Theoretical Numerical Analysis, Assignment 5

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1. Problem 4.1.4

   (a)

$$\begin{aligned}
S_n f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \sum_{j=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(jx)dx\cos(jx) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(jx)dx\sin(jx) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \sum_{j=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos(jt)\cos(jx)+\sin(jt)\sin(jx))dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \sum_{j=1}^{n}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(j(t-x))dt \\
&= \int_{-\pi}^{\pi} f(t)K_n(t-x)dt.
\end{aligned}$$

   (b)

$$\int_{-\pi}^{0} K_n(t)dt = \int_{-\pi}^{0}\frac{1}{2\pi}+\frac{1}{\pi}\sum_{i=1}^{n}\cos(jt)dt = \frac{1}{2}+\frac{1}{\pi}\sum_{i=1}^{n}\frac{1}{j}\sin(jt)\Big|_{-pi}^{0} = \frac{1}{2}.$$

   Since K_n is an even function, we know $\int_0^\pi K_n(t)dt = \frac{1}{2}$.

   When $t = 2k\pi$, we know $K_n(t) = \frac{1}{2\pi}+\frac{n}{\pi}$; When $t \neq 2k\pi$,

$$k_n(t) = \frac{1}{2\pi\sin(t/2)}(\sin(t/2)+\sum_{j=1}^{n}(\sin((j+\frac{1}{2})t)-\sin((j-\frac{1}{2})t))) = \frac{1}{2\pi\sin(t/2)}\sin((n+1/2)t)$$

   (c), (d) This are trivial.

2. Problem 4.2.1

   (4.2.5) is trivial since $\mathcal{F}$ is linear.

   (4.2.6)

$$\|\mathcal{F}(f)\|_{L^\infty} \le \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}|f|dx = (2\pi)^{-d/2}\|f\|_{L^1}.$$

   (4.2.7) This can be proven by $|\alpha|$ integration by parts, and using the property that $\partial f \to 0$ as $x \to \infty$.

   (4.2.8) This also can be proven by $|\alpha|$ integration by parts.

3. Problem 4.2.8

   Let $g(x) = \overline{f(-x)}, h = f * g$. Then

$$\mathcal{F}(h) = \mathcal{F}(f)\mathcal{F}(g) = \mathcal{F}(f)^2.$$

   Since

$$h(0) = \int_{\mathbb{R}^d} f(x)g(-x)dx = \|f\|_{L^2}^2,$$

$$h(0) = (2\pi)^{-d/2} \int_{R^d} \mathcal{F}(h)(y)dy = \|\mathcal{F}(f)\|_{L^2}^2,$$

we have $\|\mathcal{F}(f)\|_{L^2}^2 = \|f\|_{L^2}^2$.

4. Problem 4.3.2

   Notice

$$|\hat{y_k}|^2 = \hat{y_k}\overline{\hat{y_k}} = \sum_{j=0}^{n-1} y_j \sum_{l=0}^{n-1} y_l \exp(k(j-l)2\pi i/n),$$

   then

$$\begin{aligned}
\sum_{k=0}^{n-1} |\hat{y_k}|^2 &= \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} y_j \sum_{l=0}^{n-1} y_l \exp(k(j-l)2\pi i/n) \\
&= \sum_{j=0}^{n-1} y_j \sum_{l=0}^{n-1} y_l \sum_{k=0}^{n-1} \exp(k(j-l)2\pi i/n) \\
&= \sum_{j=0}^{n-1} y_j \sum_{l=0}^{n-1} y_l \frac{1-\exp((j-l)2\pi i)}{1-\exp((j-l)2\pi i/n)} \\
&= n\sum_{j=0}^{n-1} y_j^2
\end{aligned}$$

5. Problem 4.3.3

   For $0 \le k < n$,

$$(F_{2n}y)_k = (F_n y_e)_k + (D_n F_n y_o)_k = \mathcal{F}_n(\{y_{2j}\})_k + w_{2n}^{-k}\mathcal{F}_n(\{y_{2j+1}\})_k,$$

$$(F_{2n}y)_{n+k} = (F_n y_e)_k - (D_n F_n y_o)_k = \mathcal{F}_n(\{y_{2j}\})_k - w_{2n}^{-k}\mathcal{F}_n(\{y_{2j+1}\})_k.$$