Theoretical Numerical Analysis, Assignment 7

Chuan Lu

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1. Problem 6.1.1

   By Taylor expansion,

   $$af(x+h)+bf(x)+cf(x-h) = (a+b+c)f(x)+h(a-c)f'(x)+\frac{1}{2}h^2(a+c)f''(x)+\frac{1}{6}h^3(a-c)f'''(x)+O(h^4).$$

   In order to make it be an approximation of $f'(x)$, we must have

   $$a+b+c=0,\ h(a-c)=1.$$

   There are three variables with two equations, so we can further let

   $$a+c=0.$$

   Then, by letting

   $$a=\frac{1}{2h}, b=0, c=-\frac{1}{2h},$$

   we notice $a-c\neq 0$, so

   $$|af(x+h)+bf(x)+cf(x-h)-f'(x)|\leq O(h^3).$$

2. Problem 6.1.2

   The conditions now become

   $$a+b+c=0,\ a-c=0,\ \frac{1}{2}h^2(a+c)=1.$$

   There is one exact solution

   $$a=c=\frac{1}{h^2},\ b=-\frac{2}{h^2}.$$

   Notice

   $$a-c=0,\ \frac{1}{24}h^2(a+c)\neq 0,$$

   so

   $$|af(x+h)+bf(x)+cf(x-h)-f''(x)|\leq O(h^4).$$

3. Problem 6.2.2

   The operator is defined as

   $$(1+2r)C(\Delta t)v(x) = r(C(\Delta t)v(x+\Delta x)+C(\Delta t)v(x-\Delta x))+(1-2r)v(x)+r(v(x+\Delta x)+v(x-\Delta x)),$$

   where $r=\frac{\nu\Delta t}{2\Delta x^2}$.

   Then we have

   $$\|C(\Delta t)v\|\leq\frac{2r}{1+2r}\|C(\Delta t)v\|+\frac{1-2r}{1+2r}\|v\|+\frac{2r}{1+2r}\|v\|,$$

   which means

   $$\frac{1}{1+2r}\|C(\Delta t)v\|\leq\frac{1}{1+2r}\|v\|.$$

Hence,

$$\{C(\Delta t)\} \leq 1$$

is uniformly bounded. We also notice

$$\|C(\Delta t)^m\| \leq 1$$

for any m, so by Lax equivalence thm, Crank-Nicolson scheme is consistent.

4. Problem 6.3.1

   When $bc = 0$, the eignevalues are just $\lambda_j = a$. Now we let $Q = aI + \sqrt{bc}D^{-1}\Lambda D$ as suggested by the hint, then $\lambda_j = a + \sqrt{bc}\hat{\lambda}_j$, where $\hat{\lambda}$ are the eigenvalues of Λ.

   We can see that $\Lambda = \Lambda_n$ satisfies

$$\det(\lambda I - \Lambda_n) = \lambda \det(\lambda I - \Lambda_{n-1}) - \det(\lambda I - \Lambda_{n-2}),$$

   so by the roots of Chebyshev polynomials of the second kind, we know $\hat{\lambda}_j = 2\cos(\frac{j\pi}{N+1})$.

5. Problem 6.3.2