BIOS:7600 Homework 3

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1. Problem 1.8

   (a) Show that
   $$\lim_{\lambda\to 0^+}\frac{\partial}{\partial\lambda}\sum_j \mathrm{Var}(\hat{\beta}_j) = -2\frac{\sigma^2}{n}\sum_j d_j^{-2}.$$

   Proof. Let $\frac{1}{n}X^\top X = QDQ^\top$. Then
   $$\begin{aligned}\mathrm{Var}(\hat{\beta}) &= \sigma^2 WX^\top XW = \sigma^2\frac{1}{n}(QDQ^\top + n\lambda I)^{-1} nQDQ^\top\frac{1}{n}(QDQ^\top + n\lambda I)^{-1}\\ &= \sigma^2\frac{1}{n}Q(D+\lambda I)^{-1}D(D+\lambda I)^{-1}Q^{-1}\\ &= \sigma^2\frac{1}{n}Q\mathrm{diag}\{\frac{d_i}{(d_i+\lambda)^2}\}Q^{-1}.\end{aligned}$$

   Hence,
   $$\frac{\partial}{\partial\lambda}\mathrm{Var}(\hat{\beta}_j) = \frac{\partial}{\partial\lambda}\left(\frac{\sigma^2}{n}\frac{d_j}{(d_j+\lambda)^2}\right) = \frac{\sigma^2}{n}d_j\frac{-2d_j-2\lambda}{(d_j+\lambda)^4},$$
   and
   $$\lim_{\lambda\to 0^+}\frac{\partial}{\partial\lambda}\sum_j \mathrm{Var}(\hat{\beta}_j) = \lim_{\lambda\to 0^+}\sum_j\frac{\sigma^2}{n}d_j\frac{-2d_j}{d_j^4} = -2\frac{\sigma^2}{n}\sum_j d_j^{-2}.$$
   □

   (b) Show that $\lim_{\lambda\to 0^+}\frac{\partial}{\partial\lambda}\mathrm{Bias}^2(\hat{\beta}) = 0$.

   Proof. First
   $$\mathrm{MSE}(\hat{\beta}) = \sum \mathrm{Var}(\hat{\beta}_j) + \sum \mathrm{Bias}^2(\hat{\beta}_j).$$
   By (1.17) in https://arxiv.org/pdf/1509.09169;Lecture,
   $$\mathrm{MSE}(\hat{\beta}) = \sigma^2\mathrm{tr}\{\frac{1}{n}QA^{-1}DA^{-1}Q^\top\} + \beta^\top Q(A^{-1}D-I)^\top(A^{-1}D-I)Q^\top\beta,$$

where $A = D + \frac{1}{n}\lambda I$. By (a) we knowthe first term is just $\text{Var}(\hat{\beta})$, so

$$\begin{aligned}\text{Bias}^2(\hat{\beta}) &= \beta^\top Q(A^{-1}D - I)^\top (A^{-1}D - I)Q^\top \beta \\ &= \alpha^\top \text{diag}\{\frac{\frac{1}{n^2}\lambda}{(d_i + \frac{1}{n}\lambda)^2}\}\alpha,\end{aligned}$$

where $\alpha = Q^\top \beta$. Then

$$\lim_{\lambda \to 0^+} \frac{\partial}{\partial \lambda}\text{Bias}^2(\hat{\beta}) = \lim_{\lambda \to 0^+} \alpha^\top \text{diag}\{\frac{\frac{2}{n^2}\lambda(d_i + \frac{1}{n}\lambda)^2 - \frac{2}{n^3}\lambda^2(d_i + \frac{1}{n}\lambda)^2}{(d_i + \frac{1}{n}\lambda)^4}\}\alpha = 0.$$

□

2. Problem 1.10

Proof. By the definition of linear fitting models,

$$\begin{aligned}y_i - \hat{f}_{(-i)}(x_i) &= y_i - \sum_{j=1, j \neq i}^{n} \tilde{s}_{ij} y_j = y_i - \sum_{j=1, j \neq i}^{n} \frac{s_{ij}}{1 - s_{ii}} y_j \\ &= \frac{1}{1 - s_{ii}} y_i - \sum_{j=1}^{n} \frac{s_{ij}}{1 - s_{ii}} y_j \\ &= \frac{y_i - \hat{f}(x_i)}{1 - s_{ii}}.\end{aligned}$$

Hence the equality holds.

□