EE 236B HW8
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T5.18

For a given a, with introducing Lagrange multiplier $z_1 \in R^n$, we always have

$$\begin{aligned}\inf_{x \in P_1} a^T x &= \inf_x (a^T x + z_1^T (Ax - b)) \\ &= (a + A^T z_1)^T x - b z_1 \\ &= \begin{cases} -b z_1 & \text{if } A^T z_1 + a = 0 \\ -\infty & \text{otherwise} \end{cases}\end{aligned}$$

Same, with introducing Lagrange multiplier $z_1 \in R^n$, we have

$$\sup_{x \in P_2} a^T x = \begin{cases} -d z_2 & \text{if } C^T z_2 - a = 0 \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, we could find a by formulating a minimization problem and solve its dual,

$$\begin{array}{ll} \text{maximize} & \inf_{x \in P_1} a^T x - \sup_{x \in P_2} a^T x \\ \text{subject to} & ||a||_1 \leq 1 \end{array}$$

and set $\gamma = (\inf_{x \in P_1} a^T x - \sup_{x \in P_2} a^T x)/2$. Since the objective value is homogeneous in a, a bound is add. The dual is as follow:

$$\begin{array}{ll} \text{maximize} & -bz_1 + dz_2 \\ \text{subject to} & C^T z_2 - a = 0 \\ & A^T z_1 + a = 0 \\ & z_1, z_2 \geq 0. \\ & ||a||_1 \leq 1 \end{array}$$

A4.25

First we can write its equivalence,

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \alpha_i \log(\sum_{j=1}^{n} A_{ij} x_j) \\ \text{subjct to} & \sum_{i=1}^{n} \alpha_i \log x_i = 0 \end{array}$$

Thus, the Lagrangian can be formed as:

$$L(x, z) = \sum_{i=1}^{n} \alpha_i \log(\sum_{j=1}^{n} A_{ij}x_j) + z\sum_{i=1}^{n} \alpha_i \log x_i$$

Since this is differential over x, we take the partial to x_k and have

$$\sum_{i=1}^{n} \alpha_i \frac{A_{ik}x_k}{\sum_{j=1}^{n} A_{ij}x_j} = \alpha_k z, \qquad k = 1, ..., n$$

summing up $k = 1, ..., n$ and we have

$$\sum_{k=1}^{n}\sum_{i=1}^{n} \alpha_i \frac{A_{ik}x_k}{\sum_{j=1}^{n} A_{ij}x_j} = \sum_{i=1}^{n} \alpha_i \frac{\sum_{k=1}^{n} A_{ik}x_k}{\sum_{j=1}^{n} A_{ij}x_j} = \sum_{k=1}^{n} \alpha_k z$$

Therefore, $z = 1$. and we have

$$\sum_{i=1}^{n} \alpha_i \frac{A_{ik}x_k}{\sum_{j=1}^{n} A_{ij}x_j} = \alpha_k, \qquad k = 1, ..., n$$

Or in vector form,

$$\mathrm{diag}(Ax)^{-1}\mathrm{diag}(x)A^T\alpha = \alpha$$

For the original equation, we have

$$\begin{aligned}
(D_1AD_2)^Tv = D_2A^TD_1v &= \mathrm{diag}(u)^{-1}\mathrm{diag}(x)A^T\mathrm{diag}(u)\mathrm{diag}(Ax)^{-1}v \\
&= \mathrm{diag}(u)^{-1}\mathrm{diag}(Ax)^{-1}\mathrm{diag}(x)A^T\alpha \\
&= \mathrm{diag}(u)^{-1}\alpha \\
&= v
\end{aligned}$$

A4.26

(a)

$$\begin{aligned}
L(x, y, z) &= \gamma||x||_1 + ||y||_2 + z^T(Ax - b - y) \\
&= \gamma||x||_1 + z^TAx + (||y||_2 - z^Ty) - b^Tz \\
g(z) &= \inf_{x,y} L(x, y, z) \\
&= \begin{cases} \inf_x(\gamma||x||_1 + z^TAx - b^Tz), & \text{if } ||z||_2 \le 1 \\ -\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} -b^Tz, & \text{if } ||A^Tz||_\infty \le \gamma \\ -\infty & \text{otherwise} \end{cases}
\end{aligned}$$

Therefore the dual can be written as

$$\begin{array}{ll} \text{maximize} & -\,b^Tz \\ \text{subject to} & ||z||_2 \leq 1 \\ & ||A^Tz||_\infty \leq \gamma \end{array}$$

(b)

We examine the KKT conditions,

1. $x, y \text{satisfy} Ax - b = y$
2. $||A^Tz||_\infty \leq \gamma, ||z||_2 \leq 1$
3. $x^\star, y^\star$ minimize $L(x, y, z^\star)$.

Therefore for the last condition, we have

$$y^\star = \text{argmin}(||y||_2 - y^Tz^\star) = 0$$

Therefore $z^\star = y^\star/||y^\star||_2 = (Ax^\star - b)/(||Ax^\star - b||_2) = r$, i.e., r is dual optimal. and therefore we have

$$||A^Tr||_\infty \leq \gamma, \qquad r^TAx^\star + \gamma||x^\star||_1 = 0$$

With the equation followed by strong duality.

(c)

We write A as followed,

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

Where $a_i \in R^m, i = 1, ..., n$.

From the strong duality

$$\begin{aligned} (A^Tz)^Tx^\star + \gamma||x^\star||_1 &= a_1^Tzx_1^\star + \cdots + a_n^Tzx_n^\star + \gamma(|x_1^\star| + \cdots + |x_n^\star|) \\ &= (a_1^Tzx_1^\star + \gamma|x_1^\star|) + ... + (a_n^Tzx_n^\star + \gamma|x_n^\star|) \\ &\geq (-|a_1^Tz|x_1^\star + \gamma|x_1^\star|) + ... + (-|a_n^Tz|x_n^\star + \gamma|x_n^\star|) \\ &\geq (-\gamma x_1^\star + \gamma|x_1^\star|) + ... + (-\gamma x_n^\star + \gamma|x_n^\star|) \\ &= 0 \end{aligned}$$

Since $(A^Tz)^Tx^\star + \gamma||x^\star||_1 = 0$. for the i^{th} term we must either have $a_i^Tz = -|a_i^Tz| = -\gamma$ or $x_i = 0$. Since $||a_i|| < \gamma$, we have $-|a_i^Tz| \leq -||a_i||||z|| < \gamma$ and therefore we must have $x_i = 0$.

A6.5

(a)

From the derivative constraint, we must have

$$\frac{y_{i+1} - y_i}{\beta} \leq z_{i+1} - z_i \leq \frac{y_{i+1} - y_i}{\alpha}, \qquad i = 1, ..., m-1$$

Therefore we can solve the following function

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m} ||z_i - a_i^T x||_2 \\ \text{subject to} & \dfrac{y_{i+1} - y_i}{\beta} \leq z_{i+1} - z_i \leq \dfrac{y_{i+1} - y_i}{\alpha}, \qquad i = 1, ..., m-1, \end{array}$$

with variable $z \in R^m, \ x \in R^n$.

(b)

```
clc;clear;
nonlin_meas_data;
B = [-eye(m - 1), zeros(m - 1, 1)] ...
    + [zeros(m - 1, 1), eye(m - 1)];
cvx_begin
    variables x(n) z(m);
    minimize( norm( z - A*x ) );
    subject to
        (B*y)/beta <= B*z;
        B*z <= (B*y)/alpha;
cvx_end

figure(1), plot(z, y),
figure(2), scatter(A*x, y, '.');
```

A7.1

(a)

The ellipsoid $\mathcal{E} = \{Q^{1/2}y \mid ||y||_2 \leq 1\}$ is contained in C if and only if

$$\sup_{||y||_2 \leq 1} a_i^T(Q^{1/2}y) \leq ||Q^{1/2}a_i||_2 = a_i^T Q a_i \leq 1, \qquad i = 1, ..., p.$$

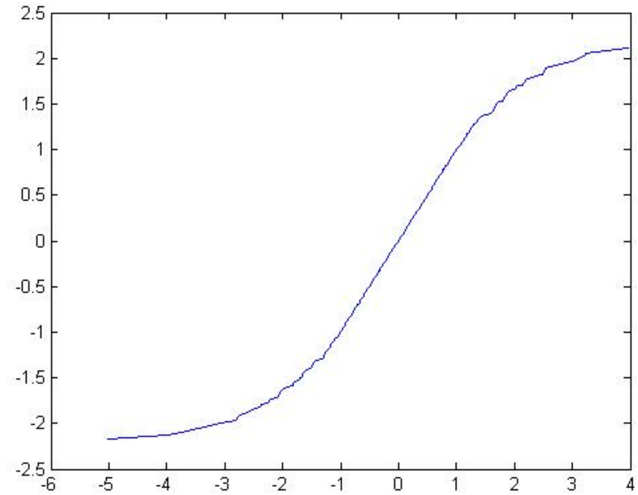

Figure 1: Estimation of $\phi(z)$

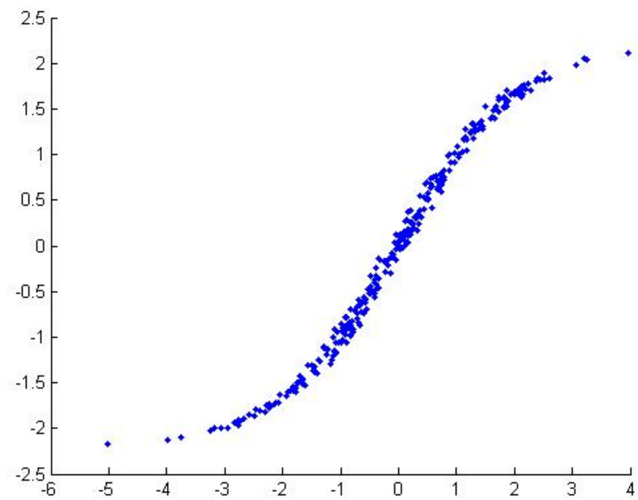

Figure 2: Estimation of $a_i^T x$

(b)

Introducing Lagrange multiplier $z \in R^P$.

$$L(Q,z) = \log\det Q^{-1} + \sum_{i=1}^{p} z_i a_i^T Q a_i - \sum_{i=1}^{p} z_i$$

$$G(z) = \inf_Q L(Q,z) =$$

$$= \begin{cases} \log\det(\sum_{i=1}^{p} z_i a_i a_i^T) + n - \sum_{i=1}^{p} z_i, & \text{if } \sum_{i=1}^{p} z_i a_i a_i^T \succ 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Therefore the dual is

$$\begin{array}{ll} \text{maximize} & \log\det(\sum_{i=1}^{p} z_i a_i a_i^T) + n - \sum_{i=1}^{p} z_i \\ \text{subject to} & z \geq 0 \end{array}$$

(c)

The KKT conditions are

1. $Q \succ 0$ and $a_i^T Q a_i \leq 1, i = 1, ..., p$
2. $z \geq 0$
3. $z_i(a_i^T Q a_i - 1) = 0, i = 1, ..., p$
4. $Q^{-1} = \sum_{i=1}^{p} z_i a_i a_i^T$

Taking the inner product of the last condition,

$$n = \mathbf{tr}(\sum_{i=1}^{p} z_i a_i^T Q a_i) = \sum_{i=1}^{p} z_i$$

Next we have,

$$x^T Q^{-1} x = x^T(\sum_{i=1}^{p} z_i a_i a_i^T)x = \sum_{i=1}^{p} z_i (a_i^T x)^2 \leq \sum_{i=1}^{p} z_i = n.$$

for $x \in C$, i.e., $|a_i^T x| leq 1$ for $i = 1, ..., p$.