Theoretical Numerical Analysis, Assignment 6

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1. Problem 5.2.2

   First, D is invertible since A is strictly diagonally dominant. For Jacobi method, define the operator $T : \mathbb{R}^m \to \mathbb{R}^m$:
   $$Tx = D^{-1}b - D^{-1}(L+U)x,$$
   then the Jacobi method can be written as
   $$x_{n+1} = Tx_n.$$
   We should show that T is contractive:
   $$\|Tx - Ty\| = \|D^{-1}(L+U)(x-y)\| \le \|D^{-1}(L+U)\|\|x-y\|.$$
   Then the convergence of Jacobi method is equavilent to
   $$r_\sigma(B) < 1,$$
   where $B = D^{-1}(L+U)$. If there is an eigenvalue λ of B such that $|\lambda| \ge 1$, consider the matrix $C = \lambda D + L + U$. We know C is also strictly diagonally dominant and thus invertible. Notice
   $$\det(\lambda I - B) = \det(D^{-1})\det(C) \ne 0,$$
   which makes a contradictory with that λ is an eigenvalue of B. Hence, $r_\sigma(B) < 1$.

   For Gauss-Seidal method, the iteration matrix is $B = (D+L)^{-1}U$. Using the same process,
   $$\det(\lambda I - B) = \det((D+L)^{-1})\det(\lambda(D+L) - U).$$
   Since $\lambda(D+L) - U$ is strictly diagonally dominant and thus invertible,
   $$\det(\lambda I - B) \ne 0,$$
   making a contradictory. Hence, $r_\sigma(B) < 1$.

2. Problem 5.2.12

   Consider the function
   $$u(s) = \exp(-\int_a^s h(r)dr)\int_a^s h(r)f(r)dr,$$
   then
   $$u'(s) = (f(s) - \int_a^s h(r)f(r)dr)h(s)\exp(-\int_a^s h(r)dr) \le g(s)h(s)\exp(-\int_a^s h(r)dr).$$
   Then
   $$u(t) \le \int_a^t g(s)h(s)\exp(-\int_a^s h(r)dr)ds,$$
   and
   $$\begin{aligned}\int_a^t h(s)f(s)ds &= u(t)\exp(-\int_a^t h(s)ds) \\ &\le \int_a^t g(s)h(s)\exp(\int_a^t h(r)dr - \int_a^s h(r)dr)ds \\ &= \int_a^t g(s)h(s)\exp(\int_s^t h(r)dr)ds.\end{aligned}$$

Hence,

$$f(t) \le g(t) + \int_a^t h(s)f(s)ds \le g(t) + \int_a^t g(s)h(s)\exp(\int_s^t h(r)dr)ds.$$

If g is nondecreasing, we know on the above

$$\begin{aligned} f(t) &\le g(t) + g(t)\int_a^t h(s)\exp(\int_s^t h(r)dr)ds \\ &= g(t) - g(t)\exp(\int_s^t h(r)dr)\Big|_{s=a}^{s=t} \\ &= g(t)\exp(\int_a^t h(s)ds). \end{aligned}$$

When $h = c$, substituting it into the equalities, we can get the results.

3. Problem 5.3.2

   Consider a vector $h = (h_1, h_2) \neq (0, 0)$. Then

   $$\lim_{t\to 0} \frac{f(u_0 + th) - f(u_0)}{t} = \lim_{t\to 0} \frac{t_1^h h_2^3}{h_1^2 + t^3 h_2^4}.$$

   When $h_1 = 0$, the limit is just 0. When $h_1 \neq 0$, the limit also goes to 0. Hence, $A \equiv 0$ and f is Gateaux differentiable.

   However,

   $$f(u_0 + h) - f(u_0) = \frac{h_1 h_2^3}{h_1^2 + h_2^4}.$$

   When $h_1 = 0$, the value $\to 0$ as $|h_2| \to 0$. When $h_1 = h_2 \neq 0$, the value $\to \frac{1}{1+h_2^2} \to 1$ as $|h_2| \to 0$. Hence, f is not Frechet differentiable.

4. Problem 5.3.7

   $$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \left(\frac{(1-\cos h)\sin x + \cos x \sin h}{h}, \frac{(1-\cos h)\cos x - \sin x \sin h}{h} \right)^\top \\ &= (\cos x, -\sin x)^\top. \end{aligned}$$

   Hence,

   $$f'(x) = (\cos x, -\sin x)^\top.$$

   In order to satisfy the provided equation,

   $$f'(2\pi\theta) = 0$$

   should be satisfied. However, $f'(x) = 0$ has no solution on $[0, 2\pi]$.

5. Problem 5.4.2

   Let

   $$F(u) = \alpha u - \mu\Delta u + (u \cdot \nabla)u - f,$$

   then the equation is $F(u) = 0$. The derivative can be computed as

   $$F'(u)(v) = \lim_{h\to 0} \frac{1}{h}(F(u + hv) - F(u)) = \alpha v - \mu\Delta v + 2\nabla u \cdot \nabla v.$$

   Hence, at each step, solve a linearized boundary value equation

   $$\begin{aligned} \alpha u_{n+1} - \mu\nabla u_{n+1} + 2\nabla(u_{n+1} - u_n)\cdot\nabla(u_{n+1} - u_n) &= f + \alpha u_n - \mu\nabla u_n, \quad \text{in } \Omega, \\ u_{n+1} &= g_0, \quad \text{on } \Gamma_0, \\ \mu\frac{\partial u_{n+1}}{\partial \nu} &= g_1, \quad \text{on } \Gamma_1. \end{aligned}$$