

A3.21

(a)

Squaring both sides and use the observation in 4.26,

$$\| [2y \quad z_1 - z_2]^T \|_2 \leq z_1 + z_2$$

Expand this and we have

$$4y^2 + z_1^2 + z_2^2 - 2z_1z_2 \leq z_1^2 + z_2^2 + 2z_1z_2$$

This is just $y^2 \leq z_1z_2$. To extend this to the case where n is a positive integer, first assume $n = 2^k$, and we have

$$\begin{aligned} y^n &\leq (y_{11}y_{12})^{2^{k-1}} \leq (y_{21}y_{22})^{2^{k-2}}(y_{23}y_{24})^{2^{k-2}} \\ &\leq \dots \leq (y_{k-1,1}y_{k-1,2})^2(y_{k-1,3}y_{k-1,4})^2 \dots (y_{k-1,2^{k-1}-1}y_{k-1,2^{k-1}})^2 \\ &\leq z_1z_2 \dots z_{2^k}, \end{aligned}$$

where we introduced variable y_{ij} , $i = 1, \dots, k-1$, $j = 1, \dots, 2^{k-1}$ and have the following inequalities:

$$\begin{aligned} y_{k-1,j} &\leq (z_{2j-1}z_{2j})^{1/2}, \quad j = 1, \dots, 2^{k-1} \\ y_{i,j} &\leq (y_{i+1,2j-1}y_{i+1,2j})^{1/2}, \quad i = 1, \dots, k-2, j = 1, \dots, 2^{i-1} \end{aligned}$$

With $2^k \leq n \leq 2^{k+1}$, we can write the original inequality as follow:

$$y^{2^{k+1}} \leq y^{2^{k+1}-n} z_1 z_2 \dots z_n$$

and applied the same method above.

(b)

We can express α as p/q , where $p, q > 0$ since α is a rational number greater than 1, and the original inequality can be written as:

$$\begin{aligned} x^\alpha &= x^{p/q} \leq t \\ x &\leq t^{q/p} = (t^q)^{1/p} \end{aligned}$$

Now we can apply the same method in (a) by setting $z_1, z_2, \dots, z_{p-1} = 1$ and $z_p = t^q$.

For the second function, again we set $\alpha = -p/q$ and write the constraint as following:

$$(x^p t^q)^{1/(p+q)} \geq 1$$

and apply the same method by setting $z_1, \dots, z_p = x$ and $z_{p+1}, \dots, z_{p+q} = t$.

Add.1

(a)

For rule 2 we have

$$\log f = \log(\max\{g_1, g_2, \dots, g_m\}) = \max\{\log g_1, \log g_2, \dots, \log g_m\}$$

This is followed by logarithm function being monotonically increasing over R_{++} , which is the domain of $f(x)$. since $\log g_i, i = 1, \dots, m$ are convex, $\log f$ is also convex since it is the supreme of a family of convex functions.

For rule 3 we have $h'(z_1, \dots, z_n) = \log h(e^{z_1}, \dots, e^{z_n})$ as convex and nondecreasing in each of its argument. Hence,

$$\begin{aligned} \log f(e^{y_1}, \dots, e^{y_n}) &= \log h(g_1(e^{y_1}, \dots, e^{y_n}), \dots, g_m(e^{y_1}, \dots, e^{y_n})) \\ &\quad h'(\log g_1(e^{y_1}, \dots, e^{y_n}), \dots, \log g_m(e^{y_1}, \dots, e^{y_n})) \end{aligned}$$

From here we can establish convexity of $\log f$ from composition rule with h' convex and nondecreasing in each of its argument and $\log g_i, i = 1, \dots, n$ as a convex function.

Add.2

The scalarized problem is clear a GP since it can be written as the following form

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^6 w_i + ut \\ \text{subject to} \quad & w_i - w_{max} + 1 \leq 1 \quad i = 1, \dots, 6 \\ & w_{min} - w_i + 1 \leq 1 \quad i = 1, \dots, 6 \\ & T_i - t + 1 \leq 1, \quad i = 1, \dots, 3 \end{aligned}$$

Where the objective function and constraints are posinomials.

```
clc;clear;
c_1 = [1.5 1 5]';
w_max = 10;
w_min = 0.1;
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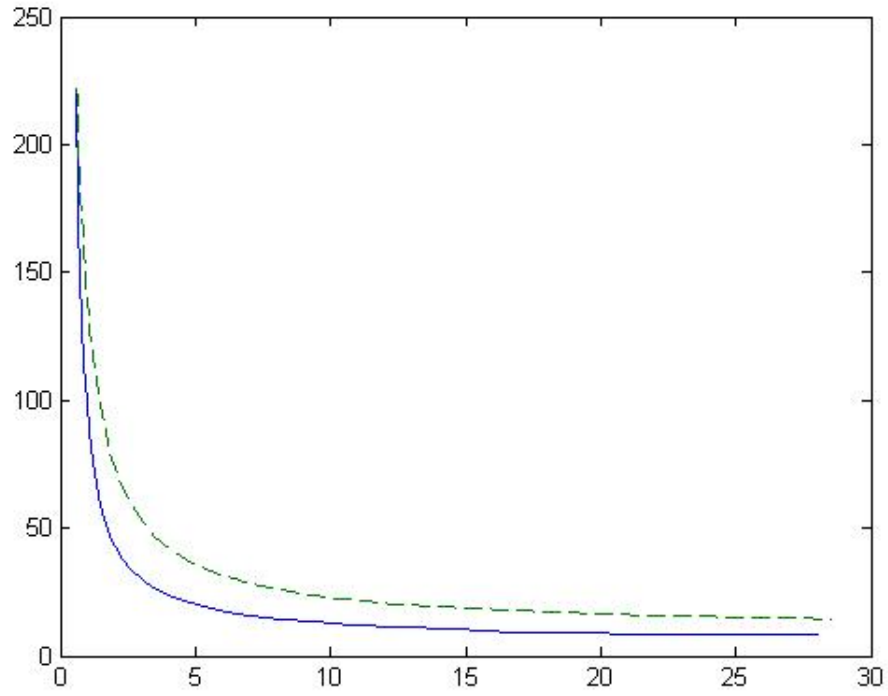


Figure 1: Comparison

```

n = 50;
u = logspace(-3,3,n);
a_1 = zeros(1,n);
d_1 = zeros(1,n);

for i=1:n
    cvx_begin gp
        variables t w(6)
        C = w;
        R = 1./w;
        minimize( sum(w) + u(i)*t )
        subject to
            w / w_max <= 1.0;
            w_min ./ w <= 1;
            (C(3) + c_1(1)) * sum(R([1,2,3])) ...
            + C(2) * sum(R([1,2])) ...
            + (sum(C([1,4,5,6])) + sum(c_1([2,3]))) * R(1) <= t;
            (C(5) + c_1(2)) * sum(R([1,4,5])) ...
            + C(4) * sum(R([1,4])) ...
            + (C(6) + c_1(3)) * sum(R([1,4])) ...

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+ (sum(C([1,2,3])) + c_1(1)) * R(1) <= t;
(C(6) + c_1(3)) * sum(R([1,4,6])) ...
+ C(4) * sum(R([1,4])) ...
+ sum(C([1,2,3]) + c_1(1)) * R(1) ...
+ (C(5) + c_1(2)) * sum(R([1,4])) <= t;
cvx_end
d_1(i) = t;
a_1(i) = sum(w);
end;

n=100;
ws = logspace(-1,1,n);
a_2 = zeros(1,n);
d_2 = zeros(1,n);
for i=1:length(ws)
    w = ws(i)*ones(6,1);
    a_2(i) = sum(w);
    C = w;
    R = 1./w;
    t1 = (C(3) + c_1(1)) * sum(R([1,2,3])) ...
+ C(2) * sum(R([1,2])) ...
+ (sum(C([1,4,5,6])) + sum(c_1([2,3]))) * R(1);
    t2 = (C(5) + c_1(2)) * sum(R([1,4,5])) ...
+ C(4) * sum(R([1,4])) ...
+ (C(6) + c_1(3)) * sum(R([1,4])) ...
+ (sum(C([1,2,3])) + c_1(1)) * R(1);
    t3 = (C(6) + c_1(3)) * sum(R([1,4,6])) ...
+ C(4) * sum(R([1,4])) ...
+ sum(C([1,2,3]) + c_1(1)) * R(1) ...
+ (C(5) + c_1(2)) * sum(R([1,4]));
    d_2(i) = max([t1, t2, t3]);
end;
plot(a_1, d_1, '- ', a_2, d_2, '--');

```

Add.3

(a)

\Rightarrow

Provided that $X \in S^n$, We introduce $\nu = [\nu_1, \nu_2]^T$, Since $X \succeq 0$, we have

$$\nu^T X \nu = [\nu_1^T, \nu_2^T] \begin{bmatrix} 0 & B \\ B^T & C \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_k \end{bmatrix} = 2\nu_1^T B \nu_2 + \nu_2^T C \nu_2 \geq 0$$

for any ν . set $\nu_1 = -tB\nu_2$ and we have

$$-2t||B\nu_2|| + \nu_2^T C \nu_2 \geq 0$$

with $t \rightarrow -\infty$, we must have $B = 0$ to satisfy the condition. This also shows $C \succeq 0$.

\Leftarrow

Provided that $B = 0$ and $C \succeq 0$, we can introduce $\nu = [\nu_1, \nu_k]^T$ and show $\nu X \nu \geq 0$.

(b)

$$\begin{aligned} AA^+ &= [Q_1 Q_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} Q_1 \Lambda_1^{-1} Q_1^T = Q_1 Q_1^T \\ A^+ A &= Q_1 \Lambda_1^{-1} Q_1^T [Q_1 Q_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} = Q_1 Q_1^T \end{aligned}$$

$$I - AA^+ = I - A^+ A = [Q_1 \ Q_2] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} - Q_1 Q_1^T = Q_2 Q_2^T$$

(c)

\Rightarrow

Let

$$A = [Q_2^T \ Q_1^T] \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} Q_2 \\ Q_1 \end{bmatrix}$$

$$\nu X \nu = [\nu_1^T, \nu_2^T, \nu_3^T] \begin{bmatrix} 0 & 0 & Q_2^T B \\ 0 & \Lambda & Q_1^T B \\ B^T Q_2 & B^T Q_1 & C \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} \succeq 0$$

Based on (a), we must have $B^T Q_2 = 0$, which is equivalent to $(I - AA^+)B = 0$. Also we must have

$$\begin{bmatrix} \Lambda & Q_1^T B \\ B^T Q_1 & C \end{bmatrix} \succeq 0$$

Apply the general *Schur Component* and we have $C - B^T Q_1 \Lambda Q_1^T B = C - B^T A^+ B \geq 0$.

A3.11

(a)

The problem can be written as:

$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && \begin{bmatrix} F(x) & c^T \\ c & t \end{bmatrix} \succeq 0 \end{aligned}$$

This is followed by *Schur Component* with $t - c^T F^{-1}(x)c \geq 0$.

(b)

The problem can be written as:

$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && \begin{bmatrix} F(x) & c^T \\ c & t \end{bmatrix} \succeq 0, \quad i = 1, \dots, K \end{aligned}$$

(c)

$$f(x) = \lambda_{\max}(F^{-1}(x)) \leq t \Leftrightarrow F^{-1}(x) \leq tI$$

Therefore the problem is formed as followed

$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && \begin{bmatrix} F(x) & I \\ I & tI \end{bmatrix} \succeq 0 \end{aligned}$$

(d)

$$\begin{aligned} E[c^T F^{-1}c] &= E[\bar{c}^T F^{-1}\bar{c} + (c - \bar{c})^T F^{-1}(c - \bar{c})] \\ &= E[\bar{c}^T F^{-1}\bar{c}] + E\left[\sum_{i=1}^m \sum_{j=1}^m f_{ij} y_i y_j\right] \\ &= \bar{c}^T F^{-1}\bar{c} + \text{tr}(F^{-1}S) \end{aligned}$$

where $y_i = c_i - \bar{c}_i$. If we factor S as $S = \sum_{k=1}^m c_k c_k^T$ the problem is equivalent to

$$\text{minimize} \quad \bar{c}^T F^{-1}\bar{c} + \sum_{k=1}^m c_k^T F(x)^{-1} c_k$$

which is equivalent to

$$\begin{array}{ll}
\text{minimize} & t_0 + \sum_k t_k \\
\text{subject to} & \begin{bmatrix} F(x) & \bar{c}^T \\ \bar{c}^T & t_0 \end{bmatrix} \succeq 0 \\
& \begin{bmatrix} F(x) & c_k^T \\ c_k^T & t_k \end{bmatrix} \succeq 0, \quad k = 1, \dots, m
\end{array}$$