HW13 637

December 19, 2024

1 Definitions

1.1 Cumulative Hierarchy

Define the *α-rank initial segment* of the Cumulative Hierarchy through the following transfinite recursion:

1. $V_0 = \varnothing$
2. $V_{\alpha+1} = P(V_\alpha)$ for every ordinal $\alpha \in \Omega$
3. $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for every limit ordinal $\lambda \in \Omega$

The Von Neumann universe is then defined as the proper class $V = \bigcup_{\alpha\in\Omega} V_\alpha$.

1.2 Constructible Universe

For a set X, define its *Definable Power Set* $D(X)$ as the set of subsets of X that are definable over $(X, \in)$ using parameters from X. The *Constructible Universe* is now defined as follows:

1. $L_0 = \varnothing$
2. $L_{\alpha+1} = D(L_\alpha)$ for any ordinal $\alpha \in \Omega$
3. $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for any limit ordinal $\lambda \in \Omega$

1.3 Satisfaction on Sets

Given a set X, and a $\in$-formula ϕ, by $X \models \phi$, we mean $(X, \in) \models \phi$, and by $(X, \in) \models \phi$, we mean $(X, \in|_X) \models \phi)$, and by $(X, \in|_X) \models \phi)$, we mean $(X, \in|_{X^2}) \models \phi)$, and by $(X, \in|_{X^2}) \models \phi)$ we mean $(X, \in \cap X^2) \models \phi$, and by $(X, \in \cap X^2) \models \phi$ we mean $(X, \in^V \cap X^2) \models \phi$, and by $(X, \in^V \cap X^2) \models \phi$, we mean $(X, \sigma, I) \models \phi$ where σ is the relational signature consisting of the single binary relation $\in$ which gets interpreted as the membership relation on X (i.e. $\alpha(\in) = 2$ and $I(\in) = \in^V \cap X^2$).

2 Actual Homework

1. Evaluate the following argument: It is clear that every element of $L_0 = \varnothing$ is definable (without parameters). Given an L_α such that every element is definable, every element of $L_{\alpha+1} = D(L_\alpha)$ is definable too by taking a formula $\varphi(x, \bar{p})$ which defines an element $a \in L_{\alpha+1}$ together with a formula $\psi(\bar{y})$ which defines $\bar{p}$ and replacing it with the formula $\exists \bar{q}; \varphi(x, \bar{q}) \wedge \psi(\bar{q})$. Lastly, the union of sets for which every element is definable will have every element be definable and so every element of $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ be definable. This makes L a pointwise definable model of ZFC.

2. A first order $\mathcal{L}$-theory T with the 0-1 property can be extended to the almost sure theory of T by adding as axioms any sentences which almost surely hold for finite models $M \models T$. Let's extend this idea to other logics: is it possible to use this same construction (or a similar one) to extend a second order/infinitary theory to its almost sure theory?

3. Characterize the cardinals κ such that $(V_\kappa, \in) \models \phi$ where ϕ can be any of the following: 'Axiom of Infinity', 'Power Set Axiom', 'Union Axiom', 'Replacement Schema', 'Axiom of Extensionality'.

4. By "$V = L$" we mean that every set occurs at some level of the constructible hierarchy. If $V = L$, is it the case that $V_\alpha = L_\alpha$ for every ordinal $\alpha \in \Omega$?

5. Suppose we have a set X which happens to model ZFC. This assumption of course entails the consistency of ZFC but is the converse true? Moreover, is every model of ZFC isomorphic to one of this form?

6. We all know and love De Morgan's Laws. Is the infinitary version of it true? That is to say, for every $\in$-formula ϕ, $\neg\forall x_0; \exists x_1; \forall x_2; ...\phi \iff \exists x_0; \forall x_1; \exists x_2; ...\neg\phi$. What if we allow ϕ to be an arbitrary $\mathcal{L}_{\infty,\infty}(\in)$ formula?

7. The construction of L is linked to definability. Is it provably the case that every definable set is in L? To clarify, you would need to prove that for every model $M \models$ ZFC of set theory, every definable $x \in M$ occurs within L^M.