Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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Outline

1. Preliminaries

1.1 GSOS

- a **framework** for specifying reduction rules and semantics

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  $\rightarrow$ rule format

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  $HX = \mathcal{P}(1 + A \times X)$

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- **GSOS rules**

$$\frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \xrightarrow{b} u[x_1 \ldots x_n, y_{i,k} \ldots]} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}, n = \text{ar } \sigma, u \in \Sigma^*, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$

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- **trace** of a term t: $\mathrm{tr}\ t$ = set of words of A^* that can be produced by t

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  - $T = \mathcal{P}$ **effectful** behaviour
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- $\text{tr}\, t \in \mathcal{P}(A^*)$

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$$\zeta : A^* \multimap BA^* \text{ or } A^* \to TBA^* \qquad \zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

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$$\zeta : A^* \multimap BA^* \text{ or } A^* \to TBA^* \qquad\qquad \zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- for any $k : X \multimap BX$,

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, dashed, "\mathrm{tr}_k"'] & BX \arrow[d, dashed, "B\mathrm{tr}_k"] \\ A^* \arrow[r, "\zeta"] & BA^* \end{tikzcd}$$

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Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?\,t \xrightarrow{\tau} t' + t''} \qquad \frac{}{a.t \xrightarrow{a} t}\,\forall a \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\,\forall a, b \qquad \frac{t \downarrow}{a.t \xrightarrow{a} t}\,\forall a$$

😕 😕 😎 😎

1.5 Trace equivalence & congruence

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\Sigma A^* \arrow[r, "{[\![-]\!]}"'] & A^* \arrow[r, "\zeta"'] & BA^*
\end{tikzcd}$$

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- Maybe $-+1 \rightsquigarrow \mathrm{Id}$

2. Result

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2.2 Sketch of the proof

- Recall

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\Sigma^* Z \arrow[r, "\llbracket - \rrbracket"', "\bullet"{pos=1}, no head] & Z
\end{tikzcd}$$

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  Remark: need dst

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$$\mathrm{tr}\,!a(b+c) = \{ab, ac, abb, \underline{abc}, \underline{acb}, acc\} \neq \{ab, ac, abb, acc\} = \mathrm{tr}\,!(ab+ac)$$

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0 & & 0
\end{tikzcd}
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\begin{tikzcd}
& !(ab+a\tau) \arrow[dl, "a"'] \arrow[dr, "a"] & \\
?\,b \arrow[d, "/" marking] & & ?\,\tau \arrow[d, "\tau"] \\
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$\longrightarrow$ observations that are “not used” 😕

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- **abstract smoothness**

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\begin{tikzcd}
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T\Sigma(X \times BX) \arrow[d, "T\rho"'] & & T\Sigma(X \times BX) \arrow[d, "T\rho"'] \\
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$$\Phi(\rho)(\sigma)(\mathrm{mix}\ X_1 \ldots\ \mathrm{mix}\ X_n) = \Phi(\rho)(\sigma)(X_1 \ldots X_n)$$

where $X_i \subset X \times BX$

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$\longrightarrow$ Can we get back information on the original system ?

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- Thank you all for welcoming me in the chair ❤️

~ The End ~

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