Solution Manual Euclid's "Elements" Redux by Daniel Callahan

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Contents

Chapter 1

Angles, Parallel Lines, Paralellograms

1.1 Proposition *CONSTRUCTING AN EQUILATERAL TRIANGLE.* (p.p. 33)

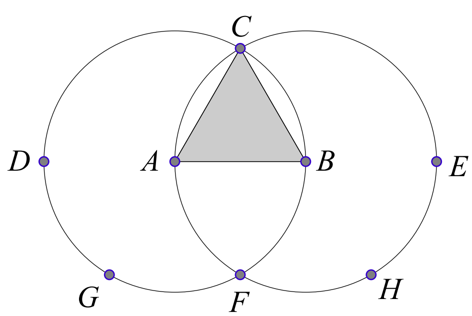

Figure 1.1.1: [1.1]

Exercise 1. If the segments $\overline{AF}$and $\overline{BF}$ are constructed, prove that the figure $\boxdot ACBF$ is a rhombus.

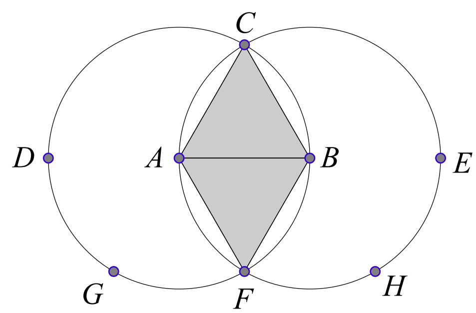

Figure 1.1.2:

Proof. Because A is the center of the circle $\bigcirc A$, $\overline{AF} = \overline{AB}$ [Def. 1.33]. Because B is the center of the circle $\bigcirc B$, $\overline{BF} = \overline{AB}$ [Def. 1.33]. Since $\overline{AF} = \overline{BF} = \overline{AB} = \overline{AC} = \overline{BC}$, it follows that $\boxdot ACBF$ is a rhombus. $\square$

Exercise 2. If $\overline{CF}$ is constructed and $\overline{AB}$ is extended to the circumferences of the circles (at points D and E), prove that the triangles $\triangle CDF$ and $\triangle CEF$ are equilateral.

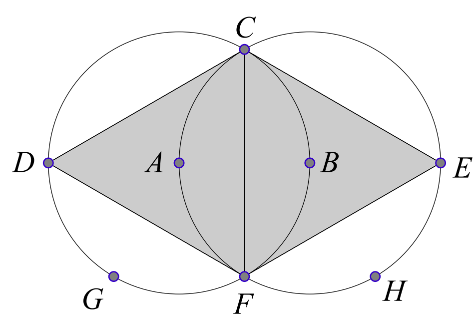

Figure 1.1.3:

Proof. Consider $\triangle CAD$, $\triangle DAF$ and $\triangle FAC$. Because A is the center of the circle $\bigcirc A$, $\overline{AC} = \overline{AD} = \overline{AF}$. Next, consider $\triangle ABC$ and $\triangle ABF$. By [1.32] and [1.32. Cor 6], $\angle CAD = \angle DAF = \angle FAC$. Since $\overline{AC} = \overline{AD} = \overline{AF}$ and $\angle CAD = \angle DAF = \angle FAC$, it follows that $\triangle CAD \cong \triangle DAF \cong \triangle FAC$ and $\overline{CD} = \overline{DF} = \overline{FC}$. So $\triangle CDF$. analogously it can be shown for $\triangle CEF$. $\square$

Exercise 3. 3. If $\overline{CA}$ and $\overline{CB}$ are extended to intersect the circumferences at points G and H, prove that the points G, F, H are collinear and that the triangle $\triangle GCH$ is equilateral.

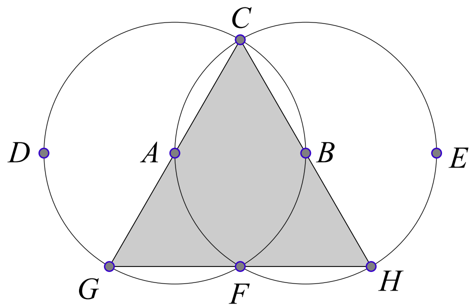

Figure 1.1.4:

Proof. By [1.1, #1], $\triangle ABF$ is equilateral. By [1.32. Cor 6], $\angle CAB = \angle BAF = \frac{2}{3}R = \angle CBA = \angle ABF$. Since $\angle CAG = 2R = \angle CBH$, it follows that $\angle GAF = \frac{2}{3}R = \angle FBH$. Consider $\triangle GAF$ and $\triangle FBH$. Because A is the center of the circle $\bigcirc A$, $\overline{AG} = \overline{AF}$ [Def. 1.33]. Because B is the center of the circle $\bigcirc B$, $\overline{BF} = \overline{BH}$ [Def. 1.33] . Since $\overline{AG} = \overline{AF}$ and $\overline{BF} = \overline{BH}$, it follows that $\triangle GAF$ and $\triangle FBH$ are isosceles. Since $\angle GAF = \frac{2}{3}R = \angle FBH$, it follows that $\angle AGF = \angle AFG = \frac{2}{3}R = \angle BFH = \angle BHF$. By [1.5, Cor 1], $\triangle GAF$ and $\triangle FBH$ are equilateral. Since $\angle AFG + \angle AFB + \angle BFH = 2R$, it follows that G, F and H are collinear. Consider $\triangle GCH$. Since $\overline{CG} = \overline{GH} = \overline{HC} = 2\overline{AB}$, it follows that $\triangle GCH$ is equilateral. □

Exercise 4. Construct $\overline{CF}$ and prove that $\overline{CF}^2 = 3 \cdot \overline{AB}^2$.

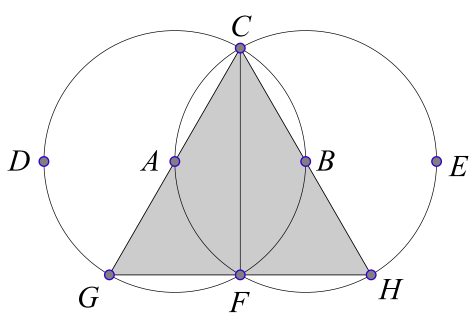

Figure 1.1.5:

Proof. Consider $\triangle CGF$ and $\triangle CFH$. Since $\overline{CG} = \overline{CH}$, $\overline{GF} = \overline{FH}$ and $\triangle CFH = \angle CHF$, it follows that $\triangle CGF \cong \triangle CFH$. Therefore, $\angle CFG = 1R = \angle CFH$ and $\triangle CGF \cong \triangle CFH$ are right triangles. By [1.47], $\overline{CF}^2 =$

$\overline{CH}^2 - \overline{FH}^2$. But, $\overline{CF} = 2{\cdot}\overline{AB}$ and $\overline{FH} = \overline{AB}$. So,

$$\begin{aligned}\overline{CF}^2 &= \overline{CH}^2 - \overline{FH}^2 \\ &= \left(2 \cdot \overline{AB}\right)^2 - \overline{AB}^2 \\ &= 4 \cdot \overline{AB}^2 - \overline{AB}^2 \\ &= 3 \cdot \overline{AB}.\end{aligned}$$

□

Exercise 5. Construct a circle in the space ACB bounded by the segment $\overline{AB}$ and the partial circumferences of the two circles.

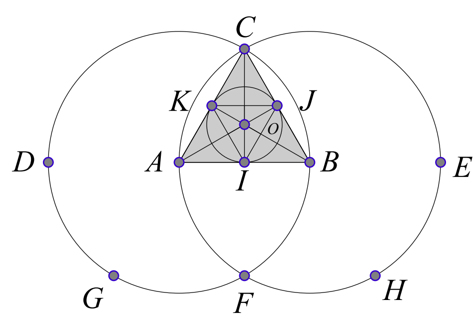

Figure 1.1.6:

Proof. Consider the midpoints I, J and K of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. Construct $\triangle IJK$. Next, construct the bisectors $\overline{CI}$, $\overline{AJ}$ and $\overline{BK}$ of $\triangle IJK$, with the vertices I, J and K respectively. Consider the circumcenter O of $\triangle IJK$. By [1.11, #5], $\overline{IO} = \overline{JO} = \overline{KO}$. Construct $\bigcirc O$ with radius $\overline{IO} = \overline{JO} = \overline{KO}$. Consider $\triangle AOI$ and $\triangle IOB$. Since $\overline{AI} = \overline{IB}$, $\angle AIO = 1R = \angle OIB$ and $\overline{IO}$ is the common side, by [1.4], $\triangle AOI \cong \triangle IOB$. Analogously for $\triangle BOJ \cong \triangle JOC \cong \triangle COK \cong \triangle KOA \cong \triangle AOI \cong \triangle IOB$. Consider $\triangle AOI$. Since $\angle AIO = 1R$, by [1.17, Cor 1], $\angle IAO < 1R$. By [1.19], $\overline{IO} < \overline{AO}$. For an arbitrary point P on $\overline{AI}$, $\overline{IO} < \overline{PO}$. Analogously for $\triangle BOJ \cong \triangle JOC \cong \triangle COK \cong \triangle KOA \cong \triangle AOI \cong \triangle IOB$. Therefore $\bigcirc O$ is inside of ACB. □

1.2 Proposition *CONSTRUCTING A LINE SEGMENT EQUAL IN LENGTH TO AN ARBITRARY LINE SEGMENT.* (p.p. 35)

Exercise 1. Prove [1.2] when A is a point on $\overline{BC}$.

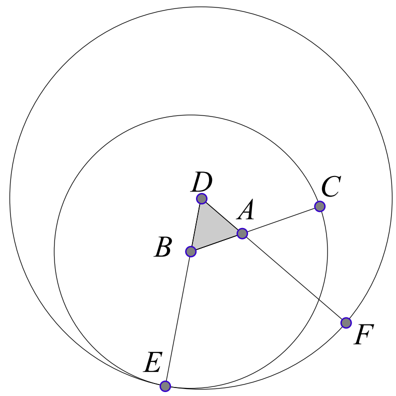

Figure 1.2.1:

Proof. By [1.1], construct the equilateral triangle $\triangle BAD$ on side $\overline{BA}$ and the circle $\bigcirc B$ with center on B and radius $\overline{BC}$ $\square$

1.3 Proposition *SUBDIVIDING A LINE SEGMENT.* (p.p. 37)

Exercise 1. Prove [Cor. 1.3.1].

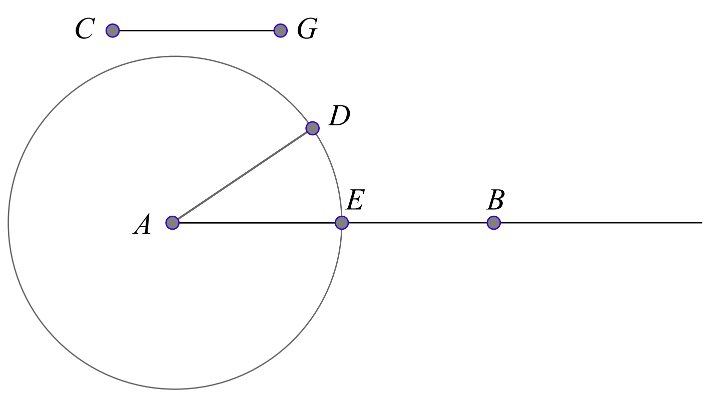

Figure 1.3.1:

Proof. The only condition for a ray $\overrightarrow{AB}$ to be this true is that $\overrightarrow{AB} > \overline{CG}$ for every arbitrary segment $\overline{CG}$. But that is true by [Def 1.4]. So the demonstration is analogous to [1.3]. $\square$

1.4 Proposition *THE "SIDE-ANGLE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES.* (p.p. 38)

Exercise 1. Prove that the line which bisects the vertical angle of an isosceles triangle also bisects the base perpendicularly.

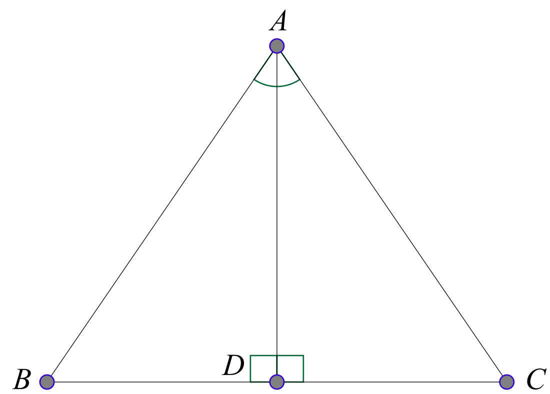

Figure 1.4.1:

Proof. Construct $\triangle ABC$ which is isosceles, $\overline{AB} = \overline{AC}$ and the bisector $\overline{AD}$ such that $\angle BAD = \angle CAD$. Consider $\triangle ABD$ and $\triangle ACD$. Since $\overline{AB} = \overline{AC}$, $\angle BAD = \angle CAD$ and AD is the common side, by [1.4], $\triangle ABD \cong \triangle ACD$. Therefore, $\overline{BD} = \overline{CD}$ and $\angle BDA = 1R = \angle CDA$. $\square$

Exercise 2. If two adjacent sides of a quadrilateral are equal in length and the diagonal bisects the angle between them, prove that their remaining sides are also equal in length.

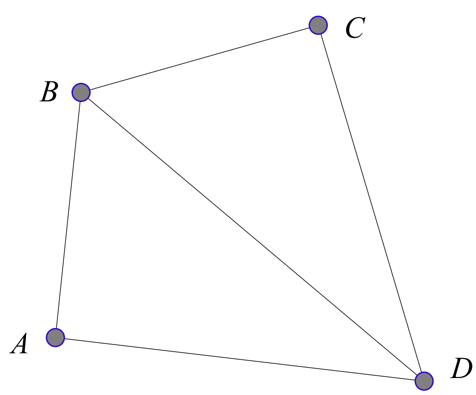

Figure 1.4.2:

Proof. Construct the quadrilateral $ABCD$ such that $\overline{AB} = \overline{BC}$ and $\angle ABD = \angle CBD$. Consider $\triangle ABD$ and $\triangle BCD$. Since $\overline{AB} = \overline{BC}$, $\angle ABD = \angle CBD$ and $\overline{BD}$ is the common side, by [1.4], $\triangle ABD \cong \triangle BCD$. Therefore, $\overline{AD} = \overline{CD}$. $\square$

Exercise 3. If two segments stand perpendicularly to each other and if each bisects the other, prove that any point on either segment is equally distant from the endpoints of the other segment.

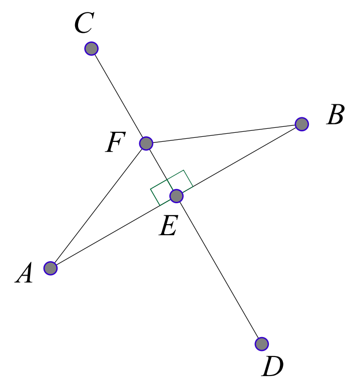

Figure 1.4.3:

Proof. Construct $\overline{AB} \perp \overline{CD}$ such that $\overline{AE} = \overline{EB}$ and $\overline{CE} = \overline{ED}$. With loss of generality, consider the arbitrary point F on CD. Consider $\triangle AFE$ and $\triangle BFE$. Since $\overline{AE} = \overline{EB}$, $\angle AEF = \angle BEF$ and $\overline{EF}$ is the common side, by [1.4], $\triangle AFE \cong \triangle BFE$. Therefore, $AF = FB$. $\square$

Exercise 4. If equilateral triangles are constructed on the sides of any triangle, prove that the distances between the vertices of the original triangle and the opposite vertices of the equilateral triangles are equal.

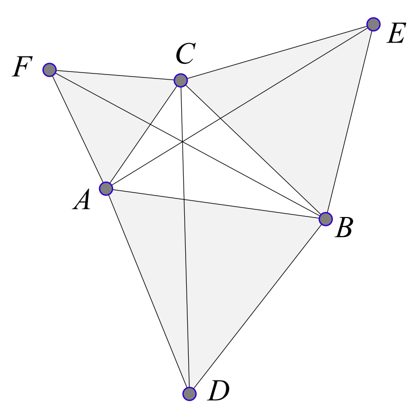

Figure 1.4.4:

Proof. Construct the triangle $\triangle ABC$ and construct the equilateral triangles $\triangle ABD$, $\triangle BCE$ and $\triangle CAF$. Construct the segments $\overline{AE}$, $\overline{BF}$ and $\overline{CD}$. Consider $\triangle BCD$ and $\triangle ABE$. Since $\overline{AB} = \overline{DB}$, $\overline{BC} = \overline{BE}$ and $\angle DBC = \angle DBA + \angle ABC = \angle ABC + \angle CBE = \angle ABE$, by [1.4], it follows that $\triangle BCD \cong \triangle ABE$. Therefore, $\overline{AE} = \overline{CD}$. Next, consider $\triangle FCB$ and $\triangle ACE$. Since $\overline{FC} = \overline{AC}$, $\overline{CB} = \overline{CE}$ and $\angle FCB = \angle FCA + \angle ACB = \angle ACB + \angle BCE = \angle ACE$, by [1.4], it follows that $\triangle FCB \cong \triangle ACE$. Therefore, $\overline{FB} = \overline{AE} = \overline{CD}$. □

1.5 Proposition *ISOSCELES TRIANGLES I.* (p.p. 40)

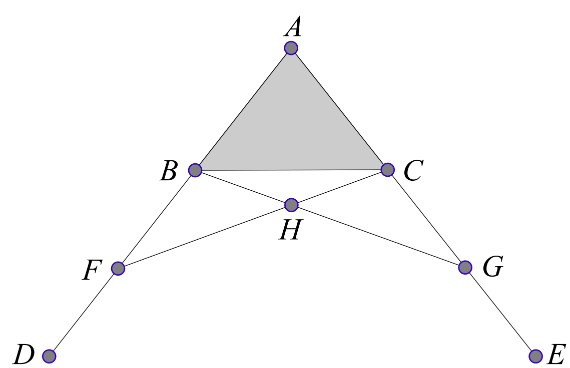

Figure 1.5.1:

Exercise 1. Prove that the angles at the base are equal without extending the sides.

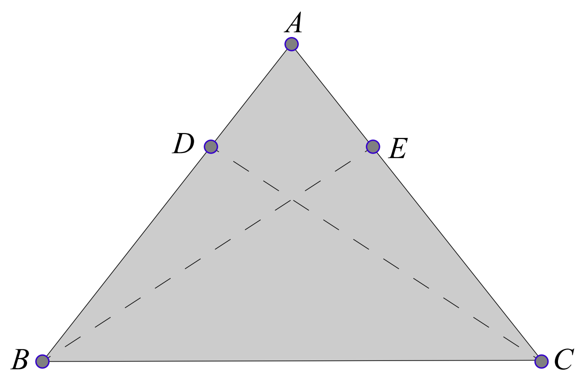

Figure 1.5.2:

Proof. By [1.3], construct on $\overline{AB}$ and $\overline{AC}$ respectively $\overline{AD} = \overline{AE}$. Consider $\triangle ADC$ and $\triangle AEB$. Since $\overline{AB} = \overline{AC}$, $\overline{AD} = \overline{AE}$ and $\angle BAC$ is the common angle, by [1.4], it follows that $\triangle ADC \cong \triangle AEB$. Therefore, $\overline{DC} = \overline{BE}$ and $\angle BEA = \angle CDA$. Since, $\angle BEA = \angle CDA$, it follows that $\angle BDC = \angle CEB$. Consider $\triangle BDC$ and $\triangle CEB$. Since $\overline{DC} = \overline{BE}$, $\overline{DB} = \overline{EC}$ and $\angle BEA = \angle CDA$, by [1.4], it follows that $\triangle BDC \cong \triangle CEB$. Therefore, $\angle BCA = \angle CBA$. □

Exercise 2. Prove that $\overleftrightarrow{AH}$ is an Axis of Symmetry of $\triangle ABC$.

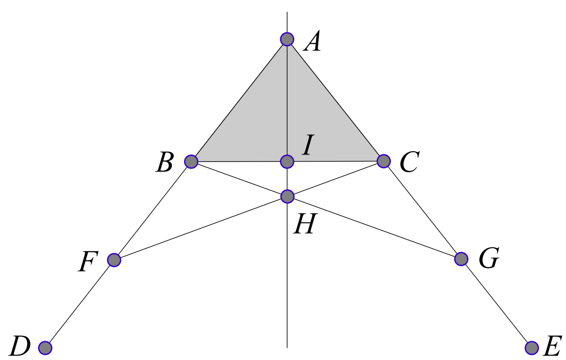

Figure 1.5.3:

Proof. Consider $\triangle BCF$ and $\triangle BCG$. By [1.5], $\angle DBC = \angle ECB$. Since $\overline{FB} = \overline{CG}$, $\angle DBC = \angle ECB$ and $\overline{BC}$ is the common side, by [1.4], it follows that $\triangle BCF \cong \triangle BCG$. Therefore, $\angle GBC = \angle FCB$. Consider $\triangle BCH$. Since $\angle GBC = \angle FCB$, by [1.6, Cor 1], it follows that the triangle $\triangle BCH$ is isosceles. Therefore, $\overline{BH} = \overline{CH}$. Construct $\overleftrightarrow{AH}$ and consider $\triangle ABH$ and $\triangle ACH$. Since

$\overline{BH} = \overline{CH}$, $\overline{AB} = \overline{AC}$ and $\angle ABH = \angle ABC + \angle GBC = \angle FCB + \angle ACB = \angle ACH$, by [1.4], it follows that $\triangle ABH \cong \triangle ACH$. Therefore, $\angle BHA = \angle CHA$. Construct the point I where $\overleftrightarrow{AH}$ intersect with $\overline{BC}$ and consider $\triangle BHI$ and $\triangle CHI$. Since $\overline{BH} = \overline{CH}$, $\angle BHA = \angle CHA$ and $\overline{HI}$ is the common side, by [1.4], it follows that $\triangle BHI \cong \triangle CHI$. Therefore, $\triangle ABI \cong \triangle ACI$ and $\overleftrightarrow{AH}$ is an Axis of Symmetry of $\triangle ABC$. □

Exercise 3. Prove that each diagonal of a rhombus is an Axis of Symmetry of the rhombus.

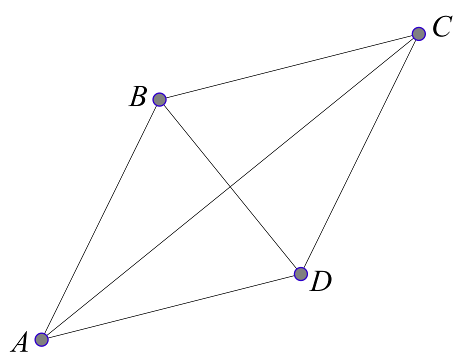

Figure 1.5.4:

Proof. Consider the rhombus $\boxdot ABCD$. Now, consider the triangles $\triangle ABC$ and $\triangle ADC$. Since $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA}$ and $\overline{AC}$ is the common side, by [1.8], it follows that $\triangle ABC \cong \triangle ADC$. Therefore, $\overline{AC}$ is an Axis of Symmetry of $\boxdot ABCD$. The proof for $\overline{BD}$ is analogous. □

Exercise 4. Take the midpoint on each side of an equilateral triangle; the segments joining them form a second equilateral triangle.

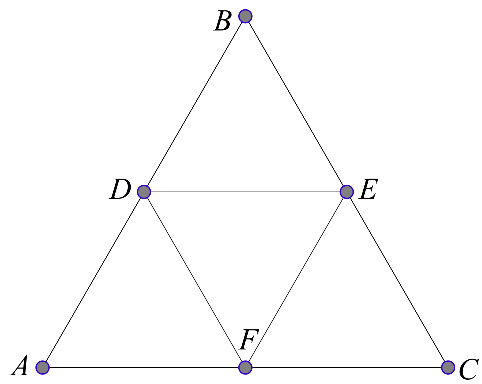

Figure 1.5.5:

Proof. Consider $\triangle ADF$, $\triangle DBE$ and $\triangle FEC$. By [1.5, Cor 1], $\angle DAF = \angle DBE = \angle ECF$. Since $AD = AF = DB = BE = EC = CF$ and $\angle DAF = \angle DBE = \angle ECF$, it follows that $\triangle ADF \cong \triangle DBE \cong \triangle DBE$. Therefore, $DE = EF = FD$ and $\triangle DEF$ is isosceles. $\square$

Exercise 5. Prove [Cor. 1.5.1].

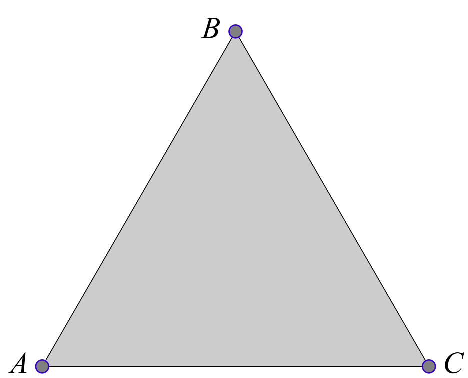

Figure 1.5.6:

Proof. Consider the equilateral triangle $\triangle ABC$. Since $AB = BC$, it follows that $\triangle ABC$ is isosceles.Therefore, by [1.5], $\angle A = \angle C$. But also $BC = CA$. Therefore, $\angle B = \angle C = \angle A$. (Claim 1)

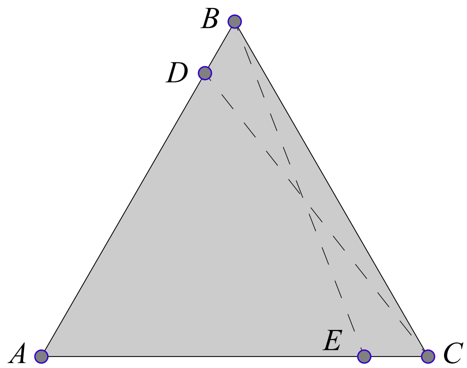

Figure 1.5.7:

Consider the equiangular triangle $\triangle ABC$. Suppose $AB > BC$. By [1.3], construct $AD = BC$. Consider $\triangle ADC$ and $\triangle ABC = \triangle ADC + \triangle DBC$. Since $AD = BC$, $\angle A = \angle C$ and AC is the common side, by [1.4], it follows that $\triangle ADC \cong \triangle ABC$. Meaning that $\triangle DBC = 0$, a contradiction. So, $AB = BC$. The proof for $AC = AB = BC$ is analogous. $\square$

1.6 Proposition *ISOSCELES TRIANGLES II.* (p.p. 42)

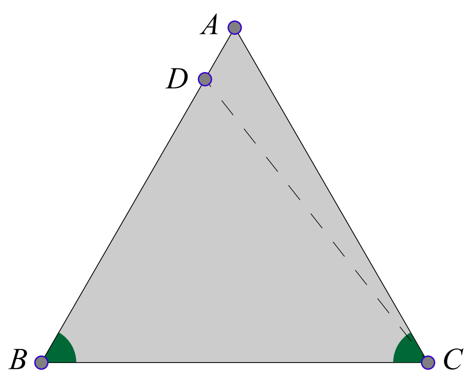

Figure 1.6.1:

Exercise 1. Prove [Cor. 1.6.1].

Proof. Consider the isosceles triangle $\triangle ABC$. By [1.5], $\angle B = \angle C$. (Claim 1).

Consider the triangle $\triangle ABC$ such that $\angle B = \angle C$. By [1.6], $AB = AC$. (Claim 2). $\square$

1.7 Proposition *DISTINCT TRIANGLES.* (p.p. 44)

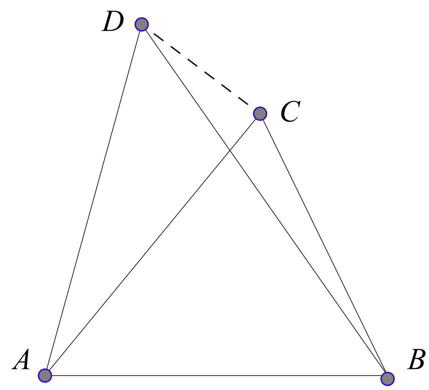

Figure 1.7.1:

Exercise 1. Prove [Cor. 1.7.1].

Proof. Consider the triangles $\triangle ABC$ and $\triangle DEF$ such that no side is equal. Suppose that $\triangle ABC \cong \triangle DEF$. But, by [1.4], means that at least have two pairs of sides equal to each other. A contradiction. $\square$

1.8 Proposition *THE "SIDE-SIDE-SIDE" THEOREM FOR THE CONGRUENCE OF TRIANGLES.* (p.p. 46)

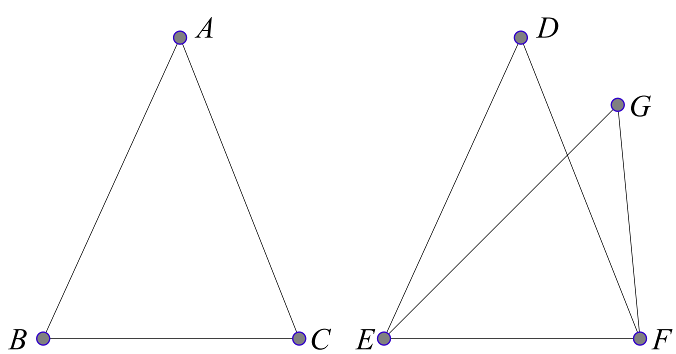

Figure 1.8.1:

1.9 Proposition *BISECTING A RECTILINEAR ANGLE.* (p.p. 47)

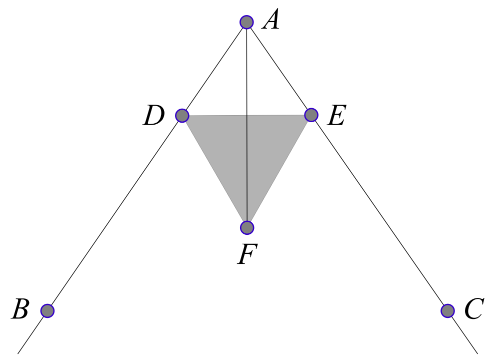

Figure 1.9.1:

Exercise 1. Prove [1.9] without using [1.8]. (Hint: use [1.5, #2].)

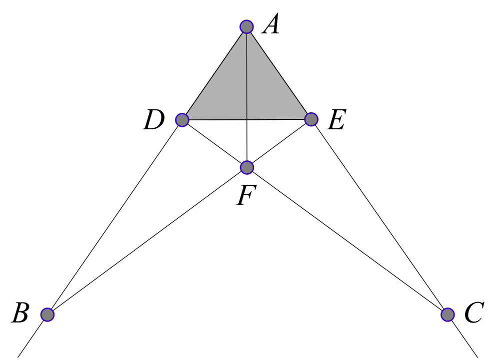

Figure 1.9.2:

Proof. By [1.3], construct $AD = AE$ and $DB = EC$. Consider $\triangle ADE$, $\triangle BEA$ and $\triangle CDA$. Since $\triangle ADE$ es isosceles and $\triangle BEA \cong \triangle CDA$, by [1.5, #2], it follows that AF is the axis of symmetry of $\triangle ADE$. Therefore, $\angle BAF = \angle CAF$. $\square$

Exercise 2. Prove that $AF \perp DE$. (Hint: use [1.5, #2].)

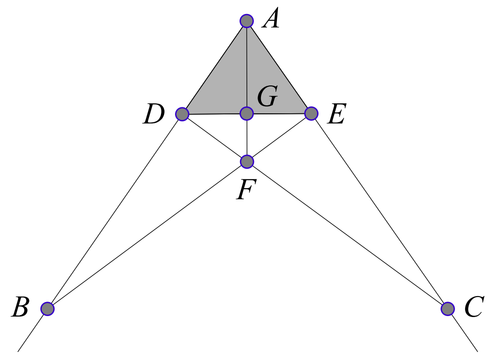

Figure 1.9.3:

Proof. By [1.9, #1], AF is the axis of symmetry of $\triangle ADE$. Therefore, $\angle DGA = \angle EGA$. Meaning that $AF \perp DE$. $\square$