随机过程 **人工智能学院**

Homework 2

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Problem 1

考虑一个交替更新过程 $\begin{cases} \text{on}, & \text{otherwise} \\ \text{off}, & Y(t) \le x \end{cases}$

$$\begin{aligned}
&\lim_{t\to\infty} P(Y(t) \le x) \\
&= \frac{E[\min(X, x)]}{E[X]} \\
&= \frac{\int_0^\infty P(\min(X, x) > y)\mathrm{d}y}{E[X]} \\
&= \frac{\int_0^x P(X > y)\mathrm{d}y}{E[X]} \\
&= \frac{\int_0^x \bar{F}(y)\mathrm{d}y}{\mu}
\end{aligned}$$

Problem 2

$$\begin{aligned}
E[N_{11\cdots1}] &= E[N_{1^k}] \\
&= E[N_{1^k|1^{k-1}}] + E[N_{1^{k-1}}] \\
&= E[N_1] + \sum_{i=2}^{k} E[N_{1^i|1^{i-1}}] \\
&= \frac{1}{p} + \sum_{i=2}^{k} \frac{1}{p^i} \\
&= \begin{cases} k, & p = 1 \\ \frac{(\frac{1}{p})^k - 1}{1-p}, & \text{otherwise} \end{cases}
\end{aligned}$$

Problem 3

$$E[N_A] = E[N_{1010}] = E[N_{1010|10}] + E[N_{10}] = \frac{1}{p^2q^2} + \frac{1}{pq} = \frac{304}{9}$$

$$E[N_B] = E[N_{0100}] = E[N_{0100|0}] + E[N_0] = \frac{1}{pq^3} + \frac{1}{q} = \frac{292}{27}$$

$$E[N_{A|B}] = E[N_{1010|0100}] = E[N_{1010}] = \frac{1}{p^2q^2} + \frac{1}{pq} = \frac{304}{9}$$

$E[N_{B|A}] = E[N_{0100|1010}] = E[N_{0100|010}] = E[N_{0100}] - E[N_{010}] = E[N_{0100}] - E[N_{010|0}] - E[N_0] = \frac{292}{27} - \frac{1}{pq^2} - \frac{1}{q} = \frac{64}{27}$

$$P(A \text{ before } B) = \frac{E[N_B] + E[N_{A|B}] - E[N_A]}{E[N_{B|A}] + E[N_{A|B}]} = \frac{\frac{292}{27} + \frac{304}{9} - \frac{304}{9}}{\frac{64}{27} + \frac{304}{9}} = \frac{73}{244}$$

Problem 4

考虑 The Ballot Problem, 候选人 A 收到 n 张票, 候选人 B 受到 m 张票, 且满足 $n > m$. 我们可以证明在投票计数过程中 A 总是领先于 B 的概率为 $P_{n,m} = (n-m)/(n+m)$.

我们通过考虑最后一张票投给谁作为条件进行全概率展开有

$$P_{n,m} = \frac{n}{n+m} P_{n-1,m} + \frac{m}{m+n} P_{n,m-1}$$

根据数学归纳法, 由于 $P_{n,0} = 1$ 与 $P_{m,m} = 0$ 可知奠基成立, 因此可以假设 $P_{n',m'} = (n' - m')/(n' + m')$ 对 $n' < n$ 或 $m' < m$ 时均成立, 则有

$$P_{n,m} = \frac{n}{n+m} \frac{n-1-m}{n-1+m} + \frac{m}{m+n} \frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

我们将其应用在对称随机游走过程中, 其中 Y_i 以 p 的概率取值为 1, $1-p$ 的概率取值 -1. 因此我们有

$$\begin{aligned}
&P(Z_1 \neq 0, Z_2 \neq 0, \ldots, Z_{2n-1} \neq 0, Z_{2n} = 0) \\
&= P(\text{first time equal } = 2n) \\
&= P(\text{first time equal } = 2n \wedge n \text{ are positive in first } 2n) \\
&= P(\text{first time equal } = 2n | n \text{ are positive in first } 2n) \binom{2n}{n} p^n (1-p)^n \\
&= P_{n,n-1} \binom{2n}{n} p^n (1-p)^n \\
&= \frac{\binom{2n}{n} p^n (1-p)^n}{2n-1}
\end{aligned}$$

由于我们有 $u_n = P(Z_{2n} = 0) = \binom{2n}{n} \frac{1}{n^{2n}}$, 我们令 $p = \frac{1}{2}$, 则可得

$$u_n = \frac{2n-1}{2n} u_{n-1}$$

以及

$$P(Z_1 \neq 0, Z_2 \neq 0, \ldots, Z_{2n-1} \neq 0, Z_{2n} = 0) = \frac{\binom{2n}{n} \left(\frac{1}{2}\right)^{2n}}{2n-1} = \frac{u_n}{2n-1}$$

因此应用上式我们可得

$$P(Z_1 \neq 0, Z_2 \neq 0, \ldots, Z_{2n} \neq 0) = 1 - \sum_{k=1}^{n} \frac{u_k}{2k-1}$$

我们只需证明

$$u_n = 1 - \sum_{k=1}^{n} \frac{u_k}{2k-1}$$

我们使用数学归纳法, 当 $n=1$ 时有 $u_1 = \frac{1}{2}$ 成立, 假设上式对 $n-1$ 时成立, 则我们有

$$\begin{aligned} 1 - \sum_{k=1}^{n} \frac{u_k}{2k-1} &= 1 - \sum_{k=1}^{n-1} \frac{u_k}{2k-1} - \frac{u_n}{2n-1} \\ &= u_{n-1} - \frac{u_n}{2n-1} \\ &= u_n \end{aligned}$$

因此可知

$$P(Z_1 \neq 0, Z_2 \neq 0, \ldots, Z_{2n} \neq 0) = u_n$$

Problem 5

$X_{N(t)+1}$ 是区间末端在时间 t 之后的第一个更新区间的长度.

证明 $P(X_{N(t)+1} \geq x) \geq \bar{F}(x)$:

$$\begin{aligned} P(X_{N(t)+1} \geq x) &= P(X_{N(t)+1} \geq x | S_{N(t)} = 0) P(S_{N(t)} = 0) \\ &\quad + \int_0^\infty P(X_{N(t)+1} \geq x | S_{N(t)} = s) \mathrm{d}F_{S_{N(t)}}(s) \\ &\geq \int_0^\infty P(X_{N(t)+1} \geq x | S_{N(t)} = s) \mathrm{d}F_{S_{N(t)}}(s) \\ &= \int_0^\infty P(X_{N(t)+1} \geq x | X_{N(t)+1} > t-s) \mathrm{d}F_{S_{N(t)}}(s) \\ &= \int_0^\infty \frac{P(X_{N(t)+1} \geq x, X_{N(t)+1} > t-s)}{P(X_{N(t)+1} > t-s)} \mathrm{d}F_{S_{N(t)}}(s) \\ &= \int_0^\infty \frac{\bar{F}(\max\{x, t-s\})}{\bar{F}(t-s)} \mathrm{d}F_{S_{N(t)}}(s) \\ &= \int_0^\infty \min\{\frac{\bar{F}(x)}{\bar{F}(t-s)}, \frac{\bar{F}(t-s)}{\bar{F}(t-s)}\} \mathrm{d}F_{S_{N(t)}}(s) \\ &= \int_0^\infty \min\{\frac{\bar{F}(x)}{\bar{F}(t-s)}, 1\} \mathrm{d}F_{S_{N(t)}}(s) \\ &\geq \int_0^\infty \bar{F}(x) \mathrm{d}F_{S_{N(t)}}(s) \\ &= \bar{F}(x) \int_0^\infty \mathrm{d}F_{S_{N(t)}}(s) \\ &= \bar{F}(x) \end{aligned}$$