Theoretical Numerical Analysis, Assignment 2

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1. Problem 3.1.5

   First, as f is continuous, we have

$$\lim_{t\to 0^+} f(-\log\frac{t}{a}) = \lim_{t\to\infty} f(t) = 0.$$

   Thus

$$g(t) = \begin{cases} f(-\log t/a), & 0 < t \le 1, \\ 0, & t = 0, \end{cases}$$

   is continuous. By Weierstrauss Approximation Theorem, for $\epsilon_n = \frac{1}{n}$, there is a polynomial p of degree d_n, s.t.

$$\|f(-\log t/a) - p(t)\|_\infty \le \epsilon_n.$$

   Let $x = -\log t/a$, then this is just

$$\|f(x) - \sum_{j=0}^{d_n} c_{n,j} e^{-jax}\|_\infty \le \epsilon_n.$$

   Then we can construct a list of desired funtions in a "stepwise" way, i.e., let $c_{m,j} = 0$ if $d_{n-1} < m \le d_n$ for q_n.

2. Problem 3.2.4

   We can see that

$$V_n(x_i) = 0, \quad 0 \le i \le n-1.$$

   Also, we can notice that V_n is a polynomial with degree up to n, so

$$V_n = c_n(x - x_0)\cdots(x - x_{n-1}).$$

   The coefficient of the highest term is

$$c_n = \det\begin{pmatrix} 1 & x_0 & \cdots & x_0^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-1} \end{pmatrix} = V_{n-1}(x_{n-1}).$$

   Thus,

$$\begin{aligned} V_n(x_n) &= V_{n-1}(x_{n-1})(x_n - x_0)\cdots(x_n - x_{n-1}) \\ &= V_{n-2}(x_{n-2})(x_{n-1} - x_0)\cdots(x_{n-1} - x_{n-2})(x_n - x_0)\cdots(x_n - x_{n-1}) \\ &= \cdots \\ &= \prod_{j>i}(x_j - x_i). \end{aligned}$$

3. Problem 3.3.8

By the properties of inner products, for $\lambda \in [0,1]$ and $u, v \in V$,

$$\begin{aligned} f(\lambda u + (1-\lambda)v) &= (\lambda u + (1-\lambda)v, \lambda u + (1-\lambda)v) \\ &= \lambda^2\|u\|^2 + (1-\lambda)^2\|v\|^2 + 2\lambda(1-\lambda)(u,v) \\ &\le \lambda^2\|u\|^2 + (1-\lambda)^2\|v\|^2 + 2\lambda(1-\lambda)\|u\|\|v\| \\ &= \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)(\|u\|^2 + \|v\|^2 - 2\|u\|\|v\|) \\ &= \lambda f(u) + (1-\lambda)f(v) - \lambda(1-\lambda)(\|u\| - \|v\|)^2 \\ &\le \lambda f(u) + (1-\lambda)f(v). \end{aligned}$$

When $u \neq v$, the last inequality is strict. Hence f is strictly convex.

4. Problem 3.3.9

   Suppose
   $$\|u+v\| = \|u\| + \|v\|,$$
   then
   $$\|u\|^2 + \|v\|^2 + 2\|u\|\|v\| = \|u\|^2 + \|v\|^2 + 2(u,v) \le \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|.$$
   By Cauchy-Schwarz inequality,
   $$u = v.$$
   Then the inner product space is strictly normed.

5. Problem 3.4.10

   $$p_n(x) = \sum_{i=0}^{n}(f,\phi_i)\phi_i = \sum_{i=0}^{n}(f,\cos jx)\cos jx + \sum_{i=1}^{n}(f,\sin jx)\sin jx,$$
   and
   $$(f,\cos jx) = \int_0^{2\pi} f(x)\cos(jx)dx,\ (f,\sin jx) = \int_0^{2\pi} f(x)\sin(jx)dx.$$
   Particularly,
   $$(f,1) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx.$$
   So the formula (3.4.9) is derived.

   For the Parseval's equality,
   $$\|f\|^2_{L^2(0,2\pi)} = \frac{1}{2}a_0^2 + \sum_{i=1}^{\infty}(a_j^2 + b_j^2).$$