数理逻辑第三次作业

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8.

定义谓词 $P(x)$: x is glitter. $Q(x)$: x is gold.

那么用一阶逻辑语言表示 "all that glitters is not gold" 可为:

$\neg(\forall x(P(x) \to Q(x)))$

9.

$$
\begin{aligned}
& FV(\forall x(P(x,y) \land \forall z \exists y(y \doteq z)) \lor (x \doteq x)) \\
= & FV(\forall x(P(x,y) \land \forall z \exists y(y \doteq z))) \cup FV(x \doteq x) \\
= & (FV(P(x,y) \land \forall z \exists y(y \doteq z)) - \{x\}) \cup \{x\} \\
= & (FV(P(x,y)) \cup FV(\forall z \exists y(y \doteq z)) - \{x\}) \cup \{x\} \\
= & (\{x,y\} \cup (FV(\exists y(y \doteq z)) - \{z\}) - \{x\}) \cup \{x\} \\
= & (\{x,y\} \cup ((FV(y \doteq z) - \{y\}) - \{z\}) - \{x\}) \cup \{x\} \\
= & (\{x,y\} \cup ((\{y,z\} - \{y\}) - \{z\}) - \{x\}) \cup \{x\} \\
= & \{x,y\}
\end{aligned}
$$

10.

$$
\begin{aligned}
&(\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z))\vee(x\doteq x))[\frac{f(x)}{y}]\\
&=(\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z)))[\frac{f(x)}{y}]\vee(x\doteq x)[\frac{f(x)}{y}]\\
&=(\forall r((P(x,y)\wedge\forall z\exists y(y\doteq z))[\frac{r}{x}][\frac{f(x)}{y}]))\vee(x[\frac{f(x)}{y}]\doteq x[\frac{f(x)}{y}])\\
&=(\forall r(P(x,y)[\frac{r}{x}][\frac{f(x)}{y}]\wedge(\forall z\exists y(y\doteq z))[\frac{r}{x}][\frac{f(x)}{y}]))\vee(x\doteq x)\\
&=(\forall r(P(r,f(x))\wedge(\forall z(\exists y(y\doteq z))[\frac{f(x)}{y}])))\vee(x\doteq x)\\
&=\forall r(P(r,f(x))\wedge\forall z\exists y(y\doteq z))\vee(x\doteq x)
\end{aligned}
$$

$$
\begin{aligned}
&(\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z))\vee(x\doteq x))[\frac{f(x)}{x}]\\
&=\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z))[\frac{f(x)}{x}]\vee(x\doteq x)[\frac{f(x)}{x}]\\
&=\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z))\vee(x[\frac{f(x)}{x}]\doteq x[\frac{f(x)}{x}])\\
&=\forall x(P(x,y)\wedge\forall z\exists y(y\doteq z))\vee(f(x)\doteq f(x))
\end{aligned}
$$

14.

定义语法 $(a=b)$, 若 $a=b$ 成立, 则 $(a=b)=T$, 否则 $(a=b)=F$

(1)

对于任意模型 (M,σ), 均有

对于任意 $a\in M$,

$(x\doteq x)_{M[\sigma[x:=a]]}=(x_{M[\sigma[x:=a]]}=x_{M[\sigma[x:=a]]})=(a=a)=T$

$\therefore\ \forall x(x\doteq x)$ 为永真式.

(2)

对于任意模型 (M,σ), 均有

对于任意 $a\in M, b\in M$,

$$
\begin{aligned}
&(x \doteq y \to y \doteq x)_{M[\sigma[x:=a][y:=b]]} \\
= &B_{\to}((x \doteq y)_{M[\sigma[x:=a][y:=b]]}, (y \doteq x)_{M[\sigma[x:=a][y:=b]]}) \\
= &B_{\to}((x_{M[\sigma[x:=a][y:=b]]} = y_{M[\sigma[x:=a][y:=b]]}), (y_{M[\sigma[x:=a][y:=b]]} = x_{M[\sigma[x:=a][y:=b]]})) \\
= &B_{\to}((a = b), (b = a)) \\
= &B_{\to}((a = b), (a = b)) \\
= &T
\end{aligned}
$$

$\therefore \forall x \forall y(x \doteq y \to y \doteq x)$ 为永真式.

(3)

对于任意模型 (M, σ), 均有

对于任意 $a \in M, b \in M, c \in M$,

$$
\begin{aligned}
&((x \doteq y \wedge y \doteq z) \to x \doteq z)_{M[\sigma[x:=a][y:=b][z:=c]]} \\
= &B_{\to}((x \doteq y \wedge y \doteq z)_{M[\sigma[x:=a][y:=b][z:=c]]}, (x \doteq z)_{M[\sigma[x:=a][y:=b][z:=c]]}) \\
= &B_{\to}(B_{\wedge}((x \doteq y)_{M[\sigma[x:=a][y:=b][z:=c]]}, (y \doteq z)_{M[\sigma[x:=a][y:=b][z:=c]]}), (x \doteq z)_{M[\sigma[x:=a][y:=b][z:=c]]}) \\
= &B_{\to}(B_{\wedge}((a = b), (b = c)), (a = c))
\end{aligned}
$$

当 $a = b$ 且 $b = c$ 时, 即 $B_{\wedge}((a = b), (b = c)) = T$ 时, 有 $a = b = c$ 即 $(a = c) = T$

$\therefore B_{\to}(B_{\wedge}((a = b), (b = c)), (a = c)) = T$

$\therefore \forall x \forall y \forall z((x \doteq y \wedge y \doteq z) \to x \doteq z)$ 为永真式.

15.

$\because (A \leftrightarrow B)$ 指 $(A \to B) \wedge (B \to A)$

定义 $B_{\leftrightarrow}(X, Y) = B_{\wedge}(B_{\to}(X, Y), B_{\to}(Y, X))$

则可得真值表

A	B	$B_{\leftrightarrow}(X, Y)$
F	F	T
F	T	F
T	F	F
T	T	T

(1)

$(\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B)))_{M[\sigma]} =$
$B_{\leftrightarrow}(B_{\neg}(B_{\wedge}(A_{M[\sigma]}, B_{M[\sigma]})), B_{\vee}(B_{\neg}(A_{M[\sigma]}), B_{\neg}(B_{M[\sigma]})))$

对于任何的 $M[\sigma]$,

列真值表如下:

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$(\neg(A \wedge B))_{M[\sigma]}$	$((\neg A) \vee (\neg B))_{M[\sigma]}$	$(\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B)))_{M[\sigma]}$
F	F	T	T	T
F	T	T	T	T
T	F	T	T	T
T	T	F	F	T

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$(\neg(A \wedge B))_{M[\sigma]}$	$((\neg A) \vee (\neg B))_{M[\sigma]}$	$(\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B)))_{M[\sigma]}$
F	F	T	T	T
F	T	F	F	T
T	F	F	F	T
T	T	F	F	T

$\therefore$ $(\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B)))$ 和 $(\neg(A \wedge B) \leftrightarrow ((\neg A) \vee (\neg B)))$ 永真.

(2)

对于任何的 $M[\sigma]$,

列真值表如下:

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$(A \wedge B)_{M[\sigma]}$	$(B \wedge A)_{M[\sigma]}$	$((A \wedge B) \leftrightarrow (B \wedge A))_{M[\sigma]}$
F	F	F	F	T
F	T	F	F	T
T	F	F	F	T
T	T	T	T	T

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$(A \vee B)_{M[\sigma]}$	$(B \vee A)_{M[\sigma]}$	$((A \vee B) \leftrightarrow (B \vee A))_{M[\sigma]}$

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$(A \vee B)_{M[\sigma]}$	$(B \vee A)_{M[\sigma]}$	$((A \vee B) \leftrightarrow (B \vee A))_{M[\sigma]}$
F	F	F	F	T
F	T	T	T	T
T	F	T	T	T
T	T	T	T	T

$\therefore ((A \wedge B) \leftrightarrow (B \wedge A))$ 和 $((A \vee B) \leftrightarrow (B \vee A))$ 永真.

(3)

对于任何的 $M[\sigma]$,

列真值表如下:

$A_{M[\sigma]}$	$(A \to A)_{M[\sigma]}$
F	T
T	T

$A_{M[\sigma]}$	$B_{M[\sigma]}$	$C_{M[\sigma]}$	$((A \to B) \wedge (B \to C))_{M[\sigma]}$	$(A \to C)_{M[\sigma]}$	$(((A \to B) \wedge (B \to C)) \to (A \to C))_{M[\sigma]}$
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	F	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	F	T	F	T	T
T	T	F	F	F	T
T	T	T	T	T	T

$\therefore (A \to A)$ 和 $(((A \to B) \wedge (B \to C)) \to (A \to C))$ 永真.

16.

$\because (A \leftrightarrow B)$ 指 $(A \to B) \wedge (B \to A)$

定义 $B_{\leftrightarrow}(X,Y)=B_{\wedge}(B_{\rightarrow}(X,Y),B_{\rightarrow}(Y,X))$

则可得真值表

A	B	$B_{\leftrightarrow}(X,Y)$
F	F	T
F	T	F
T	F	F
T	T	T

$$
\begin{aligned}
&\models (\neg\forall xA)\leftrightarrow(\exists x\neg A)\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\neg\forall xA)_{M[\sigma]},(\exists x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}(B_{\neg}(\text{ 对所有 }x\in M,\text{均有 }A_{M[\sigma]}=T),(\exists x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 并非对所有 }x\in M,\text{均有 }A_{M[\sigma]}=T),(\exists x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 对某个 }x\in M,\text{有 }A_{M[\sigma]}=F),(\exists x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 对某个 }x\in M,\text{有 }\neg A_{M[\sigma]}=T),(\exists x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\exists x\neg A)_{M[\sigma]},(\exists x\neg A)_{M[\sigma]})=T
\end{aligned}
$$

$$
\begin{aligned}
&\models (\neg\exists xA)\leftrightarrow(\forall x\neg A)\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\neg\exists xA)_{M[\sigma]},(\forall x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}(B_{\neg}(\text{ 对某个 }x\in M,\text{有 }A_{M[\sigma]}=T),(\forall x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 不可能对某个 }x\in M,\text{有 }A_{M[\sigma]}=T),(\forall x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 对所有 }x\in M,\text{均有 }A_{M[\sigma]}=F),(\forall x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\text{ 对所有 }x\in M,\text{均有 }\neg A_{M[\sigma]}=T),(\forall x\neg A)_{M[\sigma]})=T\\
\text{iff}\quad &\text{对于任意模型 }(M,\sigma)\text{ 均有 }B_{\leftrightarrow}((\forall x\neg A)_{M[\sigma]},(\forall x\neg A)_{M[\sigma]})=T
\end{aligned}
$$