随机过程

HW3

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Problem 1

There is that i is null recurrent and $i \leftrightarrow j$, now we prove that j is null recurrent.

$i \leftrightarrow j \Rightarrow d(i) = d(j) = d \geq 1$

$\pi_i = \lim_{n\to\infty} P_{ii}^{nd} = \frac{d}{\mu_{ii}} = 0$

$i \leftrightarrow j \Rightarrow \exists s, t \geq 0, P_{ij}^s > 0, P_{ji}^t > 0$

$P_{ii}^{t+s} \geq P_{ji}^t P_{ij}^s > 0 \Rightarrow d \text{ divides } t+s$

$\pi_i = \lim_{m\to\infty} P_{ii}^{t+s+md} = 0$

$P_{ii}^{t+s+md} \geq P_{ij}^s P_{jj}^{md} P_{ji}^t$

$0 = \lim_{m\to\infty} P_{ii}^{t+s+md} \geq P_{ij}^s P_{ji}^t \cdot \lim_{m\to\infty} P_{jj}^{md}$

because $P_{ij}^s > 0, P_{ji}^t > 0$ we have

$\pi_j = \lim_{m\to\infty} P_{jj}^{md} = 0$

So j is null recurrent.

Problem 2

We know $P_{00} = P_{NN} = 1, P_{i,i+1} = p = 1 - P_{i,i-1}, N = 6$

So we have

$$Q = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1-p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.4 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0 \end{pmatrix}$$

Because $M = I + QM$, we know

$$M = (I-Q)^{-1} = \begin{pmatrix} 1 & -0.4 & 0 & 0 & 0 \\ -0.6 & 1 & -0.4 & 0 & 0 \\ 0 & -0.6 & 1 & -0.4 & 0 \\ 0 & 0 & -0.6 & 1 & -0.4 \\ 0 & 0 & 0 & -0.6 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{211}{133} & \frac{130}{133} & \frac{4}{7} & \frac{40}{133} & \frac{16}{133} \\ \frac{195}{133} & \frac{325}{133} & \frac{10}{7} & \frac{100}{133} & \frac{40}{133} \\ \frac{9}{7} & \frac{15}{7} & \frac{19}{7} & \frac{10}{7} & \frac{4}{7} \\ \frac{135}{133} & \frac{225}{133} & \frac{15}{7} & \frac{325}{133} & \frac{130}{133} \\ \frac{81}{133} & \frac{135}{133} & \frac{9}{7} & \frac{195}{133} & \frac{211}{133} \end{pmatrix}$$

So we can get

$$m_{3,3} = \frac{19}{7}, m_{3,2} = \frac{15}{7}, f_{3,4} = \frac{m_{3,4}}{m_{4,4}} = \frac{\frac{10}{7}}{\frac{325}{133}} = \frac{38}{65}$$

Problem 3

Define the generating function $\phi(s) = \sum_{j=0}^{\infty} s^j P_j$

Since $P_0 + P_1 < 1$, We can get

$$\phi'(s) = \sum_{j=0}^{\infty} j s^{j-1} P_j$$

$$\phi''(s) = \sum_{j=0}^{\infty} j(j-1) s^{j-2} P_j > 0$$

for all $s \in (0, 1)$, hence $\phi(s)$ is a strictly convex function in $(0, 1)$.

We define $\psi(s) = \phi(s) - s$

Hence we have

$$\psi(0) = \phi(0) - 0 = P_0$$

$$\psi(1) = \phi(1) - 1 = 1 - 1 = 0$$

$$\psi'(s) = \phi'(s) - 1$$

$$\psi'(0) = \phi'(0) - 1 = P_1 - 1$$

and $\psi'(s)$ is a monotonically increasing function.

When $\phi'(1) \le 1$, we get

$$\psi'(s) \le \psi'(1) = \phi'(1) - 1 \le 0$$

So $\psi(s)$ is a monotonically decreasing function and

$$\psi(s) = \phi(s) - s \ge \psi(1) = 0$$

$$\phi(s) \ge s$$

When $\phi'(1) > 1$, we get

$$\psi'(0) = \phi'(0) - 1 = P_1 - 1 < 0$$

$$\psi'(1) = \phi'(1) - 1 > 0$$

Hence there is $s_1 \in (0, 1)$ where $\psi'(s_1) = 0, \psi(s_1) < \psi(1) = 0$,

and there is $s_2 \in (0, s_1)$ where $\psi(s_2) = 0$.

So $\pi_0 = s_2 < 1$ in this case.

Thus, since $\phi(\pi_0) = \pi_0, \pi_0 = 1$ if, and only if, $\phi'(1) \le 1$

The result follows, since $\phi'(1) = \sum_{j}^{\infty} j P_j = \mu$

Problem 4

because $\pi_i P_{ij} = \pi_j P^*_{ji}$

$$\sum_j \pi_j P^*_{ji} = \sum_j \pi_i P_{ij} = \pi_i \sum_j P_{ij} = \pi_i$$

So $\pi_i, i \ge 0$ are also the stationary probabilities of the reverse chain.

Problem 5

$$
\begin{aligned}
&P\big(X_k = i_k \mid X_j = i_j, \forall j \neq k\big) \\
&= \frac{P\big(X_k = i_k, X_j = i_j, \forall j \neq k\big)}{P\big(X_j = i_j, \forall j \neq k\big)} \\
&= \frac{P\big(X_j = i_j, \forall j \geq k \mid X_j = i_j, \forall j < k\big)P\big(X_j = i_j, \forall j < k\big)}{P\big(X_j = i_j, \forall j > k \mid X_j = i_j, \forall j < k\big)P\big(X_j = i_j, \forall j < k\big)} \\
&= \frac{P\big(X_j = i_j, \forall j \geq k \mid X_{k-1} = i_{k-1}\big)P\big(X_j = i_j, \forall j < k\big)}{P\big(X_j = i_j, \forall j > k \mid X_{k-1} = i_{k-1}\big)P\big(X_j = i_j, \forall j < k\big)} \\
&= \frac{P\big(X_j = i_j, \forall j \geq k \mid X_{k-1} = i_{k-1}\big)}{P\big(X_j = i_j, \forall j > k \mid X_{k-1} = i_{k-1}\big)} \\
&= \frac{P\big(X_k = i_k \mid X_{k-1} = i_{k-1}, X_j = i_j, \forall j > k\big)P\big(X_j = i_j, \forall j > k \mid X_{k-1} = i_{k-1}\big)}{P\big(X_j = i_j, \forall j > k \mid X_{k-1} = i_{k-1}\big)} \\
&= P\big(X_k = i_k \mid X_{k-1} = i_{k-1}, X_j = i_j, \forall j > k\big) \\
&= \frac{P\big(X_k = i_k, X_{k-1} = i_{k-1}, X_j = i_j, \forall j > k\big)}{P\big(X_{k-1} = i_{k-1}, X_j = i_j, \forall j > k\big)} \\
&= \frac{P\big(X_j = i_j, \forall j > k+1 \mid X_k = i_k, X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)P\big(X_k = i_k, X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)}{P\big(X_j = i_j, \forall j > k+1 \mid X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)P\big(X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)} \\
&= \frac{P\big(X_j = i_j, \forall j > k+1 \mid X_{k+1} = i_{k+1}\big)P\big(X_k = i_k, X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)}{P\big(X_j = i_j, \forall j > k+1 \mid X_{k+1} = i_{k+1}\big)P\big(X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)} \\
&= \frac{P\big(X_k = i_k, X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)}{P\big(X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\big)} \\
&= P(X_k = i_k \mid X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}))
\end{aligned}
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