Homework 1

1. Exercise 2.1

   Proof. Use induction on k. When $k=2$, by definition of convex, it is trivial. Suppose for some $k \le n$ the proposition holds, i.e., $\sum_{i=1}^{n} \theta_i x_i \in C$, then

$$\sum_{i=1}^{n+1} \theta_i x_i = \sum_{i=1}^{n-1} \theta_i x_i + (1 - \sum_{i=1}^{n-1} \theta_i) \left(\frac{\theta_n}{\theta_n + \theta_{n+1}} x_n + \frac{\theta_{n+1}}{\theta_n + \theta_{n+1}} x_{n+1} \right),$$

   and by definition $\hat{x_n} = \frac{\theta_n}{\theta_n+\theta_{n+1}} x_n + \frac{\theta_{n+1}}{\theta_n+\theta_{n+1}} x_{n+1} \in C$, hence by induction the sum is also in C.

2. Exercise 2.2

   Proof. First, if C is convex, since each line l is convex, $C \cap l$ is convex. Conversely, for any two points $x, y \in C$, let l be the line crossing both x and y, then for all θ_1, θ_2 satisfying the conditions, $\theta_1 x + \theta_2 y \in l \cap C \subset C$. Hence C is convex.

   If C is affine, then for each line l, $x_1, x_2 \in C \cap l$, and $\theta \in \mathbb{R}$, $x = \theta x_1 + (1-\theta) x_2 = x_2 + \theta(x_1 - x_2) \in l$, and by definition $x \in C$. Hence $x \in C \cap l$. The converse case is just the same with convex.

3. Exercise 2.5

   Sol. $d = \frac{|b_1 - b_2|}{\|a\|_2}$.

4. Exercise 2.7

   Sol. Notice

$$\begin{aligned}
\|x-a\|_2 - \|x-b\|_2 &= ((x-b)+(b-a))^\top((x-b)+(b-a)) - (x-b)^\top(x-b) \\
&= (x-b)^\top(b-a) + (b-a)^\top(x-b) + (b-a)^\top(b-a) \\
&= 2(b-a)^\top(x-b) + (b-a)^\top(b-a) \\
&= 2(b-a)^\top(x - \frac{a+b}{2}),
\end{aligned}$$

   Hence the set can be written as $\{x \mid (b-a)^\top(x - \frac{a+b}{2}) \le 0\}$.

5. Exercise 2.8

   (a) S is a polyhedra. It can be regarded as a parallelogram spanned by a_1 and a_2. S can be regraded as a intersect of a hyperplane and four halfspaces:

   1. The plane S_0 spanned by a_1, a_2, which can be written as $A^\top x = 0$, with $r(A) = n-2$.

   2. The twin halfspaces parallel with a_1 and perpendicular to S_0, which an be represented as $\{x + y_1 a_1 + y_2 a_2 \mid a_1^\top x = a_2^\top x = 0, -1 \le y_2 \le 1\}$.

   3. The twin halfspaces parallel with a_2 and perpendicular to S_0, which can be represented as $\{x + y_1 a_1 + y_2 a_2 \mid a_1^\top x = a_2^\top x = 0, -1 \le y_1 \le 1\}$.

   (b) S is a polyhedra (trivial by definition).

   (c) S is the intersection of R_+^n and closed unit ball, hence it cannot be described by finite number of linear inequalities.

   (d) S is just $\{x \mid \|x\|_\infty \le 1\}$, which can be described by the intersection of n hyperplanes.

6. Exercise 2.11

   Proof. Pick $x = (x_1, x_2)^\top$ and $y = (y_1, y_2)^\top$ from S, then by Jensen's inequality,

$$(\theta x_1 + (1-\theta) y_1)(\theta x_2 + (1-\theta) y_2) \ge (x_1 x_2)^\theta (y_1 y_2)^{1-\theta} \ge 1.$$

   Hence S is convex. The generalization can be shown by induction, and the details is just the same as problem 1.

7. Exercise 2.12

   (a) Convex, polyhedron.

   (b) Convex, polyhedron.

   (c) Convex, polyhedron.

   (d) Convex, halfspace.

   (e) uncertain.

   (f) Convex.

   (g) not convex, it is a ball.

8. Exercise 2.15

   (a) Convex

   (b) Convex

   (c) Convex

   (d) Convex

   (e) Convex

   (f) It is equivalent to

$$\sum_{i=1}^{n} a_i^2 p_i - \left(\sum_{i=1}^{n} a_i p_i \right)^2 \leq \alpha,$$

   which is not convex.

   (g) same as (f).