

# BIOS:7600 Homework 3

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## 1. Problem 1.8

(a) Show that

$$\lim_{\lambda \rightarrow 0^+} \frac{\partial}{\partial \lambda} \sum_j \text{Var}(\hat{\beta}_j) = -2 \frac{\sigma^2}{n} \sum_j d_j^{-2}.$$

*Proof.* Let  $\frac{1}{n} X^\top X = QDQ^\top$ . Then

$$\begin{aligned} \text{Var}(\hat{\beta}) &= \sigma^2 W X^\top X W = \sigma^2 \frac{1}{n} (QDQ^\top + n\lambda I)^{-1} n QDQ^\top \frac{1}{n} (QDQ^\top + n\lambda I)^{-1} \\ &= \sigma^2 \frac{1}{n} Q(D + \lambda I)^{-1} D(D + \lambda I)^{-1} Q^{-1} \\ &= \sigma^2 \frac{1}{n} Q \text{diag}\left\{\frac{d_i}{(d_i + \lambda)^2}\right\} Q^{-1}. \end{aligned}$$

Hence,

$$\frac{\partial}{\partial \lambda} \text{Var}(\hat{\beta}_j) = \frac{\partial}{\partial \lambda} \left( \frac{\sigma^2}{n} \frac{d_j}{(d_j + \lambda)^2} \right) = \frac{\sigma^2}{n} d_j \frac{-2d_j - 2\lambda}{(d_j + \lambda)^4},$$

and

$$\lim_{\lambda \rightarrow 0^+} \frac{\partial}{\partial \lambda} \sum_j \text{Var}(\hat{\beta}_j) = \lim_{\lambda \rightarrow 0^+} \sum_j \frac{\sigma^2}{n} d_j \frac{-2d_j}{d_j^4} = -2 \frac{\sigma^2}{n} \sum_j d_j^{-2}.$$

□

(b) Show that  $\lim_{\lambda \rightarrow 0^+} \frac{\partial}{\partial \lambda} \text{Bias}^2(\hat{\beta}) = 0$ .

*Proof.* First

$$\text{MSE}(\hat{\beta}) = \sum \text{Var}(\hat{\beta}_j) + \sum \text{Bias}^2(\hat{\beta}_j).$$

By (1.17) in <https://arxiv.org/pdf/1509.09169>; Lecture,

$$\text{MSE}(\hat{\beta}) = \sigma^2 \text{tr}\left\{\frac{1}{n} Q A^{-1} D A^{-1} Q^\top\right\} + \beta^\top Q (A^{-1} D - I)^\top (A^{-1} D - I) Q^\top \beta,$$

where  $A = D + \frac{1}{n}\lambda I$ . By (a) we know the first term is just  $\text{Var}(\hat{\beta})$ , so

$$\begin{aligned}\text{Bias}^2(\hat{\beta}) &= \beta^\top Q(A^{-1}D - I)^\top (A^{-1}D - I)Q^\top \beta \\ &= \alpha^\top \text{diag}\left\{\frac{\frac{1}{n^2}\lambda}{(d_i + \frac{1}{n}\lambda)^2}\right\}\alpha,\end{aligned}$$

where  $\alpha = Q^\top \beta$ . Then

$$\lim_{\lambda \rightarrow 0^+} \frac{\partial}{\partial \lambda} \text{Bias}^2(\hat{\beta}) = \lim_{\lambda \rightarrow 0^+} \alpha^\top \text{diag}\left\{\frac{\frac{2}{n^2}\lambda(d_i + \frac{1}{n}\lambda)^2 - \frac{2}{n^3}\lambda^2(d_i + \frac{1}{n}\lambda)^2}{(d_i + \frac{1}{n}\lambda)^4}\right\}\alpha = 0.$$

□

## 2. Problem 1.10

*Proof.* By the definition of linear fitting models,

$$\begin{aligned}y_i - \hat{f}_{(-i)}(x_i) &= y_i - \sum_{j=1, j \neq i}^n \tilde{s}_{ij} y_j = y_i - \sum_{j=1, j \neq i}^n \frac{s_{ij}}{1 - s_{ii}} y_j \\ &= \frac{1}{1 - s_{ii}} y_i - \sum_{j=1}^n \frac{s_{ij}}{1 - s_{ii}} y_j \\ &= \frac{y_i - \hat{f}(x_i)}{1 - s_{ii}}.\end{aligned}$$

Hence the equality holds.

□

## 3. Problem 1.11

(a) Based on OLS model, the most important variables are as follows:  
(ranked in descending order):

- hfa
- age
- aro
- wght
- afe
- inc
- whz

For ridge regression, we choose  $\lambda = 0.1482021$  based on the results of GCV (varies from  $10^{-5}$  to  $10^5$ ). The most important variables are:

- hfa
- age
- aro
- inc
- wght

- whz
  - rr
- (b) For OLS model, the regression coefficient for age is -0.4096032, while for ridge regression, the coefficient is -0.2893046. This is mainly from the significant correlation of **age** with other variables (which is similar with Example 1.2).
- (c) The coefficients for both variables with both models are as follows:  
I think the influence of both variables on pneumonia should look

	absu	afe
OLS	-0.02698383	0.1531287
Ridge	0.0357972	0.08959214

similar since **absu** and **afe** are similar behaviours. In this case, I think the estimates of ridge regression should be more reasonable.