Management Sciences Topics: Convex Optimization
Homework 1: Due Jan 30 (11:59 pm)

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(You can directly use any properties, theorems, examples or facts from the lectures.)

Problem 1: Are the following sets convex? You only need to answer yes or no and don't need to provide reasons.

a. $\{(x, y) \in \mathbb{R}^2 | |xy| \geq 1\}$.

   Nonconvex.

b. $\{(x, y) \in \mathbb{R}^2 | x \text{ is a postive interger}\}$.

   Nonconvex.

c. $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \text{ or } x + y \leq 0\}$.

   Nonconvex.

d. $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1 \text{ and } x + y \leq 0\}$.

   Convex.

e. $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$.

   Nonconvex.

f. The set of all copositive matrices: $\{X \in \mathbb{S}^n | \mathbf{u} \top X \mathbf{u} \geq 0 \text{ for any } \mathbf{u} \geq 0\}$.

   Convex.

g. Second order cone: $\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} | \|\mathbf{x}\|_2 \leq t\}$, where $\| \cdot \|_2$ represents the Euclidean norm.

   Convex.

h. The set of all rank-one $n \times n$ positive semi-definite matrices: $\{X \in \mathbb{S}^n | X = \mathbf{x}\mathbf{x}^\top, \mathbf{x} \in \mathbb{R}^n\}$.

   Nonconvex.

i. $\{X \in \mathbb{S}^n | -1 \leq \text{tr}X \leq 1\}$, where $\text{tr}X = \sum_{i=1}^n X_{ii}$ is the trace of X.

Convex.

Problem 2: Are the following functions convex? You need to provide reasons only if your answer is Yes.

a. $f(X,Y) = \lambda_{\max}(X - Y)$ where $X, Y \in \mathbb{S}^n$.

Yes. Since $X \in \mathbb{S}^n$, $g(X) = \lambda_{\max}(X) = \|X\|_2$ is convex, and $h(X,Y) = X - Y$ is affine. Hence $f(X,Y) = g(h(X,Y))$ is convex.

b. Hinge loss function: $f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n \max\{1 - b_i\mathbf{a}_i\top\mathbf{x}, 0\}$ where $\mathbf{a}_i \in \mathbb{R}^n$ is a feature vector and $b_i \in \{1, -1\}$ is the class label of instance i for $i = 1, \ldots, n$.

Yes. f is a composition of affine function and pointwise maximum.

c.

$$f(x) = \begin{cases} x\ln(x) & \text{if } 0 < x \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

Yes. In $(0,1]$, $f''(x) = \frac{1}{x} > 0$, and $f = \infty$ outside the interval.

d. Entropy function:

$$f(\mathbf{x}) = \begin{cases} \sum_{i=1}^n x_i \ln(x_i) & \text{if } \sum_{i=1}^n x_i = 1 \text{ and } x_i > 0 \text{ for all } i \\ +\infty & \text{otherwise} \end{cases}$$

Yes. It's just an affine combination of the function in problem c.

e. The density function of a standard univariate Gaussian distribution $\mathcal{N}(0,1)$.

No.

g. Sigmoid function: $f(x) = \frac{\exp(x)}{1+\exp(x)}$ for $x \in \mathbb{R}$.

No.

f. $f(x) = \min\{x, 0\}$ for $x \in \mathbb{R}$.

Yes. It's the pointwise min of two affine functions.

h. $f(x) = (\max\{x, 0\})^2$ for $x \in \mathbb{R}$.

Yes.

$$f(x) = \begin{cases} 0, & x < 0, \\ x^2, & x \geq 0. \end{cases}$$

And by definition we know f is convex.

i. $f(x) = (\max\{x, -1\})^2$ for $x \in \mathbb{R}$.

No.

j. $f(x) = h(g(x))$ where $g(x) = x^2$ and $h(x) = \mathbf{1}_{[1,2]}(x) = \begin{cases} 0 & \text{if } x \in [1,2] \\ +\infty & \text{otherwise} \end{cases}$,
i.e., the indicator function of $[0,1]$.

No.

Problem 3: Suppose f is an extended-real-valued function on $\mathbb{R}^n$. Show that f is convex if and only if $g_{\mathbf{x}}(t) := f(\mathbf{x} + t\mathbf{v})$ (as a function on $\mathbb{R}$) is convex for any $\mathbf{x}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

Proof. "$\Longleftarrow$": Suppose $g_x(t) = f(x + tv)$ is convex. For any $x, y \in \mathbb{R}^n$ and $0 \le \theta \le 1$, since

$$\theta g(s) + (1-\theta)g(t) \ge g(\theta s + (1-\theta)t),$$

we have

$$\theta f(x+sv) + (1-\theta)f(x+tv) \ge f(x + \theta sv + (1-\theta)tv) = f(x + v(\theta s + (1-\theta)t)).$$

Now let $s = 0, t = 1$, and $v = y - x$, we have

$$\theta f(x) + (1-\theta)f(y) \ge f(\theta x + (1-\theta)y).$$

"$\Longrightarrow$": Suppose f is convex, then for each $s, t \in \mathbb{R}$ and $0 \le \theta \le 1$,

$$\begin{aligned} \theta g(s) + (1-\theta)g(t) &= \theta f(x+sv) + (1-\theta)f(x+tv) \\ &\ge f(\theta(x+sv) + (1-\theta)(x+tv)) \\ &= f(x + v(\theta s + (1-\theta)t)) = g(\theta s + (1-\theta)t). \end{aligned}$$

Hence g is convex.

□