Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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Outline

1. Preliminaries

1.1 GSOS

- a **framework** for specifying reduction rules and semantics
  $\rightarrow$ rule format
- given a **syntax** (with endofunctor Σ)

  Example: Set of operations $\mathcal{O}$ with arity map $\mathrm{ar} : \mathcal{O} \to \mathbb{N}$, $\Sigma X = \sum_{\sigma \in \mathcal{O}} X^{\mathrm{ar}\ \sigma}$

  Example: $t ::= 0 \mid a.t\ \forall a \in A \mid t + t \mid ?\,t \rightsquigarrow \Sigma X = 1 + A \times X + X^2 + X$

  Eg. for $a, b, c, \tau \in A : a.(b.a.0 + ?\,\tau.a.c.0) = a(ba + ?\,\tau ac)$
- **behaviour** (with endofunctor H)

  Example: x **terminates** ($x \downarrow$) or **progresses** to x' with label $a \in A$ ($x \xrightarrow{a} x'$) $\rightsquigarrow$ $HX = \mathcal{P}(1 + A \times X)$

- $k : X \to HX$ a ***H*-coalgebra** $\longrightarrow$ set equipped with semantics

  Example: Let $k : X \to HX$, for $x \in X$, $x \downarrow \Leftrightarrow * \in k(x)$ and $x \xrightarrow{a} x' \Leftrightarrow (a, x') \in k(x)$

- **GSOS rules**

$$\frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \xrightarrow{b} u\left[x_1 \ldots x_n, y_{i,k} \ldots\right]} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i \in I, k \in K_i} \quad \{x_j \downarrow\}_{j \in J}}{\sigma(x_1 \ldots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}, n = \mathrm{ar}\ \sigma, u \in \Sigma^*, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$

A full example

- syntax: $t ::= 0 \mid a.t \ \forall a \in A \mid t + t \mid ?\,t$
- rules

$$\frac{}{0\downarrow} \qquad \frac{}{a.t \xrightarrow{a} t}\forall a \qquad \frac{t \xrightarrow{a} t'}{t+u \xrightarrow{a} t'}\forall a \qquad \frac{u \xrightarrow{a} u'}{t+u \xrightarrow{a} u'}\forall a \qquad \frac{t\downarrow \quad u\downarrow}{t+u\downarrow} \qquad \frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?\,t \xrightarrow{\tau} t' + t''}$$

$$\begin{tikzcd}
& a(b+c) \arrow[d, "a"'] & \\
& b+c \arrow[dl, "b"'] \arrow[dr, "c"] & \\
0 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
& ab+ac \arrow[dl, "a"'] \arrow[dr, "a"] & \\
b \arrow[d, "b"'] & & c \arrow[d, "c"'] \\
0 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
& ?\,(\tau a + \tau b) \arrow[d, "\tau"'] & \\
& a+b \arrow[dl, "a"'] \arrow[dr, "b"] & \\
0 & & 0
\end{tikzcd}$$

1.2 Abstract GSOS

- any syntax functor Σ and behaviour functor H
- rules $\rightsquigarrow$ a **natural transformation** $\rho_X : \Sigma(X \times HX) \to H\Sigma^* X$

Example: For the previous example without ? : $\Sigma X = 1 + A \times X + X^2$,

$$\rho : 1 + A \times (X \times \mathcal{P}(1 + A \times X)) + (X \times \mathcal{P}(1 + A \times X))^2 \to \mathcal{P}(1 + A \times \Sigma^* X)$$

- $\rho(*) = \{*\}$
- $\rho((a, t, T)) = \{(a, t)\}$
- $\rho((t, T), (u, U)) = \{(a, t') \mid \forall (a, t') \in T\} \cup \{(a, u') \mid \forall (a, u') \in U\} \cup \{* \mid * \in T \wedge * \in U\}$

$$\frac{t \xrightarrow{a} t'}{t + u \xrightarrow{a} t'}\forall a \qquad \frac{u \xrightarrow{a} u'}{t + u \xrightarrow{a} u'}\forall a \qquad \frac{t \downarrow \quad u \downarrow}{t + u \downarrow}$$

1.3 Trace & Kleisli categories

- **trace** of a term t: $\mathrm{tr}\ t$ = set of words of A^* that can be produced by t, defined by coinduction

$$\varepsilon \in \mathrm{tr}\ t \Leftrightarrow t\downarrow \qquad a.w \in \mathrm{tr}\ t \Leftrightarrow t \xrightarrow{a} u \wedge w \in \mathrm{tr}\ u$$

Example: $\mathrm{tr}\ a(b+c) = \{ab, ac\}$, $\mathrm{tr}\ (ab + ac) = \{ab, ac\}$, $\mathrm{tr}\ (a + b?\,c) = \{a\}$

- recall $HX = \mathcal{P}(1 + A \times X) = TBX$
  - $T = \mathcal{P}$ **effectful** behaviour $\rightsquigarrow$ powerset : non-determinism
  - $B = 1 + A \times X$ **pure** behaviour $\rightsquigarrow$ words : A^* (initial B-algebra)
- $\mathrm{tr}\, t \in \mathcal{P}(A^*)$

1.3 Trace & Kleisli categories

Trace **abstractly**

- in the **Kleisli category** of T

$$A \in \mathrm{Kl}(T) \Leftrightarrow A \in \mathbb{C} \qquad\qquad A \mathrel{-\!\!\bullet} B \in \mathrm{Kl}(T) \Leftrightarrow A \to TB \in \mathbb{C}$$

- A^* is the final B-coalgebra in $\mathrm{Kl}(T)$

$$\zeta : A^* \mathrel{-\!\!\bullet} BA^* \text{ or } A^* \to TBA^* \qquad\qquad \zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- for any $k : X \mathrel{-\!\!\bullet} BX$,

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, dashed, "\mathrm{tr}_k"'] & BX \arrow[d, dashed, "B\mathrm{tr}_k"] \\ A^* \arrow[r, "\zeta"] & BA^* \end{tikzcd}$$

1.4 Trace-GSOS

- GSOS rule

$$\rho : \Sigma(X \times TBX) \to TB\Sigma^* X$$

- **Trace**-GSOS rule

$$\rho : \Sigma(X \times BX) \mathrel{-\!\!\bullet} B\Sigma^* X$$

$\rightsquigarrow$ only pure observations

- Rules observe each variable **once and only once**

Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?\,t \xrightarrow{\tau} t' + t''} \qquad \frac{}{a.t \xrightarrow{a} t}\forall a \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\forall a, b \qquad \frac{t \downarrow}{a.t \xrightarrow{a} t}\forall a$$

😕 😕 😎 😎

1.5 Trace equivalence & congruence

- **trace equivalence**: $t \underset{\mathrm{tr}}{\equiv} u \Leftrightarrow \mathrm{tr}\ t = \mathrm{tr}\ u$

  Example: Recall $\mathrm{tr}\ a(b+c) = \{ab, ac\} = \mathrm{tr}\ (ab+ac)$. $a(b+c) \underset{\mathrm{tr}}{\equiv} ab + ac$ but not bisimilar $\rightsquigarrow \underset{\mathrm{tr}}{\equiv}$ coarsest
- **congruence**: $\forall \sigma, (\forall i, t_i \equiv u_i) \Rightarrow \sigma(t_1 \ldots t_n) \equiv \sigma(u_1 \ldots u_n)$
- prove $\mathrm{tr}(\sigma(t_1 \ldots t_n)) = [\![\sigma]\!](\mathrm{tr}\ t_1 \ldots\ \mathrm{tr}\ t_n)$

$$\begin{tikzcd}
\Sigma X \arrow[r, "i"] \arrow[d, "\Sigma \mathrm{tr}_k"'] \arrow[dr, phantom, "?"] & X \arrow[r, "k"] \arrow[d, "\mathrm{tr}_k"] \arrow[dr, phantom, "\checkmark"] & BX \arrow[d, "B\mathrm{tr}_k"] \\
\Sigma A^* \arrow[r, "{[\![-]\!]}"'] & A^* \arrow[r, "\zeta"'] & BA^*
\end{tikzcd}$$

1.6 Strong and affine monads

- **strong monad**: $\mathrm{st}_{X,Y} : X \times TY \to T(X \times Y) \rightsquigarrow \mathrm{st}' : TX \times Y \to T(X \times Y)$
- double strength $\mathrm{dst} : TX \times TY \xrightarrow{\mathrm{st}} T(TX \times Y) \xrightarrow{T\mathrm{st}'} T^2(X \times Y) \xrightarrow{\mu} T(X \times Y)$ (and dst')
- **affine monad**: $TX \times TY \xrightarrow{\mathrm{dst}} T(X \times Y) \xrightarrow{\langle T\pi_1, T\pi_2 \rangle} TX \times TY = \mathrm{id}$ or $\eta_1 : 1 \xrightarrow{\simeq} T1$
- **affine part**: greatest affine submonad

Example:

- Powerset $\mathcal{P} \rightsquigarrow \mathcal{P}_{\mathrm{ne}}$
- (Sub)distribution $\mathcal{S} \rightsquigarrow \mathcal{D}$ with $\mathcal{D}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i = 1, x_i \in X, I \text{ finite}\right\}$ and $\mathcal{S}X = \left\{\sum_{i \in I} p_i x_i \mid \sum p_i \leq 1, x_i \in X, I \text{ finite}\right\}$
- Maybe $-+1 \rightsquigarrow \mathrm{Id}$

2. Result

Theorem 2.1.1: Let $\mathbb{C}$ be a cartesian category, T be a strong **affine** *effectful* monad, B a *behaviour* endofunctor that extends to $\mathrm{Kl}(T)$, Σ a *syntax* endofunctor that extends to $\mathrm{Kl}(T)$ with all free objects $(\Sigma^* X)$, let $\zeta : Z \multimap BZ$ be the final $\overline{B}$-coalgebra (with $\exists z, \zeta = \eta \circ z$) and let $\rho : \Sigma(X \times BX) \to TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, then trace equivalence is a congruence.

2.2 Sketch of the proof

- Recall

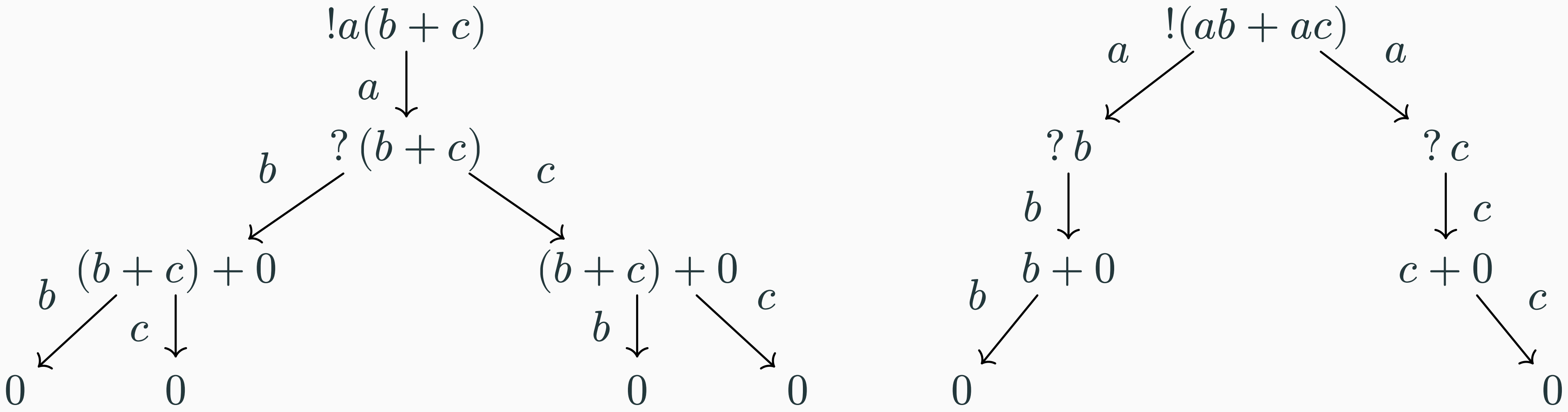
$$\begin{tikzcd} \Sigma^* X \arrow[r, "i"] \arrow[d, "\Sigma^* \mathrm{tr}_k"'] & X \arrow[d, "\mathrm{tr}_k"] \\ \Sigma^* Z \arrow[r, "[\![-]\!]"'] & Z \end{tikzcd}$$

- define $[\![-]\!]$: semantics of Z + induction + trace
- $\Sigma^* X \multimap B\Sigma^* X$ (with ρ^*) and $Z \multimap BZ$
- show $\overline{B}$-coalgebra morphisms
- $\mathrm{tr} \circ i \quad \checkmark$
- $[\![-]\!] \circ \Sigma^* \mathrm{tr}$ more complicated : naturality + smoothness + map of distributive law of ρ^*

  Remark: need dst

Example: $t ::= 0 \mid a.t \mid t+t \mid ?\,t \mid !t$ with the previous rules for $0, a., +$ and

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} ?\,t'}\forall a \qquad \frac{t \xrightarrow{a} t'}{?\,t \xrightarrow{a} t + t'}\forall a$$

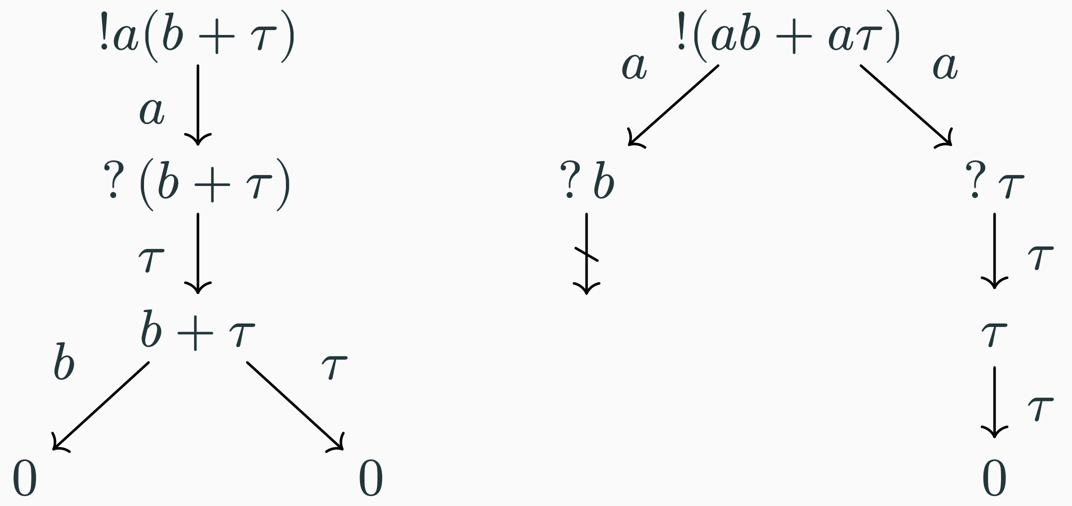


$$\mathrm{tr}\,!a(b+c) = \{ab, ac, abb, \underline{abc}, \underline{acb}, acc\} \neq \{ab, ac, abb, acc\} = \mathrm{tr}\,!(ab+ac)$$

2.3 Focus on hypothesis : Smoothness

Example: $t ::= 0 \mid a.t \mid t + t \mid ?\,t \mid !t$ with the previous rules for 0, $a.$, $+$ and

$$\frac{t \xrightarrow{a}?\, t'}{!t \xrightarrow{a}?\, t'}\ \forall a \qquad \frac{t \xrightarrow{\tau} t'}{?\,t \xrightarrow{\tau} t}$$



$$\mathrm{tr}\,!a(b+\tau) = \{\underline{a\tau b}, a\tau\tau\} \neq \{a\tau\tau\} = \mathrm{tr}\,!(ab + a\tau)$$

$\longrightarrow$ observations that are "not used" 😕

2.3 Focus on hypothesis : Smoothness

- **smoothness**
  - **linear**: if $x_i \to x_{i'}$ then not x_i and $x_{i'}$ in the target
  - if x_i in the target, the observation on x_i is **irrelevant** ie. any other observation could have been done (the same rule for each other possible observation exists)
- **abstract smoothness**

$$
\begin{tikzcd}
\Sigma T(X \times BX) \arrow[r, "\langle T\pi_1, T\pi_2 \rangle"] \arrow[d, "\lambda"'] \arrow[rr, dashed, bend left, "\text{mix}"] & \Sigma(TX \times TBX) \arrow[r, "\text{dst}"] & \Sigma T(X \times BX) \arrow[d, "\lambda"'] \\
T\Sigma(X \times BX) \arrow[d, "T\rho"'] & & T\Sigma(X \times BX) \arrow[d, "T\rho"'] \\
T^2 B\Sigma^* X \arrow[r, "\mu"] & TB\Sigma^* X & T^2 B\Sigma^* X \arrow[l, "\mu"']
\end{tikzcd}
$$

$$\Phi(\rho)(\sigma)(\text{mix } X_1 \dots \text{ mix } X_n) = \Phi(\rho)(\sigma)(X_1 \dots X_n)$$

where $X_i \subset X \times BX$

- need smoothness **for ρ^*** (terms with more than one layer)

Example: $t ::= 0 \mid a.t \mid ?\,t$ with the following smooth rules

$$\frac{}{0\downarrow} \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\forall a,b \qquad \frac{t\downarrow}{a.t \xrightarrow{a} t}\forall a \qquad \frac{t \xrightarrow{\tau} t'}{?\,t \xrightarrow{\tau} t'}$$

let $X_1 = \left\{t \xrightarrow{\tau} t', u \xrightarrow{a} u'\right\}$ then mix $X_1 = \left\{t \xrightarrow{\tau} t', t \xrightarrow{a} u', u \xrightarrow{\tau} t', u \xrightarrow{a} u'\right\}$

$$\dfrac{\dfrac{t \xrightarrow{\tau} t' \in X_1}{?\,t \xrightarrow{\tau} t'}}{b.?\,t \xrightarrow{b} ?\,t} \qquad \dfrac{\dfrac{u \xrightarrow{\tau} ? \notin X_1}{?\,u \not\rightarrow}}{b.?\,u \not\rightarrow} \qquad \dfrac{\dfrac{t \xrightarrow{\tau} t' \in \text{mix } X_1}{?\,t \xrightarrow{\tau} t'}}{b.?\,t \xrightarrow{b} ?\,t} \qquad \dfrac{\dfrac{u \xrightarrow{\tau} t' \in \text{mix } X_1}{?\,u \xrightarrow{\tau} t'}}{b.?\,u \xrightarrow{b} ?\,u}$$

$\Phi(\rho^*)(b.?\,x_1)(X_1) = \left\{\xrightarrow{b} ?\,t\right\} \neq \left\{\xrightarrow{b} ?\,t, \xrightarrow{b} ?\,u\right\} = \Phi(\rho^*)(b.?\,x_1)(\text{mix } X_1) \longrightarrow$ the **stuck** computation is messing with smoothness 😕

Theorem 2.4.1: If T is an **affine** monad then the smoothness of ρ entails the smoothness of ρ^*.

- affine part of $\mathcal{P}$ is $\mathcal{P}_{\text{ne}} \longrightarrow$ no stuckness !
- at the level of rules: give a semantics to **every situation**, nothing unspecified

Example:

$$\frac{t \xrightarrow{\tau} t'}{?\,t \xrightarrow{\tau} t'} \quad + \quad \frac{t \xrightarrow{a} t'}{?\,t\downarrow}\forall a \neq \tau \qquad \frac{t\downarrow}{?\,t\downarrow}$$

need to have some semantics eg. termination $\downarrow$

2.5 And for non affine monads ?

- still under investigation 🚧
- idea 1: add an **extra sink state** $\perp$ for stuck computations
- idea 2: map stuckness to **explicit termination** (cf. previous example) 🚨 change of semantics

$\longrightarrow$ Can we get back information on the original system ?

3. Conclusion

- For an affine monadic effect, under reasonable assumptions, trace equivalence is a congruence 🥳 !
- Can we do better ? Can we find a good reduction to the affine case for non affine monads ?
- Thank you all for welcoming me in the chair ❤️

~ The End ~

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