

Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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1. Preliminaries

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1.3 Trace & Kleisli categories

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2.3 Focus on hypothesis : Smoothness

2.4 Focus on hypothesis : Affineness

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3. Conclusion

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- **GSOS rules**

$$\frac{\left\{ x_i \xrightarrow{a_{i,k}} y_{i,k} \right\}_{i \in I, k \in K_i} \quad \left\{ x_j \downarrow \right\}_{j \in J}}{\sigma(x_1 \dots x_n) \xrightarrow{b} u[x_1 \dots x_n, y_{i,k} \dots]} \quad \text{or} \quad \frac{\left\{ x_i \xrightarrow{a_{i,k}} y_{i,k} \right\}_{i \in I, k \in K_i} \quad \left\{ x_j \downarrow \right\}_{j \in J}}{\sigma(x_1 \dots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}$, $n = \text{ar } \sigma$, $u \in \Sigma^*$, $a_{i,k}, b \in A$, $I, J, K_i \subset \llbracket 1, n \rrbracket$

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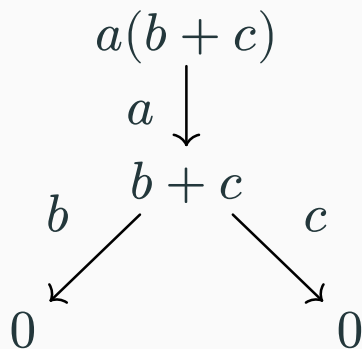
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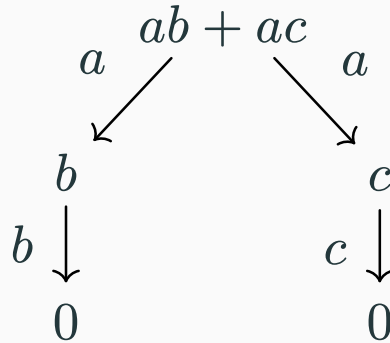
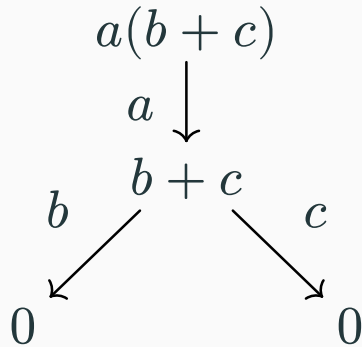
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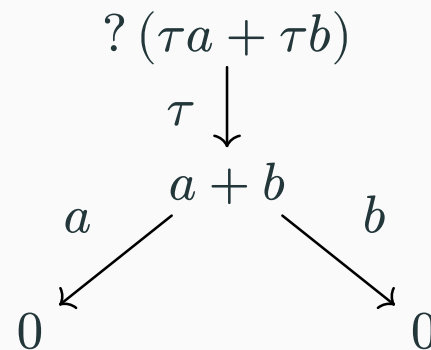
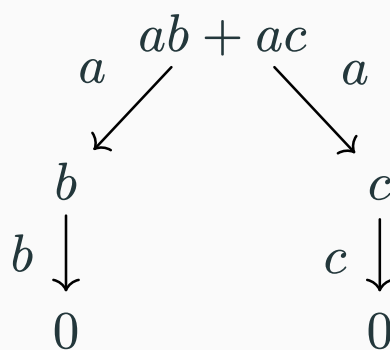
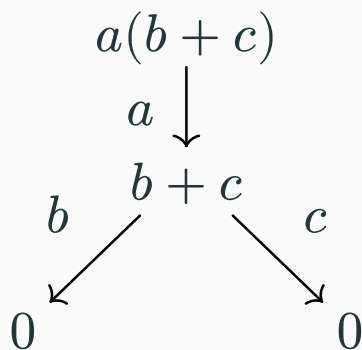
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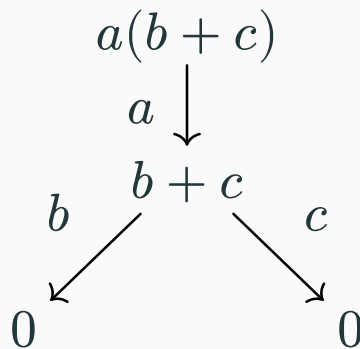
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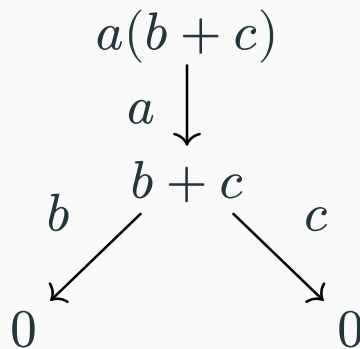


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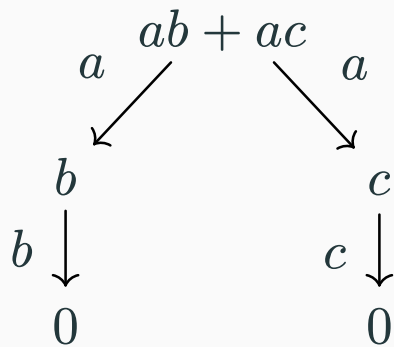


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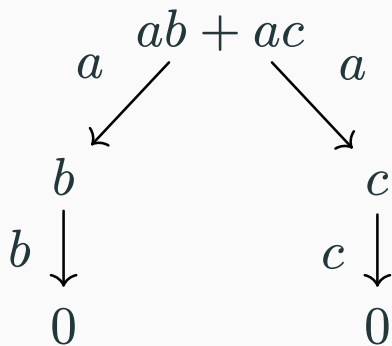


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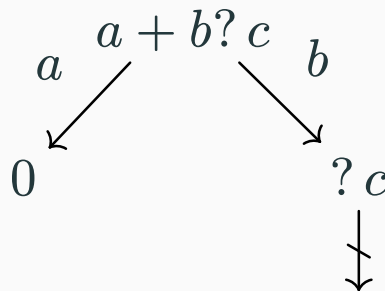


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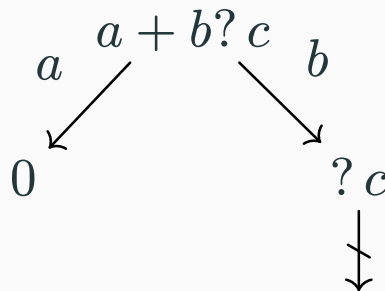


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$$\zeta : A^* \multimap BA^* \text{ or } A^* \rightarrow TBA^*$$

$$\zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

1.3 Trace & Kleisli categories

Trace **abstractly**

- in the **Kleisli category** of T

$$A \in \mathbf{Kl}(T) \Leftrightarrow A \in \mathbb{C}$$

$$A \multimap B \in \mathbf{Kl}(T) \Leftrightarrow A \rightarrow TB \in \mathbb{C}$$

- A^* is the final B -coalgebra in $\mathbf{Kl}(T)$

$$\zeta : A^* \multimap BA^* \text{ or } A^* \rightarrow TBA^*$$

$$\zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- for any $k : X \multimap BX$,

$$\begin{array}{ccc} X & \xrightarrow{k} & BX \\ \text{tr}_k \downarrow & & \downarrow B\text{tr}_k \\ A^* & \xrightarrow{\zeta} & BA^* \end{array}$$

1.4 Trace-GSOS

- GSOS rule

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- Rules observe each variable **once and only once**

Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{? t \xrightarrow{\tau} t' + t''}$$



$$\frac{}{a.t \xrightarrow{a} t} \forall a$$



$$\frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t} \forall a, b$$



$$\frac{t \downarrow}{a.t \xrightarrow{a} t} \forall a$$



1.5 Trace equivalence & congruence

- **trace equivalence:**

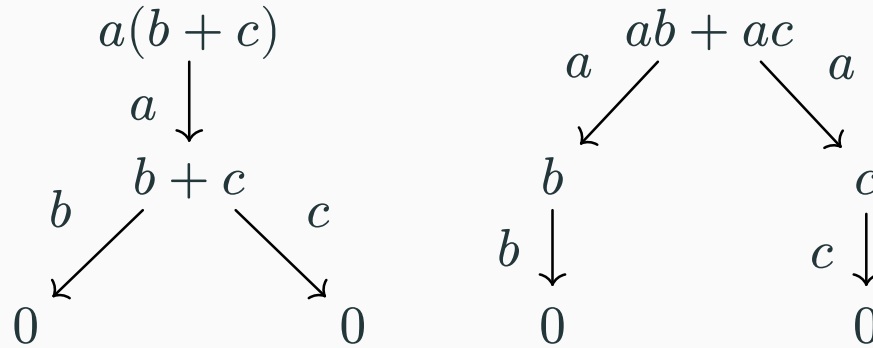
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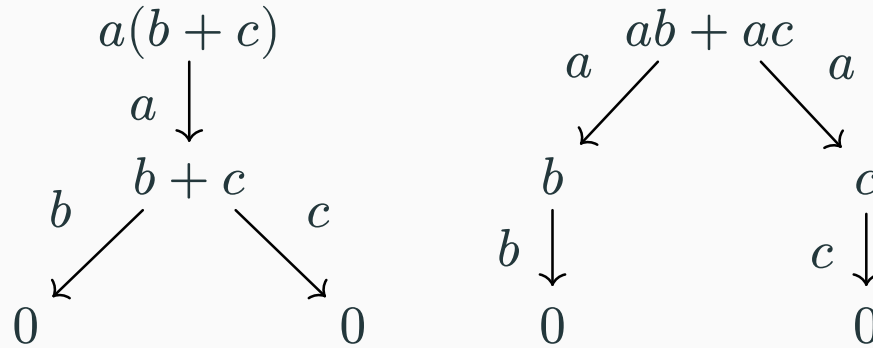
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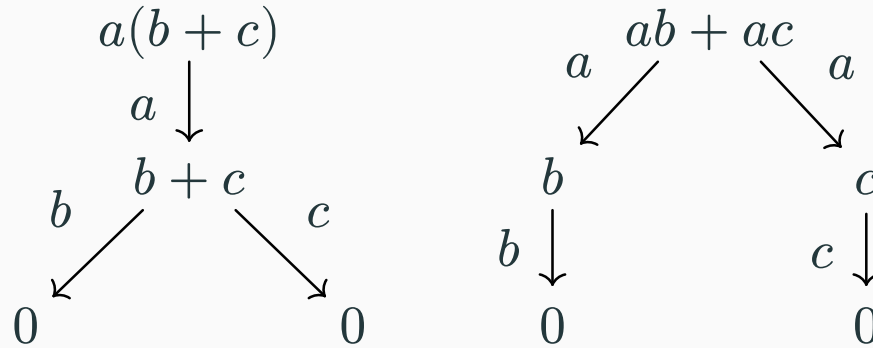


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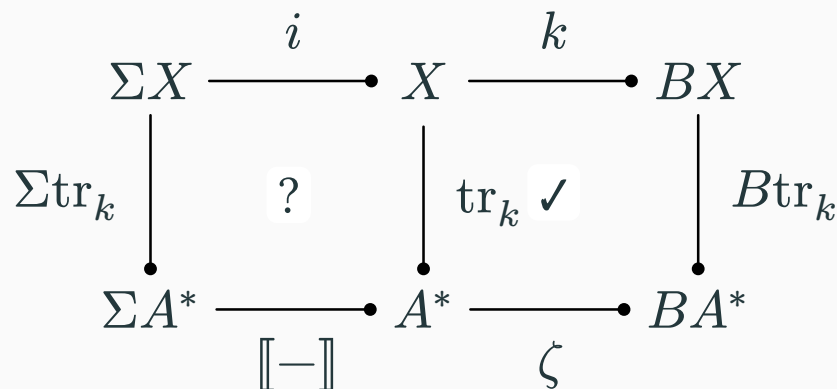
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2. Result

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- Recall

$$\begin{array}{ccc} \Sigma^* X & \xrightarrow{i} & X \\ \Sigma^* \text{tr}_k \downarrow & & \downarrow \text{tr}_k \\ \Sigma^* Z & \xrightarrow{[-]} & Z \end{array}$$

2.2 Sketch of the proof

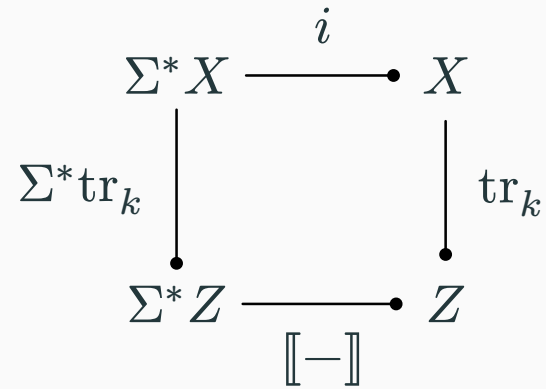
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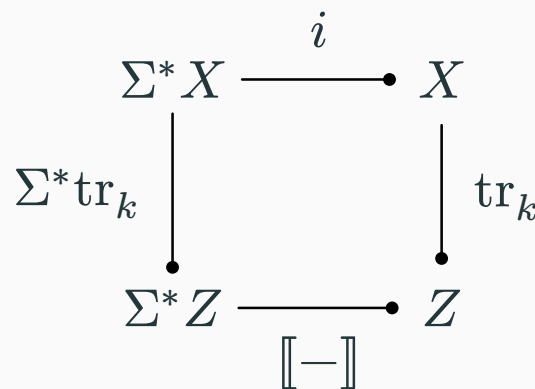
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Remark: need dst

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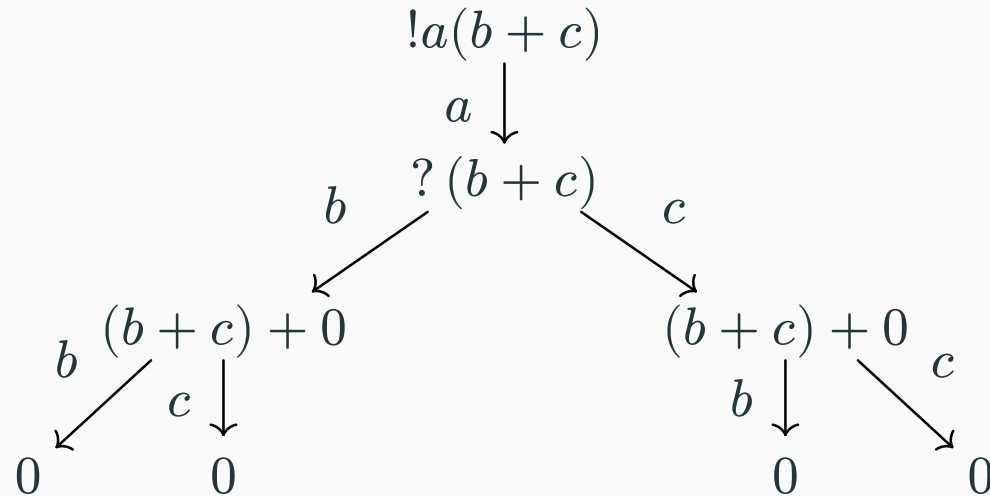
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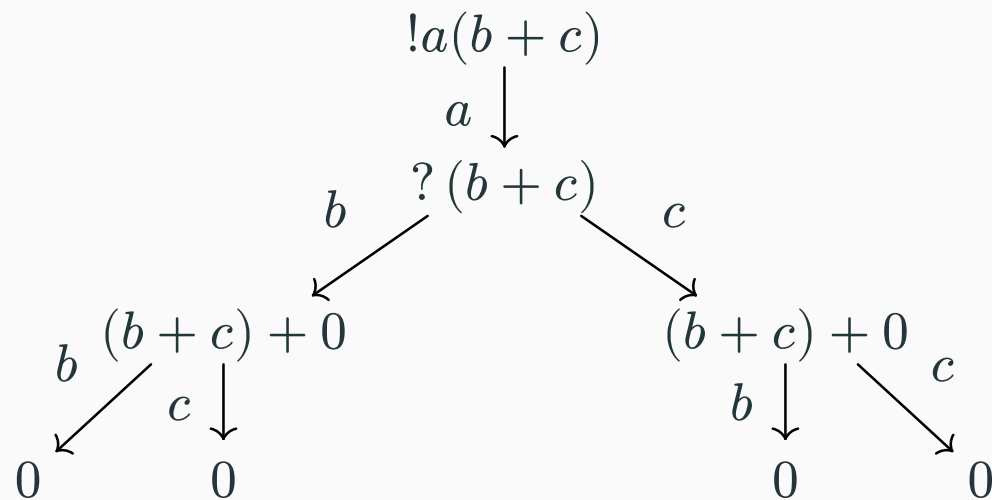
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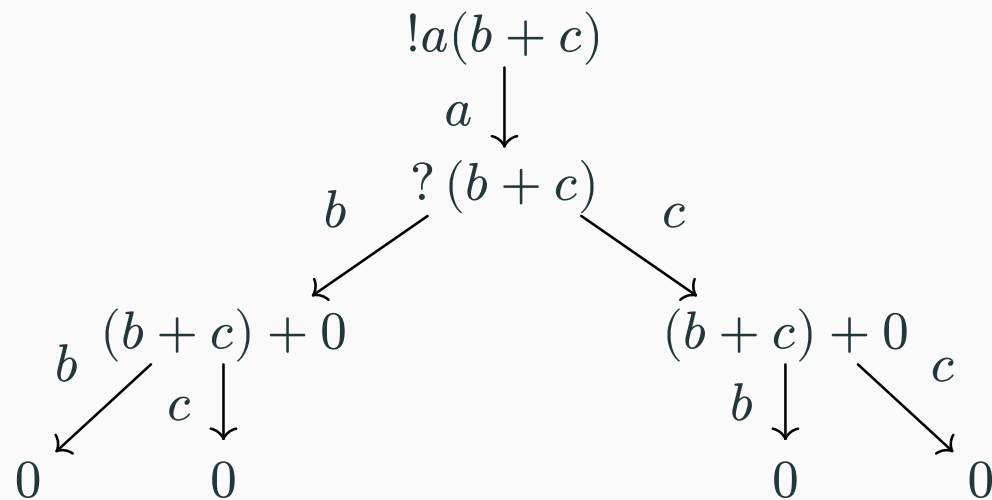


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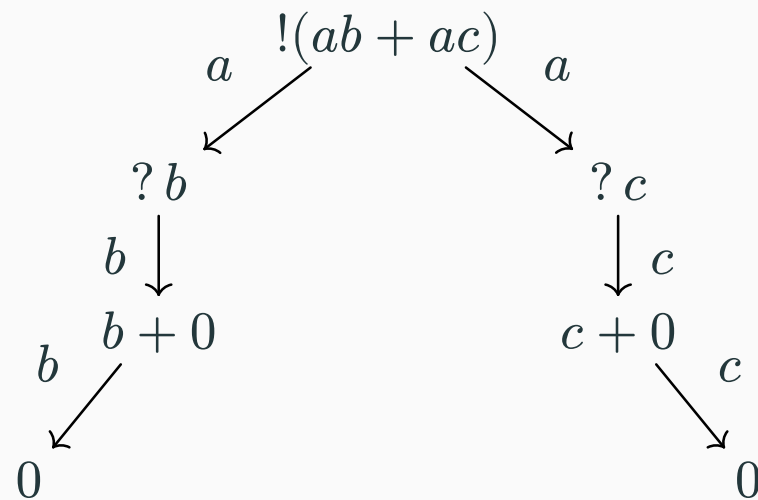
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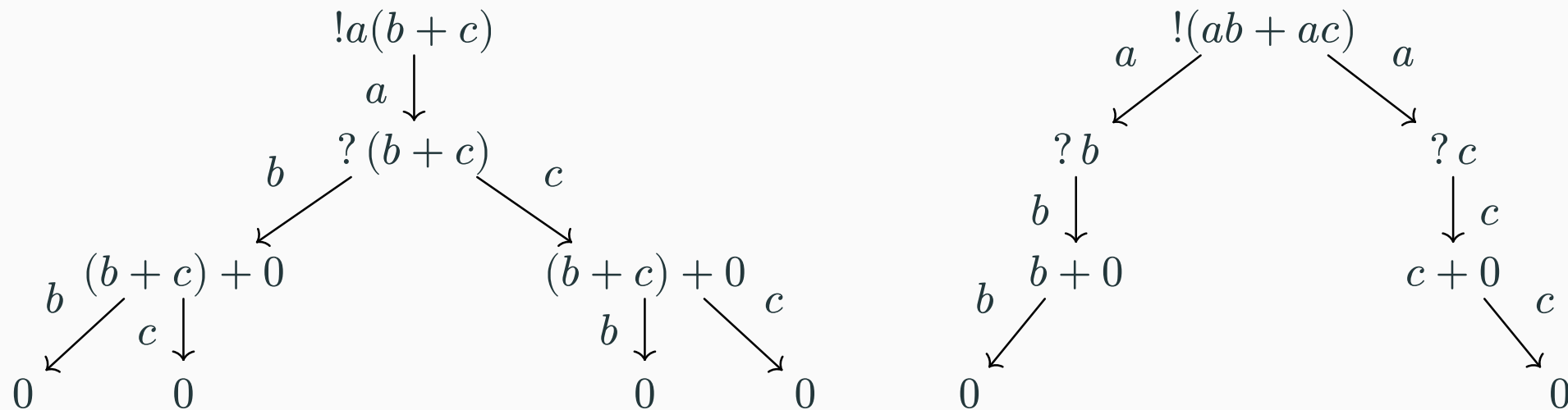
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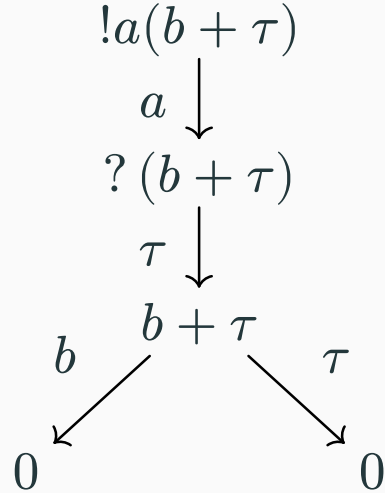
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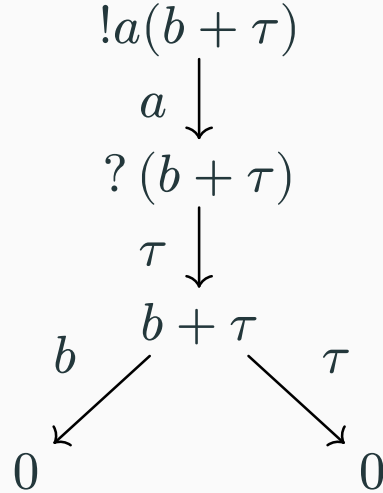
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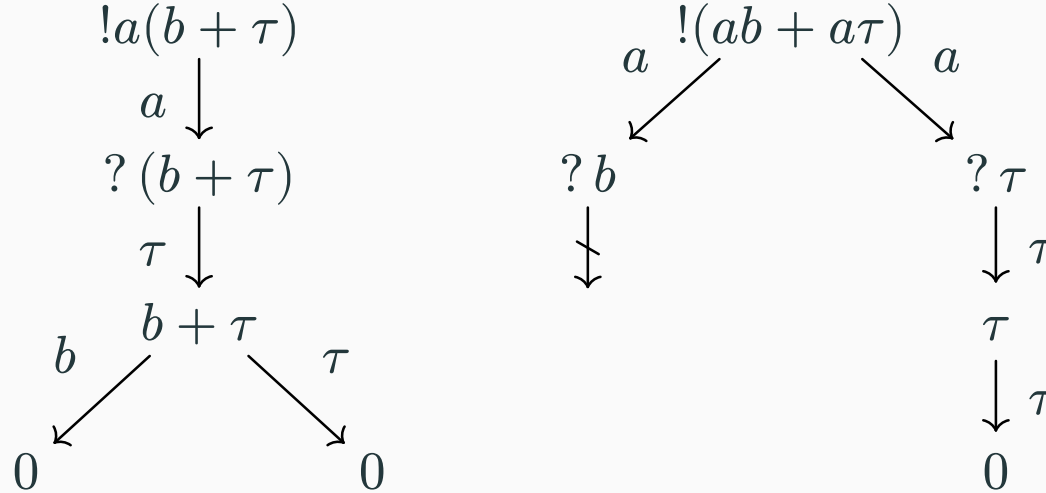


$$\text{tr } !a(b + \tau) = \{\underline{a\tau b}, a\tau\tau\}$$

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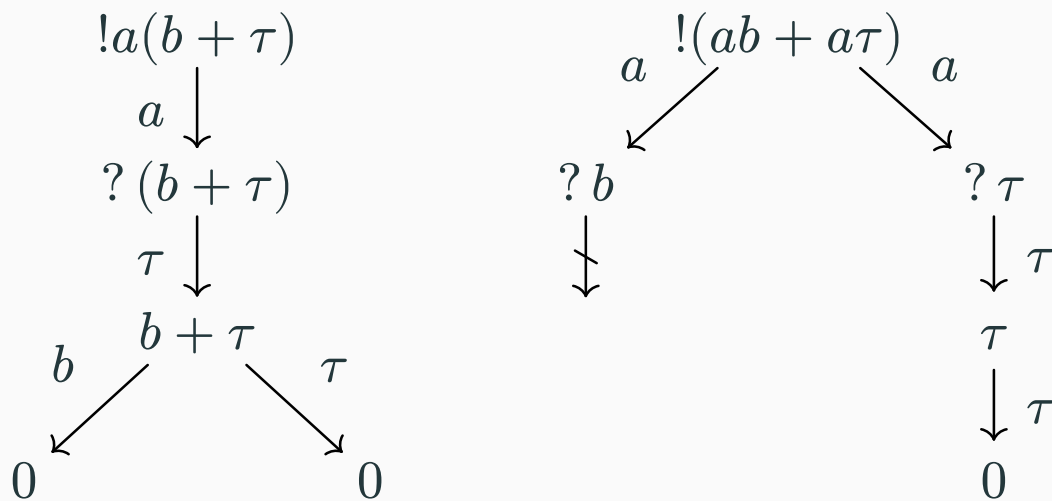


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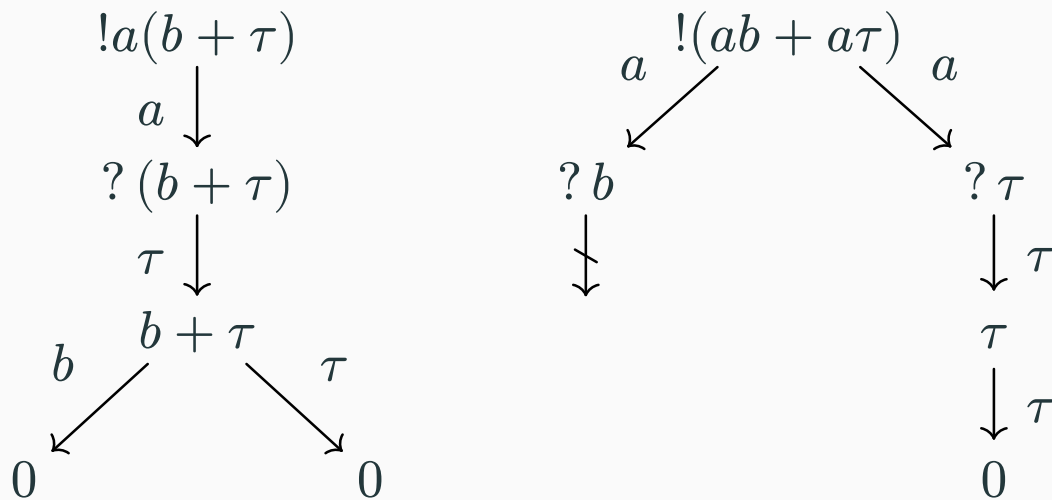


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→ observations that are “not used” 😞

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 & & \text{mix} & & \\
 & \swarrow & & \searrow & \\
 \Sigma T(X \times BX) & \xrightarrow{\langle T\pi_1, T\pi_2 \rangle} & \Sigma(TX \times TBX) & \xrightarrow{\text{dst}} & \Sigma T(X \times BX) \\
 \lambda \downarrow & & & & \lambda \downarrow \\
 T\Sigma(X \times BX) & & & & T\Sigma(X \times BX) \\
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$$\begin{aligned}
 \Phi(\rho)(\sigma)(\text{mix } X_1 \dots \text{mix } X_n) = \\
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stuck computation is messing with smoothness 😞

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- Thank you all for welcoming me in the chair ❤️

~ The End ~

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