EE 236B HW5
Zhiyuan Cao
304397496
02/16/2017

A3.21

(a)

Squaring both sides and use the observation in 4.26,

$$|| \begin{bmatrix} 2y & z_1 - z_2 \end{bmatrix}^T ||_2 \leq z_1 + z_2$$

Expand this and we have

$$4y^2 + z_1^2 + z_2^2 - 2z_1z_2 \leq z_1^2 + z_2^2 + 2z_1z_2$$

This is just $y^2 \leq z_1z_2$. To extend this to the case where n is a positive integer, first assume $n = 2^k$, and we have

$$\begin{aligned} y^n &\leq (y_{11}y_{12})^{2^{k-1}} \leq (y_{21}y_{22})^{2k-2}(y_{23}y_{24})^{2k-2} \\ &\leq ... \leq (y_{k-1,1}y_{k-1,2})^2(y_{k-1,3}y_{k-1,4})^2...(y_{k-1,2^{k-1}-1}y_{k-1,2^{k-1}})^2 \\ &\leq z_1z_2...z_{2^k}, \end{aligned}$$

where we introduced variable $y_{ij},\ i = 1, ..., k-1,\ j = 1, ..., 2^{k-1}$ and have the following inequalities:

$$\begin{aligned} &y_{k-1,j} \leq (z_{2j-1}z_{2j})^{1/2}, \qquad j = 1, ..., 2^{k-1} \\ &y_{i,j} \leq (y_{i+1,2j-1}y_{i+1,2j})^{1/2}, \qquad i = 1, ..., k-2, j = 1, ..., 2^{i-1} \end{aligned}$$

With $2^k \leq n \leq 2^{k+1}$, we can write the original inequality as follow:

$$y^{2k+1} \leq y^{2k+1-n}z_1z_2...z_n$$

and applied the same method above.

(b)

We can express α as p/q, where $p, q > 0$ since α is a rational number greater than 1, and the original inequality can be written as:

$$\begin{aligned} &x^\alpha = x^{p/q} \leq t \\ &x \leq t^{q/p} = (t^q)^{1/p} \end{aligned}$$

Now we can apply the same method in (a) by setting $z_1, z_2, ..., z_{p-1} = 1$ and $z_p = t^q$.

For the second function, again we set$\alpha = -p/q$ and write the constraint as following:

$$(x^p t^q)^{1/(p+q)} \geq 1$$

and apply the same method by setting $z_1, ..., z_p = x$ and $z_{p+1}, ..., z_{p+q} = t$.

Add.1

(a)

For rule 2 we have

$$\log f = \log(\max\{g_1, g_2, ..., g_m\}) = \max\{\log g_1, \log g_2 ..., \log g_m\}$$

This is followed by logarithm function being monotonically increasing over R_{++}, which is the domain of $f(x)$. since $\log g_i, i = 1, ..., m$ are convex, $\log f$ is also convex since it is the supreme of a family of convex functions.

For rule 3 we have $h(g(1), ..., g_m(x))$ convex since $\log g_k(e^{y_1}, ..., e^{y_n})$ is convex, and log-convex functions are closed under multiplication and addition. It is nondecreasing since it is a monotone posynomial. g is also convex since it could be written as $g(e^y) = \sum_{i=1}^{K} c_i \exp^{a_i^T y}$, which is a composition of sum over a family of exponential composited with affine functions $a_i^T y$. From here we apply the composition rule and conclude that $\log f$ is convex.

Add.2

The scalarized problem is clear a GP since it can be written as the following form

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{6} w_i + ut \\ \text{subject to} & w_i - w_{max} + 1 \leq 1 \qquad i = 1, ..., 6 \\ & w_{min} - w_i + 1 \leq 1 \qquad i = 1, ..., 6 \\ & T_i - t + 1 \leq 1, \qquad i = 1, ..., 3 \end{array}$$

Where the objective function and constraints are posinomials.

```
clc;clear;
c_l = [1.5 1 5]';
w_max = 10;
w_min = 0.1;

n = 50;
u = logspace(-3,3,n);
```

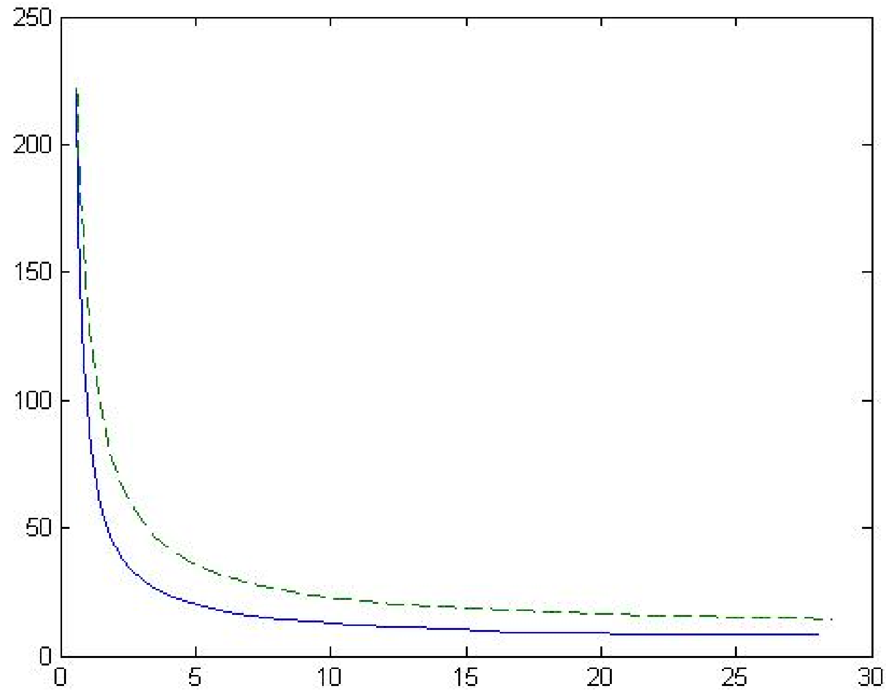

Figure 1: Comparison

```
a_1 = zeros(1,n);
d_1 = zeros(1,n);

for i=1:n
    cvx_begin gp
    variables t w(6)
    C = w;
    R = 1./w;
    minimize( sum(w) + u(i)*t )
    subject to
    w / w_max <= 1.0;
    w_min ./ w <= 1;
    (C(3) + c_l(1)) * sum(R([1,2,3])) ...
    + C(2) * sum(R([1,2])) ...
    + (sum(C([1,4,5,6])) + sum(c_l([2,3]))) * R(1) <= t;
    (C(5) + c_l(2)) * sum(R([1,4,5])) ...
    + C(4) * sum(R([1,4])) ...
    + (C(6) + c_l(3)) * sum(R([1,4])) ...
    + (sum(C([1,2,3])) + c_l(1)) * R(1) <= t;
    (C(6) + c_l(3)) * sum(R([1,4,6])) ...
    + C(4) * sum(R([1,4])) ...
```

```
    + sum(C([1,2,3]) + c_l(1)) * R(1) ...
    + (C(5) + c_l(2)) * sum(R([1,4])) <= t;
    cvx_end
    d_1(i) = t;
    a_1(i) = sum(w);
end;

n=100;
ws = logspace(-1,1,n);
a_2 = zeros(1,n);
d_2 = zeros(1,n);
for i=1:length(ws)
    w = ws(i)*ones(6,1);
    a_2(i) = sum(w);
    C = w;
    R = 1./w;
    t1 = (C(3) + c_l(1)) * sum(R([1,2,3])) ...
    + C(2) * sum(R([1,2])) ...
    + (sum(C([1,4,5,6])) + sum(c_l([2,3]))) * R(1);
    t2 = (C(5) + c_l(2)) * sum(R([1,4,5])) ...
    + C(4) * sum(R([1,4])) ...
    + (C(6) + c_l(3)) * sum(R([1,4])) ...
    + (sum(C([1,2,3])) + c_l(1)) * R(1);
    t3 = (C(6) + c_l(3)) * sum(R([1,4,6])) ...
    + C(4) * sum(R([1,4])) ...
    + sum(C([1,2,3]) + c_l(1)) * R(1) ...
    + (C(5) + c_l(2)) * sum(R([1,4]));
    d_2(i) = max([t1, t2, t3]);
end;
plot(a_1, d_1, '-', a_2, d_2, '--');
```

Add.3

(a)

$\Rightarrow$

Provided that $X \in S^n$, We introduce $\nu = [\nu_1, \nu_k]^T$, where $\nu_1 \in R$ and $\nu_k \in R^{n-1}$. Since $X \succeq 0$, we have

$$\nu X \nu = [\nu_1, \nu_k^T] \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_k \end{bmatrix} = \nu_k^T B^T \nu_1 + \nu_1 B \nu_k + \nu_k^T C \nu_k \geq 0$$

for any ν. This is equivalent to $B = 0$ and $C \succeq 0$.

$\Leftarrow$

Provided that $B = 0$ and $C \succeq 0$, we can introduce $\nu = [\nu_1, \nu_k]^T$ and show $\nu X \nu \geq 0$.

(b)

$$AA^+ = [Q1Q2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} Q_1 \Lambda_1^{-1} Q_1^T = Q_1 Q_1^T$$
$$A^+A = Q_1 \Lambda_1^{-1} Q_1^T [Q1Q2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} = Q_1 Q_1^T$$

$$I - AA^+ = I - A^+A = [Q_1 \; Q_2] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} - Q_1 Q_1^T = Q_2 Q_2^T$$

(c)

$\Rightarrow$

Assume $A \in S^n$ and $X \in S^{n+m}$. Introduce $\nu = [\nu_1^T, \nu_2^T]$, where $\nu_1 \in R^n$ and $\nu_2 \in R^m$. Since $X \succeq 0$, we have

$$\nu X \nu = [\nu_1^T, \nu_2^T] \begin{bmatrix} \Lambda & Q^T B \\ B^T Q & C \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \nu_k^T B^T \nu_1 + \nu_1 B \nu_1 + \nu_k^T C \nu_k \geq 0$$

A3.11

(a)

The problem can be written as:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & c^T \\ c & t \end{bmatrix} \succeq 0 \end{array}$$

This is followed by *Schur Component* with $t - c^T F^{-1}(x) c \geq 0$.

(b)

The problem can be written as:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & c^T \\ c & t \end{bmatrix} \succeq 0, \qquad i = 1, ..., K \end{array}$$

(c)

$$f(x) = \lambda_{max}(F^{-1}(x)) \leq t \Leftrightarrow F^{-1}(x) \leq tI$$

Therefore the problem is formed as followed

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} F(x) & I \\ I & tI \end{bmatrix} \succeq 0 \end{array}$$

(d)

$$\begin{aligned} E[c^T F^{-1} c] &= E[\bar{c}^T F^{-1} \bar{c} + (c-\bar{c})^T F^{-1} (c-\bar{c})] \\ &= E[\bar{c}^T F^{-1} \bar{c}] + E[\sum_{i=1}^{m} \sum_{j=1}^{m} f_{ij} y_i y_j] \\ &= \bar{c}^T F^{-1} \bar{c} + \mathbf{tr}(F^{-1} S) \end{aligned}$$

where $y_i = c_i - \bar{c}_i$. If we factor S as $S = \sum_{k=1}^m c_k c_k^T$ the problem is equivalent to

$$\mathit{minimize} \qquad \bar{c}^T F^{-1} \bar{c} + \sum_{k=1}^{m} c_k^T F(x)^{-1} c_k$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t_0 + \sum_k t_k \\ \text{subject to} & \begin{bmatrix} F(x) & \bar{c}^T \\ \bar{c}^T & t_0 \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} F(x) & c_k^T \\ c_k^T & t_k \end{bmatrix} \succeq 0, \qquad k = 1, ..., m \end{array}$$