

Ch02. Signals and Random Process.

2.1 Review of signals

1. Classification of Signals.

- (1) Continuous-time signal (2) Discrete-time signal.
(3) Continuous-valued signal (4) Discrete-valued signal.

{ time: x-axis
value: y-axis

Analog signal: (1)(3). Digital signal: (2)(4).

Sampled signal: (2)(3). Quantized signal: (1)(4). (P6)

2. Signal energy and power.

Given a signal $x(t)$:

(1) Energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt.$

Energy signal: E_x is finite.

(2) Power: $P_x = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \right)^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$

Power signal: P_x is finite.

Some signals are neither energy nor power signals. (P_x and E_x are infinite).

3. Some important signals.

The Rectangular Pulse: $\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$

The Triangular Pulse: $\Lambda(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ -t+1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$ $\Lambda(t) = \Pi(t) * \Pi(t)$

The Sgn Signal: $\text{Sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0. \end{cases}$

The Sinc Signal: $\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0. \end{cases}$

4. Fourier Series and Fourier Transform.

(1) A Periodic signal $x(t)$ with period T_0 . Fourier series expansion:

$$x(t) = \sum_{n=-\infty}^{+\infty} x_n e^{j2\pi \frac{n}{T_0} t}. \quad x_n = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-j2\pi \frac{n}{T_0} t} dt. \Rightarrow \text{連續周期信号. } \omega_0 = \frac{2\pi}{T_0}.$$

(2) A non-periodic signal. Fourier transform:

$$\mathcal{X}(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt. \quad x(t) = \int_{-\infty}^{+\infty} \mathcal{X}(f) e^{j2\pi f t} dt. \Rightarrow \text{連續非周期信号}$$

2.2 Review of probability and random variables.

(1) Random Variables (r.v.)

- { Discrete-valued (range is finite or countable infinite)
- { Continuous-valued (range is uncountable infinite).

The Cumulative Distribution Function (CDF): $F_X(x) = P(X \leq x)$.

The Probability Density Function (PDF): $f_X(x) = \frac{d}{dx} F_X(x)$. $F_X(x) = \int_{-\infty}^x f_X(y) dy$.

The Probability Mass Function (PMF) with discrete r.v.

(2) Some distribution: Bernoulli Distribution, Binomial Distribution, Uniform Distribution, Gaussian (Normal) Distribution.

(3) Statistic Average:

The mean or expected value of X : $m_X = E(X) = \sum_{i=1}^M x_i p_i. / \int_{-\infty}^{\infty} x f_X(x) dx$

(First moment). moment: 順序

The n^{th} moment of X : $E(X^n) = \int_{-\infty}^{+\infty} x^n f_X(x) dx$.

The mean-square value ($n=2$): $E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x) dx$

The n^{th} central moment of X : $E[(X - m_X)^n] = \int_{-\infty}^{+\infty} (x - m_X)^n f_X(x) dx$.

$\hookrightarrow n=1$ = standard deviation. $n=2$ = variance.

$\Phi_X(x) \uparrow \rightarrow$ Gaussian tail probability.

$$\Phi_X(x) = 1 - F_X(x)$$

(4) The Q Function: $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$. (Error probability analysis). The values of $Q(x)$ are fixed.

Some features: $Q(-\infty) = 1$. $Q(0) = \frac{1}{2}$. $Q(\infty) = 0$. $Q(-x) = 1 - Q(x)$.

Given $X \sim N(\mu, \sigma^2)$, then $\Pr(X > x) = Q(\frac{x-\mu}{\sigma})$.

$$\Pr(X \leq x) = \Phi(\frac{x-\mu}{\sigma}) \quad \Phi_X(x) = 1 - \Phi_X(x).$$

(5) Correlation. $R_{XY} = E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(xy) dx dy$.

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{D(X)} \cdot \sqrt{D(Y)}} = \frac{\text{cov}(X, Y)}{\sqrt{D(X)} \cdot \sqrt{D(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \in [-1, 1].$$

ρ_{XY} is a measure of the strength/direction of the linear relationship between X, Y .

Independent \Rightarrow Uncorrelated, the converse is not true (except ~~GD~~).

(b) Joint Distribution: $F_{XY}(x,y) = P(X \leq x, Y \leq y)$ $f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$.
 Marginal: $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,\beta) d\beta$. $F_X(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^x f_{XY}(\alpha,\beta) d\alpha d\beta$.
 X, Y are independent if $F_{XY}(x,y) = F_X(x)F_Y(y)$. $f_{XY}(x,y) = f_X(x)f_Y(y)$.

Joint Gaussian Random Variables:

$$f(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{\frac{n}{2}} |\det(C)|^{1/2}} \exp\left[-\frac{(X - M)^T C^{-1} (X - M)}{2}\right].$$

where $X = (x_1, x_2, \dots, x_n)^T$, $M = (m_1, \dots, m_n)^T$, $C = [C_{ij}]_{n \times n}$, $C_{ij} = E[(X_i - m_i)(X_j - m_j)]$.

Two-Variate Gaussian PDF:

$$\text{For uncorrelated } X, Y (P_{XY}=0) = f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2}\right]\right\} = f_{X_1}f_{X_2}$$

Joint Gaussian r.v.s can be completely characterized by the mean vector and the covariance matrix.

- (7) Law of Large Numbers. Let $Y = \frac{1}{n} \sum_{i=1}^n X_i$. Then $\lim_{n \rightarrow \infty} P(|Y - m_X| > \varepsilon) = 0$. $\forall \varepsilon > 0$.
 (8) Central Limit theorem. $Y = \frac{1}{n} \sum_{i=1}^n X_i$ converges to $N(m_X, \frac{\sigma_X^2}{n})$.

(9) Rayleigh Distribution

2.3 Random Processes: basic concepts.

1) Statistics of Random Process.

For an infinite collection of r.v. specified at time $t = \{X(t_1), X(t_2), \dots, X(t_n)\}$.

Joint PDF: $f(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n)$.

specified to

① First order statistics

$f(x; t) \Rightarrow$ first order density of X . (M) $E[X|t_0] = E[X|t=t_0] = \int_{-\infty}^{\infty} x f_X(x; t_0) dx = \bar{X}|t_0$.
 (V) $E[|X(t_0) - \bar{X}(t_0)|^2] = \sigma_X^2|t_0$.

② Second order statistics

$f(x_1, x_2; t_1, t_2) \Rightarrow$ second order density of $X|t$. (互相关函数与协方差)

Auto-correlation function = $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2) dx_1 dx_2$.

Example: consider $Y(t) = B \cos \omega t$, where $B \sim N(0, b^2)$. Find its mean and auto-correlation function. \Rightarrow not WSS

$$\Rightarrow E[Y(t)] = \int_{-\infty}^{\infty} B \cos \omega t f(B) dB = 0$$

$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)] = E[B^2 \cos \omega t_1 \cos \omega t_2] = E[B^2] \cdot \cos \omega t_1 \cos \omega t_2 \\ = D(B) \cos \omega t_1 \cos \omega t_2 = b^2 \cos \omega t_1 \cos \omega t_2$$

Example: consider $Y(t) = A \cos(2\pi f t + \theta)$, where θ is uniform in $(-\pi, \pi)$.

$$\Rightarrow E[X(t)] = \int_{-\pi}^{\pi} A \cos(2\pi f t + \theta) \frac{1}{2\pi} d\theta = 0. \quad \Rightarrow \text{WSS}$$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[A^2 \cos(2\pi f t_1 + \theta) \cos(2\pi f t_2 + \theta)] \\ = \frac{A^2}{2} [\cos(4\pi f t_1 + 2\pi f t_2 + 2\theta) + \cos(2\pi f t_2)] \\ = \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(4\pi f t_1 + 2\pi f t_2 + 2\theta) d\theta + \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(2\pi f t_2) d\theta \\ = 0 + \frac{A^2}{2} \cos(2\pi f t_2) = \frac{A^2}{2} \cos(2\pi f t_2).$$

12) Stationary Process

\Rightarrow For any n and $\tau = f(x_1, x_2, \dots, x_n; t_1, \dots, t_n) = f(x_1, x_2, \dots, x_n; t_1+\tau, t_2+\tau, \dots, t_n+\tau)$, $\forall n, \tau$.

① First-order statistics is independent of t = $E[X|t] = \int_{-\infty}^{\infty} x f_X(x) dx = m_X$ (not $f(x, t)$).

② Second-order statistics only depends on $\tau = t_2 - t_1$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_2 - t_1) dx_1 dx_2 = R_X(\tau).$$

③ Wide-Sense Stationary:

$$\left\{ \begin{array}{l} \text{WSS: } ① E[X|t] = m_X, \\ ② R_X(t_1, t_2) = R_X(\tau), \tau = t_2 - t_1. \end{array} \right. \quad \begin{array}{l} \text{弱平稳过程} \\ \text{强平稳过程} \end{array}$$

Strictly stationary = definition.

(3) Average and Ergodic.

① Ensemble averaging = $\bar{x}(t) = E[x(t)] = \int_{-\infty}^{\infty} xp(x; t) dx$.

统计平均

$$R_x(t_1, t_2) = E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2; t_1, t_2) dx_1 dx_2.$$

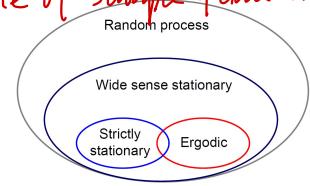
② Time averaging = $\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt$. take average over time of sample function $x(t)$

<> 时间平均

$$\langle x(t) x(t-\tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) x(t-\tau) dt.$$

Ergodic = ensemble averaging = time averaging

随机信号



(b) Frequency Domain Characteristics of Random Process.

① Power Spectral Density.

For a deterministic signal $x(t)$:

truncate to get an energy signal = $x_T(t) = \begin{cases} x(t), & |t| < T \\ 0, & \text{else} \end{cases}$ $\xrightarrow{\text{FT}} \tilde{x}_T(f) = \int_{-T}^T x(t) e^{-j2\pi ft} dt$.

Power: $P = \langle x^2(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x^2(t) dt < \infty$ \rightarrow energy spectrum. 能量谱

Energy: $E_T = \int_{-\infty}^{\infty} x_T^2(t) dt = \int_{-T}^T x^2(t) dt = \int_{-\infty}^{\infty} |\tilde{x}_T(f)|^2 df$ (Parseval).

Thus: $P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{\infty} |\tilde{x}_T(f)|^2 df = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{|\tilde{x}_T(f)|^2}{2T} df$. \rightarrow 功率谱.

Power Spectral Density of $x(t)$: $S_x(f) = \lim_{T \rightarrow \infty} \frac{|\tilde{x}_T(f)|^2}{2T}$. $\tilde{x}_T(f)$ 是 $x_T(t)$ 的 FT.
(随机信号+时间平均).

② PSD of Random Process.

Average power of r.v. $x(t)$: $P = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} \frac{E(|\tilde{x}_T(f)|^2)}{2T} df$.

PSD of $x(t)$: $S_x(f) = \lim_{T \rightarrow \infty} \frac{E(|\tilde{x}_T(f)|^2)}{2T}$

Example (P59). Given a binary-semi-random signal.

③ PSD of WSS Process.

Wiener-Khinchin theorem: $S_x(f) \leftrightarrow R_x(\tau)$

$$\begin{cases} R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f \tau} df \\ S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f \tau} d\tau \end{cases}$$

自相关函数
功率谱密度

Then $R_x(0) = \int_{-\infty}^{\infty} S_x(f) df = \text{total power.} = \boxed{\mathbb{E}[x(t)^2]}$

互相关系数
功率谱密度

Example: For random process $X(t) = A \cos(2\pi f_0 t + \theta)$. $\Rightarrow R_x(t; \tau) = \frac{A^2}{2} \cos(2\pi f_0 \tau)$.

Then $S_x(f) = \frac{A^2}{4} [\delta(f+f_0) + \delta(f-f_0)]$. $\Rightarrow P = \frac{A^2}{2}$.

Example: Given a binary random signal.

解題

(1) Random Process Transmission Through Linear Systems.

Given a linear system with impulse response $h(t)$. $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$

Mean of the output $y(t)$ = $\bar{y}(t) = \mathbb{E}(y(t)) = \int_{-\infty}^{\infty} h(\tau) \mathbb{E}[x(t-\tau)] d\tau = \int_{-\infty}^{\infty} h(\tau) \bar{x}(t-\tau) d\tau$.

If $x(t)$ is WSS = $= \bar{x} \int_{-\infty}^{\infty} h(\tau) d\tau = (\bar{x} \cdot H(0))$. $H(0) = \int_{-\infty}^{\infty} h(\tau) dt$.

Autocorrelation of $y(t)$ =

$$\begin{aligned} R_y(t, u) &= \mathbb{E}(y(t)y(u)) = \mathbb{E}\left[\int_{-\infty}^{\infty} h(\tau_1) x(t-\tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) x(u-\tau_2) d\tau_2\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) \mathbb{E}[x(t-\tau_1)x(u-\tau_2)] d\tau_1 d\tau_2 \end{aligned}$$

If $x(t)$ is WSS = $R_y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_x(t-\tau_1+\tau_2) d\tau_1 d\tau_2 = \underline{h(-\tau) * h(\tau) * R_x(\tau)}$

\Rightarrow If input $x(t)$ is WSS, then output $y(t)$ is WSS.

PSD of $y(t)$ = $S_y(f) = |H(f)|^2 S_x(f)$ (解題)

2.4 Gaussian and White Process.

For Gaussian process:

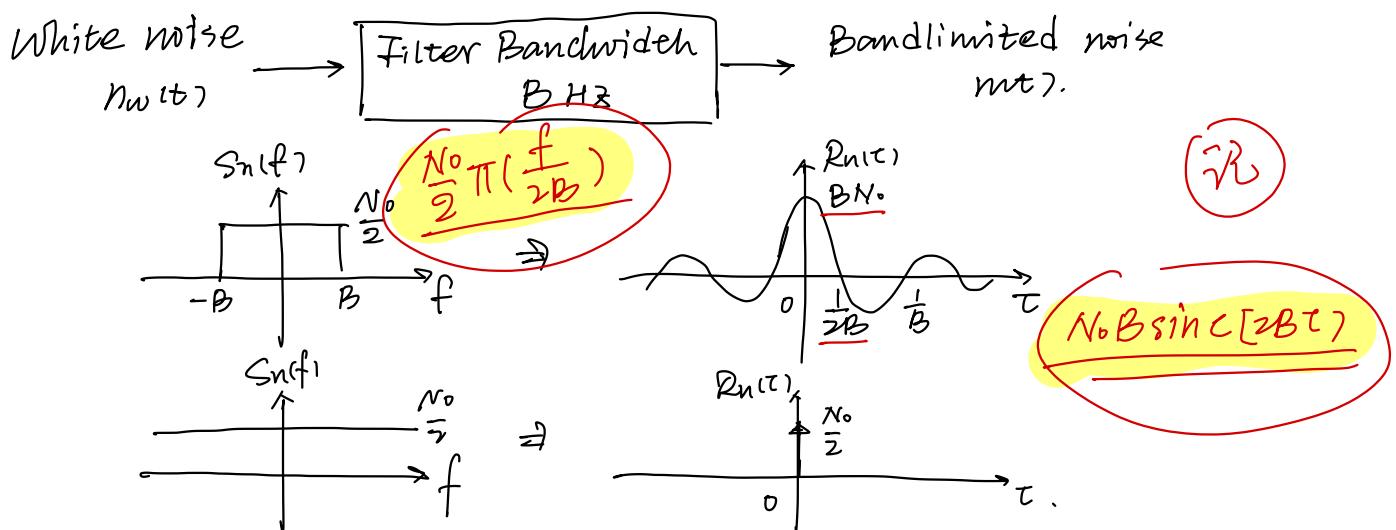
- if it is WSS, it is also strictly stationary.
- if the input to a linear system is a Gaussian process.
the output is also a Gaussian process.

Noise: Gaussian and stationary with zero mean.

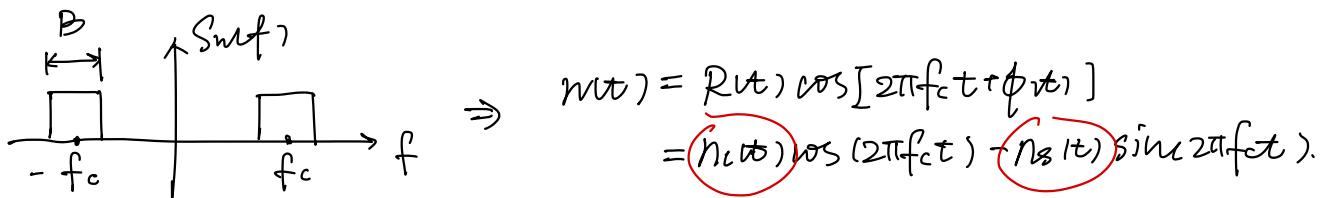
White noise: $S_n(f) = \frac{N_0}{2}$. $R_n(\tau) = \frac{N_0}{2} \delta(\tau)$.

↳ completely uncorrelated.

Bandlimited Noise:



Bandpass noise



If $n_t(t)$ is zero-mean, stationary, Gaussian noise.

$$\text{then: } S_{n_s}(f) = S_n(f) = \begin{cases} S_n(f-f_0) + S_n(f+f_0), & \text{if } |f| \leq \frac{B}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} N_0, & \text{if } |f| \leq \frac{B}{2} \\ 0, & \text{otherwise.} \end{cases}$$

chos. Analog Modulation. 模拟调制系统.

1. What is modulation? Why modulation?

→ Generate a carrier signal (usually sinusoidal) at the transmitter.

→ To translate the frequency of lowpass signal to the passband of channel.

2. Carrier signal: $c(t) = A_c \cos(2\pi f_c t + \theta_c)$.

Amplitude A_c → Amplitude modulation.

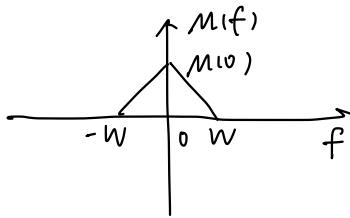
Frequency f_c } → Angle modulation.

Phase θ_c

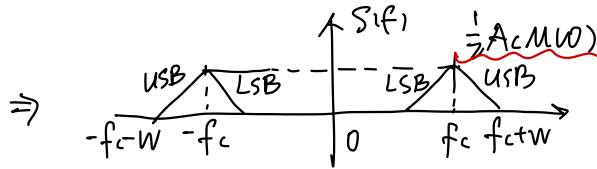
(1) AM. (DSB-SC) Double-Sideband Suppressed Carrier

Baseband signal = $m(t)$. Carrier wave = $c(t) = A_c \cos(\omega_c t + \theta_0)$.

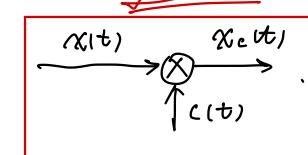
Then modulated wave: $s(t) = c(t)m(t) = A_c m(t) \cos(\omega_c t + \theta_0)$. \Rightarrow 来波器



Spectrum of message



Spectrum of DSB-SC

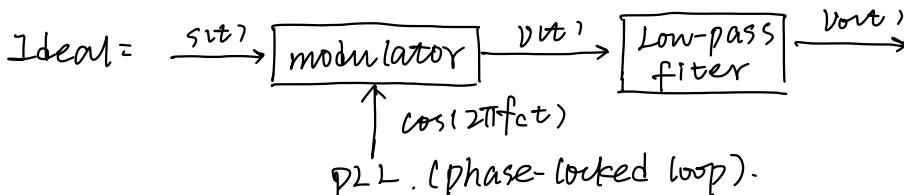


$\Rightarrow S(f) = \frac{1}{2} A_c [M(f-f_c) + M(f+f_c)]$. Translate the origin message spectrum to DSB Bandwidth = $B_c = 2W$. (基带信号带宽的两倍).

transmit power (发送信号功率) = $P_s = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} S^2(t) dt = \frac{1}{2} A_c^2 P_m$. (P_m 为基带信号平均功率). A_c 为载波信号振幅.

Demodulation of DSB-SC signals.

Phase-coherent demodulation:



\Rightarrow 乘法器和低通滤波器.
产生本地载波信号 $\cos \omega_c t$ 将已调制信号的频谱再度搬移.

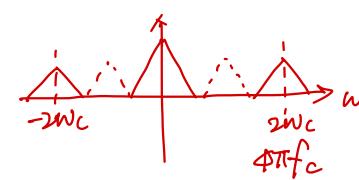
If there is a phase error ϕ , then:

$$v(t) = s(t) \cos(2\pi f_c t + \phi) = A_c \cos(2\pi f_c t) \cos(2\pi f_c t + \phi) m(t)$$

已调制信号 本地载波信号

$$= \underbrace{\frac{1}{2} A_c \cos(\phi) m(t)}_{\text{Scaled version of message signal}} + \underbrace{\frac{1}{2} A_c \cos(4\pi f_c t + \phi) m(t)}_{\text{unwanted high-frequency part}}$$

Scaled version of message signal



(2) Conventional AM.

Carrier wave = $c(t) = A_c \cos(\omega_c t + \theta_0)$. Baseband signal = $m(t) = \frac{m_{max}}{\max|m(t)|}$. Modulation index = α

$$\Rightarrow \text{Modulated wave: } s(t) = A_c [\alpha m(t)] \cos \omega_c t$$

$$\begin{cases} \alpha \leq 1 & 1 + \alpha m(t) \text{ 可正} \\ \alpha > 1 & 1 + \alpha m(t) \text{ 可负} \end{cases}$$

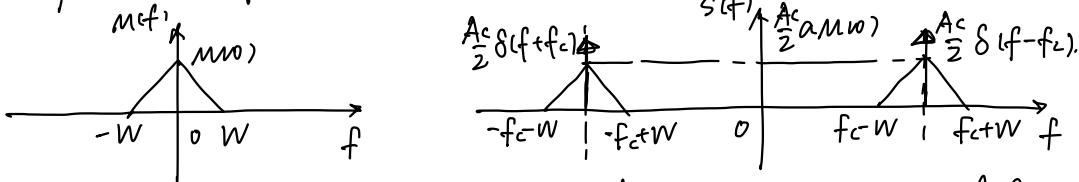
▲ Use " α " to control the power efficiency.

$$= A_c (\cos(2\pi f_c t) + \alpha \sin(\omega_m t) \cos(2\pi f_c t)).$$

式中 $m_{n(t)}$ 为假一化的基带信号, 那 $|m_{n(t)}|_{\max} = 1$. 应用包络检波器, 正确解调 AM 信号的条件为 $m \leq 1$. 当 $m > 1$ 时, AM 信号产生过调制. 此时信号包络失真, 不能使用包络检波器, 要采用同步解调.

↓
phase reversals 幅度恒正负.

Spectrum of Conventional AM:



spectrum of message signal. $S(f) = \frac{A_c}{2} [\delta(f-f_c) + \delta(f+f_c)] + \frac{A_c \alpha}{2} [M(f-f_c) + M(f+f_c)].$

Bandwidth and Power Efficiency:

\Rightarrow AM 信号的频带宽度与 DSB 信号相同. 因此也为 $B_C = 2W$.

发送AM信号的平均功率 = $P = \frac{1}{2} A_c^2 + \frac{1}{2} \alpha^2 A_c^2 P_{m,n}$. $P_{m,n} = E[m_n^2(t)]$ 为基带信号功率.
 carrier power message power.

边带功率与平均功率比值为调制效率: $E = \frac{\text{power in sideband}}{\text{total power}} = \frac{\alpha^2 P_{m,n}}{1 + \alpha^2 P_{m,n}}$
 (Modulation Efficiency).

16 hours

(textbook p151).

Example: message signal $m(t) = 3 \cos(200\pi t) + \sin(600\pi t)$.

Carrier = $c(t) = \cos(2 \times 10^5 \pi t)$. Modulation index $\alpha = 0.85$

Determine the power in the carrier component and in the sideband components of the modulated signal.

\Rightarrow power in carrier component = $\frac{A_c^2}{2} = 0.15$

power in sideband = $m(t) = 3 \cos(200\pi t) + \sin(600\pi t)$. $|m(t)|_{\max} = 3.6955$

then $m_{n(t)} = m(t) / |m(t)|_{\max} = 0.8118 \cos(200\pi t) + 0.2706 \sin(600\pi t)$

$$P_{m,n} = \frac{1}{2} [0.8118^2 + 0.2706^2] = 0.3661$$

$$\frac{A_c^2}{2} \alpha^2 P_{m,n} = \frac{1}{2} \times 0.85^2 \times 0.3661 = 0.1323.$$

△ 周期信号属于功率信号.

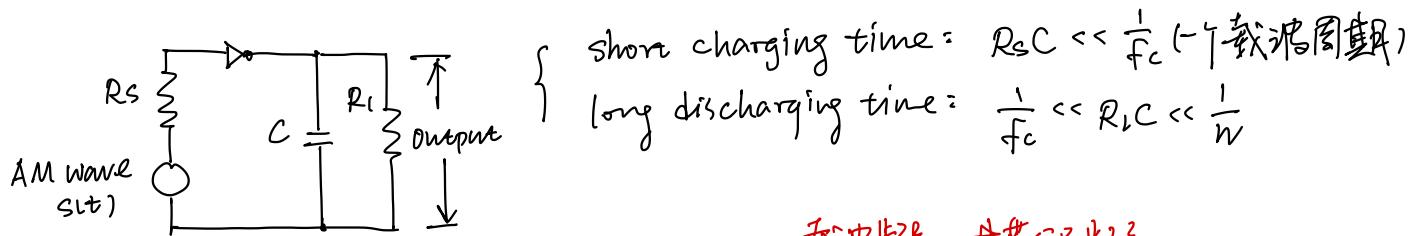
$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt \Rightarrow f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j n t} \text{ (Fourier series).}$$

$$\Rightarrow P = \sum_{n=-\infty}^{\infty} |F_n|^2 = |F_0|^2 + 2 \sum_{n=1}^{\infty} |F_n|^2.$$

直流分量 调波分量

对正弦信号 $A \cos(\omega t)$, $A \sin(\omega t)$ 来说, $P = \frac{1}{2} A^2$.

Demodulation of AM signals = Envelop Detector. (包络检波器)



短波长 基带信号部分

conditions of demodulation of AM = $m \leq 1$. $f_c \gg f_x$

(a) Single Sideband (SSB) AM. (单边带调制信号)

$$B_c = 2W(2f_x)$$

⇒ Both AM and DSB are spectrally inefficient (use twice the bandwidth of message)

The SSB use a band filter to transform DSB.



The baseband signal can be written as: $m(t) = \sum_{i=1}^n x_i \cos(2\pi f_i t + \theta_i)$, $f_i \leq f_x$

Then its USB signal (上边带信号) is: $u(t) = \frac{A_c}{2} \sum_{i=1}^n x_i \cos[2\pi(f_c + f_i)t + \theta_i]$.

After manipulation:

$$u(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t - \frac{A_c}{2} m(t) \sin 2\pi f_c t$$

$$\text{For LSB: } u(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t + \frac{A_c}{2} m(t) \sin 2\pi f_c t$$

Bandwidth = $B_c = W(f_x)$.

Power = $S_T = \frac{1}{4} A_c^2 S_x$. (带宽和功率都是DSB的一半).

(b) Vestigial Sideband: VSB.

Bandwidth = $B = W + f_v$.

(5) Signal Multiplexing

⇒ Multiplexing = independent signals are combined and transmitted in a common channel, they're demultiplexing at receiver.

TWO COMMON TYPES = TDM (时分复用). FDM (频分复用).

FDM =

LPF: ensure signal bandwidth limited to W

MOD (modulator): shift message frequency range to mutually exclusive high frequency bands

BPF: restrict the band of each modulated wave to its prescribed range

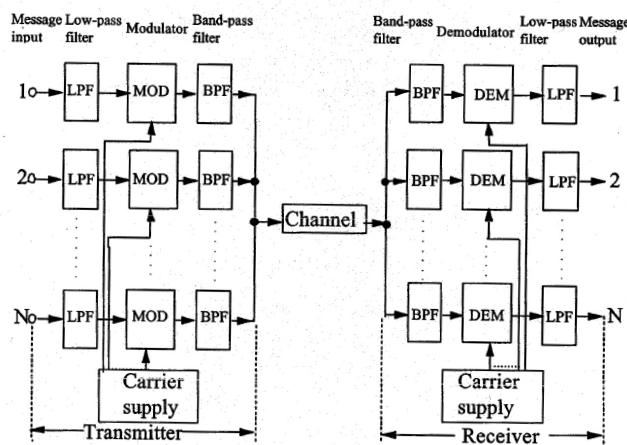


Figure 7.21 - Block diagram of FDM system.

3. Angle Modulation. (phase / frequency).

The general form: $s(t) = A_c \cos \phi(t) = A_c \cos [2\pi f_c t + \underline{\underline{\theta(t)}}]$

instantaneous phase: $\phi(t)$.

instantaneous angle frequency: $\omega_i(t) = \frac{d\phi(t)}{dt} = 2\pi f_c + \frac{d\theta(t)}{dt}$.

instantaneous frequency: $f_i(t) = f_c + \frac{1}{2\pi} \cdot \frac{d\theta(t)}{dt}$.

$2\pi f_c$, f_c 为未调制载波

instantaneous phase deviation: $(\theta(t))$

相偏

instantaneous frequency deviation: $(\frac{1}{2\pi} \cdot \frac{d\theta(t)}{dt})$

信号的频率.

A phase $\xrightarrow{\frac{d}{dt}} \int_0^t dt$ frequency

Presentation of FM and PM signals:

phase modulation (PM): $\underline{\underline{\theta(t)}} = k_p m(t)$, k_p is phase deviation constant (相偏常数)

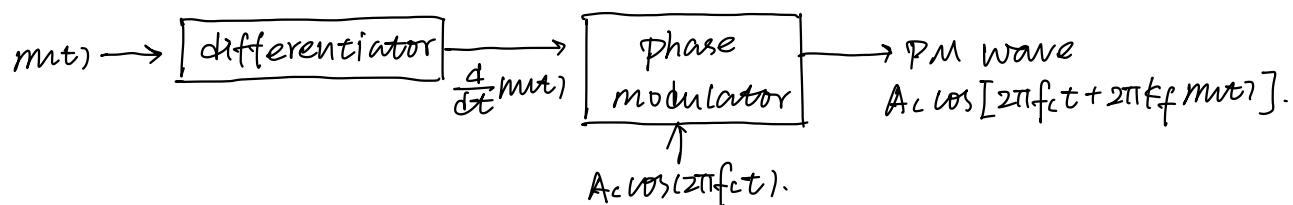
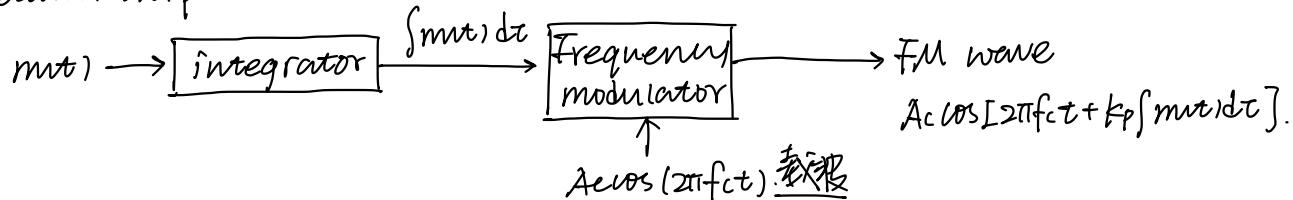
$\Rightarrow s(t) = A_c \cos [2\pi f_c t + \underline{\underline{k_p m(t)}}]$. $\rightarrow \theta(t) = 2\pi f_c \int m(t) dt$.

frequency modulation (FM): $f_i(t) - f_c = \frac{1}{2\pi} \frac{d}{dt} \underline{\underline{\theta(t)}} = k_f m(t)$.

k_f is frequency deviation constant (频偏常数)

$\Rightarrow s(t) = A_c \cos [2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau]$. frequency sensitivity.

Relationship =



Example: Sinusoidal Signal

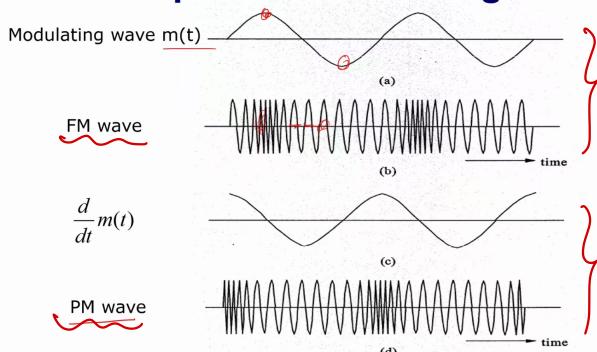


Figure 8.2 - (a) Sinusoidal modulation wave $m(t)$. (b) FM wave. (c) Derivative of $m(t)$ with respect to time. (d) PM wave.

1) FM by a Sinusoidal Signal

message = $m(t) = A_m \cos(2\pi f_m t)$ 也可能 sin.

instantaneous frequency of FM wave =

$$f_i(t) = f_c + k_f A_m \cos(2\pi f_m t) = f_c + \Delta f \cos(2\pi f_m t).$$

⇒ frequency deviation (最大频偏) = $(\Delta f = k_f A_m)$

$$\text{Carrier phase} = \theta(t) = 2\pi \int_0^t (f_i(\tau) - f_c) d\tau = \frac{\Delta f}{f_m} \sin(2\pi f_m t) = \beta \sin(2\pi f_m t).$$

⇒ Modulation index (调制指数) = $(\beta = \frac{\Delta f}{f_m})$ (α in AM)

$$\text{FM signal: } s_i(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)].$$

f_m : message 频率 (带宽)

- Example: a sinusoidal modulating wave of amplitude 5V and frequency 1kHz. is applied to a frequency modulator. The frequency sensitivity is 40Hz/V . The carrier frequency is 100kHz.

Calculate: (a) the frequency deviation. (b) the modulation index

(a) $\Rightarrow A_m = 5\text{V}, f_m = 1\text{kHz}, k_f = 40\text{Hz/V}, f_c = 100\text{kHz}$.

$$\Delta f = k_f A_m = 40\text{Hz/V} \cdot 5\text{V} = 200\text{Hz}.$$

(b) $\beta = \frac{\Delta f}{f_m} = 200\text{Hz}/1\text{kHz} = 0.2$.

Spectrum Analysis of Sinusoidal FM Wave.

FM wave in time-domain =

$$s_i(t) = A_c \operatorname{Re} \left\{ \sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + n f_m)t] \right\} = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t].$$

FM wave in frequency domain =

$$S_i(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) [\delta(f - f_c - n f_m) + \delta(f - f_c + n f_m)]. \quad J_n(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}$$



In theory, BW = ∞.

⇒ 各个冲激的强度由 $J_n(\beta)$ 决定.

12) Effective Bandwidth of FM.

How A_m and f_m affect the spectrum:

{ Fix f_m . Vary $A_m \Rightarrow \Delta f = k_f A_m$ and $\beta = \frac{\Delta f}{f_m}$ are varied.

Fix A_m . Vary $f_m \Rightarrow \Delta f = k_f A_m$ fix. $\beta = \frac{\Delta f}{f_m}$ vary.

When β is small. only focus on the first two terms. $\Rightarrow J_n(\beta)$

When β is large. more terms.

For large β . B is only slightly greater than $2\Delta f$.

For small α . the spectrum is limited to $[f_c - f_m, f_c + f_m]$. $\Rightarrow B = 2f_m$.

\Rightarrow Carson's Rule: ~~$B \approx 2\Delta f + 2f_m = 2(1+\beta) f_m$~~

理论上 FM 波有无限带宽 (即使是 NBFM). 但只考虑 $|J_n(\beta)| > 0.01$ 的部分. 就成了有限带宽.

$B \approx 2n_{max}f_m$. n_{max} is the max n that satisfies $|J_n(\beta)| > 0.01$.

13) FM by an Arbitrary Message.

给定任意调制信号 $m(t)$. 具有最高频率成分 W .

偏移比 (Modulation index) $\beta = \frac{\Delta f}{W}$

频偏 Frequency deviation: $\Delta f = k_f \cdot \max[m(t)]$.

Carson's rule: $B = 2(1+\beta)W$.

14) Narrowband FM

$s(t) = A_c \cos[2\pi f_c t + \theta(t)]$. For $m(t)$. $\theta(t) \ll 1$.

Then $s(t) = A_c \cos 2\pi f_c t \cos \theta(t) - A_c \sin 2\pi f_c t \sin \theta(t)$.
 $\approx A_c \cos 2\pi f_c t - A_c \theta(t) \sin 2\pi f_c t$.

与 conventional AM 类似. 且 $B \approx 2W$.

Example

99% bandwidth approximation

- The freq. component beyond which none of the side-freq. is greater than 1% of the unmodulated carrier amplitude
- i.e. $B \approx 2n_{max}f_m$ where n_{max} is the max n that satisfies $|J_n(\beta)| > 0.01$

β	0.1	0.3	0.5	1.0	2.0	5.0	10	20	30
$2n_{max}$	2	4	4	6	8	16	28	50	70

In north America, the maximum value of frequency deviation Δf is fixed at 75KHz for commercial FM broadcasting by radio.

Take $W = 15\text{KHz}$, typically the maximum audio frequency of interest in FM transmission, the modulation index is

$$\beta = 75/15 = 5$$

Using Carson's rule,

$$B = 2(75 + 15) = 180\text{KHz}$$

Using 99% bandwidth rule,

$$B = 16W = 16 \times 15 = 240\text{KHz}$$

$$180 < 240$$

1.5) Generation of FM waves.

First generate a narrow-band FM signal and change it to WBFM -

- Consider a narrow band FM wave

$$s_1(t) = A_1 \cos[2\pi f_1 t + \phi_1(t)]$$

where $\phi_1(t) = 2\pi k_1 \int_0^t m(\tau) d\tau$

- f_1 = carrier frequency
- k_1 = frequency sensitivity

- Given $\phi_1(t) \ll 1$ with $\beta \leq 0.3$, we may use

$$\begin{cases} \cos[\phi_1(t)] \approx 1 \\ \sin[\phi_1(t)] \approx \phi_1(t) \end{cases}$$

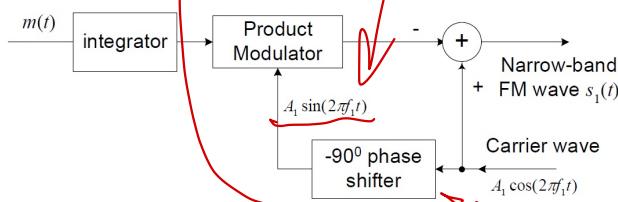
- Correspondingly, we may approximate $s_1(t)$ as

$$\begin{aligned} s_1(t) &= A_1 \cos(2\pi f_1 t) - A_1 \sin(2\pi f_1 t) \phi_1(t) \\ &= (A_1 \cos(2\pi f_1 t)) - 2\pi k (A_1 \sin(2\pi f_1 t)) \int_0^t m(\tau) d\tau \end{aligned}$$

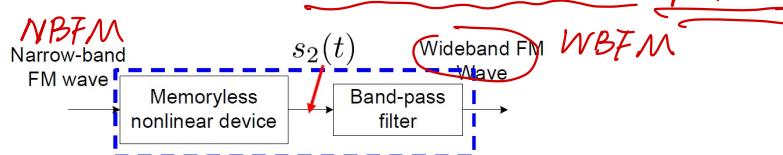
Narrow-band FM wave

NBFM

- Narrow-band frequency modulator



- Next, pass $s_1(t)$ through a frequency multiplier *倍频器*



- The input-output relationship of the non-linear device is:

$$s_2(t) = a_1 s_1(t) + a_2 s_1^2(t) + \dots + a_n s_1^n(t)$$

(n次谐波).

- The BPF is used to pass the FM wave centred at f_1 and with deviation $n\Delta f_1$ and suppress all other FM spectra

频率和带宽都放大n倍

Example: frequency multiplier with $n = 2$

- Problem:** Consider a square-law device based frequency multiplier

with $s_2(t) = a_1 s_1(t) + a_2 s_1^2(t)$

$$s_1(t) = A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau) d\tau\right)$$

s_2 是倍频器
 s_1 是 NBFM

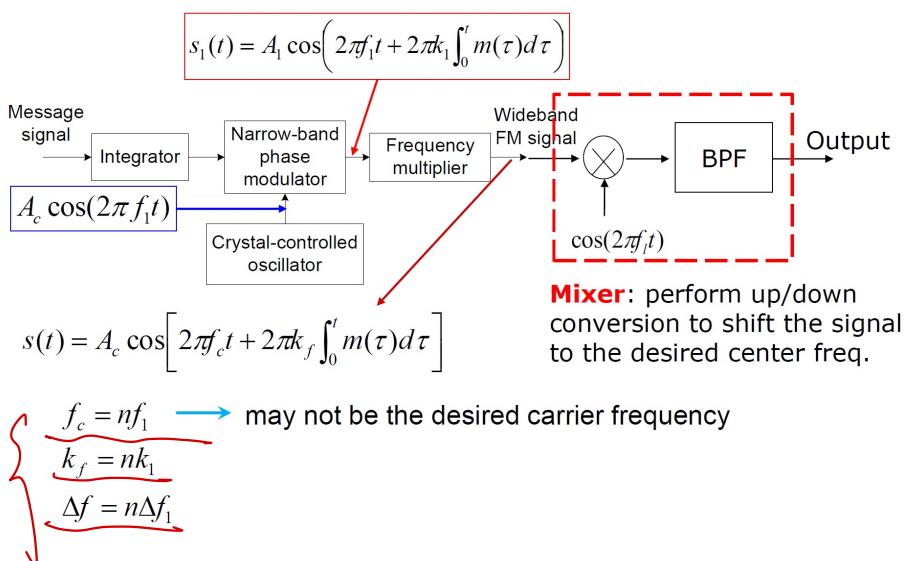
- Specify the midband freq. and bandwidth of BPF used in the freq. multiplier for the resulting freq. deviation to be twice that at the input of the nonlinear device

- Solution:**

$$\begin{aligned} s_2(t) &= a_1 A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau) d\tau\right) + a_2 A_1^2 \cos^2\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau) d\tau\right) \\ &= a_1 A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau) d\tau\right) + \frac{a_2 A_1^2}{2} + \frac{a_2 A_1^2}{2} \cos\left(4\pi f_1 t + 4\pi k_1 \int_0^t m(\tau) d\tau\right) \Rightarrow \text{保留.} \end{aligned}$$

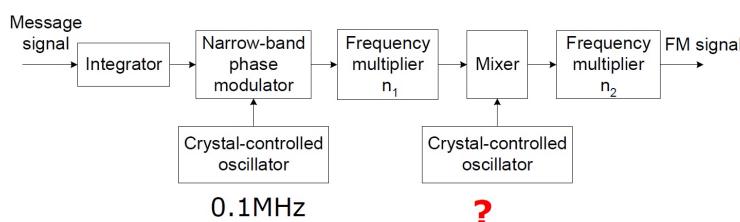
Removed by BPF with $f_c = 2f_1$ and with deviation $2\Delta f$

Generation of Wideband FM Signal



Exercise: A typical FM transmitter

- Problem:** Given the simplified block diagram of a typical FM transmitter used to transmit audio signals containing frequencies in the range 100Hz to 15kHz.
- Desired FM wave: $f_c = 100\text{MHz}$, $\Delta f = 75\text{kHz}$. $\beta = 5$
- Set $\beta_1 = 0.2$ in the narrowband phase modulation to limit harmonic distortion.
- Specify the two-stage frequency multiplier factors n_1 and n_2



Chapt. (ADC). Analog-to-Digital Conversion.

1. Sampling.

$$(1) X_8(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t-nT_s) = \underline{x(t)} \sum_{n=-\infty}^{\infty} \delta(t-nT_s). \Rightarrow \text{pulse modulation.}$$

$$\tilde{X}_8(f) = \tilde{x}(f) * \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \tilde{x}(f - \frac{n}{T_s}) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \tilde{x}(w-nws)$$

Nyquist sampling rate = $f_s = 2W$

2) Reconstruction

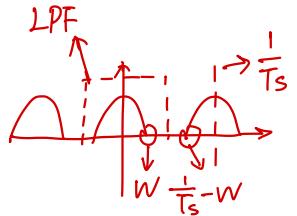
LPF with frequency response = $(H(f) = \begin{cases} T_s & |f| < W \\ 0 & |f| \geq \frac{1}{T_s} - W \end{cases})$

Ideal LPF = $(H(f) = T_s \pi (\frac{f}{2W})) \Rightarrow (h(t) = 2W' T_s \sin(2W't))$, $W \leq W' < \frac{1}{T_s} - W$.

$$x(t) = X_8(t) * 2W' T_s \sin(2W't)$$

$$= \left(\sum_{n=-\infty}^{\infty} x(nT_s) \delta(t-nT_s) \right) * 2W' T_s \sin(2W't)$$

$$= \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \sin[2W'(t-nT_s)], \Rightarrow \text{重构信号, 表示为 sinc 波形叠加.}$$



2. Quantization

$$(1) \text{Quantization regions} = \underline{R_k}, k=1, 2, \dots, N.$$

$$\text{Quantization level (representative point) of each } R_k = \underline{x_k}.$$

$$\text{For each sample } x = x(nT_s): \underline{Q(x) = x_k \text{ when } x \in R_k}.$$

2) Performance measure.

$$\text{Quantization error (noise)} = e[x, Q(x)] = \underline{x - Q(x)}$$

$$\text{Distortion (noise power)} = D = \underline{E[(x - Q(x))^2]} \quad (\text{方差}), P = R_x(D) = E[x^2]$$

$$\Rightarrow D = \int_{-x_p}^{x_p} (x - x_q)^2 f(x) dx, \text{ 计算公式}$$

Signal-to-quantization noise ratio (SQNR):

$$\underline{SQNR = \frac{E(x^2)}{E[(x - Q(x))^2]}} \quad (\text{信噪比: 信号和噪声功率之比}).$$

o Example:

The source $x(t)$ is a stationary Gaussian source with mean zero and power spectral density = $S_x(f) = \begin{cases} 2, & |f| < 100 \text{ Hz} \\ 0, & \text{otherwise} \end{cases}$

It is sampled at the Nyquist rate and each sample is quantized using the 8-level quantizer with

$$a_0 = -\infty, a_1 = -60, a_2 = -40, a_3 = -20, a_4 = 0, a_5 = 20, a_6 = 40, a_7 = 60$$

$$x_1 = -70, x_2 = -50, x_3 = -30, x_4 = -10, x_5 = 10, x_6 = 30, x_7 = 50, x_8 = 70.$$

What is the resulting distortion? the SNR? the Rate?

⇒ For this Gaussian source:

$$\mathbb{E}[x] = 0, \quad \mathbb{E}[x^2(t)] = \sigma^2 = P = R_x(\sigma=0) = \int S_x(f) df = 400. \quad R = v f_s = 600 \text{ bits/sec.}$$

$$D = \mathbb{E}[(x - Q(x))^2] = \int_{-\infty}^{-60} (x+70)^2 f(x) dx + \int_{-60}^{-40} (x+50)^2 f(x) dx + \dots + \int_{40}^{60} (x-70)^2 f(x) dx$$

$f(x)$ has mean of 0, variance of 400. So $f(x) = \frac{1}{\sqrt{2\pi 400}} e^{-\frac{x^2}{800}}$.

⇒ 查表时. $\pm a_i$ 和 $\pm x_i$ 乘 σ . D 乘 σ^2 (表中的是 $N(0,1)$ 高斯分布).
加均值

b). Uniform Quantizer.

Quantization levels = $N = 2^v$. $v \Rightarrow$ 编码长度.

All quantization regions are of equal length = $\Delta = \frac{2a}{N} = \frac{2a}{2^v} = \frac{a}{2^{v-1}}$.

Range of input samples = $x(t) \in [-a, a]$.

Quantized value = $Q(x) = \frac{x_i + x_{i-1}}{2}$.

Quantization error = $-\frac{\Delta}{2} \leq e \leq \frac{\Delta}{2}$.

Assuming = quantization error is uniformly distributed on $(-\frac{\Delta}{2}, \frac{\Delta}{2})$.

↳ noise power = $P = E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} x^2 dx = \frac{\Delta^2}{12} = \frac{a^2}{3N^2} = \frac{a^2}{3 \cdot 4^v}$.

SNR = $\frac{P_x}{E[e^2]} = 10 \log_{10} \frac{P_x}{a^2} + 6v + 4.8 \text{ (dB)}$.



0 Example:

Given a uniform quantizer with 256 levels ($V=8$ bits).

Find the SNR if the signal is $x_1(t) = \cos(2\pi ft)$.

Find the SNR if the signal is $x_2(t)$ uniformly distributed on $[0,1]$.

$$\Rightarrow P_1 = \frac{1}{2} \times 1^2 = \frac{1}{2}, \quad P_2 = E[x^2] = \int_{-1}^1 x^2 \cdot \frac{1}{2} dx = \frac{1}{3}.$$

$$N_q =$$

$$SNR_1 =$$

(Optimal).

(4) Nonuniform Quantizer. \Rightarrow 量化阶可变

\Rightarrow By relaxing the condition that the quantization regions be of equal length, we can minimize the distortion.

Lloyd-Max Conditions:

The boundaries of the quantization regions are the midpoints of the corresponding quantized values:

$$a_i = \frac{1}{2}(x_i + x_{i+1}).$$

The quantized values are the centroids of the quantization regions:

$$x_i^* = \frac{\int_{a_{i-1}}^{a_i} x f(x) dx}{\int_{a_{i-1}}^{a_i} f(x) dx}$$

$$D = \int_{-\infty}^{a_1} (x - \hat{x}_1)^2 f(x) dx + \sum_{i=1}^{N-2} \int_{a_i}^{a_{i+1}} (x - \hat{x}_{i+1})^2 f(x) dx + \int_{a_{N-1}}^{\infty} (x - \hat{x}_N)^2 f(x) dx.$$

3. Encoding. \Rightarrow map: $N=2^v$ quantization values to v bits: $x_i (b_1 b_2 \dots b_v)$.

There are v bits for each sample, and f_s samples per second.

Bit rate: $R = v f_s$ bits/second.

Speech signal usually employs 8 bits for each quantized values with 256 levels.

4. Pulse Code Modulation (PCM).



模拟信号 $x(t)$ 通常先经过带限低通滤波器使其最高频率为 f_s . 再经过采样后成为 PAM 信号. 量化/编码为 A/D 转换. $(f_s \geq 2W)$

Bit rate: if signal has a bandwidth of W and is sampled at f_s and v bits are used to encode each sampled signal: $R_b = f_s v$ bits/sec

Minimum bandwidth requirement: $BW_{req} = \frac{R_b}{2} = \frac{f_s v}{2} \geq v W$ Hz

若以浮点输入信道. 信道所需带宽最小为 vW .

Example

- For voice signals, the frequency range is 300~3400Hz.
- The international standard takes the sampling frequency as $f_s=8\text{kHz}$, and each sample value is represented by an 8-bit binary code ($v=8$)
- Thus, the bit rate of the standard voice channel is

$$R_b = 8 \times 8 = 64 \text{ kbit/s}$$

- The required minimum channel bandwidth (Nyquist \Rightarrow channel bandwidth) is

$$BW = 32 \text{ kHz}$$

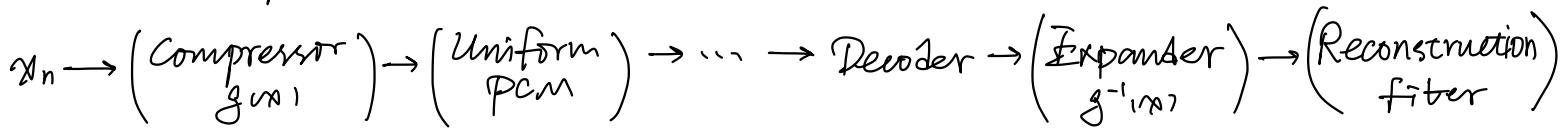
$$f_s = 8\text{kHz} \Rightarrow W = 4\text{kHz}. BW = vW = 8 \times 4 = 32\text{kHz}.$$

↑
300~3400Hz

- Example: A bandlimited signal has bandwidth 3400Hz. What Sampling Rate should be used to guarantee a guard band of 1200Hz?

$$\Rightarrow \underline{f_s = 2W + W_g} \quad \text{so} \quad f_s = 2 \times 3400 + 1200 = 8000 \text{ Hz.}$$

5. Non-uniform PCM.



\Rightarrow companding ($f_s \circ f^{-1}_{un}$) = compress + expand.

- A law compander: $g(x) = \frac{\log(1+\mu|x|)}{\log(1+\mu)} \operatorname{sgn}(x), |x| \leq 1.$

- μ law $= g(x) = \frac{1 + \log A|x|}{\log A} \operatorname{sgn}(x), |x| \leq 1$

6. Differential PCM (DPCM).

7. Delta Modulation (DM).

DM is a simplified version of DPCM having a 2-level quantizer with magnitude $\pm \Delta$.

\Rightarrow The step size Δ is critical in designing a DM system.

A Sampling Theorem.

Signal $x(t)$ have a bandwidth W . Sample $x(t)$ at interval T_s ($T_s \leq \frac{1}{2W}$). Yield the sequence: $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$.

Reconstruct the $x(t)$ by:

$$x(t) = \sum_{n=-\infty}^{\infty} 2W T_s x(nT_s) \text{sinc}[2W(t - nT_s)].$$

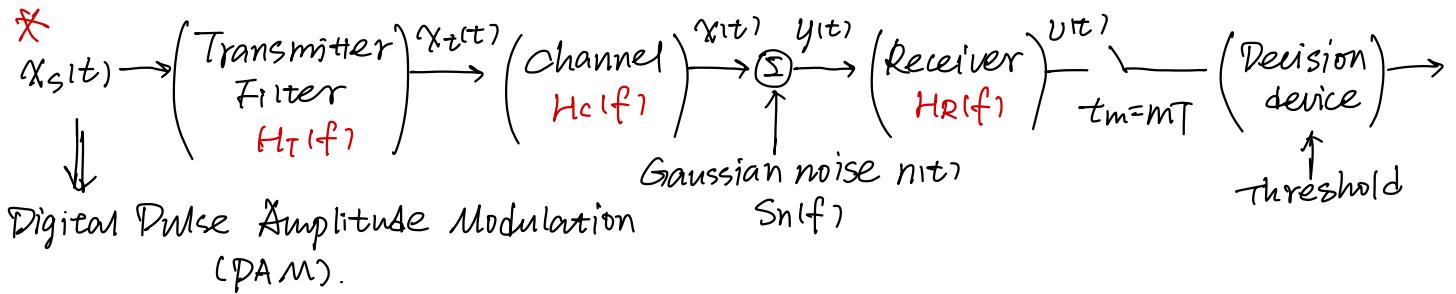
$$\text{where } W' = W \leq W' \leq \frac{1}{T_s} - W.$$

When $T_s = \frac{1}{2W}$ ($f_s = 2W$), then $W' = W = \frac{1}{2T_s}$. Reconstruction:

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}\left(\frac{t}{T_s} - n\right).$$

Chapter 05 = Digital Transmission Through Bandlimited Channels.

1. ISI = Inter-Symbol Interface



$$\hookrightarrow x_s(t) = \sum_{i=-\infty}^{\infty} A_i \delta(t-iT).$$

$$x_{t(t)} = \sum_{i=-\infty}^{\infty} A_i h_T(t-iT).$$

$$v(t) = x_{t(t)} * [h_T(t) * h_c(t) * h_R(t)] + \underbrace{n(t) * h_R(t)}$$

$$\text{Pulse response} = p(t) = h_T(t) * h_c(t) * h_R(t)$$

$$\text{Frequency response} = P(f) = H_T(f) H_c(f) H_R(f).$$

$$\text{Then} = v(t) = \sum_{k=-\infty}^{\infty} A_k p(t-kT) + n(t), \quad n(t) = n(t) * h_R(t).$$

Sample the $v(t)$ at $t_m = mT$ to detect A_m :

$$\begin{aligned} v(t_m) &= \sum_{k=-\infty}^{\infty} A_k p(mT-kT) + n(t_m). \\ &= \underbrace{A_m p(0)}_{\text{Desired signal}} + \left(\underbrace{\sum_{k \neq m}^{\infty} A_k p[(m-k)T]}_{\text{ISI}} + \underbrace{n(t_m)}_{\text{noise}} \right) \end{aligned}$$

2. The power spectrum of digital modulated signals.

$$\text{baseband transmitted signal} = x_{t(t)} = \sum_{i=-\infty}^{\infty} A_i h_T(t-iT).$$

the symbols in the sequence $\{A_m\}$ are uncorrelated:

$$\Rightarrow \text{Auto-correlation sequence } R_A[m] = E(A_i^* A_{i+m}) = \begin{cases} \sigma_A^2 + M_A^2, & m=0 \\ M_A^2, & m \neq 0. \end{cases}$$

Power spectrum (PS):

$$S_A(f) = \sum_{m=-\infty}^{\infty} R_A[m] e^{-j2\pi f m T} = \sigma_A^2 + \frac{M_A^2}{T} \sum_{m=-\infty}^{\infty} \delta(f - \frac{m}{T}).$$

The power spectral density of $x_{t(t)}$ =

$$S_x(f) = \frac{S_A(f)}{T} |H_T(f)|^2 = \left(\frac{\sigma_A^2}{T} |H_T(f)|^2 \right) + \left(\frac{M_A^2}{T^2} \sum_{m=-\infty}^{\infty} \left| H_T \left(\frac{m}{T} \right) \right|^2 \delta(f - \frac{m}{T}) \right)$$

(1) = continuous spectrum

(2) = discrete spectrum. When $M_A \equiv 0$, it becomes zero.

3. Nyquist First criterion.

Assume that bandlimited channel with no distortion: $H(f) = 1$. If $|f| \leq W$

$$\begin{aligned} v(t_m) &= \sum_{k=-\infty}^{\infty} A_k p(mT - kT) + n_0(t_m) \\ &= \text{Amp}(0) + \sum_{k \neq m}^{\infty} A_k p[(m-k)T] + n_0(t_m) \end{aligned}$$

To ensure zero ISI, the $p(nT)$ must satisfy:

$$p(nT) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

⇒ Necessary and sufficient condition =

$$P_{\Sigma}(f) = \sum_{k=-\infty}^{\infty} P(f + \frac{k}{T}) = \text{constant}. \quad (\text{Nyquist condition})$$

$$\text{with zero ISI} \Rightarrow v(t_m) = \underline{A_m + n_0(t_m)}$$

⇒ 系统总的传输函数 $P(f)$ 以 $\frac{1}{T}$ 为周期进行平移，叠加后得到一常数。
仅由此条件不能唯一确定 $P(f)$. $P(f)$ 的最小带宽应为 $\frac{1}{2T}$

Nyquist's first method uses: ($W = \frac{1}{2T}$)

$$p(f) = \begin{cases} 1, & |f| < \frac{1}{2T} \\ 0, & \text{otherwise} \end{cases} \Leftrightarrow p(t) = \text{sinc}(\frac{t}{T}) = \frac{\sin(\pi t/T)}{\pi t/T} \rightarrow \frac{1}{t}$$

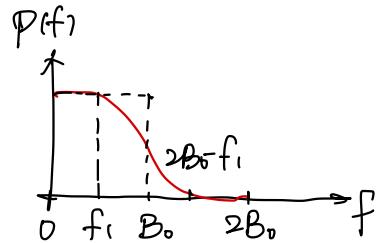
$$\begin{cases} \text{Nyquist bandwidth: } B_0 = \frac{1}{2T} = \frac{R_s}{2} \\ \text{Nyquist rate: } R_s = 2B_0. \end{cases} \Rightarrow \begin{cases} \text{给定 } R_s, \text{ 则最小信道带宽为 } \frac{R_s}{2} \\ \text{给定 } B_0, \text{ 则最大速率 } 2B_0. \end{cases}$$

- Challenges of designing such $p(t)$ or $P(f)$
 - $P(f)$ is physically unrealizable due to the abrupt transitions at $\pm B_0$
 - The tail of $p(t)$ decays as $\frac{1}{t}$, which is slow and results in little margin of error in sampling times in the receiver.
 - This demands accurate sample point timing - a major challenge in modem / data receiver design.
 - Inaccuracy in symbol timing is referred to as **timing jitter** (定时抖动).

4. Raised Cosine Spectrum.

Three parts: passband, transition band, stopband.

$$P(f) = \begin{cases} 1 & , 0 \leq |f| < f_1 \\ \frac{1}{2} \left\{ 1 + \cos \left[\frac{\pi (|f| - f_1)}{2B_0 - 2f_1} \right] \right\} & , f_1 \leq |f| < 2B_0 - f_1 \\ 0 & , |f| \geq 2B_0 - f_1 \end{cases}$$



适当放宽了 $P(f) = \begin{cases} 1, & f=0 \\ 0, & f \neq 0 \end{cases}$ 的条件，允许加入 ISI。($B_0 = \frac{1}{2T}$)

$$\text{Roll-off factor } \alpha = 1 - \frac{f_1}{B_0} \quad (\text{f}_1 \in [0, 1]) \quad \text{Total Bandwidth} = B = B_0(1 + \alpha) = \frac{1 + \alpha}{2T}$$

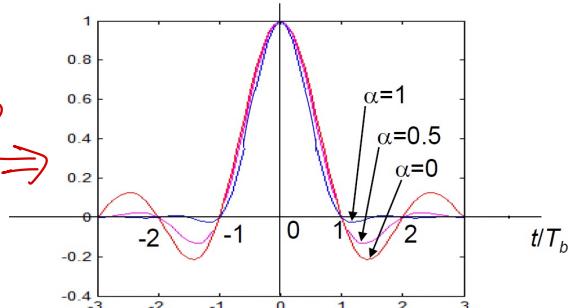
$\hookrightarrow \alpha=0$ 时，相当于 Nyquist 第一准则，增大 α ，频谱变宽，谱形变缓。

$$\Rightarrow \text{Time domain: } p(t) = \operatorname{sinc}(2B_0 t) \frac{\cos(2\pi \alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2} \rightarrow \frac{1}{t^2}$$

$\hookrightarrow \operatorname{sinc}(2B_0 t)$ 保证了在采样点处其值

为 1 ($k=m$) 或 0 ($k \neq m$)。

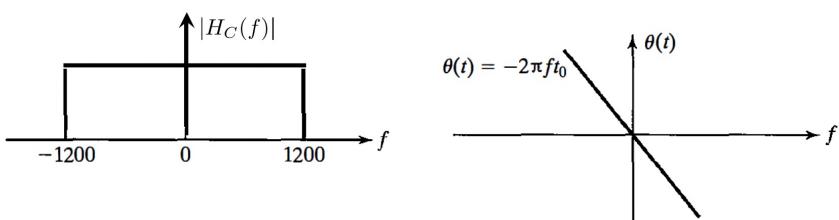
包括以 $\frac{1}{2}$ 速率下样，能够减小其造成的 ISI 影响。 α 越大越显著，但频谱会变宽。 \Rightarrow



Example:

An ideal channel has the freq. response shown below. We wish to design the transmit and receiver filters such that the overall freq. response is the raised cosine spectrum with roll-off factor $\alpha = 0.5$.

Determine the symbol rate $1/T$, and compare it with the Nyquist rate



$$\Rightarrow \text{Let } H_T(f) H_C(f) H_R(f) = R_{rc}(f)$$

$$\frac{1+\alpha}{2T} = \frac{1 + \frac{1}{2}}{2T} = 1200$$

$$\text{then } T = 1600.$$

the symbol rate $\frac{1}{T} = 1600$ symbols/sec. the Nyquist rate = 2400 symbols/sec.

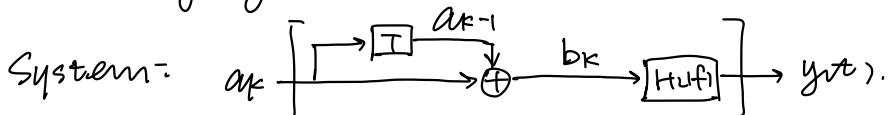
$$R_{rc}(f) = \begin{cases} T & , 0 \leq |f| \leq 400 \\ \frac{T}{2} \left[1 + \cos \left(\frac{\pi}{800} (|f| - 400) \right) \right] & , 400 \leq |f| \leq 1200 \\ 0 & , |f| > 1200 \end{cases}, f_1 = B_0(1 + \alpha) = \frac{1 + \alpha}{2T} = \underline{\underline{400}}$$

$$\Rightarrow |G_T(f)| = |G_R(f)| = \sqrt{R_{rc}(f)}, \theta_R(f) = -\theta_T(f).$$

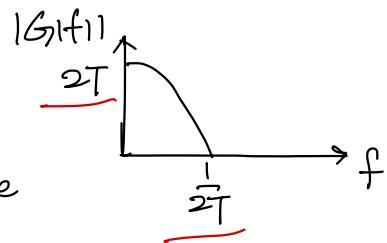
5. Duobinary Signal.

$\{a_k\}$ is the transmitted PAM binary sequence. Pulse duration is T

$$\text{Duobinary signal} = b_k = a_k + a_{k-1}$$



$$\Rightarrow G(f) = (1 + e^{-j2\pi fT}) H_L(f) = \begin{cases} 2Te^{-j\pi fT} \cos \pi fT, & |f| \leq \frac{1}{2T} \\ 0, & \text{otherwise} \end{cases}$$

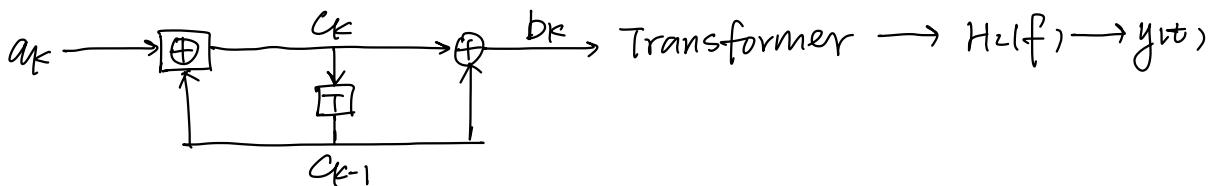


$$g(t) = [f(t) + f(t-T)] * h_L(f) = \underbrace{\sin(\frac{t}{T}) + \sin(\frac{t-T}{T})}_{= \frac{T^2}{\pi t} \cdot \frac{\sin \pi t/T}{(T-t)}} \rightarrow \frac{1}{t^2}$$

$$g(nT) = \begin{cases} 1, & n=0. \text{ (current symbol)} \\ 1, & n=1. \text{ (ISI to the next symbol)} \\ 0, & \text{otherwise} \end{cases}$$

To solve the ambiguity problem and error propagation: Precoding.

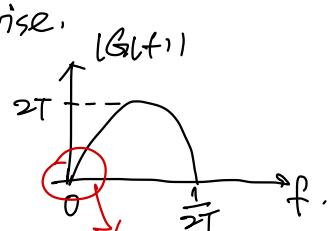
$$\Rightarrow c_k = a_k \oplus c_{k-1}, \quad b_k = c_k + c_{k-1}$$



6. Modified Duobinary signal: $b_k = a_k - a_{k-2}$.

$$\Rightarrow G(f) = (1 - e^{-j4\pi fT}) H_L(f) = \begin{cases} 2Te^{-j2\pi fT} \sin 2\pi fT, & |f| \leq \frac{1}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

$$g(t) = \sin(\frac{t}{T}) - \sin(\frac{t-2T}{T}) = -\frac{2T^2 \sin \pi t/T}{\pi t(t-2T)} \rightarrow \frac{1}{t^2}$$



$$\text{Precoding: } c_k = a_k \oplus c_{k-2}, \quad b_k = c_k - c_{k-2}$$

no DC component.

7. Optimum Transmit / Receive Filter.

$P(f) = P_{rc}(f)$. then the output $V_m = A_m + N_m$ (assume $p(0)=1$)

Consider binary PAM transmission: $A_m = \pm d$. $\left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow SNR = \frac{d^2}{\sigma^2}$.
 Variance of $N_m = \sigma^2 = \int_{-\infty}^{\infty} S_n(f) |H_R(f)|^2 df$.

After $H_c(f)$ fixed and $P(f)$ given, select $H_T(f), H_R(f)$ to maximize SNR.

Solution:
$$\left\{ \begin{array}{l} |H_T(f)| = \frac{\sqrt{P_{rc}(f)}}{|H_c(f)|^{1/2}}, \quad |f| \leq W \\ |H_R(f)| = \frac{\sqrt{P_{rc}(f)}}{|H_c(f)|^{1/2}}, \quad |f| \leq W \end{array} \right.$$

Then: $E_{av} = \int_{-\infty}^{\infty} d^2 h_T^2(t) dt = \int_{-\infty}^{\infty} d^2 H_T^2(f) df = d^2 \int_{-W}^W \frac{P_{rc}(f)}{|H_c(f)|} df$

$$d^2 = E_{av} \cdot \left[\int_{-W}^W \frac{P_{rc}(f)}{|H_c(f)|} df \right]^{-1}.$$

$$S_N(f) = S_n(f) |H_R(f)|^2. \Rightarrow \sigma^2 = \frac{N_0}{2} \int_{-W}^W |H_R(f)|^2 df = \frac{N_0}{2} \int_{-W}^W \frac{P_{rc}(f)}{|H_c(f)|} df.$$

$$\Rightarrow SNR = \frac{2E_{av}}{N_0} \cdot \left[\int_{-W}^W \frac{P_{rc}(f)}{|H_c(f)|} df \right]^{-2}$$

due to channel distortion.

Example: $\left(\frac{3}{B_{cm}}\right)$ Design transmitting and receiving filter for a given channel.

Determine the magnitude of the transmitting and receiving filter characteristics for a binary communication system that transmits data at a rate of 4800 bits/sec over a channel with frequency response

$$|C(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{W}\right)^2}}, \quad |f| \leq W,$$

where $W = 4800$ Hz. The additive noise is zero-mean white Gaussian with a spectral density

$$\underline{N_0/2} = 10^{-15} \text{ Watt/Hz}$$

\Rightarrow Symbol rate $\frac{1}{T} = 4800$ bits/sec. Nyquist bandwidth $B_0 = \frac{1}{2T} = 2400$ Hz.

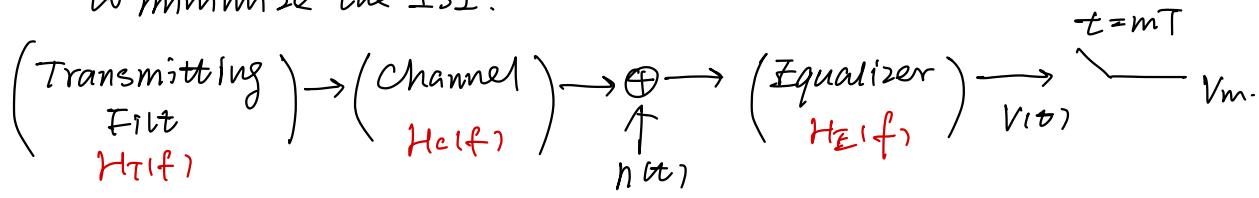
For $P_{rc}(f)$, its bandwidth $B = B_0(1+\alpha)$. $\Rightarrow \alpha=1$. $f_1=0$.

$$\text{Thus: } P_{rc}(f) = \frac{1}{2} [1 + \cos\left(\frac{\pi|f|}{2B_0}\right)] = \cos^2\left(\frac{\pi|f|}{9600}\right).$$

$$\text{Then: } |H_T(f)| = |H_R(f)| = \cos\left(\frac{\pi|f|}{9600}\right) \left[1 + \left(\frac{f}{4800}\right)^2\right]^{\frac{1}{4}}. \quad |f| \leq 4800.$$

8. Equalizer

↪ A receive filter with adjustable frequency response.
to minimize the ISI.



$$\Rightarrow H_0(f) = H_T(f) H_C(f) H_E(f). \quad \text{Nyquist criterion: } \sum_{k=-\infty}^{\infty} H_0(f + \frac{k}{T}) = \text{constant.}$$

Linear Transversal Filter (横向滤波器).

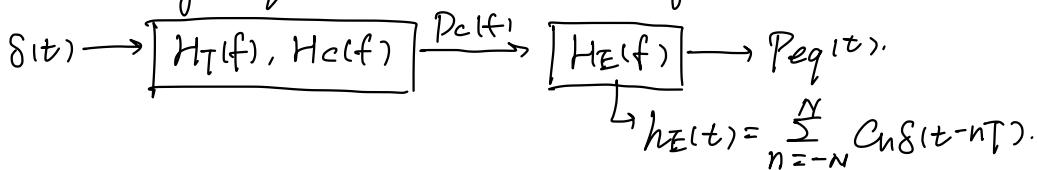
Finite impulse response filter:

$$h_E(t) = \sum_{n=-N}^{N} c_n \delta(t-nT). \quad H_E(t) = \sum_{n=-N}^{N} c_n e^{-j2\pi f n T}. \Rightarrow \underbrace{(2N+1)}_{\text{tap FIR}}$$

$\{c_n\}$ are the $2N+1$ equalizer coefficients.

N is sufficiently large to span the length of ISI.

Zero-Forcing Equalizer (零均值均衡器).



$$\Rightarrow P_{eq}(t) = P_c(t) * h_E(t) = \sum_{n=-N}^{N} c_n P_c(t-nT).$$

$$\text{Sample at } t=mT: \quad P_{eq}(mT) = \sum_{n=-N}^{N} c_n P_c[(m-n)T] = \begin{cases} 1, & m=0 \\ 0, & m=\pm 1, \dots, \pm N. \end{cases}$$

suppress $2N$ adjacent interference terms.

$$P_{eq} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdot C = \begin{pmatrix} C-N \\ \vdots \\ C-N+1 \\ \vdots \\ C-1 \\ C_0 \\ \vdots \\ C_1 \\ \vdots \\ C_N \end{pmatrix} \cdot P_c = \begin{pmatrix} P_c(0) & P_c(-1) & \cdots & P_c(-2N) \\ P_c(1) & P_c(0) & \cdots & P_c(-2N+1) \\ \vdots & \vdots & \ddots & \vdots \\ P_c(2N) & P_c(2N-1) & \cdots & P_c(0) \end{pmatrix}$$

$$\Rightarrow C = P_c^{-1} P_{eq} \text{ or the middle-column of } P_c^{-1}.$$

channel response matrix.

$$\text{equalized pulse response: } P_{eq}(m) = \sum_{n=-2}^2 c_n P_c(m-n).$$

Minimum Mean-Square Error Equalizer

Decision Feedback Equalizer

Maximum Likelihood Sequence Estimation.

Monte Carlo Simulation

Chapter 06: Signal Space Representation.

1. Vectors and space.

an n -dimensional space with unity basis vectors $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$
any vector \vec{a} in the space can be written as: $\vec{a} = \sum a_i \vec{e}_i$.

Orthonormal = mutually orthogonal and all have unity norm.

The set of basis vectors $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of a space:

① complete ② orthogonal ③ normalized.

⇒ complete orthonormal basis (完备正交基).

2. Signal space and orthonormal basis.

Signal set $\{\phi_k(t)\}_{k=1}^n$ is an orthogonal set if

$$\int_{-\infty}^{\infty} \phi_j(t) \phi_k(t) dt = \begin{cases} 0, & j \neq k \\ c_j, & j = k. \end{cases}$$

If $c_j = 1, \forall j$, then $\{\phi_k(t)\}$ is an orthonormal set.

Then: $\vec{x}(t) = \sum_{k=1}^n x_k \phi_k(t)$. $x_k = \int_{-\infty}^{\infty} \vec{x}(t) \phi_k(t) dt$. $\vec{x} = (x_1, x_2, \dots, x_n)$.

Properties: given two signals $\vec{x}(t) = \sum_{i=1}^n x_i \phi_i(t)$. $\vec{y}(t) = \sum_{i=1}^n y_i \phi_i(t)$
 $\vec{x} = (x_1, x_2, \dots, x_n)$. $\vec{y} = (y_1, y_2, \dots, y_n)$.

then $\vec{x} \cdot \vec{y} = \int_{-\infty}^{\infty} \vec{x}(t) \vec{y}(t) dt$.

Energy of signal $\vec{x}(t) = E_x = \int_{-\infty}^{\infty} \vec{x}^2(t) dt$, $E_x = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$.

3. Basic functions for a signal set.

- Consider a set of M signals (M -ary symbol) $\{s_i(t), i = 1, 2, \dots, M\}$ with finite energy. That is $\int_{-\infty}^{\infty} s_i^2(t) dt < \infty$
- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^K s_{ij} \phi_j(t) \quad \text{for } i = 1, \dots, M$$

where $K \leq M$ is the dimension of the signal space
and $\{\phi_j(t)\}_{j=1}^K$ are called the orthonormal basis functions

o Example (QPSK).

signal: $s(t) = \pm \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \pm \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$ with $t \in [0, T]$, and $f_c T \gg 1$.

basis function: $\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$, $\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$, $t \in [0, T]$.

$$\begin{aligned} \Rightarrow \int_0^T \phi_1(t) \phi_2(t) dt &= \int_0^T \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \cdot \sqrt{\frac{2}{T}} \sin(2\pi f_c t) dt \\ &= \frac{2}{T} \int_0^T \frac{1}{2} [\sin(0) + \sin(4\pi f_c t)] dt \\ &= -\frac{1}{4\pi f_c T} [\cos(4\pi f_c t)]_0^T \approx 0. \quad f_c T \gg 1. \end{aligned}$$

$$\int_0^T |\phi_1(t)|^2 dt = \int_0^T |\phi_2(t)|^2 dt = \frac{2}{T} \int_0^T \frac{1}{2} [1 + \cos(4\pi f_c t)] dt \approx 1.$$

o Notes:

- (1) Different signal sets can have the same geometric representation.
(2) The underlying geometry will determine the performance and the receiver.

4. GSO (find a complete orthonormal basis for an arbitrary signal set).

Given a signal set $\{s_1(t), s_2(t), \dots, s_M(t)\}$.

Find the basis functions $\{\phi_1(t), \phi_2(t), \dots, \phi_N(t)\}$, where $N \leq M$.

Step 1: Construct the first basis function.

$\phi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$ (Normalized by the energy $E_1 = \int_{-\infty}^{\infty} s_1^2(t) dt$).

then: $S_{11} = \int_{-\infty}^{\infty} S_1(t) \phi_1(t) dt = \sqrt{E_1}$. $S_1(t) = S_{11} \phi_1(t) = \sqrt{E_1} \phi_1(t)$.

Step 2: the correlation between $s_2(t)$ and $\phi_1(t) = S_{21} = \int_{-\infty}^{\infty} S_2(t) \phi_1(t) dt$

Subtract off the correlation portion: $g_2(t) = S_2(t) - S_{21} \phi_1(t)$

$\hookrightarrow g_2(t)$ is orthogonal to $\phi_1(t)$.

the energy = $E_{g_2} = \int_{-\infty}^{\infty} [g_2(t)]^2 dt$.

normalize: $\phi_2(t) = \frac{1}{\sqrt{E_{g_2}}} g_2(t)$.

then: $\int_{-\infty}^{\infty} S_2(t) \phi_2(t) dt = S_{22} = \sqrt{E_{g_2}}$.

Step 3: Construct the successive basis functions.

$$\text{For signal } S_k(t) = \int_{-\infty}^{\infty} s_k(t) \phi_i(t) dt, \quad i=1, 2, \dots, K-1.$$

$$g_k(t) = s_k(t) - \sum_{i=1}^{K-1} s_k \phi_i(t).$$

$$E_{g_k} = \int_{-\infty}^{\infty} [g_k(t)]^2 dt$$

k -th basis function = $\phi_k(t) = \frac{1}{\sqrt{E_{g_k}}} g_k(t)$. $S_{kF} = \sqrt{E_{g_k}} \cdot S_k(t) = \sqrt{E_{g_k}} \phi_k(t)$

- Consider a set of M signals (M-ary symbol) $\{s_i(t), i = 1, 2, \dots, M\}$ with finite energy. That is

$$\int_{-\infty}^{\infty} s_i^2(t) dt < \infty$$

- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^K s_{ij} \phi_j(t) \quad \text{for } i = 1, \dots, M$$

where $K \leq M$ is the **dimension** of the signal space
and $\{\phi_j(t)\}_{j=1}^K$ are called the **orthonormal basis functions**

Notes on GSO Procedure

- A signal set may have many different sets of basis functions.
- But the dimensionality of the signal space is independent of the selected orthonormal basis.
- A change of basis functions is essentially a rotation of the signal points around the origin.
- The order in which signals are used in the GSO procedure affects the resulting basis functions
- The choice of basis functions does not affect the performance of the modulation scheme.

Chapter 07: Optimal Receivers.

1. Statistic Decision Theory.

2. Detection Theory.

3. MAP Decision Criterion \Rightarrow 最大后验概率判决

$P(m_i | \vec{r}) = P(\text{signal } m_i \text{ was transmitted given } \vec{r} \text{ observed})$. $i=1, \dots, M$.

给定收端向量 \vec{r} 可得到一系列后验概率 $P(m_1 | \vec{r}), P(m_2 | \vec{r}), \dots, P(m_M | \vec{r})$. 在这些概率有最大值 $P(m_k | \vec{r})$. 则根据 MAP 判则. 选择 $\hat{m} = m_k$

choose $\hat{m} = m_k$ iff $P(m_k | \vec{r}) \geq P(m_i | \vec{r})$ for all $i \neq k$.

m_k 的概率.

By Bayes' Rule: $P(m_i | \vec{r}) = \frac{P_i f(\vec{r} | m_i)}{f(\vec{r})}$. then $\underline{P_k f(\vec{r} | m_k) \geq P_i f(\vec{r} | m_i)}$.

\Rightarrow 最大后验概率判决和最小差错概率是等价的 \downarrow P_i 是信号 m_i 发送的先验概率.

4. ML Decision Criterion \Rightarrow 最大似然判决.

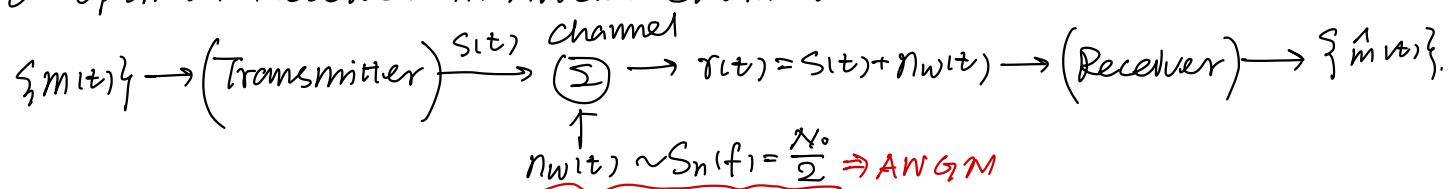
If signals are equiprobable = $P_1 = P_2 = \dots = P_M$.

then $P_k f(\vec{r} | m_k) \geq P_i f(\vec{r} | m_i) \Rightarrow \underline{f(\vec{r} | m_k) \geq f(\vec{r} | m_i)}$

choose $\hat{m} = m_k$. iff $f(\vec{r} | m_k) \geq f(\vec{r} | m_i)$. for all $i \neq k$

\hookrightarrow Find the maximum likelihood function $f(\vec{r} | m_k)$.

5. Optimal Receiver in AWGN Channel.



Transmitter transmits a sequence of symbols $\{m_1, m_2, \dots, m_M\}$ with prior probabilities $P(m_1), P(m_2), \dots, P(m_M)$. Symbols are represented by finite energy waveforms $s_1(t), s_2(t), \dots, s_M(t)$, $t \in [0, T]$

The signal space $\{s_1(t), s_2(t), \dots, s_M(t)\}$ has a dimension N ($N \leq M$)
 Signals can be represented as $s_M(t) = \sum_{k=1}^N S_{mk} \phi_k(t)$ $S_{mk} = \int_0^T s_M(t) \phi_k(t) dt$.

the noise $n_w(t)$ can be written as:

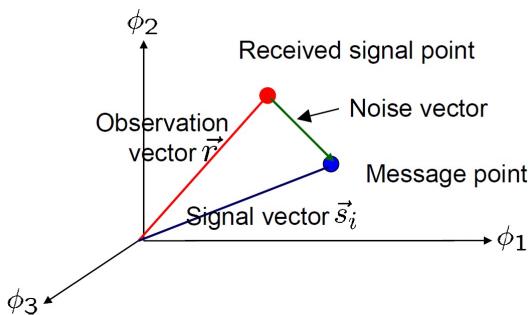
$$n_w(t) = n_o(t) + \sum_{k=1}^N n_k \phi_k(t). \quad n_k = \int_0^T n_w(t) \phi_k(t) dt.$$

\downarrow falls outside the signal space spanned by $\phi_k(t)$.

Then the received signal:

$$\begin{aligned} r(t) &= s(t) + n_w(t) = \sum_{k=1}^N S_{mk} \phi_k(t) + \sum_{k=1}^N n_k \phi_k(t) + n_o(t) \\ &= \sum_{k=1}^N (S_{mk} + n_k) \phi_k(t) + n_o(t) = \sum_{k=1}^N r_k \phi_k(t) + n_o(t). \quad r_k = S_{mk} + n_k. \end{aligned}$$

$$\vec{r} = \vec{s}_i + \vec{n}$$



6. Receiver Structure

two parts:

- { Signal demodulator = convert waveform $r(t)$ to $\vec{r} = (r_1, r_2, \dots, r_N)$
- { Detector = decide which of the M possible signals were transmitted based on the observation \vec{r} .

$$r(t) \rightarrow (\text{Signal demodulator}) \rightarrow \vec{r} \rightarrow (\text{Detector}) \rightarrow \hat{m}.$$

Signal demodulator =

- { correlation-type demodulator
- { matched-filter-type demodulator.

7. Matched Filter \Rightarrow 正配滤波器.

\hookrightarrow the optimal linear filter to maximize the output SNR.

$$x(t) = s_i(t) + n_i(t) \rightarrow \begin{pmatrix} h(t) & H(f) \end{pmatrix} \rightarrow y(t) = s_o(t) + n_o(t) \rightarrow \text{at } t=t_0 \rightarrow y(t_0)$$

the output SNR =

$$d = \frac{\mathbb{E}[s_o^2(t_0)]}{\mathbb{E}[n_o^2(t_0)]} = \frac{\left[\int_{-\infty}^{\infty} A(f) H(f) e^{j\omega_0 t_0} df \right]^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

where $A(f) = \int_{-\infty}^{\infty} s_i(t) e^{-j\omega t} dt$. $S_{n_0}(f) = \frac{N_0}{2} |H(f)|^2 \Rightarrow N_{S_{n_0}} = \mathbb{E}[n_o^2(t_0)] = \int_{-\infty}^{\infty} S_{n_0}(f) df$

\hookrightarrow Find $H(f)$ to maximize d.

$$\Rightarrow d \leq \frac{\int_{-\infty}^{\infty} |A(f)|^2 df \int_{-\infty}^{\infty} |H(f)|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} = \frac{\int_{-\infty}^{\infty} |A(f)|^2 df}{\frac{N_0}{2}} = \frac{2E}{N_0}. E \text{ is signal energy.}$$

max output SNR

when $\frac{2E}{N_0}$ is achieved: $(H_m(f) = A^*(f) e^{-j\omega_0 t_0}) \Rightarrow (h_m(t) = s_i^*(t_0 - t))$

Schwarz 不等式成立的条件.

$\Rightarrow h_m(t)$ is matched to the input signal $s_i(t)$.

$$\left\{ \begin{array}{l} \text{Frequency response: } H_m(f) = A^*(f) e^{-j\omega_0 t_0} \\ \text{Impulse response: } h_m(t) = s_i^*(t_0 - t) \end{array} \right.$$

Properties of MF =

$$\textcircled{1} \quad h_m(t) = \begin{cases} s_i(t_0 - t), & 0 \leq t \leq t_0 \\ 0, & \text{o.w.} \end{cases} \quad \text{where } \underline{t_0 > T}. \quad \hookrightarrow \text{为了满足因果性.}$$

\textcircled{2} Equivalent from Correlator 相关器.

$$y(t) = x(t) * h_m(t) = x(t) * s_i(T-t) = \int_0^T x(\tau) s_i(T-t+\tau) d\tau.$$

第一章 相关函数介绍

sampling time $t=T$: $y(T) = \int_0^T x(t) s_i(t) dt \Rightarrow$ correlation integration -

$$x(t) \xrightarrow{\text{(MF)}} \text{at } t=T \rightarrow y(T) = x(t) \xrightarrow{\text{at } t=T} \left(\int_0^T \cdot dt \right) \xrightarrow{\text{at } t=T} y(T).$$

Temporal Autocorrelation vs. Statistical Autocorrelation

1. Temporal autocorrelation function	2. Statistical autocorrelation function
$R(\tau) = \int_{-\infty}^{\infty} s(t)s(t+\tau) dt$	$R(\tau) = E[X(t)X(t+\tau)]$
Temporal cross-correlation function	Statistical cross-correlation function
$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t)s_2(t+\tau) dt$	$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$
(1) $R(\tau) = R(-\tau)$	(1) $R(\tau) = R(-\tau)$
(2) $R(0) \geq R(\tau)$	(2) $R(0) \geq R(\tau)$
(3) $R(0) = \int_{-\infty}^{\infty} s^2(t) dt = E[s^2] - \text{signal energy}$	(3) $R(0) = E[X^2(t)] = P - \text{signal average power}$
(4) $R(\tau) \leftrightarrow A(f) ^2$ - signal energy spectrum	(4) $R(\tau) \leftrightarrow S_X(f)$ - signal power spectral density

Time averaging autocorrelation function $\langle X(t)X(t+\tau) \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T X(t)X(t+\tau) dt$

③ MF output signal is the autocorrelation function of input signal.

$$S_0(t) = \int_{-\infty}^{\infty} S_i(t-u) h_m(u) du = \int_{-\infty}^{\infty} S_i(t-u) S_i(t+u) du \\ = \int_{-\infty}^{\infty} S_i(v) S_i(v+t) dv = R_{S_i}(t).$$

The peak value of $S_0(t)$ happens $t=t_0 = S_0(t_0) = R_{S_i}(0) = \int_{-\infty}^{\infty} S_i^2(v) dv = E_{S_i}$.

$S_0(t)$ is symmetric at $t=t_0 \Rightarrow$ 相关函数的性质.

$$A_0(f) = |A(f) H_m(f)| = |A(f)|^2 e^{-j\omega t_0}.$$

④ MF output noise.

The statistic autocorrelation of $n_o(t)$ =

$$R_o(\tau) = E[n_o(t)n_o(t+\tau)] = \frac{N_o}{2} \int_{-\infty}^{\infty} h_m(u) h_m(u+\tau) du = \frac{N_o}{2} \int_{-\infty}^{\infty} S_i(t) S_i(t+\tau) dt.$$

Average power:

$$E[n_o^2(t)] = R_o(0) = \frac{N_o}{2} \int_{-\infty}^{\infty} S_i^2(u) du = \frac{N_o}{2} \int_{-\infty}^{\infty} |A(f)|^2 df = \frac{N_o}{2} \int_{-\infty}^{\infty} |H_m(f)|^2 df = \frac{N_o}{2} E.$$

When the noise is colored (not white and $S_n(f) = \frac{N_o}{2}$) =

\Rightarrow choose $H_1(f)$ so that $n'(t)$ is white. $H_1(f) = H_1(f) H_2(f)$.

$S_n'(f) = |H_1(f)|^2 S_n(f) = C$. and $H_2(f)$ matches with $s'(t) = s(t) * h_1(t)$.

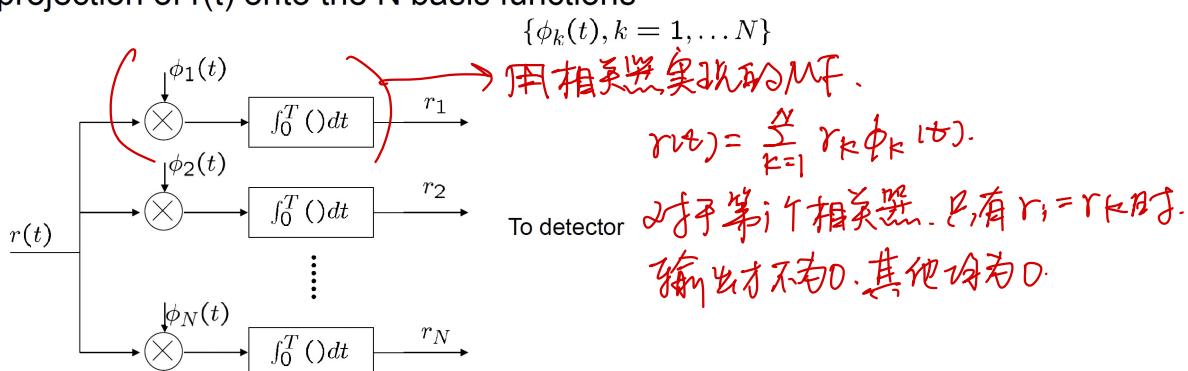
$\Rightarrow H_1(f) = |H_1(f)|^2 A^*(f) e^{-j2\pi f t_0}$ (white noise: $H_1(f) = A^*(f) e^{-j\omega t_0}$).

$$H_1(f) = \frac{A^*(f)}{S_n(f)} e^{-j2\pi f t_0} \Rightarrow \text{MF for colored noise}$$

8. Two types of demodulator

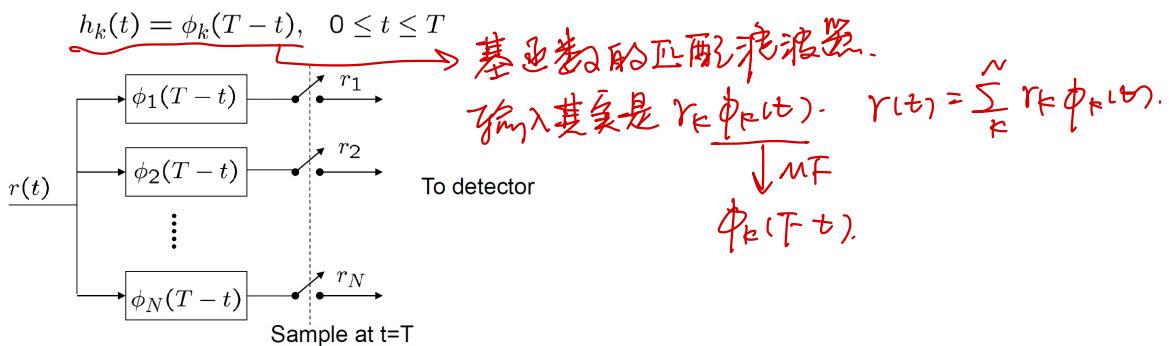
① Correlation Type Demodulator

- The received signal $r(t)$ is passed through a parallel bank of N cross correlators which basically compute the projection of $r(t)$ onto the N basis functions



② Matched-Filter type demodulator

- Alternatively, we may apply the received signal $r(t)$ to a bank of N matched filters and sample the output of filters at $t = T$. The impulse responses of the filters are



Signal demodulator 简化为判决向量 $\vec{r} = (r_1, r_2, \dots, r_N)^T$. 通过 signal detector 进行判决以达到最小误判准则.

8. Optimal Detector

① Decision Rules

$$\begin{cases} \text{MAP} = \text{choose } \hat{m} = m_k \text{ iff } P_f(\vec{r}|m_k) > P_f(\vec{r}|m_i), i \neq k. \\ \text{ML} = \text{choose } \hat{m} = m_k \text{ iff } f(\vec{r}|m_k) > f(\vec{r}|m_i), i \neq k. \end{cases}$$

② Distribution of the Noise Vector.

$n_w(t) \Rightarrow$ Gaussian random process.

$n_k = \int_0^T n_w(t) \phi_k(t) dt \Rightarrow$ Gaussian random variable.

mean: $E[n_k] = \int_0^T E[n_w(t)] \phi_k(t) dt = 0$.

correlation:

$$E[n_j n_k] = \frac{N_0}{2} \int_0^T \phi_j(\tau) \phi_k(\tau) d\tau = \begin{cases} \frac{N_0}{2}, & j=k \\ 0, & j \neq k \end{cases} \quad (\text{PPT, P36}).$$

Then: n_k are independent with zero mean and variance $\frac{N_0}{2}$.

The joint pdf of $\vec{n} = (n_1, n_2, \dots, n_K)$:

$$P(n_1, n_2, \dots, n_K) = \prod_{k=1}^K P(n_k) = (\pi N_0)^{-N/2} \exp\left(-\sum_{k=1}^K n_k^2 / N_0\right).$$

④ Likelihood function.

If m_k is transmitted. $\vec{r} = \vec{s}_k + \vec{n} \Rightarrow r_j = s_{kj} + n_j$.

$$\Rightarrow E[r_j|m_k] = s_{kj} + E[n_j] = s_{kj}.$$

$$\text{Var}[r_j|m_k] = \text{Var}[n_j] = \frac{N_0}{2}.$$

conditional pdf of $\vec{r} = (r_1, r_2, \dots, r_N) =$

$$f(\vec{r}|m_k) = \prod_{j=1}^N \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r_j - s_{kj})^2}{N_0}\right) = (\pi N_0)^{-N/2} \exp\left(-\frac{\sum_{j=1}^N (r_j - s_{kj})^2}{N_0}\right)$$

$$\Rightarrow \ln f(\vec{r}|m_k) = -\frac{N_0}{2} \ln(\pi N_0) - \frac{1}{N_0} \sum_{j=1}^N (r_j - s_{kj})^2.$$

$$D^2(\vec{r}, \vec{s}_k) = \sum_{j=1}^N (r_j - s_{kj})^2 = \|\vec{r} - \vec{s}_k\|^2 \Rightarrow \text{Euclidean distance.}$$

Optimum Detector =

$$\text{MAP: } \hat{m} = \arg \max_{\{m_1, \dots, m_M\}} f(\vec{r}|m_k) P(m_k) = \arg \min_{\{m_1, \dots, m_M\}} \left\{ \|\vec{r} - \vec{s}_k\|^2 - N_0 \ln p_k \right\}.$$

$$\text{ML: } \hat{m} = \arg \min_{\{m_1, \dots, m_M\}} \|\vec{r} - \vec{s}_k\|^2. \Rightarrow \text{Minimum distance detection}$$

$\hookrightarrow p_1 = p_2 = \dots = p_M.$

9. Optimal Receiver Structure

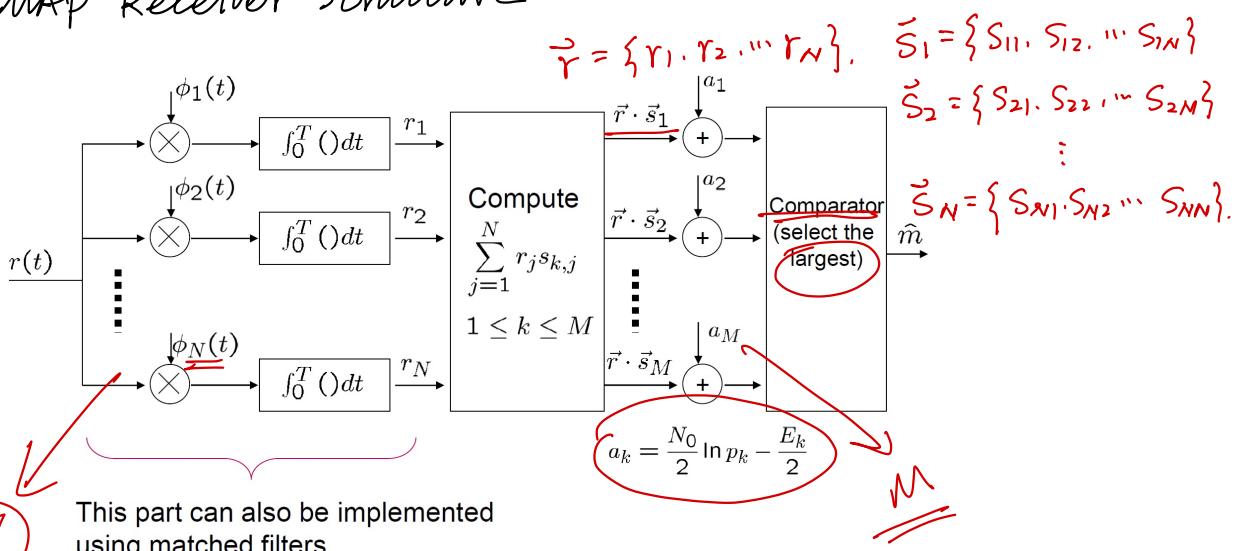
$$-\sum_{j=1}^N (r_j - s_{kj})^2 + N_0 \ln p_k = -\|\vec{r}\|^2 - \|\vec{s}_k\|^2 + 2\vec{r} \cdot \vec{s}_k + N_0 \ln p_k.$$

$$\|\vec{s}_k\|^2 = \int_0^T s_k^2(t) dt = E_k \Rightarrow \text{signal energy.}$$

$$\left\{ \begin{array}{l} \vec{r} \cdot \vec{s}_k = \int_0^T s_k(t) r(t) dt \Rightarrow \text{correlation between the received signal vector and the transmitted signal vector.} \\ \|\vec{r}\|^2 \Rightarrow \text{ignored.} \end{array} \right.$$

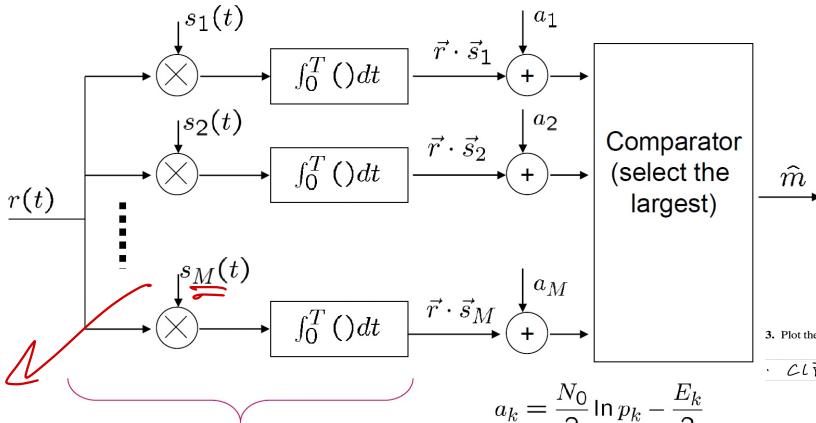
$$\Rightarrow \hat{m} = \arg \max_{\{m_1, \dots, m_M\}} \left\{ \vec{r} \cdot \vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2} \ln p_k \right\}.$$

MAP Receiver Structure =



$$\Delta C(\vec{r}, \vec{s}_m) = \int_{-\infty}^{+\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(s_m)$$

correlator receiver



\Rightarrow combine the demodulator and detector.

(M) This part can also be implemented using matched filters

$$\hat{m} = \arg \max_{m_1, \dots, m_M} \left\{ \vec{r} \cdot \vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2} \ln p_k \right\}$$

3. Plot the error probability as a function of p for $0 \leq p \leq 1$.

$$C(\vec{r}, \vec{s}_m) = \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(s_m)$$

meas

10. Graphical Interpretation of Decision Regions.

Divide the signal space into M disjoint decision regions R_1, R_2, \dots, R_M .

then if $\vec{r} \in R_k$ decide m_k was transmitted.

If $P_k = \frac{1}{M}$ for all k , then choose $\hat{m} = m_k$ iff $\|\vec{r} - \vec{s}_k\|^2$ is minimized.

Take projection of $r(t)$ in the signal space (\vec{r}). then decision is made in favor of signal that is closest to \vec{r} in sense of minimum Euclidean distance.

o Binary case =

Consider binary data transmission over AWGN channel with PSD $S_n(f) = \frac{N_0}{2}$

$$S_1(t) = -S_2(t) = \sqrt{E} \phi(t), \quad P(m_1) \neq P(m_2).$$

Determine the optimal receiver (regions).

\Rightarrow optimal decision =

$$\|\vec{r} - \vec{s}_1\| - N_0 \ln p(m_1) \leq \|\vec{r} - \vec{s}_2\| - N_0 \ln p(m_2).$$

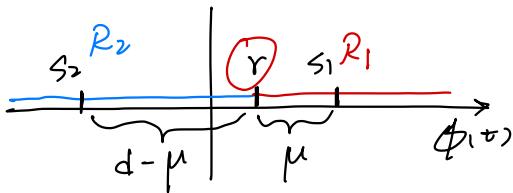
$$\text{if } d_1 = \|\vec{r} - \vec{s}_1\|, \quad d_2 = \|\vec{r} - \vec{s}_2\| \text{ then } d_1^2 - d_2^2 \leq \frac{N_0}{2} \ln \frac{P(m_1)}{P(m_2)} = C.$$

门限.

$$\text{Then } R_1 = d_1^2 - d_2^2 < C, \quad R_2 = d_1^2 - d_2^2 > C.$$

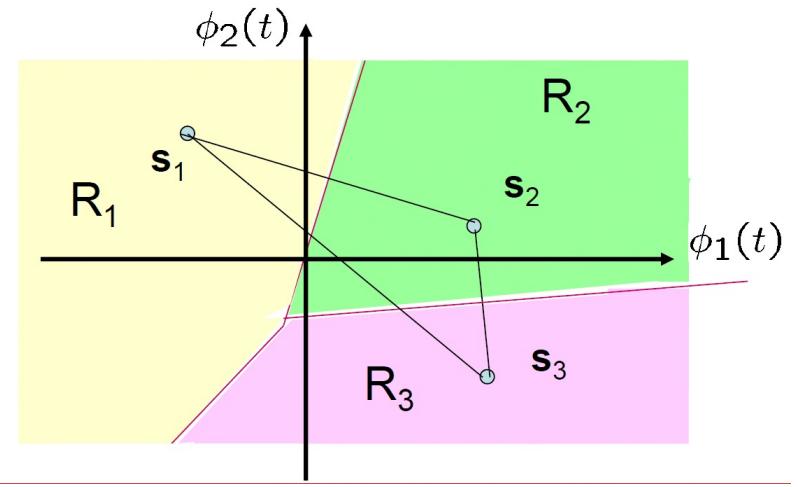
$$\begin{cases} d = d_1 + d_2 \\ d_1 = \mu \\ d_2 = d - \mu \end{cases} \Rightarrow d_1^2 - d_2^2 = 2d\mu - d^2 = C$$

$$\mu = \frac{c+d^2}{2d} = \frac{d}{2} + \frac{N_0}{2d} \ln \frac{P(m_1)}{P(m_2)} \Rightarrow \mu \begin{cases} = \frac{d}{2}, & P(m_1) = P(m_2) \\ > \frac{d}{2}, & P(m_1) > P(m_2) \\ < \frac{d}{2}, & P(m_1) < P(m_2) \end{cases}$$



先验概率 $P(m_1), P(m_2)$ 对判决区城有影响.

⇒ In general, boundaries of decision regions are perpendicular bisectors of lines joining the origin - transmitted signals.



Exercise :

Three equally probable messages m_1, m_2 , and m_3 are to be transmitted over an AWGN channel with noise power-spectral density $N_0 / 2$. The messages are

$$s_1(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \leq t \leq \frac{T}{2} \\ -1 & \frac{T}{2} \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

1. What is the dimensionality of the signal space?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure).
3. Draw the signal constellation for this problem.
4. Sketch the optimal decision regions R_1, R_2 , and R_3 .

Notes:

- Boundaries are perpendicular to a line drawn between two signal points
- If signals are equiprobable, decision boundaries lie exactly halfway in between signal points
- If signal probabilities are unequal, the region of the less probable signal will shrink

11. Probability of Error using Decision Regions.

Average probability of making correct decision: $P_c = \sum_{k=1}^M P(\vec{r} \in R_k | m_k \text{ is sent}) P(m_k)$

Average probability of error: $P_e = 1 - P_c = 1 - \sum P(\vec{r} \in R_k | m_k \text{ is sent}) P(m_k)$.

o Consider binary data transmission:

$$\begin{aligned} \text{given } m_1 \text{ is transmitted, then } P(C|S_1) &= P(\vec{r} \in R_1 | S_1) \\ &= P(S_1 + n > S_1 - \mu) = P(n > -\mu). \end{aligned}$$

n is Gaussian with zero mean and variance $N_0/2$.

$$\text{and } \mu = \frac{d}{2} + \frac{N_0}{2E} \ln \frac{P(m_1)}{P(m_2)}. \text{ then } P(C|S_1) = 1 - Q\left(\frac{\mu}{\sqrt{N_0/2}}\right) \Rightarrow Q \text{ is decreasing.}$$

$$P(C|S_2) = 1 - Q\left(\frac{d-\mu}{\sqrt{N_0/2}}\right). \quad P(C) = P(m_1)P(C|S_1) + P(m_2)P(C|S_2).$$

$$\Rightarrow P_e = 1 - P(C) = P(m_1)Q\left(\frac{\mu}{\sqrt{N_0/2}}\right) + P(m_2)Q\left(\frac{d-\mu}{\sqrt{N_0/2}}\right).$$

$$\text{where } d = 2\sqrt{E}. \quad \mu = \frac{N_0}{4\sqrt{E}} \log \left[\frac{P(m_1)}{P(m_2)} \right] + \sqrt{E}.$$

$$\text{if } P(m_1) = P(m_2), \text{ then } \mu = \sqrt{E} = \frac{d}{2}. \quad P_e = Q\left(\frac{\sqrt{2E}}{\sqrt{N_0}}\right) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right).$$

- This example demonstrates an interesting fact:

- When optimal receiver is used, P_e does not depend upon the specific waveform used

- P_e depends only on their geometrical representation in signal space

- In particular, P_e depends on signal waveforms only through their energies (distance)

$$P_e = Q\left[\frac{d/2}{\sqrt{N_0/2}}\right] = Q\left[\sqrt{\frac{d^2}{2N_0}}\right] = Q\left[\sqrt{\frac{2E}{N_0}}\right]$$

$$\text{SNR} = \frac{E}{N_0/2}$$

o General Expression for P_e .

$$P(\vec{r} \in R_k | m_k \text{ is sent}) = \int_{R_k} f(\vec{r} | m_k) d\vec{r} \Rightarrow P_e = 1 - \frac{1}{M} \sum_{k=1}^M \int_{R_k} f(\vec{r} | m_k) d\vec{r}.$$

At $j \Rightarrow \vec{r}$ is closer to \vec{s}_j than to \vec{s}_k in the signal space when $m_k(\vec{s}_{k|c})$ is sent.

$$P(e | m_k) = P(\vec{r} \notin R_k | m_k) = P\left(\bigcup_{j \neq k} A_{kj}\right) \leq \sum_{j=1}^M P(A_{kj}).$$

pair-wise error probability = $P(\vec{s}_k \rightarrow \vec{s}_j) = P(A_{kj})$.

$$P(\vec{s}_k \rightarrow \vec{s}_j) = P(n > \frac{d_{kj}}{2}) = Q\left(\frac{\sqrt{d_{kj}^2}}{\sqrt{2N_0}}\right), \quad d_{kj} = \|\vec{s}_k - \vec{s}_j\|.$$

$$\Rightarrow P_e = \frac{1}{M} \sum_{k=1}^M P(e | m_k) \leq \frac{1}{M} \sum_{k=1}^M \sum_{j=1}^M Q\left(\frac{\sqrt{d_{kj}^2}}{\sqrt{2N_0}}\right).$$

\Rightarrow Union bound (upper bound)

(8)

let $d_{\min} = \min_{k \neq j} \{d_{kj}\}$, then $\underline{P_e \leq (M-1)Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)}$. \Rightarrow simplified union bound.

Notes-

- The union bound provides a very useful bound on the error probability, particularly at high SNR
- At low SNR, the bound becomes loose and useless. More powerful bounding techniques are needed
- The union bound signifies the role of the minimum distance of a signal set on its performance, particularly at large SNRs.
- A good signal set should provide the maximum possible value of d_{\min} 这样 P_e 才会变大.
- In other words, to design a good signal set, the points the corresponding constellation should be maximally apart.

Chapter 08. Digital Modulation Techniques

1. Digital Modulation

↳ switching or keying the amplitude, frequency or phase of a sinusoidal carrier wave according to incoming digital data.

Three basic digital modulation techniques:

{ Amplitude-shift keying (ASK) — AM
Frequency-shift keying (FSK) — FM
Phase-shift keying (PSK) — PM.

use signal space approach in receiver design and performance analysis.

2. Binary Modulation.

the modulator produces one of two distinct signals in response to one bit of source data at a time.

① BPSK = Binary Phase-Shift keying.

$$"1" \Rightarrow s_{1(t)} = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t). \quad A = \sqrt{\frac{2E_b}{T_b}} \Rightarrow P = \frac{E_b}{T_b}.$$

$$"0" \Rightarrow s_{2(t)} = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t).$$

T_b = bit duration. $0 < t < T_b$. \Rightarrow bit rate $R_b = \frac{1}{T_b}$.

f_c = carrier frequency. chosen to be $\frac{n_c}{T_b}$ for fixed integer n_c . or $f_c \gg \frac{1}{T_b}$.

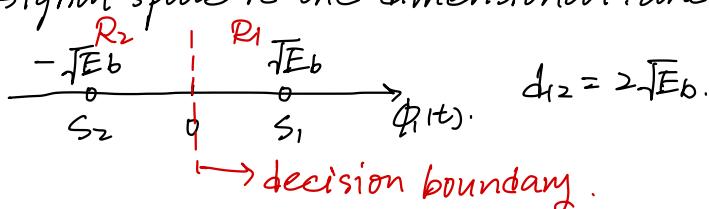
E_b = transmitted signal energy per bit.

$$\int_0^{T_b} s_1^2(t) dt = \int_0^{T_b} s_2^2(t) dt = E_b. \Rightarrow \text{power} = \frac{E_b}{T_b} = \frac{1}{2} \cdot \left(\sqrt{\frac{2E_b}{T_b}}\right)^2.$$

use one basis function = $\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$. $0 < t < T_b$. $\Rightarrow E_{\phi_1(t)} = \frac{1}{2} \cdot \left(\sqrt{\frac{2}{T_b}}\right)^2 \cdot T_b = 1$.

Then $s_{1(t)} = \sqrt{E_b} \phi_1(t)$. $s_{2(t)} = -\sqrt{E_b} \phi_1(t)$.

↳ signal space is one-dimensional. and has two message points. ($M=2$).



Decision Rule of BPSK.

assume that binary signals are equiprobable: $P(s_1) = P(s_2) = \frac{1}{2}$.

then guess $s_{1(t)}$ if received point r falls in R_1 ($r > 0$).

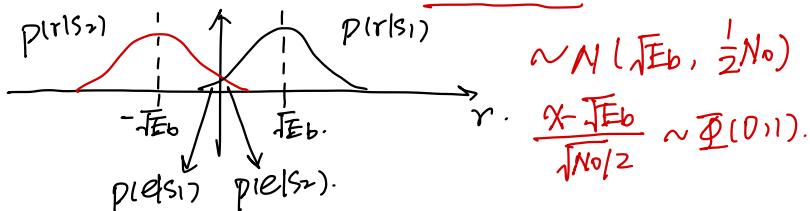
guess $s_{2(t)}$ if received point r falls in R_2 ($r \leq 0$).

Probability of Error for BPSK.

conditional probability =

if $s_1(t)$ is transmitted, then $P(e|s_1) = P(r < 0 | s_1) = \int_{-\infty}^0 \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{E_b})^2}{N_0}} dr = Q(\frac{\sqrt{E_b}}{\sqrt{N_0}})$.

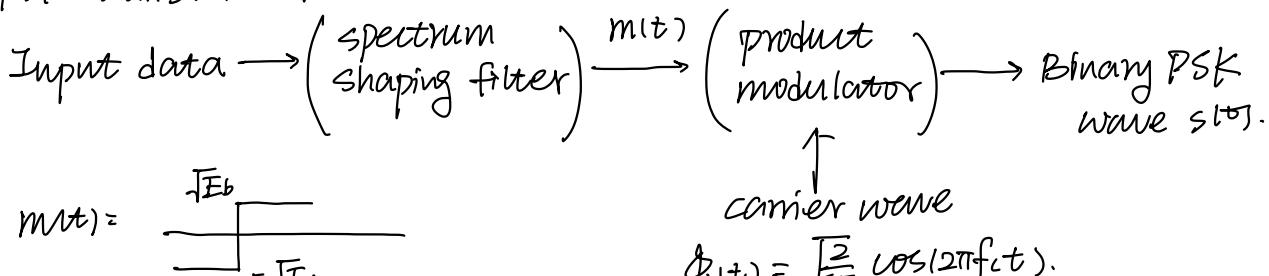
Similarly, $P(e|s_2) = P(r > 0 | s_2) = Q(\frac{\sqrt{E_b}}{\sqrt{N_0}})$.



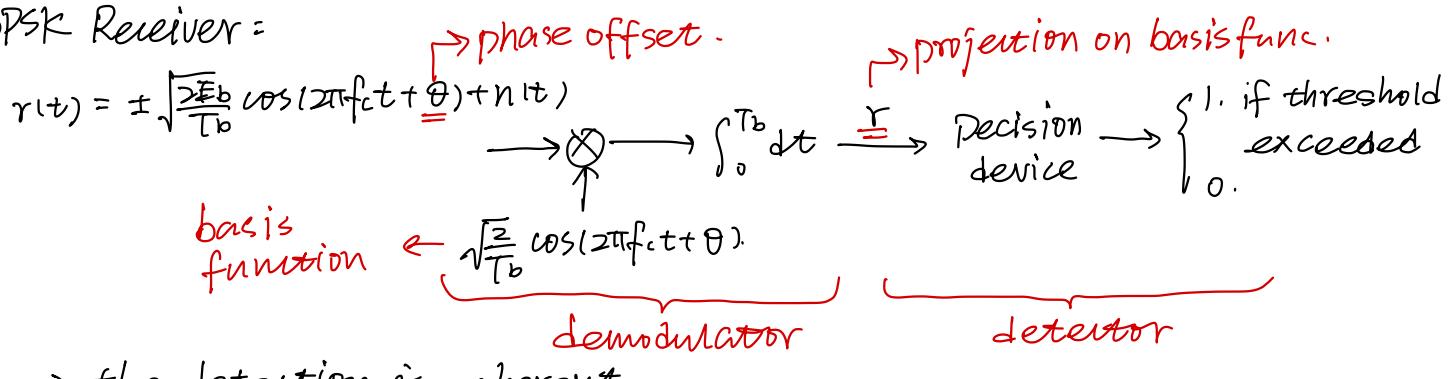
the average probability of error = $P_e = 0.5 p(e|s_1) + 0.5 p(e|s_2) = Q(\frac{\sqrt{2E_b}}{\sqrt{N_0}})$.

the ratio $(\frac{E_b}{N_0})$ is bit energy to noise energy ratio ($\frac{SNR}{bit}$) \hookrightarrow bit energy E_b .

BPSK Transmitter =



BPSK Receiver =



\Rightarrow the detection is coherent.

(2) BFSK.

"1" $\rightarrow s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t) \quad 0 < t < T_b$.

"0" $\rightarrow s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$.

f_b = transmitted energy per bit.

f_i = transmitted frequency with separation $\Delta f = f_1 - f_0$.

Δf makes $s_1(t)$ and $s_2(t)$ are orthogonal $\Rightarrow \int_0^{T_b} s_1(t) s_2(t) dt = 0$.

$$\Rightarrow f_1 = \frac{k_1}{T_b}, \quad f_2 = \frac{k_2}{T_b}, \quad \Delta f = \frac{n}{T_b} \quad \text{min of } ?$$

Two orthogonal basis functions are used:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t), \quad 0 \leq t < T_b \Rightarrow S_1(t) = \sqrt{E_b} \phi_1(t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t), \quad 0 \leq t < T_b \Rightarrow S_2(t) = \sqrt{E_b} \phi_2(t).$$

Signal space:

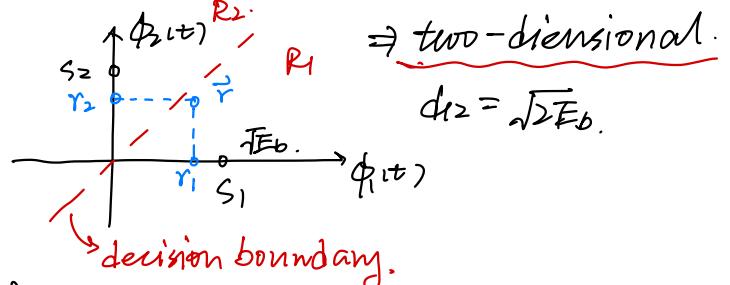
$$S_1 = [\sqrt{E_b} \ 0] \quad S_2 = [0 \ \sqrt{E_b}]$$

Observation vector $\vec{r} = [r_1 \ r_2]$.

$$r_1 = \int_0^{T_b} r(t) \phi_1(t) dt \quad \Rightarrow \text{projection on basis func.}$$

$$r_2 = \int_0^{T_b} r(t) \phi_2(t) dt$$

Decision rule: When $r_1 > r_2$, choose S_1 . When $r_1 < r_2$, choose S_2 .



Probability of Error for BFSK:

If $S_1/S_1(t)$ is transmitted: $r_1 = \sqrt{E_b} + n_1, \quad r_2 = n_2$.

then $P(e|S_1) = P(r_1 < r_2 | S_1) = P(\sqrt{E_b} + n_1 < n_2) = P(n_1 - n_2 < -\sqrt{E_b})$.

n_1 and n_2 are i.i.d with $n_1, n_2 \in N(0, \frac{N_0}{2})$. So $n = n_1 - n_2 \in N(0, N_0)$.

$$\Rightarrow P(e|S_1) = P(n < -\sqrt{E_b}) = \int_{-\infty}^{-\sqrt{E_b}} \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{x^2}{N_0}} dx = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right).$$

Similarly we can get $P(e|S_2) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right)$.

$$\begin{aligned} \text{So for BFSK: } P_e &= P(S_1)P(e|S_1) + P(S_2)P(e|S_2) \\ &= \frac{1}{2}Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right) + \frac{1}{2}Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right). \end{aligned}$$

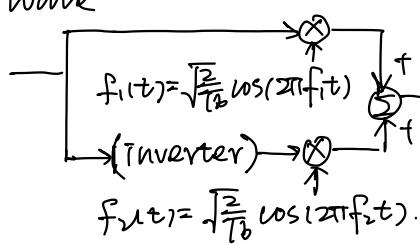
BFSK $Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0}}\right)$ is 3dB worse than BPSK $Q\left(\frac{\sqrt{2E_b}}{\sqrt{N_0}}\right)$.

To achieve the same P_e : $E_b(\text{BFSK}) = 2E_b(\text{BPSK})$.

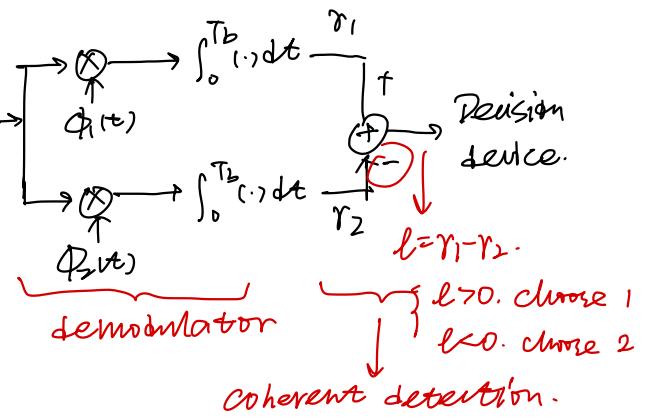
$$\hookrightarrow E_b(\text{BFSK})(\text{dB}) = E_b(\text{BPSK})(\text{dB}) + 3\text{dB}.$$

BPSK Transmitter:

Binary wave



BPSK Receiver:



② BASK.

$$E = 2E_b$$

$$\text{"1"} \rightarrow s_{1,t} = \sqrt{\frac{2E}{T_b}} \cos(2\pi f_c t), \quad 0 \leq t < T_b$$

$$\text{"0"} \rightarrow s_{0,t} = 0$$

$$\text{Average energy per bit} = E_b = \frac{E+0}{2} = \frac{1}{2}E$$

$$\text{Decision region} = \phi_{1,t} = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t) \quad s_{1,t} = \sqrt{E} \phi_{1,t}$$



$$d_{1,2} = \sqrt{2E_b}, \quad \text{BPSK} = 2\sqrt{E_b}$$

$$\text{Probability of error of BASK} = P_e = Q(\sqrt{\frac{E_b}{N_0}}) \Rightarrow \text{identical to that of BPSK}$$

\Rightarrow BPSK

BFSK

BASK

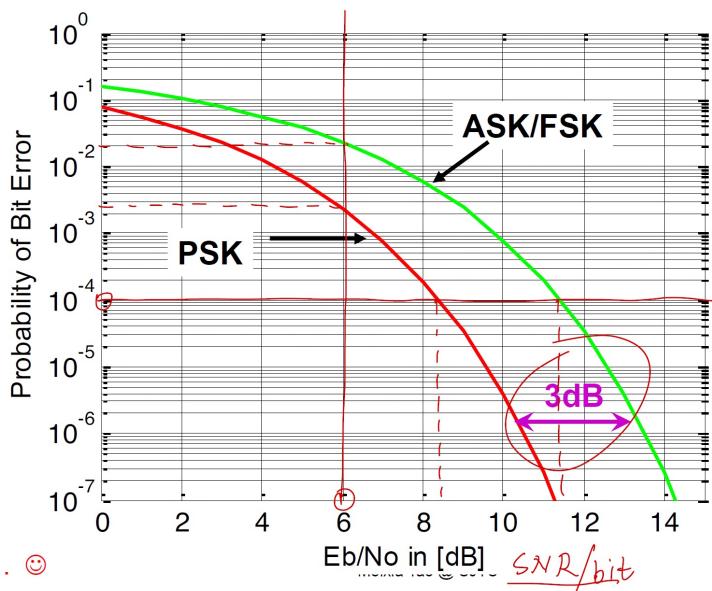
\Rightarrow

$$P_e = Q\left(\sqrt{\frac{d_{1,2}^2}{2N_0}}\right) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$$

$d_{1,2}$ 越大, P_e 越小. 比如 $d_{1,2}$.

$$d_{1,2} = 2\sqrt{E_b}, \quad d_{1,2} = \sqrt{2E_b}, \quad d_{1,2} = \sqrt{E_b}$$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right), \quad P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right), \quad P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



④ Non-coherent BFSK:

two signals of a binary FSK system:

θ_1, θ_2 = unknown random phase.

$$s_{1(t)} = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta_1) = \sqrt{\frac{2E_b}{T_b}} \cos(\pi f_1 t) \cos(\theta_1) - \sqrt{\frac{2E_b}{T_b}} \sin(\pi f_1 t) \sin(\theta_1).$$

$$s_{2(t)} = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t + \theta_2) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t) \cos(\theta_2) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_2 t) \sin(\theta_2).$$

Use four basis functions:

$$\phi_{1c}(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_1 t), \quad \phi_{1s}(t) = -\sqrt{\frac{2}{T_b}} \sin(2\pi f_1 t).$$

$$\phi_{2c}(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_2 t), \quad \phi_{2s}(t) = -\sqrt{\frac{2}{T_b}} \sin(2\pi f_2 t).$$

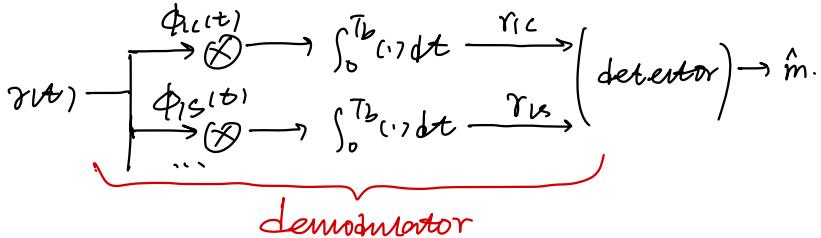
Signal space representation:

$$\vec{s}_1 = [\sqrt{E_b} \cos \theta_1 \quad \sqrt{E_b} \sin \theta_1 \quad 0 \quad 0]$$

$$\vec{s}_2 = [0 \quad 0 \quad \sqrt{E_b} \cos \theta_2 \quad \sqrt{E_b} \sin \theta_2]$$

The vector of received signal = $\vec{r} = [r_{1c} \quad r_{1s} \quad r_{2c} \quad r_{2s}]$

\hookrightarrow i.e. $r_{1c} = \int_0^{T_b} r(t) \phi_{1c}(t) dt, \quad r_{1s} = \int_0^{T_b} r(t) \phi_{1s}(t) dt$. \Rightarrow projection.



Decision rule of non-coherent BFSK.

$$\text{ML criterion } (P_1 = P_2 = \dots = P_m) = f(r|s_1) \stackrel{s_1}{>} f(r|s_2).$$

Conditional pdf:

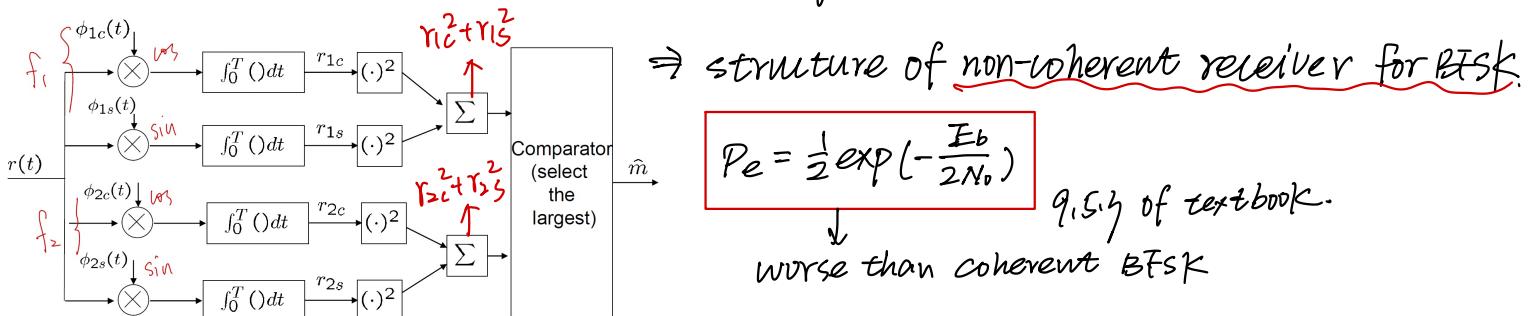
$$f(r|s_1, \theta_1) = \frac{1}{\pi N_0} \exp \left[-\frac{(r_{1c} - \sqrt{E_b} \cos \theta_1)^2 + (r_{1s} - \sqrt{E_b} \sin \theta_1)^2}{N_0} \right] \times \frac{1}{\pi N_0} \exp \left[-\frac{r_{2c}^2 + r_{2s}^2}{N_0} \right].$$

$$f(r|s_2, \theta_2) = \frac{1}{\pi N_0} \exp \left[-\frac{r_{1c}^2 + r_{1s}^2}{N_0} \right] \times \frac{1}{\pi N_0} \exp \left[-\frac{(r_{2c} - \sqrt{E_b} \cos \theta_2)^2 + (r_{2s} - \sqrt{E_b} \sin \theta_2)^2}{N_0} \right].$$

$$\Rightarrow \frac{1}{2\pi} \int_0^{\pi} \exp \left[\frac{2\sqrt{E_b} r_{1c} \cos \theta_1 + 2\sqrt{E_b} r_{1s} \sin \theta_1}{N_0} \right] d\theta_1 = I_0 \left(\frac{2\sqrt{E_b} \sqrt{r_{1c}^2 + r_{1s}^2}}{N_0} \right).$$

$$\text{choose } s_1 \text{ if } I_0 \left(\frac{2\sqrt{E_b} \sqrt{r_{1c}^2 + r_{1s}^2}}{N_0} \right) \geq I_0 \left(\frac{2\sqrt{E_b} \sqrt{r_{2c}^2 + r_{2s}^2}}{N_0} \right) \Rightarrow \sqrt{r_{1c}^2 + r_{1s}^2} \geq \sqrt{r_{2c}^2 + r_{2s}^2}.$$

\hookrightarrow Compare the energy in two frequencies and pick the larger. \Rightarrow envelop detector. carrier phase is irrelevant in decision making.



⑤ DPSK = Differential PSK. (Non-coherent version of PSK).

↳ encode the information in phase difference between successive signal transmission
to send "0", advance the phase of current signal by 180° .
to send "1", leave the phase unchanged.

{ the unknown phase θ contained in the received wave varies slowly (constant over two bit intervals) \Rightarrow the phase difference received in two successive bit intervals is independent of θ .

Generate DPSK in two steps:

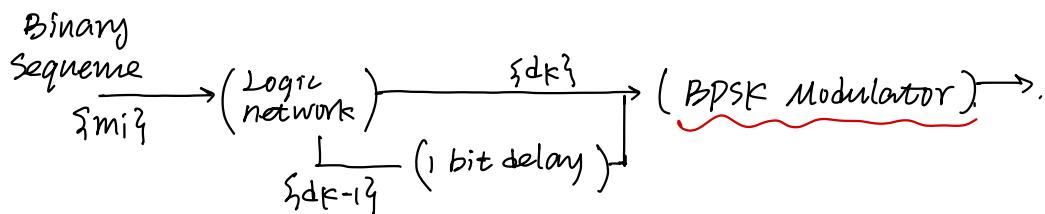
Differential encoding of the binary bits + phase shift keying.

{ Information sequence: m_i initial bit
 } encoded sequence: d_i
 { transmitted phase: θ_i \downarrow
 $d_i = \overline{d_{i-1} \oplus m_i}$
 $d_i = m_i$ if $m_i = 1$.
 $d_i = \overline{m_i}$ if $m_i = 0$

\Rightarrow similar to BPSK: $0 \rightarrow \pi$

$1 \rightarrow 0$.

DPSK Transmitter:



DPSK Detection:

$$r(t) \xrightarrow{\text{Delay } T_b} r(t-T_b) \xrightarrow{\otimes} \int_0^{T_b} r(t) dt \xrightarrow{y} \text{Decision device} \xrightarrow{l} \begin{cases} \text{choose 1 if } l > 0 \\ \text{choose 0 o.w.} \end{cases}$$

$$y = \int_0^{T_b} r(t) r(t-T_b) dt = \int_0^{T_b} \cos(\omega t + \psi_k + \theta) \cos(\omega t + \psi_{k-1} + \theta) dt \propto \cos(\psi_k - \psi_{k-1})$$

\Rightarrow unknown phase offset. $\theta_k = \theta_{k-1}$.

if $\psi_k - \psi_{k-1} = 0$ (bit 1) then $y > 0$. ($\cos \theta = 1$)

if $\psi_k - \psi_{k-1} = \pi$ (bit 0) then $y < 0$. ($\cos \pi = -1$)

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right)$$

$$\frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right)$$

2dB better than non-coherent FSK.

3. M-ary Modulation (多进制调制)

⇒ 1 symbol 包含 k bit. 可组成 $M = 2^k$ 种波形，周期为 T

- In **M-ary** modulation, the binary sequence is subdivided into blocks of k bits, called **symbols**, and each block (or symbol) is represented one of the $M = 2^k$ signal waveforms, each of duration T .
- Symbol rate: $R_s = \frac{1}{T}$ symbols/sec Δ bit rate = $kR_s = k$ Symbol rate
- Since each signal carries $k = \log_2 M$ bits, the bit rate is $R_b = kR_s = \frac{k}{T}$
- The bit interval is $T_b = \frac{1}{R_b} = \frac{T}{k}$
- These M signals are generated by changing the amplitude, phase, frequency, or combined forms of a carrier in M discrete steps.
- Thus, we have:
 - MASK
 - MPSK
 - MFSK
 - MQAM

(1) MPSK = Many Phase-Shift Keying.

The phase of the carrier takes on M possible values: $\theta_m = 2\pi \frac{m-1}{M}$, $m = 1, \dots, M$.

$$\text{Signal set: } S_m(t) = \sqrt{\frac{2E_s}{T}} \cos [2\pi f_c t + \frac{2\pi(m-1)}{M}], \quad m=1, \dots, M, \quad 0 \leq t < T,$$

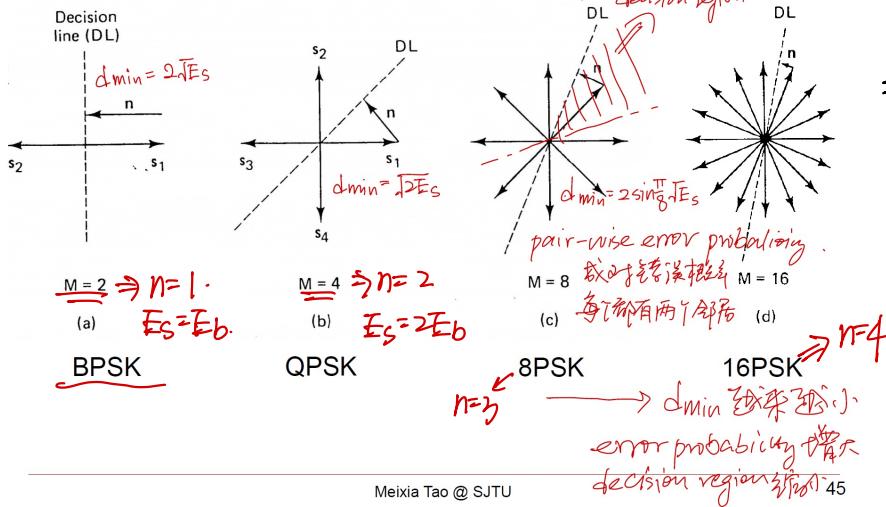
E_s = energy per symbol. T = duration of symbol. $f_c \gg \frac{1}{T}$.

$$\text{basis functions: } \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad \phi_2 = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t < T.$$

$$\text{Signal space representation: } S_m = I \sqrt{E_s} \cos \left[\frac{2\pi(m-1)}{M} \right], \quad \sqrt{E_s} \sin \left[\frac{2\pi(m-1)}{M} \right].$$

信号强度一致、受到原点距离影响都是 $\sqrt{E_s}$.

$M = 2$ 时, (S_1, S_2) 相距 $\sqrt{E_s}$.



⇒ Euclidean distance =

$$d_{min} = \sqrt{2E_s (1 - \cos \frac{2\pi(m-n)}{M})},$$

the minimum distance ($M=n=1$) =

$$d_{min} = \sqrt{2E_s (1 - \cos \frac{2\pi}{M})} = 2\sqrt{E_s} \sin \frac{\pi}{M}$$

$$M = 2 \cdot 4 \cdot 8 \cdot 6$$

$k \uparrow, M \uparrow, d_{min} \downarrow, P_e \uparrow$.

An approximation to the (symbol) error probability =

$$P_{\text{MPSK}} \approx 2Q \left(\frac{d_{min}/2}{\sqrt{N_0/2}} \right) = 2Q \left(\frac{\sqrt{2E_s} \sin \frac{\pi}{M}}{\sqrt{N_0/2}} \right).$$

→ two signal points adjacent to the transmitted signal points.

$$\frac{E_b}{N_0} (\text{dB})$$

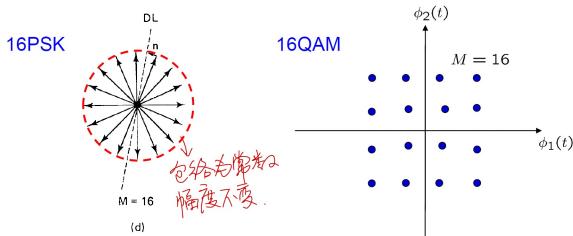
For large M , doubling the number of phases requires an additional 6dB/bit to achieve the same performance.

$$\text{Proof: } \sqrt{\frac{2E_{SM}}{N_0}} \cdot \sin \frac{\pi}{M} = \sqrt{\frac{2E_{SM}}{N_0}} \sin \frac{\pi}{2M} \Rightarrow \frac{\sqrt{E_{SM}}}{M} = \frac{\sqrt{E_{SM}}}{2M} \Rightarrow E_{SM} = 4E_{SM}$$

$$\Rightarrow E_{SM} = 4E_{SM} \cdot E_{SM}(\text{dB}) = E_{SM}(\text{dB}) + 6\text{dB}.$$

(2) MQAM = Many Quadrature Amplitude Modulation (许多幅度调制).

- In MPSK, in-phase and quadrature components are interrelated in such a way that the envelope is constant (circular constellation)
- If we relax this constraint, we get M-ary QAM



⇒ Signal set =

$$S_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}} b_i \sin(2\pi f_c t), 0 \leq t < T.$$

E_0 = the energy of signal with the lowest ampf.

{ a_i, b_i } = the sets of amplitude levels.

basis functions:

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), 0 \leq t < T.$$

⇒ signal space representation = $\vec{s}_i = [\sqrt{E_0} a_i, \sqrt{E_0} b_i]$.

⇒ 根据 { a_i, b_i } 取值的不同有 Rectangular constellation, circular constellation 等.

$M=4$ regular QAM and $M=4$ PSK are identical.

Error performance of MQAM:

upper bound of the symbol error probability = $P_e \leq \frac{1}{4} Q\left(\sqrt{\frac{2kE_b}{(M-1)N_0}}\right)$, $M=2^k$.

when k is large, if increase the number of bits from k to $k+1$, then to achieve the same performance:

$$\sqrt{\frac{3kE_b k}{(2^{k-1})N_0}} = \sqrt{\frac{3(2^k)E_b k+1}{(2^{k+1})N_0}} \Rightarrow E_{b,k+1} = \frac{2^{k+1}-1}{2^{k+1}} \cdot \frac{k}{k+1} E_{b,k} \approx 2E_{b,k} \Rightarrow 3\text{dB}.$$

$$\hookrightarrow E_{b,k+1}(\text{dB}) = E_{b,k}(\text{dB}) + 3\text{dB}$$

Four adjacent points

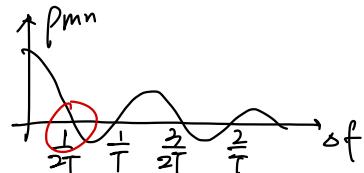
(3) MFSK

signal set = $S_m(t) = \sqrt{\frac{2E_s}{T}} \cos[2\pi(f_c + (m-1)\Delta f)t]$, $m=1, \dots, M$, $0 \leq t < T$.

$$\hookrightarrow \Delta f = f_m - f_{m-1}, f_m = f_c + m\Delta f.$$

correlation between two symbols =

$$P_{mn} = \frac{1}{E_s} \int_0^T S_m(t) S_n(t) dt = \frac{\sin[2\pi(m-n)\Delta f T]}{2\pi(m-n)\Delta f T} \quad \text{for orthogonality, minimum } \Delta f = \frac{1}{2T}.$$



选择较小的Δf以减小信号带宽.

The basis functions are = $\phi_m(t) = \sqrt{\frac{2}{T}} \cos[2\pi f_c t + m\omega_f t]$.

\Rightarrow Many orthogonal FSK has a geometric presentation as M M-dim orthogonal vectors.

$S_0 = (1, \sqrt{E_s}, 0, 0, \dots, 0)$, $S_1 = (0, \sqrt{E_s}, 1, 0, \dots, 0)$, ... $S_{M-1} = (0, 0, \dots, 0, \sqrt{E_s})$. MPSK 相反.

\hookrightarrow M \times M 维正交向量. 任意两点间距离均为 $\sqrt{E_s} = \sqrt{2E_b} = d$. 随 E_b 增大时. P_e 下降. 性能变好.

P_e is found by integrating conditional probability of error over the decision region.
 P_e depends only on the distance profile of signal constellation.

4. Bit Error.

When a symbol error occurs, all k bits could be in error.

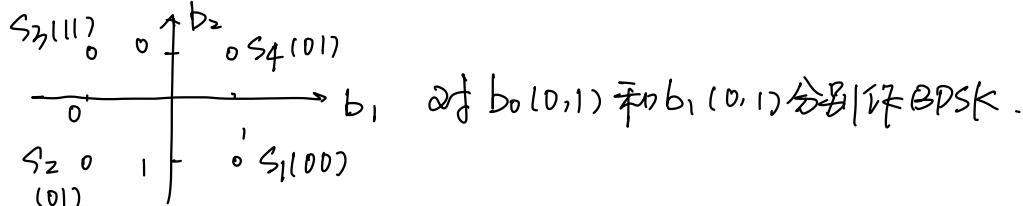
$$\text{BER (bit error rate)} = P_b = \sum_{i=1}^M P(\vec{s}_i) \sum_{j \neq i}^M \frac{n_{ij}}{\log_2 M} P(\hat{s} = \vec{s}_j | \vec{s}_i). \quad \begin{aligned} M &= 2^n, \quad n = \log_2 M \\ \frac{n_{ij}}{\log_2 M} &= \frac{n_{ij}}{n}. \end{aligned}$$

n_{ij} is the number of different bits between s_i and s_j .

Gray coding is a bit-to-symbol mapping

Example = gray coding for QPSK ($M=4$).

$$P_b = \sum_{i=1}^4 \frac{1}{4} \sum_{j \neq i}^4 \frac{n_{ij}}{\log_2 4} P(\hat{s} = \vec{s}_j | \vec{s}_i) = \frac{1}{2} P(\hat{s} = \vec{s}_1 | \vec{s}_4) + \frac{2}{2} P(\hat{s} = \vec{s}_2 | \vec{s}_4) + \frac{1}{2} P(\hat{s} = \vec{s}_3 | \vec{s}_4). \\ = [1 - Q(\sqrt{\frac{E_s}{N_0}})] \cdot Q(\sqrt{\frac{E_s}{N_0}}) + [Q(\sqrt{\frac{E_s}{N_0}})]^2 = Q(\sqrt{\frac{E_s}{N_0}}) = Q(\sqrt{\frac{2E_b}{N_0}}). \quad \begin{array}{l} \text{equal to BPSK} \\ \log_2 4 = 2. \end{array}$$



BER for MPSK and MFSK:

For MPSK with Gray coding = $P_b \approx \frac{P_e}{\log_2 M}$. $\log_2 M = n$.

For MFSK: $P_b \approx \frac{1}{2} P_e \Rightarrow$ 1 bit to baseband (M-1) bits 的距离都相同.

5. Comparison study.

{ Power utilization efficiency (energy efficiency): measured by $\frac{E_b}{N_b}$

Spectrum utilization efficiency (bandwidth efficiency): measured by $\frac{R_b}{B}$

Maximize bandwidth efficiency and minimize $\frac{E_b}{N_b}$.

$R_b = \text{bits/sec}$
 $B = \text{Hz}$.

Energy Efficiency Comparison:

MFSK:

- At fixed E_b/N_0 , increase M can provide an improvement on P_b
- At fixed P_b , increase M can provide a reduction in the E_b/N_0 requirement

MPSK

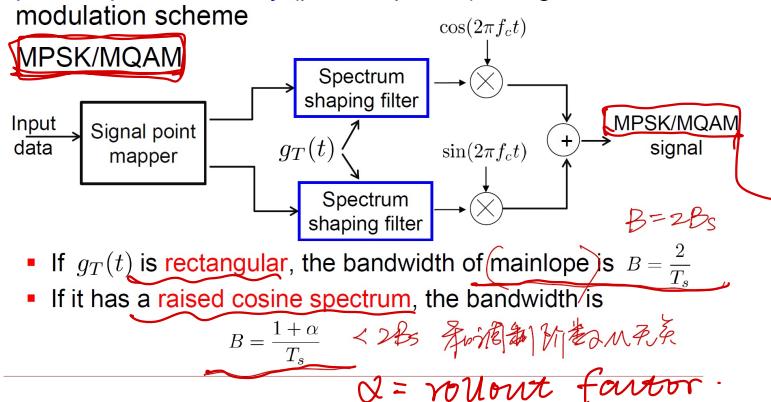
- BPSK and QPSK have the same energy efficiency
- At fixed E_b/N_0 , increase M degrades P_b
- At fixed P_b , increase M increases the Eb/No requirement

MFSK is more energy efficient than MPSK

Bandwidth Efficiency Comparison:

- To compare bandwidth efficiency, we need to know the **power spectral density** (power spectra) of a given modulation scheme

MPSK/MQAM



- If $g_T(t)$ is **rectangular**, the bandwidth of mainlobe is $B = \frac{2}{T_s}$
- If it has a **raised cosine spectrum**, the bandwidth is $B = \frac{1+\alpha}{T_s} < 2B_s$ (with α being the roll-off factor)

$$\Rightarrow \text{In general } B = \frac{1}{T_s}$$

$$\text{And } R_b = \frac{\log_2 M}{T_s} = \text{bit rate}$$

then bandwidth efficiency =

$$P = \frac{R_b}{B} = (\log_2 M) \text{ bits/sec/Hz}$$

MFSK:

- Bandwidth required to transmit MFSK signal is

$$B = \frac{M}{2T} \quad (\text{Adjacent frequencies need to be separated by } 1/2T \text{ to maintain orthogonality})$$

- Bandwidth efficiency of MFSK signal

$$\rho = \frac{R_b}{B} = \frac{2 \log_2 M}{M} \text{ (bits/s/Hz)}$$

$$\Rightarrow \text{minimum Af} = \frac{1}{2T}$$

$$\text{For } M \text{ signals, } B = M \text{Af} = \frac{M}{2T} \text{ Hz.}$$

M	2	4	8	16	32	64
(bits/s/Hz)	1	1	0.75	0.5	0.3125	0.1875

As M increases, bandwidth efficiency of MPSK/MQAM increases, but bandwidth efficiency of MFSK decreases.

$$\left. \begin{array}{l} \text{MPSK/MQAM} = P = \log_2 M \\ \text{MFSK} = P = \frac{2 \log_2 M}{M} \end{array} \right\} \uparrow$$

Channel capacity: For a bandlimited channel corrupted by AWGN, the maximum rate is given by -

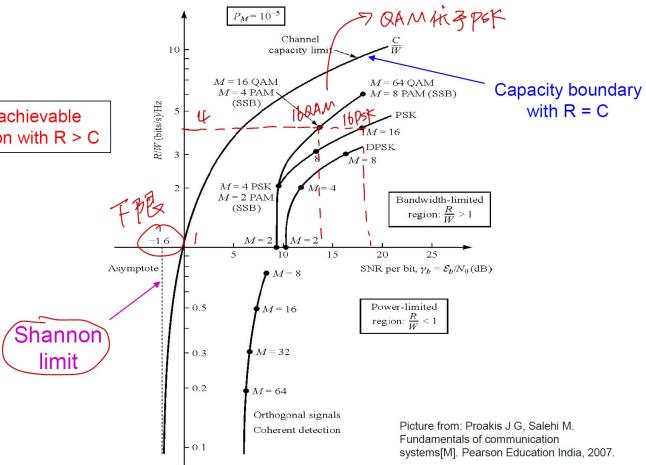
$$R \leq C = B \log_2 (1 + SNR) = B \log_2 \left(1 + \frac{P_s}{N_0 B} \right)$$

$$\left. \begin{array}{l} \text{energy per bit} \\ \uparrow \\ \text{SPs} = E_b \cdot R \\ \uparrow \\ P_s = E_b \cdot R \\ P_N = \frac{1}{2} N_0 \end{array} \right\} \rightarrow \frac{1}{2} N_0 \cdot 2B = N_0 B$$

$$\Rightarrow \frac{E_b}{N_0} \geq \frac{B}{R} (2^{\frac{R}{B}} - 1) \Rightarrow \frac{E_b}{N_0} = SNR/\text{bit} \geq \frac{2^{\frac{R}{B}} - 1}{P_s} \Rightarrow \text{energy efficiency}$$

$p = \text{bandwidth efficiency.}$

Unachievable Region with $R > C$



Picture from: Proakis J G, Salehi M.
Fundamentals of communication systems[M]. Pearson Education India, 2007.

- In the limits as R/B goes to 0, we get

$$P \rightarrow 0 : \frac{E_b}{N_0} = \ln 2 = 0.693 = -1.59 \text{ dB}$$

- This value is called the **Shannon Limit** (香农极限)
- Received E_b/N_0 must be **> 1.6 dB** to ensure reliable communications

- BPSK and QPSK require the same E_b/N_0 of **9.6 dB** to achieve $P_e = 10^{-5}$. However, **QPSK has a better bandwidth efficiency**
- MQAM is superior to MPSK**
- MPSK/MQAM increases bandwidth efficiency at the cost of lower energy efficiency**
- MFSK trades energy efficiency at reduced bandwidth efficiency.**

Chapter 09. Information Theory

1. Modeling of information source

Discrete memoryless source (DMS).

↪ A discrete-time, discrete-amplitude random process with i.i.d. rv.

2. Measure of information.

The information I that a source event \mathcal{X} can convey and the probability of the event $P(\mathcal{X})$ satisfy:

vice versa

1) $I = I[P(X)]$. 2). $P(X)$ decreases $\rightarrow I$ increases. $P(X)=1, I=0$

3) Multiple independent events $X_1, X_2 \dots$

$$I[P(X_1) P(X_2) \dots] = I[P(X_1)] + I[P(X_2)] + \dots$$

\Rightarrow Information of X : $(I = \log_a \frac{1}{P(X)} = -\log_a P(X))$ $\left\{ \begin{array}{l} a=e, \text{ nat} \\ a=2, \text{ bit} \end{array} \right.$

Entropy (信息量).

Consider a discrete source with N possible symbols.

Entropy $H(\cdot)$ = average amount of information conveyed per symbol.

$$H(X) = E[I(X_j)] = \sum_{j=1}^N P(X_j) \log_2 \frac{1}{P(X_j)} \text{ (bit/symbol). } \Rightarrow \text{信源所带信息量的期望.}$$

Binary case $\{0, 1\}$, with $P(1) = p, P(0) = 1-p$.

$$\text{then } H = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p}.$$



\Rightarrow Entropy is maximized when all symbols are equiprobable.

$$\hookrightarrow N \text{ symbols: } H = \sum_{n=1}^N \frac{1}{N} \log_2 N = \log_2 N \text{ bits/symbol.}$$

$$\hookrightarrow 0 \leq H(X) \leq \log N$$

Example =

- A source with bandwidth 4KHz is sampled at the Nyquist rate
- Assuming that the resulting sequence can be modeled by a discrete memoryless source $\{-2, -1, 0, 1, 2\}$ with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\}$
- What is the information rate of the source in bit/sec?

We have

$$\begin{aligned} H(X) &= \frac{1}{2} \log_2 2 + \frac{1}{4} \log_2 4 + \frac{1}{8} \log_2 8 + 2 \times \frac{1}{16} \log_2 16 \\ &= \frac{15}{8} \text{ bits/sample} \end{aligned}$$

Since we have 8000 samples/sec, the source produces information at a rate of 15k bits/sec.

bit/symbol · symbol rate = symbol/sec.

Joint and conditional entropy.

The joint entropy of discrete source (X, Y) =

$$H(X, Y) = - \sum_{x,y} p(x, y) \log p(x, y).$$

The conditional entropy of X given Y =

$$H(X|Y) = - \sum_{x,y} p(x, y) \log p(x|y) \quad H(X|Y=y) = - \sum_x p(x|y) \log p(x|y)$$

Using chain rule =

$$\begin{cases} H(X, Y) = H(X|Y) + H(Y) \\ H(X, Y) = H(Y|X) + H(X). \end{cases}$$

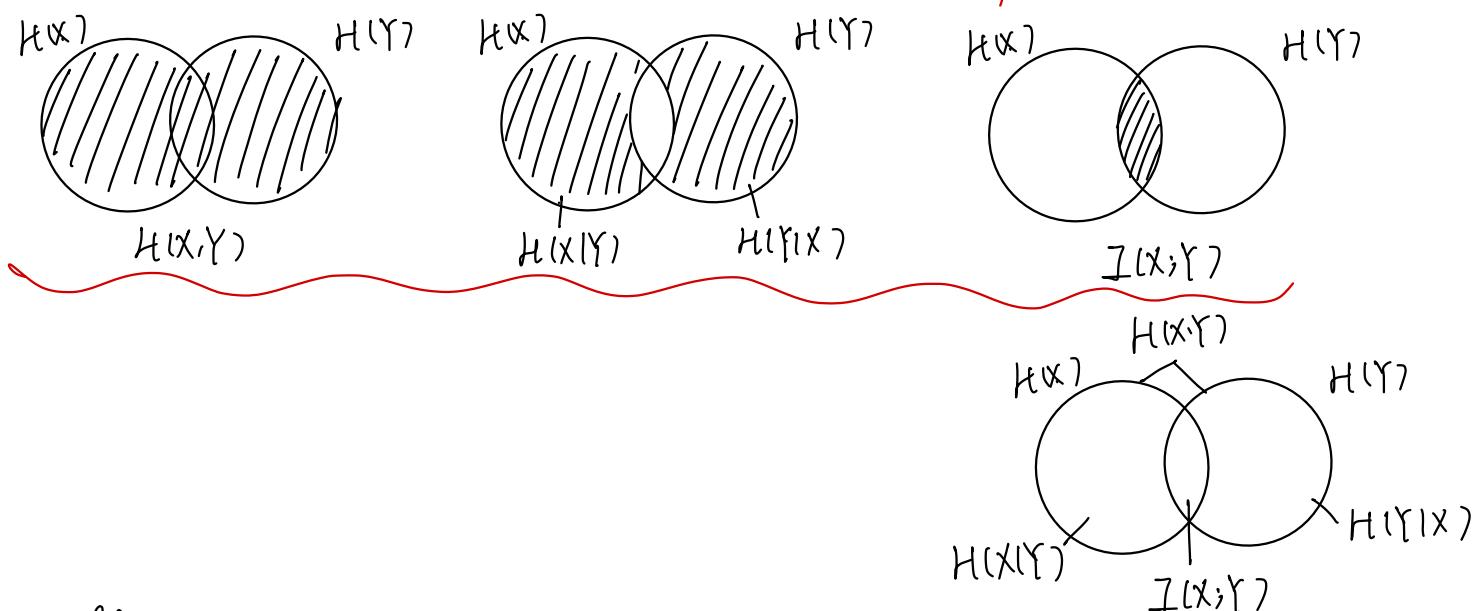
Mutual Information (互信息).

$H(X) - H(X|Y)$ denotes the amount of uncertainty of X that has been removed given Y is known.

the amount of information provided by random variable Y about r.v. X .

mutual information = $I(X; Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$.

↳ 已知信道的输出 Y 可揭示的关于 X 的信息.



Differential Entropy

The differential entropy of discrete-time continuous alphabet source X with pdf $f_X(x)$ is:

连续取值

$$h(X) = - \int_{-\infty}^{\infty} f_X(x) \log f_X(x) dx$$

Mutual information between two continuous r.v. X and Y =

$$I(X; Y) = h(X) - h(X|Y)$$

3. Source Coding Theorem. (信源编码)

A source with entropy H can be encoded with an arbitrarily small error probability at any rate R (bits/source output) as long as $R \geq H$.

H = the minimum rate at which an information source can be compressed for reliable reconstruction. $\Rightarrow R$ must $\geq H$.

变长码 = 变长编译码.

Huffman Source Coding. \Rightarrow variable-length binary coding.

\Rightarrow Map the more probable source sequences to shorter binary codewords.

Synchronization is a problem.

\hookrightarrow 在解码时会有歧义.

4. Modeling of communication channel.

(1) Binary-Symmetric Channel (BSC).

\hookrightarrow the crossover error probability is the same. $e = p(0|1) = p(1|0)$

$$p(1|0) = p(0|1) = e$$

$$e = \alpha(\sqrt{\frac{2E_b}{N_0}})$$

$$P_{00} = p(0|0) = 1 - e = P_{11} = p(1|1)$$

$$P_{10} = p(1|0) = e = P_{01} = p(0|1).$$

(2) AWGN channel = both input and output are real

$$X \xrightarrow[\text{channel}]{} Y = X + Z. \quad Z \text{ is Gaussian White Noise.}$$

the input satisfies power constraint = $\frac{1}{n} \sum_{i=1}^n x_i^2 \leq P$. 注意单位不是 bits/symbol.

(3) Channel Capacity: a maximum rate, C in bits/sec of a channel.

If $R \leq C$, theoretically guarantee almost error free transmission.

If $R > C$, reliable transmission is impossible.

The capacity of a discrete-time memoryless channel:

$$C = \max_{p(x)} I(X;Y)$$

max over all possible input distribution

Δ Entropy gives a lower bound on the rate of the codes that are capable of reproducing the source with no error. If we want to transmit a source U reliably via a channel with capacity C , then $H(U) < C$.

Binary Symmetric Channel Capacity:

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - \sum p(x)H(Y|X=x)$$

$$= H(Y) - \sum p(x)H_b(p_e) = H(Y) - H_b(p_e) \leq I(H_b(p_e))$$

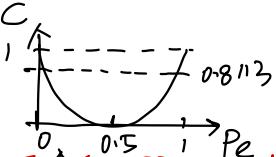
Binary channel

$$\sum p(x) = 1$$

\hookrightarrow binary source entropy.

$$H_b(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

$$\Rightarrow \text{the capacity of BSC} = C = I(H_b(p_e)). \Rightarrow$$



$\exists p_e = 0.5$ 时. 不管 X input 取什么. 输出 Y 都等于概率取的所有值. 相当于没有传递信息. $p_e = 0, 1$ 时. Y 的值都由 X 决定. 信道可将 input 完整传给 Y .

Example.

- A binary source with $P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{3}{4}$ is to be transmitted over a BSC channel with a crossover probability p_e . Assume that the channel can be used once per symbol output.
- Determine the range of p_e for reliable communication of the source.

想到香农极限

$$H(X) < C = 1 - H_b(p_e)$$

$$\Rightarrow \text{entropy } H(X) = \frac{1}{4}\log_2 4 + \frac{3}{4}\log_2 \frac{3}{4} = 0.8113. \Rightarrow p_e \in [0, 0.288) (0.972, 1].$$

\hookrightarrow bits/symbol = bits/sec = bits/channel output (binary)

AWGN - Gaussian Channel Capacity.

A discrete-time Gaussian channel = $Y = X + Z$.

Input power constraint = $\frac{1}{n} \sum_{i=1}^n x_i^2 \leq P$. Z is Gaussian White noise $\sim N(0, P_N)$.

capacity = $C = \frac{1}{2} \log(1 + \frac{P}{P_N})$ bits/channel use. proof.

Capacity of Bandlimited AWGN Channel.

A continuous-time, bandlimited AWGN channel with noise PSD $No/2$ input power constraint P . bandwidth W .

Sample it at Nyquist rate $2W$ and obtain a discrete-time channel.

The power/sample is P and the noise power/sample is WN_0 .

Shannon Formula: $P_N = \int_{-W}^W \frac{No}{2} df = WN_0$.

Thus. $C = \frac{1}{2} \log(1 + \frac{P}{WN_0})$ bits/sample. $\xrightarrow{\text{SNR}}$

bandwidth is W
 $\frac{No}{2} \frac{1}{W} \frac{1}{f} \frac{1}{df} \xrightarrow{\text{SNR}} WN_0$.

$C = \frac{1}{2} \log(1 + \frac{P}{P_N})$ bits/channel use
 $\xrightarrow{\text{usually sample at nyquist rate.}}$

Example: find the capacity of a telephone channel with bandwidth $W = 3000 \text{ Hz}$, SNR 39 dB .

$$\Rightarrow \text{SNR} = 39 \text{ dB} = 7943. C = 2 \times 3000 \text{ Hz} \times \frac{1}{2} \log(1 + 7943) \approx 38867 \text{ bits/sec}$$

\hookrightarrow sample at nyquist rate.

$$\Rightarrow C = \frac{1}{2} \log_2 \left(1 + \frac{P}{N_0 W} \right) \text{ bits/sample} = W \log_2 \left(1 + \frac{P}{N_0 W} \right) \text{ bits/sec}$$

(Nyquist rate)

5. Insights from Shannon Formula.

1. Increasing signal power P increases the capacity C

- When SNR is high enough, every doubling of P adds additional bits/s in capacity
- When P approaches infinity, so is C

$$\begin{aligned}\log_2(1+x) &\approx x \log_2 e & \text{when } x \approx 0, \\ \log_2(1+x) &\approx \log_2 x & \text{when } x \gg 1.\end{aligned}$$

2. Increasing channel bandwidth W can increase C, but cannot increase infinitely (as noise power also increases)

$$\begin{aligned}\lim_{W \rightarrow \infty} C &= \lim_{W \rightarrow \infty} \left[\frac{WN_0}{P} \log \left(1 + \frac{P}{N_0 W} \right) \right] \frac{P}{N_0} \\ &= \frac{P}{N_0} \log e = 1.44 \frac{P}{N_0}\end{aligned}$$

3. Bandwidth efficiency – energy efficiency tradeoff

- In any practical system, we must have

$$R \leq W \log_2 \left(1 + \frac{P}{N_0 W} \right)$$

- Defining $r = R/W$, the spectral bit rate

$$r = \frac{R}{W} \leq \log_2 \left(1 + \frac{P}{N_0 W} \right)$$

- Let E_b be the energy per bit, $E_b = \frac{P}{R}$

$$r \leq \log_2 \left(1 + r \frac{E_b}{N_0} \right)$$

$E_b/N_0 = \text{SNR per bit}$
 $r = \text{spectral efficiency}$

$$\frac{E_b}{N_0} = \frac{2^r - 1}{r}$$

- As $r = \frac{R}{W} \rightarrow 0$

$$\begin{aligned}\left. \frac{E_b}{N_0} \right|_{r \rightarrow 0} &= \lim_{r \rightarrow 0} \frac{1}{r} (2^r - 1) \\ &= \ln 2 \\ &= 0.693 \quad -1.6 \text{ dB} \\ &= -1.59 \text{ dB}\end{aligned}$$

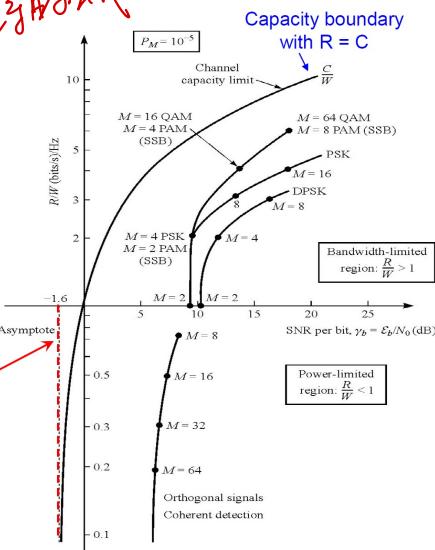
Shannon Limit, an absolute minimum for reliable communication

$$C = W \log_2 \left(1 + \frac{P}{N_0 W} \right) \text{ bits/sec}$$

① When $\text{SNR} \gg 1$, $C \propto W \log_2 \frac{P}{N_0 W}$.
 double $P \rightarrow 2P$ gets $\Delta C = W$

When $P \rightarrow \infty$, then $C \rightarrow \infty$

② When $W \gg 1$, $\text{SNR} = \frac{P}{N_0 W} \approx 0$
 $C = \frac{W N_0}{P} \log_2 \left(1 + \frac{P}{N_0 W} \right) \cdot \frac{P}{N_0}$
 $\approx \log_2 e \cdot \frac{P}{N_0} = 1.44 \frac{P}{N_0}$



Chapter 10. Channel coding. *see the slides*

Main Types of Codes: block codes and convolutional codes

① Linear block codes (线性分组码). has no memory.

② Convolutional codes (卷积码). has memory.

1. Block Codes.

An (n,k) block code is a collection of $M=2^k$ codewords of code length n .

Each block has a block of k information bits followed by a group of $r=n-k$ check bits.

message \rightarrow (channel encoder) \rightarrow n bit codewords
k bits $\qquad\qquad\qquad$ $(k+r)$ bits

o Example: Simple Parity-Check Codes

Chapter 01 .

1. Show that the unit step signal $u(t)$ is a power-type signal and find its power content.

$$2. \mathcal{F}[\cos(\pi t)] = \frac{1}{2}\delta(f+\frac{1}{2}) + \frac{1}{2}\delta(f-\frac{1}{2}), \quad \mathcal{F}[\sin(\pi t)] = \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2}).$$

5.28 Let X and Y be zero-mean jointly Gaussian random variables, each with variance σ^2 . The correlation coefficient between X and Y is denoted by ρ . Random variables Z and W are defined by

$$\begin{cases} Z = X \cos \theta + Y \sin \theta \\ W = -X \sin \theta + Y \cos \theta \end{cases}$$

where θ is a constant angle.

1. Show that Z and W are jointly Gaussian random variables.
2. For what values of θ are the random variables Z and W independent?

Chapter 02 .

5.35 Which one of the following functions can be the autocorrelation function of a random process and why?

1. $f(\tau) = \sin(2\pi f_0 \tau)$.
2. $f(\tau)$, as shown in Figure P-5.35.

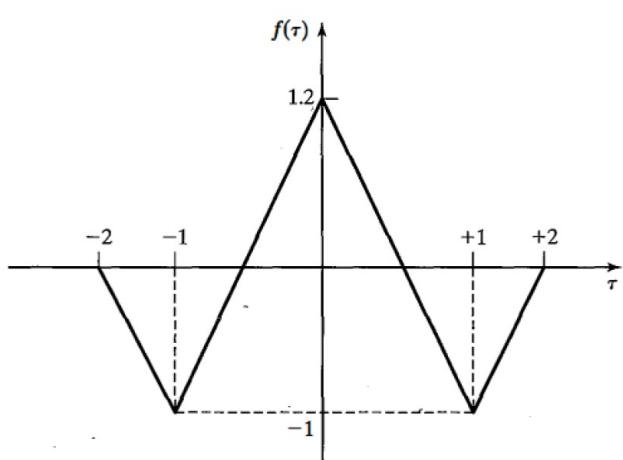


Figure P-5.35

$$(1) \quad f(\tau) = \sin(2\pi f_0 \tau)$$

$f(0) = 0$ is not its maximum value.

$$(2) \quad f(\tau) = 1.2 \Lambda(\tau) - \Lambda(\tau+1) - \Lambda(\tau-1)$$

$\Rightarrow f(0) = 0$ and it's even symmetric
its power spectral density :

$$f(\tau) \xrightarrow{\mathcal{F}} S(f) = 1.2 \operatorname{sinc}^2(f) - \sin^2(f) \cdot (e^{j2\pi f} e^{-j2\pi f})$$

$$= \operatorname{sinc}^2(f) (1.2 - 2 \cos(2\pi f)).$$

$S(0) < 0$ so $f(\tau)$ cannot be an auto-correlation function.

$$\Downarrow \Delta(\tau) \rightarrow \operatorname{sinc}^2(f).$$

功率谱密度函数在任意频率处的值不为负

- 5.40 A zero-mean white Gaussian noise process with the power spectral density of $\frac{N_0}{2}$ passes through an ideal lowpass filter with bandwidth B .

1. Find the autocorrelation of the output process $Y(t)$.
2. Assuming $\tau = \frac{1}{2B}$, find the joint probability density function of the random variables $Y(t)$ and $Y(t + \tau)$. Are these random variables independent?

1. $S_X(f) = \frac{N_0}{2}$. For the lowpass filter $H(f) = \Pi(\frac{f}{2B})$.

$$\Rightarrow S_Y(f) = |H(f)|^2 S_X(f) = \frac{N_0}{2} \Pi\left(\frac{f}{2B}\right).$$

$$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \frac{N_0}{2} \mathcal{F}^{-1}\left[\Pi\left(\frac{f}{2B}\right)\right] = N_0 B \operatorname{sinc}(2B\tau).$$

$$2. \tau = \frac{1}{2B} \Rightarrow R_Y(\tau) = R_Y\left(\frac{1}{2B}\right) = 0.$$

或者: $E[Y_{(t)}] = E[X_{(t)}] \times H(0) = 0$, $\sigma_Y^2 = R_Y(0) = N_0 B$.

The correlation coefficient of $Y_{(t)}$ and $Y_{(t+\tau)}$ is:

$$\rho = \frac{\sigma_{Y(t)} \sigma_{Y(t+\tau)}}{\sigma_{Y(t)} \sigma_{Y(t+\tau)}} = \frac{E[Y_{(t)} Y_{(t+\tau)}]}{N_0 B} = \frac{R_Y(\tau)}{N_0 B} \Rightarrow \tau = \frac{1}{2B} = \rho = 0.$$

$$SO: f = \frac{1}{2\pi N_0 B} e^{-\frac{Y^2(t) + Y^2(t+\tau)}{2N_0 B}}. \quad (\text{JPDF}).$$

\Rightarrow Independent.

Chapter 03

3.7 An AM signal has the form

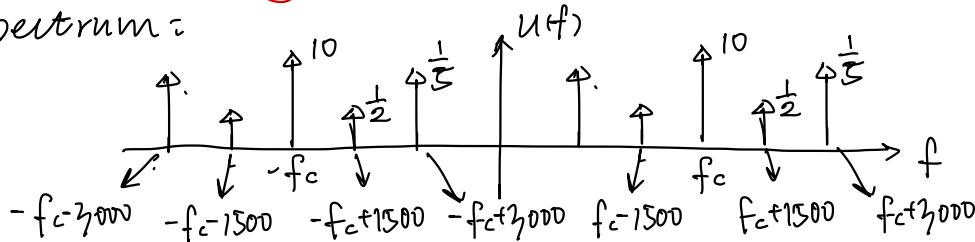
$$u(t) = [20 + 2 \cos 3000\pi t + 10 \cos 6000\pi t] \cos 2\pi f_c t,$$

where $f_c = 10^5$ Hz.

1. Sketch the (voltage) spectrum of $u(t)$.
2. Determine the power in each of the frequency components.
3. Determine the modulation index.
4. Determine the sidebands' power, the total power, and the ratio of the sidebands' power to the total power.

$$\begin{aligned} 1. \quad u(t) \xrightarrow{\text{FT}} U(f) &= \frac{20}{2} [\delta(f - f_c) + \delta(f + f_c)] \\ &+ \frac{2}{4} [\delta(f - f_c - 1500) + \delta(f - f_c + 1500) + \delta(f + f_c - 1500) + \delta(f + f_c + 1500)] \\ &+ \frac{10}{4} [\delta(f - f_c - 3000) + \delta(f - f_c + 3000) + \delta(f + f_c - 3000) + \delta(f + f_c + 3000)] \end{aligned}$$

the spectrum =



$$\begin{aligned} 2. \quad u(t) &= 20 \cos^2(2\pi f_c t) + \cos^2(2\pi(1f_c - 1500)t) + \cos^2(2\pi(1f_c + 1500)t) \\ &\quad + 25 \cos^2(2\pi(1f_c - 3000)t) + 25 \cos^2(2\pi(1f_c + 3000)t) \\ &\quad + \text{terms that are multiples of cosines.} \end{aligned}$$

或者用积化和差得：

$$u(t) = 20 \cos(2\pi f_c t) + \cos(2\pi(1f_c + 1500)t) + \cos(2\pi(1f_c - 1500)t) + 5 \cos(2\pi(1f_c + 3000)t) + 5 \cos(2\pi(1f_c - 3000)t).$$

The power of $A \cos(2\pi f_0 t + \theta)$ is $\frac{1}{2} A^2$ =

$$\Rightarrow P_{f_c} = \frac{400}{2} = 200. \quad P_{f_c-1500} = P_{f_c+1500} = \frac{1}{2}. \quad P_{f_c-3000} = P_{f_c+3000} = \frac{25}{2}.$$

$$\begin{aligned} 3. \quad u(t) &= [20 + 2 \cos(2\pi 1500t) + 10 \cos(2\pi 3000t)] \cos(2\pi f_c t) \\ &= 20 [1 + \frac{1}{10} \cos(2\pi 1500t) + \frac{1}{2} \cos(2\pi 3000t)] \cos(2\pi f_c t). \end{aligned}$$

$$\Rightarrow \underline{\alpha m_n(t)} = \frac{1}{10} \cos(2\pi 1500t) + \frac{1}{2} \cos(2\pi 3000t)$$

$$|\alpha m_n(t)|_{\max} = 0.6. \Rightarrow \alpha = 0.6. (|m_n(t)|_{\max} = 1).$$

$$4. P_{\text{total}} = P_{f_c} + P_{f_c+1500} + P_{f_c-1500} + P_{f_c+3000} + P_{f_c-3000} = 226.$$

$$P_{\text{carrier}} = \frac{1}{2} A_c^2 = \frac{1}{2} \times 200^2 = 200. \quad P_{\text{sidebands}} = 226 - 200 = 26.$$

$$\Rightarrow E = \frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226} = \frac{13}{113}.$$

4.4 An angle-modulated signal has the form

$$u(t) = 100 \cos [2\pi f_c t + 4 \sin 2000\pi t],$$

where $f_c = 10 \text{ MHz}$.

1. Determine the average transmitted power.
2. Determine the peak-phase deviation.
3. Determine the peak-frequency deviation.
4. Is this an FM or a PM signal? Explain.

$$\Rightarrow u(t) = 100 \cos [2\pi f_c t + 4 \sin 2000\pi t] \\ = 100 \cos [2\pi f_c t + 2\pi \cdot 4000 \cdot \int_0^t \cos 2000\pi \tau d\tau]$$

peak-phase deviation $\beta = 4$. peak-frequency deviation $\Delta f = k_f \cdot A_m = 4000 \text{ Hz}$

$$P = \frac{1}{2} \cdot (100)^2 = 5000.$$

$$\text{或者 } \theta(t) = 4 \sin 2000\pi t.$$

$$\Rightarrow \underline{\Delta \phi_{\max} = |\theta(t)|_{\max} = 4}. \quad \underline{\Delta f_{\max} = \left| \frac{1}{2\pi} \frac{d}{dt} \theta(t) \right|_{\max} = 4000}.$$

$$\Delta f_{\max} = \Delta \phi_{\max} \cdot f_m = 4000.$$

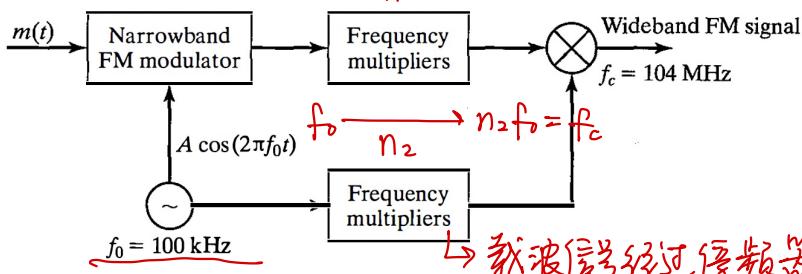
- 4.6 To generate wideband FM, we can first generate a narrowband FM signal, and then use frequency multiplication to spread the signal bandwidth. Figure P-4.6 illustrates such a scheme, which is called an Armstrong-type FM modulator. The narrowband FM signal has a maximum(angular deviation) of 0.10 radians to keep distortion under control.

$$\beta = 0.10.$$

- If the message signal has a bandwidth of 15 kHz and the output frequency from the oscillator is 100 kHz, determine the frequency multiplication that is necessary to generate an FM signal at a carrier frequency of $f_c = 104$ MHz and a frequency deviation of $f = 75$ kHz.
- If the carrier frequency for the wideband FM signal is to be within ± 2 Hz, determine the maximum allowable drift of the 100 kHz oscillator.

$$\frac{\Delta f_1}{f_0} \xrightarrow{n_1} n_1 \Delta f_1 = \Delta f$$

$$\frac{f_0}{n_1} \xrightarrow{n_1} n_1 f_0 = f_c$$



\Rightarrow (1) NW 信号经过倍频器后
其FM信号的载频增大N倍。
调制指数也增大N倍
即 $f_c' = N f_c$. $\beta' = N \beta$

\hookrightarrow 载波信号经过倍频器后频率增大 n_2 倍。

- \Rightarrow 1. The frequency deviation of the narrowband FM signal is

$$\Delta f_1 = \beta f_m = 0.1 \times 15 = 1.5 \text{ kHz}. \Rightarrow n_1 = \frac{\Delta f}{\Delta f_1} = \frac{75}{1.5} = 50.$$

To generate a FM signal with $f_c = 104$ MHz =

$$(n_1 + n_2) f_0 = f_c \Rightarrow n_2 = 990.$$

$$2. \underline{(n_1 + n_2) df} = 2 \Rightarrow df = \frac{2}{1040} = \frac{1}{520} \text{ Hz}$$

\Rightarrow If the out of the narrowband modulator is $u_1(t) = A \cos(2\pi f_0 t + \phi(t))$
then the output of the n_1 multiplier is: $u_{11}(t) = A \cos(2\pi n_1 f_0 t + n_1 \phi(t))$
the output of the n_2 multiplier: $u_{12}(t) = A \cos(2\pi n_2 f_0 t).$

Mixed output =

$$\begin{aligned} y(t) &= A^2 \cos(2\pi n_1 f_0 t + n_1 \phi(t)) \cos(2\pi n_2 f_0 t) \\ &= \frac{A^2}{2} [\cos(2\pi(n_1 + n_2)f_0 t + n_1 \phi(t)) + \cos(2\pi(n_1 - n_2)f_0 t + n_1 \phi(t))] \end{aligned}$$

- 4.10 The carrier $c(t) = A \cos 2\pi 10^6 t$ is angle modulated (PM or FM) by the sinusoid signal $m(t) = 2 \cos 2000\pi t$. The deviation constants are $k_p = 1.5 \text{ rad/V}$ and $k_f = 3000 \text{ Hz/V}$.
 $\hookrightarrow f_m = W = 1000 \text{ Hz}$.

1. Determine β_f and β_p .
2. Determine the bandwidth in each case using Carson's rule.
3. Plot the spectrum of the modulated signal in each case. (Plot only those frequency components that lie within the bandwidth derived in Part 2.)
4. If the amplitude of $m(t)$ is decreased by a factor of 2, how would your answers to Parts 1–3 change?
5. If the frequency of $m(t)$ is increased by a factor of 2, how would your answers to Parts 1–3 change?

$$\Rightarrow 1. \underbrace{\beta_f = \Delta\phi_{\max} = k_p |m(t)|_{\max}}_{= 3.} \quad \underbrace{\beta_p = \frac{Af_{\max}}{W} = \frac{k_f |m(t)|_{\max}}{W}}_{= 6.}$$

$$2. \text{ Using Carson's rule: } \underbrace{B_{PM} = 2(1 + \beta_p)W}_{= 8000} \quad \underbrace{B_{FM} = 2(1 + \beta_f)W}_{= 14000}.$$

3. The modulated PM - FM wave:

$$S_{PM}(f) = \frac{A}{2} \sum_{n=-4}^{t+4} J_n(b) [\delta(f - 10^6 - n10^3) + \delta(f + 10^6 + n10^3)],$$

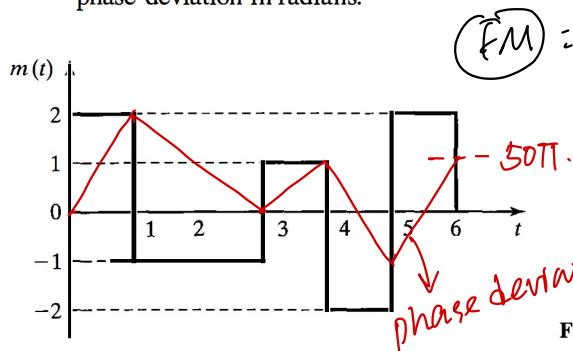
$$S_{FM}(f) = \frac{A}{2} \sum_{n=-7}^{t+7} J_n(b) [\delta(f - 10^6 - n10^3) + \delta(f + 10^6 + n10^3)].$$

\Rightarrow The FM, PM modulated signal can be written as =

$$m(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta_f) \cos [2\pi(f_c + n f_m)t]$$

$$m(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta_p) \cos [2\pi(f_c + n f_m)t].$$

- 4.16 The message signal $m(t)$ into an FM modulator with a peak frequency deviation $f_d = 25 \text{ Hz/V}$ is shown in Figure P-4.16. Plot the frequency deviation in Hz and the phase deviation in radians.



(FM) : \Rightarrow frequency deviation =

$$\Delta f = k_f m(t) = \frac{25}{2} m(t).$$

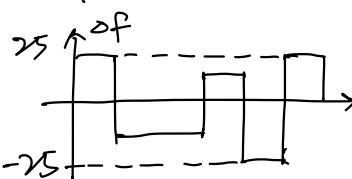


Figure P-4.16

phase deviation :

$$\Delta\phi = 2\pi k_f \int_0^t m(\tau) d\tau = 25\pi \int_0^t m(\tau) d\tau$$

\leftarrow 见书解



4.18 The modulating signal that is the input into an FM modulator is

$$m(t) = 10 \cos 16\pi t.$$

The output of the FM modulator is

$$u(t) = 10 \cos \left[4000\pi t + 2\pi k_f \int_{-\infty}^t m(\tau) d\tau \right],$$

where $k_f = 10$. (See Figure P-4.18.) If the output of the FM modulator is passed through an ideal BPF centered at $f_c = 2000$ with a bandwidth of 62 Hz, determine the power of the frequency components at the output of the filter. What percentage of the transmitter power appears at the output of the BPF?

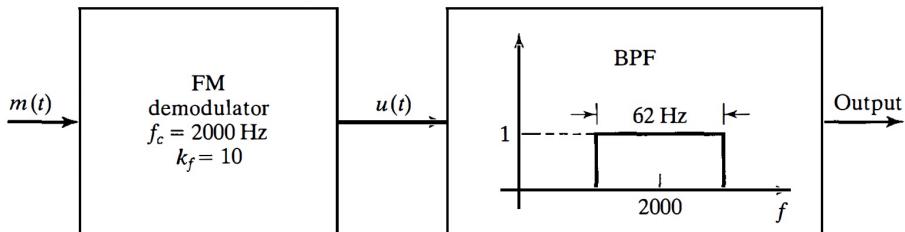


Figure P-4.18

$$\Rightarrow \text{The modulation index: } \beta = \frac{k_f |m(t)|_{\max}}{f_m} = \frac{25}{2}.$$

The output of FM modulator can be written as:

$$\begin{aligned} m(t) &= 10 \cos \left[2\pi 2000 t + 2\pi k_f \int_{-\infty}^t 10 \cos(2\pi 8\tau) d\tau \right] \\ &= \sum_{n=-\infty}^{\infty} 10 J_n(12.5) \cos [2\pi(2000 + n8)t + \phi_n]. \end{aligned}$$

Only $n = -4, \dots, 4$ terms' frequency lies within $[2000 - 32, 2000 + 32]$.

The power: $\hookrightarrow 4+8=\underline{\underline{32}}$

$$\frac{10^2}{2} J_0^2(12.5) + 2 \sum_{n=1}^4 \frac{10^2}{2} J_n^2(12.5) = 50 \times 0.2630 = 13.15.$$

$$P_{\text{total}} = \frac{1}{2} \cdot 10^2 = 50. \Rightarrow \frac{13.15}{50} = 26.30\%.$$

Chapter 04

7.18 The random process $X(t)$ is defined by $X(t) = Y \cos(2\pi f_0 t + \Theta)$, where Y and Θ are two independent random variables, Y uniform on $[-3, 3]$ and Θ uniform on $[0, 2\pi]$.

1. Find the autocorrelation function of $X(t)$ and its power spectral density.
2. If $X(t)$ is to be transmitted to maintain an SQNR of at least 40 dB using a uniform PCM system, what is the required number of bits/sample and the least bandwidth requirement (in terms of f_0)?
3. If the SQNR is to be increased by 24 dB, how many more bits/sample must be introduced, and what is the new minimum bandwidth requirement in this case?

$$\Rightarrow 1. R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E[Y^2 \cos(2\pi f_0 t_1 + \Theta) \cos(2\pi f_0 t_2 + \Theta)] \\ = \frac{1}{2} E(Y^2) E[\cos(2\pi f_0 t_1 + 2\pi f_0 t_2 + 2\Theta) + \cos(2\pi f_0(t_1 - t_2))] \\ = \frac{1}{2} E(Y^2) \left[\int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi f_0 t_1 + 2\pi f_0 t_2 + 2\theta) d\theta + \int_0^{2\pi} \frac{1}{2\pi} \cos(2\pi f_0(t_1 - t_2)) d\theta \right] \\ = \frac{1}{2} E(Y^2) \cos(2\pi f_0(t_1 - t_2))$$

$$E(Y^2) = \frac{1}{12} \cdot (3 - (-3))^2 = 3 \Rightarrow R_X(\tau) = \frac{3}{2} \cos 2\pi f_0 \tau.$$

Power spectral density = $S_X(f) = \mathcal{F}[R_X(\tau)] = \frac{3}{4} [\delta(f + f_0) + \delta(f - f_0)]$

$$\cos 2\pi f_0 \tau \xrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f + f_0) + \delta(f - f_0)]$$

$$2. \text{SQNR} = 10 \log_{10} \frac{P_x}{\sigma^2} + 6V + 4.8 \text{ (dB)}. \quad P_x = \left(\frac{1}{2} \times \left(\frac{3}{2} \right)^2 = \frac{9}{8} \right) = R_X(0) = \frac{3}{2}$$

$$\underline{a=3} \quad . \quad \underline{10 \log_{10} \text{SQNR} = 10 \log_{10} \left(\frac{3 \times 4^V \times P_x}{\sigma^2} \right) = 40} \Rightarrow V=8$$

the minimum bandwidth is $8f_0$.

$$3. \Rightarrow 6\Delta V = 24 \quad \Delta V = 4. \quad V' = V + \Delta V = 12. \Rightarrow 12f_0.$$

$$10 \log_{10} \text{SQNR} = 10 \log_{10} \left(\frac{3 \times 4^V \times P_x}{X_{\max}^2} \right)$$

7.4 The lowpass signal $x(t)$ with a bandwidth of W is sampled with a sampling interval of T_s , and the signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s) p(t - nT_s)$$

is reconstructed from the samples, where $p(t)$ is an arbitrary-shaped pulse (not necessarily time limited to the interval $[0, T_s]$).

1. Find the Fourier transform of $x_p(t)$.
2. Find the conditions for perfect reconstruction of $x(t)$ from $x_p(t)$.
3. Determine the required reconstruction filter.

$$1. \quad \mathcal{X}_p(t) = \sum_{n=-\infty}^{\infty} X(nT_s) p(t-nT_s) = p(t) * \sum_{n=-\infty}^{\infty} X(nT_s) \delta(t-nT_s).$$

$$= p(t) * X(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s).$$

$$\Rightarrow \mathcal{X}_p(t) = \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} \mathcal{X}(f - \frac{n}{T_s})$$

2. $P(f)$ should be invertible for $|f| < W$

$$3. \quad \mathcal{X}(f) = \mathcal{X}_p(f) P^{-1}(f) \Pi(\frac{f}{2W\pi}). \quad \underline{W < W_\pi < \frac{1}{T_s} - W}$$

7.9 Let $X(t)$ denote a wide-sense stationary (WSS) Gaussian process with $P_X = 10$.

1. Using Table 7.1, design a 16-level optimal uniform quantizer for this source.
2. What is the resulting distortion if the quantizer in Part 1 is employed?
3. What is the amount of improvement in SQNR (in decibels) that results from doubling the number of quantization levels from 8 to 16?

→ According to table 7.1. the optimal level spacing for a 16-level uniform quantizer is 0.3352.

then in case of $\sigma^2 = 10$. optimal level space $A = \sqrt{10} \cdot 0.3352 = 1.060$.

The quantization levels are:

$$\hat{x}_1 = -\hat{x}_{16} = -7 \times 1.060 - \frac{1}{2} \times 1.060 = -7.950$$

$$\hat{x}_2 = -\hat{x}_{15} = -6 \times 1.060 - \frac{1}{2} \times 1.060 = -6.890$$

$$\hat{x}_3 = -\hat{x}_{14} = -5 \times 1.060 - \frac{1}{2} \times 1.060 = -5.830$$

$$\hat{x}_4 = -\hat{x}_{13} = -4 \times 1.060 - \frac{1}{2} \times 1.060 = -4.770$$

$$\hat{x}_5 = -\hat{x}_{12} = -3 \times 1.060 - \frac{1}{2} \times 1.060 = -3.710$$

$$\hat{x}_6 = -\hat{x}_{11} = -2 \times 1.060 - \frac{1}{2} \times 1.060 = -2.650$$

$$\hat{x}_7 = -\hat{x}_{10} = -1 \times 1.060 - \frac{1}{2} \times 1.060 = -1.590$$

$$\hat{x}_8 = -\hat{x}_9 = -\frac{1}{2} \times 1.060 = -0.530.$$

The boundaries are:

$$a_1 = a_{15} = -7 \times 1.060 = -7.420$$

$$a_2 = a_{14} = -6 \times 1.060 = -6.360$$

$$a_3 = a_{13} = -5 \times 1.060 = -5.300$$

$$a_4 = a_{12} = -4 \times 1.060 = -4.240$$

$$a_5 = a_{11} = -3 \times 1.060 = -3.180$$

$$a_6 = a_{10} = -2 \times 1.060 = -2.120$$

$$a_7 = a_9 = -1 \times 1.060 = -1.060$$

$$a_8 = 0.$$

$$\text{Distortion} = \sigma^2 \times 0.01154 = 0.1154$$

$$D_{16} = 0.1154 \quad D_8 = \sigma^2 \times 0.03744 = 0.3744$$

$$\Rightarrow 10 \log_{10} \frac{SNR_{16}}{SNR_8} = 10 \log_{10} \frac{0.3744}{0.1154} = 5.111 \text{ dB}$$

$$SNR = \frac{E(x^2)}{E[(x-Q(x))^2]} \Rightarrow \text{distortion}.$$

Chapter 10

- 10.5 Show that a pulse having the raised cosine spectrum given by Equation (10.3.20) satisfies the Nyquist criterion given by Equation (10.3.7) for any value of the roll-off factor α .

⇒ The pulse having the raised cosine spectrum:

$$x(t) = \sin\left(\frac{t}{T}\right) \frac{\cos(\pi\alpha\frac{t}{T})}{1 - 4\alpha^2 \frac{t^2}{T^2}}$$

$$\sin\left(\frac{t}{T}\right) = 0, t = nT, n = 0, \pm 1, \dots$$

$$\frac{\cos(\pi\alpha\frac{t}{T})}{1 - 4\alpha^2 \frac{t^2}{T^2}} = \begin{cases} 1, & t=0 \\ \text{bounded}, & t \neq 0 \end{cases} \quad \text{and} \quad \lim_{dt \rightarrow \frac{\pi}{2}} \frac{\cos(\pi\alpha\frac{t}{T})}{1 - 4\alpha^2 \frac{t^2}{T^2}} = \frac{\pi}{2} < \infty$$

$$\text{Then: } x(nT) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

- 10.8 A channel has a passband characteristic in the frequency range $|f| \leq \underline{1400 \text{ Hz}}$.

- Select a symbol rate and a PAM signal constellation size to achieve a 9600 bps signal transmission.
- If a square-root raised cosine pulse is used for the transmitter pulse $g_T(t)$, select the roll-off factor. Assume that the channel has an ideal frequency-response characteristic.

1. $B = 1400 \text{ Hz}$. then the max symbol rate $= R_{S,\max} = 2W = 2800$.

$$\text{when } V=3 \quad R_S = \frac{9600}{V} = 3200 > 2800. \quad V=4, \quad R_S = 2400 < 2800$$

Then $V=4$. $R_S = 2400 \text{ symbol/sec. size} = 2^V = 16$.

$$2. \frac{1}{T} = R_S = 2400. \Rightarrow B = \frac{1+\alpha}{2T} = 1400. \text{ then } \alpha = 0.1667$$

- 10.12 The binary sequence 10010110010 is the input to the precoder whose output is used to modulate a duobinary transmitting filter. Construct a table as in Table 10.2; show the precoded sequence, the transmitted amplitude levels, the received signal levels, and the decoded sequence.

binary sequence:

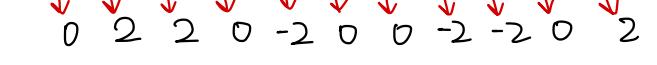
precode sequence:



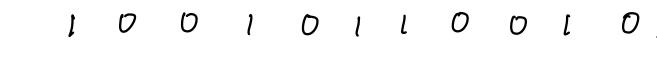
transmitted levels:



received sequence:



decoded sequence:



⇒ textbook P589.

example 10.4.1

- 10.29 The transmission of a signal pulse with a raised cosine spectrum through a channel results in the following (noise-free) sampled output from the demodulator:

$$x_k = \begin{cases} -0.5, & k = -2 \\ 0.1, & k = -1 \\ 1, & k = 0 \\ -0.2, & k = 1 \\ 0.05, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

- Determine the tap coefficients of a three-tap linear equalizer based on the zero-forcing criterion.
- For the coefficients determined in Part 1, determine the output of the equalizer for the case of the isolated pulse. Thus, determine the residual ISI and its span in time.

\Rightarrow the output of 3f-equalizer is $q_m = \sum_{n=1}^1 c_n x(m-n)$

with $q_0=1$ and $q_m=0$ for $m \neq 0$.

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_{-1} \\ x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} q_{-1} \\ q_0 \\ q_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Then: $c_{-1}=0$, $c_0=0.98$, $c_1=0.196$.

the output of the equalizer:

$$q_m = \begin{cases} 0 & , m < -4 \\ c_{-1}x_{-2} = 0 & , m = -3 \quad n = -1 \quad k = m - n = -2 \\ c_{-1}x_{-1} + c_0x_{-2} = -0.49 & , m = -2 \quad n = 0, -1, \quad k = m - n = -1, -2 \\ 0 & , m = -1 \\ 1 & , m = 0 \\ 0 & , m = 1 \\ c_0x_2 + c_1x_1 = 0.0098 & , m = 2 \\ c_1x_2 = 0.0098 & , m = 3 \\ 0 & , m > 4 \end{cases}$$

the residual ISI sequence = $\{ \dots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \dots \}$
its span is 6 symbols.

15-49)

$$1. P = \int_{-\infty}^{\infty} S_x(f) df = 4 \times 10^{-5} \times 2 \times 10^5 \times \frac{1}{2} \times 1 = 4 W$$

$$2. BW = 10^5 \text{ Hz}$$

$$3. H(f) = \begin{cases} 1, & |f| \leq 50 \text{ kHz} \\ 0, & \text{otherwise.} \end{cases} \Rightarrow S_Y(f) = |H(f)|^2 S_x(f) = \begin{cases} 4 \times 10^{-5} \Lambda \left(\frac{f}{10^5}\right), & |f| \leq 50 \text{ kHz} \\ 0, & \text{otherwise} \end{cases}$$

$$P(Y) = \int_{-\infty}^{\infty} S_Y(f) df = \left(\frac{1}{2} + 1\right) \times 5 \times 10^4 \times \frac{1}{2} \times 2 \times 4 \times 10^{-5} = 1.8 W$$

4.

$$5. R_{x(\tau)} = \int_{-\infty}^{\infty} S_x(f) e^{j 2\pi f \tau} df = \begin{cases} 10^{-2.5}, & |\tau| \leq 5 \times 10^4 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow t_0 = 0 + \tau = 5 \times 10^4.$$

15-62)

Chapter 06.

8.7

The received signal in a binary communication system that employs antipodal signals

$$r(t) = s(t) + n(t),$$

where $s(t)$ is shown in Figure P-8.7 and $n(t)$ is AWGN with power spectral density $N_0/2$ W/Hz.

1. Sketch the impulse responses of the filter matched to $s(t)$.
2. Sketch the output of the matched filter to the input $s(t)$.
3. Determine the variance of the noise of the output of the matched filter at $t = 3$.
4. Determine the probability of error as a function of A and N_0 .

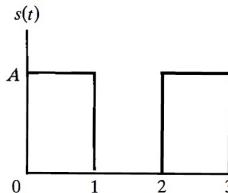


Figure P-8.7

- \Rightarrow 1. The impulse response of the filter = $h(t) = s(T-t) = s(3-t) = s(6-t)$
2. The output of the filter = $y(t) = h(t) * s(t) = s(t) * s(t) = \int_0^t s(\tau) s(t-\tau) d\tau$
 then $y(t) = \begin{cases} 0 & t < 0 \\ A^2 t, & 0 \leq t < 1 \\ A^2 (2-t), & 1 \leq t < 2 \\ 2A^2 (t-2), & 2 \leq t < 3 \\ 2A^2 (4-t), & 3 \leq t < 4 \\ A^2 (t-4), & 4 \leq t < 5 \\ A^2 (6-t), & 5 \leq t < 6 \\ 0, & t \geq 6. \end{cases}$
- Sketch =
- \Rightarrow the output of MF is the autocorrelation function of Input signal. $S_{yy}(t) = R_{ss}(t-t_0)$.
 the peak value: $S_{yy}(0) = E[y^2] = S_{yy}(3) = 2A^2$
3. $t=T=3$: $n_T = \int_0^T n(\tau) h(T-\tau) d\tau = \int_0^T n(\tau) s(3-\tau) d\tau$
 $\sigma_{n_T}^2 = E \left[\int_0^T \int_0^T n(\tau) n(\nu) s(\tau) s(\nu) d\tau d\nu \right] = \int_0^T \int_0^T n(\tau) n(\nu) s(\tau) s(\nu) d\tau d\nu$
 $= \frac{N_0}{2} \int_0^T \int_0^T s(\tau) s(\nu) \delta(\tau-\nu) d\tau d\nu = \frac{N_0}{2} \int_0^T s(\tau)^2 d\tau = N_0 A^2.$
4. $P(e) = Q \left[\sqrt{\frac{s}{N_0}} \right]$, where $\left(\frac{s}{N_0} \right)_0 = \frac{E[y^2]}{E[n_T^2]} = \frac{4A^4}{N_0 A^2} = \frac{4A^2}{N_0}$
- then $P(e) = Q \left[\sqrt{\frac{4A^2}{N_0}} \right]$.
- 直接公式 = $E[n_T^2] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2} E$
- $\Rightarrow \sigma_{n_T}^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2} \cdot 2A = N_0 A$.

the output SNR of matched filter.

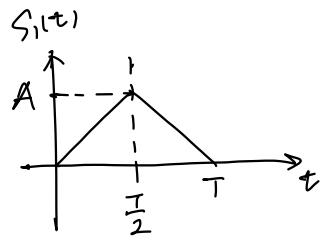
$$t=3 = y^2(t=3) = (2A^2)^2 = 4A^4.$$

For antipodal signal: OR: $SNR = \frac{2E}{N_0} = \frac{2 \cdot 2A^2}{N_0} = \frac{4A^2}{N_0}$. $P_e = Q \left(\sqrt{\frac{2E}{N_0}} \right) = Q \left(\sqrt{\frac{4A^2}{N_0}} \right)$.

X

8.17 In a binary antipodal signaling scheme, the signals are given by

$$s_1(t) = -s_2(t) = \begin{cases} \frac{2At}{T}, & 0 \leq t \leq \frac{T}{2} \\ 2A\left(1 - \frac{t}{T}\right), & \frac{T}{2} \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$



The channel is AWGN and $S_n(f) = \frac{N_0}{2}$. The two signals have prior probabilities p and $1-p$.

1. Determine the structure of the optimal receiver. *see q. optimal receiver structure*

2. Determine an expression for the error probability.

3. Plot the error probability as a function of p for $0 \leq p \leq 1$.

$$\Rightarrow 1. C(\vec{r}, \vec{s}_m) = \underbrace{\int_{-\infty}^{\infty} r(t)s_m(t)dt}_{S_1(t) = -S_2(t)} - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln p(S_m)$$

metrics.

$$\begin{aligned} \text{then: } & \int_{-\infty}^{\infty} r(t)s_1(t)dt - \int_{-\infty}^{\infty} r(t)s_2(t)dt \\ &= \int_{-\infty}^{\infty} r(t)[S_1(t) - S_2(t)]dt \stackrel{S_1}{\geq} \frac{N_0}{2} \ln \frac{P(S_2)}{P(S_1)} = \frac{N_0}{2} \ln \frac{1-p}{p} \\ &\Rightarrow \int_{-\infty}^{\infty} r(t)s_1(t)dt \stackrel{S_1}{\geq} \frac{N_0}{2} \ln \frac{1-p}{p}. \end{aligned}$$

2. If $s_1(t)$ is transmitted, the output of correlator :

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt = \int_0^T s_1^2(t)dt + \int_0^T n(t)s_1(t)dt = E_s + n.$$

n is a gaussian random variable with zero mean and variance :

$$\sigma_n^2 = \frac{N_0}{2} \int_0^T |s_1(t)|^2 dt = \frac{N_0}{2} E_s.$$

The error probability :

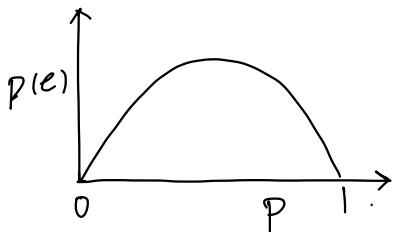
$$P(e|s_1) = \int_{-\infty}^{\frac{N_0}{4} \ln \frac{1-p}{p} - E_s} \frac{1}{\sqrt{\pi N_0 E_s}} e^{-\frac{x^2}{N_0 E_s}} dx = Q\left[\sqrt{\frac{2E_s}{N_0}} - \frac{1}{4} \sqrt{\frac{2N_0}{E_s} \ln \frac{1-p}{p}}\right].$$

$$\Rightarrow P(e|s_2) = Q\left[\sqrt{\frac{2E_s}{N_0}} + \frac{1}{4} \sqrt{\frac{2N_0}{E_s} \ln \frac{1-p}{p}}\right].$$

$$\text{Then: } P(e) = P(e|s_1) + (1-p)P(e|s_2)$$

$$= P Q\left[\sqrt{\frac{2E_s}{N_0}} - \frac{1}{4} \sqrt{\frac{2N_0}{E_s} \ln \frac{1-p}{p}}\right] + (1-p)Q\left[\sqrt{\frac{2E_s}{N_0}} + \frac{1}{4} \sqrt{\frac{2N_0}{E_s} \ln \frac{1-p}{p}}\right]$$

(b)



8.24 Consider the signal

$$u(t) = \begin{cases} \frac{A}{T}t \cos 2\pi f_c t, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}.$$

1. Determine the impulse response of the matched filter for this signal.
2. Determine the output of the matched filter at $t = T$.
3. Suppose the signal $u(t)$ is passed through a correlator that correlates the input $u(t)$ with $u(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in Part (2).

$$1. h_m(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t) \cos(2\pi f_c(T-t)), & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

2. the output of MF at $t=T$:

$$\begin{aligned} u(t) * h_m(t) \Big|_{t=T} &= \int_0^T u(T-\tau) h_m(\tau) d\tau = \frac{A^2}{T^2} \int_0^T (T-\tau)^2 \cos^2(2\pi f_c(T-\tau)) d\tau \\ &= \frac{A^2}{T^2} \left[\frac{T^3}{6} + \left(\frac{T^2}{4 \times 2\pi f_c} - \frac{1}{8 \times (2\pi f_c)^2} \right) \sin(4\pi f_c T) + \frac{T \cos(4\pi f_c T)}{4(2\pi f_c)^2} \right] \end{aligned}$$

3. the output of correlator at $t=T$:

$$\int_0^T u^2(\tau) d\tau = \frac{A^2}{T^2} \int_0^T \tau^2 \cos^2(2\pi f_c \tau) d\tau.$$

Some .



10.15 A baseband digital communication system employs the signals shown in Figure P-10.15(a) for the transmission of two equiprobable messages. It is assumed that the communication problem studied here is a “one shot” communication problem, i.e., the messages are transmitted just once and no transmission takes place afterward. The channel has no attenuation and the noise is AWG with power spectral density $\frac{N_0}{2}$.

1. Find an appropriate orthonormal basis for the representation of the signals.
2. In a block diagram, give the precise specifications of the optimal receiver using matched filters. Label the block diagram carefully. ⇒ 分 correlator type
for matched filter type.
3. Find the error probability of the optimal receiver.
4. Show that the optimal receiver can be implemented by using just one filter. [See the block diagram shown in Figure P-10.15(b).] What are the characteristics of the matched filter and the sampler and decision device?
5. Now assume the channel is not ideal, but has an impulse response of $c(t) = \delta(t) + \frac{1}{2}\delta(t - \frac{T}{2})$. Using the same matched filter as the previous part, design an optimal receiver.
6. Assume that the channel impulse response is $c(t) = \delta(t) + a\delta(t - \frac{T}{2})$, where a is a random variable uniformly distributed on $[0, 1]$. Using the same matched filter, design the optimal receiver.

$$1. \phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{o.w.} \end{cases} \quad \phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\ 0, & \text{o.w.} \end{cases}$$

$$2. r(t) \rightarrow (\phi_1(\frac{T}{2}-t)) \xrightarrow{t=\frac{T}{2}} r_1 \left(\begin{array}{l} \text{select} \\ \text{the} \\ \text{largest} \end{array} \right)$$

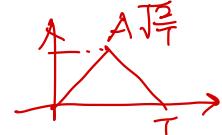
$$\rightarrow (\phi_2(T-t)) \xrightarrow{t=T} r_2$$

$$3. P(e|S_1) = P(e|S_2) = Q\left[\sqrt{\frac{A^2 T}{2N_0}}\right].$$

$$\Rightarrow P(e) = P(S_1)P(e|S_1) + P(S_2)P(e|S_2) = Q\left[\sqrt{\frac{A^2 T}{2N_0}}\right]. \quad P(S_1) = P(S_2) = \frac{1}{2}.$$

4. The two MF are the same:

$$\phi_1(\frac{T}{2}-t) = \phi_2(T-t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{o.w.} \end{cases}$$



$$\Rightarrow r(t) \rightarrow \boxed{\phi_1(t)} \xrightarrow{t=T} r_2 \xrightarrow{t=\frac{T}{2}} r_1.$$

5. If the $S_1(t)$ is transmitted, the received signal $r(t)$ is:

$$r(t) = s_1(t) + \frac{1}{2}s_1(t-\frac{T}{2}) + n(t).$$

The output of MF at $t=\frac{T}{2}$ and $t=T$:

$$r_1 = A \sqrt{\frac{2}{T}} \cdot \frac{1}{4} + \frac{3A}{2} \sqrt{\frac{2}{T}} \cdot \frac{T}{4} + n_1 = \frac{5}{2} \sqrt{\frac{A^2 T}{8}} + n_1.$$

$$r_2 = \frac{A}{2} \sqrt{\frac{2}{T}} \cdot \frac{T}{4} + n_2 = \frac{1}{2} \sqrt{\frac{A^2 T}{8}} + n_2.$$

assume that optimal receive use threshold V to make decision:

$$r_1 - r_2 \stackrel{s_1}{\geq} V$$

$$\text{then: } P(e|S_1) = P(r_1 - r_2 < V) = P(n_2 - n_1 > 2\sqrt{\frac{A^2 T}{8}} - V) = Q\left[2\sqrt{\frac{A^2 T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$$

If $s_2(t)$ transmitted, then $r(t) = s_2(t) + \frac{1}{2}s_2(t-\frac{T}{2}) + n(t)$.

the output of MF at $t=\frac{T}{2}, t=T$: $n=n_1$. $r_2 = A \sqrt{\frac{2}{T}} \cdot \frac{T}{4} + \frac{3A}{2} \sqrt{\frac{2}{T}} \cdot \frac{T}{4} + n_2 = \frac{5}{2} \sqrt{\frac{A^2 T}{8}} + n_2$

$$\text{then } P(e|S_2) = P(r_1 - r_2 > V) = P(n_1 - n_2 > \frac{5}{2} \sqrt{\frac{A^2 T}{8}} + V) = Q\left[\frac{5}{2} \sqrt{\frac{A^2 T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right].$$

$$\Rightarrow P(e) = \frac{1}{2} P(e|S_1) + \frac{1}{2} P(e|S_2) = \frac{1}{2} Q\left[2\sqrt{\frac{A^2 T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2} Q\left[\frac{5}{2} \sqrt{\frac{A^2 T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$$

$$\frac{\partial P(e)}{\partial V} = 0 \Rightarrow V = -\frac{1}{8} \sqrt{\frac{A^2 T}{2}}. \quad \frac{\partial P(e)}{\partial V} = \left(2\sqrt{\frac{A^2 T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2} \sqrt{\frac{A^2 T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$$

$$6. \text{Similar to 5). could get } V(a) = -\frac{a}{4} \sqrt{\frac{A^2 T}{2}}.$$

$$\text{The mse of } V(a) = V = \int_0^1 V(a) f(a) da = -\frac{1}{4} \sqrt{\frac{A^2 T}{2}} \int_0^1 a da = -\frac{1}{8} \sqrt{\frac{A^2 T}{2}}$$

Chapter 07

8.31 Consider the octal signal point constellations in Figure P-8.31.

1. The nearest neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles.
2. The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.
3. Determine the average transmitter powers for the two signal constellations, and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable.)

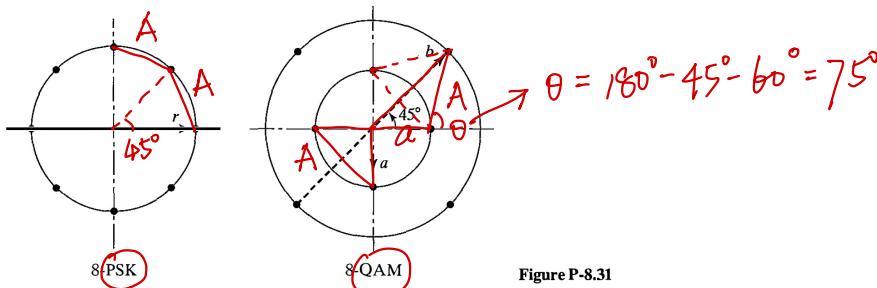


Figure P-8.31

(1) the radius of the inner circle is $a = \frac{1}{\sqrt{2}}A$.

the radius of the outer circle is $b^2 = a^2 + A^2 - 2aA \cos(180^\circ - 75^\circ) \Rightarrow b = \frac{1+\sqrt{3}}{2}A$

(2) let r be the radius of the circle :

$$A^2 = r^2 + r^2 - 2r \cos 45^\circ \Rightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}.$$

(3) the average transmitted power of PSK :

$$P_{PSK} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}} \right)^2 \Rightarrow P_{PSK} = \frac{A^2}{2-\sqrt{2}}.$$

$$\text{of the QAM} : P_{QAM} = \frac{1}{8} \left(4 \times \frac{A^2}{2} + 4 \times \frac{(1+\sqrt{3})^2}{4} A^2 \right) \Rightarrow P_{QAM} = \left[\frac{4+\sqrt{3}}{4} \right] A^2.$$

$$\text{then: } \text{gain} = \frac{P_{PSK}}{P_{QAM}} = \frac{8}{(2(1+\sqrt{3}))^2 (2-\sqrt{2})} = 1.5927 \text{ dB.}$$

the relative power advantage

星座点到原点距离的平方

$$d_{min} = A \frac{d^2}{\sqrt{M/2}}$$

$$8PSK = P_e = 2Q\left(\frac{A^2}{\sqrt{M/2}}\right).$$

$$SNR = \frac{E_s}{N_0}, \quad 8QAM = P_e = 4Q\left(\frac{A^2}{\sqrt{M/2}}\right).$$

8.32 Consider the eight-point QAM signal constellation shown in Figure P-8.31.

1. Assign three data bits to each point of the signal constellation so that the nearest (adjacent) points differ in only one bit position. ???
2. Determine the symbol rate if the desired bit rate is 90 Mbps.
3. Compare the SNR required for the eight-point QAM modulation with that required for an eight-point PSK modulation having the same error probability.
4. Which signal constellation (eight-point QAM or eight-point PSK) is more immune to phase errors? Explain the reason for your answer.

$$\{0, 0, 0, \dots, 0\}$$

(1) Assign the all zero sequence to point A, B, C and:

$$B = \{0, \dots, 0, 1, 0, \dots, 0\}, \quad C = \{0, \dots, 0, 1, 0, \dots, 0\}.$$

the position of the 1 is not the same. The the sequences of B, C differ in two bits.

8-QAM

(2) Each symbol conveys 3 bits: $R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6$ symbols/sec,

(3) The probability of error of an M-ary PSK signal: $P_m = 2Q\left[\sqrt{\frac{2E_s}{N_0}} \sin\frac{\pi}{M}\right]$.

The probability of error of an M-ary QAM signal is upper bounded by:

$$P_m = 4Q\left[\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right].$$

The two signals will achieve the same probability of error if:

$$\sqrt{2SNR_{PSK}} \sin\frac{\pi}{M} = \sqrt{2SNR_{QAM}}$$

$$M=8: \sqrt{2SNR_{PSK}} \sin\frac{\pi}{8} = \sqrt{\frac{2SNR_{QAM}}{7}} \Rightarrow \frac{SNR_{PSK}}{SNR_{QAM}} = \frac{3}{7 \times 2 \times 0.787^2} = 1.4627.$$

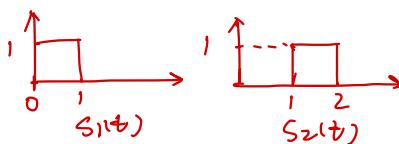
(4) The QAM constellation is more immune to phase errors.

Assuming that the magnitude of the signal points is detected correctly, then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5°. In the case of the 8-QAM constellation an error will be made if magnitude phase error exceeds 45°.

- 8.39** Two equiprobable messages are transmitted via an additive white Gaussian noise channel with a noise power spectral density of $\frac{N_0}{2} = 1$. The messages are transmitted by the signals

$$s_1(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } s_2(t) = s_1(t - 1).$$



We intended to implement the receiver using a correlation type structure, but due to imperfections in the design of the correlators, the structure shown in Figure P-8.39 has been implemented. The imperfection appears in the integrator in the upper branch where we have $\int_0^{1.5}(\cdot)$ instead of $\int_0^1(\cdot)$. The decision box, therefore, observes r_1 and r_2 ; based on this observation, it has to decide which message was transmitted. What decision rule should be adopted by the decision box for an optimal decision?

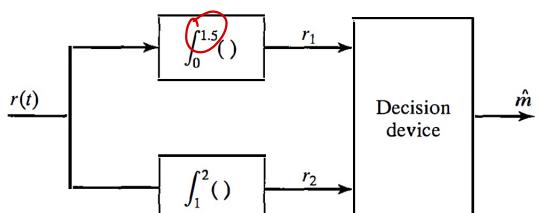


Figure P-8.39

(8-29). The $\vec{r} = [r_1, r_2]$ at the output of the integrators is: $\vec{r} = [r_1, r_2] = [\int_0^{1.5} r_1(t)dt, \int_1^2 r_2(t)dt]$.

If $s_1(t)$ is transmitted, then:

$$\int_0^{1.5} r_1(t)dt = \int_0^{1.5} [s_1(t) + n_1(t)]dt = 1 + \int_0^{1.5} n_1(t)dt = 1 + n_1$$

$$\int_1^2 r_2(t)dt = \int_1^2 [s_1(t) + n_2(t)]dt = \int_1^2 n_2(t)dt = n_2.$$

n_1 is a zero-mean Gaussian random variable with variance:

$$\sigma_{n_1}^2 = E\left[\int_0^{1.5} \int_0^{1.5} n_1(\tau) n_1(\nu) d\tau d\nu\right] = \frac{N_0}{2} \int_0^{1.5} d\tau = 1.5$$

n_2 is a zero-mean Gaussian random variable with variance:

$$\sigma_{n_2}^2 = E\left[\int_1^2 \int_1^2 n_2(\tau) n_2(\nu) d\tau d\nu\right] = \frac{N_0}{2} \int_1^2 d\tau = 1.$$

then: $\vec{r} = [1 + n_1, n_2]$.

if $s_2(t)$ is transmitted, then $\vec{r} = [0.5 + n_1, 1 + n_2]$.

the detector bases its decisions on the rule: $r_1 - r_2 \begin{cases} > T \\ \leq T \end{cases}$

The probability of error $P(e|s_1)$ is:

$$P(e|s_1) = P(r_1 - r_2 < T | s_1) = P(1 + n_1 - n_2 < T) = P(n_1 - n_2 < T - 1) = P(n < T - 1)$$

$n = n_1 - n_2$ is a zero-mean Gaussian with variance:

$$\sigma_n^2 = \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2E[n_1 n_2] = \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2 \int_1^{1.5} \frac{N_0}{2} d\tau = 1.5 + 1 - 2 \times 0.5 = 1.5$$

then: $P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx$.

similarly: $P(e|s_2) = P(n > T + 0.5) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx$

The average probability of error is:

$$P(e) = \frac{1}{2} P(e|s_1) + \frac{1}{2} P(e|s_2) = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx$$

$$\Rightarrow \frac{\partial P(e)}{\partial T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \left[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}} \right] = 0 \Rightarrow (T-1)^2 = (T+0.5)^2 \Rightarrow T = 0.25$$

Then the optimal rule: $r_1 - r_2 \begin{cases} > T \\ \leq T \end{cases}$

$$\text{signal set} = S_m(t) = \sqrt{\frac{2E_s}{T}} \cos [2\pi(f_c + (m-1)\Delta f)t].$$

- 9.10 Consider the phase-coherent demodulator for M -ary FSK signals as shown in Figure 9.11.

1. Assume that the signal

$$u_0(t) = \sqrt{\frac{2E_s}{T}} \cos 2\pi f_c t, \quad 0 \leq t \leq T$$

was transmitted; determine the output of the $M - 1$ correlators at $t = T$ that corresponds to the signals $u_m(t)$, $m = 1, 2, \dots, M - 1$, when $\hat{\phi}_m \neq \phi_m$.

2. Show that the minimum frequency separation required for the signal orthogonality at the demodulator when $\hat{\phi}_m \neq \phi_m$ is $\Delta f = \frac{1}{T}$.

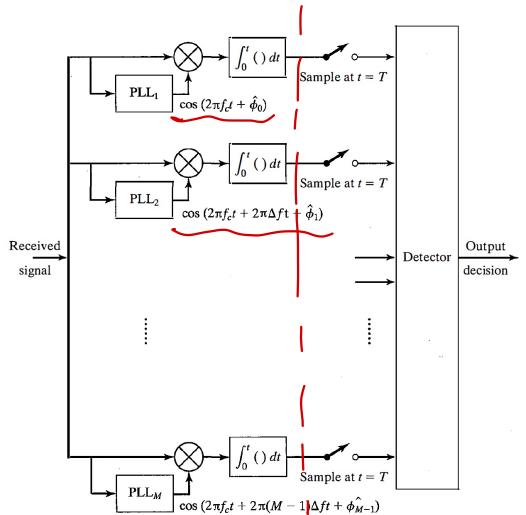


Figure 9.11 Phase-coherent demodulation of M -ary FSK signals.

(Q10).

(1) If the transmitted signal is $u(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t)$, $0 \leq t \leq T$.

then the received signal is $r(t) = \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + \phi) + n(t)$.

$E[r_m] = 0$
for signal orthogonality

In the phase-coherent demodulation of M -ary FSK signals, the output of the m th correlator is:

$$\begin{aligned} r_m &= \int_0^T r(t) \cos(2\pi f_c t + 2\pi m \Delta f t + \hat{\phi}_m) dt = \int_0^T \sqrt{\frac{2E_s}{T}} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t + 2\pi m \Delta f t + \hat{\phi}_m) dt \\ &\quad + \int_0^T n(t) \cos(2\pi f_c t + 2\pi m \Delta f t + \hat{\phi}_m) dt \\ &= \sqrt{\frac{2E_s}{T}} \frac{1}{2} \int_0^T \cos(2\pi m \Delta f t + \hat{\phi}_m - \phi) dt + n. \end{aligned}$$

n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

(2) the value of r_m , $E[r_m]$ should be zero. and since $E[n]=0$:

$$\int_0^T \cos(2\pi m \Delta f t + \hat{\phi}_m - \phi) dt = 0, \quad m \neq 0$$

the necessary condition for orthogonality is $m=1, \Delta f = \frac{1}{T}$.

本题目中的 $E[r_m]$ 表达式是这样的。

若是:

$$\gamma(t) \xrightarrow[\oplus]{\hat{\phi}_1(t)} \quad \hat{\phi}_m(t) = \sqrt{\frac{2}{T}} \cos[2\pi(f_c + m\Delta f)t].$$

$$\xrightarrow[\oplus]{\hat{\phi}_2(t)} \quad \therefore P_{mn} = \frac{1}{E_s} \int_0^T S_m(t) S_n(t) dt = \frac{\sin[2\pi(m-n)\Delta f T]}{2\pi(m-n)\Delta f T}$$

$$\Rightarrow \Delta f_{\min} = \frac{1}{2T}$$

$$\frac{1}{2\pi m \Delta f} \sin(2\pi m \Delta f t + \hat{\phi}_m - \hat{\phi}) \Big|_0^T$$

$$= \frac{1}{2\pi m \Delta f} [\sin(2\pi m \Delta f T + \hat{\phi}_m - \hat{\phi}) - \sin(\hat{\phi}_m - \hat{\phi})] = 0$$

$$\sin(2\pi m \Delta f T) \cos(\hat{\phi}_m - \hat{\phi}) + \sin(\hat{\phi}_m - \hat{\phi}) [\cos(2\pi m \Delta f T) - 1]$$

$$\text{for } m=1, \sin(2\pi \Delta f T) = 0, \cos(2\pi \Delta f T) = 1$$

$$\Delta f = 0, \frac{1}{2T}, \frac{1}{T}, \dots \quad \Delta f = 0, \frac{1}{T}, \frac{2}{T}, \dots$$

$$\Rightarrow \Delta f = \frac{1}{T}$$

$$\text{不是 } \frac{2E_s}{T_b}.$$

9.12 In on-off keying of a carrier modulated signal, the two possible signals are

$$s_0(t) = 0, \quad 0 \leq t \leq T_b$$

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t), \quad 0 \leq t \leq T_b$$

The corresponding received signals are

$$r(t) = n(t), \quad 0 \leq t \leq T_b$$

$$r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t), \quad 0 \leq t \leq T_b$$

where ϕ is the carrier phase and $n(t)$ is AWGN.

- Sketch a block diagram of the receiver (demodulator and detector) that employs noncoherent (envelope) detection.
- Determine the probability density functions for the two possible decision variables at the detector corresponding to the two possible received signals.
- Derive the probability of error for the detector.

noncoherent BPSK

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$$

$$\phi_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t).$$

$$\vec{r} = [r_c \ r_s] = [r_1 \ r_2]$$

$$S_0 = [0 \ 0]$$

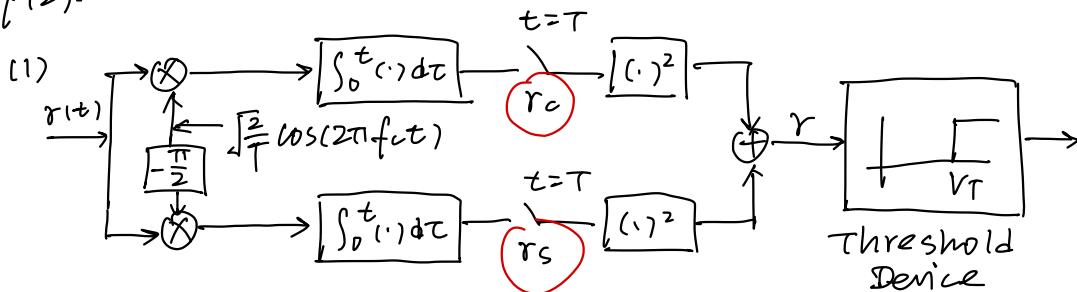
$$S_1 = [\sqrt{E_b} \ 0]$$

$$\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi)$$

$$= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \cos(\phi) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_c t) \sin(\phi)$$

$$\Rightarrow [\sqrt{E_b} \cos \phi, \sqrt{E_b} \sin \phi]$$

(9-12)-



(2) If $s_1(t)$ is transmitted, then the received signal is:

$$r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t).$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling at $t=T$:

$$r_c = \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt = \sqrt{E_b} \cos(\phi) + n_c.$$

n_c is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

Similarly, for the quadrature component: $r_s = \sqrt{E_b} \sin(\phi) + n_s$.

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{E_b + n_c^2 + n_s^2}$ is:

$$P(r|S_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+E_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{E_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+E_b}{N_0}} I_0\left(\frac{2r\sqrt{E_b}}{N_0}\right).$$

(ii) For equiprobable signals:

$$\text{P(error)} = \frac{1}{2} \int_{-\infty}^{V_T} P(r|S_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} P(r|S_0(t)) dr$$

$$= \frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+E_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{E_b}}{\sigma^2}\right) dr + \frac{1}{2} \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2+E_b}{2\sigma^2}} dr$$

~ ~ ~

- 9.13 Digital information is to be transmitted by carrier modulation through an additive Gaussian noise channel with a bandwidth of 100 kHz and $N_0 = 10^{-10}$ W/Hz. Determine the maximum rate that can be transmitted through the channel for four-phase PSK, binary FSK, and four-frequency orthogonal FSK that is detected noncoherently.

(a)

Four phase PSK

If uses a pulse shape having a raised cosine spectrum with a rolloff α , then = symbol rate $\frac{1}{T}$ $\frac{1}{2T} (1+\alpha) = 50000 \Rightarrow \frac{1}{T} = \frac{10^5}{1+\alpha} \Rightarrow$ symbol rate
 $\text{the bit rate} = \frac{2}{T} = \frac{2 \times 10^5}{1+\alpha} \text{ bps.}$

(b) Binary FSK with noncoherent detection:

Select the two frequencies to have a frequency separation of $\frac{1}{T}$. $\frac{1}{T}$ is symbol rate.
 $f_1 = f_c - \frac{1}{2T}, f_2 = f_c + \frac{1}{2T}$.

f_c is the carrier in the center of the channel band, then $\frac{1}{2T} = 50000$.
then $\frac{1}{T} = 10^5$. the bit rate is 10^5 bps \rightarrow bit rate = symbol rate

(c) M=4 FSK with noncoherent detection. four frequencies with adjacent frequencies separation $\frac{1}{T}$

$$\text{Select: } f_1 = f_c - \frac{1}{T}, f_2 = f_c - \frac{1}{2T}, f_3 = f_c + \frac{1}{2T}, f_4 = f_c + \frac{1}{T}$$

f_c is the carrier frequency and $\frac{1}{2T} = 25000$. and $\frac{1}{T} = 50000$.
the bit rate $\frac{2}{T} = 10^5$ bps.

$$\text{For MFSK: } B = M \Delta f = \frac{M}{2T} \text{ Hz.} \quad \text{BFSK: } \frac{1}{2T} = \frac{B}{M} = 50000 \text{ Hz.}$$

$$4FSK: \frac{1}{2T} = \frac{B}{M} = 25000 \text{ Hz.}$$

$$\text{For MPSK: } B = \frac{1+\alpha}{T_s} \text{ or } \frac{2}{T_s}$$

9.15 A continuous-phase FSK signal with $h = 1/2$ is represented as

$$s(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right) \cos 2\pi f_c t \pm \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right) \sin 2\pi f_c t, \Rightarrow s(t) = \pm \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t \pm \frac{\pi}{2T_b} t), 0 \leq t \leq 2T_b.$$

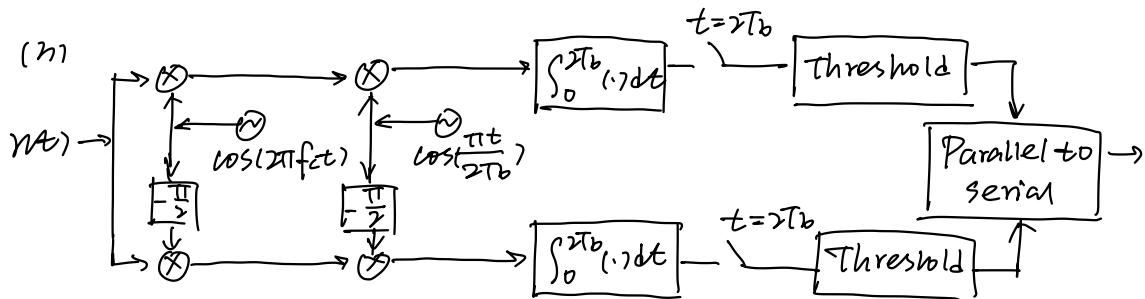
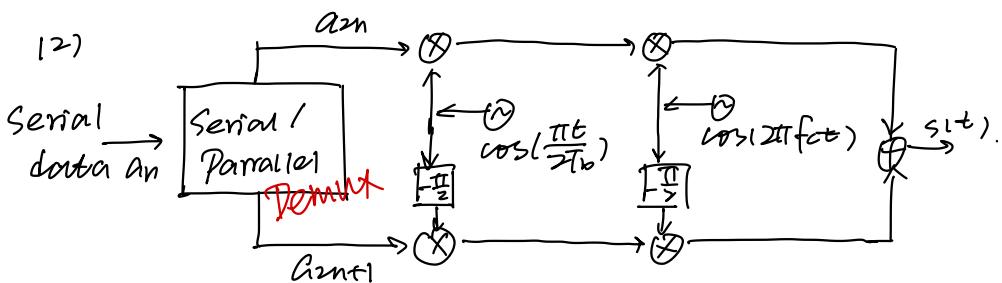
where the \pm signs depend on the information bits transmitted.

1. Show that this signal has a constant amplitude.
2. Sketch a block diagram of the modulator for synthesizing the signal.
3. Sketch a block diagram of the demodulator and detector for recovering the information.

(9.15)

(1) the envelope of the signal is =

$$|s(t)| = \sqrt{|s_{c1}(t)|^2 + |s_{c2}(t)|^2} = \sqrt{\frac{2E_b}{T_b}}. \Rightarrow \text{the signal has constant amplitude.}$$



Chapter 9 & 10

- 12.5** Let $Y = g(X)$, where g denotes a deterministic function. Show that, in general, $H(Y) \leq H(X)$. When does equality hold?

$$H(X, Y) = H(X, g(X)) = H(X) + H(g(X)|X) = H(g(X)) + H(X|g(X)).$$

Since $g(\cdot)$ is deterministic, so $H(g(X)|X) = 0$.

$$\text{Then } H(X) = H(g(X)) + H(X|g(X)) \geq H(g(X)).$$

Equality holds when $H(X|g(X)) = 0 \Rightarrow$ the values of $g(X)$ uniquely determine X .
or $g(\cdot)$ is a one-to-one mapping.

条件熵 $H(Y|X)$ 可理解为已知随机变量 X 的条件下随机变量 Y 的不确定性

$$H(Y|X) = \sum_x p(x) H(Y|X=x) = - \sum_x p(x) \sum_y p(y|x) \log p(y|x). = - \sum_{x,y} p(x,y) \log p(y|x).$$

(3)

- 12.6** An information source can be modeled as a bandlimited process with a bandwidth of 6000 Hz. This process is sampled at a rate higher than the Nyquist rate to provide a guard band of 2000 Hz. We observe that the resulting samples take values in the set $\mathcal{A} = \{-4, -3, -1, 2, 4, 7\}$ with probabilities 0.2, 0.1, 0.15, 0.05, 0.3, 0.2. What is the entropy of the discrete time source in bits per output (sample)? What is the information generated by this source in bits per second?

$$W = 6000 \text{ Hz}, f = f_s + f_g = 2W + f_g = 2 \times 6000 + 2000 = 14000 \text{ Hz}.$$

$$\text{entropy} = H = -0.2 \log_2 0.2 - 0.1 \log_2 0.1 - 0.15 \log_2 0.15 - 0.05 \log_2 0.05 - 0.3 \log_2 0.3 - 0.2 \log_2 0.2 = 2.41$$

$$\text{information} = 2.41 \text{ bits/sample} \times 14000 \text{ samples/sec} = 33721.7 \text{ bits/sec}$$

- 12.13** Let X and Y denote two jointly distributed discrete valued random variables.

1. Show that

$$H(X) = - \underbrace{\sum_{x,y} p(x,y)}_{x,y} \log p(x)$$

and

$$H(Y) = - \underbrace{\sum_{x,y} p(x,y)}_{x,y} \log p(y).$$

2. Use this result to show that

$$H(X, Y) \leq H(X) + H(Y).$$

When does the equality hold? Hint: Consider the two distributions $p(x, y)$ and $p(x)p(y)$ on the product set $\mathcal{X} \times \mathcal{Y}$, and apply the inequality proved in Problem 12.7 to $\sum_{x,y} p(x, y) \log \frac{p(x,y)}{p(x)p(y)}$.

- 11) The marginal distribution $p(x) = \sum_y p(x,y)$.

$$\Rightarrow H(X) = - \sum_x p(x) \log p(x) = - \sum_x \sum_y p(x,y) \log p(x) = - \sum_{x,y} p(x,y) \log p(x),$$

$$\text{Similarly, } H(Y) = - \sum_{x,y} p(x,y) \log p(y).$$

(2) Using the $\ln w \leq w-1$ with $w = \frac{p(x)p(y)}{p(x,y)}$:

$$\ln \frac{p(x)p(y)}{p(x,y)} \leq \frac{p(x)p(y)}{p(x,y)} - 1.$$

$$\Rightarrow \sum_{x,y} p(x,y) \ln p(x)p(y) - \sum_{x,y} p(x,y) \ln p(x,y) \leq \sum_{x,y} p(x)p(y) - \sum_{x,y} p(x,y) = 0.$$

$$\Rightarrow H(X,Y) \leq -\sum_{x,y} p(x,y) \ln p(x)p(y) = -\sum_{x,y} p(x,y) (\ln p(x) + \ln p(y))$$

$$= -\sum_{x,y} p(x,y) \ln p(x) - \sum_{x,y} p(x,y) \ln p(y) = H(X) + H(Y).$$

12.14 Use the result of Problem 12.13 to show that

$$H(X|Y) \leq H(X)$$

with equality if and only if X and Y are independent.

$$H(X|Y) = H(X|Y) + H(Y) \leq H(X) + H(Y) \Rightarrow H(X|Y) \leq H(X).$$

with equality if $H(X|Y) = H(X) + H(Y) = H(X|Y) + H(Y) \Rightarrow H(X|Y) = H(X)$.

$$H(X|Y) = -\sum_x p(x) \log p(x|y), \quad H(X) = -\sum_x p(x) \log p(x) \Rightarrow \sum_x p(x) \log \left(\frac{p(y)}{p(x|y)} \right) = 0.$$

Obviously, only $p(x) = p(x|y)$ satisfies, then x, y are independent.

X

12.36 The channel shown in Figure P-12.36 is known as the *binary erasure channel*. Find the capacity of this channel and plot it as a function of ϵ .

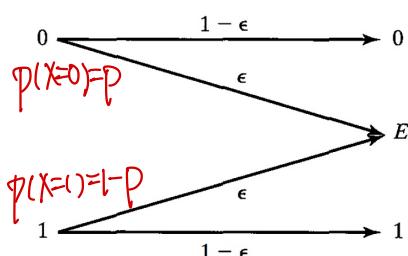


Figure P-12.36

$$C = \max_{p(x)} I(X; Y) = \max_{p(x)} [H(Y) - H(Y|X)].$$

$$\text{the probability distribution } p(x) = p(x) = \begin{cases} P, & x=0 \\ 1-P, & x=1. \end{cases}$$

$$\text{then } H(Y|X) = pH(Y|X=0) + (1-p)H(Y|X=1) = p h(\epsilon) + (1-p) h(\epsilon) = h(\epsilon)$$

where $h(\epsilon)$ is the binary entropy function $h(\epsilon) = -\epsilon \log \epsilon - (1-\epsilon) \log (1-\epsilon)$.

$\Rightarrow I(X; Y)$ is maximized when $H(Y)$ is maximized.

let V be a function of the output defined as: $V = \begin{cases} 1 & Y=E \\ 0 & \text{otherwise} \end{cases}$

clearly $H(V|Y)=0$ since Y is a deterministic function of V , then

$$H(Y,V) = H(Y) + H(V|Y) = H(Y) = H(V) + H(Y|V).$$

Because $P(V=1) = P(Y=E) = p\epsilon + (1-p)\bar{\epsilon} = \epsilon$.

$$P(V=0) = P(Y=0) + P(Y=1) = P(\bar{\epsilon}) + (1-p)(\bar{\epsilon}) = 1-\epsilon$$

$$\Rightarrow H(V) = -\epsilon \log \epsilon - (1-\epsilon) \log(1-\epsilon) = h(\epsilon).$$

$$H(Y|V) = P(V=0)H(Y|V=0) + P(V=1)H(Y|V=1).$$

And $H(Y|V=1)=0$ since no ambiguity on the output when $V=1$.

$$\begin{aligned} H(Y|V=0) &= -\sum_y P(Y|V=0) \log P(Y|V=0) \\ &= -P(Y=0|V=0) \log P(Y=0|V=0) - P(Y=1|V=0) \log P(Y=1|V=0) \end{aligned}$$

$$\text{we have } P(Y=0|V=0) = \frac{P(Y=0, V=0)}{P(V=0)} = \frac{P(\bar{\epsilon})}{1-\epsilon} = p.$$

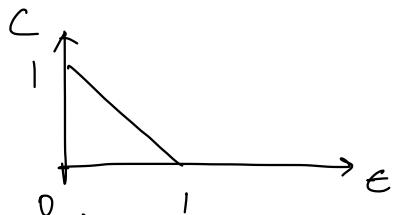
$$P(Y=1|V=0) = \frac{P(Y=1, V=0)}{P(V=0)} = \frac{(1-p)\bar{\epsilon}}{1-\epsilon} = 1-p.$$

$$\text{then } H(Y|V=0) = -p \log p - (1-p) \log(1-p) = h(p).$$

$$H(Y|V) = P(V=0)H(Y|V=0) = (1-\epsilon)h(p).$$

$$\Rightarrow C = \max_{P(X)} [H(V) + H(Y|V) - h(\epsilon)] = \max_{P(X)} [H(Y|V)] = \max_{P(X)} (1-\epsilon)h(p) = 1-\epsilon.$$

$$\text{for } P(X) = P = \frac{1}{2}, \quad h(p) = 1.$$



(12-4b).*

The capacity of non-white Gaussian noise channel is higher.

Because the noise samples are correlated, knowledge of previous noise samples provides partial information about the future noise samples and then reduces the effective variance.

(13-5)*

Parity-check matrix: $\vec{H} = [\vec{h}_1 \ \vec{h}_2 \ \dots \ \vec{h}_n]$. \vec{h}_i is an $(n-k)$ dimensional column vec.
 $\vec{c} = [c_1 \dots c_n]$ is a codeword of code C with ℓ non-zero elements $c_{i_1}, c_{i_2}, \dots, c_{i_\ell}$.
 $\Rightarrow \vec{c}^T \vec{H}^T = 0 = c_1 \vec{h}_1 + c_2 \vec{h}_2 + \dots + c_n \vec{h}_n = c_{i_1} \vec{h}_{i_1} + c_{i_2} \vec{h}_{i_2} + \dots + c_{i_\ell} \vec{h}_{i_\ell}$
 $= \vec{h}_{i_1} + \vec{h}_{i_2} + \dots + \vec{h}_{i_\ell} = 0$.

So the ℓ column vectors of matrix \vec{H} are linear dependent
there exists d_{\min} linear dependent column vectors of matrix \vec{H} .

Now assume that the minimum number of column vectors of the matrix \vec{H} that are linear dependent is d_{\min} . let $\vec{h}_{i_1}, \vec{h}_{i_2}, \dots, \vec{h}_{d_{\min}}$ be a set of linear dependent column vectors. If we form a vector \vec{c} with non-zero components at positions $i_1, i_2, \dots, i_{d_{\min}}$. then: $\vec{c}^T \vec{H}^T = c_{i_1} \vec{h}_{i_1} + \dots + c_{i_{d_{\min}}} \vec{h}_{i_{d_{\min}}} = 0$.

which implies that \vec{c} is a codeword with weight d_{\min} . So the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.

For a Hamming code the columns of the matrix \vec{H} are non-zero and distinct. No two columns \vec{h}_i, \vec{h}_j add to zero and since \vec{H} consists of all the $n-k$ tuples as its columns, the sum $\vec{h}_i + \vec{h}_j = \vec{h}_m$ should also be a column of \vec{H} . then:

$\vec{h}_i + \vec{h}_j + \vec{h}_m = 0 \Rightarrow$ the minimum distance of hamming code is 3.

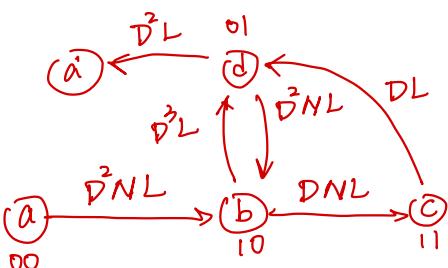
(13-9)* For (7,4) Hamming code =

Standard array =

(0,0,0,0,0,0,0)	(0,0,0,1,1,0,1)	(0,0,1,0,1,1,1)	(0,0,1,1,0,1,0)	(0,1,0,0,0,1,1)	(0,1,0,1,1,1,0)	(0,1,1,0,1,0,0)	(0,1,1,1,0,1,0)	(0,1,0,0,1,0,1)	(1,0,0,0,1,1,0)	(1,0,0,1,0,1,1)	(1,0,1,0,0,0,1)	(1,0,1,1,1,0,0)	(1,1,0,0,1,0,1)	(1,1,0,1,0,0,0)	(1,1,1,0,0,1,0)	(1,1,1,1,1,1,1)	
e_1	(1,0,0,0,0,0,0)	(1,0,0,1,1,0,1)	(1,0,1,0,1,1,1)	(1,0,1,1,0,1,0)	(1,1,0,0,0,1,1)	(1,1,0,1,1,1,0)	(1,1,1,0,1,0,0)	(1,1,1,1,0,1,0)	(0,0,0,0,1,1,0)	(0,0,0,1,0,1,1)	(0,0,1,0,0,0,1)	(0,0,1,1,0,0,0)	(0,0,1,1,0,1,0)	(0,1,0,0,1,0,1)	(0,1,0,1,0,0,0)	(0,1,1,0,0,1,0)	(0,1,1,1,1,1,1)
e_2	(0,1,0,0,0,0,0)	(0,1,0,1,1,0,1)	(0,1,1,0,1,1,1)	(0,1,1,1,0,1,0)	(0,0,0,0,1,1,0)	(0,0,0,1,1,1,0)	(0,0,1,0,1,0,0)	(0,0,1,1,0,1,0)	(0,1,0,0,1,1,0)	(0,1,0,1,0,1,1)	(0,1,1,0,0,0,1)	(0,1,1,1,0,0,0)	(1,0,0,0,1,0,1)	(1,0,0,1,0,0,0)	(1,0,1,0,0,1,0)	(1,0,1,1,1,1,1)	
e_3	(0,0,1,0,0,0,0)	(0,0,1,1,1,0,1)	(0,0,0,0,1,1,1)	(0,0,0,1,0,1,0)	(0,0,1,0,1,1,0)	(0,1,0,0,1,1,0)	(0,1,1,0,1,1,0)	(0,1,1,1,0,1,0)	(0,1,0,0,1,0,0)	(0,1,0,1,0,0,1)	(0,1,0,1,1,0,1)	(0,1,1,0,0,0,1)	(0,1,1,1,0,1,0)	(0,1,0,0,1,0,0)	(0,1,0,1,0,0,0)	(0,1,1,0,1,1,0)	(0,1,1,1,1,1,1)
e_4	(0,0,0,1,0,0,0)	(0,0,0,0,1,0,1)	(0,0,0,1,1,1,0)	(0,0,1,0,0,1,1)	(0,1,0,0,1,1,0)	(0,1,0,1,0,0,0)	(0,1,1,0,0,0,0)	(0,1,1,1,0,0,0)	(0,1,0,0,0,1,1)	(0,1,0,0,1,1,1)	(0,1,0,1,0,0,1)	(0,1,0,1,1,0,1)	(0,1,1,0,0,1,0)	(0,1,1,1,0,0,0)	(0,1,1,1,0,1,0)	(0,1,1,0,1,1,0)	(0,1,0,1,1,1,1)
e_5	(0,0,0,0,1,0,0)	(0,0,0,1,0,1,0)	(0,0,1,0,0,1,1)	(0,1,0,0,1,1,0)	(0,1,0,1,0,0,0)	(0,1,1,0,0,1,0)	(0,1,1,1,0,0,0)	(0,1,0,0,0,0,0)	(0,1,0,1,1,0,1)	(0,1,0,1,1,1,0)	(0,1,0,1,0,0,1)	(0,1,0,1,1,0,0)	(0,1,1,0,1,0,1)	(0,1,1,0,0,0,0)	(0,1,1,1,0,1,0)	(0,1,1,0,1,1,0)	(0,1,1,1,0,1,1)
e_6	(0,0,0,0,0,1,0)	(0,0,0,1,1,1,1)	(0,0,1,0,1,0,1)	(0,0,1,1,1,0,0)	(0,1,0,0,0,1,1)	(0,1,0,1,1,0,0)	(0,1,1,0,1,0,0)	(0,1,1,1,0,1,0)	(0,1,0,0,1,1,0)	(0,1,0,1,1,1,0)	(0,1,0,1,0,0,1)	(0,1,0,1,1,0,1)	(0,1,1,0,1,1,0)	(0,1,1,0,0,1,0)	(0,1,1,1,0,0,0)	(0,1,1,1,1,0,1)	(0,1,1,1,1,1,1)
e_7	(0,0,0,0,0,0,1)	(0,0,0,1,1,0,0)	(0,0,1,0,1,1,0)	(0,1,0,0,1,1,1)	(0,1,0,1,0,1,0)	(0,1,1,0,1,1,1)	(0,1,1,1,0,1,1)	(0,1,0,0,1,1,1)	(0,1,0,1,1,1,1)	(0,1,0,1,0,1,0)	(0,1,0,1,1,0,0)	(0,1,1,0,1,1,1)	(0,1,1,1,0,1,0)	(0,1,1,0,0,1,1)	(0,1,1,1,0,0,1)	(0,1,1,1,1,1,1)	(0,1,1,1,1,1,1)

$$\vec{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \cdot \vec{H}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{r} = [1, 1, 1, 0, 1, 0, 0] \Rightarrow \vec{s} = \vec{r} \vec{H}^T = [1, 1, 0]$$

$\vec{e}_1 \vec{H}^T = [1, 1, 1, 0] = \vec{s}$. Therefore $\vec{c} = \vec{r} + \vec{e}_1 = [0, 1, 1, 0, 1, 0, 0]$



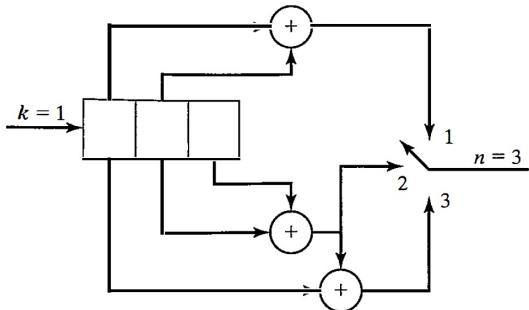
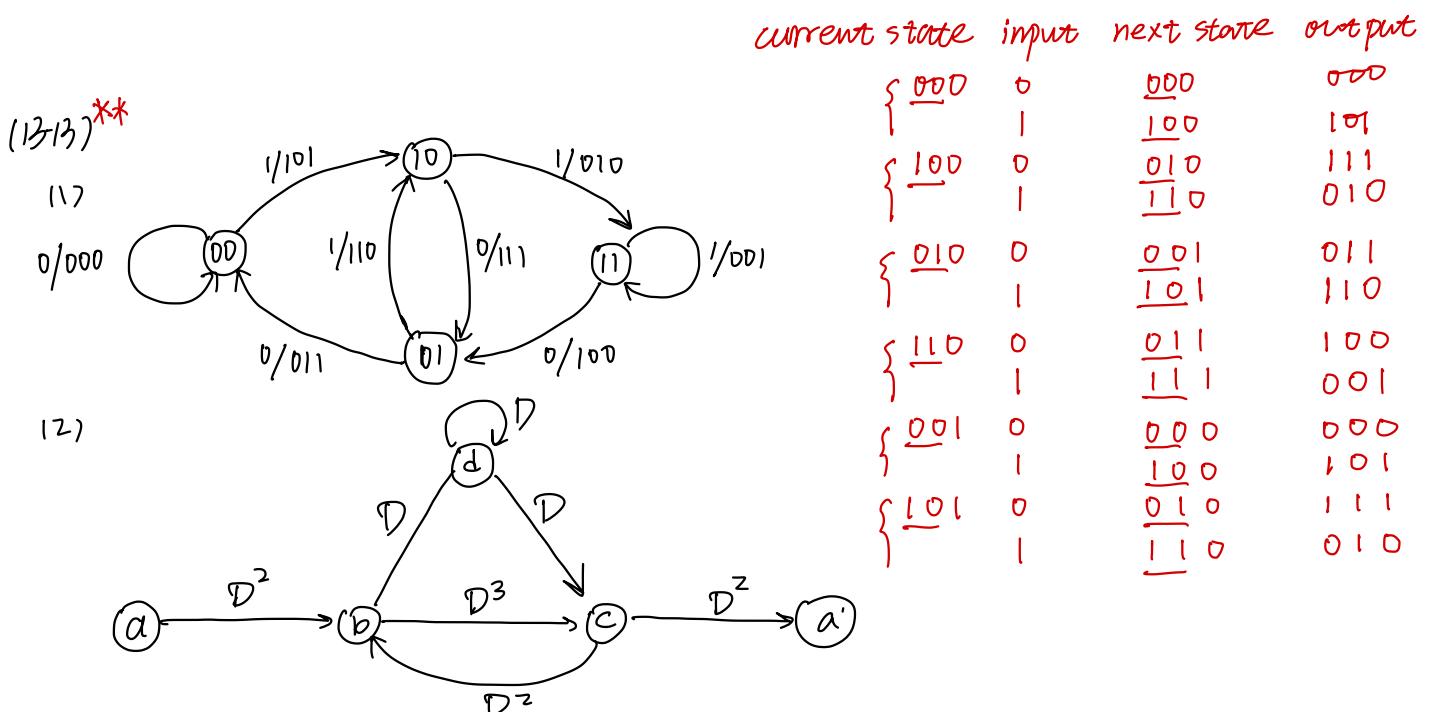


Figure P-13.13

1. Draw the state diagram for the code.
2. Find $T(D)$, the transfer function of the code.
3. What is d_{free} , the minimum free distance of the code?
4. Assume that a message has been encoded by this code and transmitted over a binary symmetric channel with an error probability of $p = 10^{-5}$. If the received sequence is $r = (110, 110, 110, 111, 010, 101, 101)$, use the Viterbi algorithm to find the transmitted bit sequence.
5. Find an upper bound to the bit error probability of the code when the preceding binary symmetric channel is employed. Make any reasonable approximations.



$$\Rightarrow \begin{cases} X_b = D^2 X_a + D^2 X_c \\ X_c = D^3 X_b + D X_d \\ X_d = D X_d + D X_b \\ X_a' = D^2 X_c \end{cases} \Rightarrow T(D) = \frac{X_a'}{X_a} = \frac{D^6 + D^7 - D^8}{1 - D - D^4 - D^5 + D^6}$$

(13) From the $T(D) = D^6 + 2D^7 + D^8 + \dots \Rightarrow d_{\text{free}} = 6$.

(14)

(15) $\overline{P}_b \leq \frac{1}{k} \cdot \left. \frac{\partial T_2(D, N)}{\partial N} \right|_{N=1, D=\sqrt{4p(1-p)}} = \frac{1}{3} (4 \times 10^{-5} (1/10^{-5}))^3 \approx 2 \times 10^{-14}$



Example 13.3.5

Assume that, in hard-decision decoding, the quantized received sequence is

$$\hat{c} = (01101111010001).$$

The convolutional code is given in Figure 13.10. Find the maximum likelihood information sequence and the number of errors.

Solution The code is a $(2, 1)$ code with $L = 3$. The length of the received sequence \hat{c} is 14. This means that $m = 7$ and we have to draw a trellis of depth 7. Also note that because the input information sequence is padded with $k(L - 1) = 2$ zeros, for the final two stages of the trellis, we will only draw the branches corresponding to all-zero inputs. This also means that the actual length of the input sequence is 5, which, after padding with two zeros, has increased to 7. The trellis diagram for this case is shown in Figure 13.18. The parsed received sequence \hat{c} is also shown in this figure. In drawing the trellis in the last two stages, we have considered only the zero inputs to the encoder. In the final two stages, there are no dashed lines corresponding to 1 inputs. Now the metric of the initial all-zero state is set to zero and the metrics of the next stage are computed. In this step, there is only one branch entering each state; therefore, there is no comparison, and the metrics, which are the Hamming distances between that part of the received sequence and the branches of the trellis, are added to the metric of the previous state. In the next stage, there is no comparison either. In the third stage, we actually have two branches entering each state. This means that a comparison has to be made, and survivors are to be chosen. From the two branches that enter each state, the one that corresponds to the least total accumulated metric remains as a survivor and the other branches are deleted (marked with $\cancel{\cdot}$ on the graph). If, at any stage, two paths result in the same metric, each one of them can be a survivor. Such cases have been marked by a "?" in the trellis diagram. The procedure is continued to the final all-zero state of the trellis. Starting from that state, we move along the surviving paths to the initial all-zero state. This path, which

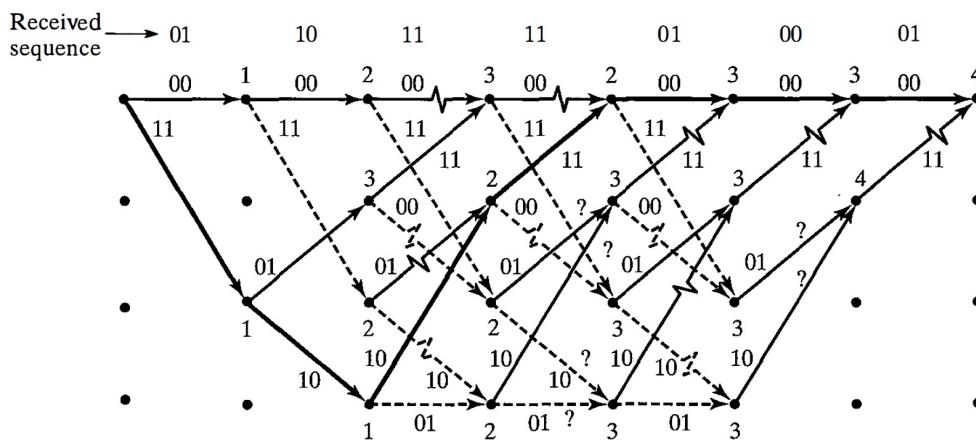


Figure 13.18 The trellis diagram for the Viterbi decoding of the sequence (01101111010001).

is denoted by a heavy path through the trellis, is the optimal path. The input bit sequence corresponding to this path is 1100000 (where the last two zeros are not information bits, but are added to return the encoder to the all-zero state). Therefore, the information sequence is 11000. The corresponding code word for the selected path is 11101011000000, which is at Hamming distance 4 from the received sequence. No other path through the trellis is at a Hamming distance less than 4 from the received \hat{c} .

For soft-decision decoding, a similar procedure is followed with squared Euclidean distance substituted for Hamming distance.