由双线性变换法得到数字微分器

Transformation: 记

$$z = e^{j\omega} = \cos\omega + j\sin\omega$$

$$= \frac{1 + j\Omega T_d/2}{1 - j\Omega T_d/2} = \frac{1 + (T_d/2)s}{1 - (T_d/2)s} \text{ 记}$$

$\Rightarrow$ $\theta = \frac{\omega}{2}$.

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}.$$

$$\Rightarrow s = \frac{2}{T_d} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}.$$

记

$$H(z) = H_c\left(\frac{2}{T_d} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

Stability:

If a pole of $H_c(s)$ is in the left half of s-plane, its image will be inside the unit circle of z-plane.

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}. \quad \text{if } s = \sigma + j\Omega.\ \sigma < 0.\ \text{then } |z| < 1.$$

Then a causal stable continuous-time filter is mapped to a causable stable discrete-time filter.

Frequency mapping: $H(e^{j\omega}) = H_c(\Omega)\big|_{\Omega = \frac{2}{T_d}\tan(\frac{\omega}{2})}$.

Pre-wrap: $\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)$. $\quad \Omega_s = \frac{2}{T_d}\tan\left(\frac{\omega_s}{2}\right)$.

Design steps:

1) Discrete-time specifications $\xrightarrow{\text{convert}}$ continuous time specification.

2) Design $H_c(s)$ prototype.

3) Transform to $H(z)$:

$$\begin{cases} H(z) = H_c(s)\Big|_{s = \frac{2}{T_d} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}}. \\ H(e^{j\omega}) = H_c(\Omega)\Big|_{\Omega = \frac{2}{T_d}\tan\frac{\omega}{2}} \end{cases}$$

$z = e^{j\omega}$

$s = j\Omega$.

6. Design of FIR filters by window method 窗函数法.

Design techniques for FIR filters are based on directly approximating the desired frequency response of the discrete time systems. Most techniques for approximating the magnitude response of an FIR system assume linear phase contraint. ⇒ constant group delay. grd $= -\frac{d\varphi}{d\omega}$.

(1) Begins with an ideal desired frequency response, inverse transform:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{H_d(e^{j\omega})}_{\text{the ideal frequency response}} e^{j\omega n}\,d\omega.$$

but it's non-causal and infinitely long.

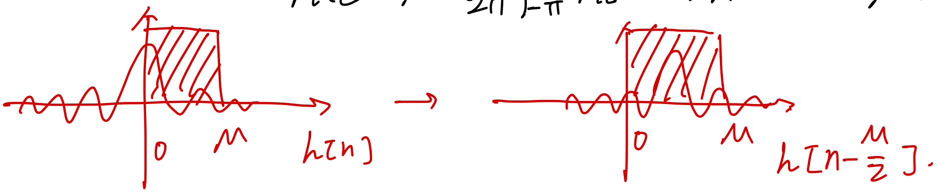
(2) To obtain a causal FIR approximation to it, truncate the ideal response:

$$h[n] = h_d[n]\,w[n]$$

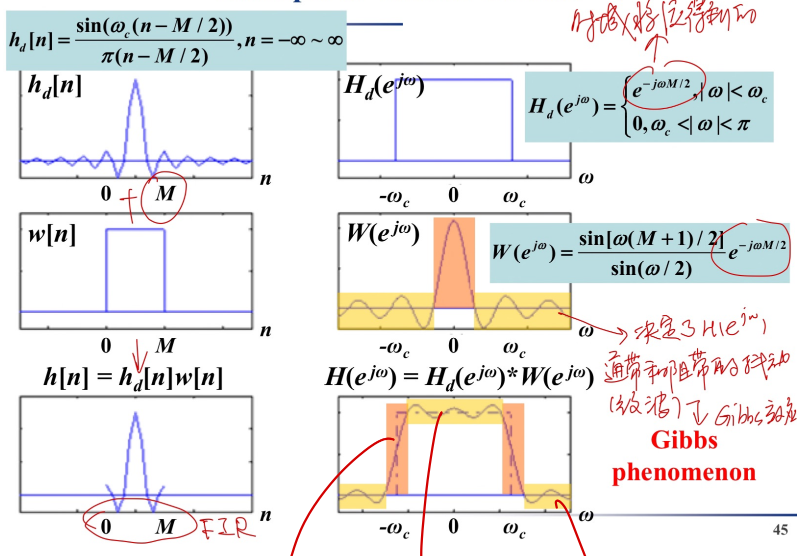
w[n] is a rectangular window or others.

Frequency domain:

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$$ → 与窗函数卷积.



Example of window method



→ 决定了 $H(e^{j\omega})$ 通带和阻带的抖动（纹波）→ Gibbs效应

Gibbs phenomenon

transition band

pass band ripples

stop band ripples

红色部分

(1) Transition band width 过渡带宽 is determined by the main lobe width 主瓣宽度 of window's spectrum.

→ 黄色部分.

(2) Pass band and stop band ripples 纹波 are determined by the side lobes 旁瓣 of window's spectrum.
The ripples in the pass band and the stop band are approximately the same, and are not dependent on *M* and can be changed only by changing the window shape.

Main lobe and side lobe of Blackman windows

Type of Window	旁瓣相对幅度(dB) Peak Side-Lobe Amplitude (Relative)	主瓣宽度 Approximate Width of Main Lobe
Rectangular	−13	$4\pi/(M+1)$
Bartlett → 三角窗.	−25	$8\pi/M$
Hanning	−31	$8\pi/M$
Hamming	−41	$8\pi/M$
Blackman	−57	$12\pi/M$
离散取值 −45dB只能选择Blackman	只与窗形相关	与窗形和窗长都相关

频率分辨率

$$\Delta\omega_{\min} = \Delta_{ml} \approx \frac{D\pi}{M}$$

$$\Delta\Omega_{\min} = \frac{\Delta\omega_{\min}}{T} \approx \frac{D}{2M}\Omega_s$$

$$\Delta f_{\min} = \frac{\Delta\Omega_{\min}}{2\pi} \approx \frac{D}{2M}f_s$$

***D* = 4, 8, 12 取决于窗形**

Kaiser window:

$$w[n] = \begin{cases} \dfrac{I_0\left(\beta\sqrt{1-[(n-\frac{M}{2})/(\frac{M}{2})]^2}\right)}{I_0(\beta)}, & 0 \le n \le M. \\ 0, & \text{o.w.} \end{cases}$$ $\Rightarrow$ params $= M, \beta$.

I_0: Zeroth-order modified Bessel function.

β: Arbitrary real number.

$\Rightarrow$ { β增大：旁瓣衰减增大，主瓣宽度增大
M增大：旁瓣衰减不变，主瓣宽度减小.

Effect of these windows to frequency response

待定参数：窗形，窗长M
旁瓣抑制：窗形决定，影响滤波器的通带纹波和阻带衰减
主瓣宽度：窗形+窗长决定，影响滤波器的过渡带宽

Blackman window: 滤波器的阻带衰减可较精确给出，但过渡带宽只能大致估计正比于窗的主瓣宽度

Kaiser window: 不用背

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.584(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 \Rightarrow \text{矩形窗} & A < 21 \end{cases}$$

阻带衰减 $A = -20\log_{10}\delta$　　过渡带宽 $\Delta\omega = \dfrac{A-8}{2.285M}$

同样的阻带衰减，采用凯泽窗比布窗过度带宽略小。

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Design steps:

(1) Obtain the ideal impulse response.

$w_c \approx (w_p + w_s)/2$.

$$H_d(e^{jw}) = A(w)e^{j\phi(w)} = \begin{cases} e^{-j\frac{M}{2}w}, & \text{pass band} \rightarrow 1\cdot e^{-j\frac{M}{2}w} \\ 0, & \text{stop band}. \end{cases}$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{jw})e^{jwn}\,dw.$$

(2) determine the window shape according to pass band and stop band ripples.

$\delta = \min(\delta_p, \delta_s)$.　$A = -20\log_{10}\delta$.

$$\Rightarrow \beta = \begin{cases} 0.1102(A-8.7) & , A > 50 \\ 0.584(A-21)^{0.4} + 0.07886(A-21) & , 21 \le A \le 50 \\ 0 & , A < 21. \end{cases}$$

(3) determine the window length according to the transition width (M is even for high-pass and band stop filter).

$\Delta W = |W_s - W_p|$ ⇒ transition bandwidth.

Blackman window: $M \approx \frac{D\pi}{2\Delta W}$. $D = 4.8.12$.

⇒ approximate width of main lobe: $2\Delta W = \frac{D\pi}{M}$

$|W_p - W_s| = \Delta W$.

Kaiser window: $M \approx \frac{A-8}{2.285\Delta W}$. $\Delta W = \frac{A-8}{2.285M}$.

(4) Truncate the ideal impulse response.

$$h[n] = h_d[n]w[n].$$

(5) verify $H(e^{jw})$ and adjust W_c. M. β.

Example of FIR filter design

Example Design a FIR low pass filter

$\omega_p = 0.4\pi,\ \omega_s = 0.6\pi,\ \alpha_p = 1\text{dB},\ \alpha_s = 40\text{dB}$

(1) Obtain ideal impulse response

$\begin{cases} \alpha \to dB \\ \delta \to 1 \end{cases}$

$$\omega_c = (\omega_p + \omega_s)/2 = 0.5\pi$$

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{\sin(0.5\pi(n - M/2))}{\pi(n - M/2)}, n = -\infty \sim \infty$$

(2) Determine window shape by ripple

$$\delta = \min(\delta_p, \delta_s) = \min(1 - 10^{-\alpha_p/20}, 10^{-\alpha_s/20}) = 0.01, A = 40dB$$

Blackman window: Hamming

Approximate width of main lobe is $\frac{8\pi}{M}$. $D = 8$.

(3) Determine window length by transition band width

过渡带宽

$$|\omega_p - \omega_s| = \frac{8\pi}{M}/2 = (0.6 - 0.4)\pi, \therefore M = 20$$

(4) Truncate the ideal impulse response

$$h[n] = \left[\frac{\sin(0.5\pi(n-10))}{\pi(n-10)}\right] \cdot \left[0.54 - 0.46\cos\left(\frac{2\pi n}{20}\right)\right] R_{21}[n]$$

20点，不矩形窗的截断.

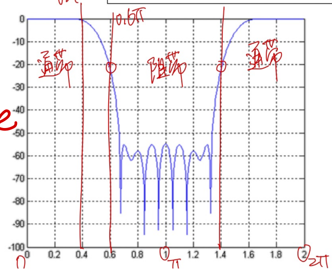
(5) Verify and adjust (by simulation)

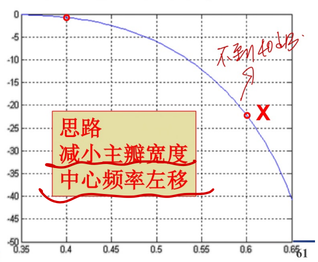
$\omega_p = 0.4\pi,\ \omega_s = 0.6\pi$
$\alpha_p = 1\text{dB},\ \alpha_s = 40\text{dB}$

滤波器: M阶数 W_c:3dB 窗函

```
h=fir1(20,0.5,hamming(21))
H=fft(h,512);   %或者freqz(h,1)
plot([0:511]/256,20*log10(abs(H)))
grid on;  hold on;      axis([0.35,0.65,-50,0]);
```

归一化



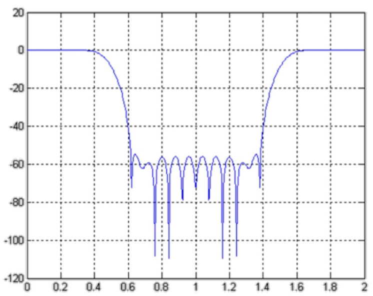


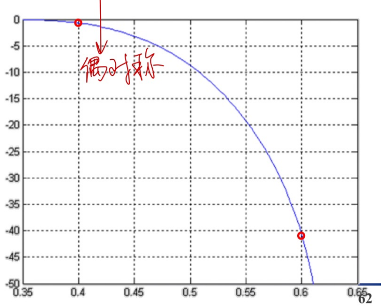
Example of FIR filter design

第?类滤波器
奇数

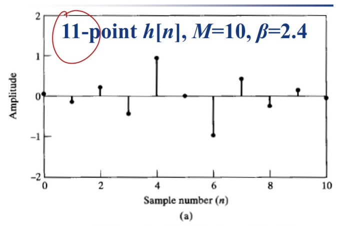
```
h=fir1(25,0.48,hamming(26))
H=fft(h,512);        plot([0:511]/256,20*log10(abs(H)));
axis([0.35,0.65,-50,0]);        grid on
```

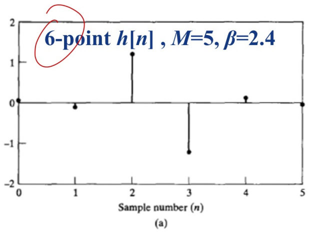
h=	-0.0000	-0.0026	-0.0005	0.0067	0.0027	-0.0161	-0.0092	0.0328
0.0251	-0.0641	-0.0684	0.1585	0.4350	0.4350	0.1585	-0.0684	-0.0641
0.0251	0.0328	-0.0092	-0.0161	0.0027	0.0067	-0.0005	-0.0026	-0.0000

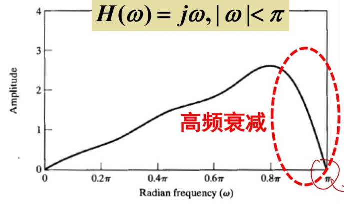


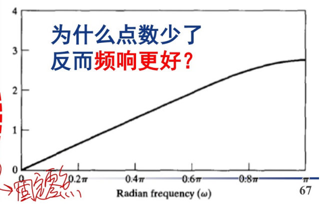


An old example: Choice of h[n] length









Chapter 09. Structures for discrete-time systems.

see lecture slides.

Δ 补充:

1. The Rectangular Pulse: $\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0. & \text{otherwise.} \end{cases}$

The Triangular Pulse: $\Lambda(t) = \begin{cases} t+1, & -1 \le t < 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$ $\Lambda(t) = \Pi(t) * \Pi(t)$

The Sgn Signal: $\operatorname{Sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0. \end{cases}$

The Sinc Signal: $\operatorname{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \ne 0 \\ 1, & t = 0. \end{cases}$

2. (a) $T\{x[n]\} = g[n]x[n]$ with $g[n]$ given: causal. linear. memoryless if $g[n]$ is bounded then stable

(b) $T\{x[n]\} = \sum_{k=n_0}^{n} x[k]$: linear.

(c) $T\{x[n]\} = \sum_{k=n-n_0}^{n+n_0} x[k]$: Always stable. linear. time invariant. if $n_0=0$ then causal. memoryless

(d) $T\{x[n]\} = x[n-n_0]$: Always stable. linear. time invariant. if $n_0 \le 0$ causal. if $n_0=0$ memoryless.

(e) $T\{x[n]\} = a x[n] + b$: Always stable. causal. time-invariant. memoryless. if $b=0$. linear.

(f) $T\{x[n]\} = x[-n]$: stable. linear.

(g) $T\{x[n]\} = x[n] + 3u[n+1]$: stable. memoryless.

3. If $x[n]$ is zero except for N consecutive points, and $h[n]$ is zero, except for M consective points. what is the maximum number of consecutive points for which $y[n]$ can be nonzero?

$\Rightarrow N+M-1$.

4. (2-31). Given the difference equation: $y[n] + \frac{1}{15}y[n-1] - \frac{2}{5}y[n-2] = x[n]$.

(a) Determine the general form of the homogenous solution to this equation.

(b) Both a causal and anticausal LTI are characterized by the given equation. Find the impulse responses of the two systems.

(c) Show that the causal LTI system is stable and the anticausal system is unstable.

(d) Find a paticular solution to the difference equation when $x[n] = (\frac{2}{5})^n u[n]$.

(a) characteristic equation: $r^2+\frac{1}{15}r-\frac{2}{5}r=0 \Rightarrow r_1=\frac{1}{5},\ r_2=-\frac{1}{3} \Rightarrow y_h[n]=C_1(\frac{1}{5})^n+C_2(-\frac{1}{3})^n$

(b) Do bilateral Z transform: $Y(z)+\frac{z^{-1}}{15}Y(z)-\frac{2z^{-2}}{5}Y(z)=X(z) \Rightarrow H(z)=\frac{z^2}{(z-\frac{1}{5})(z+\frac{1}{3})}$

$H(z)=\frac{15}{8}\cdot\left(\frac{1}{1-\frac{1}{5}z^{-1}}-\frac{1}{1+\frac{1}{3}z^{-1}}\right)$

For causal system: $|z|>\frac{1}{3}$. $\Rightarrow h[n]=\frac{15}{8}[(\frac{1}{5})^n+(-\frac{1}{3})^n]u(n)$

For anticausal system: $|z|<\frac{1}{5}$. $\Rightarrow h(n)=-\frac{15}{8}[(\frac{1}{5})^n+(-\frac{1}{3})^n]u(-n-1)$

(c) For causal system, its ROC includes the unit circle. ⇒ stable.

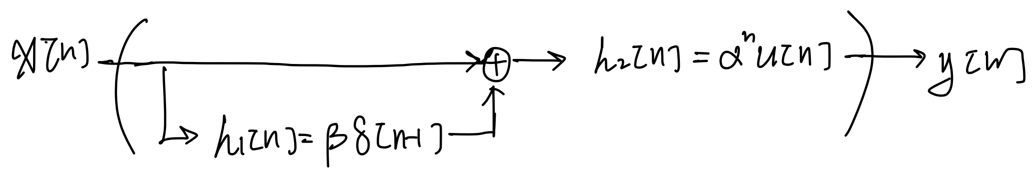
For anticausal system, its ROC doesn't includes ⇒ unstable.

(d). $x[n]=(\frac{3}{5})^n u[n] \xrightarrow{Z} X(z)=\frac{1}{1-\frac{3}{5}z^{-1}},\ |z|>\frac{3}{5}$.

Then $Y(z)=H(z)X(z)=\frac{1}{(1-\frac{1}{5}z^{-1})(1+\frac{1}{3}z^{-1})(1-\frac{3}{5}z^{-1})},\ |z|>\frac{3}{5}$.

$Y(z)=-\frac{3}{80}\cdot\frac{1}{1-\frac{1}{5}z^{-1}}-\frac{25}{264}\cdot\frac{1}{1+\frac{1}{3}z^{-1}}+\frac{27}{10}\cdot\frac{1}{1-\frac{3}{5}z^{-1}} \Rightarrow y[n]=\left[-\frac{3}{80}\cdot(\frac{1}{5})^n-\frac{25}{264}\cdot(-\frac{1}{3})^n+\frac{27}{10}\cdot(\frac{3}{5})^n\right]u(n)$

5.(2-42). Consider the system:



(a) Find the impulse response $h[n]$ of the overall system.

(b) Find the frequency response of the overall system.

(a) $h[n]=[1+h_1[n]]*h_2[n]=h_2[n]+h_1[n]*h_2[n]$

$=\alpha^n u[n]+\beta\delta[n-1]*\alpha^n u[n]=\alpha^n u[n]+\beta\alpha^{n-1}u[n-1]$

(b) $H(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}}+\frac{\beta e^{j\omega}}{1-\alpha e^{-j\omega}}=\frac{1+\beta e^{j\omega}}{1-\alpha e^{-j\omega}},\ |e^{j\omega}|>\alpha$.

(c) $Y(z)=H(z)X(z) \Rightarrow (1-\alpha z^{-1})Y(z)=(1+\beta e^{j\omega})X(z)$

DE: $y[n]-\alpha y[n-1]=x[n]+\beta x[n+1]$.

(d). causal. If $|\alpha|\le 1$ then stable.

FS: 连续周期信号: $\tilde{x}(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_s t}=\sum_{n=-\infty}^{\infty}\frac{1}{T}X_0(n\omega_s)e^{jn\omega_s t}=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\omega_s X_0(n\omega_s)e^{jn\omega_s t}$

FT: 连续非周期(由连续周期信号$T\to\infty$导致$\omega_s=\frac{2\pi}{T}\to 0$频谱连续, 级数求和变成高度为$X(n\omega_s)e^{jn\omega_s t}$宽度为$\omega_s$的小矩形求面积累加. $n=-\infty\to\infty$, $n\omega_s\to\omega$),

$$x(t)=\lim_{\omega_s\to 0}\tilde{x}(t)=\lim_{\omega_s\to 0}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\omega_s X_0(n\omega_s)e^{jn\omega_s t}=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega.$$

DFS: 离散周期信号. 周期为N. $x[n]=x[n+rN]$

DTFT: 离散时间序列信号. 时间上是离散的. 信号幅值连续. 变换产生连续信号.

DFT: 处理离散非周期信号.
对非周期信号进行周期延拓. 再进行DFS. 取其中一个周期即得到DFT.

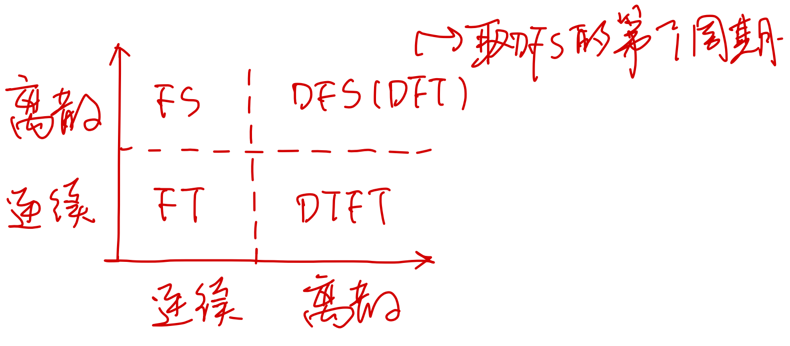
连续时间傅里叶变换CTFT:
⇒ 时域连续非周期. 频域连续非周期; 时域连续周期. 频域离散非周期. 以Ω(rad/s)为坐标轴.

离散时间傅里叶变换DTFT: $X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}$.
⇒ 时域离散序列. 频域连续周期. 以ω(rad)为坐标轴. 周期为2π.

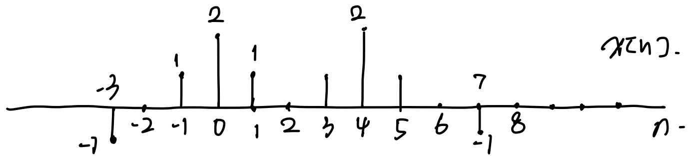
离散傅里叶变换DFT: $\tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j\frac{2\pi n}{N}k}$, ($N>$序列长度).
⇒ 时域无限长序列. 离散非周期. 频域离散. 以k为坐标轴.
⇒ DFT 相当于对DTFT进行采样.

离散傅里叶级数DFS: $\tilde{x}[n]=\sum_{k=0}^{N-1}C_k e^{j\frac{2\pi k}{N}n}$, $C_k=\frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi\frac{k}{N}n}$ (C_k也具有周期N) (N为序列周期).
⇒ 若$\tilde{x}[n]$是$x[n]$的周期延拓. 则$\tilde{X}[k]$也为$X[k]$的周期延拓. DFT和DFS是取主值与周期延拓的关系.

1. 对一个长度为N的有限长序列进行DTFT运算后. 将得到的频谱进行频率为$\omega_s=\frac{2\pi}{N}$的抽样. 结果等于对该序列进行DFT运算的结果.
2. 对长度为N的有限长序列进行周期为N的周期延拓后. 再进行DFS运算. 等同于对原序列进行DFT运算.



6.* (2-44).



(a) Evaluate $X(e^{jw})|_{w=0}$. (b) Evaluate $X(e^{jw})|_{w=\pi}$

(c) Find $\measuredangle X(e^{jw})$. (d) Evaluate $\int_{-\pi}^{\pi} X(e^{jw})\,dw$.

(e) Determine and sketch the signal whose FT is $X(e^{-jw})$.

(f) Determine and sketch the signal whose FT is $Re\{X(e^{-jw})\}$.

$\Rightarrow$ (a) $X(e^{jw})|_{w=0} = \sum_{n=-\infty}^{\infty} x[n] = -1+1+2+1+1+2+1-1 = 6$.

(b) $X(e^{jw})|_{w=\pi} = -\sum_{n=-\infty}^{\infty} x[n] = -6$.

(c) $\measuredangle X(e^{jw}) = (2\cos w + 4\cos 2w + 2\cos 3w - 2\cos 5w)(\cos 2w - j\sin 2w)$

(d) $\int_{-\pi}^{\pi} X(e^{jw})\,dw = [\int_{-\pi}^{\pi} X(e^{jw}) e^{jwn}\,dw]|_{n=0} = x[n]|_{n=0} = x[0] = 2$.

(e) $x[n+4]$.

(f) $x[n+2]$. or $\dfrac{x[n]+x[-n]}{2}$

7. (3-3). Determine the z-transform.

(1) $x_1[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise.} \end{cases}$

(2) $x_2[n] = \begin{cases} n, & 0 \le n \le N \\ 2N-n, & N+1 \le n \le 2N \\ 0, & \text{otherwise.} \end{cases}$

$\Rightarrow$ (1) $X_1(z) = \dfrac{1-z^{-N}}{1-z^{-1}}$. $z \ne 0$.

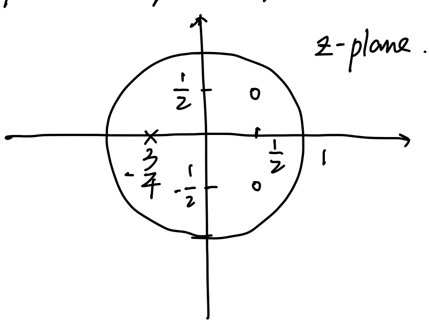
(2) $X_2[z] = z^{-1}\dfrac{(1-z^{-N})^2}{(1-z^{-1})^2}$. $z \ne 0$.

8. (3-26). Determine the inverse z-transform.

(1) $X(z) = \ln(1-4z)$. $|z| < \frac{1}{4}$.

(b) $X(z) = \dfrac{1}{1-\frac{1}{3}z^{-3}}$. $|z| > 3^{-\frac{1}{3}}$.

9. (3-32) The pole-zero diagram corresponding to the z-transform $X(z)$ of a causal sequence $x[n]$. Sketch the pole-zero diagram of $Y(z)$, where $y[n] = x[-n+3]$. Specify the region of convergence of $Y(z)$



(8-37)

8.37. Consider a finite-duration sequence $x[n]$ that is zero for $n<0$ and $n \geq N$, where N is even. The z-transform of $x[n]$ is denoted by $X(z)$. Table P8.37-1 lists seven sequences obtained from $x[n]$. Table P8.37-2 lists nine sequences obtained from $X(z)$. For each sequence in Table P8.37-1, find its DFT in Table P8.37-2. The size of the transform considered must be greater than or equal to the length of the sequence $g_k[n]$. *For purposes of illustration only,* assume that $x[n]$ can be represented by the envelope shown in Figure P8.37-1.

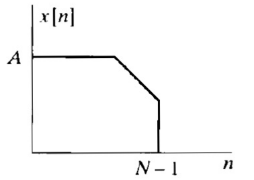


Figure P8.37-1

TABLE P8.37-2

$H_1[k] = X(e^{j2\pi k/N})$

$H_2[k] = X(e^{j2\pi k/2N})$

$H_3[k] = \begin{cases} 2X(e^{j2\pi k/2N}), & k \text{ even}, \\ 0, & k \text{ odd} \end{cases}$

$H_4[k] = X(e^{j2\pi k/(2N-1)})$

$H_5[k] = 0.5\{X(e^{j2\pi k/N}) + X(e^{j2\pi(k+N/2)/N})\}$

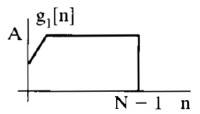
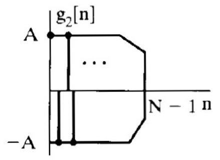
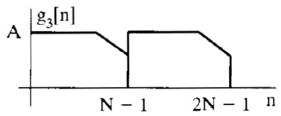
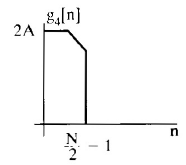
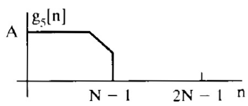
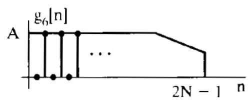
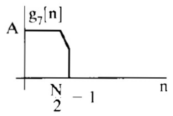
$H_6[k] = X(e^{j4\pi k/N})$

$H_7[k] = e^{j2\pi k/N}X(e^{-j2\pi k/N})$

$H_8[k] = X(e^{j(2\pi/N)(k+N/2)})$

$H_9[k] = X(e^{-j2\pi k/N})$

TABLE P8.37-1

$g_1[n] = x[N-1-n]$	
$g_2[n] = (-1)^n x[n]$	
$g_3[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ x[n-N], & N \le n \le 2N-1, \\ 0, & \text{otherwise} \end{cases}$	
$g_4[n] = \begin{cases} x[n] + x[n+N/2], & 0 \le n \le N/2-1, \\ 0, & \text{otherwise} \end{cases}$	
$g_5[n] = \begin{cases} x[n], & 0 \le n \le N-1, \\ 0, & N \le n \le 2N-1, \\ 0, & \text{otherwise} \end{cases}$	
$g_6[n] = \begin{cases} x[n/2], & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$	
$g_7[n] = x[2n]$	

(*Continued*)

(a) $g_1[n] = x[N-1-n],\ 0 \le n \le (N-1)$

$$G_1[k] = \sum_{n=0}^{N-1} x[N-1-n]\, W_N^{kn},\quad 0 \le k \le (N-1)$$

let $m = N-1-n$. then:

$$G_1[k] = \sum_{n=0}^{N-1} x[m]\, W_N^{k(N-1-m)} = W_N^{k(N-1)} \underbrace{\sum_{n=0}^{N-1} x[m]\, W_N^{-km}}_{\text{DFT}}.$$

$$W_N^{k(N-1)} = e^{-j2\pi k(N-1)/N} = e^{j2\pi k/N}.$$

$$\Rightarrow G_1[k] = e^{j2\pi k/N} X(e^{j\omega})\Big|_{\omega=-2\pi k/N} = e^{j2\pi k/N} X(e^{-j2\pi k/N}).$$

(b) $g_2[n] = \sum_{n=0}^{N-1} (-1)^n x[n]\, W_N^{kn},\ 0 \le k \le (N-1).$

$$= \sum_{n=0}^{N-1} x[n]\, W^{(\frac{N}{2})n} W_N^{kn} = \sum_{n=0}^{N-1} x[n]\, W_N^{(k+\frac{N}{2})n} = X(e^{j2\pi(k+\frac{N}{2})/N}).$$

(c) $g_3[n] = \begin{cases} x[n], & 0 \le n \le (N-1) \\ x[n-N], & N \le n \le (2N-1) \\ 0, & \text{otherwise}. \end{cases}$

$$G_3[k] = \sum_{n=0}^{2N-1} x[n] W_{2N}^{kn}, \quad 0 \le k \le (N-1).$$

$$= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{n=N}^{2N-1} x[n-N] W_{2N}^{kn}$$

$$= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{m=0}^{N-1} x[m] W_{2N}^{k(m+N)}$$

$$= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + W_{2N}^{kN} \sum_{m=0}^{N-1} x[m] W_{2N}^{km}$$

$$= \sum_{n=0}^{N-1} (1+W_{2N}^{kN}) x[n] W_{2N}^{kn} = [1+(-1)^k] X(e^{j\omega})\Big|_{\omega=\pi k/N}.$$

(d) $$g_4[n] = \begin{cases} x[n] + x[n+N/2], & 0 \le n \le (N/2-1) \\ 0 & , \text{otherwise} \end{cases}$$

$$\Rightarrow G_4[k] = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_{\frac{N}{2}}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x[n+\tfrac{N}{2}] W_{\frac{N}{2}}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_{\frac{N}{2}}^{kn} + \sum_{m=\frac{N}{2}}^{N-1} x[m] W_{\frac{N}{2}}^{k(m-\frac{N}{2})}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_{\frac{N}{2}}^{kn} + W_{\frac{N}{2}}^{-k\frac{N}{2}} \sum_{m=\frac{N}{2}}^{N-1} x[m] W_{\frac{N}{2}}^{km}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_{\frac{N}{2}}^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_{\frac{N}{2}}^{kn} = \sum_{n=0}^{N-1} x[n] W_N^{\underline{\underline{2kn}}} = X(e^{j\omega})\Big|_{\omega=4\pi k/N}$$

(e) $$g_5[n] = \begin{cases} x[n], & 0 \le n \le (N-1) \\ 0, & N \le n \le (2N-1) \\ 0, & \text{otherwise} \end{cases}$$

$$G_5[k] = \sum_{n=0}^{2N-1} x[n] W_{2N}^{kn}, \quad 0 \le k \le (N-1). \quad = \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} = X(e^{j\omega})\Big|_{\omega=\pi k/N}.$$

* (f) $$g_6[n] = \begin{cases} x[\tfrac{n}{2}], & n \text{ even}, \quad 0 \le n \le (2N-1). \\ 0, & n \text{ odd} \end{cases}$$

$$\Rightarrow G_6[k] = \sum_{n=0}^{2N-1} x[\tfrac{n}{2}] W_{2N}^{kn} = \sum_{n=0}^{N-1} x[n] W_N^{kn} = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

(g) $$g_7[n] = x[2n], \quad 0 \le n \le (\tfrac{N}{2}-1).$$

$$\Rightarrow G_7[k] = \sum_{n=0}^{\frac{N}{2}-1} x[2n] W_{\frac{N}{2}}^{kn} = \sum_{n=0}^{N-1} x[n] \left(\frac{1+(-1)^n}{2}\right) W_{\frac{N}{2}}^{kn}$$

$$= \sum_{n=0}^{N-1} x[n] \left[\frac{1+W_N^{\frac{N}{2}n}}{2}\right] W_{\frac{N}{2}}^{kn} = \frac{1}{2} \sum_{n=0}^{N-1} x[n] (W_N^{kn} + W_N^{(k+\frac{N}{2})n})$$

$$= \frac{1}{2} [X(e^{j(2\pi/N)}) + X(e^{j(2\pi/N)(k+N/2)})].$$

Homework 1

(2-1).

(a) causal. memoryless. linear. (b) linear. (c) stable. linear. time-invariant.

(d) stable. linear. time-invariant. (e). stable. causal. time-invariant. memoryless.

(f) stable. causal. time-invariant. memoryless.

(g) stable. linear. (h) stable. memoryless.

(2-2).

(a) $N_4 = N_0 + N_2$. $N_5 = N_1 + N_3$

(b) $N+M-1$.

(2-3)

$$s[n] = h[n] * u[n] = a^{-n}u[-n] * u[n] = \sum_{k=-\infty}^{\infty} a^{k} u[-k]\, u[n-k]$$

$$n<0:\ s[n] = \sum_{k=-\infty}^{-n} a^{k} = \sum_{k=n}^{\infty} a^{k} = \frac{a^{n}}{1-a}.$$

$$n\geq 0:\ s[n] = \sum_{k=-\infty}^{0} a^{-k} = \frac{1}{1-a}$$

$$\Rightarrow y[n] = \begin{cases} \frac{a^{n}}{1-a}, & n<0 \\ \frac{1}{1-a}, & n\geq 0. \end{cases}$$

(2-7)

(a) $T = \frac{2\pi}{\frac{\pi}{6}} = 12$. (b) $\frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3}$, $T=8$.

(c) not periodic (d). $\frac{2\pi}{\frac{\pi}{\sqrt{2}}} = 2\sqrt{2}$. not periodic.

Homework 2

(2-31)

(a) characteristic equation: $r^2+\frac{1}{15}r-\frac{2}{5}r=0 \Rightarrow r_1=\frac{1}{5}.\ r_2=-\frac{1}{3} \Rightarrow y_h[n]=C_1(\frac{1}{5})^n+C_2(-\frac{1}{3})^n$

(b) Do bilateral Z transform: $Y(z)+\frac{z^{-1}}{15}Y(z)-\frac{2z^{-2}}{5}Y(z)=X(z) \Rightarrow H(z)=\frac{z^2}{(z-\frac{1}{5})(z+\frac{1}{3})}$

$H(z)=\frac{15}{8}\cdot(\frac{1}{1-\frac{1}{5}z^{-1}}-\frac{1}{1+\frac{1}{3}z^{-1}})$

For causal system: $|z|>\frac{1}{3}$. $\Rightarrow h[n]=\frac{15}{8}[(\frac{1}{5})^n+(-\frac{1}{3})^n]u(n)$

For anticausal system: $|z|<\frac{1}{5}$. $\Rightarrow h(n)=-\frac{15}{8}[(\frac{1}{5})^n+(-\frac{1}{3})^n]u(-n-1)$.

(c) For causal system, its ROC inludes the unit circle. ⇒ stable.

For anticausal system, its ROC doesn't inlubes ⇒ unstable.

(d). $x[n]=(\frac{3}{5})^n u[n] \xrightarrow{Z} X(z)=\frac{1}{1-\frac{3}{5}z^{-1}}.\ |z|>\frac{3}{5}$.

Then $Y(z)=H(z)X(z)=\frac{1}{(1-\frac{1}{5}z^{-1})(1+\frac{1}{3}z^{-1})(1-\frac{3}{5}z^{-1})}.\ |z|>\frac{3}{5}$.

$Y(z)=-\frac{3}{80}\cdot\frac{1}{1-\frac{1}{5}z^{-1}}-\frac{25}{264}\cdot\frac{1}{1+\frac{1}{3}z^{-1}}+\frac{27}{10}\cdot\frac{1}{1-\frac{3}{5}z^{-1}} \Rightarrow y[n]=[-\frac{3}{80}\cdot(\frac{1}{5})^n-\frac{25}{264}\cdot(-\frac{1}{3})^n+\frac{27}{10}\cdot(\frac{3}{5})^n]$
$u(n)$

(2-42).

(a) $h[n]=[1+h_1[n]]*h_2[n]=h_2[n]+h_1[n]*h_2[n]$

$=\alpha^n u[n]+\beta\delta[n-1]*\alpha^n u[n]=\alpha^n u[n]+\beta\alpha^{n-1}u[n-1]$

(b) $H(e^{jw})=\frac{1}{1-\alpha e^{-jw}}+\frac{\beta e^{jw}}{1-\alpha e^{-jw}}=\frac{1+\beta e^{jw}}{1-\alpha e^{-jw}}$

(c) $Y(z)=H(z)X(z) \Rightarrow (1-\alpha z^{-1})Y(z)=(1+\beta e^{jw})X(z)$

DE: $y[n]-\alpha y[n-1]=x[n]+\beta x[n+1]$.

(d). causal. If $|\alpha|\leq 1$ then stable.

(2-44). ⇒ (a) $X(e^{jw})|_{w=0}=\sum_{n=-\infty}^{\infty}x[n]=-1+1+2+1+1+2+1-1=6$.

(b) $X(e^{jw})|_{w=\pi}=-\sum_{n=-\infty}^{\infty}x[n]=-6$.

(c) $\measuredangle X(e^{jw})=(2\cos w+4\cos 2w+2\cos 3w-2\cos 5w)(\cos 2w-j\sin 2w)$

(d) $\int_{-\pi}^{\pi}X(e^{jw})dw=[\int_{-\pi}^{\pi}X(e^{jw})e^{jwn}dw]|_{n=0}=2\pi x[n]|_{n=0}=4\pi$

(e) $x[n+4]$.

(f) $x[n+2]$. or $\frac{x[n]+x[-n]}{2}$

(2-46).

(a) $X(e^{jw}) = \dfrac{1-a^2}{(1-ae^{-jw})(1-ae^{jw})} = \dfrac{ae^{-jw}}{1-ae^{-jw}} + \dfrac{1}{1-ae^{jw}} = \dfrac{ae^{-jw}}{1-ae^{-jw}} - \dfrac{e^{-jw}}{a-e^{-jw}}$. $|a|<1$

$\Rightarrow x[n] = a^n u[n-1] - a^{-n} u[-n]$.

(b) $\int_{-\pi}^{\pi} X(e^{jw}) \cos w \, dw = \int_{-\pi}^{\pi} X(e^{jw}) \cdot \dfrac{e^{jw}+e^{-jw}}{2} dw$

$= \frac{1}{2} \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw \Big|_{n=1} + \frac{1}{2} \int_{-\pi}^{\pi} X(e^{-jw}) e^{jwn} dw \Big|_{n=1}$

$= \int_{-\pi}^{\pi} X(e^{jw}) e^{jwn} dw \Big|_{n=1} = 2\pi x[n]\Big|_{n=1} = 2\pi x[1] = 2\pi a$.

Homework 3

(3-1)

(a) $\frac{1}{1-\frac{1}{2}z^{-1}}$, $|z|>|\frac{1}{2}|$. (b) $\frac{1}{1-\frac{1}{2}z^{-1}}$, $|z|<|\frac{1}{2}|$.

(c) $(\frac{1}{2})^n u[-n] = \left((\frac{1}{2})^{n+1} u[-n-1]\right)\Big|_{n=n-1}$. $\Rightarrow -\frac{z^{-1}}{2-z^{-1}}$. $|z|<|\frac{1}{2}|$.

(d) 1. all z. (e) z^{-1}. $z\neq 0$ (f) z. $|z|<\infty$

(g) $(\frac{1}{2})^n(u[n]-u[n-10]) = (\frac{1}{2})^n[\delta[n]+\delta[n-1]+\dots+\delta[n-9]]$.

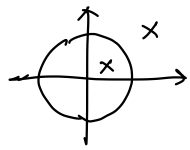
$= \delta[n]+\frac{1}{2}\delta[n-1]+(\frac{1}{2})^2\delta[n-2]+\dots+(\frac{1}{2})^9\delta[n-9]$.

$\Rightarrow 1+\frac{1}{2}z+(\frac{1}{2})^2z^2+\dots+(\frac{1}{2})^9z^9 = \frac{1-(\frac{1}{2}z)^{10}}{1-\frac{1}{2}z}$. $|z|\neq 0$

(3-3).

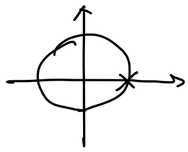
(a) $x_a[n]=\alpha^{|n|} = \alpha^n u[n]+\alpha^{-n}u[-n-1]$.

$\Rightarrow X_a(z)=\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\alpha^{-1}z^{-1}}$. $|\alpha|<|z|<|\frac{1}{\alpha}|$.



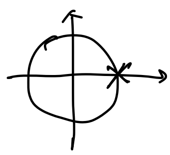
(b) $x_b[n]=u[n]-u[n-N]$

$\Rightarrow X_b(z)=\frac{1}{1-z^{-1}}(1-z^{-N})$. $|z|>1$.



(c) $x_c[n]=x_b[n]*x_b[n]$

$\Rightarrow X_c(z)=X_b(z)\cdot X_b(z) = \frac{1}{(1-z^{-1})^2}(1-z^{-N})^2$.



(3-7). (a) $X(z)=-\frac{1}{1-z^{-1}}+\frac{1}{1-\frac{1}{2}z^{-1}}$. $\frac{1}{2}<|z|<1$.

$\Rightarrow H(z)=\frac{Y(z)}{X(z)}=\frac{1-z^{-1}}{1+z^{-1}}$. $|z|>1$.

(b) $\frac{1}{2}<|z|<1$.

(c) $Y(z)=\frac{1}{3}\left(\frac{1}{1+z^{-1}}-\frac{1}{1-\frac{1}{2}z^{-1}}\right)$

$\Rightarrow y[n]=\frac{1}{3}[(-1)^n u[-n-1]+(\frac{1}{2})^n u[n])$.

(3-10). (a) $|z|>\frac{3}{4}$. converge. (b) all z. converge.

(c) $|z|<2$. converge. (d) $|z|>1$. not converge.

(e) all z. converge. (f) $\frac{1}{2}<|z|<\sqrt{13}$. converge.

(3-17). Do z transform:

$$Y(z) - \frac{5}{2}z^{-1}Y(z) + z^{-2}Y(z) = X(z) - z^{-1}X(z).$$

$$\Rightarrow H(z) = Y(z)/X(z) = \frac{1-z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} = \frac{1}{3}\cdot\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{3}\cdot\frac{1}{1-2z^{-1}}.$$

(1) $|z| < \frac{1}{2}$. $h[n] = \frac{1}{3}\cdot(\frac{1}{2})^n u[-n-1] + \frac{2}{3}\cdot 2^n\cdot u[-n-1]$. $\Rightarrow h[0] = 0$.

(2) $\frac{1}{2} < |z| < 2$. $h[n] = \frac{1}{3}\cdot(\frac{1}{2})^n u[n] + \frac{2}{3}\cdot 2^n\cdot u[-n-1] \Rightarrow h[0] = \frac{1}{3}$

(3) $|z| > 2$. $h[n] = \frac{1}{3}\cdot(\frac{1}{2})^n\cdot u[n] + \frac{2}{3}\cdot 2^n\cdot u[n] \Rightarrow h[0] = 1$.

(3-18).

(a) $H(z) = -2 + \frac{1}{3}\cdot\frac{1}{1+\frac{1}{2}z^{-1}} + \frac{8}{3}\cdot\frac{1}{1-z^{-1}}$. $|z| > 1$.

$\Rightarrow h[n] = -2\delta[n] + \frac{1}{3}(\frac{1}{2})^n u[n] + \frac{8}{3}\cdot(-1)^n u[n]$.

(b). $H(e^{j\omega}) = -2 + \frac{1}{3}\cdot\frac{1}{1+\frac{1}{2}e^{-j\omega}} + \frac{8}{3}\cdot\frac{1}{1-e^{-j\omega}}$. $\Rightarrow H(e^{j\frac{\pi}{2}}) = -2 + \frac{1}{3}\cdot\frac{1}{1-\frac{1}{2}j} + \frac{8}{3}\cdot\frac{1}{1+j} = -\frac{2}{5} - \frac{6}{5}j$

$y[n] = e^{j\frac{\pi}{2}n}(-\frac{2}{5} - \frac{6}{5}j) = \frac{6}{5} - \frac{2}{5}j$.

(3-26).

(a) $X(z) = -1 + \frac{2}{1+\frac{1}{3}z^{-1}}$. $|z| > \frac{1}{3}$. $\Rightarrow x[n] = -\delta[n] + 2\cdot(-\frac{1}{3})^n u[n]$.

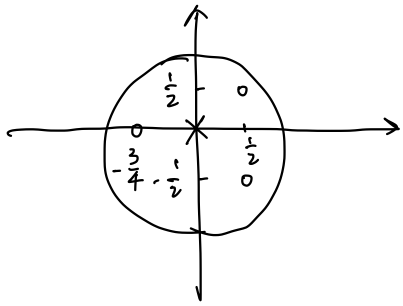
(b) $X(z) = \frac{3z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}} = \frac{z^{-1}}{1+\frac{1}{4}z^{-1}} + \frac{2z^{-1}}{1-\frac{1}{2}z^{-1}}$. $|z| > \frac{1}{2}$.

$\Rightarrow x[n] = (-\frac{1}{4})^{n-1}u[n-1] + (\frac{1}{2})^{n-2}u[n-1]$.

(c) $X(z) = \ln(1-4z) = \Sigma\frac{(-4z)^n}{n} = \Sigma\frac{(-4)^n}{n}z^n$.

(d). $X(z) = \frac{1}{1-\frac{1}{3}z^{-3}} = \frac{1}{(1-\frac{1}{\sqrt[3]{3}}z^{-1})(\frac{1}{3^{\frac{2}{3}}} - \frac{1}{\sqrt[3]{3}}z^{-1} + z^{-2})}$

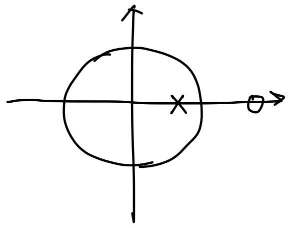
(3.32) $x[n] \Rightarrow X(z)$. $x[-n] \Rightarrow X(z^{-1})$. $x[-n+3] \Rightarrow z^{-3}X(z^{-1}) = Y(z)$. $|z|>0$.



(3.43)

(a) $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}$. $\frac{1}{2}<|z|<2$. $Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}$. $|z|>\frac{3}{4}$.

$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$. $\frac{3}{4}<|z|<2$.



(b) $H(z) = \frac{8}{3} - \frac{5}{8}\cdot\frac{1}{1-\frac{3}{4}z^{-1}}$.

$h[n] = \frac{8}{3}\delta[n] - \frac{5}{8}\cdot\left(\frac{3}{4}\right)^n u[n]$.

(c) $Y(z) = H(z)X(z) \Rightarrow Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$

$\Rightarrow y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$.

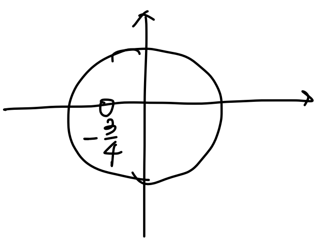
(d) causal. stable.

(3.46)

(a) $|z|>\frac{1}{2}$. (b) right-sided. (c) $|z|>\frac{3}{4}$

(d) Yes. (e) $X(z) = \frac{z-\frac{1}{4}}{(z+\frac{3}{4})(z-\frac{1}{2})}$. $x(0) = \lim_{z\to\infty} X(z) = 1$

(f) $Y(z) = \frac{z-\frac{1}{4}}{z-\frac{1}{2}} \Rightarrow H(z) = Y(z)/X(z) = z+\frac{3}{4}$



ROC = all z (g). No.

Homework 4

(5-3). $\Rightarrow \quad z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^{-1}+\frac{1}{3}z^{-2}} = \frac{z}{1+\frac{1}{3}z^{-1}}$$ ，极点，$|z| = \frac{1}{3}$.

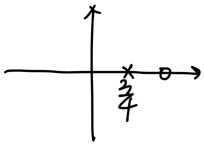
① $|z|>\frac{1}{3}$ 时. $h(n) = (-\frac{1}{3})^{n+1}u[n+1] \Rightarrow$ (a)

② $|z|<\frac{1}{3}$ 时. $h(n) = \frac{1}{3}(-\frac{1}{3})^{n}u[-n-2] \Rightarrow$ (d)

(5-4).

(a) $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}$. $\frac{1}{2}<|z|<2$. $\quad Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}$. $|z|>\frac{3}{4}$.

$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$. $|z|>\frac{3}{4}$.



(b) $H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}$. $|z|>\frac{3}{4}$.

$\Rightarrow h[n] = (\frac{3}{4})^{n}u[n] - 2(\frac{3}{4})^{n-1}u[n-1]$.

(c) $y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$.

(d) Stable. Causal.

(5-12)

(a) $|z_0| = 0.2, 3$. $\quad |z_p| = 0.9$. $\Rightarrow$ Stable.

(b). $$H(z) = \frac{(1+\frac{1}{5}z^{-1})(1-9z^{-2})}{(1+0.81z^{-2})} = \frac{(1+0.2z^{-1})(1+3z^{-1})(1-3z^{-1})}{(1+0.81z^{-2})} \cdot \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}} \cdot \frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{(1+0.2z^{-1})(1+\frac{1}{3}z^{-1})(1-\frac{1}{3}z^{-1})}{(1+0.81z^{-2})} \cdot \frac{(1+3z^{-1})(1-3z^{-1})}{(1+\frac{1}{3}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$= \frac{9(1+0.2z^{-1})(1-\frac{1}{9}z^{-2})}{(1+0.81z^{-2})} \cdot \frac{(1-9z^{-2})}{9(1-\frac{1}{9}z^{-2})}$$

$$\Rightarrow H_1(z) = \frac{9(1+0.2z^{-1})(1-\frac{1}{9}z^{-2})}{(1+0.81z^{-2})} . \quad H_{ap}(z) = \frac{(1-9z^{-2})}{9(1-\frac{1}{9}z^{-2})}$$

15-15) (a) $h[n] = 2\delta(n) + \delta(n-1) + 2\delta(n-2)$. $\Rightarrow H(e^{j\omega}) = 2 + e^{-j\omega} + 2e^{-2j\omega}$.

$\Rightarrow \alpha=1$. $\beta=0$. $A(e^{j\omega}) = 1+4\cos(\omega n)$ $\Rightarrow$ generalized linear phase filter.

(b) not generalized ... or linear phase system.

(c) $\alpha=1$. $\beta=0$. $A(e^{j\omega}) = 3+\cos(\omega n)$, Linear phase system.

(d) $\alpha=\frac{1}{2}$. $\beta=0$. $A(e^{j\omega}) = 2\cos(\frac{\omega n}{2})$. Generalized linear phase system.

(e) $\alpha=1$. $\beta=\frac{\pi}{2}$. $A(e^{j\omega}) = 2\sin(\omega n)$ Generalized linear phase system.

15-18).

(a) $H_{min}(z) = \dfrac{2(1-\frac{1}{2}z^{-1})}{1+\frac{1}{3}z^{-1}}$.

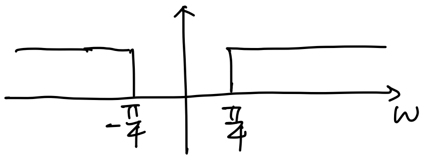
(b) $H_{min}(z) = 3(1-\frac{1}{2}z^{-1})$.

(c) $H_{min}(z) = \dfrac{9}{4} \cdot \dfrac{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})}{(1-\frac{3}{4}z^{-1})^2}$.

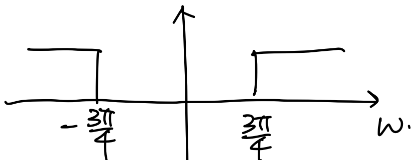
15-19). $\alpha_1 = 2$. $\alpha_2 = \frac{3}{2}$. $\alpha_3 = 2$. $\alpha_4 = 3$. $\alpha_5 = 3$. $\alpha_6 = \frac{7}{2}$

15-21)

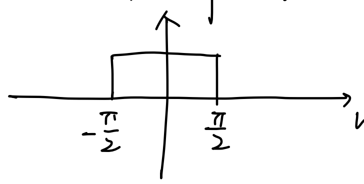
(a) Highpass filter



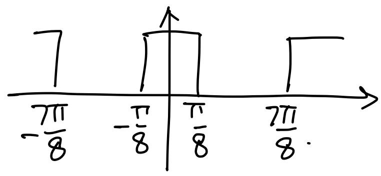
(b) Highpass filter :



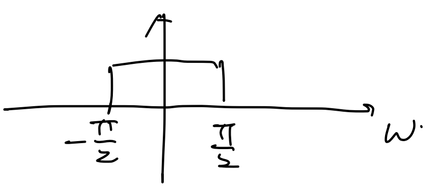
(c) Lowpass filter :



(d) bandstop :

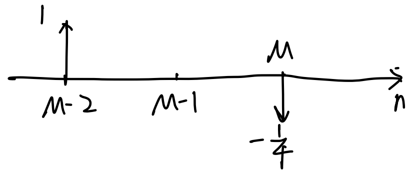


(e) lowpass :



15-30)

(a) $H(z) = \dfrac{(z+\frac{1}{2})(z-\frac{1}{2})}{z^M} = z^{-(M-2)}(1-\frac{1}{4}z^{-2}).$ $\Rightarrow$



(b) $h(n) = \delta[n-(M-2)] - \frac{1}{4}\delta[n-M]$

$w(n) = x(n) * h(n) \Rightarrow y(n) = x(2n) * h(2n). = x[n-(M-2)] - \frac{1}{4}x[n-M]$

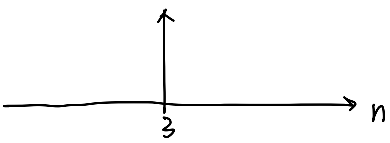
$w'(n) = x(2n). \Rightarrow g(n) = h(\frac{n}{2}). \Rightarrow G(z) = z^{-\frac{M-2}{2}} - \frac{1}{4}z^{-\frac{M}{2}}. \Rightarrow M$ is even.

15-42)

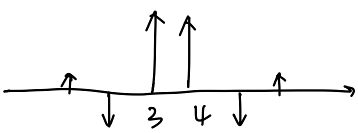
(a) $A(e^{jw}) = 1.$ $\phi(w) = -\alpha w,$ $|w| < \pi$

(b) $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha w} e^{jwn} dw = \dfrac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}$

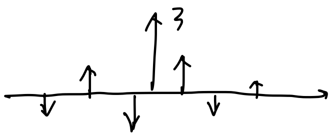
(i) $\alpha = 3$:



$\alpha = 3.5$



$\alpha = 3.25$



(c) (i) symmetric about α

(ii) symmetric about α.

(iii). not symmetric

15-48)

(1) IIR.

(2) FIR. linear phase. Stable, Generalized linear phase, Positive group delay

(3) IIR. all-pass. stable. Positive group. delay.

Homework 5

14-4).

(a) $x[n] = x_c[nT] = \sin(20\pi \cdot nT) + \cos(40\pi \cdot nT)$. $\Rightarrow$ $T = \frac{1}{100}$.

(b). Unique.

14-7)

(a). $X_c(\Omega) = S_c(\Omega) + \alpha S_c(\Omega) e^{-j\Omega\tau_d}$.

$\Rightarrow X(e^{jw}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right) + \frac{\alpha e^{-jw\frac{\tau_d}{T}}}{T}\sum_{k=-\infty}^{\infty} S_c\left(j\left(\frac{w}{T} - \frac{2\pi k}{T}\right)\right)$.

(b). $s[n] = s_c(nT)$. $r[n] = x_c(nT) = s_c(nT) + \alpha s_c(nT - \tau_d)$

$= s[n] + \alpha s_c[n - \frac{\tau_d}{T}]$.

$\Rightarrow r(e^{jw}) = s(e^{jw})[1 + \alpha e^{-jw\cdot\frac{\tau_d}{T}}]$.

Then $H(e^{jw}) = \frac{r(e^{jw})}{s(e^{jw})} = 1 + \alpha e^{-jw\frac{\tau_d}{T}}$.

(c) (i) $\tau_d = T$. $H(e^{jw}) = 1 + \alpha e^{-jw}$. $\Rightarrow h[n] = \delta[n] + \alpha\delta[n-1]$.

(ii) $\tau_d = \frac{T}{2}$. $H(e^{jw}) = 1 + \alpha e^{-\frac{1}{2}jw}$

$\Rightarrow h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 + \alpha e^{-\frac{1}{2}jw})e^{jwn}dw = \delta[n] + \frac{\alpha\sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})}$.

4-8) (a) $w = \Omega T \leq 2\pi \Rightarrow T_{max} = \frac{2\pi}{\Omega} = \frac{1}{2\times 10^4}$.

(b) $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = T\sum_{k=-\infty}^{n} x[k] = T\sum_{k=-\infty}^{\infty} x[k]u[n-k]$.

$\Rightarrow h[n] = T\cdot u[n]$.

(c) $y[n]\Big|_{n=\infty} = T\sum_{k=-\infty}^{\infty} x[k] = T\sum_{k=-\infty}^{\infty} x[k]e^{-jwk}\Big|_{w=0} = T X(0)$.

(d) $X(e^{jw}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{w}{T} + n\frac{2\pi}{T}\right)$. $\Rightarrow T\cdot X(e^{jw})\Big|_{w=0} = \sum_{n=-\infty}^{\infty} X_c\left(\frac{2\pi n}{T}\right) = y[n]\Big|_{n=\infty}$

$\Rightarrow \sum_{n=-\infty}^{\infty} X_c\left(\frac{2\pi n}{T}\right) = \int_{-\infty}^{\infty} x_c(t)dt = X_c(\Omega)\Big|_{\Omega=0}$. $\Rightarrow T \leq \frac{1}{1\times 10^4}$.

(4-11)

(a) $T=\frac{1}{40}$. not unique. $T'=\frac{9}{40}$.

(b). $T=\frac{1}{20}$. unique.

(4-15).

(a). $\Rightarrow$ bandlimited $w\in[-\frac{\pi}{4},\frac{\pi}{4}]$ $\Rightarrow$ $x_r[n]=x[n]$

(b). $\nexists$ $w\in[-\frac{\pi}{2},\frac{\pi}{2}]$. $\Rightarrow$ $x_r[n]\neq x[n]$

(c). $\exists$ $w\in[-\frac{\pi}{2},\frac{\pi}{2}]$. $\Rightarrow$ $x_r[n]=x[n]$

(4-18). $\frac{w}{L}\leq\min(\frac{\pi}{L},\frac{\pi}{M})$

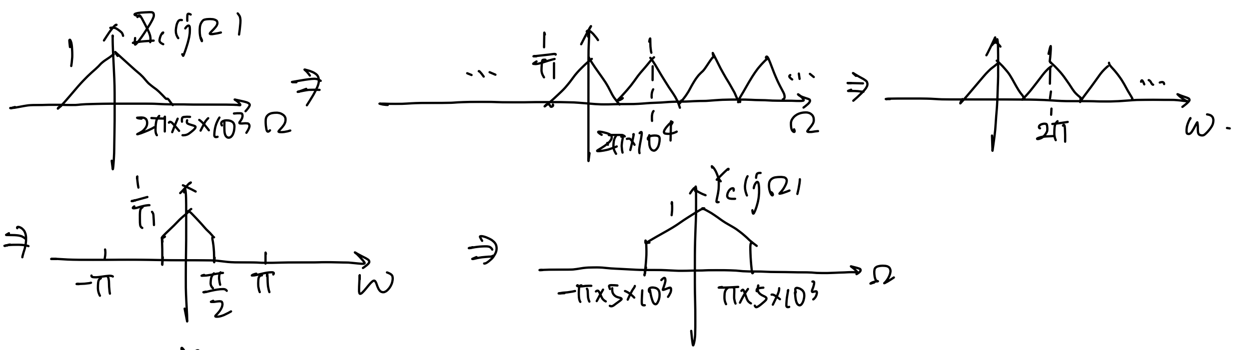
$\Rightarrow$ (a) $\frac{w_0}{2}\leq\frac{\pi}{3}$ $\Rightarrow$ $w_{0\,max}=\frac{2\pi}{5}$.

(b) $\frac{w_0}{3}\leq\frac{\pi}{5}$ $\Rightarrow$ $w_{0\,max}=\frac{3\pi}{5}$.

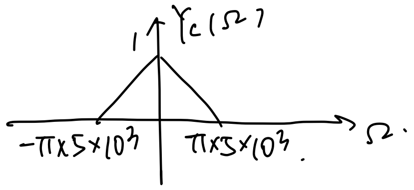
(c) $L>M$. $\Rightarrow$ $w_{0\,max}=\pi$.

(4-24)

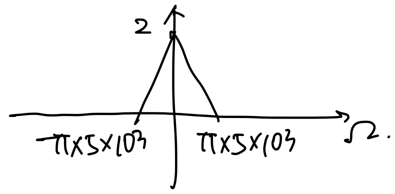
(a)



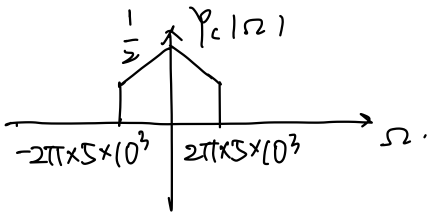
(b)



(c)



(d)



(4-29)

$X(e^{jw}) \to \boxed{\uparrow 2} \to X(e^{2jw}) \to \boxed{H_1(e^{jw})} \to X(e^{2jw})H_1(e^{jw}) \to \boxed{\downarrow 2} \to Y_1(e^{jw})$

$$Y_1(e^{jw}) = \frac{1}{2}X(e^{\frac{2jw}{2}})H_1(e^{\frac{1}{2}jw}) + \frac{1}{2}X(e^{\frac{2j(w-2\pi)}{2}})H_1(e^{\frac{j(w-2\pi)}{2}}).$$

$$= \frac{1}{2}\left(H_1(e^{\frac{1}{2}jw}) + H_1(e^{j(\frac{w}{2}-\pi)})\right)X(e^{jw}).$$

$$\Rightarrow H_2(e^{jw}) = \frac{1}{2}\left[H_1(e^{\frac{1}{2}jw}) + H_1(e^{j(\frac{w}{2}-\pi)})\right].$$

(4-37).

$s[n] \to \boxed{\uparrow 3} \to$ (Lowpass Filter, cutoff $= \frac{\pi}{3}$, gain $= 3$) $\to \boxed{\downarrow 5} \to s_1[n]$.

(4-40).

$$X(e^{jw}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{w}{T} - n\frac{2\pi}{T}\right). \Rightarrow X_e(e^{jw}) = X(e^{jwL})$$

$$H(e^{jw})X_e(e^{jw}) = \frac{e^{-jw}}{T}X_c\left(\frac{wL}{T}\right) \Rightarrow h[n]x_e[n] = \frac{1}{L}x_c[n\tfrac{T}{L}]\Big|_{n-1} = \frac{1}{L}x_c[(n-1)\tfrac{T}{L}]$$

$$\Rightarrow y[n] = \frac{1}{L}x_c[(n-1)\tfrac{T}{L}]\Big|_{nL} = \frac{1}{L}x_c[nT - \tfrac{T}{L}].$$

Homework 6

(8-1).

(a) Yes. $x[n+6] = x_c\left(\frac{n10^{-3}}{6} + 10^{-3}\right) = x_c\left(\frac{n10^{-3}}{6}\right) = x[n] \Rightarrow T = 6.$

(b) $x_c(t)$ has the maximum frequency of $f_N = \frac{9}{10^{-3}}$ Hz.

$f_s = \frac{6}{10^{-3}} < 2\times\frac{9}{10^{-3}}$. $\Rightarrow$ not the Nyquist rate.

(c) $\tilde{X}[k] = 6\sum_{r=-\infty}^{\infty} a_{k-6r}$.

(8-4)

(a) $X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$, $|\alpha|<1$.

(b) The DFS of $\tilde{x}[n]$ =

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn} = \sum_{n=0}^{N-1}\sum_{r=-\infty}^{\infty} x[n+rN]W_N^{kn}.$$

$$= \sum_{n=0}^{N-1}\sum_{r=-\infty}^{\infty}\alpha^{n+rN}u[n+rN]W_N^{kn} = \sum_{n=0}^{N-1}\sum_{r=0}^{\infty}\alpha^{n+rN}W_N^{kn}.$$

$$= \sum_{r=0}^{\infty}\alpha^{rN}\sum_{n=0}^{N-1}\alpha^n W_N^{kn} = \sum_{r=0}^{\infty}\alpha^{rN}\left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}}\right).\ |\alpha|<1.$$

$$= \frac{1}{1-\alpha^N}\left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}}\right),\ |\alpha|<1$$

$$\tilde{X}[k] = \frac{1}{1-\alpha e^{-j(2\pi k/N)}},\quad |\alpha|<1$$

(c) $\Rightarrow \tilde{X}[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$.

(8-5)

(a) $X[k] = \sum_{n=0}^{N-1}\delta[n]W_N^{kn} = 1.$

(b) $X[k] = \sum_{n=0}^{N-1}\delta[n-n_0]W_N^{kn} = e^{-j2\pi k n_0/N}.$

(c) $X[k] = \sum_{n=0}^{N-1}x[n]W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1}W_N^{2kn} = \frac{1-e^{-j2\pi k}}{1-e^{-j(4\pi k/N)}}.$

(d) $X[k] = \sum_{n=0}^{\frac{N}{2}-1}x[n]W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1}W_N^{kn} = \frac{1-e^{-j\pi k}}{1-e^{-j(2\pi k)/N}}.$

(e) $X[k] = \sum_{n=0}^{N-1}a^n W_N^{kn} = \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j(2\pi k)/N}},\ 0\le k\le(N-1).$

(8-6)

(a) $X(e^{jw}) = \sum_{n=-\infty}^{\infty} e^{jw_0 n} \cdot e^{-jwn} = \sum_{n=0}^{N-1} e^{-j(w-w_0)n} = \frac{1-e^{-j(w-w_0)N}}{1-e^{-j(w-w_0)}}$

$$= \frac{e^{-j(w-w_0)(N/2)}}{e^{-j(w-w_0)/2}} \left(\frac{\sin[(w-w_0)(N/2)]}{\sin[(w-w_0)/2]} \right)$$

$$= e^{-j(w-w_0)((N-1)/2)} \left(\frac{\sin[(w-w_0)(N/2)]}{\sin[(w-w_0)/2]} \right)$$

(b) $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{N-1} e^{jw_0 n} W_N^{kn}$

$$= \frac{1-e^{-j((2\pi k/N)-w_0)N}}{1-e^{-j((2\pi k/N)-w_0)}} = e^{-j(\frac{2\pi k}{N}-w_0)(\frac{N-1}{2})} \frac{\sin[(\frac{2\pi k}{N}-w_0)\frac{N}{2}]}{\sin[(\frac{2\pi k}{N}-w_0)/2]}.$$

(c) $w_0 = \frac{2\pi k_0}{N} = X[k] = e^{-j(2\pi/N)(k-k_0)((N-1)/2)} \frac{\sin\pi(k-k_0)}{\sin\pi(k-k_0)/N)}.$

(8-16) $\Rightarrow 4 = b+1 \Rightarrow b=3.$

(8-37)

$g_1[n] \Rightarrow H_7[k]$ $g_2[n] \Rightarrow H_8[k]$

$g_3[n] \Rightarrow H_3[k]$ $g_4[n] \Rightarrow H_6[k]$

$g_5[n] \Rightarrow H_2[k]$ $g_6[n] \Rightarrow H_1[k]$

$g_7[n] \Rightarrow H_5[k]$

Homework 7

(9-2)

(a) gain $= -W_N^2$ (b) one

(c) The gains that input sample contributes to output sample $X[2]$:

$x[0] = 1$. $x[1] = W_N^2$ $x[2] = -W_N^0 = -1$

$x[3] = -W_N^0 W_N^2 = -W_N^2$ $x[4] = W_N^0 = 1$ $x[5] = W_N^0 W_N^2 = W_N^2$

$x[6] = -W_N^0 W_N^0 = -1$. $x[7] = -W_N^0 W_N^0 W_N^2 = -W_N^2$.

Then: $X[2] = x[0] + x[1]W_8^2 + x[2]W_8^4 + x[3]W_8^6 + x[4]W_8^8 + x[5]W_8^{10} + x[6]W_8^{12} + x[7]W_8^{14}$

$= \sum_{n=0}^{7} x[n] W_8^{2n}$.

(9-3)

(a) $A[0] \sim A[7] = x[0].\ x[4].\ x[2].\ x[6]\ x[1]\ x[5]\ x[3]\ x[7]$

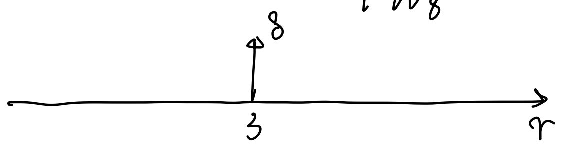
$D[0] \sim D[7] = X[0] \sim X[7]$.

(b) The DFT of $x[n] = (-W_N^n)$. $N=8$:

$X[k] = \sum_{n=0}^{7} (-W_8)^n W_8^{nk} = \sum_{n=0}^{7} (-1)^n W_8^n W_8^{nk} = \sum_{n=0}^{7} (W_8^{-4})^n W_8^n W_8^{nk}$

$= \sum_{n=0}^{7} W_8^{n(k-3)} = \frac{1-W^{k-3}}{1-W_8^{k-3}} = 8\delta[k-3]$.

$D(r) =$



(c) $D[0] = C[0] + C[4]$

$D[1] = C[1] + C[5]W_8^1$

$D[2] = C[2] + C[6]W_8^2$

$D[3] = C[3] + C[7]W_8^3$

$D[4] = C[0] - C[4]$

$D[5] = C[1] - C[5]W_8^1$

$D[6] = C[2] - C[6]W_8^2$

$D[7] = C[3] - C[7]W_8^3$

$\Rightarrow$

$C[0] = (D[0] + D[4])/2$

$C[1] = (D[1] + D[5])/2$

$C[2] = (D[2] + D[6])/2$

$C[3] = (D[3] + D[7])/2$

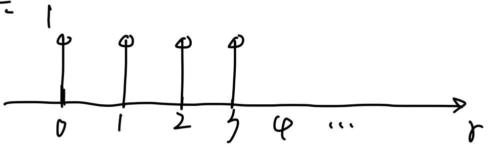
$C[4] = (D[0] - D[4])/2$

$C[5] = (D[1] - D[5])W_8^{-1}/2$

$C[6] = (D[2] - D[6])W_8^{-2}/2$

$C[7] = (D[3] - D[7])W_8^{-3}/2$

Then $C[n] =$



(9-5) $X = (A-B)D + (C-D)A = AD - BD + CA - DA = AC - BD$

$Y = (A-B)D + (C+D)B = AD - BD + BC + BD = AD - BC$

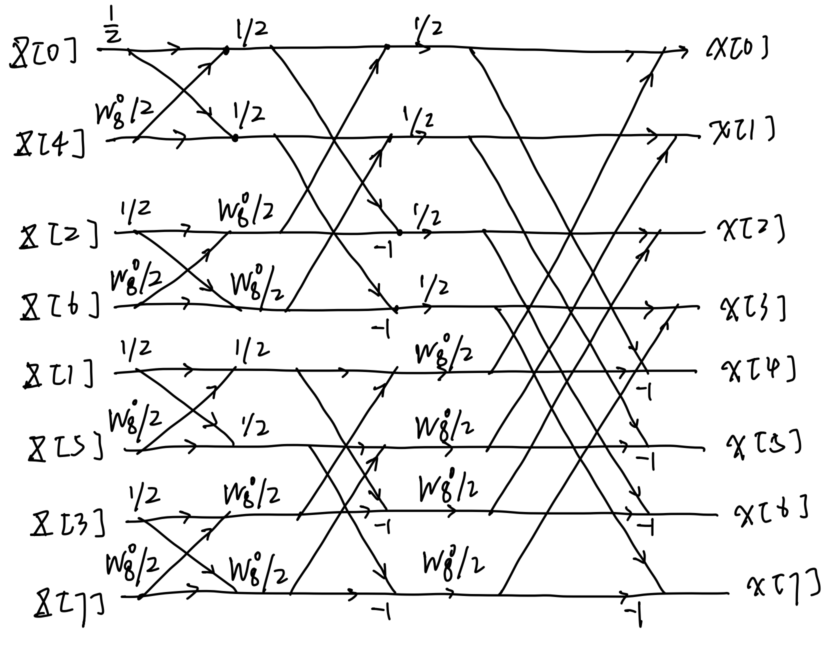
(9-11)

(0) 0000 → 0000 (0)
(1) 0001 → 1000 (8)
(2) 0010 → 0100 (4)
(3) 0011 → 1100 (12)
(4) 0100 → 0010 (2)
(5) 0101 → 1010 (10)
(6) 0110 → 0110 (6)
(7) 0111 → 1110 (14)
(8) 1000 → 0001 (1)
(9) 1001 → 1001 (9)
(10) 1010 → 0101 (5)
(11) 1011 → 1101 (13)
(12) 1100 → 0011 (3)
(13) 1101 → 1011 (11)
(14) 1110 → 0111 (7)
(15) 1111 → 1111 (15).

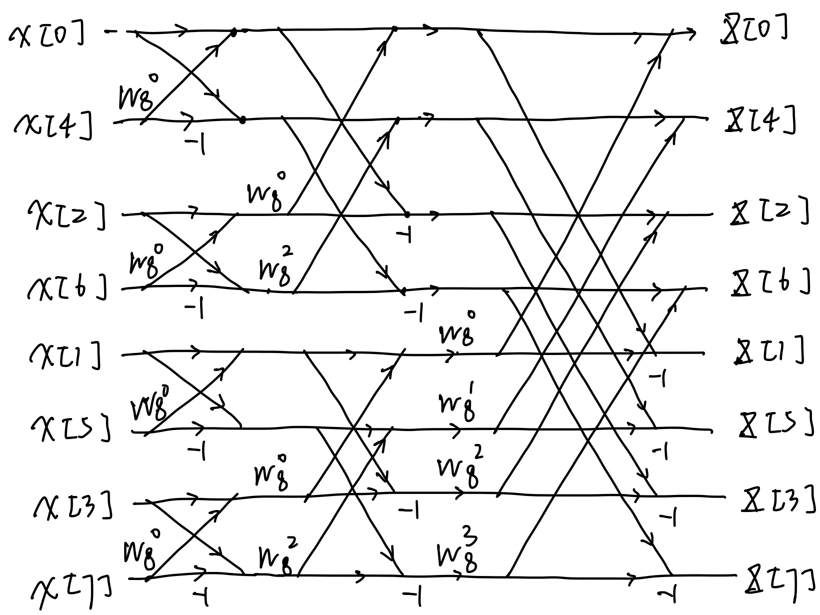
(9-23)

(a) $\begin{bmatrix} 1 & 1 \\ W_N^r & -W_N^r \end{bmatrix} \begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix} = \begin{bmatrix} X_m[p] \\ X_m[q] \end{bmatrix} \Rightarrow \begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} W_N^{-r} \\ \frac{1}{2} & -\frac{1}{2} W_N^{-r} \end{bmatrix} \begin{bmatrix} X_m[p] \\ X_m[q] \end{bmatrix}$

(b)



(c)



(d) Yes. For each decimation in time FFT there exists a decimation in frequency FFT that corresponds to interchanging the input and output and reversing the direction of all arrows.

$e^{-j\frac{2\pi}{(N/2)}(N/2)k}$

(9-20) (a) $$X[2k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi(2k)n/N} = \sum_{n=0}^{N/2-1} x[n]e^{-j\frac{2\pi}{N/2}kn} + \sum_{n=0}^{N/2-1} x[n+(N/2)]e^{-j\frac{2\pi}{(N/2)}kn}$$

$$= \sum_{n=0}^{N/2-1}(x[n]+x[n+(N/2)])e^{-j\frac{2\pi}{(N/2)}kn} = Y[k]$$

(b) The M point DFT Y[k]:

$$Y[k] = \sum_{n=0}^{M-1}\sum_{r=-\infty}^{\infty} x[n+rM]e^{-j2\pi kn/M} = \sum_{r=-\infty}^{\infty}\sum_{n=0}^{M-1} x[n+rM]e^{-j2\pi k(n+rM)/M}e^{j2\pi(rM)k/M}$$

Let $l = n+rM$. then $Y[k] = \sum_{l=-\infty}^{\infty} x[l]e^{-j2\pi kl/M} = X(e^{j2\pi k/M})$.

(c) Form the sequence:

$$y[n] = \begin{cases} (x[n]-x[n+(N/2)])e^{-j(2\pi/N)n}, & 0 \le n \le (N/2)-1 \\ 0 & \text{, o.w.} \end{cases}$$

compute the N/2 point DFT of y[n] and output Y[k]

The odd-indexed values of $X[k] = Y[(k-1)/2]$. $k = 1, 3, \ldots, N-1$.

Homework 8

17.1. (a) $H_c(s) = \dfrac{s+a}{(s+a)^2+b^2} = \dfrac{0.5}{s+a+jb} + \dfrac{0.5}{s+a-jb}$

$\Rightarrow h_c(t) = \frac{1}{2}[e^{-(a+jb)t} + e^{-(a-jb)t}]u(t)$

$h_1[n] = h_c[nT] = \frac{1}{2}[e^{-(a+jb)nT} + e^{-(a-jb)nT}]u[n]$.

$H_1(z) = \frac{1}{2}\cdot\left(\dfrac{1}{1-e^{-(a+jb)T}z^{-1}} + \dfrac{1}{1-e^{-(a-jb)T}z^{-1}}\right)$. $|z|>e^{-aT}$.

(b)* $s_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau \Rightarrow \dfrac{H_c(s)}{s} = S_c(s)$.

$\Rightarrow S_c(s) = \dfrac{s+a}{s(s+a+jb)(s+a-jb)} = \dfrac{A_1}{s} + \dfrac{A_2}{s+a+jb} + \dfrac{A_3}{s+a-jb}$.

$A_1 = \dfrac{a}{a^2+b^2}$. $A_2 = -\dfrac{0.5}{a+jb}$

from $S_2(s)$ to $S_2(z) = \dfrac{A_1}{1-z^{-1}} + \dfrac{A_2}{1-e^{-(a+jb)T}z^{-1}} + \dfrac{A_2^*}{1-e^{-(a-jb)T}z^{-1}}$

the relationship between the step response and the impulse response:

$s_2[n] = \sum_{k=-\infty}^{n} h_2[k] = \sum_{k=-\infty}^{\infty} h_2[k]u[n-k] = h_2[n] * u[n]$

$S_2[z] = \dfrac{H_2(z)}{1-z^{-1}}$.

So $H_2(z) = S_2(z)(1-z^{-1}) = A_1 + A_2\cdot\dfrac{1-z^{-1}}{1-e^{-(a+jb)T}z^{-1}} + A_2^*\dfrac{1-z^{-1}}{1-e^{-(a-jb)T}z^{-1}}$. $|z|>e^{-aT}$.

(c) $s_1[n] = \sum_{k=-\infty}^{n} h_1[k] = \frac{1}{2}\sum_{k=0}^{n}(e^{-(a+jb)kT} + e^{-(a-jb)kT}) = \frac{1}{2}\left[\dfrac{1-e^{-(a+jb)(n+1)T}}{1-e^{-(a+jb)T}} + \dfrac{1-e^{-(a-jb)(n+1)T}}{1-e^{-(a-jb)T}}\right]u[n]$

17.4.* (a) $\dfrac{1}{s+a} \longrightarrow \dfrac{T_d}{1-e^{-aT}z^{-1}}$. $\Rightarrow H_c(s) = \dfrac{2/T_d}{s+0.1} - \dfrac{1/T_d}{s+0.2} = \dfrac{1}{s+0.1} - \dfrac{0.5}{s+0.2}$.

the solution is not unique due to the periodity of $z=e^{jw}$.

$\Rightarrow H_c(s) = \dfrac{2/T_d}{s+(0.1+j\frac{2\pi k}{T_d})} - \dfrac{1/T_d}{s+(0.2+j\frac{2\pi l}{T_d})}$. k, l are integers.

(b) $z = \dfrac{1+(T_d/2)s}{1-(T_d/2)s}$

$\Rightarrow H_c(s) = \dfrac{2}{1-e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \dfrac{1}{1-e^{-0.4}\left(\frac{1-s}{1+s}\right)} = \dfrac{2(s+1)}{s(1+e^{-0.2})+(1-e^{-0.2})} - \dfrac{(s+1)}{s(1+e^{-0.4})+(1-e^{-0.4})}$

$= \left(\dfrac{2}{1+e^{-0.2}}\right)\left(\dfrac{s+1}{s+\frac{1-e^{-0.2}}{1+e^{-0.2}}}\right) - \left(\dfrac{1}{1+e^{-0.4}}\right)\left(\dfrac{s+1}{s+\frac{1-e^{-0.4}}{1+e^{-0.4}}}\right)$. $\Rightarrow$ unique.

17.9.* $w_c = \Omega_c T = [2\pi(1000)][0.0002] = 0.4\pi$ (rad).

(7-10) Using the bilinear transform frequency mapping equation:

$$w_c = 2\arctan\left(\frac{\Omega_c T}{2}\right) = 2\arctan(1.6\pi) = 0.7589\pi \text{ (rad)}.$$

(7-11) For impulse invariance technique: $w_c = \Omega_c T$.

then $\Omega_c = \frac{w_c}{T} = \frac{\pi/4}{0.0001} = 2\pi(1250)$ rad/s

(7-12). For bilinear transform technique: $w_c = 2\arctan\left(\frac{\Omega_c T}{2}\right)$.

then: $\Omega_c = \frac{2}{T}\tan\left(\frac{w_c}{2}\right) = \frac{2}{0.001}\tan\left(\frac{\pi}{4}\right) = 2000$ rad/s $= 2\pi(318.3)$ rad/s.

(7-15) From $0.95 < H(e^{jw}) < 1.05$ and $-0.1 < H(e^{jw}) < 0.1$ we can know:

$\delta_p = 0.05 \quad \delta_s = 0.1 \quad \Rightarrow \alpha_p = -26\text{dB} \quad \alpha_s = -20\text{dB}$.

Requires a window with a peak approximation error less than -26dB.
the Hanning, Blackman meet.

And $w_p = 0.25\pi$, $w_s = 0.35\pi \Rightarrow \Delta w = |w_p - w_s| = 0.1\pi$.

Hanning: $\frac{8\pi}{M} = 0.1\pi \Rightarrow M = 80$.

Hamming: $\frac{8\pi}{M} = 0.1\pi \Rightarrow M = 80$

Blackman: $\frac{12\pi}{M} = 0.1\pi \Rightarrow M = 120$.

(7-16) $w_p = 0.63\pi$. $w_s = 0.65\pi$. $\delta_p = 0.02 \quad \delta_s = 0.15$.

$\delta = \min\{\delta_p, \delta_s\} = 0.02. \Rightarrow A = -20\log_{10}\delta = 33.98$

Because $21 \le A \le 50$ so $\beta = 0.584(A-21)^{0.4} + 0.07886(A-21) = 2.652$

$\Delta w = |w_p - w_s| = 0.02\pi \Rightarrow M = \frac{A-8}{2.285\Delta w} = 180.95 \Rightarrow 181.$

Homework 9.

(6-5)

(a) $w[n] = 3w[n-1] + w[n-2] + x[n]$. $y[n] = y[n-1] + 2y[n-2] + w[n]$

(b) $W(z) = 3W(z)z^{-1} + W(z)z^{-2} + X(z)$. $Y(z) = Y(z)z^{-1} + 2Y(z)z^{-2} + W(z)$.

$$\Rightarrow \frac{W(z)}{X(z)} = \frac{1}{1-3z^{-1}-z^{-2}} \cdot \quad \frac{Y(z)}{W(z)} = \frac{1}{(1-z^{-1}-2z^{-2})}.$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1-3z^{-1}-z^{-2})(1-z^{-1}-2z^{-2})}.$$

(c) real multiplications = 2

real additions = 4.

(d) Impossible.

(6-9)* $w[n] = 8w[n-2] - w[n-1] + x[n]$, $y[n] = w[n] + 2w[n-1] + x[n-1]$.

$$\Rightarrow \frac{W(z)}{X(z)} = \frac{1}{1+z^{-1}-8z^{-2}}. \quad \frac{Y(z)}{W(z)} = 1 + z^{-1} + \frac{X(z)}{W(z)}z^{-1} = 1 + 3z^{-1} + z^{-2} - 8z^{-3}.$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+3z^{-1}+z^{-2}-8z^{-3}}{1+z^{-1}-8z^{-2}}$$

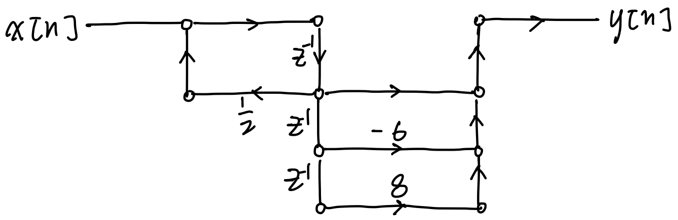
difference equation: $y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3]$.

$\Rightarrow h[n] + h[n-1] - 8h[n-2] = \delta[n] + 3\delta[n-1] + \delta[n-2] - 8\delta[n-3]$.

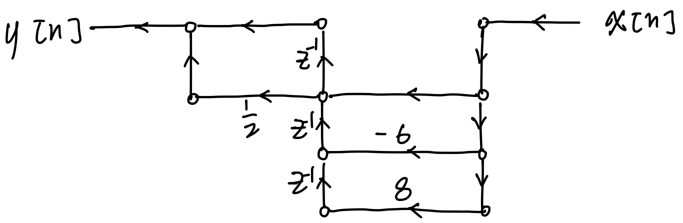
$n=0$: $h[0]=1$. $n=1$: $h[1]+h[0]=3$ $\Rightarrow h[1] = 3 - h[0] = 2$

(6-11)* $\frac{W(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}}$ $\quad \frac{Y(z)}{X(z)} = (1-2z^{-1})(1-4z^{-1})z^{-1} = z^{-1} - 6z^{-2} + 8z^{-3}$.

(a)

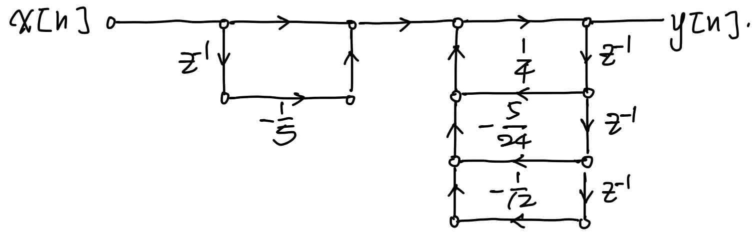


(b)

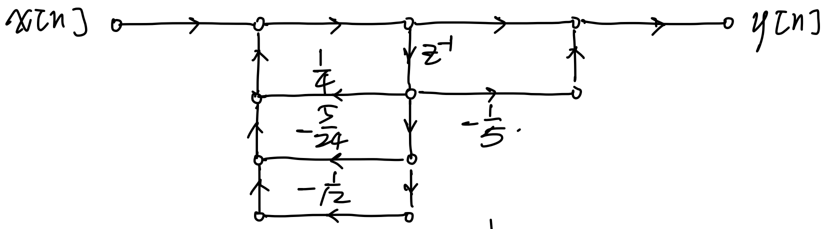


(6.23)* $\dfrac{W(z)}{X(z)} = \dfrac{1}{(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2})(1+\frac{1}{4}z^{-1})} = \dfrac{1}{1-\frac{1}{4}z^{-1}+\frac{5}{24}z^{-2}+\frac{1}{12}z^{-3}}$. $\qquad \dfrac{Y(z)}{W(z)} = 1-\frac{1}{5}z^{-1}$.

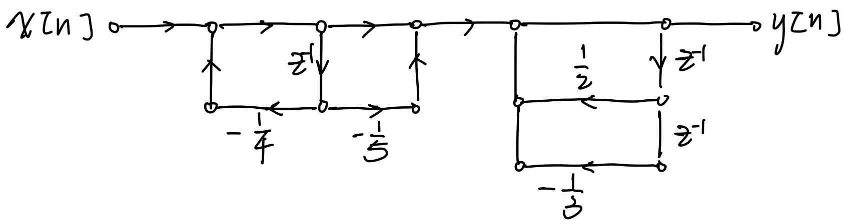
(a) (i) Direct form I:



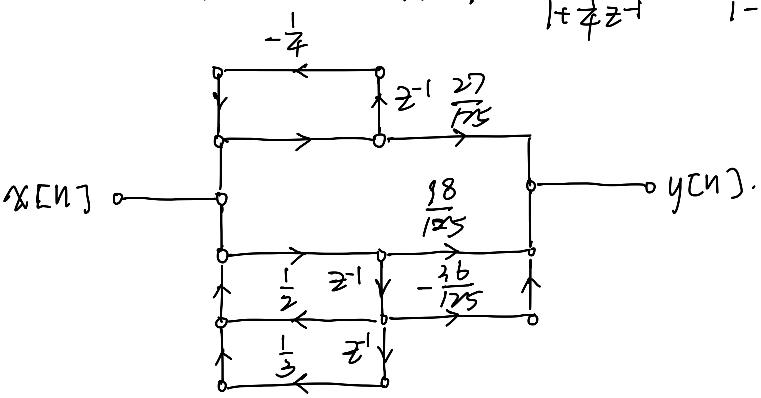
(ii) Direct form II:



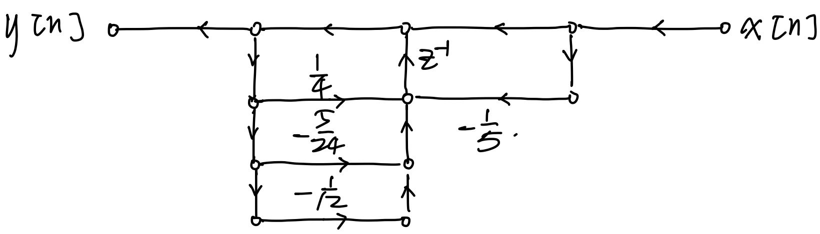
(iii)* Cascade form: $H(z) = \left(\dfrac{1-\frac{1}{5}z^{-1}}{1+\frac{1}{4}z^{-1}}\right)\left(\dfrac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}\right)$.



(iv)*. Parrallel form: $H(z) = \dfrac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}} + \dfrac{\frac{98}{125}-\frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{3}z^{-2}}$.



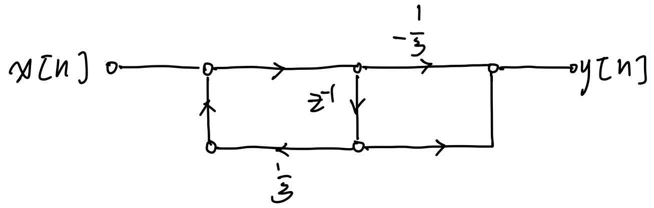
(v) Transposed direct form II:



(b) $y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n]$

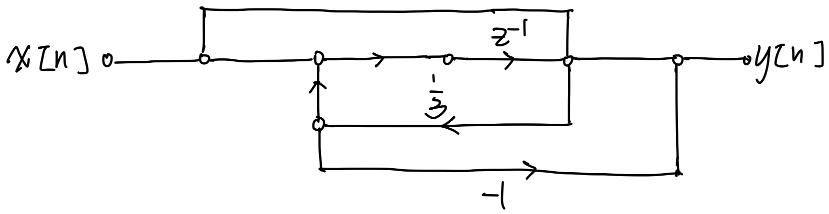
16-35)

(a)

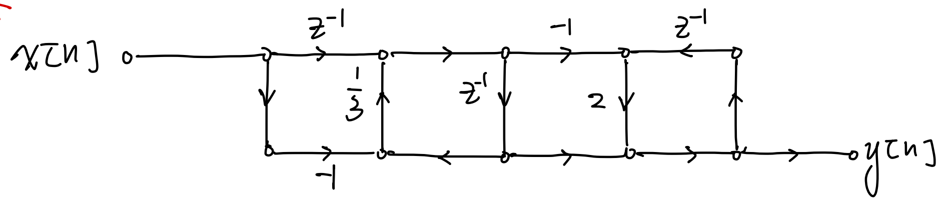


delays = 1. multipliers = 2.

(b)



(c) *



补充题目：

- For $x[n]=\dfrac{\sin(0.2\pi n)}{\pi n}$, $-\infty<n<\infty$. calculate $\sum_{n=-\infty}^{\infty}|x[n]|^2$.

$$\Rightarrow X(e^{jw})=\begin{cases}1, & |w|\le 0.2\pi\\ 0, & \text{o.w.}\end{cases} \Rightarrow \sum_{n=-\infty}^{\infty}|x[n]|^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(e^{jw})|^2dw=\frac{0.4\pi}{2\pi}=0.2$$

- For a generalized linear phase FIR system, system function is:

$$H(z)=(1+\underset{1}{\underline{2}}z^{-1}+2z^{-2})G(z)$$

Find $G(z)$ so that $H(z)$ has the lowest order.

① $G(z)=1+az^{-1}$, $H(z)=1+(2+a)z^{-1}+(2a+2)z^{-2}+2az^{-3}$.

$$\begin{cases}1=\pm 2a\\ 2+a=\pm(2a+2)\end{cases}\Rightarrow \text{无解}.$$

② $G(z)=1+az^{-1}+bz^{-2}$. $H(z)=1+(2+a)z^{-1}+(2+b+2a)z^{-2}+(2a+2b)z^{-3}+2bz^{-4}$.

$$\begin{cases}1=2b\\ 2+a=2a+2b\end{cases} \text{ or } \begin{cases}1=-2b\\ 2+a=-(2a+2b)\\ \underbrace{2+b+2a=0}_{\text{奇对称}}\end{cases}\Rightarrow\begin{cases}a=1\\ b=\frac{1}{2}\end{cases} \text{ or 无解}.$$

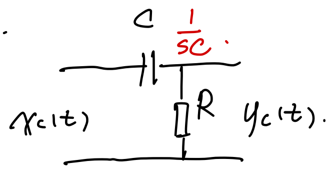
$$\Rightarrow H(z)=1+3z^{-1}+\frac{9}{2}z^{-2}+3z^{-3}+z^{-4},\quad G(z)=1+z^{-1}+\frac{1}{2}z^{-2}$$

零点共轭反演..

- For the following circuit, the input voltage is $x_c(t)$, output voltage is $y_c(t)$. C is a capaciter, R is a resistor.

(1) Derive the system function $H_c(s)$ and $H(z)$ using bilinear transform.

(2) Discuss their stability.



$$\Rightarrow (1)\quad Y(s)=\frac{R}{sC+R}X(s),\quad H(s)=\frac{Y(s)}{X(s)}=\cancel{\frac{R}{sC+R}}\quad \frac{R}{\frac{1}{sC}+R}=\frac{RCs}{1+RCs}=\frac{s}{s+1/RC}$$

for bilinear transform, $s=\dfrac{2}{T_d}\cdot\dfrac{1-z^{-1}}{1+z^{-1}}$ $\Rightarrow H(z)=H_c(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}=\dfrac{1-z^{-1}}{1+\frac{1}{RC}+(\frac{1}{RC}-1)z^{-1}}$.

$\hookrightarrow T_d=2$

(2) $H_c(s)=\dfrac{1}{s+1/RC}\Rightarrow -\dfrac{1}{RC}<0$. stable.