CO-HW-1

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1.

(1)(i) When $y-x=1$,$inf(T^*(C))$ is 2.
It exists because the distance of the round trip $\pi(1,2,...,n)$ is 2.
Now prove the optimality.
Let $\{city_i|i \geq x\}$ be denoted as $\boldsymbol{A}$ and $\{city_i|i < x\}$ be denoted as $\boldsymbol{B}$.
According to definition of matrix $\boldsymbol{C}$,we know the distance between two cities is 1 if and only if one belongs to $\boldsymbol{A}$ while the other belongs to $\boldsymbol{B}$;in all other cases, the distance is 0.
If the starting city of round trip belongs to $\boldsymbol{A}$,the salesman must enter a city in $\boldsymbol{B}$ at least once and leave for a city in $\boldsymbol{A}$ at least once for the return.The distance for both processes mentioned above is 1 and the same goes for starting from a city in $\boldsymbol{B}$.
Thus,the total distance is at least 2

(ii) When $y-x=2$,$inf(T^*(C))$ is 1.
It also exists because the distance of the round trip π is 1 in this case
Now prove the optimality.
Denote $\{city_i|i \geq x\}$ as $\boldsymbol{P}$ and $\{city_i|i \leq y\}$ as $\boldsymbol{Q}$.
According to definition of matrix $\boldsymbol{C}$,we know the distance between two cities is 1 if and only if one belongs to $\boldsymbol{P}$ while the other belongs to $\boldsymbol{Q}$;in all other cases, the distance is 0.
If the starting city of round trip belongs to $\boldsymbol{P}$,the salesman must either enter a city in $\boldsymbol{Q}$ from $\boldsymbol{P}$,or transfer from $city_{x+1}$ to a city in $\boldsymbol{Q}$ and then return to $\boldsymbol{P}$ from $\boldsymbol{Q}$,The distance for both cases mentioned above is 1,so the total distance is at least 1.The same goes for starting from $\boldsymbol{Q}$
If the starting city is $city_{x+1}$,the salesman must trip between $\boldsymbol{P}$ and $\boldsymbol{Q}$ before returning.
Therefore,the total distance is at least 1,too.

(2)(i)The matrix is

$$\mathbf{\Phi}_n = \begin{bmatrix} 0 & 1 & 1 & \ddots & 0 & 0 & 0 \\ 1 & 0 & 0 & \ddots & 1 & 0 & 0 \\ 1 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 1 \\ 0 & 0 & 1 & \ddots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ddots & 1 & 1 & 0 \end{bmatrix}$$

(ii)Just verify if the distance of π is the upper bound of $T^*(C)$.
When $y - x = 1$,the only two non-zero distances are $c_{x,y-1} = 1$ and $c_{x+1,y} = 1$,meaning the total distance is 2.
When $y - x = 2$,the only non-zero distance is $c_{x,y} = 1$,meaning the total distance is 1.
Hence the optimality.

(3)To prove π is the optimal solution,it suffices to show that the total distance of any permutation of π is greater.
Consider swapping only $city_i$ and $city_{i+1}$ in π,which results in a permutation denoted as π_1.
According to the property of symmetric Monge matrix,we have
$l_\pi = c_{\pi(1),\pi(2)} + ... + (c_{\pi(i-1),\pi(i)} + c_{\pi(i+1),\pi(i+2)}) + ... + c_{\pi(n-1),\pi(n)}$
$\leq l_{\pi_1} = c_{\pi(1),\pi(2)} + ... + (c_{\pi(i-1),\pi(i+1)} + c_{\pi(i),\pi(i+2)}) + ... + c_{\pi(n-1),\pi(n)}$
Repeat the city swapping process to generate all possible permutations...
(well...I don't know how to proceed anymore....)

2.
(1)If we exchange $c_{l+r}(> 0, r > 0)$ coins with a face value of $(l + r)$.
Consider the case that $p_1 = p_2 = ... = p_k = l$,which satisfies $\sum_{j=1}^{k} = kl \leq N = kl$.
In this case,all coins exchanged must be spent.
i.e. $\sum_{j=1}^{k} p_{ij} = c_i$ However, $p_{l+r,j}$ must be 0.(If this is not the case,then the inequality$\sum_{i=1}^{N} ip_{ij} \geq (l + r)p_{l+r,j} \geq l + r \neq l$ holds).
So $c_{l+r} = \sum_{j=1}^{k} p_{l+r,j} = \sum_{j=1}^{k} 0 = 0$,which contradicts with $c_{l+r} > 0$.

(2)If not, $\exists i_0$ s.t. $T_{i_0} < ki_0$. Consider the case that $p_1 = p_2 = ... = p_k = i_0$,which satisfies $\sum_{j=1}^{k} = ki_0 \leq N = kl$.
In this case,all the bills are supposed to be paid with coins with a face value not exceeding i_0.Obviously,these coins totaling T_{i_0} are not enough to pay those bills totaling ki_0.

(3)Building on the result from (2),we concludes that $T_i \geq ki$.

Let = holds for any i to save coins as much as possible.
We have $\sum_{j=1}^{i} jc_j = ki$.
Take $i = 1$,we get $c_1 = k$.
Take $i = 2$ and $c_1 = k$,we get $c_2 = \frac{k}{2}$.
...
Similarly,we get $c_i = \frac{k}{i}$
$\sum_{i=1}^{l} c_i = \sum_{i=1}^{l} \frac{k}{i} = k \sum_{i=1}^{l} \frac{1}{i} = kH_l$.

(4)Similar to (3),

$$c_i = \begin{cases} [\frac{k}{i}] + 1, & \text{if } c_i \notin Z, \\ \frac{k}{i} & \text{if } c_i \in Z \end{cases}$$