Homework 1

作业要求：

提交一份pdf文档，并发送到 bianc@lamda.nju.edu.cn，4月12日23:59截止。

- pdf文档命名方式："学号-姓名.pdf"，例如"MG1937000-张三.pdf"；
- 邮件标题命名："随机过程第一次作业-学号-姓名"，
  例如"随机过程第一次作业-MG1937000-张三"。

pdf可以用latex/word/markdown等方式生成，但是不要用手写证明的照片。

作业的评分主要参考以下几点：

1. 证明过程的完整性以及正确性。例如在使用之前的定理时是否充分考虑了其条件，公式推导是否完整，以及是否有错误。
2. 文档的细节。例如是否出现符号错误，文档格式是否混乱。

若发现作业出现雷同的情况，会根据相关规定给予惩罚，详情请参考课程主页中"学术诚信"的相关内容。请同学们务必独立完成作业！

Problem 1

Definition 1: The counting process $\{N(t),\ t \geq 0\}$ is said to be a Poisson process having rate $\lambda,\ \lambda > 0$, if

- $N(0) = 0$
- The process has independent increments
- The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,

$$P(N(s+t) - N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!},\ \ n = 0,1,2,\dots$$

Definition 2: The counting process $\{N(t),\ t \geq 0\}$ is said to be a Poisson process having rate $\lambda,\ \lambda > 0$, if

- $N(0) = 0$
- The process has stationary and independent increments
- $P(N(h) = 1) = \lambda h + o(h)$
- $P(N(h) \geq 2) = o(h)$

Prove: Definition 1 implies **Definition 2.**

Problem 2

X_n are iid exponential random variables having mean $1/\lambda$,

$$S_n = X_1 + X_2 + \cdots + X_n$$

Prove: S_n has a gamma distribution with parameters n and λ, i.e.,

$$P(S_n \leq t) = \sum_{k=n}^{\infty}\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Problem 3

Let $\{\boldsymbol{N_i(t), t \geq 0}\}$ be a Poisson process having rate $\boldsymbol{\lambda_i}$, where $\boldsymbol{i} \in \{\boldsymbol{1, 2, \dots, n}\}$. Suppose they are independent. Let

$$N(t) = N_1(t) + N_2(t) + \cdots + N_n(t)$$

Prove: $\{N(t), t \geq 0\}$ is a Poisson process having rate $\sum_{i=1}^{n}\lambda_i$

Problem 4

For a nonhomogeneous Poisson process $\{N(t),\, t \geq 0\}$ with intensity function $\lambda(t)$,

Prove: the number of events in interval $(t, t+s]$ is Poisson distributed with mean $m(t+s) - m(t)$. That is, for all $s, t \geq 0$,

$$P(N(t+s) - N(t) = n) = e^{-(m(t+s)-m(t))} \frac{(m(t+s) - m(t))^n}{n!}$$

where $m(t) = \int_0^t \lambda(x)\, dx$

Problem 5

Definition N2: The counting process $\{N(t),\, t \geq 0\}$ is said to be a **nonhomogeneous or nonstationary Poisson process** with intensity function $\lambda(t),\ t > 0$, if

- $N(0) = 0$
- The process has independent increments
- The number of events in $(t, t+s]$ is Poisson distributed with mean $m(t+s) - m(t)$. That is, for all $s, t \geq 0$,

  $$P(N(t+s) - N(t) = n) = e^{-(m(t+s)-m(t))} \frac{(m(t+s) - m(t))^n}{n!}$$

  where $m(t) = \int_0^t \lambda(x)\, dx$

Given a homogeneous Poisson process $\{N(t),\, t \geq 0\}$ with rate λ, where $\lambda \geq \lambda(t)$, if an event occurring at time t is counted with probability $\frac{\lambda(t)}{\lambda}$, denote the new process of counted events as $\{N'(t),\, t \geq 0\}$.

Prove: $\{N'(t),\, t \geq 0\}$ is a nonhomogeneous Poisson process with intensity $\lambda(t)$. (You should use **Definition N2**.)

Problem 6

Let $\{N^*(t), t \geq 0\}$ be a homogeneous Poisson process with rate 1, $m(t) = \int_0^t \lambda(x)\, dx$, $N(t) = N^*(m(t))$.

Prove: $\{N(t),\, t \geq 0\}$ is a nonhomogeneous Poisson process with intensity $\lambda(t)$.

Problem 7

Let $\{N(t), t \geq 0\}$ be a nonhomogeneous Poisson process with intensity $\lambda(t)$,

$m(t) = \int_0^t \lambda(x)\, dx$, $N^*(t) = N(m^{-1}(t))$.

Prove: $\{N^*(t), t \geq 0\}$ is a homogeneous Poisson process with rate 1