Theoretical Numerical Analysis, Assignment 2

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1. Problem 2.4.2

   By definition,

$$\begin{aligned}\|L_n\| &= \sup_{v\neq 0}\frac{\|L_n v\|}{\|v\|} = \sup_{v\neq 0}\frac{\|\sum_{i=0}^n w_i^{(n)} v(x_i^{(n)})\|}{\|v\|}\\ &\le \sup_{v\neq 0}\frac{\sum_{i=0}^n |w_i^{(n)|}\|v\|}{\|v\|}\\ &= \sum_{i=0}^n \||w_i^{(n)}|.\end{aligned} \tag{1}$$

   On the other hand, let $v(x) = 1$, then RHS becomes

$$\frac{\|w_i^{(n)} v\|}{\|v\|} = \sum_{i=0}^n |w_i^{(n)}|, \tag{2}$$

   so

$$\|L_n\| = \sum_{i=0}^n |w_i^{(n)}|. \tag{3}$$

2. Problem 2.4.3

   From the textbook we only need to show that

$$\sup_n \sum_{i=0}^n |w_i^{(n)}| < \infty. \tag{4}$$

   By the assumption,

$$\sup_n \sum_{i=0}^n |w_i^{(n)}| = \sup_n \sum_{i=1}^n w_i^{(n)} = \sup_n L_n u, \tag{5}$$

   where $u = u(x) \equiv 1$. Since for all n, $u \in \mathbb{P}_{d(n)}$,

$$L_n u = Lu = 1, \tag{6}$$

   we know $\sup L_n u < \infty$.

3. Problem 2.4.5

   First, consider $V = \mathbb{P}[0,1]$. For a fixed element $v \in \mathbb{P}$, we already know the error of composite trapezoidal rule is

$$E(v) = \frac{1}{12n^3}\sum_{i=0}^{n-1} v''(\xi_i) \le \frac{1}{12n^2}\|v''\|. \tag{7}$$

   Since $\|v''\| < \infty$, we know $E(v) \to 0$ as $n \to \infty$, hence $L_n v \to L_v$.

   On the other hand,

$$\|L_n\| = \sum_{i=0}^n |w_i^{(n)}| = 1 \tag{8}$$

   for all n, so $\sup\|L_n\| < \infty$. By Banach-Steinhaus theorem, $L_n v \to Lv$ for all $v \in C[0,1]$.

4. Problem 2.5.4

   By (2.5.5),
$$\ell(v) = (u, v) \tag{9}$$
   for all $v \in V$. Then let $v = e_j$, we have
$$\ell(e_j) = (u, e_j) \tag{10}$$
   for all j. Then
$$u = \sum_{j=1}^{\infty} (u, e_j) e_j = \sum_{j=1}^{\infty} \ell(e_j) e_j. \tag{11}$$

5. Problem 2.7.3

   $\Rightarrow$: If $u_n \to u$, then
$$\lim_{n\to\infty} \|u_n - u\| = 0, \tag{12}$$
   so for each $\ell inV'$,
$$\lim_{n\to\infty} \|\ell(u_n) - \ell(u)\| \leq \lim_{n\to\infty} \|\ell\| \|u_n - u\| = 0. \tag{13}$$
   Then $u_n \rightharpoonup u$. Notice $\|u_n - u\| \geq \|u_n\| - \|u\|$, and $\|u - u_n\| \geq \|u\| - \|u_n\|$ we have
$$\lim_{n\to\infty} \|u_n\| - \|u\| \leq 0, \;\; \lim_{n\to\infty} \|u\| - \|u_n\| \leq 0. \tag{14}$$
   Thus $\|u_n\| \to \|u\|$.

   $\Leftarrow$: Consider
$$\lim_{n\to\infty} \|u_n - u\|^2 = \lim_{n\to\infty} \|u_n\|^2 + \|u\|^2 - (u_n, u) - (u, u_n), \tag{15}$$
   and both $(u, \cdot)$ and $(\cdot, u)$ are linear functionals on V, we have
$$\lim_{n\to\infty} \|u_n - u\|^2 = \lim_{n\to\infty} \|u_n\|^2 + \|u\|^2 - \|u\|^2 - \|u\|^2 = 0. \tag{16}$$
   Hence $\|u - u_n\| \to 0$.