Homework 2

1. Exercise 2.28

   For $n = 1$, $x_1 \geq 0$.

   For $n = 2$, it requires $x_1 \geq 0$ and $\det(A) \geq 0$, which means $x_1x_3 - x_2^2 \geq 0$.

   For $n = 3$, $\det(A_1), \det(A_2), \det(A) \geq 0$, where $\det(A_i)$ is the i^{th} leading principle minor. So $x_1 \geq 0$, $x_1x_4 - x_2^2 \geq 0$, $x_1x_4x_6 + 2x_2x_3x_5 - x_1x_5^2 - x_2^2x_6 - x_3^2x_4 \geq 0$.

2. Exercise 2.33

   (a) i) For $x, y \in K_{m+}$, since $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$, $y_1 \geq \cdots \geq y_n \geq 0$, for each $0 \leq \theta \leq 1$, $\theta x_1 + (1-\theta)y_1 \geq \cdots \geq \theta x_n + (1-\theta)y_n \geq 0$. Hence K_{m+} is convex.

   ii) We know K_{m+} is closed because $\geq$ in $\mathbb{R}$ is preserved by limits.

   iii) For $x = \{\frac{1}{i}\}_{i=1}^n \in K_{m+}$, we can pick $r = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n})$, then $O(x, r) \subset K_{m+}$ in the sense of 2-norm. Hence K_{m+} is solid.

   iv) K_{m+} contains no line, because if $0 \neq x \in K_{m+}$, then $-x_1 < 0$, hence $-x \notin K_{m+}$.

   (b) By Abel's transformation,

   $$S = x^\top y = (x_1 - x_2)y_1 + \cdots + (x_{n-1} - x_n)\sum_{j=1}^{n-1} y_j + x_n \sum_{j=1}^{n} y_j \geq 0,$$

   for each $x \in K_{m+}$. First, we notice that if the partial sum of y, $\sum_{i=1}^k y_i \geq 0$ for each $1 \leq k \leq n$, then $S \geq 0$ since $x_i \geq x_{i+1}$. Now if there is some $y \in K_{m+}^*$ which does not have this property, then there is some k, s.t.

   $$\sum_{j=1}^{k} y_j < 0.$$

   Now we can pick $x \in K_{m+}$, which satisfies

   $$x_1 = x_2 = \cdots = x_k, \ x_{k+1} = \cdots = x_n = 0,$$

   then

   $$S = (x_k - x_{k+1})\sum_{j=1}^{k} y_k < 0,$$

   which makes a contradiction. Hence

   $$K_{m+}^* = \{y \in \mathbb{R}^n \mid \sum_{j=1}^{k} y_j \geq 0, 1 \leq k \leq n\}.$$

3. Exercise 3.2

   1) f cannot be convex. In fact, if we limit f to a line in the rightmost corner of the graph, we may find $f'' < 0$, which means f is not convex there. Hence f is not convex.

   f can be quasiconvex since all sublevel sets in the graph are convex.

   f cannot be concave or quasiconcave since superlevel sets are not convex.

   2) f cannot be convex or quasiconvex, since sublevel sets are not convex.

   f can be concave or quasiconcave.

4. Exercise 3.5

   Proof. Notice for each s, $f(sx)$ is convex in x, hence $\int_0^1 f(sx)ds$ is convex. Now let $t = sx$, then

   $$\int_0^1 f(sx)ds = \frac{1}{x}\int_0^x f(t)dt$$

   is convex.

5. Exercise 3.6

   halfspace: f satisfies $f(ax) = af(x)$ for some $a > 0$.

   convex cone: f convex

   polyhedron: f piecewise linear

6. Exercise 3.15

   (a) For each fixed $x_0 > 0$,

$$\lim_{\alpha \to 0} u_\alpha(x_0) = \lim_{\alpha \to 0} \frac{x_0^\alpha \log x_0}{1} = \log x_0.$$

   (b) It is trivial to see $u_\alpha(1) = 0$, and u_α is monotone increasing w.r.t x. Since

$$u_\alpha''(x) = (\alpha - 1)x^{\alpha-2} < 0,$$

   we can see u is concave.

7. Exercise 3.16

   (b) Hessian is the skew unit matrix, which is not positive semidefinite nor negative semidefinite. So f is not convex nor concave. By simple computation we can find $\{x | f(x) \geq \alpha\}$ is convex, so f is quasiconcave.

   (c) The hessian is

$$H = [\frac{2}{x_1^3 x_2}, \frac{1}{x_1^2 x_2^2}; \frac{1}{x_1^2 x_2^2}, \frac{1}{x_1 x_2^3}]$$

   is positive definite, so f is convex and quasiconvex.

   (d) Hessian is

$$H = [0, \frac{2x_1}{x_2^3}; -\frac{1}{x_2^2}, \frac{2x_1}{x_2^3}]$$

   is not positive or negative semidefinite, so f is not convex nor concave. By problem 3.6 we know the sublevel and superlevel sets are halfspaces, so it is quasilinear.

   (e) Hessian

$$H = [\frac{2}{x_2}, -\frac{2x_1}{x_2^2}; -\frac{2x_1}{x_2^2}, \frac{3x_1^2}{x_2^3}]$$

   is positive definite, so f is convex and quasiconvex.

8. Exercise 3.18(b)

   Proof. Considering an arbitrary line in $\mathbf{S}^n$, given by $X = Z + tV$, we define $g(t) = f(Z + tV)$ and restrict g to the interval of values of t where $Z + tV \succ 0$. We may suppose $Z \in \mathbf{S}_{++}^n$, then

$$g(t) = (\det(Z + tV))^{\frac{1}{n}} = (\det(Z^{\frac{1}{2}}(I + tZ^{-\frac{1}{2}}VZ^{-\frac{1}{2}})Z^{\frac{1}{2}}))^{\frac{1}{n}} = (\det Z)^{\frac{1}{n}} \prod_{i=1}^{n}(1 + t\lambda_i)^{\frac{1}{n}}$$

   where λ_i are the eigenvalues of $Z^{-\frac{1}{2}}VZ^{-\frac{1}{2}}$. Therefore, since geometric mean is concave, we know g is concave.

9. Exercise 3.24

   (f) Since the random variable x is discrete, we know the sublevel and superlevel set of $quatile(x)$ are convex. Hence it is quasilinear, but not convex or concave.

   (g) Similar with (f), since f is discrete, it is not convex nor concave. Since

$$f(p) \geq a \iff \sum_{i=1}^{k} p_i < 0.9,$$

   where k is the largest number less than a. We know the sum of largest numbers is convex, so the superlevel set is convex. Hence f is quasiconcave.

(h) The minimum width interval should be the form $[a_i, a_j]$, so it is discrete hence not convex nor concave. Since

$$f(p) \geq a \iff \sum_{k=i}^{j} p_k < 0.9$$

for all i, j satisfy $a_j - a_i < a$. This is convex, hence f is quasiconcave.

10. Exercise 3.36

(a) First, if $y \succ 0, 1^\top y = 1$, then it is trivial that

$$f^*(y) = \sup_x (y^\top x - f(x)) = \sum_{i=1}^{n} y_i x_i - \max(x_i) \leq 0,$$

the equality holds iff all x_i are equal.

Now suppose it is not the case. If $y \not\succeq 0$, suppose $y_1 < 0$ without loss of generality, then we can pick $x_1 < M$ for $-M$ arbitrary large, and other components of x are all equal to 0, then $f^*(x) = \infty$.

If $y \succ 0$ but $1^\top y \neq 1$, if $1^\top y > 1$ then we can pick $x = t1$ then $f^*(y) = \infty$; if $1^\top y < 1$ then pick $x = -t1$ then $f^*(y) = \infty$. Hence.

(d) For $y \geq 0$, $f^*(y) = (p-1)(y/p)^{p/(p-1)}$, for $y < 0$, $f^*(y) = 0$. When $p < 0$, $f^*(y) = (p-1)(y/p)^{p/(p-1)}$ for $y < 0$ which is the domain.