Q4)

Let $p_1 = x$ and $q_1 = 1$

For all $n > 1$ $\quad p_{n+1} = x\,p_n(x) - q_n(x)(1-x^2) \quad q_{n+1} = xq_n + p_n$

By Induction

inductive case (Assume p_n q_n work)

$$p_{n+1} = \cos((n+1)\theta) = \cos(n\theta + \theta)$$

$$\cos(n\theta)\cos(\theta) - \sin(\theta)\sin(n\theta)$$

$$p_n(x)\cos(\theta) - q_n(x)\sin^2(\theta)$$

$$x\,p_n(x) - q_n(x)(1-x^2)$$

$$q_{n+1} = \frac{\sin((n+1)\theta)}{\sin(\theta)} = \frac{\sin(n\theta + \theta)}{\sin(\theta)}$$

$$= \frac{\cos(\theta)\sin(n\theta) + \cos(n\theta)\sin(\theta)}{\sin(\theta)}$$

$$= x\,q_n + p_n$$

Base case ($n=1$)

$p_1(x) = x$ $\qquad q_1(x) = 1$

$p_1(\cos(\theta)) = \cos(\theta)$ $\qquad q_1(\sin(\theta)) = 1$

$\sin(\theta) = \sin\theta)$

Since as proven by induction this formula work $\forall n \in \mathbb{N}$ we can use them for the first 3 polynomials

1. $p_1 = x \quad q_1 = 1$
2. $p_2 = x \cdot x - 1 \cdot (1 - x^2) \qquad q_2 = x + x$

   $p_2 = 2x^2 - 1 \qquad q_n = 2x$
3. $p_3 = x(2x^2 - 1) - 2x(1 - x^2) \qquad q_3 = 2x \cdot x + 2x^2 - 1$

   $p_3 = 2x^3 - x - 2x + 2x^3 \qquad q_3 = 4x^2 - 1$

   $p_3 = 4x^3 - 3x$

4 $p_4 = x(4x^3 - 3x) - (4x^2 - 1)(1 - x^2)$ $\quad q_4 = x(4x^2 - 1) + 4x^3 - 3x$

$p_4 = 4x^4 - 3x^2 - 4x^2 + 4x^4 - x^2 + 1$ $\quad q_4 = 4x^3 + 4x^3 - x - 3x$

$p_4 = 8x^4 - 8x^2 + 1$ $\quad q_4 = 8x^3 - 4x$

5 $p_5 = x(8x^4 - 8x^2 + 1) - (8x^3 - 4x)(1 - x^2)$

$= 8x^5 - 8x^3 + x - 8x^3 + 8x^5 + 4x - 4x^3$

$= 16x^5 - 20x^3 + 5x$

$q_5 = x(8x^3 - 4x) + 8x^4 - 8x^2 + 1$

$= 8x^4 - 4x^2 + 8x^4 - 8x^2 + 1$

$= 16x^4 - 12x^2 + 1$