(Abstract) GSOS for Trace Equivalence

Séminaire LIMD

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Outline

1 Trace equivalence for concrete systems

1.1 Processes and LTSs

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Example: For $a, b, c, d \in A$

- 0 $\qquad a.0 = a \qquad d.a.0 = da$
- $a.0 + b.c.0 = a + bc \qquad ?\,(c.d.0) = ?\,cd$
- $a.(b.a.0 + ?\,(d.a.c.0)) = a(ba + ?\,dac)$

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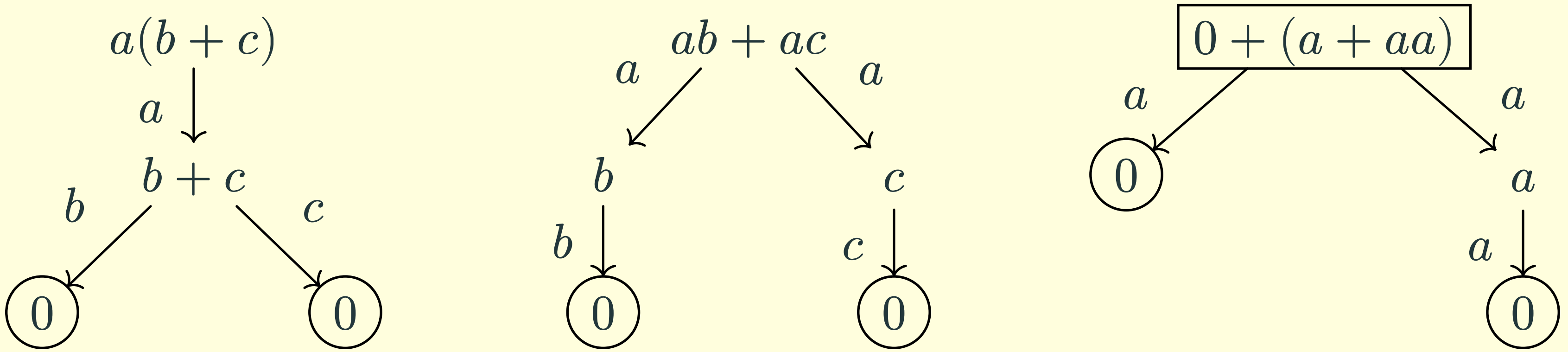

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- important property: contextual equivalence and **congruence**

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Example: $x \sim y \Rightarrow a.x \sim a.y$, or $x_1 \sim y_1 \wedge x_2 \sim y_2 \Rightarrow x_1 + x_2 \sim y_1 + y_2$

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Remark: tr is the greatest map such that $\varepsilon \in \mathrm{tr}(x) \Leftrightarrow x \downarrow$ and $a.w \in \mathrm{tr}(x) \Leftrightarrow x \xrightarrow{a} y \wedge w \in \mathrm{tr}(y)$ ("*coalgebra morphism*")

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- trace equivalence: $x \underset{\mathrm{tr}}{\sim} y \Leftrightarrow \mathrm{tr}(x) = \mathrm{tr}(y)$

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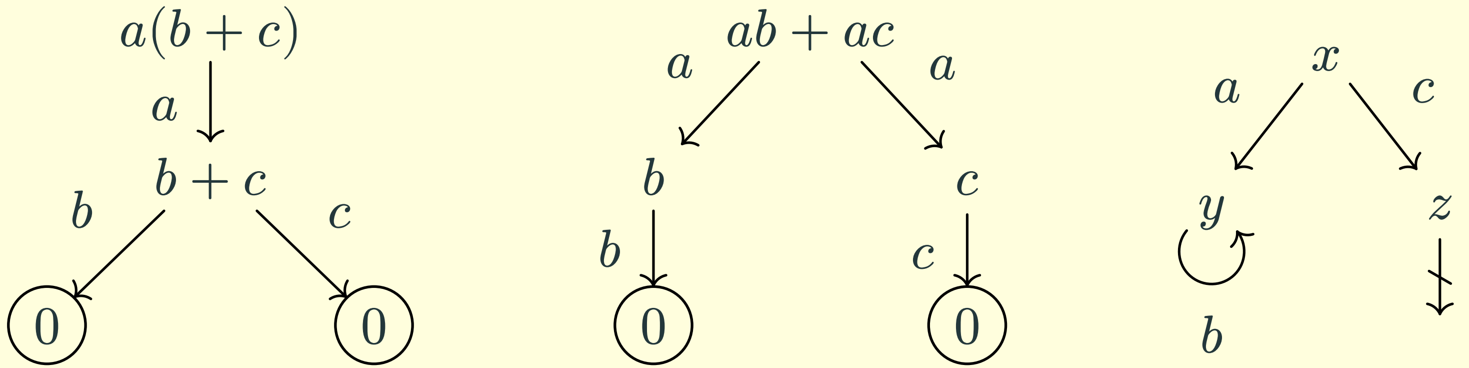

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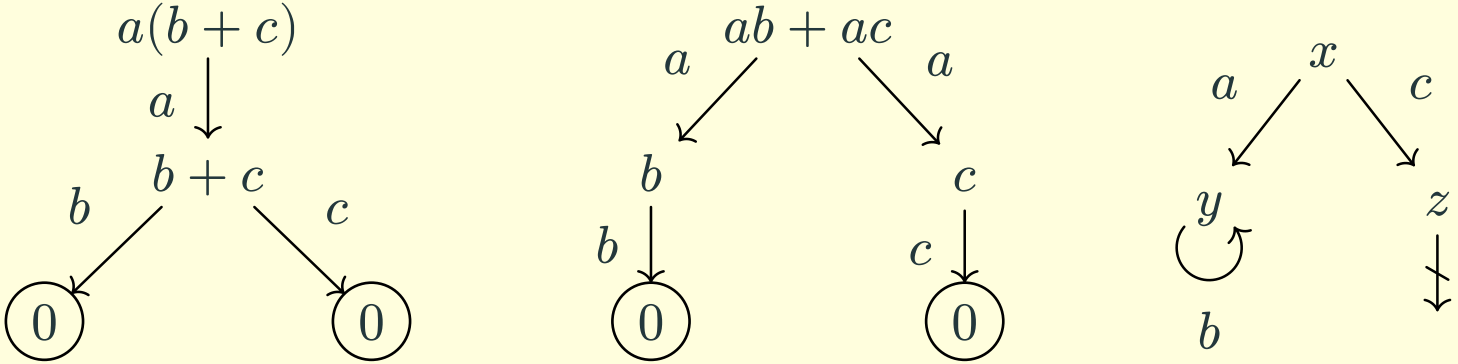

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with $\sigma \in \mathcal{O}, n = \mathrm{ar}\ \sigma, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$ and u complex term with variables in $\{x_1 \ldots x_n, y_{i,k} \cdots\}$

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- GSOS ⇒ bisimilarity is a congruence*, what about trace equivalence ?

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Remark: only **pure** observations/premises: observe each variable **once and only once**

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Example:

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  - **smoothness**: x_i in the target, the observation on x_i is **irrelevant** ie. any other observation could have been done (the same rule for each other possible observation exists)

1.6 Theorem

Theorem: Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\mathrm{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\mathrm{tr}}{\sim}$ is a congruence.

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- prove $\mathrm{tr}\ u[t_1 ... t_n] = [\![u]\!](\mathrm{tr}\ t_1 ... \ \mathrm{tr}\ t_n)$ by showing that both sides are maximal coalgebra morphisms

$\square$

1.7 Counter-examples

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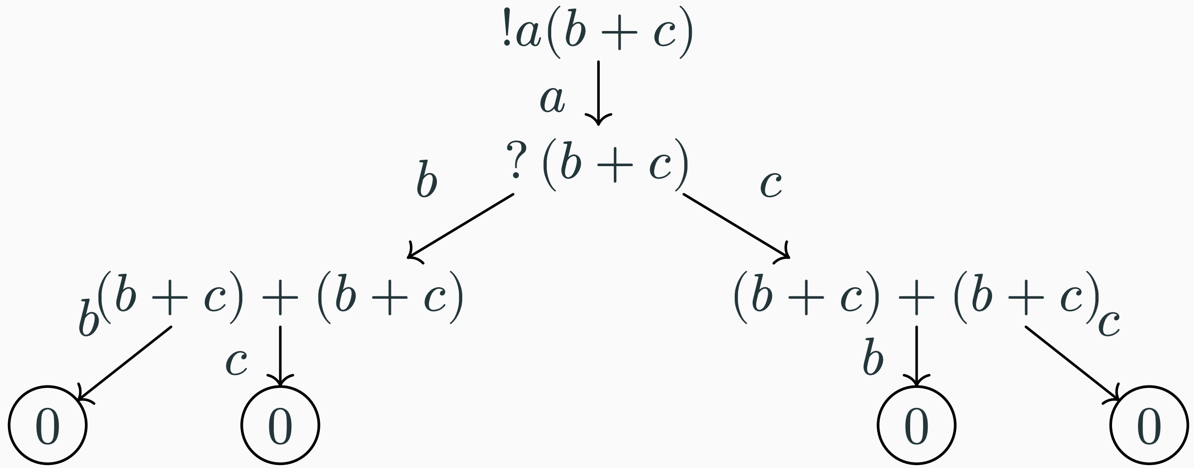

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 & (b+c)+(b+c) \arrow[dl, "b"'] \arrow[d, "c"'] & & (b+c)+(b+c) \arrow[d, "b"'] \arrow[dr, "c"] & \\
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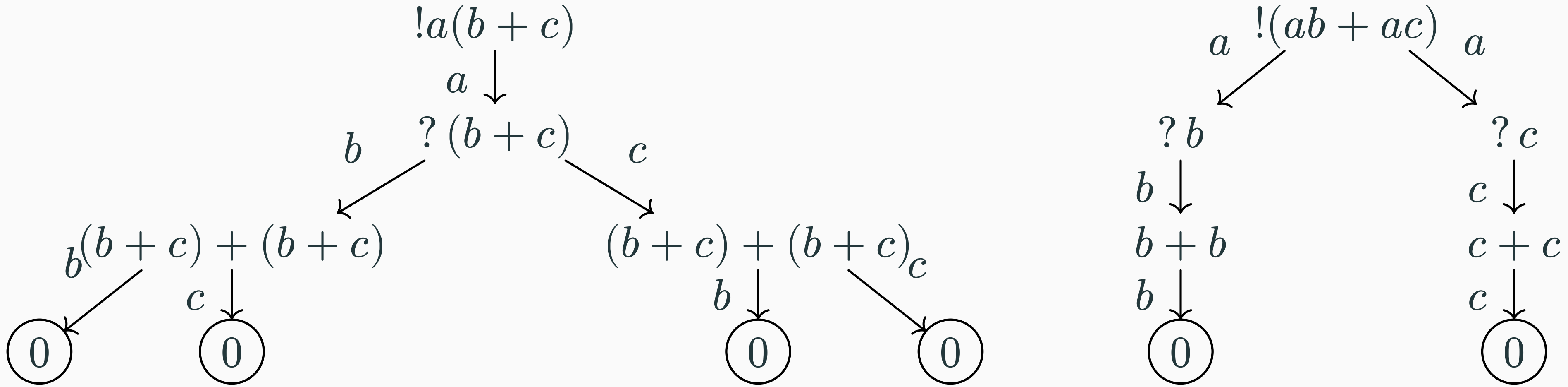

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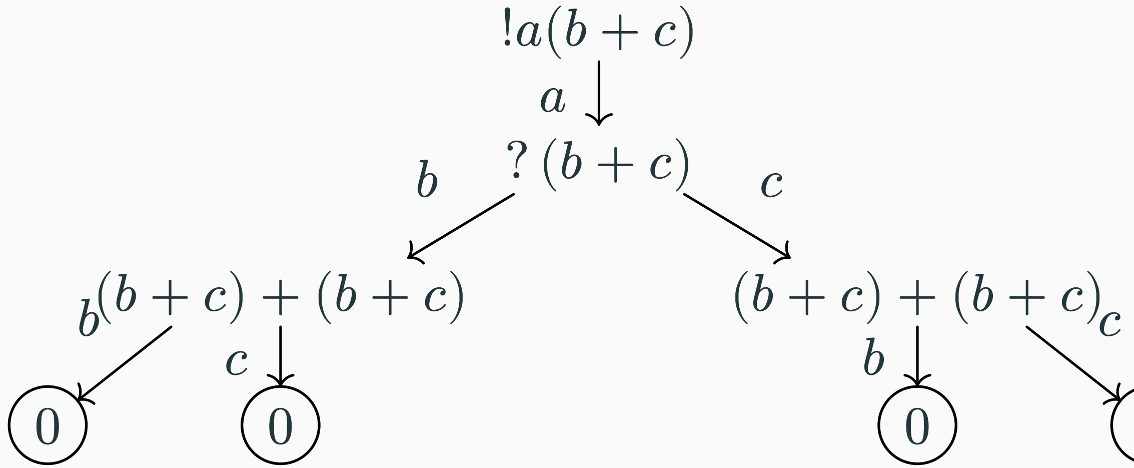

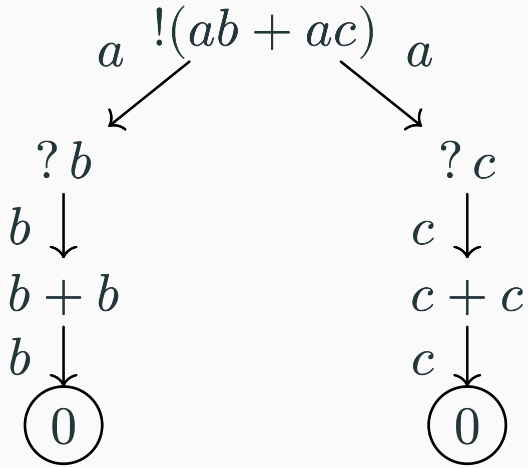

$$\mathrm{tr}\,!a(b+c) = \{abb, \underline{abc}, \underline{acb}, acc\} \neq \{abb, acc\} = \mathrm{tr}\,!(ab+ac)$$

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☢️ 🔞 WARNING 🔞 ☢️

YOU ARE ABOUT TO ENTER THE MONAD ZONE

COME IN AT YOUR PERIL

(please note that the following part of the talk is heavily populated by monads, (co)algebras, functors, natural transformations and akin, it is highly recommended to not be allergic to those if you wish to pursue your journey with us)

2 Abstraction

2.1 Algebras and coalgebras

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Example: $HX = \mathcal{P}_{\mathrm{ne}}(A \times X + 1)$

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- $B : \mathbb{C} \to \mathbb{C}$ **extends** to $\overline{B} : \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ (distributive law $\lambda^B : BT \Rightarrow TB$)

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- suppose $\mathrm{Kl}(T)$ enriched with an **order on maps with maximums**, define tr_k the **greatest $\overline{B}$-coalgebra morphism**

2.4 Trace-GSOS

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- **affineness**: ask T to be an **affine monad**

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- Maybe $-+1 \rightsquigarrow \mathrm{Id}$

2.6 Abstract smoothness

- diagrammatical condition

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$$
\begin{tikzcd}
\Sigma T(X \times BX) \arrow[r, "\langle T\pi_1, T\pi_2 \rangle"] \arrow[d, "\lambda"'] \arrow[rr, dashed, bend left, "\mathrm{mix}"] & \Sigma(TX \times TBX) \arrow[r, "\mathrm{dst}"] & \Sigma T(X \times BX) \arrow[d, "\lambda"'] \\
T\Sigma(X \times BX) \arrow[d, "T\rho"'] & & T\Sigma(X \times BX) \arrow[d, "T\rho"'] \\
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$$
\bigcup\{\rho(\sigma(\xi_1 ... \xi_n)) \mid \xi_i \in \mathrm{mix}\ \Xi_i\} = \bigcup\{\rho(\sigma(\xi_1 ... \xi_n)) \mid \xi_i \in \Xi_i\}
$$

where $\Xi_i \subset X \times BX$

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\Sigma^* X \arrow[r, "i^*", "|" marking] \arrow[d, "\Sigma^*\mathrm{tr}_k"', "|" marking] \arrow[dr, phantom, "?"] & X \arrow[r, "k", "|" marking] \arrow[d, "\mathrm{tr}_k"', "|" marking] \arrow[dr, phantom, "\checkmark"] & BX \arrow[d, "B\mathrm{tr}_k", "|" marking] \\
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- **maximality ?**

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3 Conclusion

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~ *The End*[*] ~

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[*] for today...