Ch02. Signals and Random Process.

2.1 Review of signals

1. Classification of Signals.

(1) Continous-time signal (2) Discrete-time signal.

(3) Continous-valued signal (4) Discrete-valued signal.

$\begin{cases} \text{time: x-axis} \\ \text{value: y-axis} \end{cases}$

Analog signal: (1)(3). Digital signal: (2)(4).

Sampled signal: (2)(3). Quantized signal: (1)(4). (P6)

2. Signal energy and power.

given a signal $x(t)$:

(1) Energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt.$

Energy signal: E_x is finite.

(2) Power: $P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$

Power signal: P_x is finite.

Some signals are neither energy nor power signals. (P_x and E_x are infinite).

3. Some important signals.

The Rectangular Pulse: $\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0. & \text{otherwise.} \end{cases}$

The Triangular Pulse: $\Lambda(t) = \begin{cases} t+1, & -1 \le t < 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$

$\Lambda(t) = \Pi(t) * \Pi(t)$

The Sgn Signal: $\text{Sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0. \end{cases}$

The Sinc Signal: $\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0. \end{cases}$

4. Fourier Series and Fourier Transform.

(1) A Periodic signal $x(t)$ with period T_0. Fourier series expansion:

$$x(t)=\sum_{n=-\infty}^{+\infty} x_n e^{j2\pi\frac{n}{T_0}t}. \qquad x_n=\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt.$$ ⇒ 连续周期信号 $w_0=\frac{2\pi}{T_0}$.

(2) A non-periodic signal. Fourier transform:

$$X(f)=\int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt. \qquad x(t)=\int_{-\infty}^{+\infty} X(t)e^{j2\pi ft}dt.$$ ⇒ 连续非周期信号

2.2 Review of probability and random variables.

(1) Random Variables (r.v.)

{ Discrete-valued (range is finite or countable infinite)
  Continous-valued (range is uncountable infinite).

The Cumulative Distribution Function (CDF): $F_X(x)=P(X\le x)$.

The Probability Density Function $\underbrace{\text{(PDF)}}_{C}$: $f_X(x)=\frac{d}{dx}F_X(x)$. $F_X(x)=\int_{-\infty}^{x} f_X(y)dy$.

The Probability Mass Function $\underbrace{\text{(PMF)}}_{D}$ with discrete r.v.

(2) Some distribution: Bernoulli Distribution, Binomial Distribution, Uniform Distribution.
Gaussian (Normal) Distribution.

(3) Statistic Average:

The mean or expected value of X: $m_X=E(X)=\sum_{i=1}^{M} x_i P_i$. / $\int_{-\infty}^{\infty} x f_X(x)dx$

(First moment). moment：矩

The n^{th} moment of X: $E(X^n)=\int_{-\infty}^{+\infty} x^n p_X(x)dx$.

The mean-square value $(n=2)$: $E(X^2)=\int_{-\infty}^{+\infty} x^2 p_X(x)dx$

The n^{th} central moment of X: $E[(X-m_X)^n]=\int_{-\infty}^{+\infty}(x-m_X)^n f_X(x)dx$.

↳ $n=1$: standard deviation. $n=2$: variance.

↱ Gaussian tail probability. $Q(x)=1-F_X(x)$

(4) The Q Function: $Q(x)=\int_{x}^{\infty}\frac{1}{\sqrt{2\pi}}\overset{\Phi_X(x)}{\overset{\Uparrow}{\exp}}\left(-\frac{u^2}{2}\right)du$. (Error probability analysis). The values of $Q(x)$ are fixed.

Some features: $Q(-\infty)=1$. $Q(0)=\frac{1}{2}$. $Q(\infty)=0$. $Q(-x)=1-Q(x)$.

Given $X\sim N(\mu,\sigma^2)$. ~~then~~ { $Pr(X>x)=Q\left(\frac{x-\mu}{\sigma}\right)$.
$Pr(X\le x)=\Phi\left(\frac{x-\mu}{\sigma}\right)$ $Q_X(x)=1-\Phi_X(x)$.

(5) Correlation. { $R_{XY}=E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy$.

$\sigma_{XY}=E(XY)-E(X)E(Y)=E[(X-E(X))(Y-E(Y))] \Rightarrow \rho_{XY}=\frac{cov(X,Y)}{\sqrt{D(X)}\cdot\sqrt{D(Y)}}=\frac{\sigma_{XY}}{\sigma_X\sigma_Y}\in[-1,1]$.

ρ_{XY} is a measure of the strength/direction of the linear relationship between X, Y.

Independent ⇒ Uncorrelated, the converse is not true (except GD).

(6) Joint Distribution: $F_{XY}(x,y) = P(X \le x, Y \le y)$ $\quad f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$.

Marginal: $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,\beta)\,d\beta$. $\quad F_X(x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{x} f_{XY}(\alpha,\beta)\,d\alpha\,d\beta$.

X, Y are independent if $F_{XY}(x,y) = F_X(x)F_Y(y)$. $f_{XY}(x,y) = f_X(x)f_Y(y)$.

Joint Gaussian Random Variables:

$$f(x_1, x_2, \cdots x_n) = \frac{1}{(2\pi)^{n/2}(\det(C))^{1/2}} \exp\left[-\frac{(X-m)^T C^{-1}(X-m)}{2}\right].$$

where $X = (x_1, x_2 \cdots x_n)^T$. $m = (m_1, \cdots m_n)^T$. $C = [C_{ij}]_{n\times n}$. $C_{ij} = E[(X_i - m_i)(X_j - m_j)]$

Two-Variate Gaussian PDF:

For uncorrelated X, Y ($\rho_{XY}=0$): $f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(x_1-m_1)^2}{\sigma_1^2} + \frac{(x_2-m_2)^2}{\sigma_2^2}\right]\right\} = f(x_1)f(x_2)$

Joint Gaussian r.v.s can be completely characterized by the mean vector and the covariance matrix.

(7) Law of Large Numbers. Let $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\lim_{n\to\infty} P(|Y - m_X| \ge \varepsilon) = 0$. $\forall \varepsilon > 0$.

(8) Central Limit theorem. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to $N(m_X, \frac{\sigma_X^2}{n})$.

(9) Rayleigh Distribution

2.3 Random Processes: basic concepts.

(1) Statistics of Random Process.

For an infinite collection of r.v. specified at time t: $\{X(t_1), X(t_2), \cdots X(t_n)\}$.

Joint PDF: $f(x_1, x_2, \cdots x_n; t_1, t_2, \cdots t_n)$ $\forall n$.

① First order statistics

$f(x;t) \Rightarrow$ first order density of X. (M) $E[X(t_0)] = E[X(t=t_0)] = \int_{-\infty}^{\infty} x f_X(x;t_0) = \overline{X}(t_0)$.

specified t_0

(V) $E[|X(t_0)-\overline{X}(t_0)|^2] = \sigma_X^2(t_0)$.

② Second order Statistics

$f(x_1, x_2; t_1, t_2) \Rightarrow$ second order density of $X(t)$. (自相关函数 与 偶 无关)

难点 Auto-correlation function: $R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2)\,dx_1 dx_2$.

o Example: consider $Y(t) = B\cos W_c t$. where $B \sim N(0, b^2)$. Find its mean and auto-correlation function.

$\Rightarrow$ not WSS

$\Rightarrow E[Y(t)] = \int_{-\infty}^{\infty} B\cos W_c t f(B)\,dB = 0$

$R_Y(t_1,t_2) = E[Y(t_1)Y(t_2)] = E[B^2\cos W_c t_1 \cos W_c t_2] = E(B^2)\cdot\cos W_c t_1 \cos W_c t_2$.

$= D(B)\cos W_c t_1 \cos W_c t_2 = b^2 \cos W_c t_1 \cos W_c t_2$

o Example: consider $Y(t) = A\cos(2\pi f t + \theta)$. where θ is uniform in $(-\pi, \pi)$.

$\Rightarrow$ WSS

$\Rightarrow E[X(t)] = \int_{-\pi}^{\pi} A\cos(2\pi f t+\theta)\cdot\frac{1}{2\pi}\,d\theta = 0$.

$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[A^2\cos(2\pi f t_1+\theta)\cos(2\pi f t_2+\theta)]$

$= \frac{A^2}{2}[\cos(4\pi f t_1 + 2\pi f \tau + 2\theta) + \cos(2\pi f \tau)]$

$= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f t_1 + 2\pi f\tau + 2\theta)\,d\theta + \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(2\pi f\tau)\,d\theta$

$= 0 + \frac{A^2}{2}\cos(2\pi f\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$.

(2) Stationary Process

$\Rightarrow$ For any n and τ: $f(x_1, x_2, \cdots x_n; t_1, \cdots t_n) = f(x_1, x_2 \cdots x_n; t_1+\tau, t_2+\tau, \cdots t_n+\tau)$, $\forall n, \tau$.

① First-order statistics is independent of t: $E[X(t)] = \int_{-\infty}^{\infty} x f_X(x)\,dx = m_X$ (not $f(x,t)$).

② Second-order statistics only depends on $\tau = t_2 - t_1$

$R_X(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_2-t_1)\,dx_1 dx_2 = R_X(\tau)$.

③ Wide-Sense Stationary:

WSS: ① $E[X(t)] = m_X$. ② $R_X(t_1,t_2) = R_X(\tau)$. $\tau = t_2 - t_1$. 弱平稳过程

Strictly stationary: defination. 强平稳过程

(3) Average and Ergodic.

① Ensemble averaging: $\overline{x(t)} = E[x(t)] = \int_{-\infty}^{\infty} x p(x;t)\,dx.$

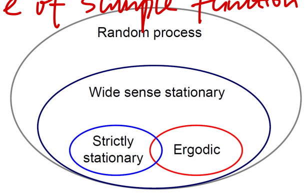
统计平均

$R_x(t_1,t_2) = E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, p(x_1,x_2;t_1,t_2)\,dx_1 dx_2.$

② Time averaging: $\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt.$ take average over time of sample function $x(t)$

$\langle\ \rangle$: 时间平均

$\langle x(t)x(t-\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)\,dt.$



Ergodic: ensemble averaging = time averaging

各态历经

(6) Frequency Domain Characteristics of Random Process.

① Power Spectral Density.

For a deterministic signal $x(t)$:

truncate to get an energy signal: $x_T(t) = \begin{cases} x(t), & |t|<T \\ 0, & \text{else} \end{cases} \xrightarrow{\mathcal{F}} X_T(f) = \int_{-T}^{T} x(t)e^{-j2\pi ft}\,dt.$

Power: $P = \langle x^2(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt < \infty$

→ energy spectrum. 能量谱

Energy: $E_T = \int_{-\infty}^{\infty} x_T^2(t)\,dt = \int_{-T}^{T} x^2(t)\,dt = \int_{-\infty}^{\infty} |X_T(f)|^2\,df$ (Parseval).

Thus: $P = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{\infty}|X_T(f)|^2\,df = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{|X_T(f)|^2}{2T}\,df.$ → 功率谱.

Power Spectral Density of $x(t)$: $S_x(f) = \lim_{T\to\infty}\frac{|X_T(f)|^2}{2T}.$

$X_T(f)$ 是 $x_T(t)$ 的 FT.
(统计平均+时间平均).

② PSD of Random Process.

Average power of r.v. $x(t)$: $P = \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{E(|X_T(f)|^2)}{2T}\,df.$

PSD of $x(t)$: $S_x(f) = \lim_{T\to\infty}\frac{E(|X_T(f)|^2)}{2T}.$

∘ Example (P59). Given a binary-semi-random signal.

③ PSD of WSS Process.

论 Wiener-Khinchin theorem: $\boxed{S_x(f) \xleftrightarrow{\mathcal{F}} R_x(\tau)}$: $\begin{cases} R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau} df \\ S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau \end{cases}$ 自相关函数 ↓ 功率谱密度 $\boxed{\text{互为傅利叶变换}}$

Then $R_x(0) = \int_{-\infty}^{\infty} S_x(f) df = \text{total power} = E[x^2(t)]$

o Example: for random process $X(t) = A\cos(2\pi f t + \theta)$. $\Rightarrow R_x(t;\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$.

Then $S_x(f) = \frac{A^2}{4}[\delta(f+f_0) + \delta(f-f_0)]$. $\Rightarrow P = \frac{A^2}{2}$.

o Example: Given a binary random signal.

重点

(7) Random Process Transmission Through Linear Systems.

Given a linear system with impulse response $h(t)$. $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$

Mean of the output $y(t)$: $\bar{y}(t) = E(y(t)) = \int_{-\infty}^{\infty} h(\tau) E[x(t-\tau)] d\tau = \int_{-\infty}^{\infty} h(\tau) \bar{x}(t-\tau) d\tau$.

If $x(t)$ is WSS: $= \bar{x} \int_{-\infty}^{\infty} h(\tau) d\tau = \bar{x} \cdot H(0)$. $H(0) = \int_{-\infty}^{\infty} h(\tau) d\tau$.

Autocorrelation of $y(t)$:

$$R_Y(t,u) = E(y(t)y(u)) = E\left[\int_{-\infty}^{\infty} h(\tau_1) x(t-\tau_1) d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2) x(u-\tau_2) d\tau_2\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) E[x(t-\tau_1) x(u-\tau_2)] d\tau_1 d\tau_2$$

If $x(t)$ is WSS: $R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_x(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 = h(-\tau) * h(\tau) * R_x(\tau)$

$\Rightarrow$ If input $x(t)$ is WSS, then output $y(t)$ is WSS.

PSD of $Y(t)$: $S_Y(f) = |H(f)|^2 S_x(f)$ 记

2.4 Gaussian and White Process.

For Gaussian process:

- if it is WSS, it is also strictly stationary.
- if the input to a linear system is a Gaussian process, the output is also a Gaussian process.

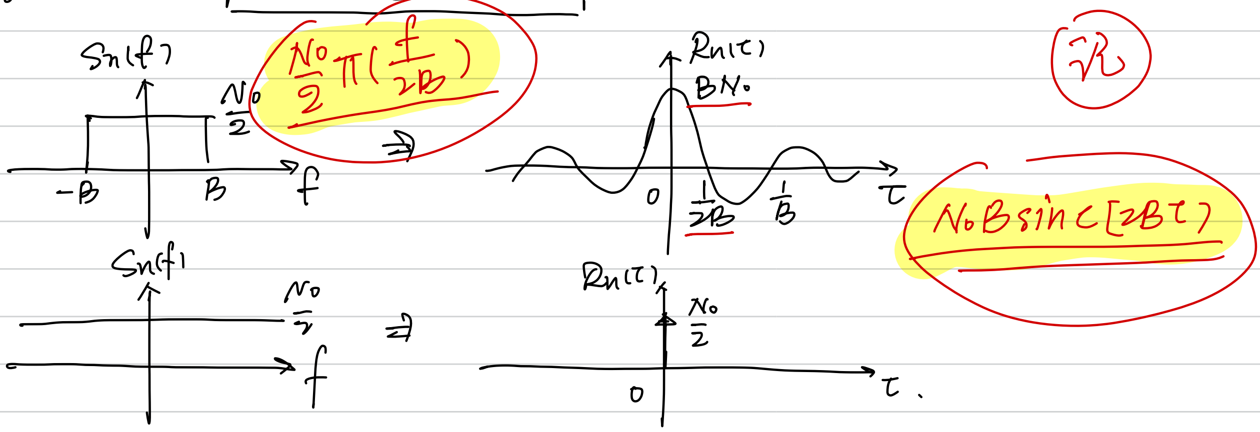
Noise: Gaussian and stationary with zero mean.

White noise: $S_n(f) = \frac{N_0}{2}$. $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$.

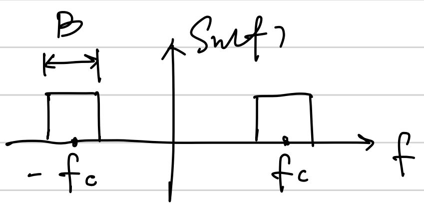
↳ completely uncorrelated.

Bandlimited Noise:

White noise $n_w(t)$ → Filter Bandwidth B Hz → Bandlimited noise $n(t)$.



Bandpass noise



$$\Rightarrow \quad n(t) = R(t)\cos[2\pi f_c t + \phi(t)]$$
$$= n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t).$$

If $n(t)$ is zero-mean, stationary, Gaussian noise.

then. $S_{n_c}(f) = S_{n_s}(f) = \begin{cases} S_n(f-f_0) + S_n(f+f_0), & |f| \le \frac{B}{2} \\ 0, & \text{otherwise} \end{cases}$

$$= \begin{cases} N_0, & |f| \le \frac{B}{2} \\ 0, & \text{otherwise.} \end{cases}$$

ch03. Analog Modulation. 模拟调制系统.

1. What is modulation? Why modulation?

→ Generate a carrier signal (usually sinusoidal) at the transmitter.

→ To translate the frequency of lowpass signal to the passband of channel.

2. Carrier signal: $c(t) = A_c\cos(2\pi f_c t + \theta_c)$.

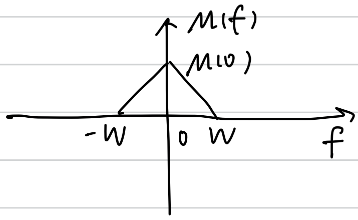
Amplitude A_c → Amplitude modulation.

$\left.\begin{array}{l}\text{Frequency } f_c\\ \text{Phase } \theta_c\end{array}\right\}$ → Angle modulation.

(1) AM. (DSB-SC) Double-Sideband Suppressed Carrier

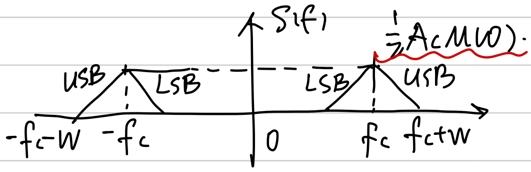
Baseband signal: $m(t)$. Carrier wave: $c(t) = A_c\cos(\omega_c t + \theta_0)$.

Then modulated wave: $s(t) = c(t)m(t) = A_c m(t)\cos(\omega_c t + \theta_0)$. ⇒ 乘法器

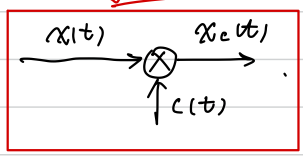


spectrum of message

⇒



spectrum of DSB-SC.



⇒ $S(f) = \frac{1}{2}A_c[M(f-f_c) + M(f+f_c)]$. translate the origin message spectrum to $\pm f_c$.

DSB BandWidth: $B_c = 2W$. (基带信号带宽的两倍).

transmit power (发送信号功率) $= P_s = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} s^2(t)\,dt = \frac{1}{2}A_c^2 P_m$. ($P_m$为基带信号平均功率). A_c为载波信号振幅.

Demodulation of DSB-SC signals.

Phase-coherent demodulation:

Ideal: $s(t) \rightarrow$ [modulator] $\xrightarrow{v(t)}$ [Low-pass filter] $\xrightarrow{v_{out}(t)}$ ⇒ 乘法器和低通滤波器.

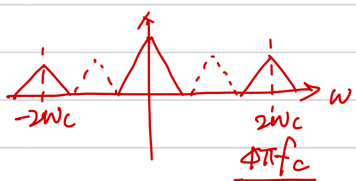
$\uparrow \cos(2\pi f_c t)$ — PLL. (phase-locked loop).

产生本地载波信号 $\cos\omega_c t$ 将已调制信号的频谱再度搬移.

If there is a phase error ϕ, then:

$v(t) = \underbrace{s(t)}_{\text{已调制信号}}\underbrace{\cos(2\pi f_c t + \phi)}_{\text{本地载波信号}} = A_c\cos(2\pi f_c t)\cos(2\pi f_c t + \phi)m(t)$

$= \underbrace{\frac{1}{2}A_c\cos(\phi)m(t)}_{\text{Scaled version of message signal}} + \underbrace{\frac{1}{2}A_c\cos(4\pi f_c t + \phi)m(t)}_{\text{unwanted 高频部分}}$



(2) Conventional AM.

Carrier wave: $c(t) = A_c\cos(\omega_c t + \theta_0)$. Baseband signal: $m_n(t) = \frac{m(t)}{\max|m(t)|}$.

Modulation index: a

⇒ Modulated wave: $s(t) = A_c[1 + a m_n(t)]\cos\omega_c t$

$\begin{cases} a \le 1: & 1 + a m_n(t) \text{ 为正} \\ a > 1: & 1 + a m_n(t) \text{ 可能为负} \end{cases}$

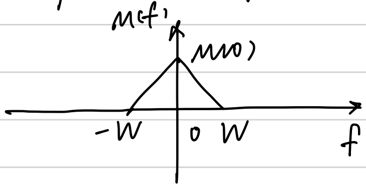
⚠ use "a" to control the power efficiency.

$= A_c\cos(2\pi f_c t) + A_c a m_n(t)\cos(2\pi f_c t)$.

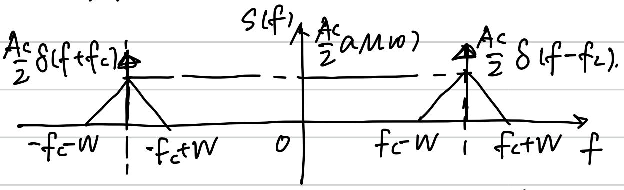
式中$m_n(t)$为归一化的基带信号，即$|m_n(t)|_{max}=1$. 应用包络检波器正确解调AM信号的条件为$m\le 1$. 当$m>1$时，AM信号产生过调制，$s(t)$信号包络失真，不能使用包络检波器，要采用同步解调.

phase reversals 幅度由正变负.

Spectrum of Conventional AM:



spectrum of message signal.



$$S(f)=\frac{A_c}{2}[\delta(f-f_c)+\delta(f+f_c)]+\frac{A_c a}{2}[M(f-f_c)+M(f+f_c)].$$

Bandwidth and Power Efficiency:

⇒ AM信号的频带宽度与DSB信号相同，因此也为 $B_c=2W$.

发送AM信号的平均功率: $P=\underbrace{\frac{1}{2}A_c^2}_{\text{carrier power}}+\underbrace{\frac{1}{2}a^2A_c^2P_{m_n}}_{\text{message power}}$. $P_{m_n}=E[m_n^2(t)]$为基带信号功率.

边带功率与平均功率比值为调制效率 (Modulation Efficiency): $E=\frac{\text{power in sideband}}{\text{total power}}=\frac{a^2P_{m_n}}{1+a^2P_{m_n}}$.

考点 (textbook P151).

Example: message signal $m(t)=3\cos(200\pi t)+\sin(600\pi t)$.

Carrier: $c(t)=\cos(2\times10^5\pi t)$. Modulation index $\alpha=0.85$

Determine the power in the carrier component and in the sideband components of the modulated signal.

⇒ power in carrier component: $\frac{A_c^2}{2}=0.5$

power in sideband: $m(t)=3\cos(200\pi t)+\sin(600\pi t)$. $|m(t)|_{max}=3.6955$

then $m_n(t)=m(t)/|m(t)|_{max}=0.8118\cos(200\pi t)+0.2706\sin(600\pi t)$

$$P_{m_n}=\frac{1}{2}[0.8118^2+0.2706^2]=0.3661$$

$$\frac{A_c^2}{2}a^2P_{m_n}=\frac{1}{2}\times0.85^2\times0.3661=0.1323.$$

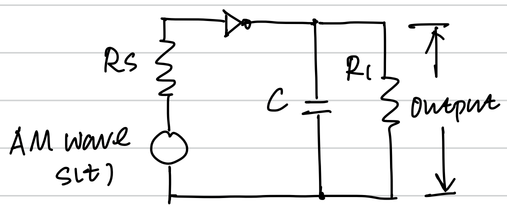
Δ 周期信号属于功率信号.

$P=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}|f(t)|^2dt$. ⇒ $f(t)=\sum_{n=-\infty}^{\infty}F_n e^{jn\omega t}$ (Fourier series).

⇒ $P=\sum_{n=-\infty}^{\infty}|F_n|^2=\underbrace{|F_0|^2}_{\text{直流分量}}+\underbrace{2\sum_{n=0}^{\infty}|F_n|^2}_{\text{谐波分量}}$.

对正弦信号 $A\cos\omega t$, $A\sin\omega t$ 来说, $P=\frac{1}{2}A^2$.

Demodulation of AM signals: Envelop Detector.(包络检波器).



short charging time: $R_sC \ll \frac{1}{f_c}$ (一个载波周期)
long discharging time: $\frac{1}{f_c} \ll R_LC \ll \frac{1}{W}$

载波频率 基带信号频率

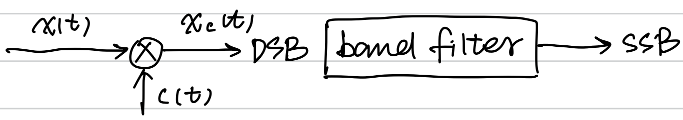
conditions of demodulation of AM: $m \le 1$. $f_c \gg f_x$

(3) Single Sideband (SSB) AM. (单边带调制信号)

$B_c = 2W\ (2f_x)$

⇒ Both AM and DBSC are spectrally inefficient (use twice the bandwidth of message)

The SSB use a band filter to transform DSB.



The baseband signal can be written as: $m(t) = \sum_{i=1}^{n} x_i \cos(2\pi f_i t + \theta_i)$. $f_i \le f_x$

Then its USB signal (上边带信号) is: $u(t) = \frac{A_c}{2} \sum_{i=1}^{n} x_i \cos[2\pi(f_c + f_i)t + \theta_i]$.

After manuplation:

$$u(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t - \frac{A_c}{2} \hat{m}(t) \sin 2\pi f_c t$$

For LSB: $u(t) = \frac{A_c}{2} m(t) \cos 2\pi f_c t + \frac{A_c}{2} \hat{m}(t) \sin 2\pi f_c t$

Bandwidth: $B_c = W\ (f_x)$.

Power: $S_T = \frac{1}{4} A_c^2 S_x$. (带宽和功率都是DSB信号的一半).

(4) Vestigial Sideband: VSB.

Bandwidth: $B = W + f_v$.

(5). Signal Multiplexing. ?

⇒ Mutiplexing: independent signals are combined and transmitted in a common channel, they're demutiplexing at receiver.

Two common types: TDM (时分复用). FDM (频分复用).

FDM:

LPF: ensure signal bandwidth limited to W

MOD (modulator): shift message frequency range to mutually exclusive high frequency bands

BPF: restrict the band of each modulated wave to its prescribed range

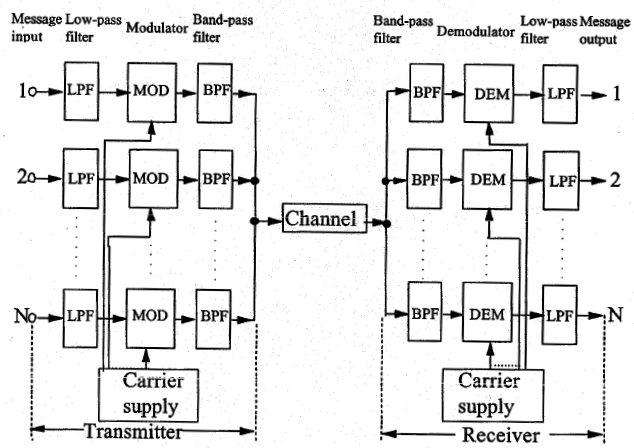


Figure 7.21 - Block diagram of FDM system.

3. Angle Modulation. (phase / frequency).

The general form: $s(t) = A_c \cos\phi(t) = A_c \cos[2\pi f_c t + \theta(t)]$

相偏

instantaneous phase: $\phi(t)$.

instantaneous angle frequency: $\omega_i(t) = \frac{d\phi(t)}{dt} = 2\pi f_c + \frac{d\theta(t)}{dt}$.

instantaneous frequency: $f_i(t) = f_c + \frac{1}{2\pi}\cdot\frac{d\theta(t)}{dt}$.

$2\pi f_c$. f_c为未调制载波信号的频率.

频偏.

instantaeous phase deviation: $\theta(t)$

instantaneous frequency deviation: $\frac{1}{2\pi}\cdot\frac{d\theta(t)}{dt}$.

Δ phase $\underset{\int_0^t d\tau}{\overset{\frac{d}{dt}}{\rightleftarrows}}$ frequency

Presentation of FM and PM signals:

phase modulation (PM): $\theta(t) = k_p m(t)$, k_p is phase deviation constant (相移常数)

$\Rightarrow s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$.

$\rightarrow \theta(t) = 2\pi k_f \int m(t)$.

Frequency modulation (FM): $f_i(t) - f_c = \frac{1}{2\pi}\frac{d}{dt}\theta(t) = k_f m(t)$.

k_f is frequency deviation constant (频偏常数)

$\Downarrow$

frequency sensitivity.

$\Rightarrow s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau]$.

Relationship:

$m(t) \longrightarrow$ [integrator] $\xrightarrow{\int m(t) dt}$ [Frequency modulator] $\longrightarrow$ FM wave

$A_c \cos[2\pi f_c t + k_p \int m(t) d\tau]$.

$\uparrow$ $A_c \cos(2\pi f_c t)$ 载波

$m(t) \longrightarrow$ [differentiator] $\xrightarrow{\frac{d}{dt}m(t)}$ [phase modulator] $\longrightarrow$ PM wave

$A_c \cos[2\pi f_c t + 2\pi k_f m(t)]$.

$\uparrow$ $A_c \cos(2\pi f_c t)$.

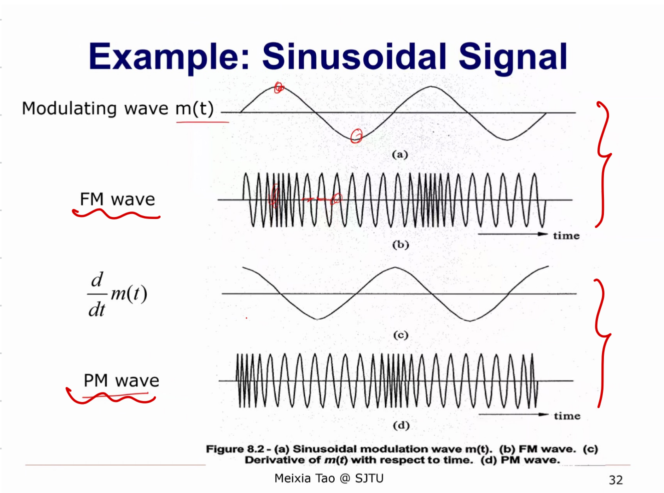


Figure 8.2 - (a) Sinusoidal modulation wave m(t). (b) FM wave. (c) Derivative of m(t) with respect to time. (d) PM wave.

(1) FM by a Sinusoidal Signal

message: $m(t) = A_m \cos(2\pi f_m t)$ 也可能是sin.

instantaneous frequency of FM wave:

$$f_i(t) = f_c + k_f A_m \cos(2\pi f_m t) = f_c + \Delta f \cos(2\pi f_m t).$$

⇒ frequency deviation (最大频偏): $\Delta f = k_f A_m$

Carrier phase: $\theta(t) = 2\pi \int_0^t (f_i(\tau) - f_c)\, d\tau = \frac{\Delta f}{f_m} \sin(2\pi f_m t) = \beta \sin(2\pi f_m t)$.

f_m: message 频率(带宽)

⇒ Modulation index (偏移比): $\beta = \frac{\Delta f}{f_m}$ (α in AM)

FM signal: $s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$.

o Example: a sinusoidal modulating wave of amplitude 5V and frequency 1 KHz is applied to a frequency modulator. The frequency sensitivity is 40Hz/V. The carrier frequency is 100kHz.

Calculate: (a) the frequency deviation. (b) the modulation index

(a) ⇒ $A_m = 5V$. $f_m = 1KHz$. $k_f = 40Hz/V$. $f_c = 100KHz$.

$$\Delta f = k_f A_m = 40Hz/V \cdot 5V = 200Hz.$$

(b) $$\beta = \frac{\Delta f}{f_m} = 200Hz / 1KHz = 0.2.$$

Spectrum Analysis of Sinusoidal FM Wave.

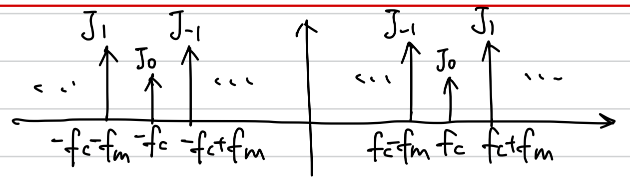
频

FM wave in time-domain:

$$s(t) = A_c \mathrm{Re}\left\{ \sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + nf_m)t] \right\} = \boxed{A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + nf_m)t]}.$$

FM wave in frequency domain:

$$\boxed{S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f - f_c + nf_m)]. \quad J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}}$$



In theory, BW = ∞.

⇒ 各个冲激的强度由 $J_n(\beta)$ 决定.

(2) Effective Bandwidth of FM.

How A_m and f_m affect the spectrum:

$\begin{cases} \text{Fix } f_m.\ \text{Vary } A_m \Rightarrow \Delta f = k_f A_m \text{ and } \beta = \frac{\Delta f}{f_m} \text{ are varied.} \\ \text{Fix } A_m.\ \text{Vary } f_m \Rightarrow \Delta f = k_f A_m \text{ fix. } \beta = \frac{\Delta f}{f_m} \text{ vary.} \end{cases}$

When β is small. only focus on the first two terms.
When β is large. more terms.
$\Rightarrow J_n(\beta)$

For large β. B is only slightly greater than $2\Delta f$.

For small α. the spectrum is limited to $[f_c - f_m, f_c + f_m]$. $\Rightarrow B = 2f_m$.

$\Rightarrow$ Carson's Rule: ~~f_c~~ $B \approx 2\Delta f + 2f_m = 2(1+\beta) f_m$

理论上FM波有无限波谱(尤其是WBFM). 但只考虑 $|J_n(\beta)| > 0.01$ 的部分. 就成了有限频带.

$B \approx 2n_{max} f_m$. n_{max} is the max n that satisfies $|J_n(\beta)| > 0.01$.

(3) FM by an Arbitrary Message.

给定任意调制信号 $m(t)$. 具有最高频率成分 W.

偏移比 (Modulation index) $\beta = \frac{\Delta f}{W}$

频偏 Frequency deviation: $\Delta f = k_f \cdot \max|m(t)|$.

Carson's rule: $B = 2(1+\beta)W$.

(4) Narrowband FM

$s(t) = A_c \cos[2\pi f_c t + \theta(t)]$. For $m(t)$. $\theta(t) \ll 1$.

Then $s(t) = A_c \cos 2\pi f_c t \cos\theta(t) - A_c \sin 2\pi f_c t \sin\theta(t)$.

$\approx A_c \cos 2\pi f_c t - A_c \theta(t) \sin 2\pi f_c t$.

与conventional AM类似. 且 $B \approx 2W$.

- 99% bandwidth approximation
  - The freq. component beyond which none of the side-freq. is greater than **1%** of the unmodulated carrier amplitude
  - i.e. $B \approx 2n_{max} f_m$ where n_{max} is the max n that satisfies $|J_n(\beta)| > 0.01$

β	0.1	0.3	0.5	1.0	2.0	5.0	10	20	30
$2n_{max}$	2	4	4	6	8	16	28	50	70

Example

- In north America, the maximum value of frequency deviation Δf is fixed at $75KHz$ for commercial FM broadcasting by radio.
- Take $W = 15KHz$, typically the maximum audio frequency of interest in FM transmission, the modulation index is

$$\beta = 75/15 = 5$$

- Using *Carson's rule*,

$$B = 2(75 + 15) = 180KHz$$

- Using 99% bandwidth rule,

$$B = 16W = 16 \times 15 = 240KHz$$

$180 < 240$

1.5) Generation of FM waves.

First generate a narrow-band FM signal and change it to WBFM.

- Consider a narrow band FM wave

$$s_1(t) = A_1 \cos\left[2\pi f_1 t + \phi_1(t)\right]$$

where $\phi_1(t) = 2\pi k_1 \int_0^t m(\tau)d\tau$ $\quad f_1$ = carrier frequency, $\quad k_1$ = frequency sensitivity

- Given $\phi_1(t) \ll 1$ with $\beta \le 0.3$, we may use

$$\begin{cases} \cos[\phi_1(t)] \approx 1 \\ \sin[\phi_1(t)] \approx \phi_1(t) \end{cases}$$

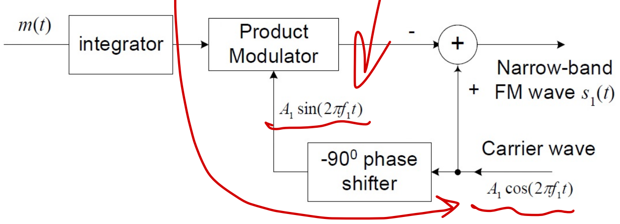
- Correspondingly, we may approximate $s_1(t)$ as

$$s_1(t) = A_1\cos(2\pi f_1 t) - A_1\sin(2\pi f_1 t)\phi_1(t)$$
$$= A_1\cos(2\pi f_1 t) - 2\pi k\left(A_1\sin(2\pi f_1 t)\right)\int_0^t m(\tau)d\tau$$

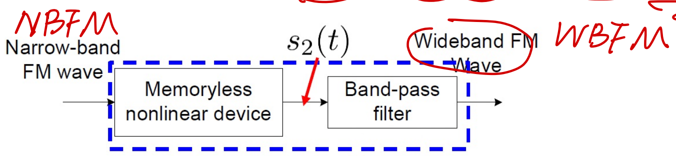
NBFM

Narrow-band FW wave

- Narrow-band frequency modulator



- Next, pass $s_1(t)$ through a frequency multiplier 倍频器



  – The input-output relationship of the non-linear device is:

$$s_2(t) = a_1 s_1(t) + a_2 s_1^2(t) + \ldots + a_n s_1^n(t) \quad (\text{各次谐波}).$$

  – The BPF is used to pass the FM wave centred at nf_1 and with deviation $n\Delta f_1$ and suppress all other FM spectra

载波和频偏都放大n倍

Example: frequency multiplier with n = 2

*

- Problem: Consider a square-law device based frequency multiplier

with
$$s_2(t) = a_1 s_1(t) + a_2 s_1^2(t)$$
$$s_1(t) = A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau\right)$$

S_2 是倍频器
S_1 是 NBFM

- Specify the midband freq. and bandwidth of BPF used in the freq. multiplier for the resulting freq. deviation to be twice that at the input of the nonlinear device
- Solution:

$$s_2(t) = a_1 A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau\right) + a_2 A_1^2 \cos^2\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau\right)$$

$$= \underbrace{a_1 A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau\right) + \frac{a_2 A_1^2}{2}}_{\text{Removed by BPF with}} + \boxed{\frac{a_2 A_1^2}{2}\cos\left(4\pi \overset{2f_1}{f_1} t + 4\pi \overset{2k_1}{k_1} \int_0^t m(\tau)d\tau\right)}$$

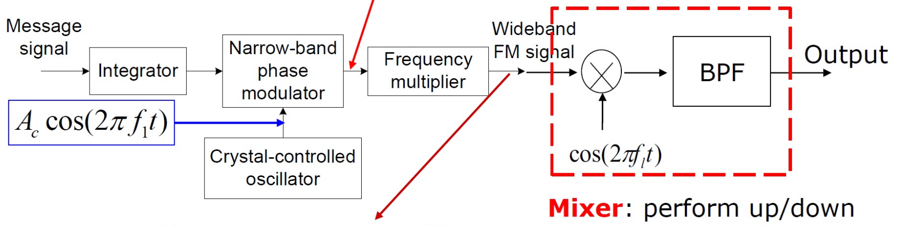
⇒ 保留.

Removed by BPF with $f_c = 2f_1$, BW > $2\Delta f = 4\Delta f_1$

centered at $2f_1$ and with deviation $2\Delta f_1$

Generation of Wideband FM Signal

$$s_1(t) = A_1 \cos\left(2\pi f_1 t + 2\pi k_1 \int_0^t m(\tau)d\tau\right)$$



$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

Mixer: perform up/down conversion to shift the signal to the desired center freq.

*

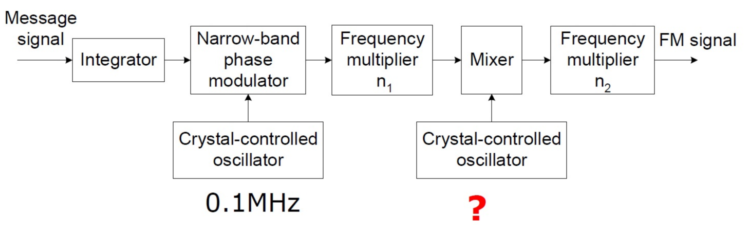
$f_c = nf_1$ → may not be the desired carrier frequency

$k_f = nk_1$

$\Delta f = n\Delta f_1$

Exercise: A typical FM transmitter

- Problem: Given the simplified block diagram of a typical FM transmitter used to transmit audio signals containing frequencies in the range 100Hz to 15kHz.
- Desired FM wave: f_c = 100MHz, Δf = 75kHz. $\beta = 5$
- Set $\beta_1 = 0.2$ in the narrowband phase modulation to limit harmonic distortion.
- Specify the two-stage frequency multiplier factors n_1 and n_2



Ch04. (ADC). Analog-to-Digital Conversion.

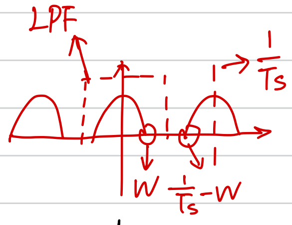
1. Sampling.

(1) $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$. $\Rightarrow$ pulse modulation.

$X_\delta(f) = X(f) * \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s}) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f-\frac{n}{T_s})$ $\frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$

Nyquist sampling rate $=$ $f_s = 2W$

(2) Reconstruction



LPF with frequency response $=$ $\left(H(f) = \begin{cases} T_s, & |f| < W \\ 0, & |f| \geq \frac{1}{T_s} - W. \end{cases}\right)$

Ideal LPF $=$ $\left(H(f) = T_s\Pi(\frac{f}{2W'})\right) \Rightarrow \left(h(t) = 2W'T_s\,\mathrm{sinc}(2W't)\right)$, $W \leq W' < \frac{1}{T_s} - W$.

$x(t) = x_\delta(t) * 2W'T_s\,\mathrm{sinc}(2W't)$

$= \left(\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)\right) * 2W'T_s\,\mathrm{sinc}(2W't)$

$= \sum_{n=-\infty}^{\infty} 2W'T_s\, x(nT_s)\,\mathrm{sinc}[2W'(t-nT_s)]$. $\Rightarrow$重构信号. 多个sinc波形叠加.

2. Quantization

(1) Quantization regions $=$ R_k, $k = 1, 2, \ldots N$.

Quantization level (representative point) of each $R_k = x_k$.

For each sample $x = x(nT_s)$: $Q(x) = x_k$ when $x \in R_k$.

(2) Performance measure.

Quantization error (noise) $=$ $e[x, Q(x)] = x - Q(x)$

Distortion (noise power) $=$ $D = E[(x-Q(x))^2]$ (方差). $P = R_x(0) = E(x^2)$

$\Rightarrow D = \int_{-x_p}^{x_p} (x-x_q)^2 f(x)\,dx$. 计算试

Signal-to-quantization noise ratio (SQNR):

$$SQNR = \frac{E(x^2)}{E[(x-Q(x))^2]}$$ (信噪比：信号和噪声功率之比).

o Example:

The source $x(t)$ is a stationary Guassian source with mean zero and power spectral density: $S_x(f) = \begin{cases} 2, & |f| < 100\,Hz \\ 0, & \text{otherwise} \end{cases}$

It is sampled at the Nyquist rate and each sample is quantized using the 8-level quantizer with

$a_0 = -\infty,\ a_1 = -60,\ a_2 = -40,\ a_3 = -20,\ a_4 = 0,\ a_5 = 20,\ a_6 = 40,\ a_7 = 60$

$x_1 = -70,\ x_2 = -50,\ x_3 = -30,\ x_4 = -10,\ x_5 = 10,\ x_6 = 30,\ x_7 = 50,\ x_8 = 70.$

What is the resulting distortion? the SQNR? the Rate?

$\Rightarrow$ For this Guassian source:

3×200

$E[x] = 0.$ ※ $E[x^2(t)] = \sigma^2 = P = R_x(\tau=0) = \int S_x(f)df = 400.$ $R = vf_s = 600$ bits/sec.

$D = E[(x - Q(x))^2] = \int_{-\infty}^{-60} (x+70)^2 f(x)dx + \int_{-60}^{-40} (x+50)^2 f(x)dx + \cdots + \int_{40}^{60} (x-70)^2 f(x)dx$

$f(x)$ has mean of 0, variance of 400. So $f(x) = \frac{1}{\sqrt{2\pi 400}} e^{-\frac{x^2}{800}}.$

$\Rightarrow$ 查表时. $\pm a_i$ 和 $\pm x_i$ 乘 σ. D 乘 σ^2 (表中的是 $N(0,1)$ 高斯分布).
加均值

(3). Uniform Quantizer.

Quantization levels: $N = 2^v$. $v \Rightarrow$ 编码长度.

All quantization regions are of equal length: $\Delta = \frac{2a}{N} = \frac{2a}{2^v} = \frac{a}{2^{v-1}}.$

Range of input samples: $x(t) \in [-a, a]$.

Quantizatized value: $Q(x) = \frac{x_i + x_{i-1}}{2}.$

Quantization error: $-\frac{\Delta}{2} \le e \le \frac{\Delta}{2}.$

Assuming: quatization error is uniformly distributed on $(-\frac{\Delta}{2}, \frac{\Delta}{2})$.

↳ noise power: $P = E[e^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} x^2 dx = \frac{\Delta^2}{12} = \frac{a^2}{3N^2} = \frac{a^2}{3 \cdot 4^v}.$

重点

$SQNR = \frac{P_x}{E[e^2]} = 10\log_{10}\frac{P_x}{a} + 6v + 4.8$ (dB).

○ Example:

Given a uniform quantizer with 256 levels ($\nu = 8$ bits).

Find the SQNR if the signal is $x_1(t) = \cos(2\pi f t)$.

Find the SQNR if the signal is $x_2(t)$ uniformly distributed on $[-1, 1]$.

$\Rightarrow P_1 = \frac{1}{2} \times 1^2 = \frac{1}{2}$. $\quad P_2 = E[x^2] = \int_{-1}^{1} x^2 \cdot \frac{1}{2} dx = \frac{1}{3}$.

$N_q =$

$SQNR_1 =$

(Optimal).

(4) Nonuniform Quantizer. ⇒量化阶可变.

⇒ By relaxing the condition that the quantization regions be of equal length, we can minimize the distortion.

Lloyd-Max Conditions:

The boundraies of the quantization regions are the midpoints of the corresponding quantized values:

$$a_i = \frac{1}{2}(x_i + x_{i+1}).$$

The quatized values are the centroids of the quantization regions:

$$x_i = \frac{\int_{a_{i-1}}^{a_i} x f(x) dx}{\int_{a_{i-1}}^{a_i} f(x) dx}$$

$$D = \int_{-\infty}^{a_1} (x - \hat{x}_1)^2 f_X(x) dx + \sum_{i=1}^{N-2} \int_{a_i}^{a_{i+1}} (x - \hat{x}_{i+1}) f_X(x) dx + \int_{a_{N-1}}^{\infty} (x - \hat{x}_N)^2 f_X(x) dx.$$

3. Encoding. $\Rightarrow$ map: $N=2^{\nu}$ quatization values to ν bits: $x_i\ (b_1 b_2 \cdots b_\nu)$.

There are ν bits for each sample, and f_s samples per second.

Bit rate: $R=\nu f_s$ bits/second.

Speech signal usually employs 8 bits for each quatized values with 256 levels.

4. Pulse Code Modulation (PCM).

$$x(t) \longrightarrow \boxed{\text{Sampler}} \xrightarrow{x_n} \boxed{\text{Quantizer}} \xrightarrow{\hat{x}_n} \boxed{\text{Encoder}} \longrightarrow \text{bits stream.}$$

模拟信号x(t)通常先经过带限低通滤波器使其最高频率为f_s. 再经过采样后成为PAM信号. 量化/编码为A/D转换.

Bit rate: if signal has a bandwidth of W and is sampled at f_s $(f_s \geq 2W)$ and ν bits are used to encode each sampled signal: $R_b = f_s \nu$ bits/sec

Minimum bandwidth requeriement: $BW_{req} = \frac{R_b}{2} = \frac{f_s \nu}{2} \geq \nu W$ Hz

$f_s \geq 2W$

将比特流输入信道. 信道所需带宽最小为 νW.

Example

- For voice signals, the frequency range is 300～3400Hz.
- The international standard takes the sampling frequency as fs=8kHz, and each sample value is represented by an 8-bit binary code (v=8)
- Thus, the bit rate of the standard voice channel is

$$R_b = 8 \times 8 = 64 \text{ kbits/s}$$

- The required minimum channel bandwidth (Nyquist $\Rightarrow$ channel. bandwidth) is

$$BW = 32 \text{ kHz}$$

$f_s = 8\text{kHz} \Rightarrow W = 4\text{kHz}$. $BW = \nu W = 8 \times 4 = 32\text{kHz}$.

300～3400Hz

∘ Example : A bandlimited signal has bandwidth 3400Hz. What Sampling rate should be used to guarantee a guard band of 1200Hz ?

$\Rightarrow$ $f_s = 2W + W_G$ so $f_s = 2 \times 3400 + 1200 = 8000$ HZ.

5. Non-uniform PCM.

$x_n \longrightarrow \begin{pmatrix} \text{Compressor} \\ g(x) \end{pmatrix} \rightarrow \begin{pmatrix} \text{Uniform} \\ \text{PCM} \end{pmatrix} \rightarrow \cdots \rightarrow \text{Decoder} \rightarrow \begin{pmatrix} \text{Expander} \\ g^{-1}(x) \end{pmatrix} \rightarrow \begin{pmatrix} \text{Reconstruction} \\ \text{filter} \end{pmatrix}$

$\Rightarrow$ companding (压扩器) : compress + expand.

∘ A law compander : $g(x) = \dfrac{\log(1+\mu|x|)}{\log(1+\mu)} \operatorname{sgn}(x)$. $|x| \le 1$.

∘ μ law : $g(x) = \dfrac{1+\log A|x|}{1+\log A} \operatorname{sgn}(x)$. $|x| \le 1$

6. Differential PCM (DPCM).

7. Delta Modulation (DM).

DM is a simplified version of DPCM having a 2-level quantizer with magnitude $\pm\Delta$.

$\Rightarrow$ The step size Δ is critical in designing a DM system.

A Sampling Theorem.

Signal $x(t)$ have a bandwidth W. Sample $x(t)$ at interval T_s ($T_s \leq \frac{1}{2W}$).

yield the sequence: $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$.

Reconstruct the $x(t)$ by:

$$x(t) = \sum_{n=-\infty}^{\infty} 2W'T_s\, x(nT_s)\operatorname{sinc}[2W'(t-nT_s)].$$

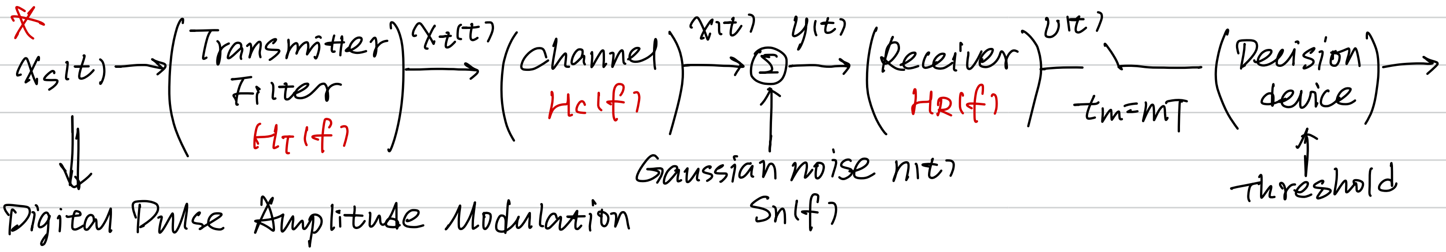
where W': $W \leq W' \leq \frac{1}{T_s} - W$.

When $T_s = \frac{1}{2W}$ ($f_s = 2W$). then $W' = W = \frac{1}{2T_s}$. Reconstruction:

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\operatorname{sinc}\left(\frac{t}{T_s} - n\right).}$$

Chapter 05: Digital Transmission Through Bandlimited Channels.

1. ISI: Inter-Symbol Interface



Digital Pulse Amplitude Modulation (PAM).

↳ $x_s(t) = \sum_{j=-\infty}^{\infty} A_j \delta(t - jT)$.

$x_t(t) = \sum_{j=-\infty}^{\infty} A_j h(t - jT)$.

$v(t) = x_s(t) * [h_T(t) * h_c(t) * h_R(t)] + n(t) * h_R(t)$

Pulse response: $p(t) = h_T(t) * h_c(t) * h_R(t)$

Frequence response: $P(f) = H_T(f) H_c(f) H_R(f)$.

Then: $v(t) = \sum_{k=-\infty}^{\infty} A_k p(t - kT) + n_o(t)$, $n_o(t) = n(t) * h_R(t)$.

Sample the $v(t)$ at $t_m = mT$ to detect A_m:

$$v(t_m) = \sum_{k=-\infty}^{\infty} A_k p(mT - kT) + n_o(t_m).$$

$$= \underbrace{A_m p(0)}_{\substack{k=m \\ \text{Desired signal}}} + \left(\underbrace{\sum_{k \neq m}^{\infty} A_k p[(m-k)T]}_{\text{ISI}} + \underbrace{n_o(t_m)}_{\text{noise}} \right)$$

2. The power spectrum of digital modulated signals.

baseband transmitted signal: $x_t(t) = \sum_{j=-\infty}^{\infty} A_j h_T(t - jT)$.

the symbols in the sequence $\{A_m\}$ are uncorrelated:

$\Rightarrow$ Auto-correlation sequence $R_A[m] = E(A_j^* A_{j+m}) = \begin{cases} \sigma_A^2 + m_A^2, & m = 0 \\ m_A^2, & m \neq 0. \end{cases}$

Power spectrum (FS):

$$S_A(f) = \sum_{m=-\infty}^{\infty} R_A[m] e^{-j2\pi f mT} = \sigma_A^2 + \frac{m_A^2}{T} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T}\right).$$

The power spectral density of $x_t(t)$:

$$S_x(f) = \frac{S_A(f)}{T} |H_T(f)|^2 = \underbrace{\left(\frac{\sigma_A^2}{T} |H_T(f)|^2 \right)}_{①} + \underbrace{\left(\frac{m_A^2}{T^2} \sum_{m=-\infty}^{\infty} \left|H_T\left(\frac{m}{T}\right)\right|^2 \delta\left(f - \frac{m}{T}\right) \right)}_{②}.$$

①: continous spectrum

②: discrete spectrum. when $m_A = 0$. it becomes zero.

3. Nyquist First criterion.

Assume that bandlimited channel with no distortion: $H_c(f) = 1$. $|f| \le W$.

$$\boxed{\begin{aligned} v(t_m) &= \sum_{k=-\infty}^{\infty} A_k p(mT - kT) + n_0(t_m). \\ &= A_m p(0) + \sum_{k\neq m}^{\infty} A_k p[(m-k)T] + n_0(t_m) \end{aligned}}$$

To ensure zero ISI, the $p(nT)$ must satisfy:

$$p(nT) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0. \end{cases}$$

$\Rightarrow$ Necessary and sufficient condition:

$$P_\Sigma(f) = \sum_{k=-\infty}^{\infty} P(f + \frac{k}{T}) = \text{constant}. \quad \text{(Nyquist condition)}$$

with zero ISI $\Rightarrow$ $\underline{v(t_m) = A_m + n_0(t_m)}$

$\Rightarrow$ 系统总的传输函数 $P(f)$ 以 $\frac{1}{T}$ 为周期进行平移，叠加后得到一常数。
但用此条件不能唯一确定 $P(f)$。 $\underline{P(f)\text{ 的最小带宽应为 } \frac{1}{2T}}$

Nyquist's first method uses: ($W = \frac{1}{2T}$)

$$P(f) = \begin{cases} 1, & |f| < \frac{1}{2T} \\ 0, & \text{otherwise} \end{cases} \quad \Leftrightarrow \quad p(t) = \mathrm{sinc}(\frac{t}{T}) = \frac{\sin(\pi t/T)}{\pi t/T} \rightarrow \frac{1}{t}$$

$$\begin{cases} \text{Nyquist bandwidth: } \underline{B_0 = \frac{1}{2T} = \frac{R_s}{2}} \\ \text{Nyquist rate: } \underline{R_s = 2B_0.} \end{cases}$$

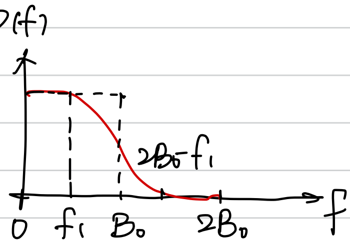
$\Rightarrow \begin{cases} \text{给定 } R_s\text{，则最小信道带宽为 } \frac{R_s}{2} \\ \text{给定 } B_0\text{，则最大速率为 } 2B_0 \end{cases}$

- Challenges of designing such $p(t)$ or $P(f)$
  - $P(f)$ is physically unrealizable due to the abrupt transitions at $\pm B_0$
  - The tail of $p(t)$ decays as $\frac{1}{t}$, which is slow and results in little margin of error in sampling times in the receiver.
  - This demands accurate sample point timing - a major challenge in modem / data receiver design.
  - Inaccuracy in symbol timing is referred to as timing jitter（定时抖动）.

4. Raised Cosine Spectrum.

Three parts: passband, transition band, stopband.

$$P(f)=\begin{cases}1, & 0\le |f|<f_1\\ \frac{1}{2}\left\{1+\cos\left[\frac{\pi(|f|-f_1)}{2B_0-2f_1}\right]\right\}, & f_1\le |f|<2B_0-f_1\\ 0, & |f|\ge 2B_0-f_1\end{cases} \Rightarrow$$



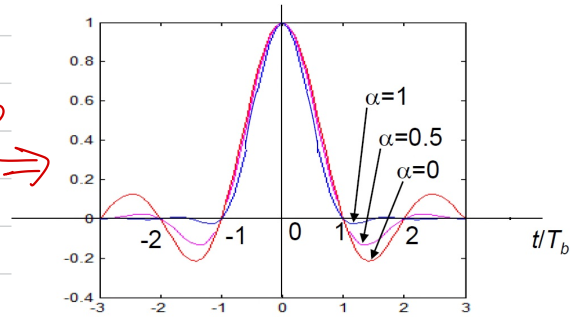
适当放宽了 $p(f)=\begin{cases}1, & f=0\\ 0, & f\neq 0\end{cases}$ 的条件，允许加入一点ISI。 $(B_0=\frac{1}{2T})$

Roll-off factor: $\alpha=1-\frac{f_1}{B_0}$ $\in[0,1]$. Total Bandwidth: $f_1=B_0(1-\alpha)$ $B=B_0(1+\alpha)=\frac{1+\alpha}{2T}$.

↳ $\alpha=0$时，相当于Nyquist第一准则。增大α，频谱变宽，谱形变缓。

$\Rightarrow$ Time domain: $p(t)=\text{sinc}(2B_0t)\frac{\cos(2\pi\alpha B_0 t)}{1-16\alpha^2B_0^2t^2}$. $\to \frac{1}{t^2}$. $\to \frac{1}{t^3}$

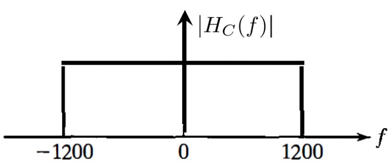
↳ $\text{sinc}(2B_0t)$保证了在采样点处其值为1 $(k=m)$ 或0 $(k\neq m)$。
包络以 t^3 速率衰减，能够减小其造成的ISI影响。α越大越显著，但频谱会变宽。$\Rightarrow$

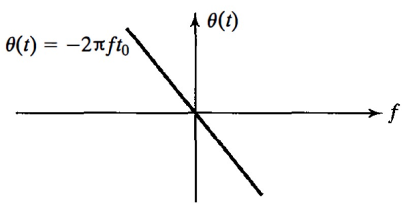


Example:

An ideal channel has the freq. response shown below. We wish to design the transmit and receiver filters such that the overall freq. response is the raised cosine spectrum with roll-off factor $\alpha = 0.5$.

Determine the symbol rate 1/T, and compare it with the Nyquist rate





$\Rightarrow$ Let $H_T(f)H_c(f)H_R(f)=X_{rc}(f)$

$$\frac{1+\alpha}{2T}=\frac{1+\frac{1}{2}}{2T}=1200$$

then $T=1600$.

the symbol rate $\frac{1}{T}=1600$ symbols/sec. the Nyquist rate: 2400 symbols/sec.

$$X_{rc}(f)=\begin{cases}T, & 0\le |f|\le 400\\ \frac{T}{2}\left[1+\cos\left(\frac{\pi}{800}(|f|-400)\right)\right], & 400\le |f|\le 1200\\ 0, & |f|\ge 1200\end{cases}, \quad f_1=B_0(1-\alpha)=\frac{1-\alpha}{2T}=400.$$

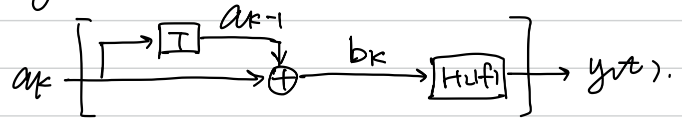
$\Rightarrow |G_T(f)|=|G_R(f)|=\sqrt{X_{rc}(f)}$. $\theta_R(f)=-\theta_T(f)$.

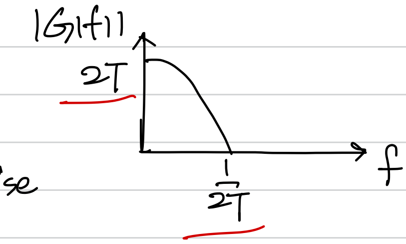
5. Duobinary Signal.

$\{a_k\}$ is the transmitted PAM binary sequence. Pulse duration is T

Duobinary signal: $b_k = a_k + a_{k-1}$.

System:





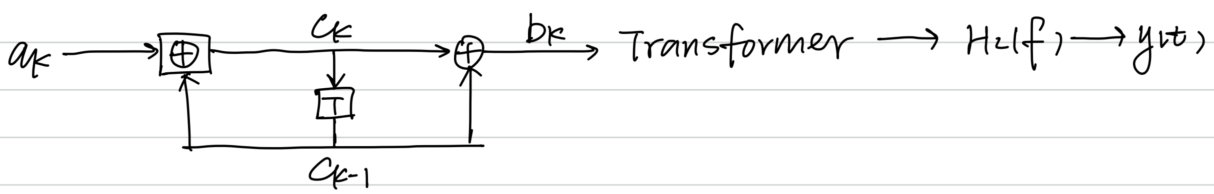
$$\Rightarrow G(f) = (1+e^{-j2\pi fT})H_2(f) = \begin{cases} 2Te^{-j\pi fT}\cos\pi fT, & |f| \le \frac{1}{2T} \\ 0 & , \text{otherwise} \end{cases}$$

$$g(t) = [\delta(t)+\delta(t-T)] * h_2(f) = \operatorname{sinc}\left(\frac{t}{T}\right) + \operatorname{sinc}\left(\frac{t-T}{T}\right) = \frac{T^2}{\pi t} \cdot \frac{\sin \pi t/T}{(T-t)} \rightarrow \frac{1}{t^2}$$

$$g(nT) = \begin{cases} 1, & n=0. \text{ (current symbol)} \\ 1, & n=1. \text{ (ISI to the next symbol)} \\ 0, & \text{otherwise} \end{cases}$$

To solve the ambiguity problem and error propagation: Precoding.

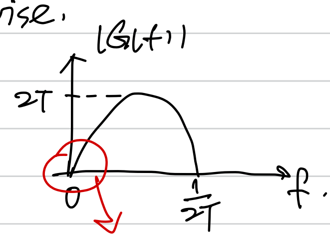
$\Rightarrow c_k = a_k \oplus c_{k-1}$. $\quad b_k = c_k + c_{k-1}$.



6. Modified Duobinary signal: $b_k = a_k - a_{k-2}$.

$$\Rightarrow G(f) = (1-e^{-j4\pi fT})H_2(f) = \begin{cases} 2Tje^{-j2\pi fT}\sin 2\pi ft. & |f| \le \frac{1}{2T}. \\ 0 & . \text{otherwise}. \end{cases}$$

$$g(t) = \operatorname{sinc}\left(\frac{t}{T}\right) - \operatorname{sinc}\left(\frac{t-2T}{T}\right) = -\frac{2T^2\sin\pi t/T}{\pi t(t-2T)} \rightarrow \frac{1}{t^2}.$$



no DC component.

Precoding: $c_k = a_k \oplus c_{k-2}$. $\quad b_k = c_k - c_{k-2}$

7. Optimum Transmit / Receive Filter.

$P(f) = P_{rc}(f)$. then the output $V_m = A_m + N_m$ (assume $p(0)=1$)

Consider binary PAM transmission: $A_m = \pm d$.

Variance of N_m: $\sigma^2 = \int_{-\infty}^{\infty} S_n(f)\, |H_R(f)|^2 df$

$\Big\} \Rightarrow SNR = \dfrac{d^2}{\sigma^2}$.

↳ After $H_c(f)$ fixed and $p(f)$ given, select $H_T(f) \cdot H_R(f)$ to maximize SNR.

Solution:

$$\begin{cases} |H_T(f)| = \dfrac{\sqrt{P_{rc}(f)}}{|H_c(f)|^{1/2}}, & |f| \le W \\ |H_R(f)| = \dfrac{\sqrt{P_{rc}(f)}}{|H_c(f)|^{1/2}}, & |f| \le W. \end{cases}$$

Then: $\mathcal{E}_{av} = \int_{-\infty}^{\infty} d^2 h_T^2(t)\, dt = \int_{-\infty}^{\infty} d^2 H_T^2(f)\, df = d^2 \int_{-W}^{W} \dfrac{P_{rc}(f)}{|H_c(f)|} df$

$$d^2 = \mathcal{E}_{av} \cdot \left[\int_{-W}^{W} \frac{P_{rc}(f)}{|H_c(f)|} df \right]^{-1}.$$

$$S_\nu(f) = S_n(f)\, |H_R(f)|^2. \Rightarrow \sigma^2 = \frac{N_0}{2} \int_{-W}^{W} |H_R(f)|^2 df = \frac{N_0}{2} \int_{-W}^{W} \frac{P_{rc}(f)}{|H_c(f)|} df.$$

$$\Rightarrow SNR = \frac{2\mathcal{E}_{av}}{N_0} \cdot \left[\int_{-W}^{W} \frac{P_{rc}(f)}{|H_c(f)|} df \right]^{-2}$$

↳ due to channel distortion.

∘ Example: [illegible] Design transmitting and receiving filter for a given channel.

Determine the magnitude of the transmitting and receiving filter characteristics for a binary communication system that transmits data at a rate of 4800 bits/sec over a channel with frequency response

$$H_c(f) \qquad |C(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{W}\right)^2}}, \qquad |f| \le W,$$

where $W = 4800$ Hz. The additive noise is zero-mean white Gaussian with a spectral density $N_0/2$. $= 10^{-15}$ Watt/Hz

$\Rightarrow$ Symbol rate $\frac{1}{T} = 4800$ bits/sec. Nyquist bandwidth $B_0 = \frac{1}{2T} = 2400$ Hz.

For $P_{rc}(f)$, its bandwidth $B = B_0(1+\alpha)$. $\Rightarrow \alpha = 1$. $f_1 = 0$.

Thus: $P_{rc}(f) = \frac{1}{2}\left[1 + \cos\left(\frac{\pi |f|}{2B_0}\right)\right] = \cos^2\left(\frac{\pi |f|}{9600}\right)$.

Then: $|H_T(f)| = |H_R(f)| = \cos\left(\frac{\pi |f|}{9600}\right) \left[1 + \left(\frac{f}{4800}\right)^2\right]^{\frac{1}{4}}$. $|f| \le 4800$.

8. Equalizer

↳ A receive filter with adjustable frequeme response.
to minimize the ISI.

$$\begin{pmatrix}\text{Transmitting Filt}\\ H_T(f)\end{pmatrix} \rightarrow \begin{pmatrix}\text{Channel}\\ H_c(f)\end{pmatrix} \rightarrow \underset{\uparrow\, n(t)}{\oplus} \rightarrow \begin{pmatrix}\text{Equalizer}\\ H_E(f)\end{pmatrix} \xrightarrow{V(t)} \overset{t=mT}{\diagdown} \text{——} V_m.$$

$\Rightarrow H_0(f) = H_T(f)\, H_c(f)\, H_E(f)$. Nyquist criterion: $\sum_{k=-\infty}^{\infty} H_0(f+\frac{k}{T}) = \text{constant}$.

Linear Transversal Filter (横向滤波器).

Finite impulse response filter:

$$h_E(t) = \sum_{n=-N}^{N} C_n \delta(t-nT). \qquad H_E(t) = \sum_{n=-N}^{N} C_n e^{-j2\pi f nT}. \Rightarrow \text{(2N+1)-tap FIR}$$

$\{C_n\}$ are the $2N+1$ equalizer coefficients.
N is sufficiently large to span the length of ISI.

Zero-Forcing Equalizer (迫零均衡器).

$$\delta(t) \rightarrow \boxed{H_T(f),\ H_c(f)} \xrightarrow{P_c(f)} \boxed{H_E(f)} \rightarrow P_{eq}(t).$$

↳ $h_E(t) = \sum_{n=-N}^{N} C_n \delta(t-nT)$.

$$\Rightarrow P_{eq}(t) = P_c(t) * h_E(t) = \sum_{n=-N}^{N} C_n P_c(t-nT).$$

Sample at $t=mT$: $P_{eq}(mT) = \sum_{n=-N}^{N} C_n P_c[(m-n)T] = \begin{cases} 1, & m=0 \\ 0, & m=\pm 1, \cdots \pm N. \end{cases}$

⇓

suppress 2N adjacent interference terms.

$$P_{eq} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.\quad C = \begin{pmatrix} C_{-N} \\ C_{-N+1} \\ \vdots \\ C_{-1} \\ C_0 \\ C_1 \\ \vdots \\ C_N \end{pmatrix}.\quad P_c = \begin{pmatrix} P_c(0) & P_c(-1) & \cdots & P_c(-2N) \\ P_c(1) & P_c(0) & \cdots & P_c(-2N+1) \\ \vdots & \vdots & & \vdots \\ P_c(2N) & P_c(2N-1) & \cdots & P_c(0) \end{pmatrix}$$

⇓

Channel response matrix.

$\Rightarrow C = P_c^{-1} P_{eq}$ or the middle-column of P_c^{-1}.

equalized pulse response: $P_{eq}(m) = \sum_{n=-2}^{2} C_n P_c(m-n)$.

Minimum Mean-Square Error Equalizer
Decision Feedback Equalizer
Maximum Likehood Sequence Estimation.
Monte Carlo Simulation

Chapter 06: Signal Space Representation.

1. vectors and space.

an n-dimensional space with unity basis vectors $\{\vec{e}_1, \vec{e}_2, \cdots \vec{e}_n\}$
any vector $\vec{a}$ in the space can be written as: $\vec{a} = \sum a_i \vec{e}_i$.
Orthonormal: mutually orthogonal and all have unity norm.

The set of basis vectors $\{\vec{e}_1, \vec{e}_2, \cdots \vec{e}_n\}$ of a space:
① complete ② orthogonal ③ normalized.
⇒ complete orthonormal basis (完备正交基).

2. signal space and orthonormal basis.

signal set $\{\phi_k(t)\}_{k=1}^{n}$ is an orthogonal set if

$$\int_{-\infty}^{\infty} \phi_j(t)\phi_k(t)\,dt = \begin{cases} 0, & j \neq k \\ c_j, & j = k. \end{cases}$$

if $c_j \equiv 1, \forall j$. then $\{\phi_k(t)\}$ is an orthonormal set.

Then: $x(t) = \sum_{k=1}^{n} x_k \phi_k(t)$. $x_k = \int_{-\infty}^{\infty} x(t)\phi_k(t)\,dt$, $\vec{x} = (x_1, x_2, \cdots x_n)$.

o Properties: given two signals $x(t) = \sum_{i=1}^{n} x_i \phi_i(t)$. $y(t) = \sum_{i=1}^{n} y_i \phi_i(t)$
$\vec{x} = (x_1, x_2, \cdots x_n)$. $\vec{y} = (y_1, y_2, \cdots y_n)$.

then $\vec{x}\cdot\vec{y} = \int_{-\infty}^{\infty} x(t)y(t)\,dt$.

Energy of signal $x(t)$: $E_x = \int_{-\infty}^{\infty} x^2(t)\,dt$, $E_x = \vec{x}\cdot\vec{x} = \|x\|^2$.

3. Basic functions for a signal set.

- Consider a set of M signals (M-ary symbol) $\{s_i(t), i = 1, 2, ..., M\}$ with finite energy. That is

$$\int_{-\infty}^{\infty} s_i^2(t)\,dt < \infty$$

- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^{K} s_{ij}\phi_j(t) \qquad \text{for } i = 1, \ldots, M$$

where K ≤ M is the dimension of the signal space and $\{\phi_j(t)\}_{j=1}^{K}$ are called the orthonormal basis functions

○ Example (QPSK).

signal: $s(t) = \pm\sqrt{\frac{2}{T}}\cos(2\pi f_c t) \pm \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$ with $t \in [0, T)$. and $f_c T \gg 1$.

basis function: $\phi_1(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t)$. $\phi_2(t) = \sqrt{\frac{2}{T}}\sin(2\pi f_c t)$. $t \in [0, T)$.

$$\Rightarrow \int_0^T \phi_1(t)\phi_2(t)\,dt = \int_0^T \sqrt{\frac{2}{T}}\cos(2\pi f_c t)\cdot\sqrt{\frac{2}{T}}\sin(2\pi f_c t)\,dt$$

$$= \frac{2}{T}\int_0^T \frac{1}{2}[\sin(0) + \sin(4\pi f_c t)]\,dt$$

$$= -\frac{1}{4\pi f_c T}[\cos(4\pi f_c t)]_0^T \approx 0.\ f_c T \gg 1.$$

$$\int_0^T |\phi_1(t)|^2 dt = \int_0^T |\phi_2(t)|^2 dt = \frac{2}{T}\int_0^T \frac{1}{2}[1 + \cos(4\pi f_c t)]\,dt \approx 1.$$

○ notes:

① Different signal sets can have the same geometric representation.

② The underlying geometry will determine the performance and the receiver.

4. GSO (find a complete orthonormal basis for an arbitrary signal set).

Given a signal set $\{s_1(t), s_2(t) \cdots s_M(t)\}$.

Find the basis functions $\{\phi_1(t)\ \phi_2(t) \cdots \phi_N(t)\}$. where $N \le M$

Step 1: Construct the first basis function.

$\phi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$ (Normalised by the energy $E_1 = \int_{-\infty}^{\infty} s_1^2(t)\,dt$).

then: $s_{11} = \int_{-\infty}^{\infty} s_1(t)\phi_1(t)\,dt = \sqrt{E_1}$. $s_1(t) = s_{11}\phi_1(t) = \sqrt{E_1}\phi_1(t)$.

Step 2: the correlation between $s_2(t)$ and $\phi_1(t) = s_{21} = \int_{-\infty}^{\infty} s_2(t)\phi_1(t)\,dt$

Subtract off the correlation portion: $g_2(t) = s_2(t) - s_{21}\phi_1(t)$

↳ $g_2(t)$ is orthogonal to $\phi_1(t)$.

the energy $= E_{g_2} = \int_{-\infty}^{\infty} [g_2(t)]^2 dt$.

normalize: $\phi_2(t) = \frac{1}{\sqrt{E_{g_2}}} g_2(t)$.

then: $\int_{-\infty}^{\infty} s_2(t)\phi_2(t)\,dt = s_{22} = \sqrt{E_{g_2}}$.

Step 3: Construct the successive basis functions.

For signal $s_k(t)$: $s_{ki} = \int_{-\infty}^{\infty} s_k(t)\phi_i(t)dt$, $i=1,2,\dots k-1$.

$$g_k(t) = s_k(t) - \sum_{i=1}^{k-1} s_{ki}\phi_i(t).$$

$$E_{g_k} = \int_{-\infty}^{\infty} [g_k(t)]^2 dt$$

k-th basis function: $\phi_k(t) = \frac{1}{\sqrt{E_{g_k}}} \cancel{s_k}\,g_k(t)$. $s_{kk} = \sqrt{E_{g_k}}$. $s_k(t) = \sqrt{E_{g_k}}\phi_k(t)$

- Consider a set of M signals (M-ary symbol) $\{s_i(t), i = 1, 2, ..., M\}$ with finite energy. That is

$$\int_{-\infty}^{\infty} s_i^2(t)dt < \infty$$

- Then, we can express each of these waveforms as weighted linear combination of orthonormal signals

$$s_i(t) = \sum_{j=1}^{K} s_{ij}\phi_j(t) \qquad \text{for } i = 1, \dots, M$$

where K ≤ M is the dimension of the signal space and $\{\phi_j(t)\}_{j=1}^{K}$ are called the orthonormal basis functions

Notes on GSO Procedure

- A signal set may have many different sets of basis functions.
- But the dimensionality of the signal space is independent of the selected orthonormal basis.
- A change of basis functions is essentially a rotation of the signal points around the origin.
- The order in which signals are used in the GSO procedure affects the resulting basis functions
- The choice of basis functions does not affect the performance of the modulation scheme.

Chapter 07: Optimal Receivers.

1. Statistic Decision Theory.

2. Detection Theory.

3. MAP Decision Criterion ⇒ 最大后验概率准则

$P(m_i|\vec{r}) = P(\text{signal } m_i \text{ was transmitted given } \vec{r} \text{ observed})$. $i=1,\cdots M$.

给定此观测向量 $\vec{r}$ 后，可得到一系列后验概率 $P(m_1|\vec{r}).\ P(m_2|\vec{r}) \cdots P(m_M|\vec{r})$.

在这些概率有最大值 $P(m_k|\vec{r})$. 则根据 MAP 准则. 选择 $\hat{m}=m_k$

choose $\hat{m}=m_k$ iff $P(m_k|\vec{r}) \geq P(m_i|\vec{r})$ for all $i \neq k$.

似然函数.

By Bayes' Rule: $P(m_i|\vec{r}) = \dfrac{P_i f(\vec{r}|m_i)}{f(\vec{r})}$. then: $P_k f(\vec{r}|m_k) \geq P_i f(\vec{r}|m_i)$.

⇒ 最大后验概率准则和最小差错概率是等价的 P_i是信号m_i发送的先验概率.

4. ML Decision Criterion ⇒ 最大似然准则.

If signals are equiprobable: $P_1 = P_2 = \cdots = P_M$.

then $P_k f(\vec{r}|m_k) \geq P_i f(\vec{r}|m_i) \Rightarrow f(\vec{r}|m_k) \geq f(\vec{r}|m_i)$

choose $\hat{m}=m_k$. iff $f(\vec{r}|m_k) \geq f(\vec{r}|m_i)$. for all $i \neq k$

↳ Find the maximum liklihood function $f(\vec{r}|m_k)$.

5. Optimal Receiver in AWGN Channel.

$\{m(t)\} \rightarrow$ (Transmitter) $\xrightarrow{s(t)}$ channel $(\Sigma) \rightarrow r(t) = s(t) + n_w(t) \rightarrow$ (Receiver) $\rightarrow \{\hat{m}(t)\}$

$n_w(t) \sim S_n(f) = \dfrac{N_0}{2}$ ⇒ AWGN

Transmitter transmits a sequence of symbols $\{m_1, m_2 \cdots m_M\}$ with prior probabilities $P(m_1), P(m_2) \cdots P(m_M)$. Symbols are represented by finite energy waveforms $s_1(t), s_2(t), \cdots s_M(t)$. $t \in [0, T]$

The signal space $\{s_1(t), s_2(t) \cdots s_M(t)\}$ has a dimension N $(N \le M)$

Signals can be represented as: $s_m(t) = \sum_{k=1}^{N} s_{mk}\phi_k(t)$ $\quad s_{mk} = \int_0^T s_m(t)\phi_k(t)dt$.

the noise $n_w(t)$ can be written as:

$$n_w(t) = n_0(t) + \sum_{k=1}^{N} n_k\phi_k(t). \quad n_k = \int_0^T n_w(t)\phi_k(t)dt.$$

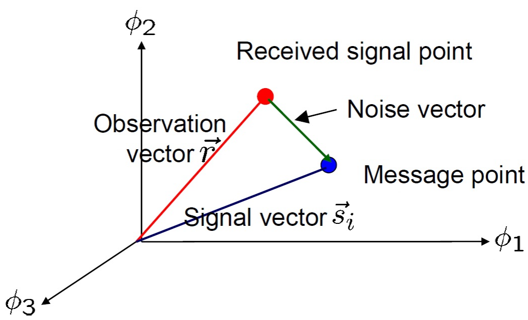
$\hookrightarrow$ falls outside the signal space spanned by $\phi_k(t)$.

Then the received signal:

$$r(t) = s(t) + n_w(t) = \sum_{k=1}^{N} s_{mk}\phi_k(t) + \sum_{k=1}^{N} n_k\phi_k(t) + n_0(t)$$

$$= \sum_{k=1}^{N} (s_{mk} + n_k)\phi_k(t) + n_0(t) = \sum_{k=1}^{N} r_k\phi_k(t) + n_0(t). \quad r_k = s_{mk} + n_k.$$

$$\vec{r} = \vec{s}_i + \vec{n}$$



6. Receiver Structure.

two parts:
- Signal demodulator: convert waveform $r(t)$ to $\vec{r} = (r_1, r_2 \cdots r_N)$
- Detector: decide which of the M possible signals were transmitted based on the observation $\vec{r}$.

$r(t) \longrightarrow$ (Signal demodulator) $\longrightarrow \vec{r} \longrightarrow$ (Detector) $\longrightarrow \hat{m}$.

Signal demodulator:
- correlation-Type demodulator
- matched-filter-type demodulator.

7. Matched Filter ⇒ 匹配滤波器.

↳ the opitmal linear filter to maximize the output SNR.

$x(t) = s_i(t) + n_i(t) \longrightarrow \left(h(t) \quad H(f) \right) \longrightarrow y(t) = s_o(t) + n_o(t) \longrightarrow \underset{t = t_0}{\text{switch}} \longrightarrow y(t_0)$

the output SNR =

$$d = \frac{s_o^2(t_0)}{E[n_o^2(t)]} = \frac{\left[\int_{-\infty}^{\infty} A(f) H(f) e^{j\omega t_0} df \right]^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df}$$

where $A(f) = \int_{-\infty}^{\infty} s_i(t) e^{-j\omega t} dt$. $S_{n_o}(f) = \frac{N_0}{2} |H(f)|^2 \Rightarrow N_{S_{n_o}} = E[n_o^2(t)] = \int_{-\infty}^{\infty} S_{n_o}(f) df$

↳ Find $H(f)$ to maximize d.

$$\Rightarrow d \leq \frac{\int_{-\infty}^{\infty} |A(f)|^2 df \int_{-\infty}^{\infty} |H(f)|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} = \frac{\int_{-\infty}^{\infty} |A(f)|^2 df}{\frac{N_0}{2}} = \underbrace{\frac{2E}{N_0}}_{\text{max output SNR}}.$$

E is signal energy.

When $\frac{2E}{N_0}$ is achieved: $\underbrace{H_m(f) = A^*(f) e^{-j\omega t_0}}_{\text{Schwarz不等式成立的条件.}} \Rightarrow h_m(t) = s_i^*(t_0 - t)$.

⇒ $h_m(t)$ is matched to the input signal $s_i(t)$.

$$\begin{cases} \text{Frequency response: } H_m(f) = A^*(f) e^{-j\omega t_0}. \\ \text{Impulse response: } h_m(t) = s_i^*(t_0 - t) \end{cases}$$

Properties of MF:

① $h_m(t) = \begin{cases} s_i(t_0 - t). & 0 \leq t < t_0 \\ 0 & \text{o.w.} \end{cases}$ where $\underline{t_0 > T}$.

↳ 为了满足因果性.

② Equivalent from Correlator 相关器.

$y(t) = x(t) * h_m(t) = x(t) * s_i(T - t) = \int_0^T x(\tau) s_i(T - t + \tau) d\tau$.

第一章相关已有介绍

⇑

sampling time $t = T$: $y(T) = \int_0^T x(t) s_i(t) dt$. ⇒ correlation integration.

$x(t) \longrightarrow (MF) \xrightarrow{t = T} y(T) = x(t) \longrightarrow \otimes \longrightarrow \left(\int_0^T (\cdot) dt \right) \longrightarrow y(T)$.

(input to $\otimes$: $s_i(t)$)

Temporal Autocorrelation vs. Statistical Autocorrelation

1.Temporal autocorrelation function	2.statistical autocorrelation function
$R(\tau) = \int_{-\infty}^{\infty} s(t) s(t+\tau) dt$	$R(\tau) = E[X(t) X(t+\tau)]$
(Temporal crsss-correlation function $R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t) s_2(t+\tau) dt$)	(statistical crosscorrelation function $R_{XY}(\tau) = E[X(t) Y(t+\tau)]$)
(1) $R(\tau) = R(-\tau)$	(1) $R(\tau) = R(-\tau)$
(2) $R(0) \geq R(\tau)$	(2) $R(0) \geq R(\tau)$
(3) $R(0) = \int_{-\infty}^{\infty} s^2(t) dt = E$ – signal energy	(3) $R(0) = E[X^2(t)] = P$ – signal average power
(4) $R(\tau) \leftrightarrow \lvert A(f) \rvert^2$ – signal energy spectrum	(4) $R(\tau) \leftrightarrow S_X(f)$ – signal power spectral density

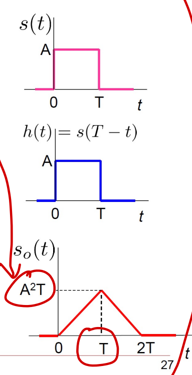
Time averaging autocorrelation function $\langle X(t) X(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t) X(t+\tau) dt$

Example: MF for a rectangular pulse

- Consider a rectangular pulse $s(t)$

  $E_s = A^2 T$
- The impulse response of a filter matched to $s(t)$ is also a rectangular pulse
- The output of the matched filter $s_0(t)$ is $h(t) * s(t)$
- The output SNR is

  $(SNR)_o = \frac{2}{N_0} \int_0^T s^2(t) dt = \frac{2 A^2 T}{N_0}$



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③ MF output signal is the autocorrelation function of input signal.

$$s_o(t) = \int_{-\infty}^{\infty} s_i(t-u) h_m(u) du = \int_{-\infty}^{\infty} s_i(t-u) s_i(t_0-u) du$$

$$= \int_{-\infty}^{\infty} s_i(v) s_i(v+t-t_0) dv = R_{s_i}(t-t_0).$$

The peak value of $s_o(t)$ happens $t=t_0$: $s_o(t_0) = R_{s_i}(0) = \int_{-\infty}^{\infty} s_i^2(v) dv = E_{s_i}$.

$s_o(t)$ is symmetric at $t=t_0$ ⇒ 自相关函数的性质.

$$A_o(f) = A(f) H_m(f) = |A(f)|^2 e^{-j\omega t_0}.$$

④ MF output noise. $n_o(t) = n_w(t) * h_m(t) = \int_0^T n_w(t) h_m(T-t) dt = \int_0^T n_w(t) s_i(t) dt.$

The statistic autocorrelation of $n_o(t)$:

$$R_o(\tau) = E[n_o(t) n_o(t+\tau)] = \frac{N_0}{2} \int_{-\infty}^{\infty} h_m(u) h_m(u+\tau) du = \frac{N_0}{2} \int_{-\infty}^{\infty} s_i(t) s_i(t-\tau) d\tau.$$

Average power:

$$E[n_o^2(t)] = R_{n_o}(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} s_i^2(u) du = \frac{N_0}{2} \int_{-\infty}^{\infty} |A(f)|^2 df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H_m(f)|^2 df = \frac{N_0}{2} E.$$

When the noise is colored (not white and $S_n(f) = \frac{N_0}{2}$):

⇒ choose $H_1(f)$ so that $n'(t)$ is white. $H(f) = H_1(f) H_2(f)$.

$S_n'(f) = |H_1(f)|^2 S_n(f) = C$. and $H_2(f)$ matches with $s'(t) = s(t) * h_1(t)$.

⇒ $H(f) = |H_1(f)|^2 A^*(f) e^{-j2\pi f t_0}$ (white noise: $H(f) = A^*(f) e^{-j\omega t_0}$).

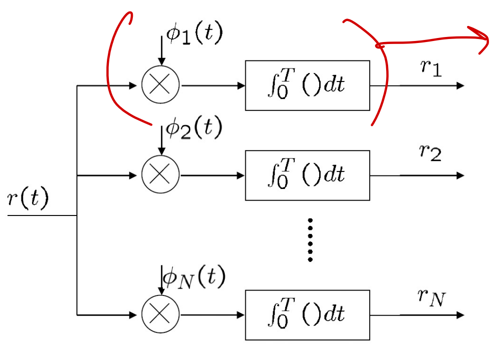
$$H(f) = \frac{A^*(f)}{S_n(f)} e^{-j2\pi f t_0}$$ ⇒ MF for colored noise

8. Two types of demodulator

① Correlation Type Demodulator.

- The received signal r(t) is passed through a parallel bank of N cross correlators which basically compute the projection of r(t) onto the N basis functions $\{\phi_k(t), k = 1, \ldots N\}$

用相关器实现的MF.



To detector

$r(t) = \sum_{k=1}^{N} r_k \phi_k(t).$

对于第i个相关器. 只有 $r_i = r_k$ 时. 输出才不为0. 其他均为0.

② Matched-Filter type demodulator

- Alternatively, we may apply the received signal r(t) to a bank of N matched filters and sample the output of filters at t = T. The impulse responses of the filters are

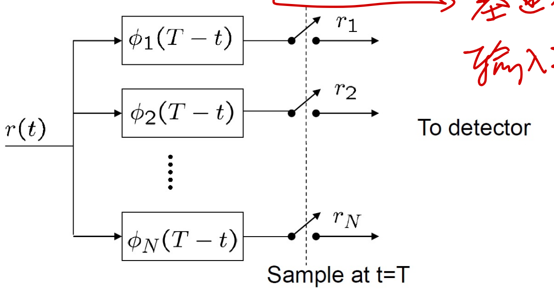
$$h_k(t) = \phi_k(T-t), \quad 0 \le t \le T$$

→ 基函数的匹配滤波器.

输入其实是 $r_k \phi_k(t)$. $r(t) = \sum_k^N r_k \phi_k(t)$.

$\downarrow$ MF

$\phi_k(T-t)$.



Signal demodulator 输出观测向量 $\vec{r} = (r_1, r_2, \cdots r_N)$ 后. 通过 signal detector 进行判决以达到最小误差准则.

8. Optimal Detector

① Decision Rules

$$\begin{cases} \text{MAP}: \text{ choose } \hat{m} = m_k \text{ iff } P_k f(\vec{r}|m_k) > P_i f(\vec{r}|m_i).\ i \ne k. \\ \text{ML}: \text{ choose } \hat{m} = m_k \text{ iff } f(\vec{r}|m_k) > f(\vec{r}|m_i).\ i \ne k. \end{cases}$$

② Distribution of the Noise Vector.

$n_w(t) \Rightarrow$ Gaussian random process.

$n_k = \int_0^T n_w(t)\phi_k(t)dt \Rightarrow$ Gaussian random variable.

↳ mean: $E[n_k] = \int_0^T E[n_w(t)]\phi_k(t)dt = 0.$

correlation:

$$E[n_j n_k] = \frac{N_0}{2}\int_0^T \phi_j(\tau)\phi_k(\tau)d\tau = \begin{cases} \frac{N_0}{2}, & j = k \\ 0, & j \ne k \end{cases}$$ (PPT. P36).

Then: n_k are independent with zero mean and variance $\frac{N_0}{2}$.

The joint pdf of $\vec{n} = (n_1, n_2 \cdots n_k)$:

$$p(n_1, n_2 \cdots n_k) = \prod_{k=1}^N p(n_k) = (\pi N_0)^{-N/2} \exp\left(-\sum_{k=1}^N n_k^2 / N_0\right).$$

③ Likelihood function.

If m_k is transmitted. $\vec{r} = \vec{s}_k + \vec{n} \Rightarrow r_j = s_{kj} + n_j.$

$\Rightarrow E[r_j|m_k] = s_{kj} + E[n_j] = s_{kj}.$

$Var[r_j|m_k] = Var[n_j] = \frac{N_0}{2}.$

conditional pdf of $\vec{r} = (r_1, r_2 \cdots r_N)$:

$$f(\vec{r}\,|\,m_k) = \prod_{j=1}^{N} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r_j - S_{kj})^2}{N_0}\right) = (\pi N_0)^{-N/2} \exp\left(-\frac{\sum_{j=1}^{N}(r_j - S_{kj})^2}{N_0}\right)$$

$$\Rightarrow \ln f(\vec{r}\,|\,m_k) = -\frac{N_0}{2}\ln(\pi N_0) - \frac{1}{N_0}\sum_{j=1}^{N}(r_j - S_{kj})^2.$$

$$D^2(\vec{r}, \vec{s}_k) = \sum_{j=1}^{N}(r_j - S_{kj})^2 = \underline{\|\vec{r} - \vec{s}_k\|^2} \Rightarrow \text{Euclidean distance.}$$

Optimum Detector:

MAP: $\hat{m} = \arg\max_{\{m_1 \cdots m_M\}} f(\vec{r}\,|\,m_k)P(m_k) = \arg\min_{\{m_1 \cdots m_M\}} \{\|\vec{r} - \vec{s}_k\|^2 - N_0 \ln P_k\}.$

ML: $\hat{m} = \arg\min_{\{m_1 \cdots m_M\}} \|\vec{r} - \vec{s}_k\|^2.$ $\Rightarrow$ Minimum distance detection

$\hookrightarrow P_1 = P_2 = \cdots = P_M.$

9. Optimal Receiver Structure

$$-\sum_{j=1}^{N}(r_j - S_{kj})^2 + N_0 \ln P_k = \underline{-\|\vec{r}\|^2 - \|\vec{s}_k\|^2 + 2\vec{r}\cdot\vec{s}_k + N_0 \ln P_k}.$$

$\|\vec{s}_k\|^2 = \int_0^T s_k^2(t)\,dt = E_k \Rightarrow$ signal energy.

$\vec{r}\cdot\vec{s}_k = \int_0^T s_k(t)r(t)\,dt \Rightarrow$ correlation between the received signal vector and the transmitted signal vector.

$\|\vec{r}\|^2 \Rightarrow$ ignored.

$$\Rightarrow \hat{m} = \arg\max_{\{m_1 \cdots m_M\}} \left\{\vec{r}\cdot\vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2}\ln P_k\right\}.$$

MAP Receiver Structure:

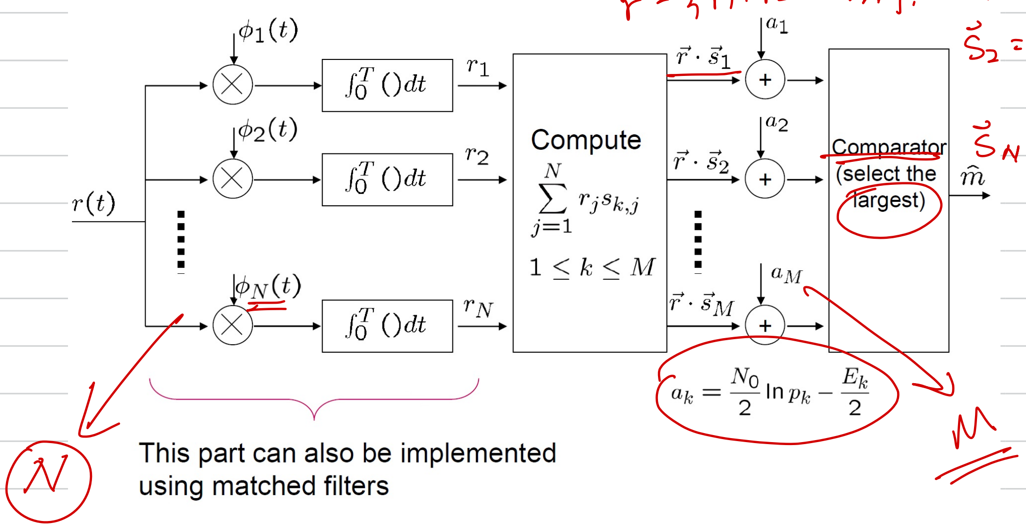
$\vec{r} = \{r_1, r_2, \cdots r_N\}$.

$\vec{s}_1 = \{S_{11}, S_{12}, \cdots S_{1N}\}$

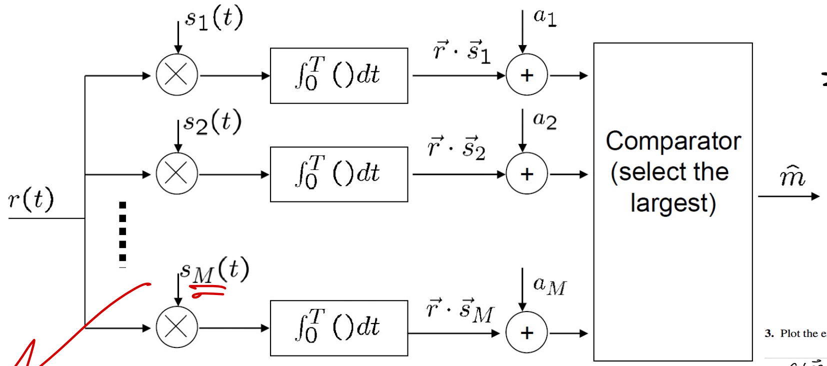
$\vec{s}_2 = \{S_{21}, S_{22}, \cdots S_{2N}\}$

$\vdots$

$\vec{s}_N = \{S_{N1}, S_{N2} \cdots S_{NN}\}$.



A $C(\vec{r}, \vec{s}_m) = \int_{-\infty}^{+\infty} r(t) s_m(t)\,dt - \frac{1}{2}\int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2}\ln p(s_m)$ correlator receiver



⇒ combine the demodulator and detector.

M

This part can also be implemented using matched filters

$$a_k = \frac{N_0}{2}\ln p_k - \frac{E_k}{2}$$

$$\hat{m} = \arg\max_{m_1,\ldots,m_M}\left\{\vec{r}\cdot\vec{s}_k - \frac{E_k}{2} + \frac{N_0}{2}\ln P_k\right\}$$

3. Plot the error probability as a function of p for $0 \le p \le 1$.

$C(\vec{r}, \vec{s}_m) = \int_{-\infty}^{\infty} r(t)s_m(t)\,dt - \frac{1}{2}\int_{-\infty}^{\infty}|s_m(t)|^2 dt + \frac{N_0}{2}\ln p(s_m)$ → metric

10. Graphical Interpretation of Decision Regions.

Divide the signal space into M disjoint decision regions $R_1, R_2 \cdots R_M$.

then if $\vec{r} \in R_k$ decide m_k was transmitted.

If $P_k = \frac{1}{M}$ for all k. then choose $\hat{m} = m_k$ iff $\|\vec{r} - \vec{s}_k\|^2$ is minimized.

Take projection of $r(t)$ in the signal space ($\vec{r}$). then decision is made in favor of signal that is closest to $\vec{r}$ in sense of minimum Euclidean distance.

o Binary case:

Consider binary data transmission over AWGN channel with PSD $S_n(f) = \frac{N_0}{2}$

$s_1(t) = -s_2(t) = \sqrt{E}\phi(t)$. $P(m_1) \neq P(m_2)$.

Determine the optimal receiver (regions).

⇒ optimal decision:

$$\|\vec{r} - \vec{s}_1\|^2 - N_0 \ln p(m_1) \underset{m_2}{\overset{m_1}{\lessgtr}} \|\vec{r} - \vec{s}_2\|^2 - N_0 \ln p(m_2).$$

if $d_1 = \|\vec{r} - \vec{s}_1\|$, $d_2 = \|\vec{r} - \vec{s}_2\|$ then: $\quad d_1^2 - d_2^2 \underset{m_2}{\overset{m_1}{\lessgtr}} \left(N_0 \ln \frac{P(m_1)}{P(m_2)}\right) \equiv C.$

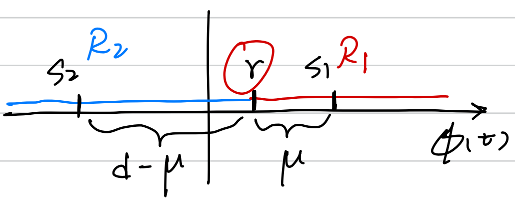
↳门限.

Then $R_1: d_1^2 - d_2^2 < C$. $\quad R_2: d_1^2 - d_2^2 > C$.

$$\begin{cases} d = d_1 + d_2 \\ d_1 = \mu \\ d_2 = d - \mu \end{cases} \Rightarrow \quad d_1^2 - d_2^2 = 2d\mu - d^2 \equiv C$$

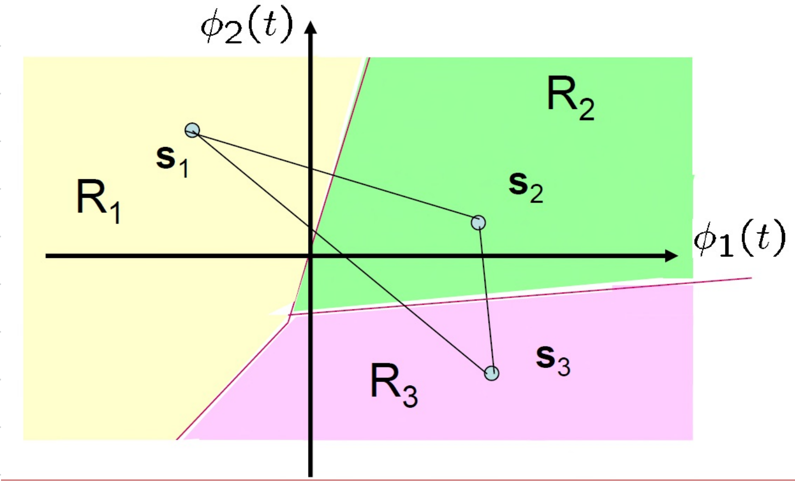
$$\mu = \frac{C + d^2}{2d} = \frac{d}{2} + \frac{N_0}{2d}\ln\frac{P(m_1)}{P(m_2)} \Rightarrow \mu \begin{cases} = \frac{d}{2}. & p(m_1) = p(m_2) \\ > \frac{d}{2}. & p(m_1) > p(m_2) \\ < \frac{d}{2}. & p(m_1) < p(m_2) \end{cases}$$

⇓



先验概率 $p(m_1)\cdot p(m_2)$ 对决策区域的影响.

⇒ In general, boundaries of decision regions are perpendicular bisectors of lines joining the origin ~~transmitted~~ signals.



○ Exercise:

Three equally probable messages m1, m2, and m3 are to be transmitted over an AWGN channel with noise power-spectral density $N_0/2$. The messages are

$$s_1(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & otherwise \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \le t \le \frac{T}{2} \\ -1 & \frac{T}{2} \le t \le T \\ 0 & otherwise \end{cases}$$

1. What is the dimensionality of the signal space ?
2. Find an appropriate basis for the signal space (Hint: You can find the basis without using the Gram-Schmidt procedure).
3. Draw the signal constellation for this problem.
4. Sketch the optimal decision regions R1, R2, and R3.

Notes:

- Boundaries are perpendicular to a line drawn between two signal points
- If signals are equiprobable, decision boundaries lie exactly halfway in between signal points
- If signal probabilities are unequal, the region of the less probable signal will shrink

11. Probability of Error using Decision Regions.

Average probability of making correct decision: $P_c = \sum_{k=1}^{M} P(\vec{r} \in R_k \mid m_k \text{ is sent}) P(m_k)$

Averge probability of error: $P_e = 1 - P_c = 1 - \sum P(\vec{r} \in R_k \mid m_k \text{ is sent}) P(m_k)$.

o Consider binary data transmission:

given m_1 is transmitted, then $p(c|s_1) = p(r \in R_1 | s_1)$

$= p(s_1 + n > s_1 - \mu) = p(n > -\mu)$.

n is Gaussian with zero mean and variance $N_0/2$.

and $\mu = \frac{d}{2} + \frac{N_0}{2d} \ln \frac{p(m_1)}{p(m_2)}$. then $p(c|s_1) = 1 - Q\left(\frac{\mu}{\sqrt{N_0/2}}\right)$. ⇒ Q 函数.

$p(c|s_2) = 1 - Q\left(\frac{d-\mu}{\sqrt{N_0/2}}\right)$. $p(c) = p(m_1) p(c|s_1) + p(m_2) p(c|s_2)$.

$\Rightarrow P_e = 1 - p(c) = p(m_1) Q\left(\frac{\mu}{\sqrt{N_0/2}}\right) + p(m_2) Q\left(\frac{d-\mu}{\sqrt{N_0/2}}\right)$.

where $d = 2\sqrt{E}$. $\mu = \frac{N_0}{4\sqrt{E}} \log\left[\frac{p(m_1)}{p(m_2)}\right] + \sqrt{E}$.

if $p(m_1) = p(m_2)$. then $\mu = \sqrt{E} = \frac{d}{2}$. $P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$.

$SNR = \frac{E}{N_0/2}$

- This example demonstrates an interesting fact:
  - When optimal receiver is used, P_e does not depend upon the specific waveform used
  - P_e depends only on their geometrical representation in signal space
  - In particular, P_e depends on signal waveforms only through their energies (distance)

$$\boxed{P_e = Q\left[\frac{d/2}{\sqrt{N_0/2}}\right] = Q\left[\sqrt{\frac{d^2}{2N_0}}\right] = Q\left[\sqrt{\frac{2E}{N_0}}\right]}$$

o General Expression for P_e.

$p(\vec{r} \in R_k \mid m_k \text{ is sent}) = \int_{R_k} f(\vec{r} \mid m_k) d\vec{r} \Rightarrow P_e = 1 - \frac{1}{M} \sum_{k=1}^{M} \int_{R_k} f(\vec{r} \mid m_k) d\vec{r}$.

$A_{kj} \Rightarrow \vec{r}$ is closer to $\vec{s}_j$ than to $\vec{s}_k$ in the signal space when $m_k (\vec{s}_k)$ is sent.

$P(e|m_k) = p(\vec{r} \notin R_k \mid m_k) = P\left(\bigcup_{j \neq k} A_{kj}\right) \leq \sum_{\substack{j=1 \\ j \neq k}}^{M} P(A_{kj})$.

pair-wise error probability: $p(\vec{s}_k \to \vec{s}_j) = p(A_{kj})$.

$p(\vec{s}_k \to \vec{s}_j) = p\left(n > \frac{d_{kj}}{2}\right) = Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right)$, $d_{kj} = \|\vec{s}_k - \vec{s}_j\|$.

$\Rightarrow P_e = \frac{1}{M} \sum_{k=1}^{M} p(e|m_k) \leq \frac{1}{M} \sum_{k=1}^{M} \sum_{\substack{j=1 \\ j \neq k}}^{M} Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right)$. ⇒ union bound (upper bound) let $d_{min} = \min_{k \neq j} \{d_{kj}\}$, then $P_e \leq (M-1) Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right)$ ⇒ simplified union bound.

Notes:

- The union bound provides a very useful bound on the error probability, particularly at high SNR
- At low SNR, the bound becomes loose and useless. More powerful bounding techniques are needed
- The union bound signifies the role of the minimum distance of a signal set on its performance, particularly at large SNRs.
- A good signal set should provide the maximum possible value of d_{min} 给出 Pe 的最大值，
- In other words, to design a good signal set, the points the corresponding constellation should be maximally apart.

Chapter 08. Digital Modulation Techniques

1. Digital Modulation

↳ switching or keying the amplitude, frequency or phase of a sinusoidal carrier wave according to incoming digital data.

Three basic digital modulation techniques:

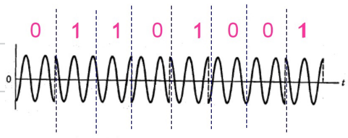
- Amplitude-shift keying (ASK) —— AM
- Frequence-shift keying (FSK) —— FM
- Phase-shift keying (PSK) —— PM.

Use signal space approach in receiver design and performance analysis.

2. Binary Modulation.

the modulator produces one of two distinct signals in response to one bit of source data at a time.

① BPSK = Binary Phase-Shift Keying.



"1" ⇒ $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$. $A = \sqrt{\frac{2E_b}{T_b}} \Rightarrow P = \frac{E_b}{T_b}$.

"0" ⇒ $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$.

T_b = bit duration. $0<t<T_b$. ⇒ bit rate $R_b = \frac{1}{T_b}$.

f_c = carrier frequency. chosen to be $\frac{n_c}{T_b}$ for fixed integer n_c. or $f_c \gg \frac{1}{T_b}$.

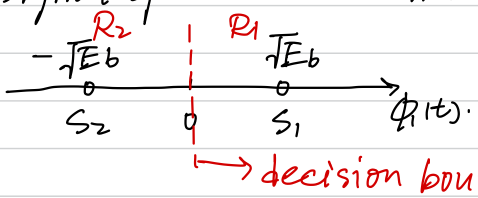
E_b = transmitted signal energy per bit.

$$\int_0^{T_b} s_1^2(t)\,dt = \int_0^{T_b} s_2^2(t)\,dt = E_b. \Rightarrow \text{power} = \frac{E_b}{T_b} = \frac{1}{2}\cdot\left(\sqrt{\frac{2E_b}{T_b}}\right)^2.$$

use one basis function = $\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$. $0<t<T_b$. ⇒ $E_{\phi_1(t)} = \frac{1}{2}\cdot\left(\sqrt{\frac{2}{T_b}}\right)^2 \cdot T_b = 1$.

Then $s_1(t) = \sqrt{E_b}\,\phi_1(t)$. $s_2(t) = -\sqrt{E_b}\,\phi_1(t)$.

↳ signal space is one-dimensional. and has two message points. ($M=2$).



$d_{12} = 2\sqrt{E_b}$.

Decision Rule of BPSK.

assume that binary signals are equiprobable: $P(s_1) = P(s_2) = \frac{1}{2}$.

then guess $s_1(t)$ if received point r falls in R_1 ($r>0$).

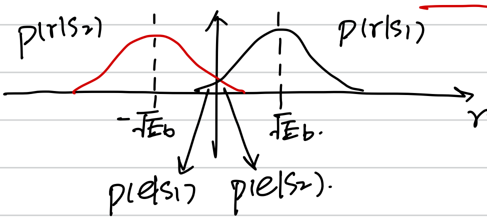
guess $s_2(t)$ if received point r falls in R_2 ($r\le 0$).

Probability of Error for BPSK.

conditional probability =

if $s_1(t)$ is transmitted. then $p(e|s_1) = p(r<0|s_1) = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{E_b})^2}{N_0}} dr = \underline{Q(\sqrt{\frac{2E_b}{N_0}})}$.

Similarly. $p(e|s_2) = p(r>0|s_2) = \underline{Q(\sqrt{\frac{2E_b}{N_0}})}$.



$\sim N(\sqrt{E_b}, \frac{1}{2}N_0)$

$\frac{x-\sqrt{E_b}}{\sqrt{N_0/2}} \sim \Phi(0,1)$.

the average probability of error = $P_e = 0.5\, p(e|s_1) + 0.5\, p(e|s_2) = \underline{Q(\sqrt{\frac{2E_b}{N_0}})}$.

the ratio $(\frac{E_b}{N_0})$ is bit energy to noise energy ratio (SNR/bit)

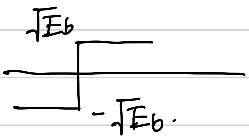
↳ bit energy E_b.

BPSK Transmitter =

Input data → (spectrum shaping filter) $\xrightarrow{m(t)}$ (product modulator) → Binary PSK wave $s(t)$.

↑ carrier wave $\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$.

$m(t) =$



BPSK Receiver =

→ phase offset.

$r(t) = \pm\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \underline{\theta}) + n(t)$

→ projection on basis func.

$\longrightarrow \otimes \longrightarrow \int_0^{T_b} dt \xrightarrow{\underline{r}}$ Decision device $\longrightarrow \begin{cases} 1. & \text{if threshold exceeded} \\ 0. & \end{cases}$

↑ $\underbrace{\sqrt{\frac{2}{T_b}} \cos(2\pi f_c t + \theta)}_{\text{demodulator}}$. ← basis function

$\underbrace{\qquad\qquad\qquad}_{\text{detector}}$

⇒ the detection is coherent.

② BFSK.

"1" → $s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t)$. $0<t<T_b$.

"0" → $s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_2 t)$.

E_b = transmitted energy per bit.

f_i = transmitted frequency with separation $\Delta f = f_1 - f_0$.

$\underline{\Delta f}$ makes $s_1(t)$ and $s_2(t)$ are orthogonal ⇒ $\int_0^{T_b} s_1(t) s_2(t) dt = 0$.

⇒ $f_1 = \frac{k_1}{T_b}$. $f_2 = \frac{k_2}{T_b}$. $\Delta f = \frac{n}{T_b}$. min of ?

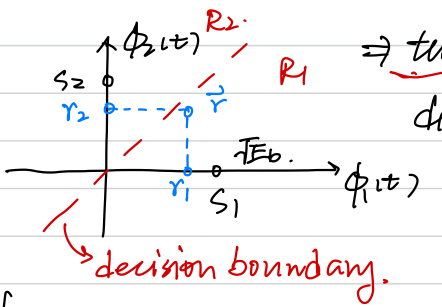
Two orthogonal basis functions are used:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t), \quad 0 \le t < T_b \qquad \Rightarrow \qquad s_1(t) = \sqrt{E_b}\,\phi_1(t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t), \quad 0 \le t < T_b \qquad\qquad s_2(t) = \sqrt{E_b}\,\phi_2(t).$$

signal space:

$s_1 = [\sqrt{E_b}\ \ 0]$ $\quad s_2 = [0.\ \sqrt{E_b}]$



$\Rightarrow$ two-diensional.

$d_{12} = \sqrt{2E_b}$.

observation vector $\vec{r} = [r_1\ r_2]$.

$$r_1 = \int_0^{T_b} r(t)\phi_1(t)\,dt$$
$$r_2 = \int_0^{T_b} r(t)\phi_2(t)\,dt$$
$\Rightarrow$ projection on basis func.

decision rule: when $r_1 > r_2$. choose s_1. when $r_1 < r_2$. choose s_2.

Probability of Error for BFSK:

if $s_1/s_1(t)$ is transmitted: $r_1 = \sqrt{E_b} + n_1$. $r_2 = n_2$.

then $p(e|s_1) = p(r_1 < r_2 | s_1) = p(\sqrt{E_b} + n_1 < n_2) = p(n_1 - n_2 < -\sqrt{E_b})$.

n_1 and n_2 are i.i.d with $n_1 . n_2 \in N(0, \frac{N_0}{2})$. so $n = n_1 - n_2 \in N(0, N_0)$.

$$\Rightarrow p(e|s_1) = p(n < -\sqrt{E_b}) = \int_{-\infty}^{-\sqrt{E_b}} \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{x^2}{N_0 2}}\,dx = Q\left(\sqrt{\frac{E_b}{N_0}}\right).$$

Similarly we can get $p(e|s_2) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$.

$\begin{cases} \Phi\left(\frac{n}{\sqrt{N_0}}\right) 于标准正态分布 \\ p\left(\frac{n}{\sqrt{N_0}} < -\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right). \end{cases}$

So for BFSK:

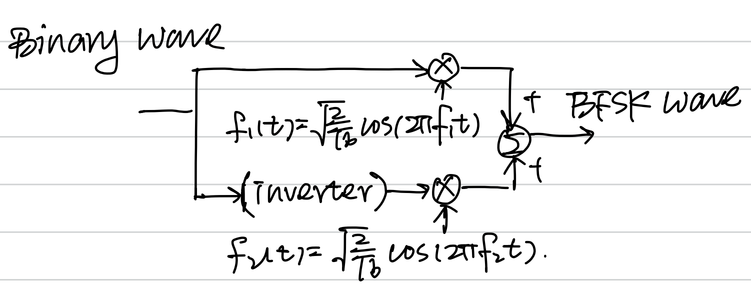
$$P_e = p(s_1)p(e|s_1) + p(s_2)p(e|s_2)$$
$$= \frac{1}{2}Q\left(\sqrt{\frac{E_b}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right).$$

BFSK $Q\sqrt{\frac{E_b}{N_0}}$ is 3dB worse than BPSK $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$.

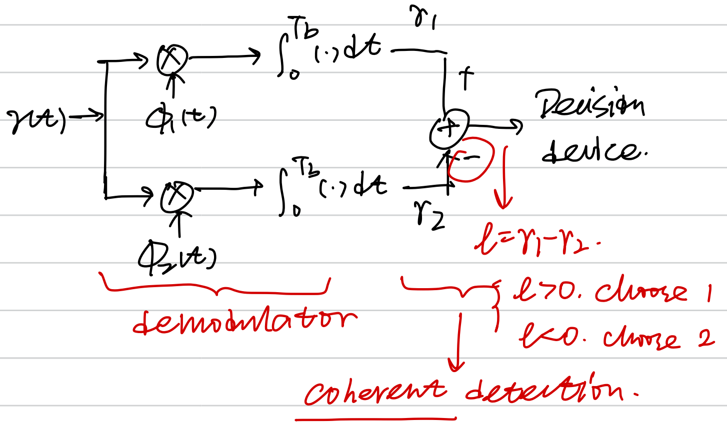
To achieve the same P_e. $E_b(\text{BFSK}) = 2E_b(\text{BPSK})$.

$\hookrightarrow E_b(\text{BFSK})(dB) = E_b(\text{BPSK})(dB) + 3dB$.

BFSK Transmitter:



BFSK Receiver:



② BASK.

"1" $\longrightarrow s_1(t) = \sqrt{\frac{2E}{T_b}}\cos(2\pi f_c t)$. $0 \le t < T_b$ ($E = 2E_b$.)

"0" $\longrightarrow s_2(t) = 0$.

Average energy per bit: $E_b = \frac{E+0}{2} = \frac{1}{2}E$.

Decision region: $\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ $s_1(t) = \sqrt{E}\,\phi_1(t)$.

s_2: 0 $\sqrt{\frac{E_b}{2}}$ s_1: $\sqrt{2E_b}$ $\phi_1(t)$ → decision boundary

$d_{12} = \sqrt{2E_b}$. BPSK: $2\sqrt{E_b}$

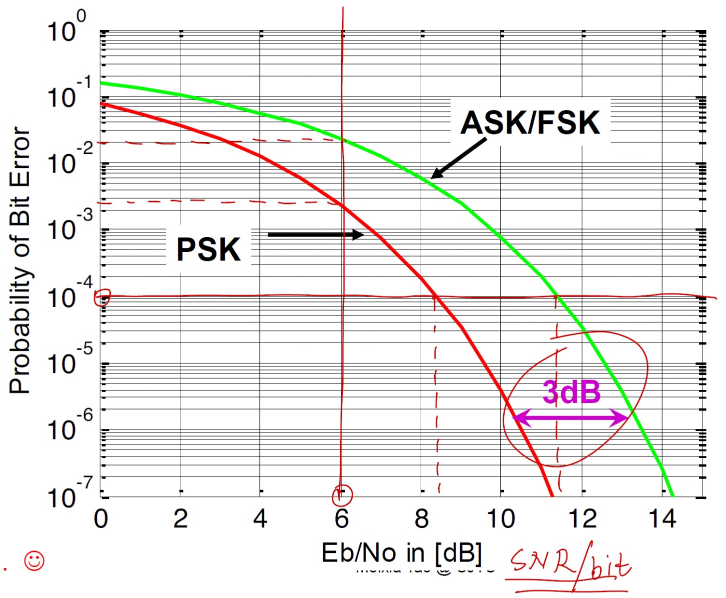
Probability of error of BASK: $P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$. $\Rightarrow$ identical to that of BFSK.

$\Rightarrow$

BPSK	BFSK	BASK
$d_{12} = 2\sqrt{E_b}$	$d_{12} = \sqrt{2E_b}$	$d_{12} = \sqrt{2E_b}$
$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$	$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$

$\Rightarrow P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right)$.

d_{12} 越大，P_e 越小，性能越好.



☺

④ Non-coherent BFSK:

two signals of a binary FSK system: θ_1, θ_2 = unknown random phase.

$$S_1(t)=\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t+\theta_1)=\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t)\cos(\theta_1)-\sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_1 t)\sin(\theta_1).$$

$$S_2(t)=\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_2 t+\theta_2)=\sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_2 t)\cos(\theta_2)-\sqrt{\frac{2E_b}{T_b}}\sin(2\pi f_2 t)\sin(\theta_2).$$

Use four basis functions:

$$\phi_{1c}(t)=\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t).\quad \phi_{1s}(t)=-\sqrt{\frac{2}{T_b}}\sin(2\pi f_1 t).$$

$$\phi_{2c}(t)=\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t).\quad \phi_{2s}(t)=-\sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t).$$

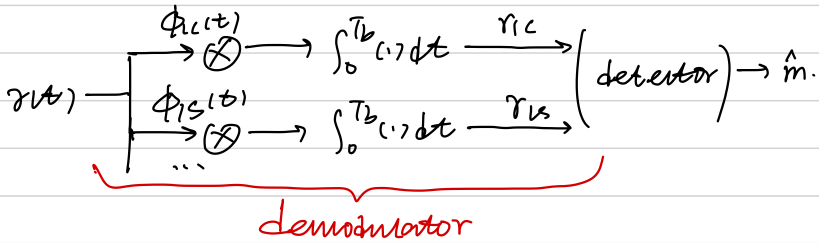
Signal space representation:

$$\vec{s}_1=[\sqrt{E_b}\cos\theta_1\quad \sqrt{E_b}\sin\theta_1\quad 0\quad 0]$$

$$\vec{s}_2=[0\quad 0\quad \sqrt{E_b}\cos\theta_2\quad \sqrt{E_b}\sin\theta_2]$$

the vector of received signal: $\vec{r}=[r_{1c}\ r_{1s}\ r_{2c}\ r_{2s}]$

↳ i.e. $r_{1c}=\int_0^{T_b} r(t)\phi_{1c}(t)dt$. $r_{1s}=\int_0^{T_b} r(t)\phi_{1s}(t)dt$. ⇒ projection.



Decision rule of non-coherent BFSK.

ML criterion ($P_1=P_2=\cdots=P_M$): $f(r|s_1)\ \underset{s_2}{\overset{s_1}{\gtrless}}\ f(r|s_2)$.

Conditional pdf:

$$f(r|s_1,\theta_1)=\frac{1}{\pi N_0}\exp\left[-\frac{(r_{1c}-\sqrt{E_b}\cos\theta_1)^2+(r_{1s}-\sqrt{E_b}\sin\theta_1)^2}{N_0}\right]\times\frac{1}{\pi N_0}\exp\left[-\frac{r_{2c}^2+r_{2s}^2}{N_0}\right].$$

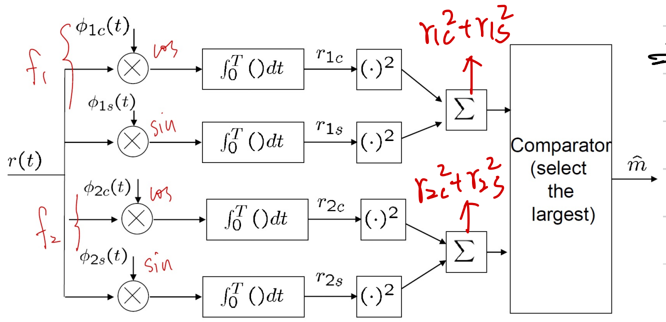
$$f(r|s_2,\theta_2)=\frac{1}{\pi N_0}\exp\left[-\frac{r_{1c}^2+r_{1s}^2}{N_0}\right]\times\frac{1}{\pi N_0}\exp\left[-\frac{(r_{2c}-\sqrt{E_b}\cos\theta_2)^2+(r_{2s}-\sqrt{E_b}\sin\theta_2)^2}{N_0}\right].$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi}\exp\left[\frac{2\sqrt{E}r_{1c}\cos(\theta_1)+2\sqrt{E}r_{1s}\sin(\theta_1)}{N_0}\right]d\theta_1=I_0\left(\frac{2\sqrt{E(r_{1c}^2+r_{1s}^2)}}{N_0}\right).$$

metrics

choose s_1 if: $I_0\left(\frac{2\sqrt{E(r_{1c}^2+r_{1s}^2)}}{N_0}\right)\geq I_0\left(\frac{2\sqrt{E(r_{2c}^2+r_{2s}^2)}}{N_0}\right)\Rightarrow \sqrt{r_{1c}^2+r_{1s}^2}\geq\sqrt{r_{2c}^2+r_{2s}^2}$.

↳ Compare the energy in two frequencies and pick the larger. ⇒ envelop detector.
carrier phase is irrelavent in decision making.



⇒ structure of non-coherent receiver for BFSK.

$$\boxed{P_e=\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)}$$

9.5.4 of textbook.

↓

worse than coherent BFSK

⑤ DPSK = Differential PSK. (Non-coherent version of PSK).

↳ encode the information in phase difference between successive signal transmission

to send "0", advance the phase of current signal by 180°.

to send "1", leave the phase unchanged.

{ the unknown phase θ contained in the received wave varies slowly (constant over two bit intervals) ⇒ the phase difference received in two successive bit intervals is independent of θ.

Generat DPSK in two steps:

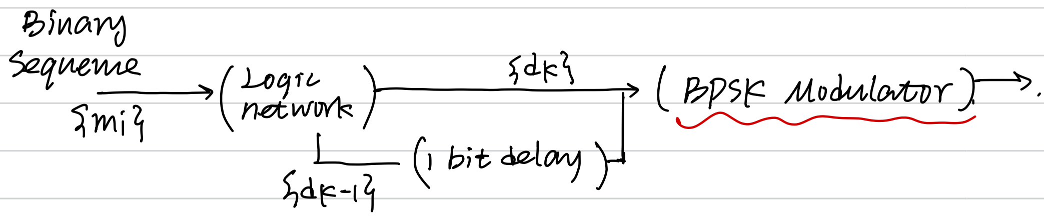
Differential encoding of the binary bits + phase shift keying.

initial bit

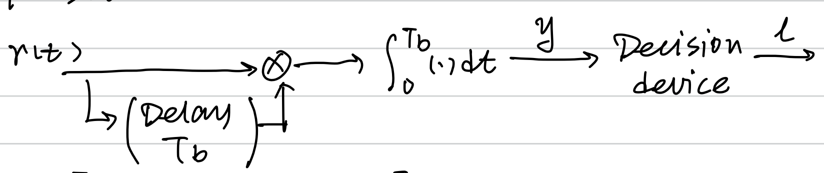
Information sequence:		1	0	0	1	0	0	1	1	$\{m_i\}$		$d_i = \overline{d_{i-1} \oplus m_i}$
encodeded sequence:	1	1	0	1	1	0	1	1	1	$\{d_i\}$.		$\begin{cases} d_i = m_i. & \text{if } m_i = 1. \\ d_i = \overline{m_i}. & \text{if } m_i = 0 \end{cases}$
transmitted phase:	0	0	π	0	0	π	0	0	0			

↳ similar to BPSK: $0 \to \pi$, $1 \to 0$.

DPSK Transmitter:



DPSK Detection:



$\begin{cases} \text{choose 1 if } l > 0 \\ \text{choose 0 o.w.} \end{cases}$

$$y = \int_0^{T_b} r(t)\, r(t-T_b)\, dt = \int_0^{T_b} \cos(\omega_c t + \psi_k + \theta) \cos(\omega_c t + \psi_{k-1} + \theta)\, dt \propto \cos(\psi_k - \psi_{k-1})$$

↳ unknown phase offset. $\theta_k = \theta_{k-1}$.

if $\psi_k - \psi_{k-1} = 0$ (bit 1) then $y > 0$. ($\cos 0 = 1$)

if $\psi_k - \psi_{k-1} = \pi$ (bit 0) then $y < 0$. ($\cos\pi = -1$)

$$\boxed{P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)}$$

⇓

3dB better than non-coherent FSK. $\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$

3. M-ary Modulation (多进制调制)

⇒ 1个symbol包含 k bit，一共可组成 $M=2^k$ 种波形，周期为T

- In *M*-ary modulation, the binary sequence is subdivided into blocks of *k* bits, called symbols, and each block (or symbol) is represented one of the $M = 2^k$ signal waveforms, each of duration T.
- Symbol rate: $R_s = \frac{1}{T}$ symbols/sec △ bit rate $= kR_s = k \cdot$ Symbol rate
- Since each signal carries $k = \log_2 M$ bits, the bit rate is $R_b = kR_s = \frac{k}{T}$
- The bit interval is ↪ $T_b = \frac{1}{R_b} = \frac{T}{k}$
- These M signals are generated by changing the amplitude, phase, frequency, or combined forms of a carrier in M discrete steps.
- Thus, we have:
  - MASK MPSK MFSK MQAM

(1) MPSK: M-ary Phase-Shift Keying.

The phase of the carrier takes on M possible values: $\theta_m = 2\pi \cdot \frac{m-1}{M}$. $m = 1, \cdots M$.

Signal set: $s_m(t) = \sqrt{\frac{2E_s}{T}} \cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right]$. $m=1,\cdots M$. $0 \le t < T$.

E_s: energy per symbol. T: duration of symbol. $f_c \gg \frac{1}{T}$.

basis functions: $\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$. $\phi_2 = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$. $0 \le t < T$.

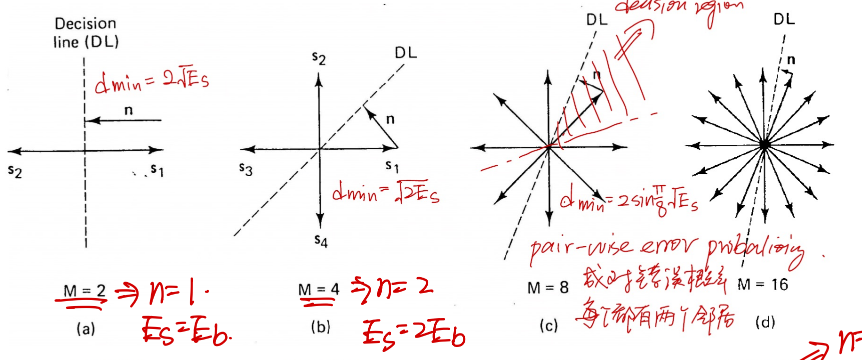
Signal space representation: $s_m = \left[\sqrt{E_s} \cos\left(\frac{2\pi(m-1)}{M}\right), \sqrt{E_s} \sin\left(\frac{2\pi(m-1)}{M}\right)\right]$.

⇓

M=2时，(s_1, s_2) 两个点，

M=16时，$(s_1, s_2, \cdots s_{16})$，16个点。

信号强度一致，点到原点的距离都是 $\sqrt{E_s}$.



M = 2 ⇒ n=1. $E_s = E_b$. (a) BPSK

M = 4 ⇒ n=2 $E_s = 2E_b$ (b) QPSK

M = 8 (c) 8PSK n=3

M = 16 (d) 16PSK ⇒ n=4

⟶ d_{min} 越来越小，error probability 增大，decision region 缩小

⇒ Euclidean distance:

$d_{min} = \sqrt{2E_s\left(1-\cos\frac{2\pi(m-n)}{M}\right)}$.

the minimum distance $(m-n=1)$:

$d_{min} = \sqrt{2E_s\left(1-\cos\frac{2\pi}{M}\right)} = 2\sqrt{E_s}\sin\frac{\pi}{M}$

M = 2, 4, 8, 16

↓

$k\uparrow, M\uparrow, d_{min}\downarrow, P_e\uparrow$.

An approximation to the symbol error probability:

$$P_{MPSK} \approx 2Q\left(\frac{d_{min}/2}{\sqrt{N_0/2}}\right) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right).$$ 背

↪ two signal points adjacent to the transmitted signal points.

$\frac{E_b}{N_0}$ (dB)

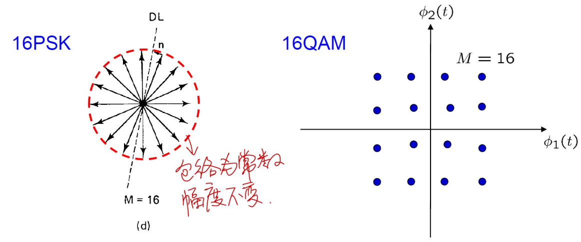
For large M, doubling the number of phases requires an additional 6dB/bit to achieve the same performance.

proof: $\sqrt{\frac{2E_{sM}}{N_0}} \cdot \sin\frac{\pi}{M} = \sqrt{\frac{2E_{s2M}}{N_0}} \sin\frac{\pi}{2M} \Rightarrow \frac{\sqrt{E_{sM}}}{M} = \frac{\sqrt{E_{s2M}}}{2M} \Rightarrow E_{s2M} = 4E_{sM}$

$\Rightarrow E_{b2M} = 4E_{bM}$. $E_{b2M}(dB) = E_{bM}(dB) + 6dB$.

(2) MQAM: M-ary Quadrature Amplitude Modulation (正交幅度调制).

- In MPSK, in-phase and quadrature components are interrelated in such a way that the envelope is constant (circular constellation)
- If we relax this constraint, we get M-ary QAM



$\Rightarrow$ Signal set:

$s_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_0}{T}} b_i \sin(2\pi f_c t)$. $0 \le t < T$.

E_0: the energy of signal with the lowest amp.

$\{a_i, b_i\}$: the sets of amplitude levels.

basis functions:

$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$. $\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$. $0 \le t < T$.

$\Rightarrow$ signal space representation: $\vec{s}_i = [\sqrt{E_0} a_i, \sqrt{E_0} b_i]$.

$\Rightarrow$ 根据 $\{a_i, b_i\}$ 取值的不同有 Rectangular constellation, circlar constellation 等.

$M=4$ regular QAM and $M=4$ PSK are identical.

Error performance of MQAM:

Four adjacent points.

upper bound of the symbol error probability: $P_e \le 4Q\left(\sqrt{\frac{3kE_b}{(M-1)N_0}}\right)$. $M = 2^k$.

when k is large, if increase the number of bits from k to $k+1$. then to achieve the same performance:

$$\sqrt{\frac{3kE_{bk}}{(2^k-1)N_0}} = \sqrt{\frac{3(k+1)E_{bk+1}}{(2^{k+1}-1)N_0}} \Rightarrow E_{b,k+1} = \frac{2^{k+1}-1}{2^k-1} \cdot \frac{k}{k+1} E_{b,k} \approx 2E_{bk} \Rightarrow 3dB.$$

$\hookrightarrow E_{b,k+1}(dB) = E_{b,k}(dB) + 3dB$

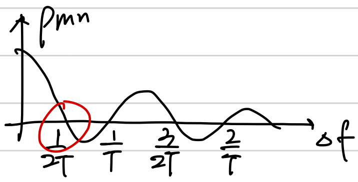
(3) MFSK

Signal set: $s_m(t) = \sqrt{\frac{2E_s}{T}} \cos[2\pi(f_c + (m-1)\Delta f)t]$. $m = 1, \dots, M$. $0 \le t < T$.

$\hookrightarrow \Delta f = f_m - f_{m-1}$, $f_m = f_c + m\Delta f$.

correlation between two symbols:

$\rho_{mn} = \frac{1}{E_s} \int_0^T s_m(t) s_n(t)\, dt = \frac{\sin[2\pi(m-n)\Delta f T]}{2\pi(m-n)\Delta f T}$. For orthogonality, minimum $\Delta f = \frac{1}{2T}$.



选择较小的Δf以减小信号带宽.

The basis functions are: $\phi_m(t)=\sqrt{\frac{2}{T}}\cos[2\pi(f_c+m\Delta f)t]$.

$\Rightarrow$ M-ary orthogonal FSK has a geometric presentation as M M-dim orthogonal vectors.

$s_0=(\sqrt{E_s},0,0,\cdots,0)$. $s_1=(0,\sqrt{E_s},0,\cdots,0)$. $\cdots$ $s_{M-1}=(0,0,\cdots,0,\sqrt{E_s})$.

↳ M个M维正交向量. 任意两点间距离均为 $\sqrt{2E_s}=\sqrt{2kE_b}=d$. 当k增大时($E_b$不变). Pe下降. 性能变好(与MPSK相反).

Pe is found by integrating conditional probability of error over the decision region.

Pe depends only on the distance profile of signal constellation.

4. Bit Error.

When a symbol error occurs, all k bits could be in error.

BER (bit error rate) = $P_b=\sum_{i=1}^{M}P(\vec{s}_i)\sum_{j=1,j\neq i}^{M}\frac{n_{ij}}{\log_2 M}P(\hat{\vec{s}}=\vec{s}_j\,|\,\vec{s}_i)$.

$M=2^n$. $n=\log_2 M$

$\frac{n_{ij}}{\log_2 M}=\frac{n_{ij}}{n}$.

n_{ij} is the number of different bits between s_i and s_j.

Gray coding is a bit-to-symbol mapping

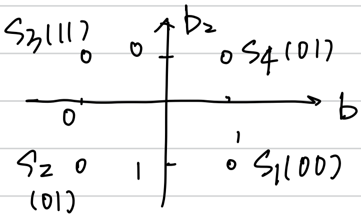
∘ Example: gray coding for QPMK (M=4).

$$P_b=\sum_{i=1}^{4}\frac{1}{4}\sum_{j=1,j\neq i}^{4}\frac{n_{ij}}{\log_2 4}P(\hat{\vec{s}}=\vec{s}_j|\vec{s}_i)=\frac{1}{2}P(\hat{\vec{s}}=\vec{s}_1|\vec{s}_4)+\frac{2}{2}P(\hat{\vec{s}}=\vec{s}_2|\vec{s}_4)+\frac{1}{2}P(\hat{\vec{s}}=\vec{s}_3|\vec{s}_4).$$

$$=\left[1-Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]\cdot Q\left(\sqrt{\frac{E_s}{N_0}}\right)+\left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2=Q\left(\sqrt{\frac{E_s}{N_0}}\right).=Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

→ equal to BPSK

→ $\log_2 4=2$.



对 $b_0(0,1)$ 和 $b_1(0,1)$ 分别作BPSK.

BER for MPSK and MFSK:

For MPSK with Gray coding: $P_b\approx\frac{P_e}{\log_2 M}$. $\log_2 M=n$.

For MFSK: $P_b\doteqdot\frac{1}{2}P_e$ $\Rightarrow$ 任何一个bit到其他(M-1)bits的距离都相同.

5. Comparsion study.

{ Power utilization efficiency (energy efficiency): measured by $\frac{E_b}{N_b}$

Spectrum utilization efficiency (bandwidth efficiency): measured by $\frac{R_b}{B}$ — R_b= bits/sec, B=Hz.

Maximize bandwidth efficiency at minimal required $\frac{E_b}{N_b}$.

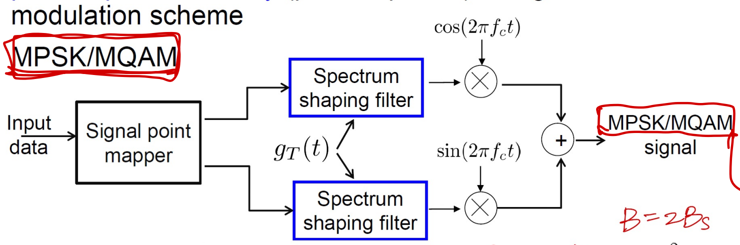
Energy Efficiency Comparison:

- MFSK:
  - At fixed E_b/N_o, increase M can provide an improvement on P_b
  - At fixed P_b increase M can provide a reduction in the E_b/N_o requirement
- MPSK
  - BPSK and QPSK have the same energy efficiency
  - At fixed E_b/N_o, increase M degrades Pb
  - At fixed Pb, increase M increases the Eb/No requirement

MFSK is more energy efficient than MPSK

Bandwidth Efficiency Comparison:

- To compare bandwidth efficiency, we need to know the power spectral density (power spectra) of a given modulation scheme
- MPSK/MQAM



$B=2B_s$

  - If $g_T(t)$ is rectangular, the bandwidth of mainlope is $B = \frac{2}{T_s}$
  - If it has a raised cosine spectrum, the bandwidth is

$$B = \frac{1+\alpha}{T_s} < 2B_s \quad \text{和调制阶数M无关}$$

α = rollout factor.

$\Rightarrow$ In general: $B = \frac{1}{T_s}$.

And $R_b = \frac{\log_2 M}{T_s}$ = bit rate

then bandwidth efficiency =

$$\rho = \frac{R_b}{B} = \log_2 M \ \text{bits/sec/Hz}.$$

- MFSK:
  - Bandwidth required to transmit MFSK signal is

$$\boxed{B = \frac{M}{2T}}$$ (Adjacent frequencies need to be separated by 1/2T to maintain orthogonality)

$\Rightarrow$ minimum $\Delta f = \frac{1}{2T}$.

For M signals. $B = M\Delta f = \frac{M}{2T}$ Hz.

  - Bandwidth efficiency of MFSK signal

$$\boxed{\rho = \frac{R_b}{B} = \frac{2\log_2 M}{M} \ \text{(bits/s/Hz)}}$$

$R_b = \frac{\log_2 M}{T}$

$B = \frac{M}{2T}$.

M	2	4	8	16	32	64
ρ (bits/s/Hz)	1	1	0.75	0.5	0.3125	0.1875

As M increases, bandwidth efficiency of MPSK/MQAM increases, but bandwidth efficiency of MFSK decreases.

$$\begin{cases} \text{MPSK/MQAM}: \rho = \log_2 M. \ \uparrow \\ \text{MFSK}: \rho = \frac{2\log_2 M}{M}. \ \downarrow \end{cases}$$

Channel capacity: For a bandlimited channel corrupted by AWGN, the maximum rate is given by:

$$R \le C = B\log_2(1+SNR) = B\log_2\left(1+\frac{P_s}{N_o B}\right).$$

$$\begin{cases} P_s = E_b \cdot R, \\ P_N = \frac{1}{2}N_o. \Rightarrow \frac{1}{2}N_o \cdot 2B = N_o B. \end{cases}$$

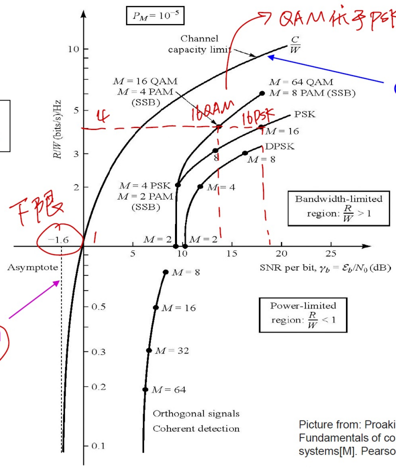
energy per bit

bits/sec

$$\Rightarrow \frac{E_b}{N_o} \ge \frac{B}{R}\left(2^{\frac{R}{B}}-1\right) \Rightarrow \frac{E_b}{N_o} = SNR/bit \ge \frac{2^{\rho}-1}{\rho}.$$

$\Rightarrow$ energy efficiency

ρ: bandwidth efficiency.



Picture from: Proakis J G, Salehi M. Fundamentals of communication systems[M]. Pearson Education India, 2007.

- In the limits as R/B goes to 0, we get

$$R \to 0: \quad \frac{E_b}{N_0} = \ln 2 = 0.693 = -1.59dB \quad -1.6dB.$$

  - This value is called the Shannon Limit （香农极限）
  - Received E_b/N_0 must be **>-1.6dB** to ensure reliable communications

- BPSK and QPSK require the same E_b/N_0 of 9.6 dB to achieve $P_e = 10^{-5}$. However, QPSK has a better bandwidth efficiency
- MQAM is superior to MPSK
- MPSK/MQAM increases bandwidth efficiency at the cost of lower energy efficiency
- MFSK trades energy efficiency at reduced bandwidth efficiency.

Chapter 09. Information Theory.

1. Modeling of information source

Discrete memoryless source (DMS).

↳ A discrete-time, discrete-amplitude random process with i.i.d. rv.

2. Measure of information.

The information I that a source event X can convey and the probability of the event $P(X)$ satisfy:

1) $I = I[P(X)]$.   2). $P(X)$ decreases → I increases. vice versa $P(X)=1.\ I=0$

3) Mutiple independent events $X_1, X_2 \cdots$

$$I[P(X_1)P(X_2)\cdots] = I[P(X_1)] + I[P(X_2)] + \cdots$$

⇒ Information of X: $I = \log_a \frac{1}{P(X)} = -\log_a P(X)$ $\begin{cases} a=e. & \text{nat} \\ a=2. & \text{bit} \end{cases}$

Entropy (熵).

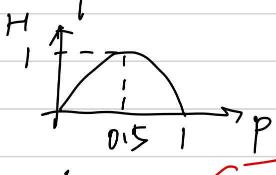
Consider a discrete soure with N possible symbols.

Entropy $H(\cdot)$: average amount of information convoyed per symbol.

$$H(X) \hat{=} E[I(X_j)] = \sum_{j=1}^{N} P(X_j)\log_2 \frac{1}{P(X_j)} \ (\text{bit/symbol}).$$ ⇒信源所带信息量的期望. 注意单位是bit/symbol.

Binary case $\{0,1\}$. with $P(1)=p$. $P(0)=1-p$.

then $H = p\log_2\frac{1}{p} + (1-p)\log_2\frac{1}{1-p}$. ⇒



⇒ Entropy is maximized when all symbols are equiprobable.

↳ N symbols: $H = \sum_{n=1}^{N}\frac{1}{N}\log_2 N = \log_2 N$ bits/symbol.

↳ $0 \le H(X) \le \log N$

∘ Example:

- A source with bandwidth *4KHz* is sampled at the Nyquist rate
- Assuming that the resulting sequence can be modeled by a discrete memoryless source $\{-2,-1,0,1,2\}$ with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\}$
- What is the information rate of the source in bit/sec?

- We have

$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + 2\times\frac{1}{16}\log_2 16$$
$$= \frac{15}{8} \text{ bits/sample}$$

- Since we have 8000 samples/sec, the source produces information at a rate of 15k bits/sec.

bit/symbol. symbol rate: symbol/sec.

Joint and conditional entropy.

The joint entropy of discrete source (X,Y) =

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y).$$

The conditional entropy of X given Y =

$$H(X|Y) = -\sum_{x,y} p(x,y) \log p(x|y) \qquad H(X|Y=y) = -\sum_{x} p(x|y) \log p(x|y)$$

Using chain rule:

$$\begin{cases} H(X,Y) = H(X|Y) + H(Y) \\ H(X,Y) = H(Y|X) + H(X). \end{cases}$$

⇒ 区分 $H(X,Y)$ 和 $I(X;Y)$.

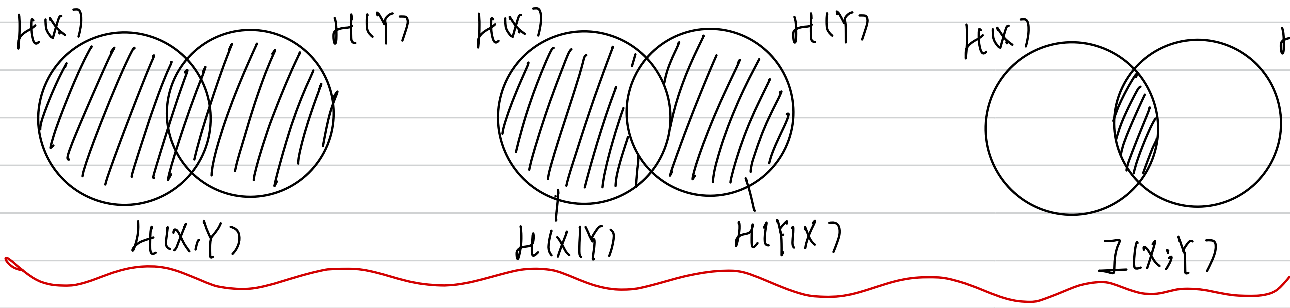
Mutual Information (互信息).

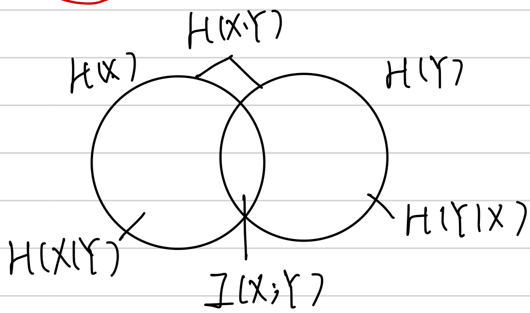
$H(X) - H(X|Y)$ denotes the amount of uncertainty of X that has been removed given Y is known.

the amount of information provided by random variable Y about r.v. X.

mutual information: $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X).$

↳ 已知信道的输出 Y, 可揭示的关于 X 的信息.





Differential Entropy

The differential entropy of discrete-time continuous alphabet source X with pdf (X) is: 连续取值

$$h(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\, dx$$

Mutual information between two continuous r.v. X and Y =

$$I(X;Y) = h(X) - h(X|Y)$$

3. Source Coding Theorem. (信源编码)

A source with entropy H can be encoded with an arbitrarily small error probability at any rate R (bits/source output) as long as $R > H$.

H = the minimum rate at which an information source can be compressed for reliable reconstruction. $\Rightarrow$ R must $\geq H$.

变长度二进制编码.

Huffman Source Coding. $\Rightarrow$ variable-length binary coding.

$\Rightarrow$ Map the more probable source sequences to shorter binary codeworks.

Synchronization is a problem.

↳ 在解码时会有歧义.

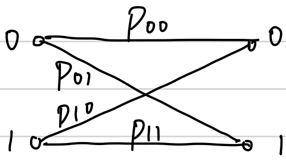
4. Modeling of communication channel.

(1) Binary-Symmetric Channel (BSC).

↳ the crossover error probabilty is the same. $e = p(0|1) = p(1|0)$

$p(1/0) = p(0/1) = e$

$e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$.



$P_{00} = p(0/0) = 1-e = P_{11} = p(1/1)$

$P_{10} = p(1/0) = e = P_{01} = p(0/1)$.

(2) AWGN channel: both input and output are real

$$X \longrightarrow \underset{\text{Channel}}{\oplus} \longrightarrow Y = X + Z.$$ (Z enters at $\oplus$)

Z is Gaussian White Noise.

the input satisfies power constraint: $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P$.

注意单位不是bits/symbol.

(3) Channel Capasity: a maximum rate, C in bits/sec of a channel.

If $R \leq C$, theroetically guarantee almost error free transmission.

If $R > C$, reliable transmission is impossible.

The capacity of a discrete-time memoryless channel:

$$C = \max_{p(x)} I(X;Y).$$ max over all possible input distribution

△ Entropy gives a lower bound on the rate of the codes that are capable of reproducing the source with no error. If we want to transmit a source U reliably via a channel with capacity C, then $H(U) < C$.

Binary Symmetric Channel Capacity:

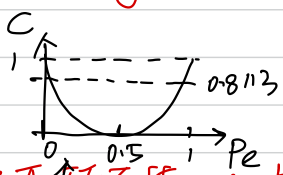
$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - \underbrace{\sum p(x) H(Y|X=x)}$$

$$= H(Y) - \underbrace{\sum p(x)}_{\sum p(x) = 1} H_b(P_e) = H(Y) - \underbrace{H_b(P_e)}_{\text{binary source entropy.}} \le 1 - H_b(P_e)$$

Binary channel

$$H_b(p) = -p\log_2 p - (1-p)\log_2(1-p).$$

$\Rightarrow$ the capacity of BSC: $C = 1 - H_b(P_e)$. $\Rightarrow$



当Pe=0.5时.不管X input取什么.输出Y都等概率取所有值.相当于没有传递信息. Pe=0,1时.Y的值都由X决定.信道可将input完整传输.

○ Example.

- A binary source with $P(X=0)=\frac{1}{4}$, $P(X=1)=\frac{3}{4}$ is to be transmitted over a BSC channel with a crossover probability p_e. Assume that the channel can be used once per symbol output. $\Rightarrow$ per channel output, per symbol.
- Determine the range of p_e for reliable communication of the source.

想到香农极限

$H(x) < C = 1 - H_b(P_e)$

$\Rightarrow$ entropy $H(x) = \frac{1}{4}\log_2 4 + \frac{3}{4}\log_2 \frac{4}{3} = 0.8113$. $\Rightarrow$ $P_e \in [0, 0.288)$ $(0.972, 1]$.

↳ bits/symbol = bits/sec = bits/channel output (binary)

AWGN. Gaussian Channel Capacity.

A discrete-time Gaussian channel: $Y = X + Z$.

Input power constraint: $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \le P$. Z is Gaussian White noise $\sim N(0, P_N)$.

capacity: $C = \frac{1}{2}\log\left(1 + \frac{P}{P_N}\right)$ bits/channel use. proof.

Capacity of Bandlimited AWGN Channel.

A continuous-time bandlimited AWGN channel with noise PSD $N_0/2$

input power constraint P. bandwidth W.

Sample it at Nyquist rate ($2W$) and obtain a discrete-time channel.

The power/sample is P and the noise power/sample is WN_0.

Shannon Formula: $P_N = \int_{-W}^{W} \frac{N_0}{2} df = WN_0$.

Thus. $C = \frac{1}{2}\log\left(1 + \frac{P}{N_0 W}\right)$ bits/sample. $\longrightarrow$ SNR

↳ usually sample at nyquist rate.

bandwidth is W

$\Rightarrow WN_0$.

$C = \frac{1}{2}\log\left(1 + \frac{P}{P_N}\right)$ bits/channel use

○ Example: find the capacity of a telephone channel with bandwidth $W = 3000$ Hz, SNR 39 dB.

$\Rightarrow$ SNR = 39 dB = 7943. $C = \underbrace{2 \times 3000\,\text{Hz}}_{\text{sample at nyquist rate.}} \times \frac{1}{2}\log(1+7943) \approx 38867$ bits/sec

$\Rightarrow C = \frac{1}{2}\log(1+\frac{P}{N_0W})$ bits/sample $= W\log(1+\frac{P}{N_0W})$ bits/sec
(nyquist rate)

5. Insights from Shannon Formula.

1. Increasing signal power P increases the capacity C
   - When SNR is high enough, every doubling of P adds additional B bits/s in capacity
   - When P approaches infinity, so is C

$$\boxed{\begin{aligned} \log_2(1+x) &\approx x\log_2 e \quad \text{when } x \approx 0, \\ \log_2(1+x) &\approx \log_2 x \quad \text{when } x \gg 1. \end{aligned}}$$

$C = W\log_2(1+\frac{P}{N_0W})$ bits/sec

① When SNR ≫1. $C \approx W\log_2\frac{P}{N_0W}$.
double $P' = 2P$ gets $\underline{\Delta C = W}$

When $P\to\infty$. then $C\to\infty$

2. Increasing channel bandwidth W can increase C, but cannot increase infinitely (as noise power also increases)

$$\lim_{W\to\infty} C = \lim_{W\to\infty}\left[\frac{WN_0}{P}\log\left(1+\frac{P}{N_0W}\right)\right]\frac{P}{N_0}$$

$$= \frac{P}{N_0}\log e = 1.44\frac{P}{N_0}$$

② When $W \gg 1$. $SNR = \frac{P}{N_0W} \approx 0$

$C = \frac{WN_0}{P}\log_2(1+\frac{P}{N_0W})\cdot\frac{P}{N_0}$

$\approx \log_2 e\cdot\frac{P}{N_0} = \underline{1.44\frac{P}{N_0}}$

3. Bandwidth efficiency – energy efficiency tradeoff
   - In any practical system, we must have

$$R \le W\log_2\left(1+\frac{P}{N_0W}\right)$$

R = bit rate (bits/sec)
$C = W\log_2(1+\frac{P}{N_0W})$ is the maximum rate

   - Defining r=R/W, the spectral bit rate

$$r = \frac{R}{W} \le \log_2(1+\frac{P}{N_0W})$$

   - Let Eb be the energy per bit, $E_b = \frac{P}{R}$
   - Then, $\boxed{r \le \log_2\left(1+r\frac{E_b}{N_0}\right)}$

E_b/N_0 = SNR per bit → 能量效率.
r = spectral efficiency 频谱效率.

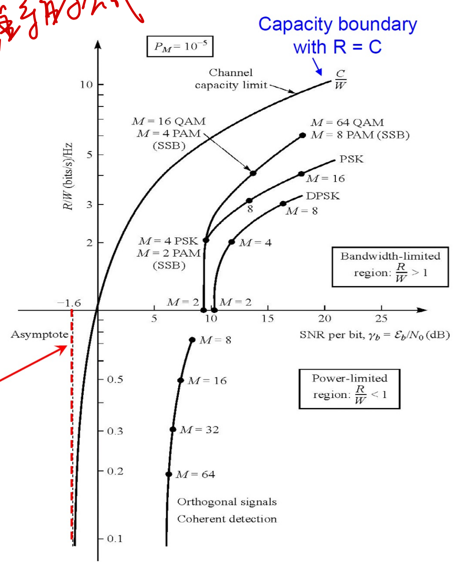
新推导的公式.

$$\frac{E_b}{N_0} = \frac{2^r-1}{r}$$

- As $r = \frac{R}{B} \to 0$

$$\left.\frac{E_b}{N_0}\right|_{r\to 0} = \lim_{r\to 0}\frac{1}{r}(2^r-1)$$
$$= \ln 2$$
$$= 0.693 \quad -1.6dB$$
$$= -1.59\text{dB}$$

Shannon Limit, an absolute minimum for reliable communication



Chapter 10. Channel coding. see the slides

Main Types of Codes: block codes and convolutional codes

① Linear block codes (线性分组码). has no memory.

② Convolutional codes (卷积码). has memory.

1. Block Codes.

An (n,k) block code is a collection of $M=2^k$ codeworks of code length n.

Each block has a block of k information bits followed by a group of $r=n-k$ check bits.

$$\underset{k\text{ bits}}{\text{message}} \longrightarrow (\text{channel encoder}) \longrightarrow \underset{(k+r)\text{ dits}}{n\text{ bit codewords}}$$

- Example: Simple Parity-Check Codes

Chapter 01.

1. Show that the unit step signal $u(t)$ is a power-type signal and find its power content.

2. $\mathcal{F}[\cos(\pi t)] = \frac{1}{2}\delta(f+\frac{1}{2}) + \frac{1}{2}\delta(f-\frac{1}{2})$. $\mathcal{F}[\sin \pi t)] = \frac{j}{2}\delta(f+\frac{1}{2}) - \frac{j}{2}\delta(f-\frac{1}{2})]$.

5.28 Let X and Y be zero-mean jointly Gaussian random variables, each with variance σ^2. The correlation coefficient between X and Y is denoted by ρ. Random variables Z and W are defined by

$$\begin{cases} Z = X\cos\theta + Y\sin\theta \\ W = -X\sin\theta + Y\cos\theta \end{cases},$$

where θ is a constant angle.

1. Show that Z and W are jointly Gaussian random variables.
2. For what values of θ are the random variables Z and W independent?

Chapter 02.

5.35 Which one of the following functions can be the autocorrelation function of a random process and why?

1. $f(\tau) = \sin(2\pi f_0 \tau)$.
2. $f(\tau)$, as shown in Figure P-5.35.

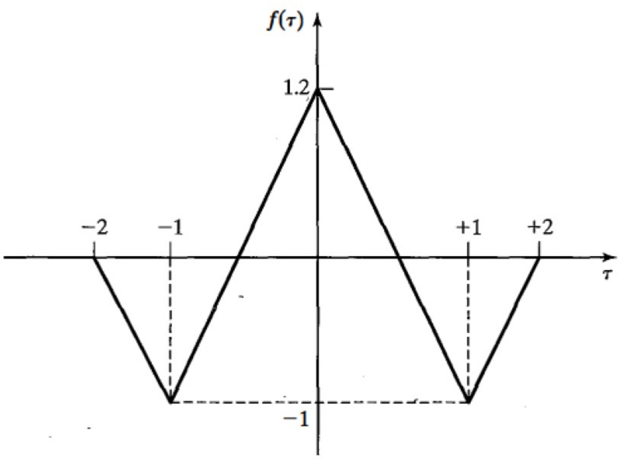


Figure P-5.35

(1) $f(\tau) = \sin(2\pi f_0 \tau)$
$f(0) = 0$ is not its maximum value.

(2) $f(\tau) = 1.2\Lambda(t) - \Lambda(\tau+1) - \Lambda(\tau-1)$
$\Rightarrow f(0) = 0$ and it's even symmetric
its power spectral density:
$f(\tau) \xrightarrow{\mathcal{F}} S(f) = 1.2\,\text{sinc}^2(f) - \sin^2(f)\,(e^{j2\pi f} + e^{-j2\pi f})$.
$= \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$.
$S(0) < 0$ so $f(\tau)$ cannot be an auto-correlation function.

$\Lambda(\tau) \longrightarrow \text{sinc}^2(f)$.

$\Downarrow$

功率谱密度函数在任意频率处的值不可能为负数.

5.40 A zero-mean white Gaussian noise process with the power spectral density of $\frac{N_0}{2}$ passes through an ideal lowpass filter with bandwidth B.

1. Find the autocorrelation of the output process $Y(t)$.
2. Assuming $\tau = \frac{1}{2B}$, find the joint probability density function of the random variables $Y(t)$ and $Y(t+\tau)$. Are these random variables independent?

1、 $S_X(f) = \frac{N_0}{2}$. For the lowpass filter: $H(f) = \Pi(\frac{f}{2B})$.

$\Rightarrow S_Y(f) = |H(f)|^2 S_X(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$.

$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \frac{N_0}{2}\mathcal{F}^{-1}[\Pi(\frac{f}{2B})] = N_0 B\,\mathrm{sinc}(2B\tau)$.

2. $\tau = \frac{1}{2B} \Rightarrow R_Y(\tau) = R_Y(\frac{1}{2B}) = 0$.

或者: $E[Y(t)] = E[X(t)] \times H(0) = 0$, $\sigma_Y^2 = R_Y(0) = N_0 B$.

The correlation coefficient of $Y(t)$ and $Y(t+\tau)$ is:

$$\rho = \frac{\sigma_{Y(t)Y(t+\tau)}}{\sigma_{Y(t)}\sigma_{Y(t+\tau)}} = \frac{E[Y(t)Y(t+\tau)]}{N_0 B} = \frac{R_Y(\tau)}{N_0 B} \Rightarrow \tau = \frac{1}{2B}: \rho = 0.$$

So: $f = \frac{1}{2\pi N_0 B} e^{-\frac{Y^2(t)+Y^2(t+\tau)}{2N_0 B}}$. (JPDF).

$\Rightarrow$ Independent.

Chapter 03

3.7 An AM signal has the form

$$u(t) = [20 + 2\cos 3000\pi t + 10\cos 6000\pi t]\cos 2\pi f_c t,$$

where $f_c = 10^5\text{Hz}$.

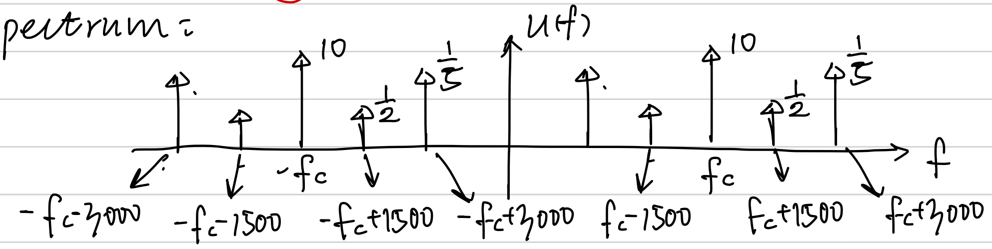
1. Sketch the (voltage) spectrum of $u(t)$.
2. Determine the power in each of the frequency components.
3. Determine the modulation index.
4. Determine the sidebands' power, the total power, and the ratio of the sidebands' power to the total power.

1. $u(t) \xrightarrow{\mathcal{F}} U(f) = \frac{20}{2}[\delta(f-f_c)+\delta(f+f_c)]$

$+\frac{2}{4}[\delta(f-f_c-1500)+\delta(f-f_c+1500)+\delta(f+f_c-1500)+\delta(f+f_c+1500)]$.

$+\frac{10}{4}[\delta(f-f_c-3000)+\delta(f-f_c+3000)+\delta(f+f_c-3000)+\delta(f+f_c+3000)]$

The spectrum:



2. $u^2(t) = 400\cos^2(2\pi f_c t) + \cos^2(2\pi(f_c-1500)t) + \cos^2(2\pi(f_c+1500)t)$

$+25\cos^2(2\pi(f_c-3000)t) + 25\cos^2(2\pi(f_c+3000)t)$

+ terms that are multiples of cosines.

或者由积化和差得:

$+5\cos(2\pi(f_c-3000)t)$.

$u(t) = 20\cos(2\pi f_c t) + \cos(2\pi(f_c+1500)t) + \cos(2\pi(f_c-1500)t) + 5\cos(2\pi(f_c+3000)t)$

The power of $A\cos(2\pi f_0 t+\theta)$ is $\frac{1}{2}A^2$:

$\Rightarrow P_{f_c} = \frac{400}{2} = 200.$ $P_{f_c-1500} = P_{f_c+1500} = \frac{1}{2}.$ $P_{f_c-3000} = P_{f_c+3000} = \frac{25}{2}.$

3. $u(t) = [20 + 2\cos(2\pi 1500t) + 10\cos(2\pi 3000t)]\cos(2\pi f_c t)$

$= 20[1+\frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t)]\cos(2\pi f_c t)$.

$\Rightarrow \alpha m_n(t) = \frac{1}{10}\cos(2\pi 1500t) + \frac{1}{2}\cos(2\pi 3000t)$

$|\alpha m_n(t)|_{max} = 0.6.$ $\Rightarrow \alpha = 0.6.$ $(|m_n(t)|_{max} = 1)$.

4. $P_{total} = P_{f_c} + P_{f_c+1500} + P_{f_c-1500} + P_{f_c+3000} + P_{f_c-3000} = 226.$

$P_{carrier} = \frac{1}{2} A_c^2 = \frac{1}{2} \times 200^2 = 200.$ $\quad P_{sidebands} = 226 - 200 = 26.$

$$\Rightarrow E = \frac{P_{sidebands}}{P_{total}} = \frac{26}{226} = \frac{13}{113}.$$

4.4 An angle-modulated signal has the form

$$u(t) = 100 \cos\left[2\pi f_c t + 4 \sin 2000\pi t\right],$$

where $f_c = 10$ MHz.

1. Determine the average transmitted power.
2. Determine the peak-phase deviation.
3. Determine the peak-frequency deviation.
4. Is this an FM or a PM signal? Explain.

$\Rightarrow u(t) = 100 \cos[2\pi f_c t + 4 \sin 2000\pi t]$

$= 100 \cos[2\pi f_c t + 2\pi \cdot 4000 \cdot \int_0^t \cos 2000\pi \tau \, d\tau]$

peak-phase deviation $\beta = 4$. peak-frequency deviation $\Delta f = k_f \cdot A_m = 4000\text{Hz}$

$P = \frac{1}{2} \cdot 100^2 = 5000.$

試看：$\theta(t) = 4 \sin 2000\pi t.$

$\Rightarrow \Delta\phi_{max} = |\theta(t)|_{max} = 4.$ $\quad \Delta f_{max} = \left|\frac{1}{2\pi} \frac{d}{dt} \theta(t)\right|_{max} = 4000.$

$\Delta f_{max} = \Delta\phi_{max} \cdot f_m = 4000.$

4.6 To generate wideband FM, we can first generate a narrowband FM signal, and then use frequency multiplication to spread the signal bandwidth. Figure P-4.6 illustrates such a scheme, which is called an Armstrong-type FM modulator. The narrowband FM signal has a maximum angular deviation of 0.10 radians to keep distortion under control.

↳ $\beta = 0.10$.

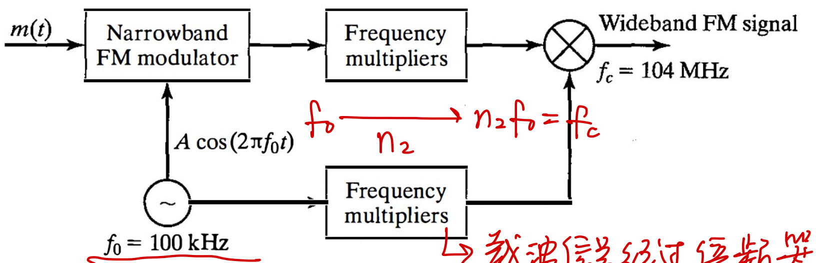
1. If the message signal has a bandwidth of 15 kHz and the output frequency from the oscillator is 100 kHz, determine the frequency multiplication that is necessary to generate an FM signal at a carrier frequency of $f_c = 104$ MHz and a frequency deviation of $f = 75$ kHz.

2. If the carrier frequency for the wideband FM signal is to be within ±2 Hz, determine the maximum allowable drift of the 100 kHz oscillator.

⇒(1) NW信号经过倍频器后其FM信号的载频增大N倍. 调制指数也增大N倍 即 $f_c' = Nf_c$, $\beta' = N\beta_1'$

$\Delta f_1 \longrightarrow n_1 \Delta f_1 = \Delta f$

$f_0 \xrightarrow{n_1} n_1 f_0 = f_c$



$f_0 \xrightarrow{n_2} n_2 f_0 = f_c$

↳ 载波信号经过倍频器后频率增大n_2倍.

⇒ 1. The frequency deviation of the narrowband FM signal is

$$\Delta f_1 = \beta f_m = 0.1 \times 15 = 1.5\text{kHz}. \Rightarrow n_1 = \frac{\Delta f}{\Delta f_1} = \frac{75}{1.5} = 50.$$

To generate a FM signal with $f_c = 104$ MHz:

$$(n_1 + n_2) f_0 = f_c \Rightarrow n_2 = 990.$$

2. $(n_1+n_2)df = 2 \Rightarrow df = \frac{2}{1040} = \frac{1}{520}$ Hz

⇒ If the out of the narrowband modulator is: $u(t) = A\cos(2\pi f_0 t + \phi(t))$

then the output of the n_1 multiplier is: $u_1(t) = A\cos(2\pi n_1 f_0 t + n_1\phi(t))$

the output of the n_2 mutiplier: $u_2(t) = A\cos(2\pi n_2 f_0 t)$.

Mixed output:

$$y(t) = A^2\cos(2\pi n_1 f_0 t + n_1\phi(t))\cos(2\pi n_2 f_0 t)$$

$$= \frac{A^2}{2}[\cos(2\pi(n_1+n_2)f_0 t + n_1\phi(t)) + \cos(2\pi(n_1-n_2)f_0 t + n_1\phi(t))].$$

4.10 The carrier $c(t) = A\cos 2\pi 10^6 t$ is angle modulated (PM or FM) by the sinusoid signal $m(t) = 2\cos 2000\pi t$. The deviation constants are $k_p = 1.5$ rad/V and $k_f = 3000$ Hz/V. $\hookrightarrow f_m = W = 1000\,Hz.$

1. Determine β_f and β_p.
2. Determine the bandwidth in each case using Carson's rule.
3. Plot the spectrum of the modulated signal in each case. (Plot only those frequency components that lie within the bandwidth derived in Part 2.)
4. If the amplitude of $m(t)$ is decreased by a factor of 2, how would your answers to Parts 1–3 change?
5. If the frequency of $m(t)$ is increased by a factor of 2, how would your answers to Parts 1–3 change?

$\Rightarrow$ 1. $\beta_f = \Delta\phi_{max} = k_p\,|m(t)|_{max} = 3.$ $\quad \beta_p = \dfrac{\Delta f_{max}}{W} = \dfrac{k_f\,|m(t)|_{max}}{W} = 6.$

2. Using Carson's rule: $B_{PM} = 2(1+\beta_p)W = 8000$

$B_{FM} = 2(1+\beta_f)W = 14000.$

3. The modulated PM · FM wave:

$$S_{PM}(f) = \frac{A}{2}\sum_{n=-4}^{+4} J_n(3)\left[\delta(f-10^6-n10^3) + \delta(f+10^6+n10^3)\right].$$

$$S_{FM}(f) = \frac{A}{2}\sum_{n=-7}^{+7} J_n(6)\left[\delta(f-10^6-n10^3) + \delta(f+10^6+n10^6)\right].$$

$\Rightarrow$ The FM. PM modulated signal can be written as:

$$u(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta_f)\cos[2\pi(f_c+nf_m)t]$$

$$u(t) = \sum_{n=-\infty}^{\infty} A J_n(\beta_p)\cos[2\pi(f_c+nf_m)t].$$

4.16 The message signal $m(t)$ into an FM modulator with a peak frequency deviation $f_d = 25$ Hz/V is shown in Figure P-4.16. Plot the frequency deviation in Hz and the phase deviation in radians.

FM :

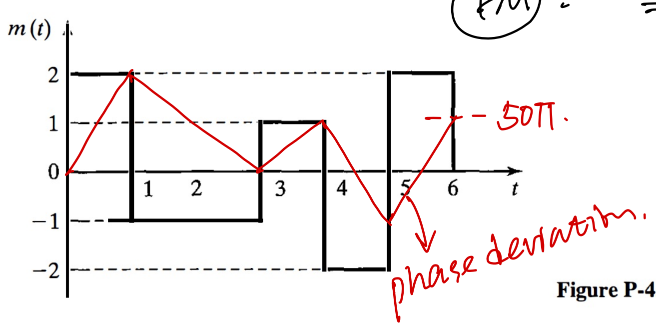
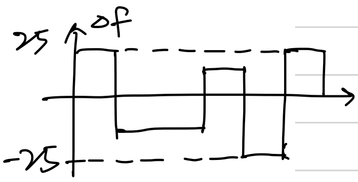


Figure P-4.16

$\Rightarrow$ frequency deviation:

$\Delta f = k_f\, m(t) = \frac{25}{2} m(t).$



phase deviation:

$\Delta\phi = 2\pi k_f \int_0^t m(\tau)d\tau = 25\pi\int_0^t m(\tau)d\tau$

← 见红线

4.18 The modulating signal that is the input into an FM modulator is

$$m(t) = 10\cos 16\pi t.$$

The output of the FM modulator is

$$u(t) = 10\cos\left[4000\pi t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right],$$

where $k_f = 10$. (See Figure P-4.18.) If the output of the FM modulator is passed through an ideal BPF centered at $f_c = 2000$ with a bandwidth of 62 Hz, determine the power of the frequency components at the output of the filter. What percentage of the transmitter power appears at the output of the BPF?

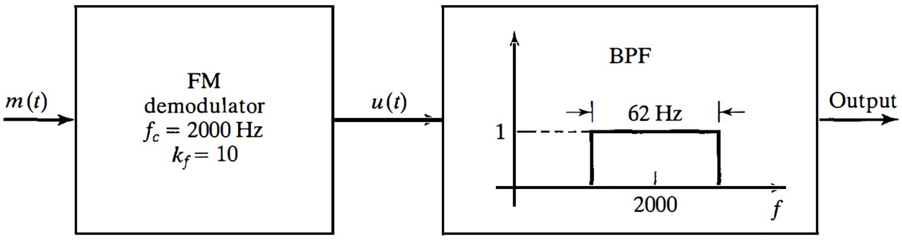


Figure P-4.18

⇒ The modulation index: $\beta = \dfrac{k_f\,|m(t)|_{max}}{f_m} = \dfrac{25}{2}$.

The output of FM modulator can be written as:

$$u(t) = 10\cos\left[2\pi 2000 t + 2\pi k_f \int_{-\infty}^{t} 10\cos(2\pi 8\tau)\,d\tau\right]$$

$$= \sum_{n=-\infty}^{\infty} 10 J_n(12.5)\cos[2\pi(2000+n8)t + \phi_n].$$

Only $n = -4, \dots, 4$ terms' frequency lies within $[2000-32, 2000+32]$.

↳ $4 \times 8 = 32$

The power:

$$\frac{10^2}{2} J_0^2(12.5) + 2\sum_{n=1}^{4} \frac{10^2}{2} J_n^2(12.5) = 50 \times 0.2630 = 13.15.$$

$P_{total} = \frac{1}{2}\cdot 10^2 = 50$. ⇒ $\dfrac{13.15}{50} = 26.30\%$.

Chapter 04

7.18 The random process $X(t)$ is defined by $X(t) = Y\cos(2\pi f_0 t + \Theta)$, where Y and Θ are two independent random variables, Y uniform on $[-3, 3]$ and Θ uniform on $[0, 2\pi]$.

1. Find the autocorrelation function of $X(t)$ and its power spectral density.
2. If $X(t)$ is to be transmitted to maintain an SQNR of at least 40 dB using a uniform PCM system, what is the required number of bits/sample and the least bandwidth requirement (in terms of f_0)?
3. If the SQNR is to be increased by 24 dB, how many more bits/sample must be introduced, and what is the new minimum bandwidth requirement in this case?

$\Rightarrow$ 1. $R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[Y^2\cos(2\pi f_0 t_1+\theta)\cos(2\pi f_0 t_2+\theta)]$

$= \frac{1}{2}E(Y^2)E[\cos(2\pi f_0 t_1 + 2\pi f_0 t_2 + 2\theta) + \cos(2\pi f_0(t_1-t_2))]$

$= \frac{1}{2}E(Y^2)[\int_0^{2\pi}\frac{1}{2\pi}\cos(2\pi f_0 t_1 + 2\pi f_0 t_2 + 2\theta)d\theta + \int_0^{2\pi}\frac{1}{2\pi}\cos(2\pi f_0(t_1-t_2))d\theta]$

$= \frac{1}{2}E(Y^2)\cos[2\pi f_0(t_1-t_2)]$

$E(Y^2) = \frac{1}{12}\cdot(3-(-3))^2 = 3 \Rightarrow R_X(\tau) = \frac{3}{2}\cos 2\pi f_0\tau.$

power spectral density: $S_X(f) = \mathcal{F}[R_X(\tau)] = \frac{3}{4}[\delta(f+f_0)+\delta(f-f_0)]$

$\cos 2\pi f_0 t \xrightarrow{\mathcal{F}} \frac{1}{2}[\delta(f+f_0)+\delta(f-f_0)]$

2. $SQNR = 10\log_{10}\frac{P_X}{a} + 6v + 4.8$ (dB). $P_X = \frac{1}{2}\times(\cancel{\frac{3}{2}})^2 = \frac{9}{8}$. $= R_X(0) = \frac{3}{2}$

$a=3$. $10\log_{10}SQNR = 10\log_{10}\left(\frac{3\times 4^v\times P_X}{a^2}\right) = 40 \Rightarrow v=8$

the minimum bandwidth is $8f_0$.

3. $\Rightarrow 6\Delta v = 24$ $\Delta v = 4.$ $v' = v+\Delta v = 12.$ $\Rightarrow 12f_0.$

$10\log_{10}SQNR = 10\log_{10}\left(\frac{3\times 4^v\times P_X}{X^2_{max}}\right).$

7.4 The lowpass signal $x(t)$ with a bandwidth of W is sampled with a sampling interval of T_s, and the signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)p(t-nT_s)$$

is reconstructed from the samples, where $p(t)$ is an arbitrary-shaped pulse (not necessarily time limited to the interval $[0, T_s]$).

1. Find the Fourier transform of $x_p(t)$.
2. Find the conditions for perfect reconstruction of $x(t)$ from $x_p(t)$.
3. Determine the required reconstruction filter.

1. $x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s) p(t-nT_s) = p(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s).$

$= p(t) * x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s).$

$\Rightarrow X_p(f) = \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} X(f-\frac{n}{T_s})$

2. $P(f)$ should be invertible for $|f|<W$

3. $X(f) = X_p(f) P^{-1}(f) \Pi(\frac{f}{2W_\Pi})$. $W < W_\Pi < \frac{1}{T_s} - W$

7.9 Let $X(t)$ denote a wide-sense stationary (WSS) Gaussian process with $P_X = 10$.

1. Using Table 7.1, design a 16-level optimal uniform quantizer for this source. → 区别于 optimal nonuniform.
2. What is the resulting distortion if the quantizer in Part 1 is employed?
3. What is the amount of improvement in SQNR (in decibels) that results from doubling the number of quantization levels from 8 to 16?

⇒ According to table 7.1. the optimal level spacing for a 16-level uniform quatizer is 0.3352.

then in case of $\sigma^2=10$. optimal level space $\Delta = \sqrt{10} \cdot 0.3352 = 1.060$.

The quantization levels are:

$\hat{x}_1 = -\hat{x}_{16} = -7 \times 1.060 - \frac{1}{2} \times 1.060 = -7.950$

$\hat{x}_2 = -\hat{x}_{15} = -6 \times 1.060 - \frac{1}{2} \times 1.060 = -6.890$

$\hat{x}_3 = -\hat{x}_{14} = -5 \times 1.060 - \frac{1}{2} \times 1.060 = -5.830$

$\hat{x}_4 = -\hat{x}_{13} = -4 \times 1.060 - \frac{1}{2} \times 1.060 = -4.770$

$\hat{x}_5 = -\hat{x}_{12} = -3 \times 1.060 - \frac{1}{2} \times 1.060 = -3.710$

$\hat{x}_6 = -\hat{x}_{11} = -2 \times 1.060 - \frac{1}{2} \times 1.060 = -2.650$

$\hat{x}_7 = -\hat{x}_{10} = -1 \times 1.060 - \frac{1}{2} \times 1.060 = -1.590$

$\hat{x}_8 = -\hat{x}_9 = -\frac{1}{2} \times 1.060 = -0.530.$

The boundaries are:

$a_1 = a_{15} = -7 \times 1.060 = -7.420$

$a_2 = a_{14} = -6 \times 1.060 = -6.360$

$a_3 = a_{13} = -5 \times 1.060 = -5.300$

$a_4 = a_{12} = -4 \times 1.060 = -4.240$

$a_5 = a_{11} = -3 \times 1.060 = -3.180$

$a_6 = a_{10} = -2 \times 1.060 = -2.120$

$a_7 = a_9 = -1 \times 1.060 = -1.060$

$a_8 = 0.$

Distortion $= \sigma^2 \times 0.01154 = 0.1154$

$D_{16} = 0.1154 \quad D_8 = \sigma^2 \times 0.03744 = 0.3744$

$\Rightarrow 10\log_{10} \frac{SQNR_{16}}{SQNR_8} = 10\log_{10} \frac{0.3744}{0.1154} = 5.111\,dB$

$SQNR = \frac{E(X^2)}{E[(X-Q(X))^2]} \Rightarrow$ distortion.

Chapter 05

10.5 Show that a pulse having the raised cosine spectrum given by Equation (10.3.20) satisfies the Nyquist criterion given by Equation (10.3.7) for any value of the roll-off factor α.

$\Rightarrow$ The pulse having the raised cosine spectrum:

$$x(t) = \operatorname{sinc}\left(\frac{t}{T}\right) \frac{\cos\left(\pi\alpha \frac{t}{T}\right)}{1-4\alpha^2 \frac{t^2}{T^2}}.$$

$\operatorname{sinc}\left(\frac{t}{T}\right)=0.\ t=nT,\ n=0, \pm 1, \cdots$

$$\frac{\cos\left(\pi\alpha \frac{t}{T}\right)}{1-4\alpha^2 \frac{t^2}{T^2}} = \begin{cases} 1, & t=0 \\ \text{bounded}, & t\neq 0 \end{cases} \quad \text{and} \quad \lim_{\alpha t \to \frac{\pi}{2}} \frac{\cos\left(\pi\alpha \frac{t}{T}\right)}{1-4\alpha^2 \frac{t^2}{T^2}} = \frac{\pi}{2} < \infty$$

Then: $x(nT) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0. \end{cases}$

10.8 A channel has a passband characteristic in the frequency range $|f| \leq 1400$ Hz. $H_c(f)$

1. Select a symbol rate and a PAM signal constellation size to achieve a 9600 bps signal transmission.
2. If a square-root raised cosine pulse is used for the transmitter pulse $g_T(t)$, select the roll-off factor. Assume that the channel has an ideal frequency-response characteristic.

1. $B = 1400$ Hz. then the max symbol rate: $R_{s\,max} = 2W = 2800$.
   when $\nu=3$ $R_s = \frac{9600}{\nu} = 3200 > 2800$. $\nu=4$. $R_s = 2400 < 2800$
   Then $\nu=4$. $R_s = 2400$ symbol/sec. size $= 2^\nu = 16$.
2. $\frac{1}{T} = R_s = 2400. \Rightarrow B = \frac{1+\alpha}{2T} = 1400$. then $\alpha = 0.1667$.

10.12 The binary sequence 10010110010 is the input to the precoder whose output is used to modulate a duobinary transmitting filter. Construct a table as in Table 10.2; show the precoded sequence, the transmitted amplitude levels, the received signal levels, and the decoded sequence. $\Rightarrow$ textbook P589. example 10.4.1

binary sequence:		1	0	0	1	0	1	1	0	0	1	0
precode sequence:	0	1	1	1	0	0	1	0	0	0	1	1
transmitted levels:	-1	1	1	1	-1	-1	1	-1	-1	-1	1	1
received sequence:		0	2	2	0	-2	0	0	-2	-2	0	2
decoded sequence:		1	0	0	1	0	1	1	0	0	1	0.

10.29 The transmission of a signal pulse with a raised cosine spectrum through a channel results in the following (noise-free) sampled output from the demodulator:

$$x_k = \begin{cases} -0.5, & k=-2 \\ 0.1, & k=-1 \\ 1, & k=0 \\ -0.2, & k=1 \\ 0.05, & k=2 \\ 0, & \text{otherwise} \end{cases}.$$

1. Determine the tap coefficients of a three-tap linear equalizer based on the zero-forcing criterion.
2. For the coefficients determined in Part 1, determine the output of the equalizer for the case of the isolated pulse. Thus, determine the residual ISI and its span in time.

$\Rightarrow$ the output of zf-equalizer is: $q_m = \sum_{n=-1}^{1} c_n \underbrace{x(m-n)}_{x(k)}$

with $q_0=1$ and $q_m=0$ for $m\neq 0$. :

$$\begin{pmatrix} x_0 & x_{-1} & x_{-2} \\ x_1 & x_0 & x_{-1} \\ x_2 & x_1 & x_0 \end{pmatrix}\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} q_{-1} \\ q_0 \\ q_1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix}\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Then: $c_{-1} = 0.$ $c_0 = 0.98$. $c_1 = 0.196.$

the output of the equalizer:

$$q_m = \begin{cases} 0 & , m<-4 \\ c_{-1}x_{-2} = 0 & , m=-3 \quad n=-1 \quad k=m-n=-2 \\ c_{-1}x_{-1} + c_0 x_{-2} = -0.49 & , m=-2 \quad n=0,-1, \quad k=m-n=-1,-2. \\ 0 & , m=-1 \\ 1 & , m=0 \\ 0 & , m=1 \\ c_0x_2 + c_1x_1 = 0.0098 & . \ m=2 \\ c_1x_2 = 0.0098 & , m=3 \\ 0 & , m\geq 4. \end{cases}$$

the residual ISI sequence = $\{\dots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0\dots\}$

its span is 6 symbols.

15-49)

1. $P = \int_{-\infty}^{\infty} S_X(f)\,df = 4\times10^{-5} \times 2\times10^{5} \times \frac{1}{2} \times 1 = 4\ W$

2. $BW = 10^5\ HZ$

3. $H(f) = \begin{cases} 1, & |f| \le 50kHZ \\ 0, & \text{otherwise.} \end{cases} \Rightarrow S_Y(f) = |H(f)|^2 S_X(f) = \begin{cases} 4\times10^{-5}\Lambda(\frac{f}{10^5}), & |f| < 50kHZ \\ 0 & \text{otherwise} \end{cases}$

$P(Y) = \int_{-\infty}^{\infty} S_Y(f)\,df = (\frac{1}{2}+1) \times 5\times10^4 \times \frac{1}{2} \times 2 \times 4\times10^{-5} = 1.8\ W$

4.

5. $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau}\,df = \begin{cases} 10^{-2.5}, & |\tau| \le 5\times10^4 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow t_0 = 0 + \tau = 5\times10^4.$

15-62)

Chapter 06.

8.7 The received signal in a binary communication system that employs antipodal signals is

$$r(t) = s(t) + n(t),$$

where $s(t)$ is shown in Figure P-8.7 and $n(t)$ is AWGN with power spectral density $N_0/2$ W/Hz.

1. Sketch the impulse responses of the filter matched to $s(t)$.
2. Sketch the output of the matched filter to the input $s(t)$.
3. Determine the variance of the noise of the output of the matched filter at $t = 3$.
4. Determine the probability of error as a function of A and N_0.

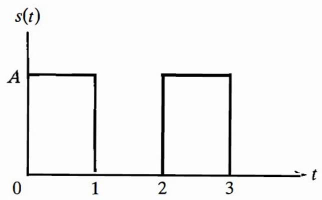


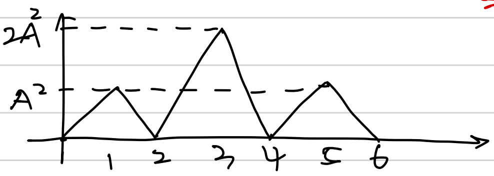
Figure P-8.7

⇒ 1. The impulse response of the filter: $h(t) = s(T-t) = s(3-t) = s(t)$

2. The output of the filter: $y(t) = h(t) * s(t) = s(t) * s(t) = \int_0^t s(\tau)s(t-\tau)d\tau$

then $y(t) = \begin{cases} 0, & t<0 \\ A^2 t, & 0 \le t < 1 \\ A^2(2-t), & 1 \le t < 2 \\ 2A^2(t-2), & 2 \le t < 3 \\ 2A^2(4-t), & 3 \le t < 4 \\ A^2(t-4), & 4 \le t < 5 \\ A^2(6-t), & 5 \le t < 6 \\ 0, & t \ge 6. \end{cases}$

Sketch:



⇒ the output of MF is the autocorrelation function of input signal, $S_o(t) = R_{s_i}(t-t_0)$.
the peak value: $S_o(t_0) = E_{s_i}$ $S_o(3) = 2A^2$

3. $t = T = 3$: $n_T = \int_0^T n(\tau)h(T-\tau)d\tau = \int_0^T n(\tau)s(\tau)d\tau$

$$\sigma_{n_T}^2 = E\left[\int_0^T\int_0^T n(\tau)n(v)s(\tau)s(v)\,d\tau dv\right] = \int_0^T\int_0^T n(\tau)n(v)s(\tau)s(v)\,d\tau dv$$

$$= \frac{N_0}{2}\int_0^T\int_0^T s(\tau)s(v)\delta(\tau-v)\,d\tau dv = \frac{N_0}{2}\int_0^T s^2(\tau)\,d\tau = N_0A^2.$$

4. $P(e) = Q\left[\sqrt{\left(\frac{S}{N}\right)_0}\right]$, where $\left(\frac{S}{N}\right)_0 = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0A^2} = \frac{4A^2}{N_0}$.

then $P(e) = Q\left[\sqrt{\frac{4A^2}{N_0}}\right]$.

直接公式: $\boxed{E[n_o^2(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df = \frac{N_0}{2}E}$

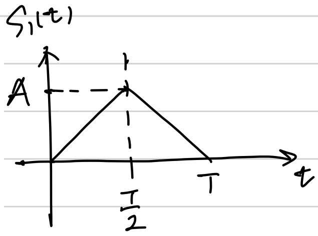
⇒ $\sigma_{n_T}^2 = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df = \frac{N_0}{2}\cdot 2A = N_0A.$

the output SNR of matched filter.

$t=3$: $y^2(t=3) = (2A^2)^2 = 4A^4.$

For antipodal signal: or: $\boxed{SNR = \frac{2E}{N_0}} = \frac{2\cdot 2A^2}{N_0} = \frac{4A^2}{N_0}$. $\boxed{P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)} = Q\left(\sqrt{\frac{4A^2}{N_0}}\right)$.

8.17 In a binary antipodal signaling scheme, the signals are given by



$$s_1(t) = -s_2(t) = \begin{cases} \frac{2At}{T}, & 0 \le t \le \frac{T}{2} \\ 2A\left(1 - \frac{t}{T}\right), & \frac{T}{2} \le t \le T. \\ 0, & \text{otherwise} \end{cases}$$

The channel is AWGN and $\mathcal{S}_n(f) = \frac{N_0}{2}$. The two signals have prior probabilities p and $1-p$.

1. Determine the structure of the optimal receiver. see 9. optimal receiver structure
2. Determine an expression for the error probability.
3. Plot the error probability as a function of p for $0 \le p \le 1$.

$\Rightarrow$ 1. $$\underbrace{C(\vec{r}, \vec{s}_m) = \int_{-\infty}^{\infty} r(t) s_m(t)\,dt - \frac{1}{2}\int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2}\ln p(s_m)}_{\text{metrics}}.$$

$s_1(t) = -s_2(t) \Rightarrow E(s_1) = E(s_2)$.

then: $$\int_{-\infty}^{\infty} r(t)s_1(t)\,dt - \int_{-\infty}^{\infty} r(t)s_2(t)\,dt$$

$$= \int_{-\infty}^{\infty} r(t)[s_1(t) - s_2(t)]\,dt \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{2}\ln\frac{p(s_2)}{p(s_1)} = \frac{N_0}{2}\ln\frac{1-p}{p}.$$

$$\Rightarrow \int_{-\infty}^{\infty} r(t)s_1(t)\,dt \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4}\ln\frac{1-p}{p}.$$

2. If $s_1(t)$ is transmitted, the output of correlator:

$$\int_{-\infty}^{\infty} r(t)s_1(t)\,dt = \int_0^T s_1^2(t)\,dt + \int_0^T n(t)s_1(t)\,dt = E_s + n.$$

n is a gaussian random variable with zero mean and variance:

$$\sigma_n^2 = \frac{N_0}{2}\int_0^T |s_1(\tau)|^2 d\tau = \frac{N_0}{2}E_s.$$

The error probability:

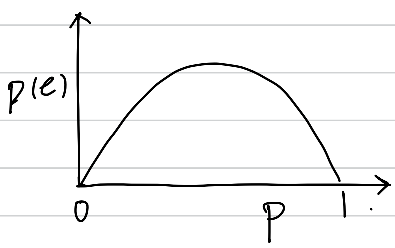
$$p(e|s_1) = \int_{-\infty}^{\frac{N_0}{4}\ln\frac{1-p}{p} - E_s} \frac{1}{\sqrt{\pi N_0 E_s}} e^{-\frac{x^2}{N_0 E_s}}\,dx = Q\left[\sqrt{\frac{2E_0}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{E_s}}\ln\frac{1-p}{p}\right].$$

$$\Rightarrow p(e|s_2) = Q\left[\sqrt{\frac{2E_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{E_s}}\ln\frac{1-p}{p}\right].$$

Then: $p(e) = pP(e|s_1) + (1-p)P(e|s_2)$

$$= pQ\left[\sqrt{\frac{2E_0}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{E_s}}\ln\frac{1-p}{p}\right] + (1-p)Q\left[\sqrt{\frac{2E_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{E_s}}\ln\frac{1-p}{p}\right]$$

(3)



8.24 Consider the signal

$$u(t) = \begin{cases} \frac{A}{T} t \cos 2\pi f_c t, & 0 \le t < T \\ 0, & \text{otherwise} \end{cases}.$$

1. Determine the impulse response of the matched filter for this signal.
2. Determine the output of the matched filter at $t = T$.
3. Suppose the signal $u(t)$ is passed through a correlator that correlates the input $u(t)$ with $u(t)$. Determine the value of the correlator output at $t = T$. Compare your result with that in Part (2).

1. $h_m(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)). & 0 \le t \le T. \\ 0 & , \text{o.w.} \end{cases}$

2. the output of MF at $t=T$:

$$u(t) * h_m(t)\Big|_{t=T} = \int_0^T u(T-\tau) h_m(\tau)d\tau = \frac{A^2}{T^2}\int_0^T (T-\tau)^2 \cos^2(2\pi f_c(T-\tau))d\tau$$

$$= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]$$

3. the output of correlator at $t=T$:

$$\int_0^T u^2(\tau)d\tau = \frac{A^2}{T^2}\int_0^T \tau^2 \cos^2(2\pi f_c \tau)d\tau.$$

Same.

书 **10.15** A baseband digital communication system employs the signals 2 shown in Figure P-10.15(a) for the transmission of two equiprobable messages. It is assumed that the communication problem studied here is a "one shot" communication problem, i.e., the messages are transmitted just once and no transmission takes place afterward. The channel has no attenuation and the noise is AWG with power spectral density $\frac{N_0}{2}$.

1. Find an appropriate orthonormal basis for the representation of the signals.
2. In a block diagram, give the precise specifications of the optimal receiver using matched filters. Label the block diagram carefully. ⇒注意区分 correlator type 和 matched filter type.
3. Find the error probability of the optimal receiver.
4. Show that the optimal receiver can be implemented by using just *one* filter. [See the block diagram shown in Figure P-10.15(b).] What are the characteristics of the matched filter and the sampler and decision device?
5. Now assume the channel is not ideal, but has an impulse response of $c(t) = \delta(t) + \frac{1}{2}\delta(t - \frac{T}{2})$. Using the same matched filter as the previous part, design an optimal receiver.
6. Assume that the channel impulse response is $c(t) = \delta(t) + a\delta(t - \frac{T}{2})$, where a is a random variable uniformly distributed on [0, 1]. Using the same matched filter, design the optimal receiver.

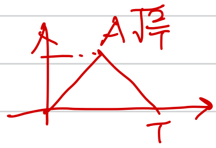
1. $\phi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{o.w.} \end{cases}$ $\quad \phi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \le t < T \\ 0, & \text{o.w.} \end{cases}$

2. $r(t) \to \begin{cases} \phi_1(\frac{T}{2}-t) \xrightarrow{t=\frac{T}{2}} r_1 \\ \phi_2(T-t) \xrightarrow{t=T} r_2 \end{cases} \to$ (select the largest).

3. $p(e|s_1) = p(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$.

$\Rightarrow p(e) = p(s_1)p(e|s_1) + p(s_2)p(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$. $p(s_1) = p(s_2) = \frac{1}{2}$.

4. The two MF are the same:



$$\phi_1(\tfrac{T}{2}-t) = \phi_2(T-t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0. & \text{o.w.} \end{cases}$$

$\Rightarrow r(t) \longrightarrow \boxed{\phi_1(t)} \to \begin{cases} \xrightarrow{t=T} r_2 \\ \xrightarrow{t=\frac{T}{2}} r_1. \end{cases}$

5. If the $s_1(t)$ is transmitted, the received signal $r(t)$ is:

$$r(t) = s_1(t) + \frac{1}{2}s_1(t-\frac{T}{2}) + n(t).$$

The output of MF at $t=\frac{T}{2}$ and $t=T$:

$$r_1 = A\sqrt{\frac{2}{T}}\cdot\frac{T}{4} + \frac{3A}{2}\sqrt{\frac{2}{T}}\cdot\frac{T}{4} + n_1 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_1.$$

$$r_2 = \frac{A}{2}\sqrt{\frac{2}{T}}\cdot\frac{T}{4} + n_2 = \frac{1}{2}\sqrt{\frac{A^2T}{8}} + n_2.$$

assume that optimal receive use threshold V to make decision:

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} V.$$

then: $p(e|s_1) = p(r_1-r_2<V) = p(n_2-n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$

If $s_2(t)$ transmitted, then $r(t) = s_2(t) + \frac{1}{2}s_2(t-\frac{T}{2}) + n(t)$.

the output of MF at $t=\frac{T}{2}$, $t=T$: $r_1 = n_1$. $r_2 = A\sqrt{\frac{2}{T}}\cdot\frac{T}{4} + \frac{3A}{2}\sqrt{\frac{2}{T}}\cdot\frac{T}{4} + n_2 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2$

then $p(e|s_2) = p(r_1-r_2 > V) = p(n_1-n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$.

$\Rightarrow p(e) = \frac{1}{2}p(e|s_1) + \frac{1}{2}p(e|s_2) = \frac{1}{2}Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$

$\frac{\partial p(e)}{\partial V} = 0 \Rightarrow V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$. $\quad \frac{\partial p(e)}{\partial V} = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$

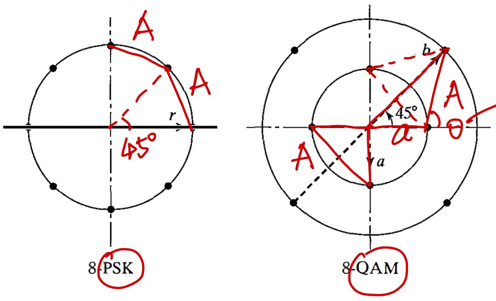
6. Similar to (5). could get $V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$.

The mse of $V(a)$: $V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}}\int_0^1 a\,da = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$

Chapter 07

8.31 Consider the octal signal point constellations in Figure P-8.31.

1. The nearest neighbor signal points in the 8-QAM signal constellation are separated in distance by A units. Determine the radii a and b of the inner and outer circles.
2. The adjacent signal points in the 8-PSK are separated by a distance of A units. Determine the radius r of the circle.
3. Determine the average transmitter powers for the two signal constellations, and compare the two powers. What is the relative power advantage of one constellation over the other? (Assume that all signal points are equally probable.)



$\theta = 180^\circ - 45^\circ - 60^\circ = 75^\circ$

Figure P-8.31

(1) the radius of the inner circle is $a = \frac{1}{\sqrt{2}}A$.

the radius of the outter circle is $b^2 = a^2 + A^2 - 2aA\cos(180^\circ - 75^\circ) \Rightarrow b = \frac{1+\sqrt{3}}{2}A$

(2) let r be the radius of the circle:

$$A^2 = r^2 + r^2 - 2r\cos 45 \Rightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}.$$

(3) the average transmitted power of PSK:

$$P_{PSK} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Rightarrow P_{PSK} = \frac{A^2}{2-\sqrt{2}}.$$

of the QAM: $P_{QAM} = \frac{1}{8}\left(4 \times \frac{A^2}{2} + 4 \times \frac{(1+\sqrt{3})^2}{4}A^2\right) \Rightarrow P_{QAM} = \left[\frac{4+\sqrt{3}}{4}\right]A^2.$

then: $\text{gain} = \frac{P_{PSK}}{P_{QAM}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927\,\text{dB}.$

$\Downarrow$

星座点到原点距离的平方

the relative power advantage

$d_{min} = A \quad \frac{d^2}{\sqrt{N_0/2}}$

$8PSK: P_e = 2Q\left(\frac{A^2}{\sqrt{N_0/2}}\right).$

$SNR = \frac{E_s}{N_0}. \quad 8QAM: P_e = 4Q\left(\frac{A^2}{\sqrt{N_0/2}}\right).$

8.32 Consider the eight-point QAM signal constellation shown in Figure P-8.31.

1. Assign three data bits to each point of the signal constellation so that the nearest (adjacent) points differ in only one bit position. ???
2. Determine the symbol rate if the desired bit rate is 90 Mbps.
3. Compare the SNR required for the eight-point QAM modulation with that required for an eight-point PSK modulation having the same error probability.
4. Which signal constellation (eight-point QAM or eight-point PSK) is more immune to phase errors? Explain the reason for your answer.

$\{0,0,0,\ldots,0\}$

(1) Assign the all zero sequence to point A, B, C and:

$B=\{0,\ldots,0,1,0,\ldots,0\}$, $C=\{0,\ldots,0,1,0,\ldots,0\}$.

the position of the 1 is not the same. The the sequences of B, C differ in two bits.

(2) Each symbol conveys 3 bits (8-QAM): $R_s=\frac{90\times10^6}{3}=30\times10^6$ symbols/sec.

(3) The probability of error of an M-ary PSK signal: $P_M=2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right]$.

the probability of error of an M-ary QAM signal is upper bounded by:

$$P_M=4Q\left[\sqrt{\frac{3E_{av}}{(M-1)N_0}}\right].$$

the two signals will achieve the same probality of error if:

$$\sqrt{2SNR_{PSK}}\sin\frac{\pi}{M}=\sqrt{\frac{3SNR_{QAM}}{M-1}}$$

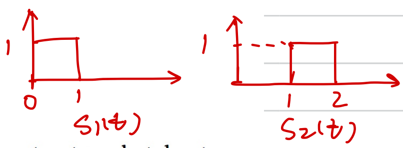
$k=3$

$M=8$: $\sqrt{2SNR_{PSK}}\sin\frac{\pi}{8}=\sqrt{\frac{3SNR_{QAM}}{7}}\Rightarrow\frac{SNR_{PSK}}{SNR_{QAM}}=\frac{3}{7\times2\times0.3827^2}=1.4627$.

(4) the QAM constellation is more immune to phase errors. Assuming that the magnitude of the signal points is detected correctly. then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5°. In the case of the 8-QAM constellation an error will be made if magnitude phase error exceeds 45°.

8.39 Two equiprobable messages are transmitted via an additive white Gaussian noise channel with a noise power spectral density of $\frac{N_0}{2}=1$. The messages are transmitted by the signals

$$s_1(t)=\begin{cases}1 & 0\le t\le 1\\ 0 & \text{otherwise}\end{cases}$$



and $s_2(t)=s_1(t-1)$.

We intended to implement the receiver using a correlation type structure, but due to imperfections in the design of the correlators, the structure shown in Figure P-8.39 has been implemented. The imperfection appears in the integrator in the upper branch where we have $\int_0^{1.5}$ instead of $\int_0^1$. The decision box, therefore, observes r_1 and r_2; based on this observation, it has to decide which message was transmitted. What decision rule should be adopted by the decision box for an optimal decision?

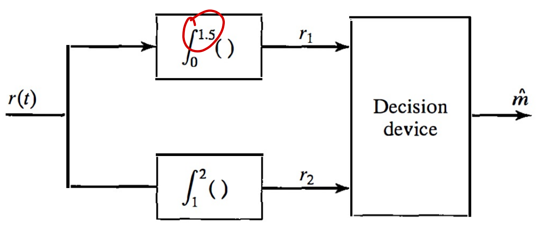


Figure P-8.39

(8-29). The $\vec{r}=[r_1, r_2]$ at the output of the integrators is: $\vec{r}=[r_1, r_2]=[\int_0^{1.5} r(t)dt, \int_1^2 r(t)dt]$.

If $s_1(t)$ is transmitted then:

$$\int_0^{1.5} r(t)dt = \int_0^{1.5}[s_1(t)+n(t)]dt = 1+\int_0^{1.5} n(t)dt = 1+n_1$$

$$\int_1^2 r(t)dt = \int_1^2 [s_1(t)+n(t)]dt = \int_1^2 n(t)dt = n_2.$$

n_1 is a zero-mean Gaussian random variable with variance:

$$\sigma_{n_1}^2 = E[\int_0^{1.5}\int_0^{1.5} n(\tau)n(v)d\tau dv] = \frac{N_0}{2}\int_0^{1.5} d\tau = 1.5$$

n_2 is a zero-mean Gaussian random variable with variance:

$$\sigma_{n_2}^2 = E[\int_1^2\int_1^2 n(\tau)n(v)d\tau dv] = \frac{N_0}{2}\int_1^2 d\tau = 1.$$

then: $\vec{r} = [1+n_1, n_2]$.

if $s_2(t)$ is transmitted, then $\vec{r}=[0.5+n_1, 1+n_2]$.

the detector bases its decisions on the rule: $r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} T$.

The probability of error $P(e|s_1)$ is:

$$P(e|s_1) = p(r_1-r<T|s_1) = P(1+n_1-n_2<T) = p(n_1-n_2<T-1) = p(n<T-1)$$

$n=n_1-n_2$ is a zero-mean Gaussian with variance:

$$\sigma_n^2 = \sigma_{n_1}^2+\sigma_{n_2}^2-2E[n_1n_2] = \sigma_{n_1}^2+\sigma_{n_2}^2 - 2\int_1^{1.5}\frac{N_0}{2}d\tau = 1.5+1-2\times 0.5 = 1.5$$

then: $P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}}\int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}}dx$.

similarly: $P(e|s_2) = p(n_1-n_2>T+0.5) = \frac{1}{\sqrt{2\pi\sigma_n^2}}\int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}}dx$

The average probability of error is:

$$p(e) = \frac{1}{2}p(e|s_1)+\frac{1}{2}p(e|s_2) = \frac{1}{2\sqrt{2\pi\sigma_n^2}}\int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}}dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}}\int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}}dx$$

$$\Rightarrow \frac{\partial p(e)}{\partial T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}}[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}}] = 0 \Rightarrow (T-1)^2 = (T+0.5)^2 \Rightarrow T=0.25$$

Then the optimal rule: $r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} 0.25$.

※: signal set: $S_m(t)=\sqrt{\frac{2E_s}{T}}\cos[2\pi(f_c+(m-1)\Delta f)t]$.

9.10 Consider the phase-coherent demodulator for M-ary FSK signals as shown in Figure 9.11.

1. Assume that the signal

$$u_0(t)=\sqrt{\frac{2\mathcal{E}_s}{T}}\cos 2\pi f_c t,\quad 0\le t\le T$$

was transmitted; determine the output of the $M-1$ correlators at $t=T$ that corresponds to the signals $u_m(t)$, $m=1,2,\ldots,M-1$, when $\hat{\phi}_m\neq\phi_m$.

2. Show that the minimum frequency separation required for the signal orthogonality at the demodulator when $\hat{\phi}_m\neq\phi_m$ is $\Delta f=\frac{1}{T}$.

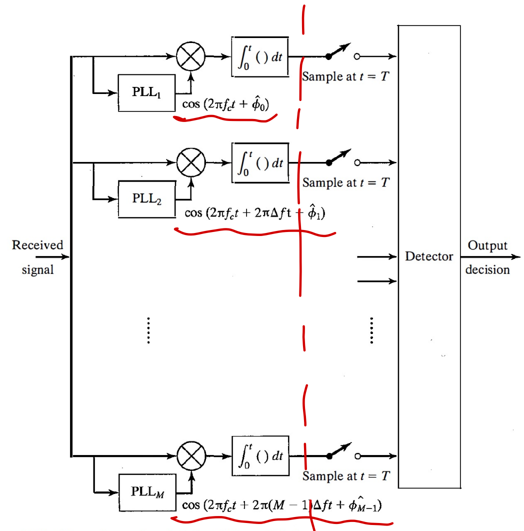


Figure 9.11 Phase-coherent demodulation of M-ary FSK signals.

$E[r_m]=0$ for signal orthogonality

(9.10).

(1) If the transmitted signal is: $u_0(t)=\sqrt{\frac{2E_s}{T}}\cos(2\pi f_c t)$. $0\le t\le T$.

then the received signal is: $r(t)=\sqrt{\frac{2E_s}{T}}\cos(2\pi f_c t+\phi)+n(t)$.

In the phase-coherent demodulation of M-ary FSK signals, the output of the m^{th} correlator is:

$$r_m=\int_0^T r(t)\cos(2\pi f_c t+2\pi m\Delta f t+\hat{\phi}_m)dt=\int_0^T\sqrt{\frac{2E_s}{T}}\cos(2\pi f_c t+\phi)\cos(2\pi f_c t+2\pi m\Delta f t+\hat{\phi}_m)dt$$

$$+\int_0^T n(t)\cos(2\pi f_c t+2\pi m\Delta f t+\hat{\phi}_m)dt$$

$$=\sqrt{\frac{2E_s}{T}}\,\frac{1}{2}\int_0^T\cos(2\pi m\Delta f t+\hat{\phi}_m-\phi)dt+n.$$

n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

(2) the value of r_m. $E[r_m]$ should be zero. and since $E[n]=0$: 条件

$$\int_0^T\cos(2\pi m\Delta f t+\hat{\phi}_m-\phi)dt=0.\quad\forall m\neq 0$$

the necessiary condition for orthogonality is: $m=1$, $\Delta f=\frac{1}{T}$.

本题目中的 $E[r_m]$ 表达式是选择的.

若是:

$r(t)$ → $\phi_1(t)$ ⊗, → $\phi_2(t)$ ⊗, ……

$\phi_m(t)=\sqrt{\frac{2}{T}}\cos[2\pi(f_c+m\Delta f)t]$.

那 $\rho_{mn}=\frac{1}{E_s}\int_0^T S_m(t)S_n(t)dt=\frac{\sin[2\pi(m-n)\Delta f T]}{2\pi(m-n)\Delta f T}$

$\Rightarrow \Delta f_{min}=\frac{1}{2T}$

$$\frac{1}{2\pi m\Delta f}\sin(2\pi m\Delta f t+\hat{\phi}_m-\hat{\phi})\Big|_0^T$$

$$=\frac{1}{2\pi m\Delta f}[\sin(2\pi m\Delta f T+\hat{\phi}_m-\hat{\phi})-\sin(\hat{\phi}_m-\hat{\phi})]=0$$

$$\sin(2\pi m\Delta f T)\cos(\hat{\phi}_m-\hat{\phi})+\sin(\hat{\phi}_m-\hat{\phi})[\cos(2\pi m\Delta f T)-1]$$

for $m=1$. $\sin(2\pi\Delta f T)=0$ $\quad\cos(2\pi\Delta f T)=1$

$\hookrightarrow \Delta f=0,\frac{1}{2T},\frac{1}{T},\cdots$ $\quad\hookrightarrow \Delta f=0,\frac{1}{T},\frac{2}{T},\cdots$

$\Rightarrow \Delta f=\frac{1}{T}$.

不是 $\sqrt{\frac{2E_s}{T_b}}$.

9.12 In on--off keying of a carrier modulated signal, the two possible signals are

$$s_0(t) = 0, \qquad 0 \le t \le T_b$$
$$s_1(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos 2\pi f_c t, \qquad 0 \le t \le T_b.$$

The corresponding received signals are

$$r(t) = n(t), \qquad 0 \le t \le T_b$$
$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t), \quad 0 \le t \le T_b,$$

where ϕ is the carrier phase and $n(t)$ is AWGN.

1. Sketch a block diagram of the receiver (demodulator and detector) that employs noncoherent (envelope) detection.
2. Determine the probability density functions for the two possible decision variables at the detector corresponding to the two possible received signals.
3. Derive the probability of error for the detector.

noncoherent BFSK

$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$

$\phi_2(t) = -\sqrt{\frac{2}{T}} \sin(2\pi f_c t)$.

$\vec{r} = [r_c \ \ r_s] = [r_1 \ \ r_2]$

$s_0 = [0 \ \ 0]$

$s_1 = [\sqrt{E_b} \ \ 0]$

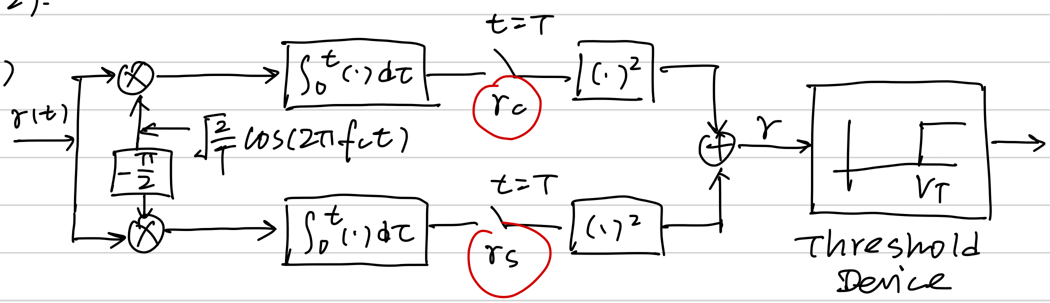
$\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi)$

$= \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)\cos(\phi) - \sqrt{\frac{2E_b}{T_b}} \sin(2\pi f_c t)\sin(\phi)$

$\Rightarrow [\sqrt{E_b}\cos\phi, \ \sqrt{E_b}\sin\phi]$

(9-12).

(1)



(2) If $s_1(t)$ is transmitted, then the received signal is:

$$r(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t).$$

crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and sampling at $t=T$:

$$r_c = \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t)\,dt = \sqrt{E_b}\cos(\phi) + n_c.$$

n_c is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

similarly, for the quardrature component: $r_s = \sqrt{E_b}\sin(\phi) + n_s$.

The PDF of the random variable $r = \sqrt{r_s^2 + r_c^2} = \sqrt{E_b + n_c^2 + n_s^2}$ is:

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+E_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{E_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+E_b}{N_0}} I_0\left(\frac{2r\sqrt{E_b}}{N_0}\right).$$

(3) For equiprobable signals:

$$p(\text{error}) = \frac{1}{2}\int_{-\infty}^{V_T} p(r|s_1(t))\,dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))\,dr$$
$$= \frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+E_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{E_b}}{\sigma^2}\right) dr + \frac{1}{2}\int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}\,dr$$

???

9.13 Digital information is to be transmitted by carrier modulation through an additive Gaussian noise channel with a bandwidth of 100 kHz and $N_0 = 10^{-10}$ W/Hz. Determine the maximum rate that can be transmitted through the channel for four-phase PSK, binary FSK, and four-frequency orthogonal FSK that is detected noncoherently.

(9-13)

(a) Four phase PSK

If uses a pulse shape having a raised cosine spectrum with a rolloff α, then:

symbol rate $\frac{1}{T}$

$K=2$ $\quad \frac{1}{2T}(1+\alpha) = 50000. \Rightarrow \frac{1}{T} = \frac{10^5}{1+\alpha} \Rightarrow$ symbol rate

the bit rate $= \frac{2}{T} = \frac{2\times 10^5}{1+\alpha}$ bps.

(b) Binary FSK with noncoherent detection:

Select the two frequencies to have a frequency separation of $\frac{1}{T}$. $\frac{1}{T}$ is symbol rate.

$$f_1 = f_c - \frac{1}{2T}. \quad f_2 = f_c + \frac{1}{2T}.$$

f_c is the carrier in the center of the channel band, then $\frac{1}{2T} = 50000$.

then $\frac{1}{T} = 10^5$. $K=1$ the bit rate is 10^5 bps $\rightarrow$ bit rate = symbol rate

(c) $M=4$ FSK with noncoherent detection. four frequencies with adjacent frequencies seperation $\frac{1}{T}$

select: $f_1 = f_c - \frac{1.5}{T}$. $f_2 = f_c - \frac{1}{2T}$. $f_3 = f_c + \frac{1}{2T}$. $f_4 = f_c + \frac{1.5}{T}$.

f_c is the carrier frequency and $\frac{1}{2T} = 25000$. and $\frac{1}{T} = 50000$.

$K=2$

the the bit rate $\frac{2}{T} = 10^5$ bps.

For MFSK: $B = M\Delta f = \frac{M}{2T}$ Hz. $\quad$ BFSK: $\frac{1}{2T} = \frac{B}{M} = 50000$ Hz.

4FSK: $\frac{1}{2T} = \frac{B}{M} = 25000$ Hz.

For MPSK / MQAM: $B = \frac{1+\alpha}{T_s}$ or $\frac{2}{T_s}$

9.15 A continuous-phase FSK signal with $h = 1/2$ is represented as

$$s(t) = \pm\sqrt{\frac{2\mathcal{E}_b}{T_b}}\cos\left(\frac{\pi t}{2T_b}\right)\cos 2\pi f_c t \pm \sqrt{\frac{2\mathcal{E}_b}{T_b}}\sin\left(\frac{\pi t}{2T_b}\right)\sin 2\pi f_c t, \quad 0 \le t \le 2T_b$$

$\Rightarrow s(t) = \pm\sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_c t \pm \frac{\pi}{2T_b}t\right). \; 0 \le t \le 2T_b.$

where the $\pm$ signs depend on the information bits transmitted.

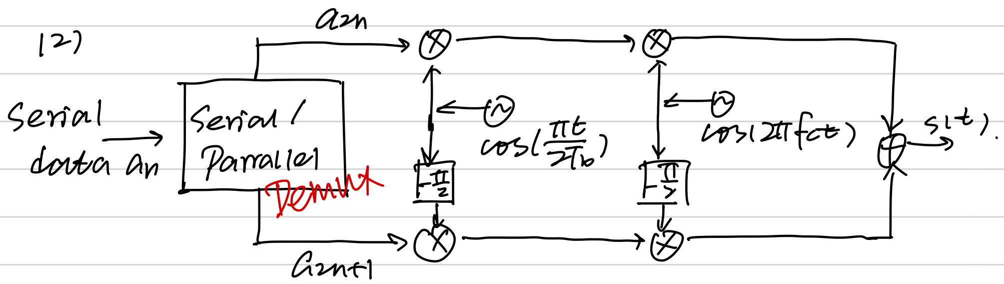
1. Show that this signal has a constant amplitude.
2. Sketch a block diagram of the modulator for synthesizing the signal.
3. Sketch a block diagram of the demodulator and detector for recovering the information.

(9.15)

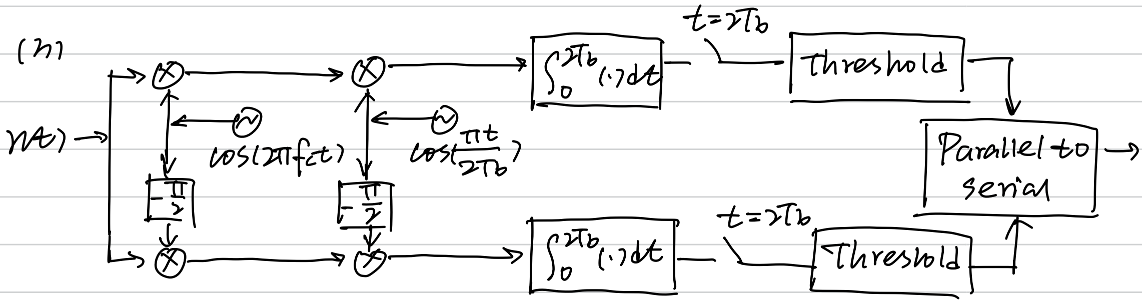
(1) the envelope of the signal is:

$|s(t)| = \sqrt{|s_c(t)|^2 + |s_s(t)|^2} = \sqrt{\frac{2E_b}{T_b}}$. $\Rightarrow$ the signal has constant amplitude.

(2)



(3)



Chapter 09 & 10

12.5 Let $Y = g(X)$, where g denotes a deterministic function. Show that, in general, $H(Y) \le H(X)$. When does equality hold?

$H(X,Y) = H(X, g(X)) = H(X) + H(g(X)|X) = H(g(X)) + H(X|g(X))$.

Since $g(\cdot)$ is deterministic, so $H(g(X)|X) = 0$.

Then $H(X) = H(g(X)) + H(X|g(X)) \ge H(g(X))$.

Equality holds when $H(X|g(X)) = 0 \Rightarrow$ the values of $g(x)$ uniquely determin X.

or $g(\cdot)$ is a one-to-one mapping.

条件熵 $H(Y|X)$ 可理解为已知随机变量X的条件下随机变量Y的不确定性

$H(Y|X) = \sum_x p(x) H(Y|X=x) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x) = -\sum_{x,y} p(x,y) \log p(y|x)$.

12.6 An information source can be modeled as a bandlimited process with a bandwidth of 6000 Hz. This process is sampled at a rate higher than the Nyquist rate to provide a guard band of 2000 Hz. We observe that the resulting samples take values in the set $\mathcal{A} = \{-4, -3, -1, 2, 4, 7\}$ with probabilities 0.2, 0.1, 0.15, 0.05, 0.3, 0.2. What is the entropy of the discrete time source in bits per output (sample)? What is the information generated by this source in bits per second?

$W = 6000\,Hz$. $f = f_s + f_g = 2W + f_g = 2 \times 6000 + 2000 = 14000\,Hz$.

entropy: $H = -0.2\log_2 0.2 - 0.1\log_2 0.1 - 0.15\log_2 0.15 - 0.05\log_2 0.05 - 0.3\log_2 0.3 - 0.2\log_2 0.2 = 2.41$ bits/sample

information: 2.41 bits/sample $\times$ 14000 samples/sec $= 33721.7$ bits/sec

12.13 Let X and Y denote two jointly distributed discrete valued random variables.

1. Show that

$$H(X) = -\sum_{x,y} p(x,y) \log p(x)$$

and

$$H(Y) = -\sum_{x,y} p(x,y) \log p(y).$$

2. Use this result to show that

$$H(X,Y) \le H(X) + H(Y).$$

When does the equality hold? **Hint**: Consider the two distributions $p(x,y)$ and $p(x)p(y)$ on the product set $\mathcal{X} \times \mathcal{Y}$, and apply the inequality proved in Problem 12.7 to $\sum_{x,y} p(x,y) \log \frac{p(x)p(y)}{p(x,y)}$.

(1) The marginal distribution $p(x) = \sum_y p(x,y)$.

$\Rightarrow H(X) = -\sum_x p(x) \log p(x) = -\sum_x \sum_y p(x,y) \log p(x) = -\sum_{x,y} p(x,y) \log p(x)$,

Similarly, $H(Y) = -\sum_{x,y} p(x,y) \log p(y)$.

(2) Using the $\ln w \le w-1$ with $w = \frac{p(x)p(y)}{p(x,y)}$:

$$\ln \frac{p(x)p(y)}{p(x,y)} \le \frac{p(x)p(y)}{p(x,y)} - 1.$$

$$\Rightarrow \sum_{x,y} p(x,y) \ln p(x)p(y) - \sum_{x,y} p(x,y) \ln p(x,y) \le \sum_{x,y} p(x)p(y) - \sum_{x,y} p(x,y) = 0.$$

$$\Rightarrow H(X,Y) \le -\sum_{x,y} p(x,y) \ln p(x)p(y) = -\sum_{x,y} p(x,y)(\ln p(x) + \ln p(y))$$

$$= -\sum_{x,y} p(x,y) \ln p(x) - \sum_{x,y} p(x,y) \ln p(y) = H(X) + H(Y).$$

12.14 Use the result of Problem 12.13 to show that

$$H(X|Y) \le H(X)$$

with equality if and only if X and Y are independent.

$H(X,Y) = H(X|Y) + H(Y) \le H(X) + H(Y) \Rightarrow H(X|Y) \le H(X).$

with equality if $H(X,Y) = H(X) + H(Y) = H(X|Y) + H(Y) \Rightarrow H(X|Y) = H(X).$

$H(X|Y) = -\sum_x p(x) \log p(x|y)$, $H(X) = -\sum_x p(x) \log p(x) \Rightarrow \sum_x p(x) \log\left(\frac{p(x)}{p(x|y)}\right) = 0.$

Obviously, only $p(x) = p(x|y)$ satisfies, then x, y are independent.

12.36 The channel shown in Figure P-12.36 is known as the *binary erasure channel*. Find the capacity of this channel and plot it as a function of ϵ.

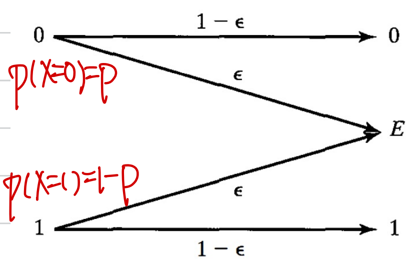


Figure P-12.36

$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)].$

the probability distribution $p(x) = p(x) = \begin{cases} p, & x=0 \\ 1-p, & x=1. \end{cases}$

then $H(Y|X) = pH(Y|X=0) + (1-p)H(Y|X=1) = ph(\epsilon) + (1-p)h(\epsilon) = h(\epsilon)$

where $h(\epsilon)$ is the binary entropy function $h(\epsilon) = -\epsilon\log\epsilon - (1-\epsilon)\log(1-\epsilon).$

$\Rightarrow I(X;Y)$ is maximized when $H(Y)$ is maximized.

Let V be a function of the output defined as: $V = \begin{cases} 1, & Y=E \\ 0, & \text{o/w.} \end{cases}$

clearly $H(V|Y)=0$ since Y is a determinstic function of Y, then

$$H(Y,V) = H(Y) + H(V|Y) = H(Y) = H(V) + H(Y|V).$$

Because $P(V=1) = P(Y=E) = p\epsilon + (1-p)\epsilon = \epsilon$.

$$p(V=0) = p(Y=0) + p(Y=1) = p(1-\epsilon) + (1-p)(1-\epsilon) = 1-\epsilon$$

$$\Rightarrow H(V) = -\epsilon\log\epsilon - (1-\epsilon)\log(1-\epsilon) = h(\epsilon).$$

$$H(Y|V) = p(V=0)H(Y|V=0) + p(V=1)H(Y|V=1).$$

And $H(Y|V=1)=0$ since no ambiguity on the output when $V=1$.

$$H(Y|V=0) = -\sum_y p(y|V=0)\log p(y|V=0)$$

$$= -p(Y=0|V=0)\log p(Y=0|V=0) - p(Y=1|V=0)\log p(Y=0|V=0)$$

we have $p(Y=0|V=0) = \dfrac{p(Y=0, V=0)}{p(V=0)} = \dfrac{p(1-\epsilon)}{1-\epsilon} = p.$

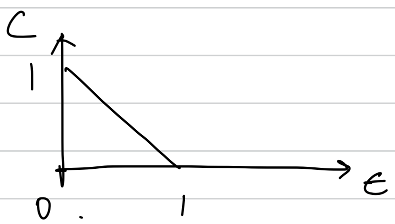
$$p(Y=1|V=0) = \frac{p(Y=1, V=0)}{p(V=0)} = \frac{(1-p)(1-\epsilon)}{1-\epsilon} = 1-p.$$

then $H(Y|V=0) = -p\log p - (1-p)\log(1-p) = h(p).$

$$H(Y|V) = p(V=0)H(Y|V=0) = (1-\epsilon)h(p).$$

$$\Rightarrow C = \max_{p(x)}[H(V) + H(Y|V) - h(\epsilon)] = \max_{p(x)}[H(Y|V)] = \max_{p(x)}(1-\epsilon)h(p) = 1-\epsilon.$$

for $p(x) = p = \frac{1}{2}$. $h(p) = 1$.



(12-46).*

The capacity of non-white Gaussian noise channel is higher.

Becouse the noise samples are correlated, knowledge of previous noise samples provides partial information about the future noise samples and then redues the effective variance.

(13-5)*

Parity-check matrix: $\vec{H} = [\vec{h}_1, \vec{h}_2, \cdots \vec{h}_n]$. $\vec{h}_i$ is an $(n-k)$ dimensional colomn vec.

$\vec{c} = [c_1 \cdots c_n]$ is a codeword of code C with ℓ non-zero elements $c_{i_1}, c_{i_2}, \cdots c_{i_\ell}$.

$$\Rightarrow \vec{c}\vec{H}^t = 0 = c_1\vec{h}_1 + c_2\vec{h}_2 + \cdots + c_n\vec{h}_n = c_{i_1}\vec{h}_{i_1} + c_{i_2}\vec{h}_{i_2} + \cdots + c_{i_\ell}\vec{h}_{i_\ell}$$
$$= \vec{h}_{i_1} + \vec{h}_{i_2} + \cdots + \vec{h}_{i_\ell} = 0.$$

So the ℓ column vectors of matrix $\vec{H}$ are linear dependent there exists d_{min} linear dependent column vectors of matrix $\vec{H}$.

Now assume that the minimum number of column vectors of the matrix $\vec{H}$ that are linear dependent is d_{min}. let $\vec{h}_{i_1}, \vec{h}_{i_2}, \cdots \vec{h}_{d_{min}}$ be a set of linear dependent column vectors. If we form a vector $\vec{c}$ with non-zero components at positions $i_1, i_2 \cdots i_{d_{min}}$. then: $\vec{c}\vec{H}^t = c_{i_1}\vec{h}_{i_1} + \cdots + c_{i_{d_{min}}}\vec{h}_{d_{min}} = 0$.

Which implies that $\vec{c}$ is a codeword with weight d_{min}. So the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.

For a Hamming code the columns of the matrix $\vec{H}$ are non-zero and distinct. No two columns $\vec{h}_i, \vec{h}_j$ add to zero and since $\vec{H}$ consists of all the $n-k$ tuples as its columns, the sum $\vec{h}_i + \vec{h}_j = \vec{h}_m$ should also be a column of $\vec{H}$. then:

$\vec{h}_i + \vec{h}_j + \vec{h}_m = 0$. $\Rightarrow$ the minimum distance of hamming code is 3.

(13-9)* For (7,4) Hamming code:

Standard array:

	(0,0,0,0,0,0,0)	(0,0,0,1,1,0,1)	(0,0,1,0,1,1,1)	(0,0,1,1,0,1,0)	(0,1,0,0,0,1,1)	(0,1,0,1,1,1,0)	(0,1,1,0,1,0,0)	(0,1,1,1,0,0,1)
e_1	(1,0,0,0,0,0,0)	(1,0,0,1,1,0,1)	(1,0,1,0,1,1,1)	(1,0,1,1,0,1,0)	(1,1,0,0,0,1,1)	(1,1,0,1,1,1,0)	(1,1,1,0,1,0,0)	(1,1,1,1,0,0,1)
e_2	(0,1,0,0,0,0,0)	(0,1,0,1,1,0,1)	(0,1,1,0,1,1,1)	(0,1,1,1,0,1,0)	(0,0,0,0,0,1,1)	(0,0,0,1,1,1,0)	(0,0,1,0,1,0,0)	(0,0,1,1,0,0,1)
e_3	(0,0,1,0,0,0,0)	(0,0,1,1,1,0,1)	(0,0,0,0,1,1,1)	(0,0,0,1,0,1,0)	(0,1,1,0,0,1,1)	(0,1,1,1,1,1,0)	(0,1,0,0,1,0,0)	(0,1,0,1,0,0,1)
e_4	(0,0,0,1,0,0,0)	(0,0,0,0,1,0,1)	(0,0,1,1,1,1,1)	(0,0,1,0,0,1,0)	(0,1,0,1,0,1,1)	(0,1,0,0,1,1,0)	(0,1,1,1,1,0,0)	(0,1,1,0,0,0,1)
e_5	(0,0,0,0,1,0,0)	(0,0,0,1,0,0,1)	(0,0,1,0,0,1,1)	(0,0,1,1,1,1,0)	(0,1,0,0,1,1,1)	(0,1,0,1,0,1,0)	(0,1,1,0,0,0,0)	(0,1,1,1,1,0,1)
e_6	(0,0,0,0,0,1,0)	(0,0,0,1,1,1,1)	(0,0,1,0,1,0,1)	(0,0,1,1,0,0,0)	(0,1,0,0,0,1,1)	(0,1,0,1,1,0,0)	(0,1,1,0,1,1,0)	(0,1,1,1,0,1,1)
e_7	(0,0,0,0,0,0,1)	(0,0,0,1,1,0,0)	(0,0,1,0,1,1,0)	(0,0,1,1,0,1,1)	(0,1,0,0,0,1,0)	(0,1,0,1,1,1,1)	(0,1,1,0,1,0,1)	(0,1,1,1,0,0,0)

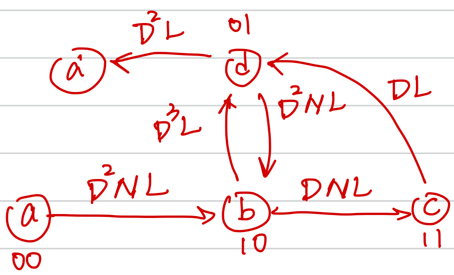
	(1,0,0,0,1,1,0)	(1,0,0,1,0,1,1)	(1,0,1,0,0,0,1)	(1,0,1,1,1,0,0)	(1,1,0,0,1,0,1)	(1,1,0,1,0,0,0)	(1,1,1,0,0,1,0)	(1,1,1,1,1,1,1)
e_1	(0,0,0,0,1,1,0)	(0,0,0,1,0,1,1)	(0,0,1,0,0,0,1)	(0,0,1,1,1,0,0)	(0,1,0,0,1,0,1)	(0,1,0,1,0,0,0)	(0,1,1,0,0,1,0)	(0,1,1,1,1,1,1)
e_2	(1,1,0,0,1,1,0)	(1,1,0,1,0,1,1)	(1,1,1,0,0,0,1)	(1,1,1,1,1,0,0)	(1,0,0,0,1,0,1)	(1,0,0,1,0,0,0)	(1,0,1,0,0,1,0)	(1,0,1,1,1,1,1)
e_3	(1,0,1,0,1,1,0)	(1,0,1,1,0,1,1)	(1,0,0,0,0,0,1)	(1,0,0,1,1,0,0)	(1,1,1,0,1,0,1)	(1,1,1,1,0,0,0)	(1,1,0,0,0,1,0)	(1,1,0,1,1,1,1)
e_4	(1,0,0,1,1,1,0)	(1,0,0,0,0,1,1)	(1,0,1,1,0,0,1)	(1,0,1,0,1,0,0)	(1,1,0,1,1,0,1)	(1,1,0,0,0,0,0)	(1,1,1,1,0,1,0)	(1,1,1,0,1,1,1)
e_5	(1,0,0,0,0,1,0)	(1,0,0,1,1,1,1)	(1,0,1,0,1,0,1)	(1,0,1,1,0,0,0)	(1,1,0,0,0,0,1)	(1,1,0,1,1,0,0)	(1,1,1,0,1,1,0)	(1,1,1,1,0,1,1)
e_6	(1,0,0,0,1,0,0)	(1,0,0,1,0,0,1)	(1,0,1,0,0,1,1)	(1,0,1,1,1,1,0)	(1,1,0,0,1,1,1)	(1,1,0,1,0,1,0)	(1,1,1,0,0,0,0)	(1,1,1,1,1,0,1)
e_7	(1,0,0,0,1,1,1)	(1,0,0,1,0,1,0)	(1,0,1,0,0,0,0)	(1,0,1,1,1,0,1)	(1,1,0,0,1,0,0)	(1,1,0,1,0,0,1)	(1,1,1,0,0,1,1)	(1,1,1,1,1,1,0)

$$\vec{G} = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right], \quad \vec{H}^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\vec{r} = [1,1,1,0,1,0,0]$.

$\Rightarrow \vec{s} = \vec{r}\vec{H}^T = [1, 1, 0]$

$\vec{e}_1 H^T = [1,1,0] = \vec{s}$. Therefore $\vec{c} = \vec{r} + \vec{e}_1 = [0,1,1,0,1,0,0]$



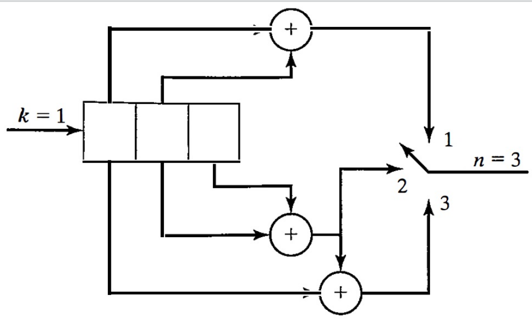
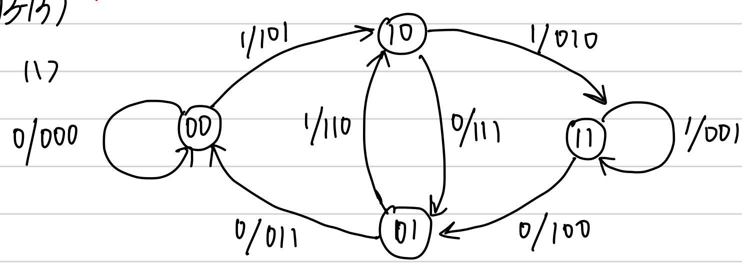


Figure P-13.13

1. Draw the state diagram for the code.
2. Find $T(D)$, the transfer function of the code.
3. What is d_{free}, the minimum free distance of the code?
4. Assume that a message has been encoded by this code and transmitted over a binary symmetric channel with an error probability of $p = 10^{-5}$. If the received sequence is $\boldsymbol{r} = (110, 110, 110, 111, 010, 101, 101)$, use the Viterbi algorithm to find the transmitted bit sequence.
5. Find an upper bound to the bit error probability of the code when the preceding binary symmetric channel is employed. Make any reasonable approximations.

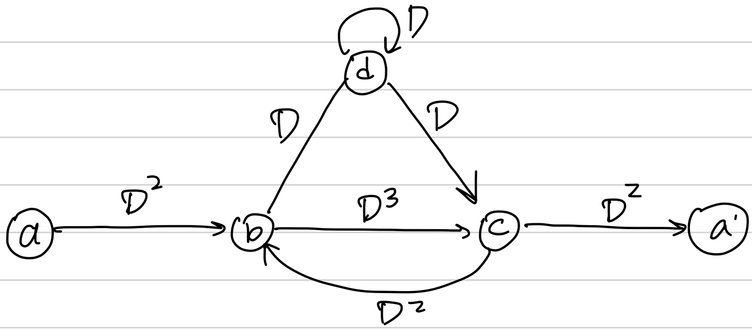
(13.13)**

(1)



current state	input	next state	output
000	0	000	000
	1	100	101
100	0	010	111
	1	110	010
010	0	001	011
	1	101	110
110	0	011	100
	1	111	001
001	0	000	000
	1	100	101
101	0	010	111
	1	110	010

(2)



$$\Rightarrow \begin{cases} X_b = D^2X_a + D^2X_c \\ X_c = D^3X_b + DX_d \\ X_d = DX_d + DX_b \\ X_{a'} = D^2X_c \end{cases} \Rightarrow T(D) = \frac{X_{a'}}{X_a} = \frac{D^6+D^7-D^8}{1-D-D^4-D^5+D^6}.$$

(3) From the $T(D)$: $T(D) = D^6 + 2D^7 + D^8 + \cdots$. $\Rightarrow d_{free} = 6$.

(4)

(5) $\bar{P}_b \le \frac{1}{k} \cdot \frac{\partial T_2(D,N)}{\partial N}\Big|_{N=1,\ D=\sqrt{4p(1-p)}} = \frac{1}{3}\left(4\times10^{-5}(1-10^{-5})\right)^3 \approx 2\times10^{-14}$.

Example 13.3.5

Assume that, in hard-decision decoding, the quantized received sequence is

$$\hat{c} = (01101111010001).$$

The convolutional code is given in Figure 13.10. Find the maximum likelihood information sequence and the number of errors.

Solution The code is a (2, 1) code with $L = 3$. The length of the received sequence $\hat{c}$ is 14. This means that $m = 7$ and we have to draw a trellis of depth 7. Also note that because the input information sequence is padded with $k(L-1) = 2$ zeros, for the final two stages of the trellis, we will only draw the branches corresponding to all-zero inputs. This also means that the actual length of the input sequence is 5, which, after padding with two zeros, has increased to 7. The trellis diagram for this case is shown in Figure 13.18. The parsed received sequence $\hat{c}$ is also shown in this figure. In drawing the trellis in the last two stages, we have considered only the zero inputs to the encoder. In the final two stages, there are no dashed lines corresponding to 1 inputs. Now the metric of the initial all-zero state is set to zero and the metrics of the next stage are computed. In this step, there is only one branch entering each state; therefore, there is no comparison, and the metrics, which are the Hamming distances between that part of the received sequence and the branches of the trellis, are added to the metric of the previous state. In the next stage, there is no comparison either. In the third stage, we actually have two branches entering each state. This means that a comparison has to be made, and survivors are to be chosen. From the two branches that enter each state, the one that corresponds to the least total accumulated metric remains as a survivor and the other branches are deleted (marked with ⩘ on the graph). If, at any stage, two paths result in the same metric, each one of them can be a survivor. Such cases have been marked by a "?" in the trellis diagram. The procedure is continued to the final all-zero state of the trellis. Starting from that state, we move along the surviving paths to the initial all-zero state. This path, which

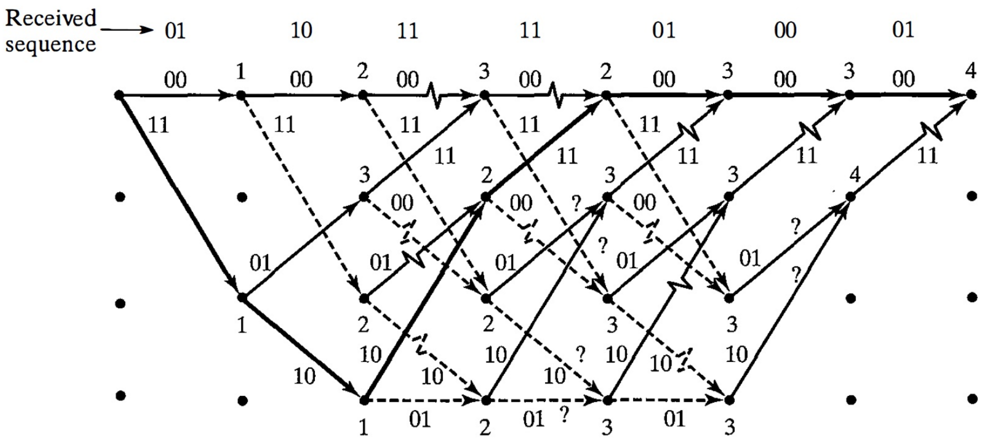


Figure 13.18 The trellis diagram for the Viterbi decoding of the sequence (01101111010001).

is denoted by a heavy path through the trellis, is the optimal path. The input bit sequence corresponding to this path is 1100000 (where the last two zeros are not information bits, but are added to return the encoder to the all-zero state). Therefore, the information sequence is 11000. The corresponding code word for the selected path is 11101011000000, which is at Hamming distance 4 from the received sequence. No other path through the trellis is at a Hamming distance less than 4 from the received $\hat{c}$. ■

For soft-decision decoding, a similar procedure is followed with squared Euclidean distance substituted for Hamming distance.