HW12 637

October 8, 2024

1 Introduction

So as your new ~~TA classmate~~ interloper, I've made–under the advice of [REDACTED]–your first homework assignment! You really ought to take this one seriously seeing as this will make up 100% of your grade.

2 Definitions

Apparently, you guys have learned elementary equivalence but not elementary substructures.

2.1 Elementary Diagram

Given an $\mathcal{L}$-structure M, its elementary diagram is obtained by adding to $\mathcal{L}$ constant symbols for every element of M with $c_a^M = a$ for any $a \in M$. The (complete/full) theory of this new structure is the *elementary diagram* $\Delta(M)$ of M.

2.2 Elementary Substructures

Two $\mathcal{L}$-structures $A \subseteq B$ (I use $\subseteq$ for the substructure relation) can be elementarily equivalent but we want a stronger property. We say that A is an *elementary substructure* of B, written $A \preceq B$, whenever the structures A' and B' obtained by extending the languages of A and B with (appropriately interpreted) constant symbols for every element of A are elementarily equivalent (i.e. $B' \models \Delta(A)$). We alternatively say that A is elementary in B, or that B is an elementary extension of A. More pithily, $A \subseteq B$ and $\Delta(A) \subseteq \Delta(B)$.

3 Theorem of the Day

3.1 Downward Löwenheim–Skolem Theorem

Let A be an $\mathcal{L}$-structure. For every cardinal $|\mathcal{L}| \leq \kappa \leq |A|$ there exists an elementary substructure $B \preceq A$ with cardinality κ.

4 Actual Homework

1. The cardinalities involved in the Downward Löwenheim–Skolem Theorem are bounded from below by the cardinality of the language. Find a language $\mathcal{L}$ and an $\mathcal{L}$-theory T whose models all have cardinalities greater than or equal to the cardinality of the language.

2. Skolem's paradox is the following: the Downward Löwenheim–Skolem Theorem implies that if there is a model of set theory, then there is a countable one. This countable model must contain an uncountable element (Cantor's theorem) as well as all of its elements. However, it cannot contain uncountably many elements. Resolve this paradox.

3. Consider and evaluate the following argument: the language of set theory is countable–i.e. there are only countable many $\in$-formulae. There could therefore only be countably many definitions yet there are uncountably many sets which means that there are some sets (in fact, most sets) which are so infinitely complicated that no string of characters produced by mortals could describe it.

4. Let $R = C^\infty(\mathbb{R})$ be the ring of smooth functions on $\mathbb{R}$ (with pointwise operations) augmented with a unary function symbol δ with $\delta^R = \frac{d}{dx}$ the derivative operator on R, and constant symbols $a^R = \sin$, $b^R = \cos$, $c^R = \exp$, $d^R = \text{sinc}$. Compute the following terms: $\delta(((a \cdot a) \cdot b) + (d \cdot c))$, $\delta(a + b)$, $\delta(b \cdot c) \cdot \delta(d)$. This problem was suggested by the aforementioned [REDACTED].

5. Consider the empty theory in the empty signature. What is its class of models (it's not the universe of sets).

6. Consider the class Δ of definable sets. Is it definable? If so, provide a definition (in the language of set theory). Consider the class of ordinals.

7. Quantifiers are oftentimes thought of as infinitary connectives (by amateurs)– e.g. $\exists x; \phi(x) \iff \bigvee_x \phi(x)$. For instance, a text might have "$\bigcup_{n\in\mathbb{N}} A_n = A_0 \cup A_1 \cup A_2 \cup \cdots$". Is this conventional wisdom accurate? If it is not, explain why and provide a sentence which demonstrates–possibly using multiple structures–why this is the case.

8. Consider the following first order tautology: $\forall x; \phi \to \exists x; \phi$. Connectives are (correctly) thought of as being equivalent to quantifiers over finite sets. Does this mean that we have the following hold for any finite index set I: $\bigwedge_{i\in I} \phi_i \to \bigvee_{i\in I} \phi_i$?

9. Elementary substructures $A \preceq B$ are both substructures $A \subseteq B$ and elementarily equivalent $A \equiv B$, but is this sufficient to characterize elementarity? What if the two structures are isomorphic? Give counterexamples whenever possible.