EE 236A HW6
Zhiyuan Cao
304397496
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Problem 1

(1, 2)

See attached sheet for Steiner tree graphs.

(3)

Let $T = \{T_1, T_2, ..., T_m\}$ be the Steiner trees found in graph G. Each tree T_i has a flow f_i. Then an LP could be formed as followed:

$$\begin{array}{ll} \textit{maximum} & \sum_{i \in T} f_i \\ \textit{subject to} & f_i \geq 0, \; i = 1, ..., m \\ & \sum_{i: e \in T_i} f_i \leq C_e, \; \forall e \in E \end{array}$$

(4)

The corresponding dual of this problem could be formed as followed:

$$\begin{array}{ll} \textit{minimize} & \sum_{e \in E} c_e d_e \\ \textit{subject to} & d_e \geq 0, \; \forall e \in E \\ & \sum_{e: e \in T_i} d_e \geq 1, \; i = 1, ..., m \end{array}$$

This is similar to max-flow, min-cut problem with variable being the path except here the variable is the Steiner tree. Therefore the dual variable d_e can be seen as the distance between s and r with its minimum value being 1. This is to say that any cut given by the dual will seperate s and r.

Problem 2

For this problem, we assume an indicator variable $a_{ij}, \; i = 1, ..., n, \; j = 1, ..., k$.

$$a_{ij} = \begin{cases} 1 & v_i \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

Also assume $|V| = n$, an ILP can be formed:

$$\begin{aligned}
&maximum && \sum_{(i,l)\in E} \omega_{il} \sum_{m,n=1,m\neq n}^{k} |a_{im} - a_{ln}| \\
&subject\ to && a_{ij} \in \{0,1\},\ i = 1,...n, j = 1,...,k \\
& && \sum_{j=1}^{k} a_{ij} = 1, i = 1,...,n
\end{aligned}$$

Notice that in the objective function, $|a_{im} - a_{ln}| = 0$ for vertices $i \in m$ and $l \in n$ and 1 otherwise, which is exactly the opposite we are looking for. However, this results in calculating ω_{ij} for k-1 times which is a constant and therefore this pose no effect on the solution.

Problem 3

(1)

First we introduce a variable x_e such that:

$$x_e = \begin{cases} 1 & \text{e belongs to the matching} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore an ILP can be formed as follow:

$$\begin{aligned}
&maximize && \sum_{e\in E} x_e \\
&subject\ to && \sum_{e\in\delta(u)} x_e \leq 1,\ \forall u \in V \\
& && x_e \in \{0,1\},\ \forall u \in E
\end{aligned}$$

Since the constraint matrix is TUM the ILP can be relaxed into LP as followed:

$$\begin{aligned}
&maximize && \sum_{e\in E} x_e \\
&subject\ to && \sum_{e\in\delta(u)} x_e \leq 1,\ \forall u \in V \\
& && x_e \geq 0,\ \forall u \in E
\end{aligned}$$

Therefore, the dual can be written as:

$$\begin{aligned}
&minimize && \sum_{u\in V} \lambda_u \\
&subject\ to && \lambda_u + \lambda_v \geq 1,\ \forall e = (u,v) \in E \\
& && \lambda_u \geq 0
\end{aligned}$$

Again, since the constraint matrix is TUM we can write the corresponding ILP as:

$$\begin{array}{ll} \textit{minimize} & \sum_{u \in V} \lambda_u \\ \textit{subject to} & \lambda_u + \lambda_v \geq 1, \ \forall e = (u, v) \in E \\ & \lambda_u \in \{0, 1\} \end{array}$$

(2)

The dual is a vertex cover problem and the objective function tries to minimize the size of the vertex cover. The first constraint says that for a certain edge e, at least one of the vertices is in the set. λ_u is the indicator function such that:

$$\lambda_u = \begin{cases} 1 & \text{vertex u is in the cover} \\ 0 & \text{otherwise.} \end{cases}$$

(3)

Since $M < |V_1|$, there is no perfect matching for this graph. Assume $|V_1| = |V_2| = n$, the size of the maximum matching is $\leq n - 1$, and so is the size of the maximum cut is also $\leq n - 1$. Let's call this cut C. We introduce $L_1 := C \cap V_1$, $L_2 := V_1 - C$, $R_1 := C \cap V_2$, $R_2 := V_2 - C$.

The capacity of cut C can be written as:

$$\textit{capacity}(S) = |L_2| + |R_1| + |(L1, R2)|$$

Therefore we have:

$$n - 1 = |L_2| + |R_1| + |(L1, R2)|$$

Also since $|L_1| = n - |L_2|$,

$$|L_1| > |L_2| + |R_1| + |(L1, R2)| + 1$$

We also have $|N(L)| < |L_2| + |R_1| + |(L1, R2)|$ because the neighbor of L_1 can at most include $|(L1, R2)|$ vertices in $|R_2|$. To conclude, we have:

$$|L_1| > |N(L)| + 1$$

Therefore we have found such a set.

(4)

If we have $M = |V_1|$, that means the solution returned by (1) is a perfect matching. Therefore for any subset S, there is at least one edge that match vertices in S to its neighbor $N(S)$, and therefore $|S| \leq |N(S)|$ holds.

Problem 4

(1)

First, we assume there are k students and $m-k$ pizza options. We notice that these can be seen as two sets of points, call them S_s and S_p. And all preferences (assume there are n of them in total) of students can be seen as edges connecting these two sets. We construct a graph $G=(V,E)$, where $|V|=m$, $|E|=n$. The problem can therefore be seen as a matching problem with objective to minimize $|S_p'|$ where $S_p' \subseteq S_p$, while having each of the element in S_s at least one edge connection to $|S_p'|$. We introduce the following variable x_e in solving the problem:

$$x_e = \begin{cases} 1 & \text{e belongs to the matching} \\ 0 & \text{otherwise.} \end{cases}$$

To simplify notation, let $M = [M_1^T M_2^T]^T$ where M_1 is the edge adjacent matrix of S_s and M_2 is the edge adjacent matrix of S_p. An ILP can be formed as follow:

$$\begin{array}{ll} \textit{minimize} & \mathbf{1}^T min\{M_2 x, \mathbf{1}\} \\ \textit{subject to} & x \in \{0,1\} \\ & M_1 x \geq 1 \end{array}$$

We introduce an axillary variable t and the problem can be reformed as followed:

$$\begin{array}{ll} \textit{minimize} & \mathbf{1}^T t \\ \textit{subject to} & x \in \{0,1\} \\ & t \in \{0,1\} \\ & M_2 x \geq t \\ & M_1 x \geq 1 \end{array}$$

(2)

The LP relaxation of the problem can be written as:

$$\begin{array}{ll} \textit{minimize} & \mathbf{1}^T t \\ \textit{subject to} & x \geq 0 \\ & t \geq 0 \\ & M_2 x \geq t \\ & M_1 x \geq 1 \end{array}$$

In matrix form:

$$\begin{array}{ll} \textit{minimize} & \mathbf{1}^T t \\ \textit{subject to} & \begin{bmatrix} \text{-}M_1 & 0 \\ \text{-}M_2 & I \\ \text{-}I & 0 \\ 0 & \text{-}I \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq \begin{bmatrix} \mathbf{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

Therefore the dual can be written as:

$$\begin{array}{ll} \textit{maximize} & \mathbf{1}^T \lambda_1 \\ \textit{subject to} & M_1^T \lambda_1 + M_2^T \lambda_2 \leq \mathbf{1} \\ & \lambda_2 \leq \mathbf{1} \\ & \lambda_1,\ \lambda_2 \geq 0 \end{array}$$

The λ_1 and λ_2 represent vertices being selected in set S_p and S_s.

Problem 5

First of all, since every vertex in both A and B has a degree of k, we have $A = B$ from $kA = kB$. Assume a perfect assignment o maximize the sum of satisfaction, we can assume an indicator variable $a_{ij},\ i = 1, ..., A,\ j = 1, ..., B$.

$$a_{ij} = \begin{cases} 1 & \text{Applicant } i \text{ gets job } j \\ 0 & \text{otherwise.} \end{cases}$$

An ILP can be formed as followed:

$$\begin{array}{ll} \textit{maximum} & \sum_{i=1}^{A} \sum_{j=1}^{B} c_{ij} a_{ij} \\ \textit{subject to} & a_{ij} \in \{0, 1\},\ i = 1, ...A, j = 1, ..., B \\ & \sum_{j=1}^{B} a_{ij} = 1, i = 1, ..., A \\ & \sum_{i=1}^{A} a_{ij} = 1, j = 1, ..., B \end{array}$$

This is equivalent to the following LP:

$$
\begin{aligned}
\textit{maximum} \quad & \sum_{i=1}^{A}\sum_{j=1}^{B} c_{ij}a_{ij} \\
\textit{subject to} \quad & a_{ij} \geq 0,\ \ i = 1, ...A, j = 1, ..., B \\
& \sum_{j=1}^{B} a_{ij} = 1,\ \ i = 1, ..., A \\
& \sum_{i=1}^{A} a_{ij} = 1,\ \ j = 1, ..., B
\end{aligned}
$$

We can first relax the ILP into LP by making $a_{ij} \in \{0, 1\}$ to $0 \leq a_{ij} \leq 1$. The constraint matrix can be proved to be a TUM. Also since we have the second and third constraint, $a_{ij} \leq 1$ is redundant. Therefore these two problems are equivalent.

Notice that in the LP there's no constraint on whether applicant i is capable of job j. One method is simply set all $a_{ij} = 0$ for such cases. Assume $G = (V, E)$ where $(i, j) \in E$ represents all job j satisfied by applicant i. Another method to resolve such problem is to manually reform the satisfaction index c_{ij} into some relative large negative number for such cases, and the rest remaining the same. This would deceive the solver to provide us with a solution that fits the registration record.

$$
\begin{aligned}
\textit{maximum} \quad & \sum_{i=1}^{A}\sum_{j=1}^{B} c_{ij}a_{ij} \\
\textit{subject to} \quad & a_{ij} \geq 0,\ \ i = 1, ...A, j = 1, ..., B \\
& a_{ij} = 0,\ \ \forall (i, j) \notin E \\
& \sum_{j=1}^{B} a_{ij} = 1,\ \ i = 1, ..., A \\
& \sum_{i=1}^{A} a_{ij} = 1,\ \ j = 1, ..., B
\end{aligned}
$$