PHYS:5905 Homework 8

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1. HW 8a, Problem 1

   (c) Let $u_j^n = \xi^n e^{ikj\Delta x}$, then

   $$\frac{1}{2\Delta t}(\xi^{n+1}e^{ikj\Delta x} - \xi^{n-1}e^{ikj\Delta x}) = -c\frac{1}{2\Delta x}(\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}).$$

   Then

   $$\xi^2 + c\frac{\Delta t}{\Delta x}2i\sin k\Delta x\xi - 1 = 0,$$

   we have

   $$\xi = -c\frac{\Delta t}{\Delta x}i\sin k\Delta x \pm \sqrt{1 - c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x}.$$

   (d) When

   $$1 - c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x \geq 0,$$

   $$|\xi| = \sqrt{c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x + 1 - c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x} = 1,$$

   so the leapfrog scheme is stable when

   $$c\frac{\Delta t}{\Delta x} \leq 1.$$

   On the other hand, when $c\frac{\Delta t}{\Delta x} > 1$, then for some $k > 0$,

   $$1 - c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x < 0,$$

   then

   $$|\xi| = |c\frac{\Delta t}{\Delta x}\sin k\Delta x| + |\sqrt{c^2\frac{\Delta t^2}{\Delta x^2}\sin^2 k\Delta x - 1}| > 1,$$

   so the algorithm is unstable. Hence, the stability condition is

   $$c\frac{\Delta t}{\Delta x} \leq 1.$$

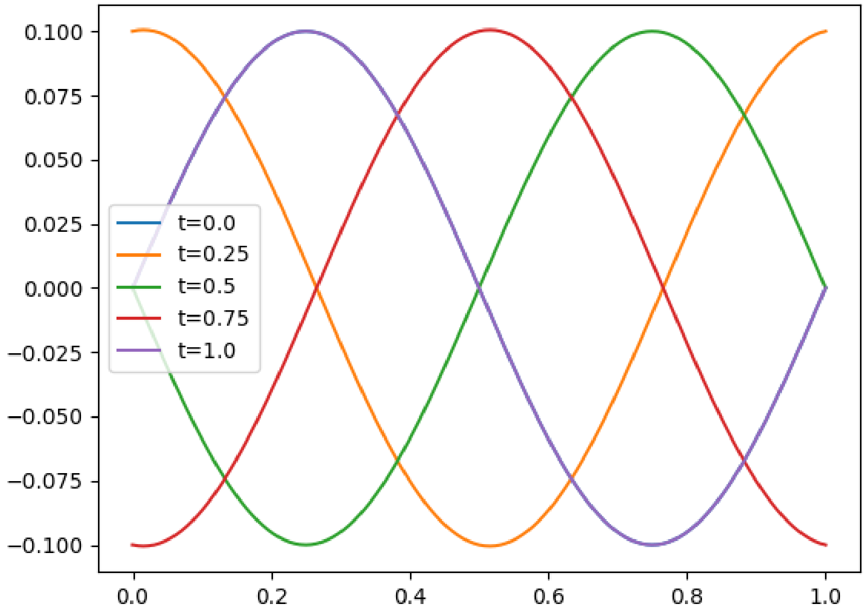

Figure 1: $u(x)$ at $t = 0, 0.25, 0.5, 0.75, 1.0$ with $n_x = 128$, $\Delta t = \frac{1}{128}$.

2. HW 8a, Problem 2

   (e) The result is shown in Figure 1.

   (f) The result is shown in Figure 2.

   (g) Since the stability condition for leapfrog method is

   $$|c|\frac{\Delta t}{\Delta x} \leq 1,$$

   which is satisfied for the different configurations, so the algorithm is stable.

   (h) Now we pick $n_x = 128$ and $\Delta t = 1/100$, and the plot is shown in Figure 3.

3. HW 8b, Problem 1

   (d) The plot is shown in Figure 4.

   (e) The graph is shown in Figure 5, where the fit is done by `scipy.optimize.curve_fit`.

   (f) The plot is in Figure 6.

   (g) The plot is in Figure 7.

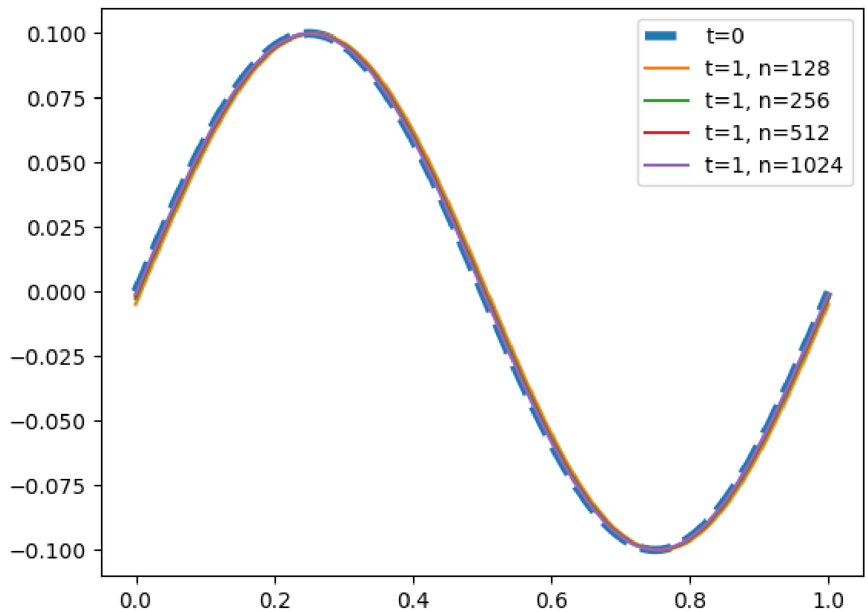

Figure 2: $u(x)$ at $t = 0$, and at $t = 1$ with $\Delta t = \frac{1}{128}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}$, while $n_x = 128$.

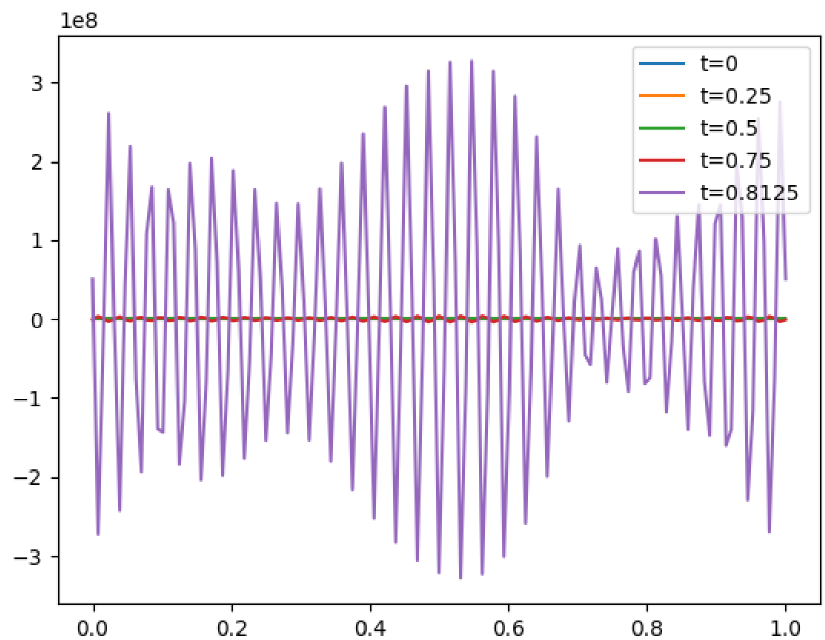

Figure 3: $u(x)$ at $t = 0, 0.25, 0.5, 0.75, 0.8125$ with $\Delta t = \frac{1}{100}$ and $n_x = 128$.

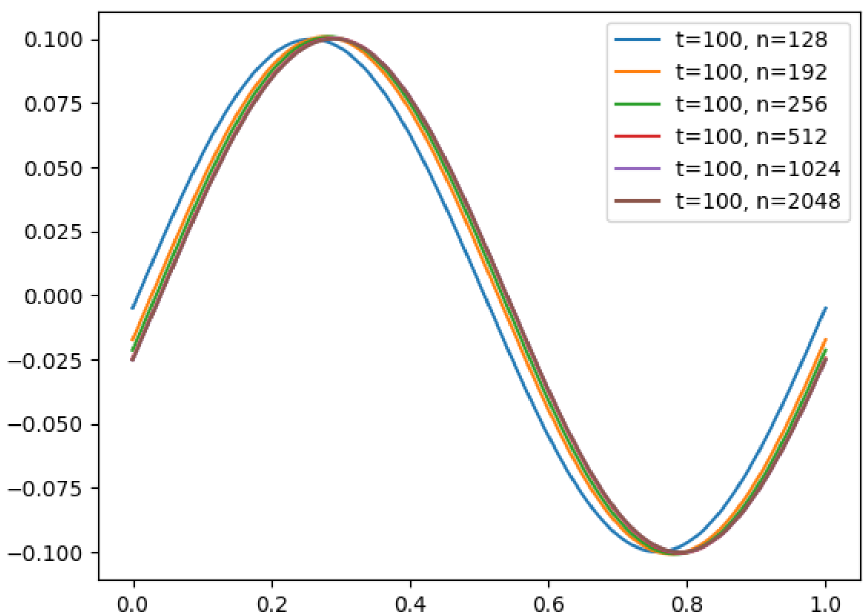

Figure 4: $u(x)$ at $t = 100$ with $\Delta t = \frac{1}{128}, \frac{1}{192}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048}$ and $n_x = 128$.

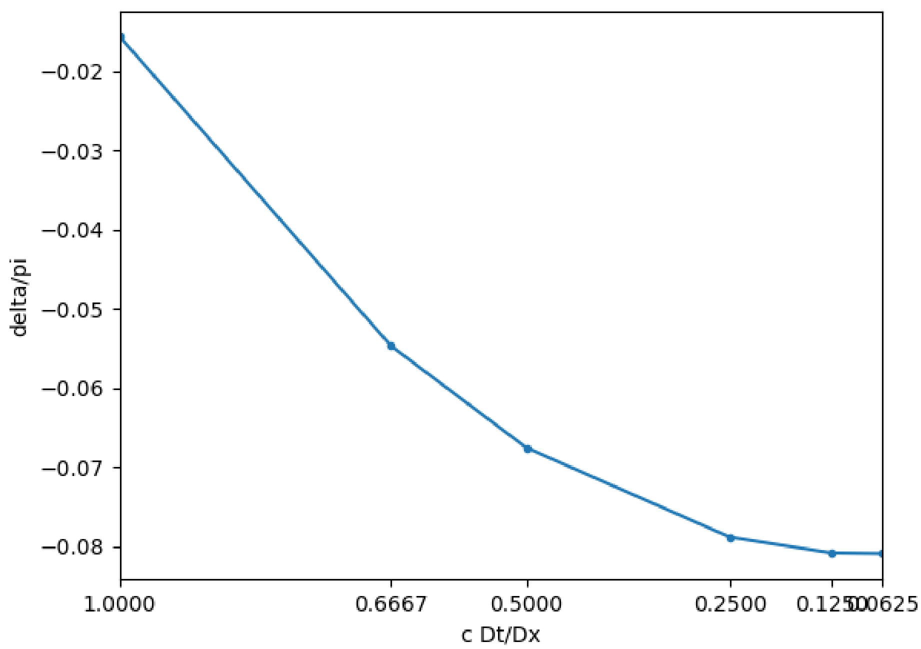

Figure 5: δ/π vs. $c\Delta t/\Delta x$ for $\Delta t = \frac{1}{128}, \frac{1}{192}, \frac{1}{256}, \frac{1}{512}, \frac{1}{1024}, \frac{1}{2048}$ and $n_x = 128$.

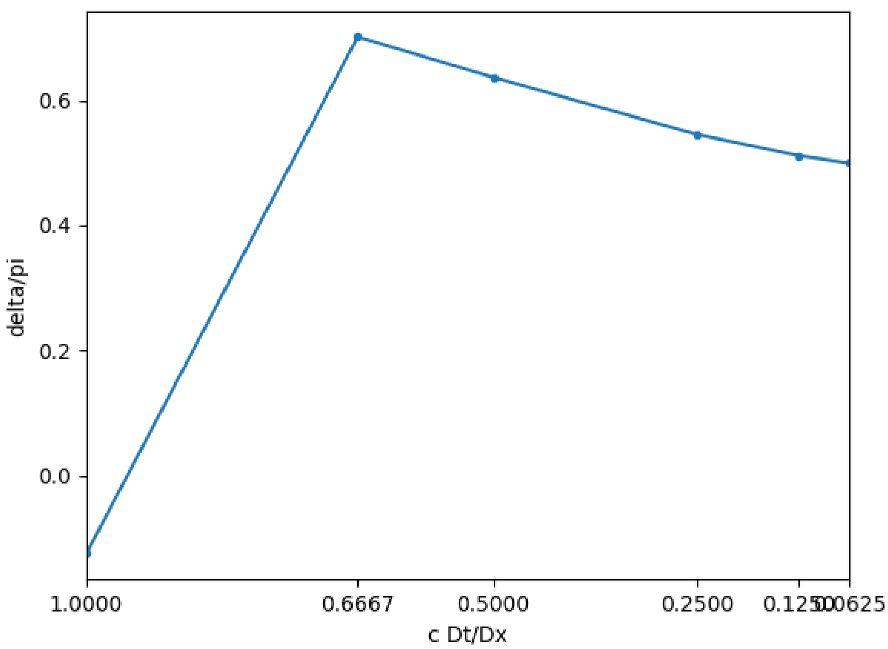

Figure 6: δ/π vs. $c\Delta t/\Delta x$ for $\Delta t = \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$ and $n_x = 16$.

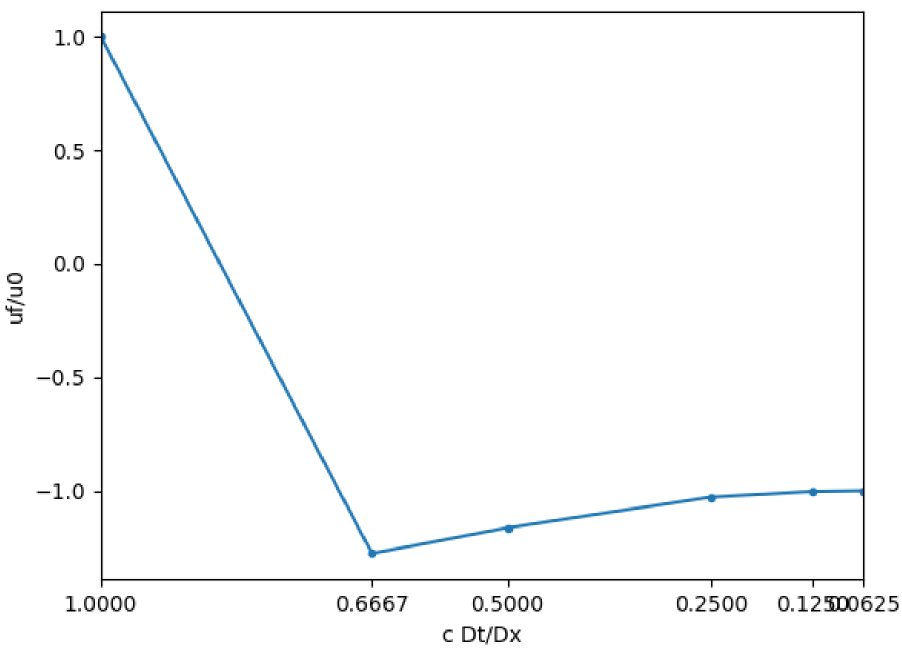

Figure 7: u_f/u_0 vs. $c\Delta t/\Delta x$ for $\Delta t = \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$ and $n_x = 16$.