(Abstract) GSOS for Trace Equivalence

Séminaire LIMD

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Outline

1 Trace equivalence for concrete systems

1.1 Processes and LTSs

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Example. For $a, b, c, d \in A$

- 0 — $a.0 = a$ — $d.a.0 = da$
- $a.0 + b.c.0 = a + bc$ — $?\,(c.d.0) = ?\,cd$
- $a.(b.a.0 + ?\,(d.a.c.0)) = a(ba + ?\,dac)$

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- important property: contextual equivalence and **congruence**

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Example. $x \sim y \Rightarrow a.x \sim a.y$, or $x_1 \sim y_1 \wedge x_2 \sim y_2 \Rightarrow x_1 + x_2 \sim y_1 + y_2$

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- trace equivalence: $x \underset{\text{tr}}{\sim} y \Leftrightarrow \operatorname{tr}(x) = \operatorname{tr}(y)$

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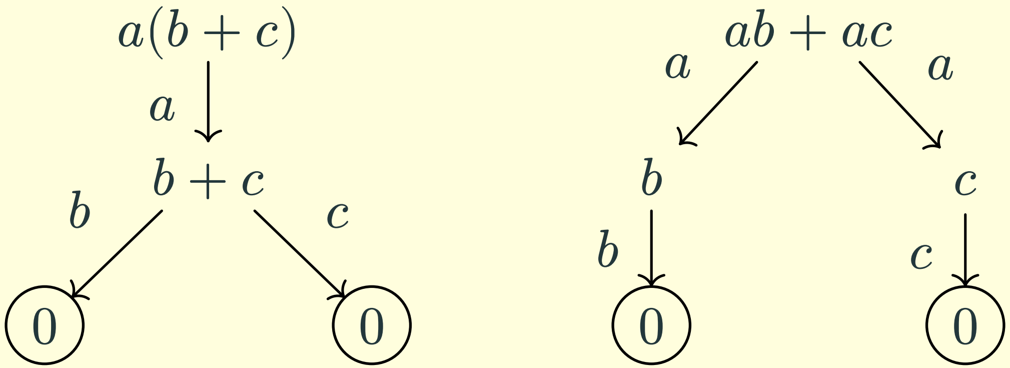

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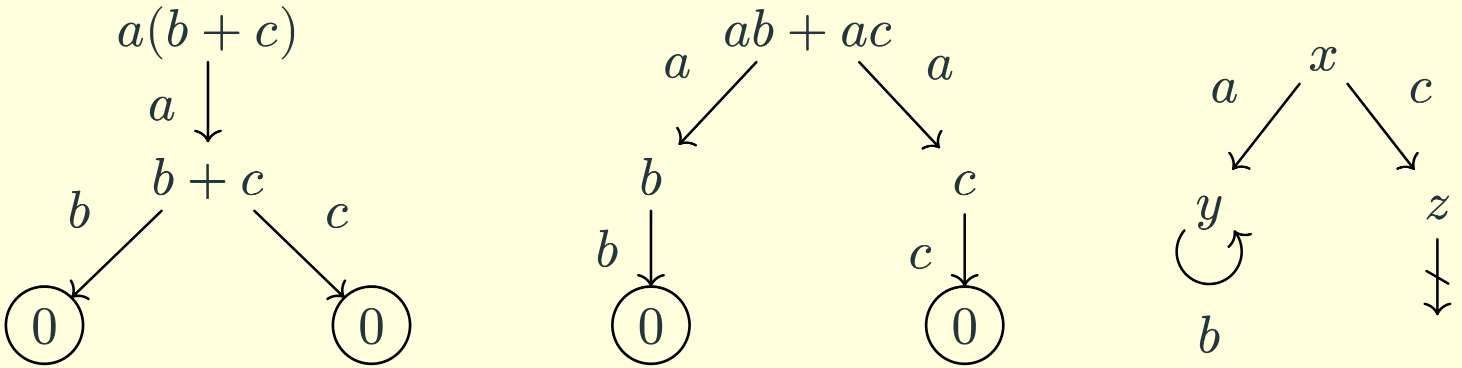

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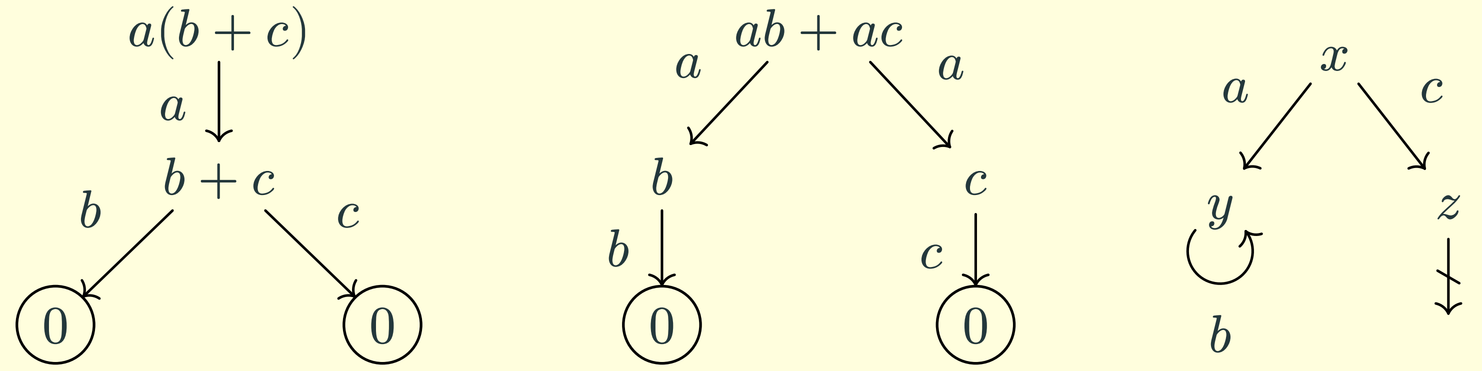

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- GSOS ⇒ bisimilarity is a congruence*, what about trace equivalence ?

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Remark. only **pure** observations/premises: observe each variable **once and only once**

Example.

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  - **affineness**: for each term, there is a least one rule that can apply

Example.

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  - **smoothness**: x_i in the target, the observation on x_i is **irrelevant** ie. any other observation could have been done (the same rule for each other possible observation exists)

1.6 Theorem

Theorem 1. Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\text{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\text{tr}}{\sim}$ is a congruence.

1.6 Theorem

Theorem 2. Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\mathrm{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\mathrm{tr}}{\sim}$ is a congruence.

Proof. Show that the trace of a complex term can be obtained from the traces of its subterms.

Theorem 3. Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\text{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\text{tr}}{\sim}$ is a congruence.

Proof. Show that the trace of a complex term can be obtained from the traces of its subterms.

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Theorem 4. Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\text{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\text{tr}}{\sim}$ is a congruence.

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Theorem 5. Let $\mathcal{R}$ be a smooth and affine set of Trace-GSOS rules. Let X be a set of terms equipped with behaviour $k : X \to \mathcal{P}_{\mathrm{ne}}(A \times X + \{\star\})$ induced by $\mathcal{R}$. Then trace equivalence $\underset{\mathrm{tr}}{\sim}$ is a congruence.

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- prove $\mathrm{tr}\ u[t_1 ... t_n] = [\![u]\!](\mathrm{tr}\ t_1 ...\ \mathrm{tr}\ t_n)$ by showing that both sides are maximal coalgebra morphisms

$\square$

1.7 Counter-examples

Affineness, smoothness and sublinearity are **necessary**

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1.7.1 Sublinearity

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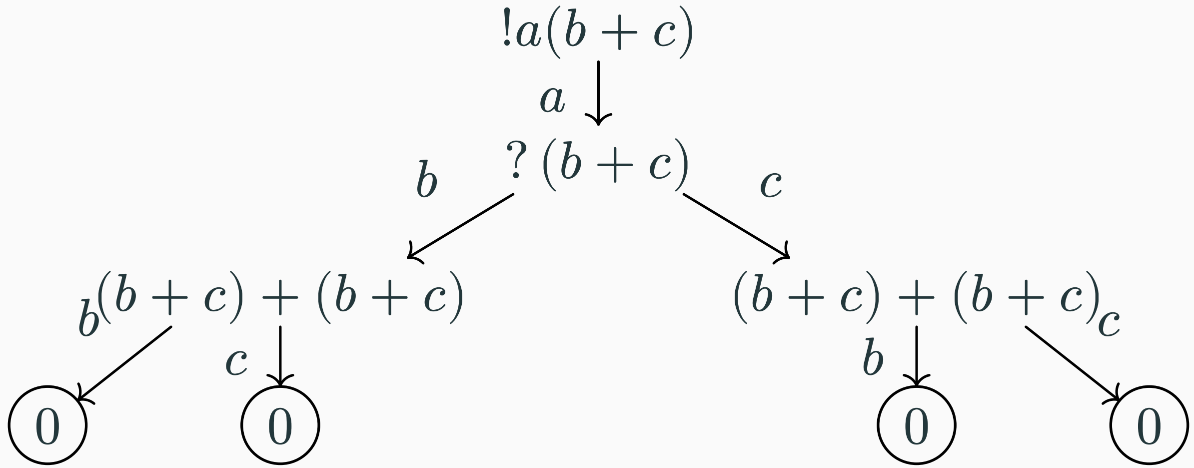

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 & & !a(b+c) \arrow[d, "a"'] & & \\
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 & (b+c)+(b+c) \arrow[dl, "b"'] \arrow[d, "c"'] & & (b+c)+(b+c) \arrow[d, "b"'] \arrow[dr, "c"] & \\
0 & 0 & & 0 & 0
\end{tikzcd}$$

$\mathrm{tr}\,!a(b+c) = \{abb, \underline{abc}, \underline{acb}, acc\}$

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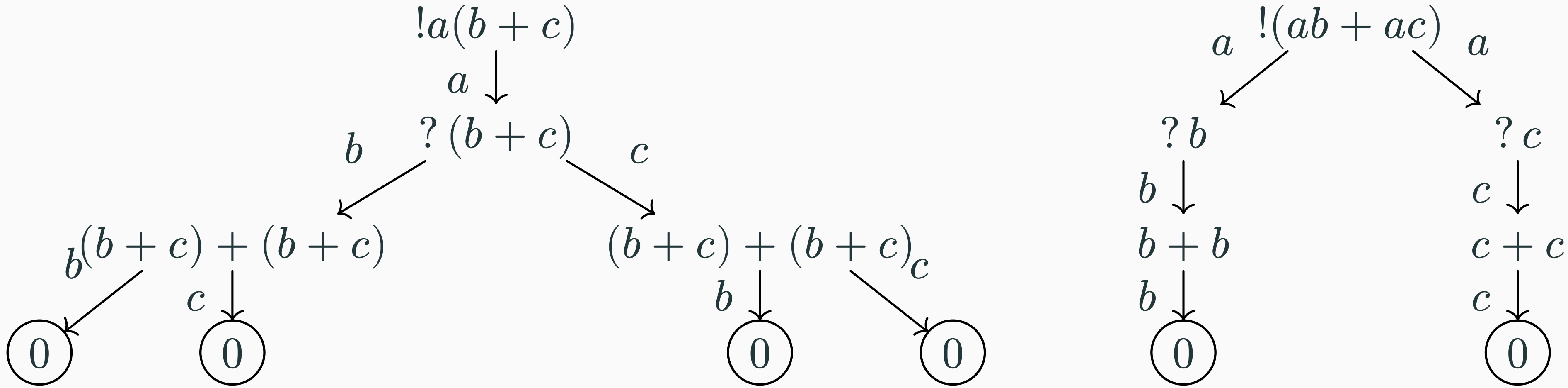

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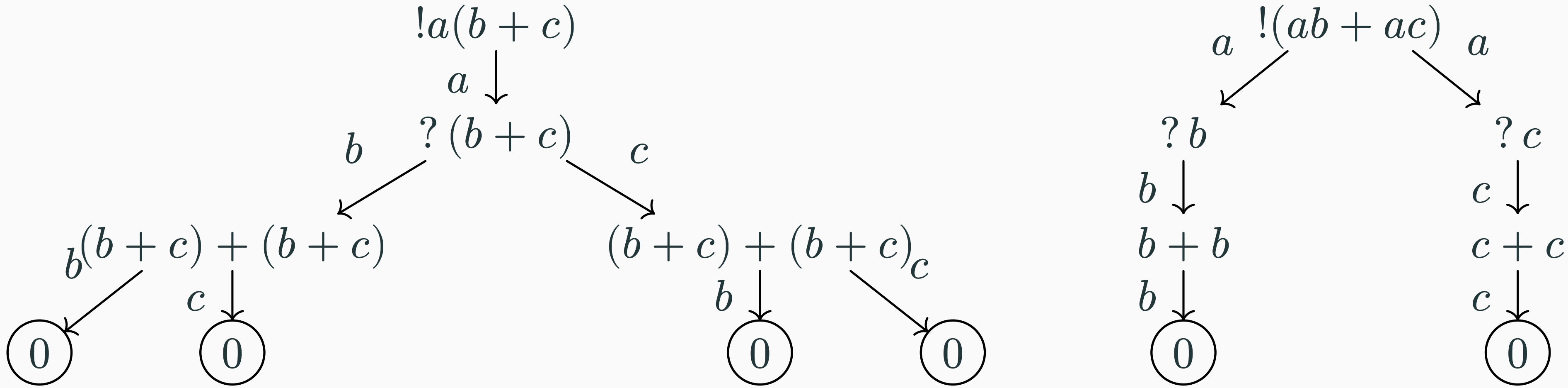

$$\mathrm{tr}\, !a(b+c) = \{abb, \underline{abc}, \underline{acb}, acc\} \neq \{abb, acc\} = \mathrm{tr}\, !(ab+ac)$$

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☢️ 🔞 WARNING 🔞 ☢️

YOU ARE ABOUT TO ENTER THE MONAD ZONE

COME IN AT YOUR PERIL

(please note that the following part of the talk is heavily populated by monads, (co)algebras, functors, natural transformations and akin, it is highly recommended to not be allergic to those if you wish to pursue your journey with us)

2 Abstraction

2.1 Algebras and coalgebras

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Example. $HX = \mathcal{P}_{\mathrm{ne}}(A \times X + 1)$

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- set of rules $\mathcal{R} \rightsquigarrow$ a **natural transformation** $\rho_X : \Sigma(X \times HX) \to H\Sigma^* X$

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$$A \in \mathrm{Kl}(T) \Leftrightarrow A \in \mathbb{C} \qquad\qquad A \nrightarrow B \in \mathrm{Kl}(T) \Leftrightarrow A \to TB \in \mathbb{C}$$

- $B : \mathbb{C} \to \mathbb{C}$ **extends** to $\overline{B} : \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ (distributive law $\lambda^B : BT \Rightarrow TB$)

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$$\zeta : Z \nrightarrow BZ \text{ or } Z \to TBZ \qquad\qquad \zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- $Z = A^\infty$ is a B-coalgebra in $\mathrm{Kl}(T)$

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- for any $k : X \nrightarrow BX$, $h : X \nrightarrow Z$ is a $\overline{B}$-coalgebra morphism if

$$\begin{tikzcd} X \arrow[r, "k", "\vert"'] \arrow[d, dashed, "h"', "\vert"] & BX \arrow[d, dashed, "\overline{B}h", "\vert"'] \\ Z \arrow[r, "\zeta", "\vert"'] & BZ \end{tikzcd}$$

2.3 Kleisli trace semantics

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- suppose $\mathrm{Kl}(T)$ enriched with an **order on maps with maximums**, define tr_k the **greatest $\overline{B}$-coalgebra morphism**

2.4 Trace-GSOS

- GSOS rule

$$\rho : \Sigma(X \times HX) \to H\Sigma^* X$$

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$\rightsquigarrow$ only pure observations

- B and Σ extend to $\mathrm{Kl}(T)$ (and Σ^*) but not $+$ 😕
- **affineness**: ask T to be an **affine monad**

2.5 Strong and affine monads

- **strong monad**: $\mathrm{st}_{X,Y} : X \times TY \to T(X \times Y)$

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- Powerset $\mathcal{P} \rightsquigarrow \mathcal{P}_{\mathrm{ne}}$
- (Sub)distribution $\mathcal{S} \rightsquigarrow \mathcal{D}$ with $\mathcal{D}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i = 1, x_i \in X, I \text{ finite}\right\}$ and $\mathcal{S}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i \leq 1, x_i \in X, I \text{ finite}\right\}$

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- Maybe $- + 1 \rightsquigarrow \mathrm{Id}$

2.6 Abstract smoothness

- diagrammatical condition

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$$
\begin{tikzcd}
\Sigma T(X \times BX) \arrow[r, "\langle T\pi_1, T\pi_2 \rangle"] \arrow[rr, dashed, bend left, "\mathrm{mix}"] \arrow[d, "\lambda"'] & \Sigma(TX \times TBX) \arrow[r, "\mathrm{dst}"] & \Sigma T(X \times BX) \arrow[d, "\lambda"'] \\
T\Sigma(X \times BX) \arrow[d, "T\rho"'] & & T\Sigma(X \times BX) \arrow[d, "T\rho"'] \\
T^2 B\Sigma^* X \arrow[r, "\mu"] & TB\Sigma^* X & T^2 B\Sigma^* X \arrow[l, "\mu"']
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\end{tikzcd}
$$

$$
\bigcup\{\rho(\sigma(\xi_1 \ldots \xi_n)) \mid \xi_i \in \mathrm{mix}\ \Xi_i\} = \bigcup\{\rho(\sigma(\xi_1 \ldots \xi_n)) \mid \xi_i \in \Xi_i\}
$$

$$
\text{where } \Xi_i \subset X \times BX
$$

2.7 Sketch of the proof

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\begin{tikzcd}
\Sigma^* X \arrow[r, "i^*", "\shortmid" marking] \arrow[d, "\Sigma^* \mathrm{tr}_k"', "\shortmid" marking] \arrow[dr, phantom, "?"] & X \arrow[r, "k", "\shortmid" marking] \arrow[d, "\mathrm{tr}_k"', "\shortmid" marking] \arrow[dr, phantom, "\checkmark"] & BX \arrow[d, "B\mathrm{tr}_k", "\shortmid" marking] \\
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  - $\mathrm{tr} \circ i$ $\checkmark$
  - $g \circ \Sigma^*\mathrm{tr}$ more complicated : naturality + smoothness + map of distributive law of ρ^*

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  - $g \circ \Sigma^*\mathrm{tr}$ more complicated : naturality + smoothness + map of distributive law of ρ^*
- **maximality ?**

Theorem 6. Let $\mathbb{C}$ be a cartesian category,

2.8 To sum up: the theorem

Theorem 7. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*,

2.8 To sum up: the theorem

Theorem 8. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$,

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Theorem 9. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$,

2.8 To sum up: the theorem

Theorem 10. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and

Theorem 11. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules*

Theorem 12. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws,

Theorem 13. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, **and that is sublinear (?)**

2.8 To sum up: the theorem

Theorem 14. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, **and that is sublinear (?)** then trace equivalence is a congruence.

Theorem 15. Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** monad *for effects*, B an endofunctor *for behaviour* that extends to $\mathrm{Kl}(T)$, Σ a endofunctor *for syntax* that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let Z be the final B-algebra, suppose there is an infinitary trace situation, and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, **and that is sublinear (?)** then trace equivalence is a congruence. (Hopefully !)

3 Conclusion

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- concrete proof ✓

*

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- abstract:
  - missing the "sublinearity" $\rightsquigarrow$ using presheaves?

*

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  - moving to Eilenberg-Moore to have *finite*, *partial* and trace semantics as a **final coalgebra**

*

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- thank you and bravo if you are still there !

*

3 Conclusion

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~ *The End*[*] ~

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[*] for today…