EE 236B HW2
Zhiyuan Cao
304397496
01/26/2017

T2.37

(b)

$\Rightarrow$

With $p(t) \geq 0$ we can rewrite it as:

$$\begin{aligned}p(t) &= r(t)^2 + s(t)^2 = (r_1 + r_2 t + ... + r_{k+1}t^k)^2 + (s_1 + s_2 t + ... + s_{k+1}t^k)^2 \\ &= (r_1^2 + s_1^2) + (r_1 r_2 + s_1 s_2)t + ... + (r_1 r_{k+1} + s_1 s_{k+1})t^k + \\ &\quad (r_1 r_2 + s_1 s_2)t + (r_2^2 + s_2^2)t^2 + ... + (r_2 r_{k+1} + s_2 s_{k+1})t^{k+1} + \\ &\quad ... \\ &\quad + (r_1 r_{k+1} + s_1 s_{k+1})t^k + (r_2 r_{k+1} + s_2 s_{k+1})t^{k+1} + ... + (r_{k+1}^2 + s_{k+1}^2)t^{2k} \\ &= \nu^t Y \nu\end{aligned}$$

Where $\nu = [1, t, t^2 ... t^k] \in R^{k+1}$ and

$$Y = \begin{bmatrix} r_1^2 + s_1^2 & r_1 r_2 + s_1 s_2 & ... & r_1 r_{k+1} + s_1 s_{k+1} \\ r_1 r_2 + s_1 s_2 & r_2^2 + s_2^2 & ... & r_2 r_{k+1} + s_2 s_{k+1} \\ ... & ... & ... & ... \\ r_1 r_{k+1} + s_1 s_{k+1} & r_2 r_{k+1} + s_2 s_{k+1} & ... & r_{k+1}^2 + s_{k+1}^2 \end{bmatrix}$$

Since $\nu^T Y \nu \geq 0$, $Y \in S_+^{k+1}$.

$\Leftarrow$

Conversely, with $x_i = \sum_{m+n=i+1} Y_{mn}$, we can rewrite $p(t)$ as

$$\begin{aligned}p(t) &= Y_{11} + (Y_{12} + Y_{21})t + (Y_{13} + Y_{22} + Y_{31})t^2 + ... + Y_{k+1,k+1}t^{2k} \\ &= \nu^t Y \nu\end{aligned}$$

Where $\nu = [1, t, t^2 ... t^k] \in R^{k+1}$ and since $Y \in S_+^{k+1}$, we have $p(t) \geq 0$.

(c)

From $K_{pol}^* = \{z \mid x^T z \geq 0, \forall x \in K_{pol}\}$ we have

$$\sum_{i=1}^{2k+1} z_i \sum_{m+n=i+1} Y_{mn} = \sum_{m,n=1}^{k+1} Y_{mn} z_{m+n-1} = \mathrm{tr}(YH(z)) \geq 0$$

Therefore $H(z) \succeq 0$.

T3.1

(a)

$x \in [a, b]$ is equivalent to $x = \theta a + (1-\theta)b$, $0 \leq \theta \leq 1$. Plug in the inequality and we get:

$$f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta)f(b)$$

Which is the definition of a convex function.

(b)

Again, set $x = \theta a + (1-\theta)b$ and we have:

$$\frac{f(b)-f(x)}{b-x} = \frac{f(b)-f(\theta a+(1-\theta)b)}{b-\theta a-(1-\theta)b} \geq \frac{f(b)-\theta f(a)-(1-\theta)f(b)}{\theta(b-a)} = \frac{f(b)-f(a)}{b-a}$$

Where we applied result from (a) at the first inequality. Same argument can be made to the left inequality, i.e.

$$\frac{f(x)-f(a)}{x-a} = \frac{f(\theta a+(1-\theta)b)-f(a)}{\theta a+(1-\theta)b-a} \leq \frac{\theta f(a)+(1-\theta)f(b)-f(a)}{(1-\theta)(b-a)} = \frac{f(b)-f(a)}{b-a}$$

(c)

By taking the limit of $x \to a$ we have:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \leq \frac{f(b)-f(a)}{b-a}$$

We can also obtain the right inequality by taking the limit of $x \to b$.

(d)

From (c) we have:

$$\frac{f'(b)-f'(a)}{b-a} \geq 0$$

By taking the limit of $b \to a$ we get $f''(a) \geq 0$ and by taking the limit of $a \to b$ we get $f''(b) \geq 0$.

T3.2

The first function could be quasiconvex because the sublevel sets appear to be convex. It is not concave or quasiconcave since the superlevel sets are not convex. It is not convex since the function alone the blue line is not convex.

For the second function it could be concave and therefore quasiconcave. It is not convex nor quasiconvex because the sublevel sets are not convex.

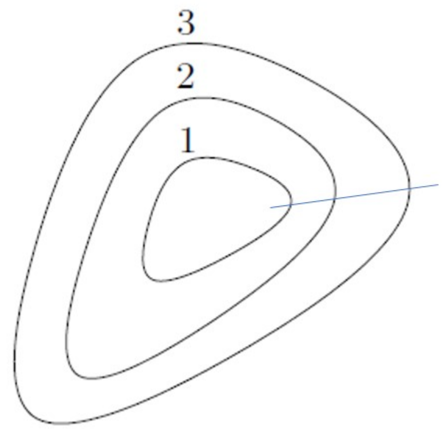

Figure 1: Quasiconvex

A2.10

The Hessian $\nabla^2 f(x)$ is given by

$$\frac{\partial^2 f(x)}{\partial x_k^2} = \frac{\alpha_k(\alpha_k - 1)}{x_k^2} f(x), \qquad \frac{\partial^2 f(x)}{\partial x_k \partial x_l} = \frac{\alpha_k \alpha_l}{x_k x_l} f(x) \qquad \text{for } k \neq l$$

and can be expressed as

$$\nabla^2 f(x) = -f(x)(\mathbf{diag}(z_k^2/\alpha_k) - zz^T)$$

Where $z_k = \alpha_k/x_k$. We must show $\nabla^2 f(x) \preceq 0$, i.e, that

$$\nu^T \nabla^2 f(x) \nu = -f(x)(\sum_{i=1}^{n} \frac{\nu_i^2 z_i^2}{\alpha_i} - (\sum_{i=1}^{n} v \nu_i z_i)^2) \leq 0$$

With

$$\sum_{i=1}^{n} \frac{\nu_i^2 z_i^2}{\alpha_i} \geq \sum_{i=1}^{n} \frac{\nu_i^2 z_i^2}{\max\{\alpha_k\}} = \frac{1}{\max\{\alpha_k\}} \sum_{i=1}^{n} \nu_i^2 z_i^2 \geq \sum_{i=1}^{n} \nu_i^2 z_i^2$$

since $\alpha_k \geq 0$, $k = 1, ..., n$ and $\sum_k \alpha_k = 1$, apply the Cauchy-Schwarz inequity $(a^T a)(b^T b) \geq (a^T b)^2$, where $a = \nu$ and $b = x$, and we get the result.

A5.8

(a)

We can write it into a convex optimization form with

$$A = \begin{bmatrix} g(t_1)^T \\ g(t_2)^T \\ \dots \\ g(t_N)^T \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_N \end{bmatrix}$$

To guarantee that $x^T g(t)$ is convex in t on $[\alpha_0, \alpha_M]$, because the cubic spline is piece-wise polynomial, its second derivative is piece-wise linear and therefore $f''(t) \geq 0$ on $[\alpha_0, \alpha_M]$ if $f''(t) \geq 0$ for $t = \alpha_0,\ \alpha_1, ..., \alpha_M$. Therefore

$$G = -\begin{bmatrix} g''(\alpha_0)^T \\ g''(\alpha_1)^T \\ \dots \\ g(\alpha_M)^T \end{bmatrix}$$

(b)

See the following figure and M-code:

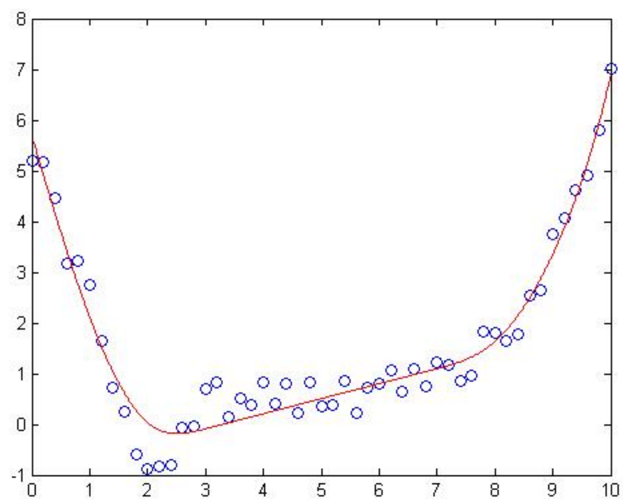

Figure 2: Cubic spline fit with convex constraint

```
clc;clear;
%%------------Setup--------------------
[t, y]=spline_data;
G=[];
A=[];
```

```
n=13;
for i=0:10
    [~, ~, gpp] = bsplines(i);
    G=[G; gpp'];
end

for i=1:length(t)
    [g, ~, ~] =bsplines(t(i));
    A=[A; g'];
end
%%------------CVX----------------------
cvx_begin
    variable x(n)
    minimize norm(A*x - y)
    subject to
        G*x >= 0
cvx_end
%%-----------Plot----------------------
fp=0:.01:10;
fm=[];
for i=1:length(fp)
    [g, ~, ~] =bsplines(fp(i));
    fm=[fm; g'];
end
f=fm*x;
plot(t, y, 'o'), hold on,
plot(fp,f, 'r')
hold off;
```