EE 236A HW3
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Problem 1

(a)

With the rank method we have:

$$J = \{1, 2, 3, 4\}$$

and corresponding A_J as:

$$A_J = \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix}$$

A_J is full rank and therefore $\tilde{x}$ is an extreme point of P

(b)

To find the vector c, notice that $\tilde{x}$ is the unique solution to linear equation $A_J x = b$ since $N(A_J) = \{0\}$. Therefore we have $\forall x \in P$, $A_J x \leq b$. If we set c as any non-positive combination of the rows in A_J we would have:

$$c = -A_J^T * d$$

where $d \in \mathbf{R}_+^4$. One example is to set $d = [1, 1, 1, 1]$, and we would have $c^T x \geq -13$, and only when $x = \tilde{x}$ we have $c^T \tilde{x} = -13$.

Problem 2

By using the same rank method we have:

$$J = \{2, 3, 4\}$$

and corresponding A_J as:

$$A_J = \begin{bmatrix} -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix}$$

$R(\begin{bmatrix} A_J \\ C \end{bmatrix}) = 4$ and therefore $\hat{x}$ is a extreme point of P.

Problem 3

(a)

The equivalent could be written as:

$$\begin{array}{ll} \textit{minimize} & \mathbf{1}^T y \\ \textit{subject to} & -y_k\mathbf{1} \leq A_k^T x - b_k \leq y_k\mathbf{1}, \quad k = 1, ..., m, \end{array}$$

with $y \in \mathbf{R}^m$ as our auxiliary variable.

(b)

The Lagrangian could be formed as follow:

$$\begin{aligned} & L(x,\ y,\ \lambda,\ \mu) \\ &= \sum_{i=1}^{m} [\mathbf{1}^T y_k + \lambda_k^T (A_k^T x - b_k - y_k\mathbf{1}) + \mu_k^T(-y_k\mathbf{1} - A_k^T x + b_k)] \\ &= \sum_{i=1}^{m} \{(\mathbf{1} - \lambda_k - \mu_k)^T y_k + [A_k^T(\lambda_k - \mu_k)]^T x - (\lambda_k - \mu_k)^T(b_k)\} \end{aligned}$$

Therefore the Lagrangian dual function:

$$\begin{aligned} g(\lambda,\ \mu) &= \inf_{x,\ y} L(x,\ y,\ \lambda,\ \mu) \\ &= \inf_{x,\ y} \{\sum_{i=1}^{m} [(\mathbf{1} - \lambda_k - \mu_k)^T y_k + [A_k^T(\lambda_k - \mu_k)]^T x - (\lambda_k - \mu_k)^T(b_k)]\} \\ &= \begin{cases} \sum_{i=1}^{m} b_k^T(\lambda_k - \mu_k) & \mathbf{1} - \lambda_k - \mu_k = 0 \ \&\ A_k^T(\lambda_k - \mu_k) = 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

And the dual problem and therefore be formed as:

$$\begin{array}{ll} \textit{maximize} & \sum_{i=1}^{m} b_k^T(\lambda_k - \mu_k) \\ \textit{subject to} & \mathbf{1}^T(\lambda_k + \mu_k) = 1 \\ & \sum_{i=1}^{m} A_k^T(\lambda_k - \mu_k) = 0 \\ & \lambda_k \geq 0, \quad \mu_k \geq 0, \quad k = 1, ..., m. \end{array}$$

To have the simplified form we only have to set $z_k = \lambda_k - \mu_k$.

$$\begin{aligned} &maximize && \sum_{i=1}^{m} b_k^T z_k \\ &subject\ to && \sum_{i=1}^{m} A_k^T z_k = 0 \\ & && ||z_k|| \leq 1 \quad k = 1, ..., m. \end{aligned}$$

(c)

For least square solution x_{ls}, it satisfies normal equation, i.e.,

$$A^T A x_{ls} = A^T b$$

or in our case:

$$\sum_{i=1}^{m} A_k^T (A_k x_{ls} - b_k) = 0$$

i.e.,

$$\sum_{i=1}^{m} A_k^T r_{ls} = 0$$

Therefore r_{ls} satisfy the first constraint of the dual problem. To find out a lower bound of the primal problem we need a feasible value of the dual problem and therefore we set:

$$z_k = -\frac{r_k}{max_{k=1,...,m}\ ||r_k||_1}$$

Obviously $||z_k||_1 \leq 1$, Therefore we have:

$$\sum_{i=1}^{m} b_k^T z_k \leq \sum_{i=1}^{m} (A_k x_{ls} - b_k) z_k = \frac{r_k^2}{max_{k=1,...,m}\ ||r_k||_1}$$

Problem 4

(a)

The standard LP:

$$\begin{aligned} &minimize && y \\ &subject\ to && -y\mathbf{1} \leq x - x_0 \leq y\mathbf{1} \\ & && A^T x \leq b \end{aligned}$$

(b)

The Lagrangian could be formed as follow:

$$
\begin{aligned}
&L(x,\ y,\ \lambda,\ \mu,\ \omega)\\
&= y + \lambda^T(x - x_0 - y\mathbf{1}) + \mu^T(\lambda - \mu + A^T\omega) + \omega^T(A^Tx - b)\\
&= (1 - \mathbf{1}^T\lambda - \mathbf{1}^T\mu)y + (\lambda - \mu + A^T\omega)^Tx + (\mu - \lambda)^Tx_0 - b^T\omega
\end{aligned}
$$

Therefore the Lagrangian dual function:

$$
\begin{aligned}
&g(\lambda,\ \mu,\ \omega) = \inf_{x,\ y} L(x,\ y,\ \lambda,\ \mu,\ \omega)\\
&= \inf_{x,\ y}\{[(1 - \mathbf{1}^T\lambda - \mathbf{1}^T\mu)y + (\lambda - \mu + A^T\omega)^Tx + (\mu - \lambda)^Tx_0 - b^T\omega]\}\\
&= \begin{cases} (\mu - \lambda)^Tx_0 - b^T\omega & \mathbf{1} - \lambda_k - \mu_k = 0\ \&\ \lambda - \mu + A^T\omega = 0\\ -\infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

Then the dual problem could be formed as follow:

$$
\begin{aligned}
&\textit{maximize} && (\mu - \lambda)^Tx_0 - b^T\omega\\
&\textit{subject to} && \mathbf{1}^T(\lambda_k + \mu_k) = 1\\
& && A^T\omega = \mu - \lambda\\
& && \lambda \geq 0, \quad \mu \geq 0, \quad \omega \geq 0
\end{aligned}
$$

Using the second constraint $A^T\omega = \mu - \lambda$ and a simplified version could be formed as follow:

$$
\begin{aligned}
&\textit{maximize} && (A^T\omega - b)^Tx_0\\
&\textit{subject to} && ||A^T\omega||_1 \leq 1\\
& && \omega \geq 0
\end{aligned}
$$