EE 236A HW5
Zhiyuan Cao
304397496
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Problem 1

(1)

We could rewrite the max-flow LP in matrix form as followed:

$$\underbrace{\begin{bmatrix} I_{N-1} \; 0 \\ M \\ -I_N \end{bmatrix}}_{\text{A}} f = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}$$

We proved that M is a TUM during the class. Now consider following situations with u being a square sub-matrix of A.

1. If all rows belong to M: u is TUM since M is TUM.
2. If some of the rows belongs to the first/third Identity matrices in A
   - If it contains a row with all zero, the determinant is 0.
   - Otherwise, expand along columns that contains 1/-1, and the determinant will falls into one of the situations discussed and eventually ends up in either 0, 1 or -1.

Therefore we could conclude that A is a TUM.

(2)

First, we relax the ILP into LP as followed:

$$\begin{aligned} &\textit{minimize} && c^T d \\ &\textit{subject to} && p_s - p_t \geq 1 \\ &&& d_{ij} - p_i + p_j \geq 0 \\ &&& 0 \leq d \leq 1 \\ &&& 0 \leq p \leq 1 \end{aligned}$$

This is similar to the max-flow LP with two extra constraints: $d \leq 1\&p \leq 1$. To prove equivalence we prove that these two constraints are redundant.

First, the optimal solution should satisfy the first constraint with equality. if $p_s - p_t = c \geq 1$, then we could acquire a better solution with p/c and d/c. Moreover, we could prove that at optimal point, $p_s = 1$ and $p_t = 0$ using the same method.

Second, if there is an edge from i to j, then we would have $p_i < p_j$. With the optimal solution, we always have $d_{ij} = max(0, p_i - p_j)$. If an optimal solution has $p_i < p_j$, then $d_ij = 0$ and we could have $p_i = p_j$ without losing optimality.

Third, given $0 \leq p_i \leq 1$ we can conclude that $0 \leq d_{ij} \leq 1$ from $d_{ij} = max(0, p_i - p_j)$.

Therefore we can conclude that those two constraints are redundant. To prove these two problems have the same optimal value, recall that the feasible polyhedron has integer vertices due to TUM property.

Problem 2

For case 1, to solve the problem with undirected graph $G = (V, E)$. We could replace each edge with two edges pointing toward the opposite direction the with same capacity constraint. An LP can be formulated as follow with the new graph $G = (V, E')$:

$$
\begin{aligned}
&maximum && f_{st} \\
&subject\ to && f_{ij} \leq c_{ij}, \quad \forall (i,j)\ in\ E' \\
& && \sum_{j:(i,j)\in E'} f_{ij} - \sum_{k:(i,k)\in E'} f_{jk} \leq 0,\ \forall i \in V \\
& && f_{i,j} \geq 0, \forall (i,j) \in E'
\end{aligned}
$$

For case 2, we could simply add an additional constraint on top of case 1:

$$
\begin{aligned}
&maximum && f_{st} \\
&subject\ to && f_{ij} \leq c_{ij}, \quad \forall (i,j)\ in\ E' \\
& && \sum_{j:(i,j)\in E'} f_{ij} - \sum_{k:(i,k)\in E'} f_{jk} \leq 0,\ \forall i \in V \\
& && f_{i,j} \geq 0, \forall (i,j) \in E' \\
& && f_{ij} + f_{ji} \leq c_{ij},\ \forall (i,j) \in E'
\end{aligned}
$$

Problem 3

(1)

Let's name the LP with the edge setup as ① and path setup as ②. To prove equivalence:

① $\Rightarrow$ ②

Given a feasible solution of ①, we can acquire a feasible set of f_i through flow decomposition:

1. Start with path P_1. For this path P_1 select the edge e^{min} such that $e^{min} = arg \min_{e \in P_1} f_e^\star$. Set the flow of the path as $f_{e^{min}}$.

2. Subtract $f_{e^{min}}$ from all edges that belongs to P_1.
3. Remove edges from the graph that only belong to P_1.
4. Repeat the process with path index +1 until all edges with flows greater than zero are assigned and removed.

Such set P of path is clearly feasible because of the flow conservation constraint in ①. To see ① and ② have the same optimal value, notice that each path ends up in the sink t, and therefore for each P_i in P, there exist j such that $(j, t) \in P_i$.

$$\sum f_i = \sum_{j:(j,t)\in E} f_{jt} = f_{ds}$$

② ⇒ ①

Given a set of viable path P. We can simply add them up to acquire a set E of edge flow. Such set E satisfies the capacity constraint because of $\sum_{(i:e\in P_i)} f_i \leq c_e$. The flow conservation constraint also holds because for $i \in V$, for all paths that go through V_i we have $f_{in} = f_{out}$. And by summing up we acquire the flow conservation constraint. Same argument could be use to prove that the objective function value of the two problems are the same.

(2, 3)

To prove the statement. We first derive the dual of ② as follow:

$$\begin{aligned} &\textit{minimize} && \sum C_e d_e \\ &\textit{subject to} && \sum_{e\in P_i} d_e \geq 1, \ \forall P_i \\ & && d_e \geq 0 \end{aligned}$$

The dual of ② is a min-cut problem with d_e representing as an indicator variable whether edge e is in the cut, i.e.,

$$d_e = \begin{cases} 1 & e \in C^\star(s,t) \\ 0 & \text{otherwise.} \end{cases}$$

Now statement in (2) is obviously true. When we have $C_e = 1, \ \forall e \in E$, the objective function becomes $\sum d_e = k$ and therefore for the min-cut $C^\star(s,t)$ there is k edge-disjoint paths from s to t.

Problem 4

1. If the optimal solution is unique

This problem is trivial since the max-flow solution is the solution with minimum penalty.

2. If the optimal solution is not unique

   To acquire a max-flow solution that also has the minimum penalty, we could first solve the max-flow problem with optimal value $p^\star$. Then we solve the following LP:

$$\begin{array}{ll} \textit{minimize} & \sum p_e f_e \\ \textit{subject to} & f_{ij} \le c_{ij},\ \forall (i,j) \in E \\ & \sum\limits_{j:(i,j)\in E} f_{ij} - \sum\limits_{k:(i,k)\in E} f_{jk} \le 0,\ \forall i \in V \\ & f_{i,j} \ge 0, \forall (i,j) \in E \\ & \sum\limits_{i:(s,i)\in E} f_{si} = p^\star \end{array}$$

   The first three constraints are exactly the same ones used in solving the max-flow problem to guarantee a same solution space. The last constraint is added to guarantee the max-flow is reached while minimizing the penalty.

Problem 5

The problem states that Bob wants to visit each house H exactly once with the shortest total distance. This is similar to traveling salesman problem that could be formed into an ILP.

Assume we have n homes, with distance between home i and j as c_{ij}, and define an auxiliary variable u_i, where $i,\ j = 1, ..., n$. Define variable x_{ij} as follow:

$$x_{ij} = \begin{cases} 1 & \text{the path goes from i to j} \\ 0 & \text{otherwise.} \end{cases}$$

An ILP can be formed as follow:

$$\begin{array}{lll} \textit{minimize} & \sum\limits_{i=0`}^{n} \sum\limits_{j\ne i, j=1}^{n} c_{ij} x_{ij} & \\ \textit{subject to} & \sum\limits_{i=0, i\ne j}^{n} x_{ij} = 1 & j = 1, ..., n \\ & \sum\limits_{j=0, j\ne i}^{n} x_{ij} = 1 & i = 1, ..., n \\ & x_{ij} \in \{0,1\} & i,j = 1, ..., n \\ & u_i - u_j + n x_{ij} \le n-1 & 1 \le i \ne j \le n \end{array}$$

The first two constraints guarantee that each home can serve as the destination and starting point for exactly once. The third constraint restrict potential value of x_{ij} to 0 and 1. The last constraint enforces that there is only one single tour, i.e., no disjointed tours. To see why the last constraint holds, it is sufficient to show that every subtour in a feasible solution pass through home 0 (restaurant). If we sum all inequalities corresponding to $x_{ij} = 1$ for any subtour of k steps without passing through home 0, we have $nk \leq (n-1)k$ which is a contradiction. Therefore, there exist only one single tour.