

CO-HW-1

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1.

(1)(i) When $y - x = 1, \inf(T^*(C))$ is 2.

It exists because the distance of the round trip $\pi(1, 2, \dots, n)$ is 2.

Now prove the optimality.

Let $\{city_i | i \geq x\}$ be denoted as **A** and $\{city_i | i < x\}$ be denoted as **B**.

According to definition of matrix **C**, we know the distance between two cities is 1 if and only if one belongs to **A** while the other belongs to **B**; in all other cases, the distance is 0.

If the starting city of round trip belongs to **A**, the salesman must enter a city in **B** at least once and leave for a city in **A** at least once for the return. The distance for both processes mentioned above is 1 and the same goes for starting from a city in **B**.

Thus, the total distance is at least 2

(ii) When $y - x = 2, \inf(T^*(C))$ is 1.

It also exists because the distance of the round trip π is 1 in this case

Now prove the optimality.

Denote $\{city_i | i \geq x\}$ as **P** and $\{city_i | i \leq y\}$ as **Q**.

According to definition of matrix **C**, we know the distance between two cities is 1 if and only if one belongs to **P** while the other belongs to **Q**; in all other cases, the distance is 0.

If the starting city of round trip belongs to **P**, the salesman must either enter a city in **Q** from **P**, or transfer from $city_{x+1}$ to a city in **Q** and then return to **P** from **Q**. The distance for both cases mentioned above is 1, so the total distance is at least 1. The same goes for starting from **Q**.

If the starting city is $city_{x+1}$, the salesman must trip between **P** and **Q** before returning.

Therefore, the total distance is at least 1, too.

(2)(i)The matrix is

$$\Phi_n = \begin{bmatrix} 0 & 1 & 1 & \ddots & 0 & 0 & 0 \\ 1 & 0 & 0 & \ddots & 1 & 0 & 0 \\ 1 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 & 1 \\ 0 & 0 & 1 & \ddots & 0 & 0 & 1 \\ 0 & 0 & 0 & \ddots & 1 & 1 & 0 \end{bmatrix}$$

(ii)Just verify if the distance of π is the upper bound of $T^*(C)$.

When $y - x = 1$,the only two non-zero distances are $c_{x,y-1} = 1$ and $c_{x+1,y} = 1$,meaning the total distance is 2.

When $y - x = 2$,the only non-zero distance is $c_{x,y} = 1$,meaning the total distance is 1.

Hence the optimality.

(3)To prove π is the optimal solution,it suffices to show that the total distance of any permutation of π is greater.

Consider swapping only $city_i$ and $city_{i+1}$ in π ,which results in a permutation denoted as π_1 .

According to the property of symmetric Monge matrix,we have

$$l_\pi = c_{\pi(1),\pi(2)} + \dots + (c_{\pi(i-1),\pi(i)} + c_{\pi(i+1),\pi(i+2)}) + \dots + c_{\pi(n-1),\pi(n)}$$

$$\leq l_{\pi_1} = c_{\pi(1),\pi(2)} + \dots + (c_{\pi(i-1),\pi(i+1)} + c_{\pi(i),\pi(i+2)}) + \dots + c_{\pi(n-1),\pi(n)}$$

Repeat the city swapping process to generate all possible permutations...

(well...I don't know how to proceed anymore....)

2.

(1)If we exchange $c_{l+r}(> 0, r > 0)$ coins with a face value of $(l + r)$.

Consider the case that $p_1 = p_2 = \dots = p_k = l$,which satisfies $\sum_{j=1}^k = kl \leq N = kl$.

In this case,all coins exchanged must be spent.

i.e. $\sum_{j=1}^k p_{ij} = c_i$ However, $p_{l+r,j}$ must be 0.(If this is not the case,then the inequality $\sum_{i=1}^N ip_{ij} \geq (l+r)p_{l+r,j} \geq l+r \neq l$ holds).

So $c_{l+r} = \sum_{j=1}^k p_{l+r,j} = \sum_{j=1}^k 0 = 0$,which contradicts with $c_{l+r} > 0$.

(2)If not, $\exists i_0$ s.t. $T_{i_0} < ki_0$. Consider the case that $p_1 = p_2 = \dots = p_k = i_0$,which satisfies $\sum_{j=1}^k = ki_0 \leq N = kl$.

In this case,all the bills are supposed to be paid with coins with a face value not exceeding i_0 .Obviously,these coins totaling T_{i_0} are not enough to pay those bills totaling ki_0 .

(3)Building on the result from (2),we concludes that $T_i \geq ki$.

Let $=$ holds for any i to save coins as much as possible.

We have $\sum_{j=1}^i jc_j = ki$.

Take $i = 1$, we get $c_1 = k$.

Take $i = 2$ and $c_1 = k$, we get $c_2 = \frac{k}{2}$.

...

Similarly, we get $c_i = \frac{k}{i}$

$\sum_{i=1}^l c_i = \sum_{i=1}^l \frac{k}{i} = k \sum_{i=1}^l \frac{1}{i} = kH_l$.

(4) Similar to (3),

$$c_i = \begin{cases} [\frac{k}{i}] + 1, & \text{if } c_i \notin Z, \\ \frac{k}{i} & \text{if } c_i \in Z \end{cases}$$