Homework 3

1. Exercise 3.42

   Proof. This is to show the superlevel sets of W are convex. We know

$$W(x) \geq a \iff |\sum_{i=1}^{n} x_i f_i(t) - f_0(t)| \leq \epsilon, \forall 0 \leq t < a.$$

   Then for $0 \leq \theta \leq 1$,

$$|(\theta x^1 + (1-\theta)x^2)^\top f(t) - f_0(t)| \leq \theta |x^{1\top} f(t) - f_0(t)| + (1-\theta)|x^{2\top} f(t) - f_0(t)| \leq \epsilon.$$

   Hence.

2. Exercise 3.54

   (a) By definition

$$f'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ f''(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2},$$

   so

$$f''(x)f(x) = -\frac{x}{2\pi} e^{-x^2/2} \int_{-\infty}^{x} e^{-t^2/2} dt \leq 0 \leq f'(x)^2$$

   for $x > 0$.

   (b) Trivial by mean value inequality if x and t have the same sign. If $xt < 0$, then left side ≥ 0, and right side < 0, so the inequality also holds.

   (c) The first inequality is trivial since $\exp(x)$ is monotone increasing. For fixed $x < 0$, since the equality holds for any x, t, we can just integrate from $-\infty$ to x and we will get the second inequality.

   (d)

$$f''(x)f(x) \leq -\frac{x}{2\pi} e^{-x^2/2} e^{x^2/2} \int_{-\infty}^{x} e^{-xt} dt = \frac{1}{2\pi} e^{-x^2} = f'(x)^2.$$

3. Exercise 3.57

   Proof. We need to show for each fixed $y \in \mathbb{R}^n$, $f(X) = y^\top X^{-1} y$ is convex. This can be shown by Example 3.4.

4. Exercise 4.1

   (a) $\{(\frac{2}{5}, \frac{1}{5})\}$, $\frac{3}{5}$

   (b) unbounded below.

   (c) $\{(0, x_2), x_2 \geq 1\}$, 0

   (d) $\{(\frac{1}{3}, \frac{1}{3})\}$, $\frac{1}{3}$

   (e) $\{(\frac{1}{2}, \frac{1}{6})\}$, since ∇f_0 is perpendicular to the boundary at that point.

5. Exercise 4.4

   (a) We may notice, for each fixed Q_i, the orbit $\{Q_i Q_j\}$ is just G. Hence for any $x \in \mathbb{R}^n$,

$$Q_i \bar{x} = \frac{1}{k} Q_i \sum_{j=1}^{k} Q_j x = \frac{1}{k} \sum_{j=1}^{k} Q_i Q_j x = \frac{1}{k} \sum_{i=1}^{k} Q_i x = \bar{x}.$$

   (b) Since f is convex,

$$f(\bar{x}) = f(\frac{1}{k} \sum_{i=1}^{k} Q_i x) \leq \frac{1}{k} \sum_{i=1}^{k} f(Q_i x) = \frac{1}{k} \sum_{i=1}^{k} f(x) = f(x).$$

(c) Suppose x_0 is the optimal point of the problem, then by (a), $\bar{x_0}$ is feasible, and by (b) $f_0(\bar{x_0}) \leq f_0(x_0)$. So $\bar{x_0}$ is optimal.

(d) By (a), (b), (c), we notice for a minimizer x_0 of this problem,

$$f(\frac{1}{n!}\sum_P Px_0) \leq f(x_0).$$

But $\frac{1}{n!}\sum_P Px_0 = \alpha 1$. Hence.

6. Exercise 4.8

(a) i) If the constraint is not feasible, i.e., $Ax = b$ has no solutions, then the optimal result is ∞.

ii) Now let $c = A^\top c_1 + c_2$, where $Ac_2 = 0$. Then $c^\top x = c_1^\top b + c_2^\top x$. If $c_2 = 0$ then $c^\top x \equiv c_1^\top b$. If $c_2 \neq 0$, then pick $\hat{x} = x - tc_2$, we have $A\hat{x} = b$ and $c_2^\top \hat{x} = -t|c_2|^2$, which means it is not bounded below, so the optimal result is $-\infty$.

(b) Let $c = ka + c_1$, where $c_1^\top a = 0$. Then $c^\top x = ka^\top x + c_1^\top x$. If $c_1 = 0$, if $k > 0$, pick $x = -ta$, then $a^\top x \leq b$ when $t \to -\infty$, and $c^\top x = -kt|a|^2$ is unbounded below. If $k \leq 0$, then $ka^\top x \geq kb$, so $\min f_0 = kb$. If $c_1 \neq 0$, pick $x = ba - tc_1$ and let $t \to -\infty$, the function is not bounded below.

(c) We can minimize w.r.t. each component seperately. For each i, if $c_i > 0$, then $x_i^* = l_i$; if $c_i = 0$, then any $l_i \leq x_i \leq u_i$ is optimal. if $c_i < 0$, then $x_i^* = u_i$. Hence.

(d) Notice

$$c^\top x \geq \min\{c_i\} 1^\top x = \min\{c_i\}.$$

If constraint is replaced, then

$$c^\top x \geq \min\{0, c_i\}.$$

(e) First suppose $c_1 \leq c_2 \leq \cdot \leq c_n$. Then

$$c^\top x \geq \sum_{i=1}^{\alpha} c_i.$$

If α is not an integer,

$$c^\top x \geq \sum_{i=1}^{\lfloor\alpha\rfloor} c_i + (\alpha - \lfloor\alpha\rfloor)c_{\lfloor\alpha\rfloor+1}.$$

If is replaced with inequality, then

$$c^\top x \geq \sum_{i=1}^{k} c_i,$$

where k satisfies $k \leq \alpha$ and

$$c_1 \leq \cdots \leq c_k \leq 0.$$

7. Exercise 4.17

The problem can be written as

$$\begin{aligned}&\text{maximize } \sum_{i=}^{n} r_j(x_j)\\ &\text{subject to } x \succcurlyeq 0, Ax \preccurlyeq c^{max}\end{aligned}$$

This is a convex optimization problem. Notice

$$r_j(x_j) = \min\{p_j x_j, p_j q_j + p_j^{disc}(x_j - q_j)\}.$$

then

$$r_j(x_j) \geq t \iff p_j x_j \geq t,\ p_j q_j + p_j^{disc}(x_j - q_j) \geq t.$$

Hence the LP should be

$$\begin{aligned}&\text{maximize } 1^\top t\\ &\text{subject to } x \succcurlyeq 0,\ Ax \preccurlyeq c^{max},\ p_j x_j \geq t_j,\ p_j q_j + p_j^{disc}(x_j - q_j) \geq t_j.\end{aligned}$$