Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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Outline

1. Preliminaries

1.1 GSOS

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- **GSOS rules**

$$\frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i\in I, k\in K_i} \quad \{x_j\downarrow\}_{j\in J}}{\sigma(x_1 \ldots x_n) \xrightarrow{b} u[x_1 \ldots x_n, y_{i,k} \ldots]} \quad \text{or} \quad \frac{\left\{x_i \xrightarrow{a_{i,k}} y_{i,k}\right\}_{i\in I, k\in K_i} \quad \{x_j\downarrow\}_{j\in J}}{\sigma(x_1 \ldots x_n)\downarrow}$$

with $\sigma \in \mathcal{O}, n = \text{ar } \sigma, u \in \Sigma^*, a_{i,k}, b \in A, I, J, K_i \subset [\![1, n]\!]$

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- for any $k : X \multimap BX$,

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, dashed, "\mathrm{tr}_k"'] & BX \arrow[d, dashed, "B\mathrm{tr}_k"] \\ A^* \arrow[r, "\zeta"] & BA^* \end{tikzcd}$$

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Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?\,t \xrightarrow{\tau} t' + t''} \qquad \frac{}{a.t \xrightarrow{a} t}\,\forall a \qquad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t}\,\forall a, b \qquad \frac{t \downarrow}{a.t \xrightarrow{a} t}\,\forall a$$

😕 😕 😎 😎

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1.5 Trace equivalence & congruence

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\begin{tikzcd}
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\Sigma A^* \arrow[r, "{[\![-]\!]}"'] & A^* \arrow[r, "\zeta"'] & BA^*
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- Maybe $-+1 \rightsquigarrow \mathrm{Id}$

2. Result

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2.2 Sketch of the proof

- Recall

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\begin{tikzcd}
\Sigma^* X \arrow[r, "i", -{Circle}] \arrow[d, "\Sigma^* \mathrm{tr}_k"', -{Circle}] & X \arrow[d, "\mathrm{tr}_k", -{Circle}] \\
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  Remark: need dst

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0 & & 0
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2.3 Focus on hypothesis : Smoothness

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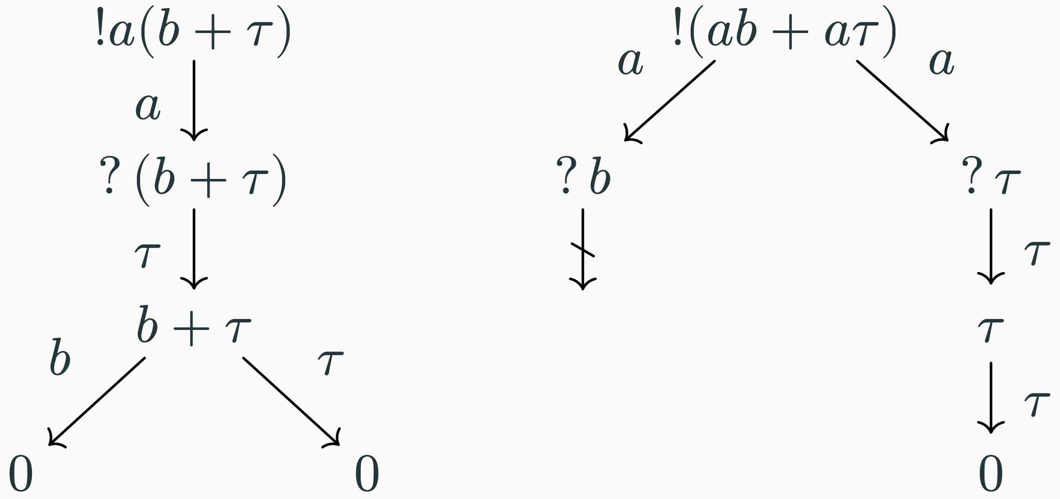

$$\operatorname{tr} !a(b+\tau) = \{\underline{a\tau b}, a\tau\tau\} \neq \{a\tau\tau\} = \operatorname{tr} !(ab+a\tau)$$

$\longrightarrow$ observations that are “not used” 😕

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- **abstract smoothness**

$$
\begin{tikzcd}
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$$\Phi(\rho)(\sigma)(\mathrm{mix}\ X_1 \ldots\ \mathrm{mix}\ X_n) = \Phi(\rho)(\sigma)(X_1 \ldots X_n)$$
where $X_i \subset X \times BX$

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$\longrightarrow$ Can we get back information on the original system ?

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- Thank you all for welcoming me in the chair ❤️

~ The End ~

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