

Trace Equivalence in Abstract GSOS

Oberseminar des Lehrstuhls für Theoretische Informatik

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1. Preliminaries

1.1 GSOS

- a **framework** for specifying reduction rules and semantics
→ rule format
- given a **syntax** (with endofunctor Σ)

Example: Set of operations \mathcal{O} with arity map $\text{ar} : \mathcal{O} \rightarrow \mathbb{N}$, $\Sigma X = \sum_{\sigma \in \mathcal{O}} X^{\text{ar } \sigma}$

Example: $t ::= 0 \mid a.t \mid \forall a \in A \mid t + t \mid ?t \rightsquigarrow \Sigma X = 1 + A \times X + X^2 + X$

Eg. for $a, b, c, \tau \in A$: $a.(b.a.0 + ?\tau.a.c.0) = a(ba + ?\tau ac)$

- **behaviour** (with endofunctor H)

Example: x **terminates** ($x \downarrow$) or **progresses** to x' with label $a \in A$ ($x \xrightarrow{a} x'$) \rightsquigarrow
 $HX = \mathcal{P}(1 + A \times X)$

1.1 GSOS

- $k : X \rightarrow HX$ a **H-coalgebra** \longrightarrow set equipped with semantics

Example: Let $k : X \rightarrow HX$, for $x \in X$, $x \downarrow \Leftrightarrow * \in k(x)$ and $x \xrightarrow{a} x' \Leftrightarrow (a, x') \in k(x)$

- **GSOS rules**

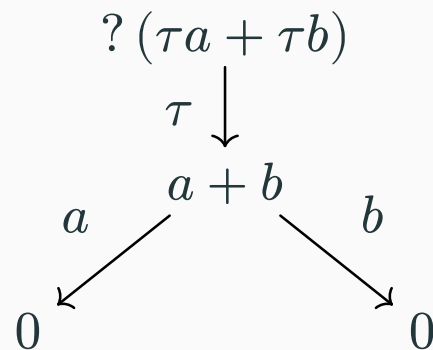
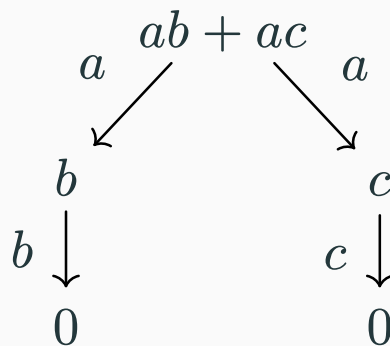
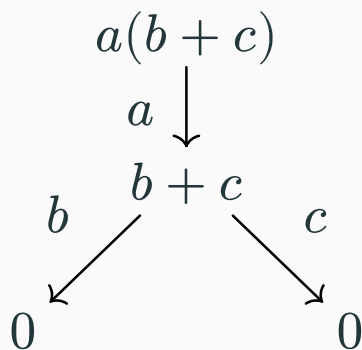
$$\frac{\left\{ x_i \xrightarrow{a_{i,k}} y_{i,k} \right\}_{i \in I, k \in K_i} \quad \left\{ x_j \downarrow \right\}_{j \in J}}{\sigma(x_1 \dots x_n) \xrightarrow{b} u[x_1 \dots x_n, y_{i,k} \dots]} \quad \text{or} \quad \frac{\left\{ x_i \xrightarrow{a_{i,k}} y_{i,k} \right\}_{i \in I, k \in K_i} \quad \left\{ x_j \downarrow \right\}_{j \in J}}{\sigma(x_1 \dots x_n) \downarrow}$$

with $\sigma \in \mathcal{O}$, $n = \text{ar } \sigma$, $u \in \Sigma^*$, $a_{i,k}, b \in A$, $I, J, K_i \subset \llbracket 1, n \rrbracket$

A full example

- syntax: $t ::= 0 \mid a.t \mid \forall a \in A \mid t + t \mid ?t$
- rules

$$\begin{array}{c}
 \frac{}{0 \downarrow} \quad \frac{}{a.t \xrightarrow{a} t} \forall a \quad \frac{t \xrightarrow{a} t'}{t + u \xrightarrow{a} t'} \forall a \quad \frac{u \xrightarrow{a} u'}{t + u \xrightarrow{a} u'} \forall a \quad \frac{t \downarrow \quad u \downarrow}{t + u \downarrow} \quad \frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?t \xrightarrow{\tau} t' + t''}
 \end{array}$$



1.2 Abstract GSOS

- any syntax functor Σ and behaviour functor H
- rules \rightsquigarrow a **natural transformation** $\rho_X : \Sigma(X \times HX) \rightarrow H\Sigma^* X$

Example: For the previous example without $?$: $\Sigma X = 1 + A \times X + X^2$,

$$\rho : 1 + A \times (X \times \mathcal{P}(1 + A \times X)) + (X \times \mathcal{P}(1 + A \times X))^2 \rightarrow \mathcal{P}(1 + A \times \Sigma^* X)$$

- $\rho(*) = \{*\}$
- $\rho((a, t, T)) = \{(a, t)\}$
- $\rho((t, T), (u, U)) = \{(a, t') \mid \forall (a, t') \in T\} \cup \{(a, u') \mid \forall (a, u') \in U\} \cup \{* \mid * \in T \wedge * \in U\}$

$$\frac{t \xrightarrow{a} t'}{t + u \xrightarrow{a} t'} \forall a$$

$$\frac{u \xrightarrow{a} u'}{t + u \xrightarrow{a} u'} \forall a$$

$$\frac{t \downarrow \quad u \downarrow}{t + u \downarrow}$$

1.3 Trace & Kleisli categories

- **trace** of a term t : $\text{tr } t =$ set of words of A^* that can be produced by t , defined by coinduction

$$\varepsilon \in \text{tr } t \Leftrightarrow t \downarrow \quad a.w \in \text{tr } t \Leftrightarrow t \xrightarrow{a} u \wedge w \in \text{tr } u$$

Example: $\text{tr } a(b + c) = \{ab, ac\}$, $\text{tr } (ab + ac) = \{ab, ac\}$, $\text{tr } (a + b? c) = \{a\}$

- recall $HX = \mathcal{P}(1 + A \times X) = TBX$
 - $T = \mathcal{P}$ **effectful** behaviour \leadsto powerset : non-determinism
 - $B = 1 + A \times X$ **pure** behaviour \leadsto words : A^* (initial B -algebra)
- $\text{tr } t \in \mathcal{P}(A^*)$

1.3 Trace & Kleisli categories

Trace **abstractly**

- in the **Kleisli category** of T

$$A \in \mathbf{Kl}(T) \Leftrightarrow A \in \mathbb{C}$$

$$A \multimap B \in \mathbf{Kl}(T) \Leftrightarrow A \rightarrow TB \in \mathbb{C}$$

- A^* is the final B -coalgebra in $\mathbf{Kl}(T)$

$$\zeta : A^* \multimap BA^* \text{ or } A^* \rightarrow TBA^*$$

$$\zeta(\varepsilon) = \{*\}, \quad \zeta(a.w) = \{(a, w)\}$$

- for any $k : X \multimap BX$,

$$\begin{array}{ccc} X & \xrightarrow{k} & BX \\ \text{tr}_k \downarrow & & \downarrow B\text{tr}_k \\ A^* & \xrightarrow{\zeta} & BA^* \end{array}$$

1.4 Trace-GSOS

- GSOS rule

$$\rho : \Sigma(X \times TBX) \rightarrow TB\Sigma^*X$$

- Trace**-GSOS rule

$$\rho : \Sigma(X \times BX) \rightarrow B\Sigma^*X$$

\rightsquigarrow only pure observations

- Rules observe each variable **once and only once**

Example:

$$\frac{t \xrightarrow{\tau} t' \quad t \xrightarrow{\tau} t''}{?t \xrightarrow{\tau} t' + t''}$$



$$\frac{}{a.t \xrightarrow{a} t} \forall a$$



$$\frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t} \forall a, b$$



$$\frac{t \downarrow}{a.t \xrightarrow{a} t} \forall a$$

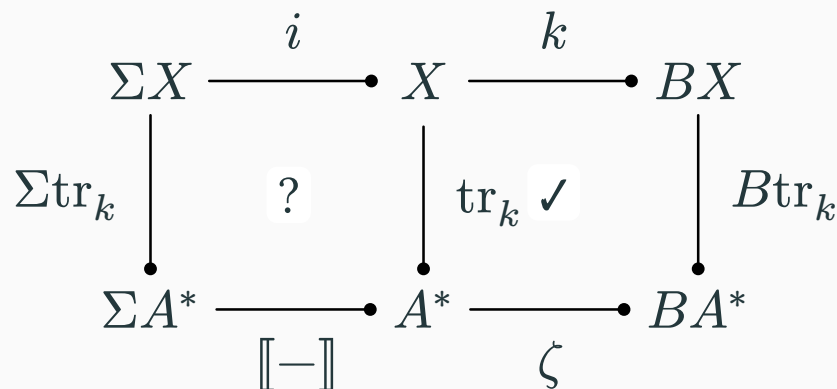


1.5 Trace equivalence & congruence

- trace equivalence:** $t \equiv_{\text{tr}} u \Leftrightarrow \text{tr } t = \text{tr } u$

Example: Recall $\text{tr } a(b + c) = \{ab, ac\} = \text{tr } (ab + ac)$. $a(b + c) \equiv_{\text{tr}} ab + ac$ but not bisimilar $\sim_{\text{tr}} \equiv$ coarsest

- congruence:** $\forall \sigma, (\forall i, t_i \equiv u_i) \Rightarrow \sigma(t_1 \dots t_n) \equiv \sigma(u_1 \dots u_n)$
- prove $\text{tr}(\sigma(t_1 \dots t_n)) = \llbracket \sigma \rrbracket(\text{tr } t_1 \dots \text{tr } t_n)$



1.6 Strong and affine monads

- **strong monad**: $\text{st}_{X,Y} : X \times TY \rightarrow T(X \times Y) \rightsquigarrow \text{st}' : TX \times Y \rightarrow T(X \times Y)$
- double strength $\text{dst} : TX \times TY \xrightarrow{\text{st}} T(TX \times Y) \xrightarrow{T\text{st}'} T^2(X \times Y) \xrightarrow{\mu} T(X \times Y)$ (and dst')
- **affine monad**: $TX \times TY \xrightarrow{\text{dst}} T(X \times Y) \xrightarrow{\langle T\pi_1, T\pi_2 \rangle} TX \times TY = \text{id}$ or $\eta_1 : 1 \xrightarrow{\simeq} T1$
- **affine part**: greatest affine submonad

Example:

- Powerset $\mathcal{P} \rightsquigarrow \mathcal{P}_{\text{ne}}$
- (Sub)distribution $\mathcal{S} \rightsquigarrow \mathcal{D}$ with $\mathcal{D}X = \left\{ \sum_{i \in I} p_i x_i \mid \sum p_i = 1, x_i \in X, I \text{ finite} \right\}$
and $\mathcal{S}X = \left\{ \sum_{i \in I} p_i x_i \mid \sum p_i \leq 1, x_i \in X, I \text{ finite} \right\}$
- Maybe $-+1 \rightsquigarrow \text{Id}$

2. Result

2.1 The theorem

Theorem 2.1.1: Let \mathbb{C} be a cartesian category, T be a strong **affine** *effectful* monad, B a *behaviour* endofunctor that extends to $\mathbf{Kl}(T)$, Σ a *syntax* endofunctor that extends to $\mathbf{Kl}(T)$ with all free objects $(\Sigma^* X)$, let $\zeta : Z \rightarrowtail BZ$ be the final \overline{B} -coalgebra (with $\exists z, \zeta = \eta \circ z$) and let $\rho : \Sigma(X \times BX) \rightarrow TB\Sigma^* X$ be a natural transformation *representing Trace-GSOS rules* such that ρ is **smooth** and is a map of distributive laws, then trace equivalence is a congruence.

2.2 Sketch of the proof

- Recall

$$\begin{array}{ccc} \Sigma^* X & \xrightarrow{i} & X \\ \Sigma^* \text{tr}_k \downarrow & & \downarrow \text{tr}_k \\ \Sigma^* Z & \xrightarrow{\llbracket - \rrbracket} & Z \end{array}$$

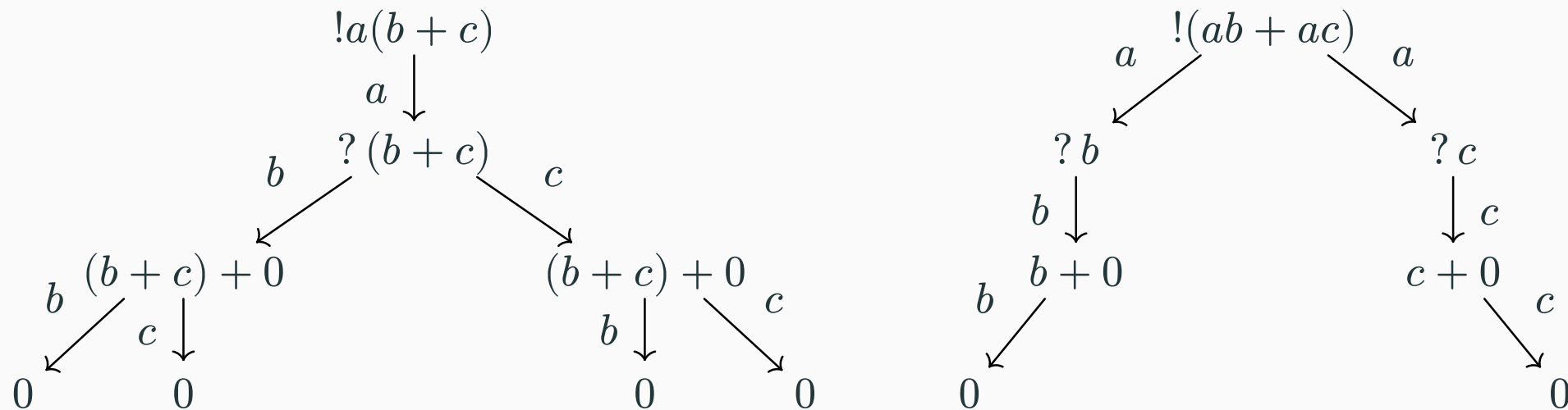
- define $\llbracket - \rrbracket$: semantics of Z + induction + trace
- $\Sigma^* X \rightarrow \bullet B\Sigma^* X$ (with ρ^*) and $Z \rightarrow \bullet BZ$
- show \overline{B} -coalgebra morphisms
- $\text{tr} \circ i \quad \checkmark$
- $\llbracket - \rrbracket \circ \Sigma^* \text{tr}$ more complicated : naturality + smoothness + map of distributive law of ρ^*

Remark: need dst

2.3 Focus on hypothesis : Smoothness

Example: $t ::= 0 \mid a.t \mid t + t \mid ?t \mid !t$ with the previous rules for 0, $a.$, $+$ and

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} ?t'} \forall a \qquad \frac{t \xrightarrow{a} t'}{?t \xrightarrow{a} t + t'} \forall a$$

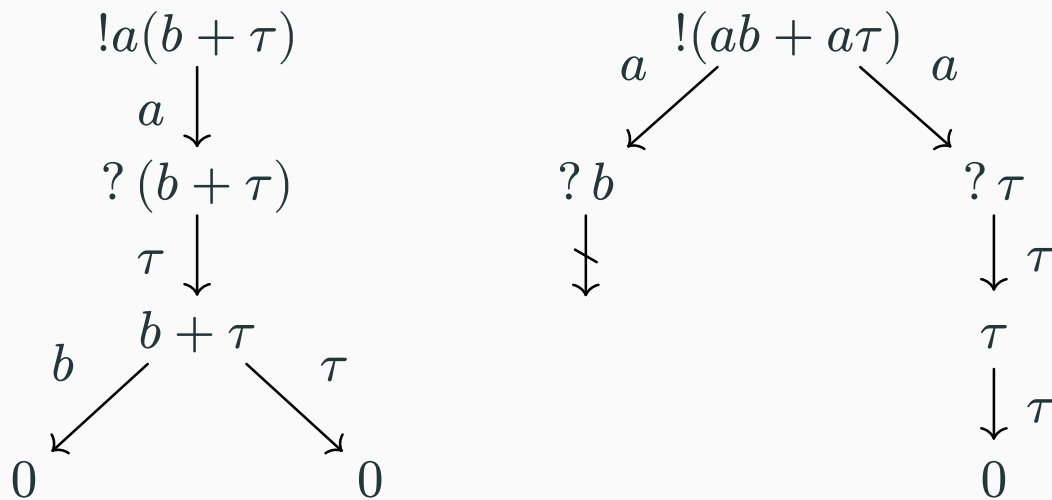


$$\text{tr } !a(b+c) = \{ab, ac, abb, \underline{abc}, \underline{acb}, acc\} \neq \{ab, ac, abb, acc\} = \text{tr } !(ab+ac)$$

2.3 Focus on hypothesis : Smoothness

Example: $t ::= 0 \mid a.t \mid t + t \mid ?t \mid !t$ with the previous rules for $0, a., +$ and

$$\frac{t \xrightarrow{a} t'}{!t \xrightarrow{a} ?t'} \forall a \qquad \frac{t \xrightarrow{\tau} t'}{?t \xrightarrow{\tau} t}$$



$$\text{tr } !a(b + \tau) = \{\underline{a\tau b}, a\tau\tau\} \neq \{a\tau\tau\} = \text{tr } !(ab + a\tau)$$

→ observations that are “not used” 😞

2.3 Focus on hypothesis : Smoothness

- **smoothness**
 - ▶ **linear**: if $x_i \rightarrow x_{i'}$, then not x_i and $x_{i'}$ in the target
 - ▶ if x_i in the target, the observation on x_i is **irrelevant** ie. any other observation could have been done (the same rule for each other possible observation exists)
- **abstract smoothness**

$$\begin{array}{ccc}
 & \text{mix} & \\
 & \text{-----} & \\
 \Sigma T(X \times BX) & \xrightarrow{\langle T\pi_1, T\pi_2 \rangle} & \Sigma(TX \times TBX) \xrightarrow{\text{dst}} \Sigma T(X \times BX) \\
 \lambda \downarrow & & \lambda \downarrow \\
 T\Sigma(X \times BX) & & T\Sigma(X \times BX) \\
 T\rho \downarrow & & T\rho \downarrow \\
 T^2 B\Sigma^* X & \xrightarrow{\mu} & TB\Sigma^* X \xleftarrow{\mu} T^2 B\Sigma^* X
 \end{array}$$

$$\begin{aligned}
 \Phi(\rho)(\sigma)(\text{mix } X_1 \dots \text{mix } X_n) = \\
 \Phi(\rho)(\sigma)(X_1 \dots X_n) \\
 \text{where } X_i \subset X \times BX
 \end{aligned}$$

2.3 Focus on hypothesis : Smoothness

- need smoothness **for** ρ^* (terms with more than one layer)

Example: $t ::= 0 \mid a.t \mid ?t$ with the following smooth rules

$$\frac{}{0 \downarrow} \quad \frac{t \xrightarrow{b} t'}{a.t \xrightarrow{a} t} \forall a, b \quad \frac{t \downarrow}{a.t \xrightarrow{a} t} \forall a \quad \frac{t \xrightarrow{\tau} t'}{?t \xrightarrow{\tau} t'}$$

let $X_1 = \{t \xrightarrow{\tau} t', u \xrightarrow{a} u'\}$ then $\text{mix } X_1 = \{t \xrightarrow{\tau} t', t \xrightarrow{a} u', u \xrightarrow{\tau} t', u \xrightarrow{a} u'\}$

$$\frac{t \xrightarrow{\tau} t' \in X_1}{?t \xrightarrow{\tau} t'} \quad \frac{u \xrightarrow{\tau} ? \notin X_1}{?u \nrightarrow} \quad \frac{t \xrightarrow{\tau} t' \in \text{mix } X_1}{?t \xrightarrow{\tau} t'} \quad \frac{u \xrightarrow{\tau} t' \in \text{mix } X_1}{?u \xrightarrow{\tau} t'}$$

$$\frac{?t \xrightarrow{\tau} t'}{b.?t \xrightarrow{b} ?t} \quad \frac{?u \nrightarrow}{b.?u \nrightarrow} \quad \frac{?t \xrightarrow{\tau} t'}{b.?t \xrightarrow{b} ?t} \quad \frac{?u \xrightarrow{\tau} t'}{b.?u \xrightarrow{b} ?u}$$

$\Phi(\rho^*)(b.?x_1)(X_1) = \left\{ \xrightarrow{b} ?t \right\} \neq \left\{ \xrightarrow{b} ?t, \xrightarrow{b} ?u \right\} = \Phi(\rho^*)(b.?x_1)(\text{mix } X_1) \rightarrow$ the

stuck computation is messing with smoothness 😞

2.4 Focus on hypothesis : Affineness

Theorem 2.4.1: If T is an **affine** monad then the smoothness of ρ entails the smoothness of ρ^* .

- affine part of \mathcal{P} is $\mathcal{P}_{\text{ne}} \longrightarrow$ no stuckness !
- at the level of rules: give a semantics to **every situation**, nothing unspecified

Example:

$$\frac{t \xrightarrow{\tau} t'}{? t \xrightarrow{\tau} t'} \quad + \quad \frac{t \xrightarrow{a} t'}{? t \downarrow} \quad \forall a \neq \tau \quad \frac{t \downarrow}{? t \downarrow}$$

need to have some semantics eg. termination \downarrow

2.5 And for non affine monads ?

- still under investigation 🚧
 - idea 1: add an **extra sink state** \perp for stuck computations
 - idea 2: map stuckness to **explicit termination** (cf. previous example) 🚨 change of semantics
- Can we get back information on the original system ?

3. Conclusion

3. Conclusion

- For an affine monadic effect, under reasonable assumptions, trace equivalence is a congruence 🎉 !
- Can we do better ? Can we find a good reduction to the affine case for non affine monads ?
- Thank you all for welcoming me in the chair ❤️

~ The End ~

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