设 $\{X(t)\}$ 为 Gauss 平稳过程，均值为 0，$R_X(\tau)=Ae^{-a|\tau|}cos\beta\tau$ ，令 $Y(t)=X^2(t)$，
验证 $R_Y(\tau)=A^2e^{-2a|\tau|}(1+cos2\beta\tau)$，并求出 $S_Y(\omega)$。
解：

$$EY(t)=EX^2(t)=R_X(0)+[EX(t)]^2=A$$

因为，若 (X_1,X_2,X_3,X_4) 是均值为 0 的联合正态随机变量，则：

$$EX_1X_2X_3X_4=Cov(X_1,X_2)Cov(X_3,X_4)+Cov(X_1,X_3)Cov(X_2,X_4)+Cov(X_1,X_4)Cov(X_2,X_3)$$

设 $Z(t)=X(t)X(t+\tau)$

$$\begin{aligned}R_Z(\tau_1)&=EZ(t+\tau_1)Z(t)-EZ(t+\tau_1)EZ(t)\\&=EX(t+\tau_1)X(t+\tau+\tau_1)X(t)X(t+\tau)-R_X^2(\tau)\\&=Cov(X(t+\tau_1),X(t+\tau+\tau_1))Cov(X(t),X(t+\tau))\\&+Cov(X(t+\tau_1),X(t))Cov(X(t+\tau+\tau_1),X(t+\tau))\\&+Cov(X(t+\tau_1),X(t+\tau))Cov(X(t+\tau+\tau_1),X(t))-R_X^2(\tau)\\&=R_X^2(\tau)+R_X^2(\tau_1)+R_X(\tau_1-\tau)R_X(\tau_1+\tau)-R_X^2(\tau)\\&=R_X^2(\tau_1)+R_X(\tau_1-\tau)R_X(\tau_1+\tau)\end{aligned}\tag{1}$$

$$\begin{aligned}R_Y(\tau)&=E(Y(t)-A)(Y(t+\tau)-A)\\&=E(Y(t)Y(t+\tau))-AE(Y(t))-AE(Y(t+\tau))+A^2\\&=E(Y(t)Y(t+\tau))-A^2\\&=E(X^2(t)X^2(t+\tau))-A^2\\&=Var(Z(t))+[E(X(t)X(t+\tau))]^2-A^2\\&=A^2+R_X^2(\tau)+R_X^2(\tau)-A^2\\&=2R_X^2(\tau)\\&=A^2e^{-2a|\tau|}(1+cos2\beta\tau)\end{aligned}\tag{2}$$

$$
\begin{aligned}
S_Y(\omega) &= \int_{-\infty}^{+\infty} R(\tau)e^{-i\omega\tau}d\tau \\
&= \int_{0}^{+\infty} A^2 e^{-(2a+i\omega)\tau}(1+2cos2\beta\tau)d\tau + \int_{-\infty}^{0} A^2 e^{(2a-i\omega)\tau}(1+2cos2\beta\tau)d\tau \\
&= \int_{0}^{+\infty} A^2 e^{-(2a+i\omega)\tau} \\
&+ \int_{0}^{+\infty} \frac{A^2}{2} e^{-(2a+i\omega)\tau}(e^{-2i\beta\tau} + e^{2i\beta\tau})d\tau \\
&- \int_{-\infty}^{0} A^2 e^{-(2a-i\omega)\tau} \\
&- \int_{-\infty}^{0} \frac{A^2}{2} e^{-(2a-i\omega)\tau}(e^{-2i\beta\tau} + e^{2i\beta\tau})d\tau \\
&= A^2\frac{1}{2a+i\omega} + \frac{A^2}{2}(\frac{1}{2a+2\beta+i\omega} + \frac{1}{2a-2\beta+i\omega}) \\
&- A^2\frac{1}{2a-i\omega} - \frac{A^2}{2}(\frac{1}{2a+2\beta-i\omega} + \frac{1}{2a-2\beta-i\omega}) \\
&= A^2\frac{-2i\omega}{4a^2+\omega^2} + \frac{A^2}{2}(\frac{-2i\omega}{(2a+2\beta)^2+\omega^2} + \frac{-2i\omega}{(2a-2\beta)^2+\omega^2})
\end{aligned} \tag{3}
$$