EE 236A HW4
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Problem 1

Let $A_{ij} = \bar{A}_{ij} + \alpha\Delta A_{ij}$, where $-1 \le \alpha \le 1$. Since $\Delta A_{ij} \ge 0$, we have

$$\bar{A}_{ij} - \Delta A_{ij} \le \bar{A}_{ij} + \alpha\Delta A_{ij} \le \bar{A}_{ij} + \Delta A_{ij}$$
$$\bar{A}_{ij}x - \Delta A_{ij}|x| \le (\bar{A}_{ij} + \alpha\Delta A_{ij})x = \bar{A}_{ij}x + \alpha\Delta A_{ij}x \le \bar{A}_{ij}x + \Delta A_{ij}|x|$$

To make sure x is feasible for all possible values of A, the following inequality should be established:

$$\bar{A}_{ij}x + \Delta A_{ij}|x| \le b$$

And therefore an LP can be formed as follow:

$$\begin{array}{ll} \textit{minimize} & c^T x \\ \textit{subject to} & \bar{A}_{ij}x + \Delta A_{ij}|x| \le b \end{array}$$

Problem 2

The given system could be reformed as follow:

$$\begin{aligned} &(P - I)y = 0 \\ &\mathbf{1}^T y = 1 \\ &y \le 0 \end{aligned}$$

From Farkas's Lemma, to prove the above system to be feasible, we can prove the following alternative system to be infeasible, i.e., there exist no $z \in R^n,\ \omega \in R$ such that satisfy:

$$\begin{aligned} &(P - I)z + \mathbf{1}\omega = 0 \\ &\omega < 0 \end{aligned}$$

Combine the two and we have

$$(P - I)z > 0$$

or,

$$Pz > z \tag{1}$$

This is impossible due to the fact that p_{ij} is positive and columns of P sum to one and we should have

$$(Pz)_i \leq \max\{z_j\}, \qquad i, j = 1, ..., n$$

Which contradicts to (1), and therefore we can prove that the original system is feasible.

Problem 3

An LP can be formed as follow:

$$\begin{array}{ll} \textit{maximize} & \displaystyle\sum_{i:a_i \geq \alpha} p_i \\ \textit{subject to} & \displaystyle\sum_{i=1}^{n} p_i a_i = b \\ & \displaystyle\sum_{i=1}^{n} p_i = 1 \\ & -p \leq 0 \end{array}$$

or in matrix form:

$$\begin{array}{ll} \textit{maximize} & -d^T p \\ \textit{subject to} & G^T p + c = 0 \\ & p \geq 0 \end{array}$$

Where $d \in R^n$, and $d_i = 0$ for $a_i < \alpha$, and -1 otherwise. $G = \begin{bmatrix} a & \mathbf{1} \end{bmatrix}$, $c = \begin{bmatrix} -b & -1 \end{bmatrix}$. This is clearly a dual of a LP in inequality form, i.e.,

$$\begin{array}{ll} \textit{minimize} & -bx_1 - x_2 \\ \textit{subject to} & a_i x_1 + x_2 \leq d_i \end{array}$$

let $\lambda = -x_1$, and $\nu = -x_2$, and replace d_i with 0 and -1, we have:

$$\begin{array}{ll} \textit{minimize} & \lambda b + \nu \\ \textit{subject to} & \lambda a_i + \nu \geq 0, \text{ for all } a_i < \alpha \\ & \lambda a_i + \nu \geq 1, \text{ for all } a_i \geq \alpha \end{array}$$

The constraints of primal problem states that for $b \geq \bar{a}$, we have $f(b) = \lambda b + \nu \geq 1$ and for $b \leq \bar{a}$, $f(b) \geq 0$. Since the dual problem is a sum of probability and it's optimal value $d^\star \leq 1$, for $b \geq \bar{a}$ we have

$$p^\star = f(b^\star) = 1$$

Here we simply pick $\nu = 1,\ \lambda = 0$, which makes $f(b) = 1$.

For $b \leq \bar{a}$, to minimize the function value, we should have $f(a_1) = 0$ and $f(\bar{a}) = 1$, from here we have $\nu = -a_1/(\bar{a} - a_1)$ and $\lambda = 1/(\bar{a} - a_1)$. Therefore we have:

$$p^\star = f(b^\star) = (b - a_1)/(\bar{a} - a_1)$$

Problem 4

Plug in $x = (1, 1, 1, 1)$ and we have $Ax = [7\ 13\ {-4}\ 27\ {-18}]^T$, which states that the 2^{rd} to 5^{th} inequality constraints are tight. According to complementary slackness, $z = [0\ z_2\ z_3\ z_4\ z_5]$, and by solving $A^T z + c = 0$ we have $z = [0\ 2\ 1\ 2\ 2]^T$. Obviously z is feasible in dual and $-b^T z = -40 = c^T x$. Therefore $x^\star = x = (1\ 1\ 1\ 1)$.

$x^\star = (1\ 1\ 1\ 1)$ is a non-degenerate vertex of the primal problem and therefore it is unique, i.e., there are no other primal optimal solutions. The dual optimal must satisfy the complementary slackness and since the $z^\star = [0\ 2\ 1\ 2\ 2]^T$ is the unique solution of $A^T z + c = 0$, we can conclude that no other feasible z satisfy the complementary slackness and therefore no other dual optimal solutions.

Problem 5

For the dual optimal $\omega^\star$, we have $(Ax_0 - b)^T \omega = ||x^\star - x_0||_\infty > 0$ for $x_0 \notin P$ and $x^\star \in P$. Let $(Ax_0 - b)^T \omega^\star = d$ and $A^T \omega^\star = c$. We have:

$$c^T x = \omega^{\star T} Ax \leq \omega^{\star T} b = (A^T \omega^\star)^T x_0 - d = c^T x_0 - d < c^T x_0 - d/2$$

For x_0, we always have $c^T x_0 > c^T x_0 - d/2$ and therefore $c^T x = c^T x_0 - d/2$ defines a hyperplane that separates x_0 from P.