(Abstract) GSOS for Trace Equivalence

Séminaire LIMD

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Outline

1 Trace equivalence for concrete systems

1.1 Processes and LTSs

- study the **behaviour of processes**
- running example: **labelled transition systems** (LTS) with explicit termination
  - set of actions/labels A
  - set of processes/states X with some operations
    - a special state $0 \in X$
    - for each action $a \in A$ and process $x \in X$, a process $a.x \in X$
    - and at will: a binary operation $+$, unary operations $!$, $?$ ...

Example: For $a, b, c, d \in A$

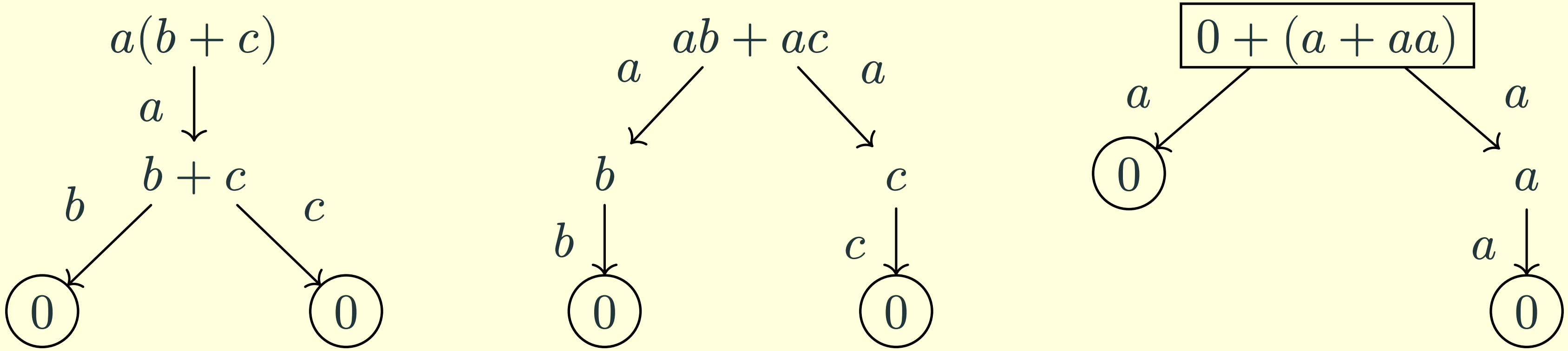
- 0 — $a.0 = a$ — $d.a.0 = da$
- $a.0 + b.c.0 = a + bc$ — $?\,(c.d.0) = ?\,cd$
- $a.(b.a.0 + ?\,(d.a.c.0)) = a(ba + ?\,dac)$

- **behaviour** of a process x:
  - can progress emitting a label $a \in A$ and continuing with process y: "$x \xrightarrow{a} y$" $\rightsquigarrow (a, y) \in k(x)$
  - or terminate: "$x \downarrow$" $\rightsquigarrow \star \in k(x)$
  - collect everything in a map $k : X \to \mathcal{P}(A \times X + \{\star\})$
- given by the **rules**:

$$\frac{}{0 \downarrow} \qquad \frac{}{a.t \xrightarrow{a} t}\forall a \qquad \frac{t \xrightarrow{a} t'}{t + u \xrightarrow{a} t'}\forall a \qquad \frac{u \xrightarrow{a} u'}{t + u \xrightarrow{a} u'}\forall a \qquad \frac{t \downarrow}{t + u \downarrow} \qquad \frac{u \downarrow}{t + u \downarrow}$$

$$\frac{}{0\downarrow} \qquad \frac{}{a.t \xrightarrow{a} t}\,\forall a \qquad \frac{t \xrightarrow{a} t'}{t+u \xrightarrow{a} t'}\,\forall a \qquad \frac{u \xrightarrow{a} u'}{t+u \xrightarrow{a} u'}\,\forall a \qquad \frac{t\downarrow}{t+u\downarrow} \qquad \frac{u\downarrow}{t+u\downarrow}$$

Example:



1.2 Program equivalences

- how to compare programs? what does it mean to be "the same" program?

Example: $f(x) := x + x$ and $g(x) := 2 \times x$

- many different notions (linear time/branching time spectrum): bisimilarity, trace equivalence
- important property: contextual equivalence and **congruence**

$$(\forall i, x_i \sim y_i) \Rightarrow \sigma(x_1..x_n) \sim \sigma(y_1...y_n)$$

Example: $x \sim y \Rightarrow a.x \sim a.y$, or $x_1 \sim y_1 \land x_2 \sim y_2 \Rightarrow x_1 + x_2 \sim y_1 + y_2$

1.3 Trace and trace equivalence

- different flavors: partial vs. complete, finite vs. infinite
- complete infinitary traces:

  $$\mathrm{tr}(x) = \bigcup_{n \in \mathbb{N}} \left\{ w \in A^n \,\middle|\, x \xrightarrow{w_1} x_1 \xrightarrow{w_2} \dots \xrightarrow{w_n} x_n \downarrow \right\} \cup \left\{ w \in A^\omega \,\middle|\, x \xrightarrow{w_1} x_1 \xrightarrow{w_2} \dots \right\} \subseteq A^\infty$$

  Set of finite and infinite words with letters in A corresponding to all possible (terminating or infinite) executions.

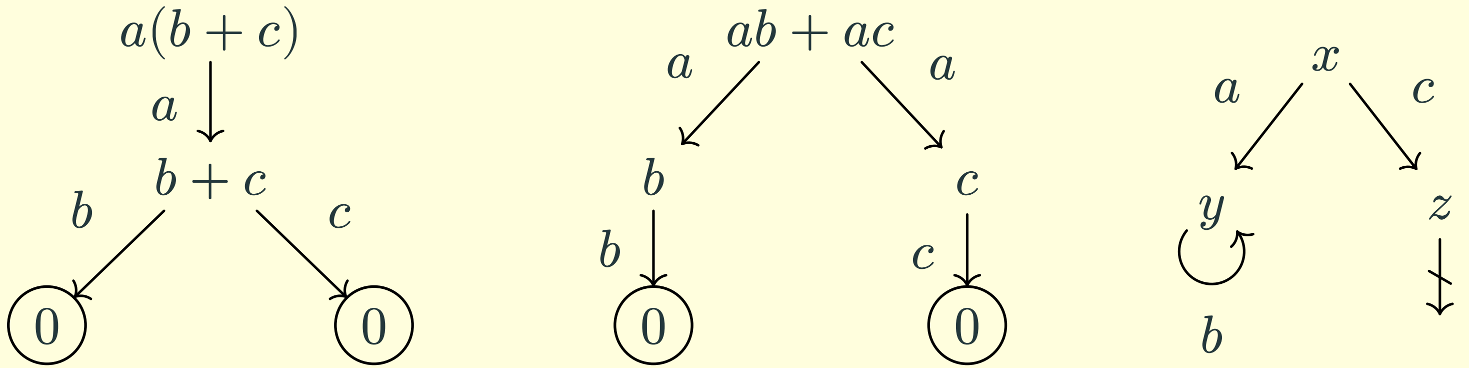
  Remark: tr is the greatest map such that $\varepsilon \in \mathrm{tr}(x) \Leftrightarrow x \downarrow$ and $a.w \in \mathrm{tr}(x) \Leftrightarrow x \xrightarrow{a} y \wedge w \in \mathrm{tr}(y)$ ("*coalgebra morphism*")

- trace equivalence: $x \underset{\mathrm{tr}}{\sim} y \Leftrightarrow \mathrm{tr}(x) = \mathrm{tr}(y)$

1.3 Trace and trace equivalence

$$\mathrm{tr}(x) = \bigcup_{n \in \mathbb{N}} \left\{ w \in A^n \,\middle|\, x \xrightarrow{w_1} x_1 \xrightarrow{w_2} \dots \xrightarrow{w_n} x_n \downarrow \right\} \cup \left\{ w \in A^\omega \,\middle|\, x \xrightarrow{w_1} x_1 \xrightarrow{w_2} \dots \right\}$$

Example: $\mathrm{tr}\ a(b+c) = \{ab, ac\}$, $\mathrm{tr}\ (ab + ac) = \{ab, ac\}$, $\mathrm{tr}\ x = \{a.b^\omega\}$



- a **framework** for reduction rules → rule format
- given a **syntax** : set of operations $\mathcal{O} = \{0, a., b., ..., +, !, ...\}$ with arity map ar : $\mathcal{O} \to \mathbb{N}$
- **GSOS rules**

$$\frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \xrightarrow{b} u} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}, n = \text{ar } \sigma, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$ and u complex term with variables in $\{x_1 \ldots x_n, y_{i,k} \cdots\}$

- GSOS ⇒ bisimilarity is a congruence[*], what about trace equivalence ?

[*] D. Turi et G. Plotkin, « Towards a mathematical operational semantics », 1997

1.5 Trace-GSOS

- recall behaviour $k : X \to \mathcal{P}(A \times X + \{\star\})$
  - $\mathcal{P}$: **effectful** behaviour (non-determinism)
  - $A \times \_ + \{\star\}$: **pure** behaviour (emit labels or terminate)
- **Trace**-GSOS rules

$$\frac{\left\{x_i \xrightarrow{a_i} y_i\right\}_{i \in I} \quad \left\{x_j \downarrow\right\}_{j \notin I}}{\sigma(x_1 ... x_n) \xrightarrow{b} u} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_i} y_i\right\}_{i \in I} \quad \left\{x_j \downarrow\right\}_{j \notin I}}{\sigma(x_1 ... x_n) \downarrow}$$

with u complex term with variables in $\{x_1 ... x_n, y_i ...\}$ with **at most one of x_i / y_i for each i** (*sublinearity*)

Remark: only **pure** observations/premises: observe each variable **once and only once**

Example:

$$\frac{t \xrightarrow{a} t' \quad t \xrightarrow{a} t''}{?t \xrightarrow{a} t + t'} \qquad \frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} t + t} \qquad \frac{}{a.t \xrightarrow{a} t}\,\forall a \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\,\forall a, b \qquad \frac{t \downarrow}{a.t \xrightarrow{a} t}\,\forall a$$

😕 😕 😕 😎 😎

- require 2 extra conditions on the set of rules:
  - **affineness**: for each term, there is a least one rule that can apply $\rightsquigarrow k : X \to \mathcal{P}_{\mathrm{ne}}(A \times X + \{\star\})$
  - **smoothness**: x_i in the target, the observation on x_i is **irrelevant** ie. any other observation could have been done (the same rule for each other possible observation exists)

1.6 Theorem

Theorem: Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\mathrm{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\mathrm{tr}}{\sim}$ is a congruence.

Proof: Show that the trace of a complex term can be obtained from the traces of its subterms.

- consider complex terms with words of A^∞ as leaves, and behaviour induced by $\mathcal{R}$ with $a.w \xrightarrow{a} w$ and $\varepsilon \downarrow$
- extend the trace function of this system to maps $[\![u]\!] : (\mathcal{P}_{\mathrm{ne}} A^\infty)^n \to \mathcal{P}_{\mathrm{ne}} A^\infty$ for each complex term u with n free variables
- prove $\mathrm{tr}\ u[t_1 \ldots t_n] = [\![u]\!](\mathrm{tr}\ t_1 \ldots\ \mathrm{tr}\ t_n)$ by showing that both sides are maximal coalgebra morphisms

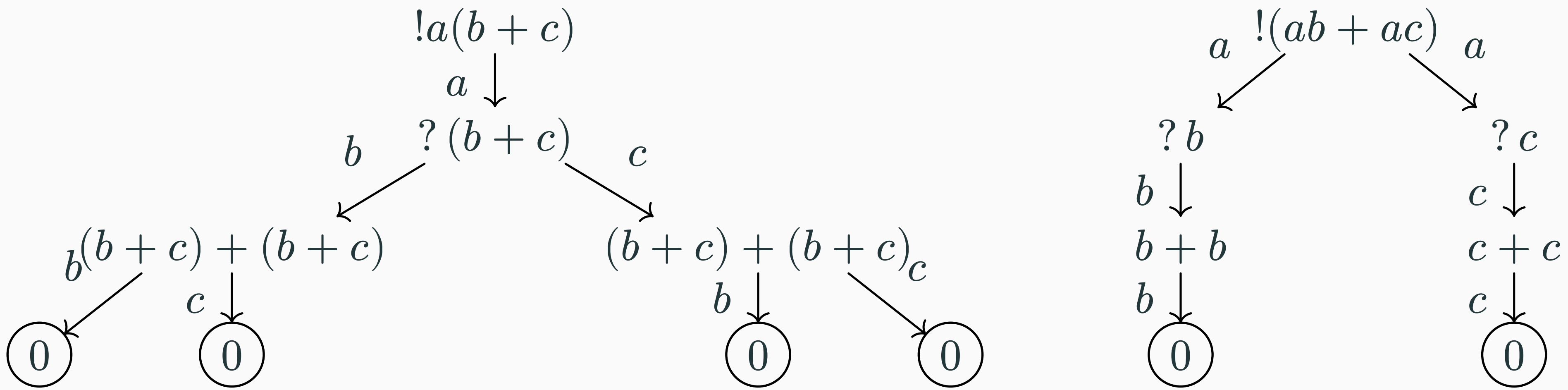
$\square$

1.7 Counter-examples

Affineness, smoothness and sublinearity are **necessary**

1.7.1 Sublinearity

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} ?\,t'}\,\forall a \qquad \frac{t \xrightarrow{a} t'}{?\,t \xrightarrow{a} t + t}\,\forall a$$

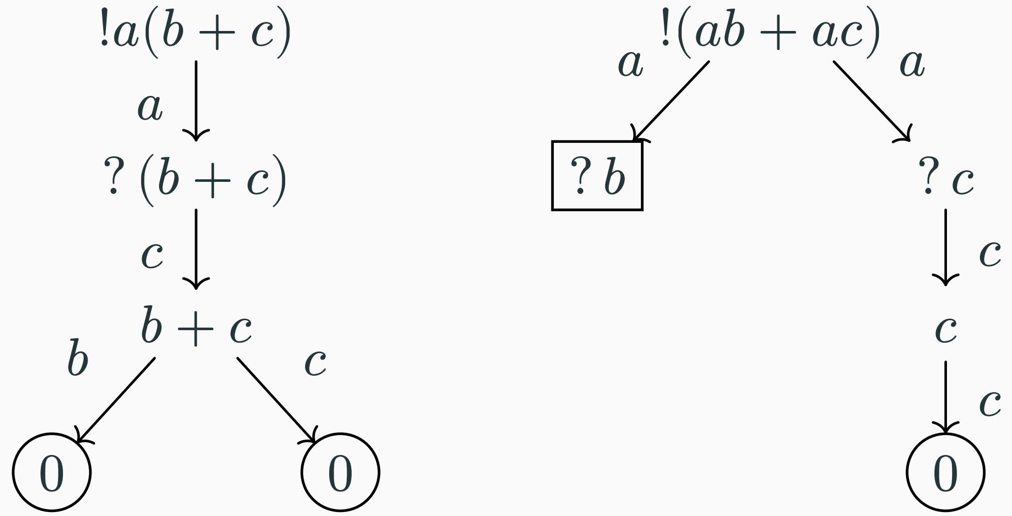


$$\mathrm{tr}\,!a(b+c) = \{abb, \underline{abc}, \underline{acb}, acc\} \neq \{abb, acc\} = \mathrm{tr}\,!(ab+ac)$$

1.7 Counter-examples

1.7.2 Smoothness

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} ?\,t'}\,\forall a \qquad \frac{t \xrightarrow{c} t'}{?\,t \xrightarrow{c} t} \qquad \frac{t \xrightarrow{a} t'}{?\,t \downarrow}\,\forall a \neq c$$



$$\mathrm{tr}\,!a(b+c) = \{\underline{acb}, acc\} \neq \{\underline{a}, acc\} = \mathrm{tr}\,!(ab+ac)$$

1.7.3 Affineness

- the proof need *deep smoothness*: rules on complex terms obtained by stacking rules of $\mathcal{R}$ need to be smooth
- without affineness, no deep smoothness:

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} b.?\,t'}\,\forall a \qquad \frac{t \xrightarrow{c} t'}{?\,t \xrightarrow{c} t'}$$

$$\dfrac{\dfrac{t \xrightarrow{c} t'}{?\,t \xrightarrow{c} t'}}{a.?\,t \xrightarrow{a} ?\,t} = \frac{t \xrightarrow{c} t'}{a.?\,t \xrightarrow{a} ?\,t} \quad \text{for smoothness, need} \quad \frac{t \xrightarrow{b} t'}{a.?\,t \xrightarrow{a} ?\,t} \quad \text{but} \quad \dfrac{\dfrac{t \xrightarrow{b} t'}{?\,t \not\rightarrow}}{a.?\,t \not\rightarrow}$$

☢️ 🔞 WARNING 🔞 ☢️

YOU ARE ABOUT TO ENTER THE MONAD ZONE

COME IN AT YOUR PERIL

(please note that the following part of the talk is heavily populated by monads, (co)algebras, functors, natural transformations and akin, it is highly recommended to not be allergic to those if you wish to pursue your journey with us)

2 Abstraction

2.1 Algebras and coalgebras

- $\mathbb{C}$ a category with products (eg. Set)
- **syntax**: endofunctor $\Sigma : \mathbb{C} \to \mathbb{C}$
  - Σ-algebra $i : \Sigma X \to X$

Example: $\Sigma X = \coprod_{\sigma \in \mathcal{O}} X^{\text{ar } \sigma}$

For $t ::= 0 \mid a.t \mid t + t \mid !t$, $\Sigma X = 1 + A \times X + X^2 + X$

- **behaviour**: endofunctor $H : \mathbb{C} \to \mathbb{C}$
  - H-coalgebra $k : X \to HX$

Example: $HX = \mathcal{P}_{\text{ne}}(A \times X + 1)$

- set of rules $\mathcal{R}$ ⇝ a **natural transformation** $\rho_X : \Sigma(X \times HX) \to H\Sigma^* X$

Example: For the previous example without $!$: $\Sigma X = 1 + A \times X + X^2$,

$$\rho : 1 + A \times (X \times \mathcal{P}(1 + A \times X)) + (X \times \mathcal{P}(1 + A \times X))^2 \to \mathcal{P}(1 + A \times \Sigma^* X)$$

- $\rho(*) = \{*\}$
- $\rho((a, t, T)) = \{(a, t)\}$
- $\rho((t, T), (u, U)) = \{(a, t') \mid \forall (a, t') \in T\} \cup \{(a, u') \mid \forall (a, u') \in U\} \cup \{* \mid * \in T\} \cup \{* \mid * \in U\}$

$$\frac{t \xrightarrow{a} t'}{t + u \xrightarrow{a} t'}\forall a \qquad \frac{u \xrightarrow{a} u'}{t + u \xrightarrow{a} u'}\forall a \qquad \frac{t \downarrow}{t + u \downarrow} \qquad \frac{u \downarrow}{t + u \downarrow}$$

2.3 Kleisli trace semantics

- recall $HX = \mathcal{P}_{\text{ne}}(1 + A \times X) = TBX$
  - $T = \mathcal{P}_{\text{ne}}$ **effectful** behaviour $\rightsquigarrow$ non empty powerset : non-determinism
  - $B = 1 + A \times X$ **pure** behaviour $\rightsquigarrow$ words : $Z = A^\infty$ (final B-algebra)
- $\text{tr}\, x \in \mathcal{P}(A^\infty) = TZ$
- T is a **monad**
- **Kleisli category** of T

$$A \in \text{Kl}(T) \Leftrightarrow A \in \mathbb{C} \qquad\qquad A \nrightarrow B \in \text{Kl}(T) \Leftrightarrow A \to TB \in \mathbb{C}$$

- $B : \mathbb{C} \to \mathbb{C}$ **extends** to $\overline{B} : \text{Kl}(T) \to \text{Kl}(T)$ (distributive law $\lambda^B : BT \Rightarrow TB$)

2.3 Kleisli trace semantics

- $Z = A^\infty$ is a B-coalgebra in $\mathrm{Kl}(T)$

$$\zeta : Z \nrightarrow BZ \text{ or } Z \to TBZ \qquad\qquad \zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- for any $k : X \nrightarrow BX$, $h : X \nrightarrow Z$ is a $\overline{B}$-coalgebra morphism if

$$\begin{tikzcd}
X \arrow[r, "k", "|" marking] \arrow[d, dashed, "h"', "|" marking] & BX \arrow[d, dashed, "\overline{B}h", "|" marking] \\
Z \arrow[r, "\zeta", "|" marking] & BZ
\end{tikzcd}$$

- suppose $\mathrm{Kl}(T)$ enriched with an **order on maps with maximums**, define tr_k the **greatest $\overline{B}$-coalgebra morphism**

2.4 Trace-GSOS

- GSOS rule

$$\rho : \Sigma(X \times TBX) \to TB\Sigma^* X$$

- **Trace**-GSOS rule

$$\rho : \Sigma(X \times BX) \nrightarrow B\Sigma^* X$$

$\rightsquigarrow$ only pure observations

- B and Σ extend to $\mathrm{Kl}(T)$ (and Σ^*) but not $+$ 😕
- **affineness**: ask T to be an **affine monad**

2.5 Strong and affine monads

- **strong monad**: $\mathrm{st}_{X,Y} : X \times TY \to T(X \times Y) \rightsquigarrow \mathrm{st}' : TX \times Y \to T(X \times Y)$
- double strength $\mathrm{dst} : TX \times TY \xrightarrow{\mathrm{st}} T(TX \times Y) \xrightarrow{T\mathrm{st}'} T^2(X \times Y) \xrightarrow{\mu} T(X \times Y)$ (and dst')
- **affine monad**: $TX \times TY \xrightarrow{\mathrm{dst}} T(X \times Y) \xrightarrow{\langle T\pi_1, T\pi_2 \rangle} TX \times TY = \mathrm{id}$ or $\eta_1 : 1 \xrightarrow{\simeq} T1$
- **affine part**: greatest affine submonad

Example:

- Powerset $\mathcal{P} \rightsquigarrow \mathcal{P}_{\mathrm{ne}}$
- (Sub)distribution $\mathcal{S} \rightsquigarrow \mathcal{D}$ with $\mathcal{D}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i = 1, x_i \in X, I \text{ finite}\right\}$ and $\mathcal{S}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i \leq 1, x_i \in X, I \text{ finite}\right\}$
- Maybe $-+1 \rightsquigarrow \mathrm{Id}$

- diagrammatical condition

$$
\begin{tikzcd}
\Sigma T(X \times BX) \arrow[r, "\langle T\pi_1, T\pi_2 \rangle"] \arrow[d, "\lambda"'] \arrow[rr, bend left, dashed, "\mathrm{mix}"] & \Sigma(TX \times TBX) \arrow[r, "\mathrm{dst}"] & \Sigma T(X \times BX) \arrow[d, "\lambda"'] \\
T\Sigma(X \times BX) \arrow[d, "T\rho"'] & & T\Sigma(X \times BX) \arrow[d, "T\rho"'] \\
T^2 B\Sigma^* X \arrow[r, "\mu"] & TB\Sigma^* X & T^2 B\Sigma^* X \arrow[l, "\mu"']
\end{tikzcd}
$$

$$
\bigcup\{\rho(\sigma(\xi_1 ... \xi_n)) \mid \xi_i \in \mathrm{mix}\ \Xi_i\} = \bigcup\{\rho(\sigma(\xi_1 ... \xi_n)) \mid \xi_i \in \Xi_i\}
$$
$$
\text{where } \Xi_i \subset X \times BX
$$

2.7 Sketch of the proof

- Recall congruence $\forall \sigma, (\forall i, t_i \sim u_i) \Rightarrow \sigma(t_1 ... t_n) \sim \sigma(u_1 ... u_n)$
  Prove $\mathrm{tr}(\sigma(t_1 ... t_n)) = [\![\sigma]\!](\mathrm{tr}\ t_1 ... \ \mathrm{tr}\ t_n)$

$$
\begin{tikzcd}
\Sigma^* X \arrow[r, "i^*", "\shortmid"{marking}] \arrow[d, "\Sigma^* \mathrm{tr}_k"', "\shortmid"{marking}] \arrow[dr, phantom, "?"] & X \arrow[r, "k", "\shortmid"{marking}] \arrow[d, "\mathrm{tr}_k", "\shortmid"{marking}] \arrow[dr, phantom, "\checkmark"] & BX \arrow[d, "B\mathrm{tr}_k", "\shortmid"{marking}] \\
\Sigma^* Z \arrow[r, "g"', "\shortmid"{marking}] & Z \arrow[r, "\zeta"', "\shortmid"{marking}] & BZ
\end{tikzcd}
$$

- define $[\![-]\!]/g$: semantics of Z (B-coalgebra) + induction + trace
- $\Sigma^* X \nrightarrow B\Sigma^* X$ (with ρ^*) and $Z \nrightarrow BZ$
- show $\overline{B}$-coalgebra morphisms:
  - $\mathrm{tr} \circ i$ ✓
  - $g \circ \Sigma^* \mathrm{tr}$ more complicated : naturality + smoothness + map of distributive law of ρ^*
- **maximality ?**

Theorem: Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, **and that is sublinear (?)** then trace equivalence is a congruence. (Hopefully !)

3 Conclusion

3 Conclusion

- concrete proof ✓
- abstract:
  - missing the "sublinearity" $\rightsquigarrow$ using presheaves?
  - moving to Eilenberg-Moore to have *finite*, *partial* and trace semantics as a **final coalgebra**
- thank you and bravo if you are still there !

~ *The End** ~

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*for today…