Theoretical Numerical Analysis, Assignment 4

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1. Problem 3.5.3

   Consider an homomorphism $[a, b]$ to $[-1, 1]$, where $x \to \frac{2x-a-b}{b-a}$. Then the polynomials

   $$\tilde{L}_n(x) = L_n(\frac{2x - a - b}{b - a})$$

   are the Legendre polynomials on $[a, b]$, where L_n are the Legendre polynomials on $[-1, 1]$.
   Similarly,

   $$\tilde{T}_n(x) = T_n(\frac{2x - a - b}{b - a})$$

   are the Chebyshev polynomials on $[a, b]$, where T_n are the Chebyshev polynomials on $[-1, 1]$. The weight function should be

   $$\tilde{w}(x) = \frac{1}{\sqrt{1 - (\frac{2x-a-b}{b-a})^2}}$$

2. Problem 3.5.5

   Let

   $$xp_n(x) = \sum_{i=0}^{n+1} \alpha_i p_i(x),$$

   then take inner product $(\cdot, \cdot)_{0,w}$ with p_j on both sides,

   $$(xp_n, p_j) = \alpha_j \|p_j\|^2.$$

   For $j \le n - 2$, xp_j is a linear combination of p_k for $0 \le k \le n - 1$,

   $$(xp_n, p_j) = \int xp_n p_j w dx = \int p_n \sum_{k=0}^{n-1} \beta_k p_k w dx = \sum_{k=0}^{n-1} \beta_k \int p_n p_k w dx = 0.$$

   Then $\alpha_j = 0$ for $j \le n - 2$. So

   $$xp_n = \alpha_{n-1} p_{n-1} + \alpha_n p_n + \alpha_{n+1} p_{n+1},$$

   which is

   $$p_{n+1}(x) = \frac{x - \alpha_n}{\alpha_{n+1}} p_n - \frac{\alpha_{n-1}}{\alpha_{n+1}} p_{n-1}.$$

3. Problem 3.5.8

   Let

   $$T_n(x) = \cos(n \arccos x) = 0,$$

   we have

   $$x = \cos(\frac{1}{n}(k\pi + \frac{1}{2}\pi)),\ k \in \mathbb{Z},\ \frac{1}{n}(k + \frac{1}{2})\pi \in [-1, 1].$$

   Then $k = [-\frac{n}{\pi} - \frac{1}{2}, \frac{n}{\pi} - \frac{1}{2}] \cap \mathbb{Z}$.

Let

$$T_n(x) = \cos(n \arccos x) = \pm 1,$$

we have

$$x = \cos(\frac{k\pi}{n}),\ k \in \mathbb{Z},\ \frac{k\pi}{n} \in [-1, 1].$$

Then $k = [-\frac{n}{\pi}, \frac{n}{\pi}] \cap \mathbb{Z}$.

4. Problem 3.6.2

Consider $v \in V_1^\perp$ and $v_1 \in V_1$,

$$(v, v_1) = \int_{-1}^{1} vv_1 dx = \int_{0}^{1} vv_1 dx = 0$$

for all $v_1 \in V_1$. First, consider $v \in W = \{v \in V \mid v(x) = 0\text{a.e. in}(0,1)\}$, then each $v \in W$ satisties the condition above. On the other hand, if $\exists v \in V_1^\perp \setminus W$, consider $u(x) \in V_1$, s.t.

$$u(x) = \begin{cases} 0, x \in (-1, 0], \\ sgn(v(x)), x \in (0, 1), \end{cases}$$

then since v is not 0 a.e. on $(0,1)$, $(u, v) > 0$. Hence $V_1^\perp = W$.

5. Problem 3.7.5

$$\begin{aligned} \phi_j(x) &= \frac{2}{2n+1} D_n(x - x_j) = \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{j=1}^{n} \cos(j(x - x_j)) \\ &= \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{j=1}^{n} \cos jx_j \cos jx + \frac{2}{2n+1} \sum_{j=1}^{n} \sin jx_j \sin jx \in \mathbb{T}_n, \end{aligned}$$

and

$$\phi_j(x_j) = \frac{1}{2n+1} + \frac{2}{2n+1} \sum_{i=1}^{n} 1 = 1,$$

$$\phi_j(x_k) =$$