Linear Algebra and its Applications 446 (2014) 269–280

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The inverse, rank and product of tensors

Changjiang Bu[a], Xu Zhang[a], Jiang Zhou[a,b], Wenzhe Wang[a], Yimin Wei[c]

[a] College of Science, Harbin Engineering University, Harbin 150001, PR China
[b] College of Computer Science and Technology, Harbin Engineering University, Harbin 150001, PR China
[c] School of Mathematical Sciences and Shanghai Key Laboratory of Contemporary Applied Mathematics, Fudan University, Shanghai 200433, PR China

A R T I C L E   I N F O

Article history:
Received 25 September 2013
Accepted 9 December 2013
Available online 7 January 2014
Submitted by R. Brualdi

MSC:
15A69
15A09
15A03

Keywords:
Tensor
Left (right) inverse
Tensor rank
Product
Hypergraph

A B S T R A C T

In this paper, we give some basic properties for the left (right) inverse, rank and product of tensors. The existence of order 2 left (right) inverses of tensors is characterized. We obtain some equalities and inequalities on the tensor rank. We also show that the rank of a uniform hypergraph is independent of the ordering of its vertices, and the Laplacian tensor and the signless Laplacian tensor have the same rank for odd-bipartite even uniform hypergraphs.

1. Introduction

For a positive integer n, let $[n] = \{1, \ldots, n\}$. An order k tensor $\mathcal{A} = (a_{i_1 \cdots i_k}) \in \mathbb{C}^{n_1 \times \cdots \times n_k}$ is a multidimensional array with $n_1 \cdots n_k$ entries, where $i_j \in [n_j]$, $j = 1, \ldots, k$.

E-mail address: buchangjiang@hrbeu.edu.cn (C. Bu).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.12.015

We sometimes write $a_{i_1\cdots i_k}$ as $a_{i_1\alpha}$, where $\alpha = i_2\cdots i_k$. When $k=2$, $\mathcal{A}$ is an $n_1\times n_2$ matrix. If $n_1=\cdots=n_k=n$, then $\mathcal{A}$ is an order k dimension n tensor. Recently the research on tensors has attracted extensive attention [1,2,4–6,10–13].

Now we introduce the following product of tensors.

Definition 1.1. Let $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ and $\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$ be order $m\geqslant 2$ and $k\geqslant 1$ tensors, respectively. The product $\mathcal{AB}$ is the following tensor $\mathcal{C}$ of order $(m-1)(k-1)+1$ with entries:

$$c_{i\alpha_1\ldots\alpha_{m-1}} = \sum_{i_2,\ldots,i_m\in[n_2]} a_{ii_2\ldots i_m} b_{i_2\alpha_1}\cdots b_{i_m\alpha_{m-1}},$$

where $i\in[n_1]$, $\alpha_1,\ldots,\alpha_{m-1}\in[n_3]\times\cdots\times[n_{k+1}]$.

In the above definition, if $n_1=n_2=\cdots=n_{k+1}=n$, then $\mathcal{AB}$ is the tensor product introduced in [4,11]. The tensor product defined in Definition 1.1 has the following properties:

(1) $(\mathcal{A}_1+\mathcal{A}_2)\mathcal{B}=\mathcal{A}_1\mathcal{B}+\mathcal{A}_2\mathcal{B}$, where $\mathcal{A}_1,\mathcal{A}_2\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$, $\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$.
(2) $A(\mathcal{B}_1+\mathcal{B}_2)=A\mathcal{B}_1+A\mathcal{B}_2$, where $A\in\mathbb{C}^{n_1\times n_2}$, $\mathcal{B}_1,\mathcal{B}_2\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$.
(3) $\mathcal{A}I_{n_2}=\mathcal{A}$, $I_{n_2}\mathcal{B}=\mathcal{B}$, where $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$, $\mathcal{B}\in\mathbb{C}^{n_2\times\cdots\times n_{k+1}}$, I_{n_2} is the identity matrix of order n_2.
(4) $\mathcal{A}(\mathcal{BC})=(\mathcal{AB})\mathcal{C}$, where $\mathcal{A}\in\mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$, $\mathcal{B}\in\mathbb{C}^{n_2\times n_3\times\cdots\times n_3}$, $\mathcal{C}\in\mathbb{C}^{n_3\times\cdots\times n_r}$.

Clearly parts (1)–(3) follow from Definition 1.1. Part (4) will be proved at the beginning of Section 2.

The *unit tensor* of order m and dimension n is the tensor $\mathcal{I}=(\delta_{i_1i_2\cdots i_m})$ such that $\delta_{i_1i_2\cdots i_m}=1$ if $i_1=i_2=\cdots=i_m$, and $\delta_{i_1i_2\cdots i_m}=0$ otherwise.

Definition 1.2. Let $\mathcal{A}$ be a tensor of order m and dimension n, and let $\mathcal{B}$ be a tensor of order k and dimension n. If $\mathcal{AB}=\mathcal{I}$, then $\mathcal{A}$ is called an order m left inverse of $\mathcal{B}$, and $\mathcal{B}$ is called an order k right inverse of $\mathcal{A}$.

The *Segre outer product* of $a_1\in\mathbb{C}^{n_1},\ldots,a_k\in\mathbb{C}^{n_k}$, denoted by $a_1\otimes\cdots\otimes a_k$, is the tensor $\mathcal{A}\in\mathbb{C}^{n_1\times\cdots\times n_k}$ with entries $a_{i_1\cdots i_k}=(a_1)_{i_1}\cdots(a_k)_{i_k}$. A tensor $\mathcal{A}\in\mathbb{C}^{n_1\times\cdots\times n_k}$ is said to have *rank one* if there exist nonzero $a_i\in\mathbb{C}^{n_i}$ $(i=1,\ldots,k)$ such that $\mathcal{A}=a_1\otimes\cdots\otimes a_k$. The *rank* of a tensor $\mathcal{A}$, denoted by $\operatorname{rank}(\mathcal{A})$, is defined to be the smallest r such that $\mathcal{A}$ can be written as a sum of r rank one tensors. If $\mathcal{A}=0$, then $\operatorname{rank}(\mathcal{A})=0$ (see [8]).

In this paper, some basic properties for order 2 left (right) inverse and product of tensors are given. We also obtain some results on rank of tensors and hypergraphs.

2. Preliminaries

In this section, we first use a method similar with the proof of [11, Theorem 1.1] to show the associative law $\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\mathcal{C}$, where $\mathcal{A} \in \mathbb{C}^{N_1\times N_2\times\cdots\times N_2}$, $\mathcal{B} \in \mathbb{C}^{N_2\times N_3\times\cdots\times N_3}$ and $\mathcal{C} \in \mathbb{C}^{N_3\times N_4\times\cdots\times N_{r+3}}$ are order $m+1$, $k+1$ and $r+1$ tensors, respectively. In order to prove $\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\mathcal{C}$, we need the following identity (see Eq. (1.4) in [11]):

$$\prod_{j=1}^{m} \sum_{t_{j1},\ldots,t_{jk}\in[N_3]} f(j,t_{j1},\ldots,t_{jk}) = \sum_{t_{jh}\in[N_3](1\leqslant j\leqslant m;\ 1\leqslant h\leqslant k)} \prod_{j=1}^{m} f(j,t_{j1},\ldots,t_{jk}), \qquad (1)$$

where $f(j,t_{j1},\ldots,t_{jk})$ is a complex-valued function with respect to indices $j,t_{j1},\ldots,t_{jk}$.

For $\beta_1,\ldots,\beta_m \in ([N_4]\times\cdots\times[N_{r+3}])^k$, we write $\beta_1 = \theta_{11}\cdots\theta_{1k},\ldots,\beta_m = \theta_{m1}\cdots\theta_{mk}$, where $\theta_{ij} \in [N_4]\times\cdots\times[N_{r+3}]$, $i=1,\ldots,m$, $j=1,\ldots,k$. By Definition 1.1, we have

$$\begin{aligned}
&\big(\mathcal{A}(\mathcal{B}\mathcal{C})\big)_{i\beta_1\ldots\beta_m}\\
&\quad= \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\left(\prod_{j=1}^{m}(\mathcal{B}\mathcal{C})_{i_j\beta_j}\right)\\
&\quad= \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\left(\prod_{j=1}^{m}(\mathcal{B}\mathcal{C})_{i_j\theta_{j1}\cdots\theta_{jk}}\right)\\
&\quad= \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\left(\prod_{j=1}^{m}\sum_{t_{j1},\ldots,t_{jk}\in[N_3]} b_{i_jt_{j1}\cdots t_{jk}}(c_{t_{j1}\theta_{j1}}\cdots c_{t_{jk}\theta_{jk}})\right)\\
&\quad= \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\sum_{t_{jh}\in[N_3](1\leqslant j\leqslant m;\ 1\leqslant h\leqslant k)}\left(\prod_{j=1}^{m} b_{i_jt_{j1}\cdots t_{jk}}(c_{t_{j1}\theta_{j1}}\cdots c_{t_{jk}\theta_{jk}})\right), \qquad (2)
\end{aligned}$$

where the last equation follows from Eq. (1).

For $\alpha_1,\ldots,\alpha_m \in [N_3]^k$, we write $\alpha_1 = t_{11}\cdots t_{1k},\ldots,\alpha_m = t_{m1}\cdots t_{mk}$ ($t_{ij}\in[N_3]$, $i=1,\ldots,m$; $j=1,\ldots,k$). By Definition 1.1, we have

$$\begin{aligned}
&\big((\mathcal{A}\mathcal{B})\mathcal{C}\big)_{i\beta_1\ldots\beta_m}\\
&\quad= \sum_{\alpha_1,\ldots,\alpha_m\in[N_3]^k} (\mathcal{A}\mathcal{B})_{i\alpha_1\ldots\alpha_m}\left(\prod_{j=1}^{m}(c_{t_{j1}\theta_{j1}}\cdots c_{t_{jk}\theta_{jk}})\right)\\
&\quad= \sum_{t_{jh}\in[N_3](1\leqslant j\leqslant m;1\leqslant h\leqslant k)}\ \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\left(\prod_{j=1}^{m} b_{i_j\alpha_j}\right)\left(\prod_{j=1}^{m}(c_{t_{j1}\theta_{j1}}\cdots c_{t_{jk}\theta_{jk}})\right)\\
&\quad= \sum_{i_1,\ldots,i_m\in[N_2]} a_{ii_1\cdots i_m}\sum_{t_{jh}\in[N_3](1\leqslant j\leqslant m;\ 1\leqslant h\leqslant k)}\left(\prod_{j=1}^{m} b_{i_jt_{j1},\ldots,t_{jk}}(c_{t_{j1}\theta_{j1}}\cdots c_{t_{jk}\theta_{jk}})\right). \qquad (3)
\end{aligned}$$

Comparing Eqs. (2) and (3), we get $\mathcal{A}(\mathcal{B}\mathcal{C}) = (\mathcal{A}\mathcal{B})\mathcal{C}$.

Lemma 2.1. *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ be an order m tensor, and let $\mathcal{I}$ be the unit tensor of dimension n_2. If $\mathcal{A}\mathcal{I} = 0$, then $\mathcal{A} = 0$.*

Proof. By Definition 1.1, we have

$$(\mathcal{A}\mathcal{I})_{i\alpha_1 \ldots \alpha_{m-1}} = \begin{cases} a_{ii_2 \cdots i_m} & \alpha_j = i_{j+1} \cdots i_{j+1}\ , j = 2, \ldots, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mathcal{A}\mathcal{I} = 0$ implies that $\mathcal{A} = 0$. $\square$

Lemma 2.2. *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times \cdots \times n_k}$ be a tensor, and let $\mathcal{I}$ be the unit tensor of order m and dimension n_1. If $\mathcal{I}\mathcal{A} = 0$, then $\mathcal{A} = 0$.*

Proof. By the way of contradiction, suppose that $\mathcal{A}$ has a nonzero entry $a_{i_1 \cdots i_k}$. By Definition 1.1, $\mathcal{C} = \mathcal{I}\mathcal{A}$ has a nonzero entry $c_{i_1\alpha \cdots \alpha} = a_{i_1 \cdots i_k}^{m-1}$, $\alpha = i_2 \cdots i_k$, a contradiction. Hence $\mathcal{I}\mathcal{A} = 0$ implies that $\mathcal{A} = 0$. $\square$

Let $\mathcal{A}$ be an order m dimension n tensor, and let $x = (x_1, \ldots, x_n)^\top$. Then $\mathcal{A}x$ is a vector in $\mathbb{C}^n$ whose i-th component is (see [11, Example 1.1])

$$(\mathcal{A}x)_i = \sum_{i_2, \ldots, i_m \in [n]} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

Moreover, $\mathcal{A}x = 0$ is a homogeneous equations of degree $m - 1$ with n variables. The *determinant* of $\mathcal{A}$, denoted by $\det(\mathcal{A})$, is the resultant of $\mathcal{A}x = 0$ (see [11]).

Lemma 2.3. *(See [11].) For any unit tensor $\mathcal{I}$, we have $\det(\mathcal{I}) = 1$.*

Lemma 2.4. *(See [12].) Let $\mathcal{A}$ be an order m dimension n tensor, and let $\mathcal{B}$ be an order k dimension n tensor. Then $\det(\mathcal{A}\mathcal{B}) = \det(\mathcal{A})^{(k-1)^{n-1}} \det(\mathcal{B})^{(m-1)^n}$.*

Lemma 2.5. *(See [11].) Let $\mathcal{A}$ be an order m dimension n tensor. Then $\det(\mathcal{A}) = 0$ if and only if $\mathcal{A}x = 0$ has a nonzero solution.*

Lemma 2.6. *Let $f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n)$ be homogeneous polynomials of degree m, and let $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n))^\top$. If $r < n$, then $F(x_1, \ldots, x_n) = 0$ has a nonzero solution.*

Proof. Let $\mathcal{A} = (a_{i_1 \cdots i_{m+1}})$ be an order $m + 1$ dimension n tensor such that

$$f_i(x_1, \ldots, x_n) = \sum_{i_2, \ldots, i_{m+1} \in [n]} a_{ii_2 \cdots i_{m+1}} x_{i_2} \cdots x_{i_{m+1}}, \quad i \in [r],$$

and $a_{i_1\cdots i_{m+1}} = 0$ for any $i_1 \geqslant r+1$. Then $F(x_1,\ldots,x_n) = 0$ and $\mathcal{A}x = 0$ have the same solution, where $x = (x_1,\ldots,x_n)^\top$. By [5, Corollary 2.5], we get $\det(\mathcal{A}) = 0$. By Lemma 2.5, $F(x_1,\ldots,x_n) = 0$ has a nonzero solution. □

For a matrix P, let $P(:,k)$ denote the k-th column of P.

Lemma 2.7. *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ be an order m rank one tensor. Then the following statements hold:*

(1) *For any $P \in \mathbb{C}^{n_2\times n_3}$, we have* $\operatorname{rank}(\mathcal{A}P) \leqslant 1$.
(2) *For any nonsingular matrix $P \in \mathbb{C}^{n_2\times n_2}$, we have* $\operatorname{rank}(\mathcal{A}P) = 1$.

Proof. There exist nonzero column vectors $\alpha_1,\ldots,\alpha_m$ such that $\mathcal{A} = \alpha_1 \otimes \cdots \otimes \alpha_m$. By Definition 1.1, $\mathcal{C} = \mathcal{A}P$ is a tensor with entry

$$\begin{aligned} c_{ik_1\ldots k_{m-1}} &= \sum_{i_2,\ldots,i_m\in[n_2]} a_{ii_2\ldots i_m} p_{i_2k_1}\cdots p_{i_mk_{m-1}} \\ &= \sum_{i_2,\ldots,i_m\in[n_2]} (\alpha_1)_i(\alpha_2)_{i_2}\cdots(\alpha_m)_{i_m} p_{i_2k_1}\cdots p_{i_mk_{m-1}} \\ &= (\alpha_1)_i\big[\alpha_2^\top P(:,k_1)\big]\cdots\big[\alpha_m^\top P(:,k_{m-1})\big]. \end{aligned}$$

Hence $\mathcal{C} = \mathcal{A}P = \alpha_1 \otimes P^\top\alpha_2 \otimes \cdots \otimes P^\top\alpha_m$, which implies that $\operatorname{rank}(\mathcal{A}P) \leqslant 1$. So part (1) holds. If P is nonsingular, then $P^\top\alpha_2,\ldots,P^\top\alpha_m$ are nonzero column vectors. In this case, $\operatorname{rank}(\mathcal{A}P) = 1$. So part (2) holds. □

3. Main results

Let $\mathcal{A}\{R_k\}$ denote the set of all order k right inverses of a tensor $\mathcal{A}$.

Theorem 3.1. *Let $\mathcal{A}$ be an order m dimension n diagonal tensor. Then the following statements hold:*

(1) *$\mathcal{A}$ has an order k left inverse if and only if $a_{i\cdots i} \neq 0$, $i = 1,\ldots,n$. Moreover, an order k diagonal tensor $\mathcal{D}$ with diagonal entry $d_{i\cdots i} = a_{i\cdots i}^{-(k-1)}$ is the unique order k left inverse of $\mathcal{A}$.*
(2) *$\mathcal{A}$ has an order k right inverse if and only if $a_{i\cdots i} \neq 0$, $i = 1,\ldots,n$. In this case, we have*

$$\mathcal{A}\{R_k\} = \big\{\mathcal{B} \mid \mathcal{B} \textit{ is an order } k \textit{ diagonal tensor and } b_{i\cdots i}^{m-1} = a_{i\cdots i}^{-1},\ i = 1,\ldots,n\big\}.$$

Proof. If $\mathcal{A}$ has an order k left inverse, then there exists an order k dimension n tensor $\mathcal{D}$ such that $\mathcal{D}\mathcal{A} = \mathcal{I}$. Since $\mathcal{A}$ is diagonal, by Definition 1.1, $\mathcal{A}$ has an order k left inverse if and only if $a_{i\cdots i} \neq 0$, $i = 1, \ldots, n$. By $\mathcal{D}\mathcal{A} = \mathcal{I}$ we know that $\mathcal{D}$ is diagonal and $d_{i\cdots i} = a_{i\cdots i}^{-(k-1)}$. Hence part (1) holds.

If $\mathcal{A}$ has an order k right inverse, then there exists an order k dimension n tensor $\mathcal{B}$ such that $\mathcal{A}\mathcal{B} = \mathcal{I}$. Since $\mathcal{A}$ is diagonal, by Definition 1.1, $\mathcal{A}$ has an order k right inverse if and only if $a_{i\cdots i} \neq 0$, $i = 1, \ldots, n$. By $\mathcal{A}\mathcal{B} = \mathcal{I}$ we know that $\mathcal{B}$ is diagonal and $a_{i\cdots i} b_{i\cdots i}^{m-1} = 1$. Hence part (2) holds. $\square$

Theorem 3.2. *Let $\mathcal{A}$ be an order m dimension n tensor. Then $\mathcal{A}$ has an order 2 left inverse if and only if there exists a nonsingular matrix P such that $\mathcal{A} = P\mathcal{I}$. Moreover, P^{-1} is the unique order 2 left inverse of $\mathcal{A}$.*

Proof. If $\mathcal{A} = P\mathcal{I}$ for a nonsingular matrix P, then $\mathcal{A}$ has an order 2 left inverse P^{-1}. If X is an order 2 left inverse of $\mathcal{A}$, then $X\mathcal{A} = \mathcal{I}$. Lemmas 2.3 and 2.4 imply that X is nonsingular. So $\mathcal{A} = X^{-1}\mathcal{I}$. Suppose that Y is also an order 2 left inverse of $\mathcal{A}$, we can also get $\mathcal{A} = Y^{-1}\mathcal{I}$. Hence $X^{-1}\mathcal{I} = Y^{-1}\mathcal{I}$, $(X^{-1} - Y^{-1})\mathcal{I} = 0$. By Lemma 2.1, we have $X = Y$. So X is the unique order 2 left inverse of $\mathcal{A}$. $\square$

For an order m dimension n tensor $\mathcal{A} = (a_{j_1\cdots j_m})$, $\Phi_{\mathcal{A}}(\lambda) = \det(\lambda\mathcal{I} - \mathcal{A})$ is called the *characteristic polynomial* of $\mathcal{A}$. Eigenvalues of tensors are defined in [7,9]. It is known that eigenvalues of $\mathcal{A}$ are exactly roots of $\Phi_{\mathcal{A}}(\lambda)$ (see [9, Theorem 1]). Let $\mathcal{A}^{(L_2)}$ denote the order 2 left inverse of a tensor $\mathcal{A}$. Let $\mathcal{A}(p,q) = (a_{j_1\cdots j_m}) \in \mathbb{C}^{(n-1)\times\cdots\times(n-1)}$ be a sub-tensor of $\mathcal{A}$, where $j_1 \neq p$, $j_2, \ldots, j_m \neq q$.

Theorem 3.3. *Let $\mathcal{A}$ be an order m dimension n tensor such that $\mathcal{A}^{(L_2)}$ exists. Then the following hold:*

(1) $\det(\mathcal{A}) = \det(\mathcal{A}^{(L_2)})^{-(m-1)^{n-1}}$.
(2) $\Phi_{\mathcal{A}}(\lambda) = [\Phi_{[\mathcal{A}^{(L_2)}]^{-1}}(\lambda)]^{(m-1)^{n-1}}$.
(3) $(\mathcal{A}^{(L_2)})_{ij} = (-1)^{i+j} \frac{\det[\mathcal{A}(j,i)]^{(m-1)^{-(n-2)}}}{\det(\mathcal{A})^{(m-1)^{-(n-1)}}}$.

Proof. Since $\mathcal{A}^{(L_2)}\mathcal{A} = \mathcal{I}$, by Lemmas 2.3 and 2.4, we have

$$\det(\mathcal{A}^{(L_2)})^{(m-1)^{n-1}} \det(\mathcal{A}) = 1, \quad \det(\mathcal{A}) = \det(\mathcal{A}^{(L_2)})^{-(m-1)^{n-1}}.$$

So part (1) holds.

By Theorem 3.2, we have $\mathcal{A} = [\mathcal{A}^{(L_2)}]^{-1}\mathcal{I}$. Then $\Phi_{\mathcal{A}}(\lambda) = \det(\lambda\mathcal{I} - \mathcal{A}) = \det(\lambda\mathcal{I} - [\mathcal{A}^{(L_2)}]^{-1}\mathcal{I}) = \det((\lambda I - [\mathcal{A}^{(L_2)}]^{-1})\mathcal{I})$, where I is the identity matrix. By Lemma 2.4, we get $\Phi_{\mathcal{A}}(\lambda) = [\det(\lambda I - [\mathcal{A}^{(L_2)}]^{-1})]^{(m-1)^{n-1}} = [\Phi_{[\mathcal{A}^{(L_2)}]^{-1}}(\lambda)]^{(m-1)^{n-1}}$. So part (2) holds.

Since $\mathcal{A}^{(L_2)}$ exists, by Theorem 3.2, there exists a nonsingular matrix P such that $\mathcal{A} = P\mathcal{I}$ and $\mathcal{A}^{(L_2)} = P^{-1}$. So we have

$$(\mathcal{A}^{(L_2)})_{ij} = (P^{-1})_{ij} = (-1)^{i+j} \frac{\det[P(j,i)]}{\det(P)}.$$

Since $\mathcal{A} = P\mathcal{I}$, by Lemmas 2.3 and 2.4, we have $\det \mathcal{A} = \det(P)^{(m-1)^{(n-1)}}$. By Definition 1.1, we get $\mathcal{A}(j,i) = P(j,i)\mathcal{I}(i,i)$. Note that $\mathcal{I}(i,i)$ is an unit tensor of dimension $n-1$. By Lemmas 2.3 and 2.4, we have $\det[\mathcal{A}(j,i)] = \det[P(j,i)]^{(m-1)^{(n-2)}}$. Hence

$$\left(\mathcal{A}^{(L_2)}\right)_{ij} = (-1)^{i+j}\frac{\det[P(j,i)]}{\det(P)} = (-1)^{i+j}\frac{\det[\mathcal{A}(j,i)]^{(m-1)^{-(n-2)}}}{\det(\mathcal{A})^{(m-1)^{-(n-1)}}}.$$

So part (3) holds. $\square$

Here we give an example for Theorem 3.3. For any order 3 dimension 2 tensor $\mathcal{A}$, its determinant is (see [3])

$$\begin{aligned}\det(\mathcal{A}) = {} & a_{111}^2a_{222}^2 + a_{112}^2a_{221}^2 + a_{121}^2a_{212}^2 + a_{211}^2a_{122}^2 \\ & - 2a_{111}a_{112}a_{221}a_{222} - 2a_{111}a_{121}a_{212}a_{222} - 2a_{111}a_{122}a_{211}a_{222} \\ & - 2a_{112}a_{121}a_{212}a_{221} - 2a_{112}a_{122}a_{221}a_{211} - 2a_{121}a_{122}a_{212}a_{211} \\ & + 4a_{111}a_{122}a_{212}a_{221} + 4a_{112}a_{121}a_{211}a_{222}.\end{aligned}$$

Let $\mathcal{B}$ be an order 3 dimension 2 tensor such that $b_{111} = 1$, $b_{122} = 2$, $b_{211} = -1$, $b_{222} = 1$ and the other entries of $\mathcal{B}$ are 0. Then we have

$$\begin{aligned}&\det(\mathcal{B}) = 9, \qquad \det\left[\mathcal{B}(1,1)\right] = b_{222} = 1, \qquad \det\left[\mathcal{B}(1,2)\right] = b_{211} = -1, \\ &\det\left[\mathcal{B}(2,1)\right] = b_{122} = 2, \qquad \det\left[\mathcal{B}(2,2)\right] = b_{111} = 1.\end{aligned}$$

If $\mathcal{B}^{(L_2)}$ exists, then by Theorem 3.3, $B_1 = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$ or $B_2 = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$ is an order 2 left inverse of $\mathcal{B}$. By computation, we get $B_1\mathcal{B} = \mathcal{I}$. Since $\mathcal{B}^{(L_2)}$ is unique, we have $\mathcal{B}^{(L_2)} = B_1$ and B_2 is not an order 2 left inverse of $\mathcal{B}$. Since $B_1^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, by Theorem 3.3, we have

$$\Phi_{\mathcal{B}}(\lambda) = \left[\Phi_{B_1^{-1}}(\lambda)\right]^2 = \left(\lambda^2 - 2\lambda + 3\right)^2.$$

For $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$, let $x^{[m]} = (x_1^m, \ldots, x_n^m)^\top$.

Theorem 3.4. *Let $\mathcal{A}$ be an order m dimension n tensor. For the polynomial equations $\mathcal{A}x = b$, if there exists an $n \times n$ matrix P such that $(\mathcal{A}P)^{(L_2)}$ exists, then $x = P[(\mathcal{A}P)^{(L_2)}b]^{[(m-1)^{-1}]}$.*

Proof. Since $(\mathcal{A}P)^{(L_2)}\mathcal{A}P = \mathcal{I}$, by Lemmas 2.3 and 2.4, we get $\det(\mathcal{A}P) \neq 0$, $\det(P) \neq 0$. Let $y = P^{-1}x$, then $\mathcal{A}Py = b$, $\mathcal{I}y = (\mathcal{A}P)^{(L_2)}b$. By $\mathcal{I}y = y^{[m-1]}$, we have $y = [(\mathcal{A}P)^{(L_2)}b]^{[(m-1)^{-1}]}$. Hence $x = Py = P[(\mathcal{A}P)^{(L_2)}b]^{[(m-1)^{-1}]}$. $\square$

Let I denote the identity matrix.

Theorem 3.5. *Let $\mathcal{A}$ be an order m dimension n tensor. Then $\mathcal{A}$ has an order 2 right inverse if and only if there exists a nonsingular matrix Q such that $\mathcal{A} = \mathcal{I}Q$. In this case, we have*

$$\mathcal{A}\{R_2\} = \{Q^{-1}Y \mid Y^{m-1} = I,\ Y \text{ is a diagonal matrix}\}.$$

Proof. If $\mathcal{A} = \mathcal{I}Q$ for some nonsingular matrix Q, then $\mathcal{A}$ has an order 2 right inverse Q^{-1}. If X is an order 2 right inverse of $\mathcal{A}$, then $\mathcal{A}X = \mathcal{I}$. Lemmas 2.3 and 2.4 imply that X is a nonsingular matrix. So $\mathcal{A} = \mathcal{I}X^{-1}$. Hence if $\mathcal{A}$ has an order 2 right inverse, then there exists a nonsingular matrix Q such that $\mathcal{A} = \mathcal{I}Q$.

Let P be any order 2 right inverse of $\mathcal{A}$, then $\mathcal{A}P = \mathcal{I}QP = \mathcal{I}$. Let $Y = QP$, then $\mathcal{I} = \mathcal{I}Y$. By Definition 1.1, we know that Y is diagonal and $Y^{m-1} = I$. So $P = Q^{-1}Y$. □

Theorem 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be tensors such that $\mathcal{A}\mathcal{B} = 0$. Then the following hold:*

(1) *If $\mathcal{A}^{(L_2)}$ (resp. $\mathcal{B}^{(L_2)}$) exists, then $\mathcal{B} = 0$ (resp. $\mathcal{A} = 0$).*
(2) *If $\mathcal{A}\{R_2\} \neq \emptyset$ (resp. $\mathcal{B}\{R_2\} \neq \emptyset$), then $\mathcal{B} = 0$ (resp. $\mathcal{A} = 0$).*

Proof. If $\mathcal{A}^{(L_2)}$ exists, then by Theorem 3.2, there exists a nonsingular matrix P such that $P\mathcal{I}\mathcal{B} = 0$. So $\mathcal{I}\mathcal{B} = 0$. By Lemma 2.2, we get $\mathcal{B} = 0$. If $\mathcal{B}^{(L_2)}$ exists, then by Theorem 3.2, there exists a nonsingular matrix P such that $\mathcal{A}P\mathcal{I} = 0$. By Lemma 2.1, we get $\mathcal{A}P = 0$, $\mathcal{A} = 0$. Hence part (1) holds.

If $\mathcal{A}\{R_2\} \neq \emptyset$, then by Theorem 3.5, there exists a nonsingular matrix Q such that $\mathcal{I}Q\mathcal{B} = 0$. By Lemma 2.2, we get $Q\mathcal{B} = 0$, $\mathcal{B} = 0$. If $\mathcal{B}\{R_2\} \neq \emptyset$, then by Theorem 3.5, there exists a nonsingular matrix Q such that $\mathcal{A}\mathcal{I}Q = 0$. So $\mathcal{A}\mathcal{I} = 0$. By Lemma 2.1, we get $\mathcal{A} = 0$. Hence part (2) holds. □

For $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{n_1\times\cdots\times n_m}$, it can be partitioned as a block tensor $\mathcal{A} = \binom{\mathcal{A}_1}{\mathcal{A}_2}$, where $\mathcal{A}_1 = (a_{ii_2\ldots i_m}) \in \mathbb{C}^{r\times n_2\times\cdots\times n_m}$, $i = 1,\ldots,r$, $\mathcal{A}_2 = (a_{ji_2\ldots i_m}) \in \mathbb{C}^{(n_1-r)\times n_2\times\cdots\times n_m}$, $j = r+1,\ldots,n_1$.

Theorem 3.7. *Let $\mathcal{A} = \binom{\mathcal{A}_1}{\mathcal{A}_2}$ ($\mathcal{A}_i \in \mathbb{C}^{r_i\times n_2\times\cdots\times n_m}$, $i = 1,2$) be a block tensor, and let $B = (\,B_1 \quad B_2\,)$ be a block matrix such that B_i has r_i ($i = 1,2$) columns. Then*

$$B\mathcal{A} = (\,B_1 \quad B_2\,)\begin{pmatrix}\mathcal{A}_1\\ \mathcal{A}_2\end{pmatrix} = B_1\mathcal{A}_1 + B_2\mathcal{A}_2.$$

Proof. By Definition 1.1, we have

$$\begin{aligned}(B\mathcal{A})_{i_1i_2\ldots i_m} &= \sum_{j=1}^{r_1+r_2} b_{i_1j}a_{ji_2\ldots i_m} = \sum_{j=1}^{r_1} b_{i_1j}a_{ji_2\ldots i_m} + \sum_{j=r_1+1}^{r_1+r_2} b_{i_1j}a_{ji_2\ldots i_m}\\ &= (B_1\mathcal{A}_1)_{i_1i_2\ldots i_m} + (B_2\mathcal{A}_2)_{i_1i_2\ldots i_m}. \qquad \square\end{aligned}$$

We can obtain the following result from Definition 1.1.

Theorem 3.8. *Let* $\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_2}$ *be a block tensor. For any* $\mathcal{B} \in \mathbb{C}^{n_2\times\cdots\times n_{k+1}}$, *we have*

$$\begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} \mathcal{B} = \begin{pmatrix} \mathcal{A}_1\mathcal{B} \\ \mathcal{A}_2\mathcal{B} \end{pmatrix}.$$

Theorem 3.9. *Let* $\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) \in \mathbb{C}^{n\times n\times\cdots\times n}$ *be a block tensor, where* $\mathcal{A}_1 \in \mathbb{C}^{r\times n\times\cdots\times n}$. *For any* $B \in \mathbb{C}^{(n-r)\times r}$, *we have* $\det\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$.

Proof. By Theorems 3.7 and 3.8, we get $\left(\begin{smallmatrix} I & 0 \\ B & I \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$. By Lemma 2.4, we have $\det\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \det\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$. □

Theorem 3.10. *Let* $\mathcal{A} \in \mathbb{C}^{n_1\times\cdots\times n_k}$ *be a tensor. Then the following hold:*

(1) *For any* $P \in \mathbb{C}^{m\times n_1}$, *we have* $\operatorname{rank}(P\mathcal{A}) \leqslant \operatorname{rank}(\mathcal{A})$.
(2) *For any nonsingular matrix* $P \in \mathbb{C}^{n_1\times n_1}$, *we have* $\operatorname{rank}(P\mathcal{A}) = \operatorname{rank}(\mathcal{A})$.

Proof. There exist rank one tensors $\mathcal{A}_1,\dots,\mathcal{A}_r$ such that $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$, where $r = \operatorname{rank}(\mathcal{A})$. So $\operatorname{rank}(P\mathcal{A}) \leqslant \operatorname{rank}(P\mathcal{A}_1) + \cdots + \operatorname{rank}(P\mathcal{A}_r)$. In order to prove part (1), we need to prove $\operatorname{rank}(P\mathcal{B}) \leqslant 1$ for any rank one tensor $\mathcal{B} \in \mathbb{C}^{n_1\times\cdots\times n_k}$.

For any rank one tensor $\mathcal{B} \in \mathbb{C}^{n_1\times\cdots\times n_k}$, there exist nonzero column vectors $b_1,\dots,b_k$ such that $\mathcal{B} = b_1 \otimes \cdots \otimes b_k$. From Definition 1.1, we get $P\mathcal{B} = Pb_1 \otimes \cdots \otimes b_k$. Hence $\operatorname{rank}(P\mathcal{B}) \leqslant 1$, which implies that $\operatorname{rank}(P\mathcal{A}) \leqslant r = \operatorname{rank}(\mathcal{A})$. So part (1) holds. If P is nonsingular, then $Pb_1 \neq 0$. Hence $\operatorname{rank}(P\mathcal{B}) = 1$ for any nonsingular matrix P and rank one tensor $\mathcal{B}$.

Next we only consider the case that P is nonsingular. If $\operatorname{rank}(P\mathcal{A}) = s < r$, then there exist rank one tensors $\mathcal{C}_1,\dots,\mathcal{C}_s$ such that $P\mathcal{A} = \mathcal{C}_1 + \cdots + \mathcal{C}_s$. Since P is nonsingular, we get $\mathcal{A} = P^{-1}\mathcal{C}_1 + \cdots + P^{-1}\mathcal{C}_s$. Since $P^{-1}\mathcal{C}_1,\dots,P^{-1}\mathcal{C}_s$ are rank one tensors, we get $\operatorname{rank}(\mathcal{A}) \leqslant s < r = \operatorname{rank}(\mathcal{A})$, a contradiction. Hence $\operatorname{rank}(P\mathcal{A}) \geqslant r = \operatorname{rank}(\mathcal{A})$. From part (1) we have $\operatorname{rank}(P\mathcal{A}) = \operatorname{rank}(\mathcal{A})$. So part (2) holds. □

Theorem 3.11. *Let* $\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) \in \mathbb{C}^{n_1\times\cdots\times n_k}$ *be a block tensor, where* $\mathcal{A}_1 \in \mathbb{C}^{r\times n_2\times\cdots\times n_k}$. *For any* $B \in \mathbb{C}^{(n_1-r)\times r}$, *we have* $\operatorname{rank}\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \operatorname{rank}\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$.

Proof. By Theorems 3.7 and 3.8, we get $\left(\begin{smallmatrix} I & 0 \\ B & I \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$. By Theorem 3.10, we have $\operatorname{rank}\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{smallmatrix}\right) = \operatorname{rank}\left(\begin{smallmatrix} \mathcal{A}_1 \\ \mathcal{A}_2+B\mathcal{A}_1 \end{smallmatrix}\right)$. □

Theorem 3.12. *Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times \cdots \times n_2}$ be an order m tensor. Then the following hold:*

(1) *For any $P \in \mathbb{C}^{n_2 \times n_3}$, we have* $\operatorname{rank}(\mathcal{A}P) \leqslant \operatorname{rank}(\mathcal{A})$.
(2) *For any nonsingular matrix $P \in \mathbb{C}^{n_2 \times n_2}$, we have* $\operatorname{rank}(\mathcal{A}P) = \operatorname{rank}(\mathcal{A})$.

Proof. There exist rank one tensors $\mathcal{A}_1, \dots, \mathcal{A}_r$ such that $\mathcal{A} = \mathcal{A}_1 + \cdots + \mathcal{A}_r$, where $r = \operatorname{rank}(\mathcal{A})$. So $\operatorname{rank}(\mathcal{A}P) \leqslant \operatorname{rank}(\mathcal{A}_1 P) + \cdots + \operatorname{rank}(\mathcal{A}_r P)$. By Lemma 2.7, we get $\operatorname{rank}(\mathcal{A}P) \leqslant r = \operatorname{rank}(\mathcal{A})$. Hence part (1) holds.

Next we only consider the case that P is nonsingular. If $\operatorname{rank}(\mathcal{A}P) = s < r$, then there exist rank one tensors $\mathcal{C}_1, \dots, \mathcal{C}_s$ such that $\mathcal{A}P = \mathcal{C}_1 + \cdots + \mathcal{C}_s$. Since P is nonsingular, we get $\mathcal{A} = \mathcal{C}_1 P^{-1} + \cdots + \mathcal{C}_s P^{-1}$. By Lemma 2.7, $\mathcal{C}_1 P^{-1}, \dots, \mathcal{C}_s P^{-1}$ are rank one tensors. So we have $\operatorname{rank}(\mathcal{A}) \leqslant s < r = \operatorname{rank}(\mathcal{A})$, a contradiction. Hence $\operatorname{rank}(\mathcal{A}P) \geqslant r = \operatorname{rank}(\mathcal{A})$. From part (1) we have $\operatorname{rank}(\mathcal{A}P) = \operatorname{rank}(\mathcal{A})$. So part (2) holds. $\square$

We say that two tensors $\mathcal{A}$, $\mathcal{B}$ are *equivalent* if there exist nonsingular matrices P, Q such that $\mathcal{A} = P\mathcal{B}Q$. The following result follows from Theorems 3.10 and 3.12.

Theorem 3.13. *Equivalent tensors have the same rank.*

Theorem 3.14. *Let $\mathcal{A}$ be an order m dimension n tensor. If* $\operatorname{rank}(\mathcal{A}) < n$, *then* $\det(\mathcal{A}) = 0$.

Proof. Let $r = \operatorname{rank}(\mathcal{A})$. There exist nonzero column vectors α_{ki} ($k \in [r], i \in [m]$) such that $\mathcal{A} = \sum_{k=1}^{r} \alpha_{k1} \otimes \cdots \otimes \alpha_{km}$. By Lemma 2.5, $\det(\mathcal{A}) = 0$ if and only if $\mathcal{A}x = 0$ has a nonzero solution, where $x = (x_1, \dots, x_n)^\top$. From the proof of Lemma 2.7, we have

$$\mathcal{A}x = \sum_{k=1}^{r} \alpha_{k1} \left(x^\top \alpha_{k2}\right) \cdots \left(x^\top \alpha_{km}\right) = BF(x_1, \dots, x_n),$$

where $B = (\alpha_{11}, \dots, \alpha_{r1})$, $F(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n))^\top$, $f_k(x_1, \dots, x_n) = (x^\top \alpha_{k2}) \cdots (x^\top \alpha_{km})$, $1 \leqslant k \leqslant r$.

Since $B \in \mathbb{C}^{n \times r}$ and $r < n$, there exist nonsingular matrix P and permutation matrix Q such that $PBQ = \left(\begin{smallmatrix} B_1 & B_2 \\ 0 & 0 \end{smallmatrix}\right)$, where $B_1 \in \mathbb{C}^{s \times s}$ is nonsingular and $s = \operatorname{rank}(B) < n$. Let $G(x_1, \dots, x_n) = Q^\top F(x_1, \dots, x_n)$, then $\mathcal{A}x = BF(x_1, \dots, x_n) = 0$ and $\left(\begin{smallmatrix} B_1 & B_2 \\ 0 & 0 \end{smallmatrix}\right) G(x_1, \dots, x_n) = 0$ have the same solution. Since $\left(\begin{smallmatrix} B_1 & B_2 \\ 0 & 0 \end{smallmatrix}\right) G(x_1, \dots, x_n) = 0$ is a homogeneous equations with n variables and s ($s < n$) equations, by Lemma 2.6, it has a nonzero solution. Hence $\mathcal{A}x = 0$ has a nonzero solution. By Lemma 2.5, we get $\det(\mathcal{A}) = 0$. $\square$

Let $\mathcal{E}_{i \cdots i}$ be the diagonal tensor whose the i-th diagonal entry is 1, and the other entries are 0. Clearly $\mathcal{E}_{i \cdots i}$ is a rank one tensor.

Theorem 3.15. *Let $\mathcal{A}$ be an order m dimension n tensor. If $\mathcal{A}^{(L_2)}$ exists or $\mathcal{A}\{R_2\} \neq \emptyset$, then* $\operatorname{rank}(\mathcal{A}) = n$.

Proof. If $\mathcal{A}^{(L_2)}$ exists or $\mathcal{A}\{R_2\} \neq \emptyset$, then by Theorems 3.2 and 3.5, $\mathcal{A}$ is equivalent to a dimension n unit tensor $\mathcal{I}$. By Theorem 3.13, we have $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{I})$. So we need to show that $\operatorname{rank}(\mathcal{I}) = n$.

Since $\mathcal{I} = \sum_{i=1}^{n} \mathcal{E}_{i\cdots i}$, we have $\operatorname{rank}(\mathcal{I}) \leqslant n$. By Lemma 2.3 and Theorem 3.14, we get $\operatorname{rank}(\mathcal{I}) \geqslant n$. Hence $\operatorname{rank}(\mathcal{I}) = n$. $\square$

Let $H = (V, E)$ be a k-uniform hypergraph. The *adjacency tensor* of H is an order k dimension $|V|$ tensor $\mathcal{A}_H$ with entries $a_{i_1 i_2 \cdots i_k} = \frac{1}{(k-1)!}$ if $i_1 i_2 \cdots i_k \in E$ and $a_{i_1 i_2 \cdots i_k} = 0$ otherwise. Let $\mathcal{D}_H$ be the diagonal tensor whose diagonal entries are vertex degrees of H. Then $\mathcal{L}_H = \mathcal{D}_H - \mathcal{A}_H$ and $\mathcal{Q}_H = \mathcal{D}_H + \mathcal{A}_H$ are called the *Laplacian tensor* and the *signless Laplacian tensor* of H, respectively (see [6,10]). We say that the rank of $\mathcal{A}_H$ is the *rank* of H.

Recently, Shao proved that the spectrum of a hypergraph is independent of the ordering of its vertices (see [11, Theorem 2.2]). This property also holds for the rank of a hypergraph.

Theorem 3.16. *Isomorphic hypergraphs have the same rank.*

Proof. Let G and H be isomorphic uniform hypergraphs. Then there exists a permutation matrix P such that $\mathcal{A}_G = P\mathcal{A}_H P^\top$ (see [11]). By Theorem 3.13, G and H have the same rank. $\square$

Let $H = (V, E)$ be a k-uniform hypergraph. If the vertex set V has a disjoint partition $V = V_1 \cup V_2$ such that every edge in E intersects V_1 with exactly an odd number of vertices, then H is called *odd-bipartite* (see [6]).

Theorem 3.17. *Let $H = (V, E)$ be a k-uniform hypergraph. If k is even and H is odd-bipartite, then* $\operatorname{rank}(\mathcal{L}_H) = \operatorname{rank}(\mathcal{Q}_H)$.

Proof. Since H is odd-bipartite, V has a partition $V = V_1 \cup V_2$ such that every edge in E intersects V_1 with exactly an odd number of vertices. Let P be the diagonal matrix whose i-th diagonal entry is 1 if $i \in V_1$ and -1 if $i \in V_2$. By computation, we have $P\mathcal{L}_H P = P(\mathcal{D}_H - \mathcal{A}_H)P = P\mathcal{D}_H P - P\mathcal{A}_H P = \mathcal{D}_H + \mathcal{A}_H = \mathcal{Q}_H$. By Theorem 3.13, we have $\operatorname{rank}(\mathcal{L}_H) = \operatorname{rank}(\mathcal{Q}_H)$. $\square$

Acknowledgements

The authors would like to thank the referee for giving helpful suggestions. This work is supported by the National Natural Science Foundation of China (No. 11371109 and No. 11271084), and the Fundamental Research Funds for the Central Universities.

References

[1] K.C. Chang, K. Pearson, T. Zhang, On eigenvalue problems of real symmetric tensors, J. Math. Anal. Appl. 350 (2009) 416–422.

[2] W. Ding, L. Qi, Y. Wei, $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors, Linear Algebra Appl. 439 (2013) 3264–3278.

[3] I. Gelfand, M. Kapranov, A. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[4] S. Hu, Z. Huang, C. Ling, L. Qi, E-determinants of tensors, arXiv:1109.0348v3, 2011.

[5] S. Hu, Z. Huang, C. Ling, L. Qi, On determinants and eigenvalue theory of tensors, J. Symbolic Comput. 50 (2013) 508–531.

[6] S. Hu, L. Qi, The eigenvectors of the zero Laplacian and signless Laplacian eigenvalues of a uniform hypergraph, arXiv:1303.4048, 2013.

[7] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'05), vol. 1, 2005, pp. 129–132.

[8] L.-H. Lim, Tensors and hypermatrices, in: L. Hogben (Ed.), Handbook of Linear Algebra, CRC Press, Boca Raton, FL, 2013.

[9] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40 (2005) 1302–1324.

[10] L. Qi, H^{+}-eigenvalues of Laplacian and signless Laplacian tensors, Commun. Math. Sci. (2013), in press, arXiv:1303.2186.

[11] J.Y. Shao, A general product of tensors with applications, Linear Algebra Appl. 439 (2013) 2350–2366.

[12] J.Y. Shao, H.Y. Shan, L. Zhang, On some properties of the determinants of tensors, Linear Algebra Appl. 439 (2013) 3057–3069.

[13] J. Xie, A. Chang, On the Z-eigenvalues of the adjacency tensors for uniform hypergraphs, Linear Algebra Appl. 439 (2013) 2195–2204.