Theoretical Numerical Analysis, Assignment 1

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1. Problem 1.4.13

   Let

$$f(x) = \begin{cases} -1, & x \leq -1 \\ \dfrac{e^{\frac{1}{x-1}} - e^{\frac{1}{x+1}}}{e^{\frac{1}{x-1}} + e^{\frac{1}{x+1}}}, & -1 < x < 1 \\ 1, & x \geq 1 \end{cases} \tag{1}$$

   Then by Exercise 1.4.11, we know $f \in C^\infty(\mathbb{R})$.

2. Problem 1.5.4

   First, by Jensen's inequality we have the given inequality. Then

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q = \log a + \log b = \log ab. \tag{2}$$

   Since $\log(\cdot)$ is an increasing function,

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab. \tag{3}$$

3. Problem 1.5.9

$$\begin{aligned} |(f * g)(x)| &= \left|\int_{\mathbb{R}^d} f(y)g(x-y)dy\right| \leq \int_{\mathbb{R}^d} |f(y)||g(x-y)|dy \\ &= \int_{\mathbb{R}^d} |f(y)||g(x-y)|^{\frac{1}{p}}|g(x-y)|^{\frac{1}{q}}dy \\ &\leq \|f(y)g(x-y)^{\frac{1}{p}}\|_{L^p}\|g(x-y)^{\frac{1}{q}}\|_{L^q}. \end{aligned} \tag{4}$$

   This comes from that

$$fg^{\frac{1}{p}} \in L^p,\ g^{\frac{1}{q}} \in L^q. \tag{5}$$

   Then

$$\begin{aligned} \|f * g\|_{L^p} &\leq \left\| \|f(y)g(x-y)^{\frac{1}{p}}\|_{L^p}\|g(x-y)^{\frac{1}{q}}\|_{L^q} \right\|_{L^p} \\ &\leq \|f\|_{L^p} \left\| \|g^{\frac{1}{p}}\|_{L^p}\|g^{\frac{1}{q}}\|_{L^q} \right\|_{L^p} \\ &= \|f\|_{L^p}\|g\|_{L^1}^{\frac{1}{p}}\|g\|_{L^1}^{\frac{1}{q}} = \|f\|_{L^p}\|g\|_{L^1}. \end{aligned} \tag{6}$$

4. Problem 1.5.10

   Let

$$u(x) = u_n,\ v(x) = v_n,\ n \leq x < n+1. \tag{7}$$

   Then $u \in L^p,\ v \in L^q$, and

$$\begin{aligned} \sum_{n=0}^{\infty} u_n v_n &= \int_{\mathbb{R}} u(x)v(x)dx \leq \int_{\mathbb{R}} |u(x)||v(x)|dx \\ &\leq \|u\|_{L^p}\|v\|_{L^q} = \|u\|_{\ell^p}\|v\|_{\ell^q}. \end{aligned} \tag{8}$$

5. Problem 1.5.12

   First, consider $\Omega = [1, \infty), p = 1, q = 2$, and $f(x) = \frac{1}{x} \in L^q$, but $f(x) \notin L^p$. Thus L^p does not belong to L^q.

   On the other hand, consider $\Omega = [1, \infty), p = 1, q = 2$, and

$$f(x) = \begin{cases} n, & n \leq x < n + \dfrac{1}{n^3} \\ 0, & n + \dfrac{1}{n^3} \leq x < n + 1 \end{cases} \tag{9}$$

   for each $n \in \mathbb{Z}$. Then $f \in L^p$ but $f \notin L^q$. Thus L^q does not belong to L^p.