#### DE FINETTI FOR MATHEMATICS UNDERGRADUATES

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ABSTRACT. In 1931 de Finetti proved what is known as his Dutch Book Theorem. This result implies that the finite additivity axiom for the probability of the disjunction of two incompatible events becomes a consequence of de Finetti's logic-operational consistency notion. Working in the context of boolean algebras we prove de Finetti's theorem. The mathematical background required is little more than that which is taught in high school. As a preliminary step we prove what de Finetti called "the Fundamental Theorem of Probability", his main contribution both to Boole's probabilistic inference problem and to its modern reformulation known as the optimization version of the probabilistic satisfiability problem. In a final section we give a self-contained combinatorial proof of de Finetti's exchangeability theorem.

#### Introduction

In his 1931 paper [3], de Finetti introduced his celebrated consistency notion (also known as "coherence"), and proved what today is known as his Dutch Book Theorem. In  $[3, \S 16, page 328]$  he summarized his results as follows (italics by de Finetti):

Dimostrate le proprietá fondamentali del calcolo classico delle probabilitá, ne scende che tutti i risultati di tale calcolo non sono che conseguenze della definizione che abbiamo data della coerenza.

(Having proved the fundamental properties of the classical probability calculus, it follows that all its results are nothing else but *consequences* of the definition of *consistency* given in this paper.)

Indeed, a main consequence of his Dutch Book theorem is that the finite additivity axiom for the probability of the disjunction of two incompatible events becomes a consequence of de Finetti's consistency<sup>1</sup> notion. Working in the context of boolean algebras, we offer a self-contained proof of de Finetti's theorem. As a preliminary step we prove what de Finetti called "the Fundamental Theorem of Probability", his main contribution both to Boole's probabilistic inference problem and to its modern reformulation known as the optimization version of the probabilistic satisfiability problem.

In the final part of this paper we will give a self-contained proof of de Finetti's exchangeability theorem.

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<sup>&</sup>lt;sup>1</sup>The Italian adjective "coerente" (resp., the French "cohérent") adopted by de Finetti in his original paper [3] (resp., in his paper [4]), is translated "consistent" in the present paper—for the sake of terminological consistency. As a matter of fact, "logical consistency" is translated "coerenza logica" in Italian. Moreover, Corollary 2.3 shows that de Finetti's (probabilistic) consistency is a generalization of logical consistency.

1. De Finetti's "Fundamental Theorem of Probability"

In his paper [3] de Finetti writes (in his own italics):

Un evento E è una proposizione, un'affermazione, che non sappiamo ancora se sia vera o falsa

An event E is a proposition, a statement, which we do not yet know whether it is true or false

De Finetti, [3, §7 p. 307]

Un individuo è coerente nel valutare le probabilità di certi eventi se qualunque gruppo di puntate  $S_1, S_2, \ldots, S_n$  un competitore faccia su un insieme qualunque di eventi  $E_1, E_2, \ldots, E_n$  fra quelli che egli ha considerato, non è possibile che il guadagno G del competitore risulti in ogni caso positivo.

An individual is consistent in evaluating the probabilities of certain events if for any set of stakes  $S_1, S_2, \ldots, S_n$  a competitor places on any set of events  $E_1, E_2, \ldots, E_n$  among those he has considered, it is not possible for the competitor's G gain to be positive in any case.

This expository style in presenting two of the most basic notions of de Finetti's theory differs from the style adopted in this paper. The next few pages will be devoted to the *definition* of events and their outcomes in the context of boolean algebras, the *motivation* of these definitions, and the *representation* of events in euclidean finite-dimensional space. In Section 1.3 we will give concrete examples of events and their outcomes in our algebraic framework.

1.1. Boolean algebras and their homomorphisms. While groups are a mathematical counterpart to "symmetries", boolean algebras provide a rigorous approach to the imprecise notion of "event". Furthermore, the homomorphisms of boolean algebras into the two-element boolean algebra  $\{0,1\}$  provide a convenient formal counterpart to the "possible outcomes" of these events.

Boolean algebras also provide an algebraic counterpart to "propositions" in boolean logic, equipped with the connectives of negation, conjunction and disjunction. Although logic is not the subject of this paper, the reader will have various opportunities to see the mutual relationships between these two interpretations of boolean algebras.

An algebra is a nonempty set equipped with distinguished constants and operations.

Following standard practice, the mathematical neologism "iff" stands for "if and only if". Thus for example, an even number is prime iff it equals 2. Sometimes we write  $\Leftrightarrow$  instead of "iff".

**Definition 1.1.** A lattice is an algebra  $L = (L, \wedge, \vee)$  equipped with a partial order such that any two elements  $x, y \in L$  have a greatest lower bound (alias the infimum, or meet)  $x \wedge y$  and a least upper bound (also known as the supremum, or join)  $x \vee y$ . We say that L is distributive if for all  $x, y, z \in L$ ,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
 and  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ .

The underlying order of L is defined by the stipulation  $x \leq y$  iff  $x \wedge y = x$ .

A boolean algebra  $A = (A, 0, 1, \neg, \wedge, \vee)$  is a distributive lattice  $(A, \wedge, \vee)$  with a smallest element 0 and a largest element 1, equipped with an operation  $\neg$  (called complementation) satisfying  $x \wedge \neg x = 0$  and  $x \vee \neg x = 1$ .<sup>2</sup>

<sup>&</sup>lt;sup>2</sup>All boolean algebras in this paper will satisfy the nontriviality condition  $0 \neq 1$ .

An element  $b \in A$  is said to be an *atom* if it is a nonzero minimal element, in symbols,

$$b \in at(A)$$
.

**Example 1.2.** The two-element boolean algebra  $\{0,1\} = (\{0,1\},\neg,\wedge,\vee)$  is defined by  $\neg 0 = 1$ ,  $\neg 1 = 0$ ,  $a \wedge b = \min(a,b)$ ,  $a \vee b = \max(a,b)$  for all  $a,b \in \{0,1\}$ .

"Events" and their mutual relations will be understood as elements of a boolean algebra A acted upon by the operations of A.

**Definition 1.3.** A homomorphism of a boolean algebra  $A = (A, 0, 1, \neg, \wedge, \vee)$  into a boolean algebra  $A' = (A', 0', 1', \neg', \wedge', \vee')$  is a function  $\eta: A \to A'$  that preserves the complemented lattice structure:

$$\eta(\neg x) = \neg' x, \ \eta(x \land y) = \eta(x) \land' \eta(y), \ \eta(x \lor y) = \eta(x) \lor' \eta(y).$$

It follows that  $\eta(0) = 0'$  and  $\eta(1) = 1'$ .

An isomorphism of A onto A' is a one-one function  $\theta$  of A onto A' such that both  $\theta$  and its inverse are homomorphisms.

Until further notice,  $A = (A, 0, 1, \neg, \lor, \land)$  denotes a finite boolean algebra

This restriction will be removed in Section 2.4.

Elements a, b of a boolean algebra A are said to be incompatible if  $a \wedge b = 0$ .

### **Proposition 1.4.** Let $x, y \in A$ .

- (i) If  $x \neq 0$  then x dominates some atom  $b \in at(A)$ . In symbols,  $x \geq b$ .
- (ii) For every  $a \in at(A)$  exactly one of  $a \le x$  or  $a \le \neg x$  holds.
- (iii)  $x \le y$  iff  $x \land \neg y = 0$ .
- (iv) If x < y then  $y \land \neg x \neq 0$ .
- (v)  $\neg x$  is the largest element of A incompatible with x. Symmetrically,  $\neg x$  is the the smallest element  $z \in A$  such that  $x \lor z = 1$ .
- (vi) For a function  $\phi: A \to B$  to be an isomorphism of boolean algebras A and B, it is necessary and sufficient that  $\phi$  is onto B and for all  $x, y \in A$ ,

$$x \le y \quad iff \quad \phi(x) \le \phi(y).$$
 (1)

- *Proof.* (i) If x is an atom we are done. If x is not an atom, by definition of minimality there is a nonzero  $x_1 < x$ . If  $x_1$  is an atom we are done. Otherwise there is nonzero  $x_2 < x_1$ . Since A is finite, any such chain  $x > x_1 > x_2 > \ldots$  will end after finitely many steps with a nonzero minimal element.
- (ii) If (absurdum hypothesis)  $a \le x$  and  $a \le \neg x$  then by definition of infimum,  $a \le x \land \neg x = 0$  which contradicts  $0 \ne a \in \operatorname{at}(A)$ . Thus at most one of the inequalities  $a \le x$  and  $a \le \neg x$  holds. To show that at least one inequality holds, let us assume  $a \not\le x$ , i.e.,  $a \land x < a$ , with the intent of proving  $a \le \neg x$ . Since a is minimal nonzero,

$$a \wedge x = 0. (2)$$

Furthermore,

$$a \land \neg x \neq 0.$$
 (3)

For otherwise, combining  $a \wedge \neg x = 0$  with condition (2) and the distributivity property of A, we obtain  $0 = (a \wedge x) \vee (a \wedge \neg x) = a \wedge (x \vee \neg x) = a \wedge 1 = a$ , which is impossible.

By (i) and (3) there is an atom  $b \le a \land \neg x$ , i.e.,  $b \le \neg x$  and  $b \le a$ . It follows that b = a, whence  $a \le \neg x$ , as desired.

- (iii) ( $\Rightarrow$ ) From the assumption  $x \wedge y = x$  we have  $x \wedge \neg y = x \wedge y \wedge \neg y = 0$ . ( $\Leftarrow$ ) We first write  $y \vee (\neg y \wedge x) = (y \vee \neg y) \wedge (y \vee x) = 1 \wedge (y \vee x) = y \vee x$ . From our standing hypothesis  $x \wedge \neg y = 0$  we obtain  $y \vee x = y \vee (\neg y \wedge x) = y \vee 0 = y$  whence  $y \geq x$ .
- (iv) By (iii),  $x \neq y$  iff either  $x \land \neg y \neq 0$  or  $y \land \neg x \neq 0$ . Therefore, x < y iff  $y \neq x$  and  $x \leq y$  iff (either  $x \land \neg y \neq 0$  or  $y \land \neg x \neq 0$ ) and  $x \land \neg y = 0$  iff  $y \land \neg x \neq 0$  and  $x \land \neg y = 0$ , which implies  $y \land \neg x \neq 0$ .
- (v) Distributivity ensures that the supremum s of (the finite set of) all elements of A incompatible with x is also incompatible with x. Therefore, s is the greatest element of A incompatible with x. In particular, since  $\neg x$  is incompatible with x,  $s \ge \neg x$ . By way of contradiction, assume  $s > \neg x$ . Then from (iv) we can write  $0 \ne s \land \neg \neg x = s \land x = 0$ , which is impossible. So  $s = \neg x$ .

The rest is proved similarly.

(vi) If  $\phi$  is an isomorphism then it is onto B, and both  $\phi$  and its inverse preserve the lattice-order structure, whence (1) is satisfied. Conversely, assume  $\phi \colon A \to B$  is onto B and satisfies (1). It follows that  $\phi$  is one-one. Further, both  $\phi$  and  $\phi^{-1}$  preserve the lattice operations. By the characterization of  $\neg x$  in (v),  $\phi$  and  $\phi^{-1}$  also preserve the  $\neg$  operation.

The symbol  $\square$  stands for the end of a proof.

Let  $\emptyset$  be shorthand for the empty set. For any boolean algebra A let

$$pow(at(A))$$
 (read: "the powerset of  $at(A)$ ") (4)

be the boolean algebra of all subsets of  $\operatorname{at}(A)$ , where  $0 = \emptyset$ ,  $1 = \operatorname{at}(A)$  and for all subsets X, Y of  $\operatorname{at}(A)$ ,

$$X \vee Y = X \cup Y$$
,  $X \wedge Y = X \cap Y$ ,  $\neg X = \operatorname{at}(A) \setminus X = \operatorname{the complement of } X$ .

**Proposition 1.5.** For any  $x \in A$  let  $\downarrow x$  be the set of atoms dominated by x,

$$\downarrow x = \{ a \in \operatorname{at}(A) \mid a \le x \}.$$

We then have

- $(i) \downarrow y = \emptyset$  iff y = 0. If  $\downarrow x = \{b_1, \dots, b_l\} \neq \emptyset$  then  $x = b_1 \vee \dots \vee b_l$ .
- (ii) The function  $\downarrow$  is an isomorphism of A onto pow(at(A)).

Proof. By Proposition 1.4(i), the first statement is trivial. For the second statement, by definition of supremum,  $x \geq \bigvee_{i=1}^l b_i$ . By way of contradiction, assume  $x > \bigvee_{i=1}^l b_i$ . By Proposition 1.4(iv),  $x \land \neg \bigvee_{i=1}^l b_i \neq 0$ , whence by Proposition 1.4(i) some atom b is dominated by both x and  $\neg \bigvee_{i=1}^l b_i$ . Since  $\{b_1, \ldots, b_l\}$  is the list of all atoms dominated by x, we may safely assume  $b = b_1$ . Then  $b_1 \leq \neg \bigvee_{i=1}^l b_i = \bigwedge_{i=1}^l \neg b_i$ , whence  $b_1 \leq \neg b_1$ . Thus  $b_1 \leq b_1 \land \neg b_1 = 0$ , which is impossible.

(ii) From (i) we have

for any 
$$\{b_1, \dots, b_l\} \subseteq \operatorname{at}(A), \quad \downarrow (b_1 \vee \dots \vee b_l) = \{b_1, \dots, b_l\}.$$
 (5)

Thus the function  $\downarrow$  is onto pow(at(A)). In the light of Proposition 1.4(vi), there remains to be proved that for all  $x, y \in A$ ,  $x \leq y$  iff  $\downarrow x \subseteq \downarrow y$ . If  $x \leq y$  then every atom dominated by x is also dominated by y, and hence,  $\downarrow x \subseteq \downarrow y$ . Conversely, if  $x \nleq y$  then  $x \land \neg y \neq 0$  by Proposition 1.4(ii). Proposition 1.4(i) yields an atom a such that  $a \leq x \land \neg y$ . Since  $a \leq \neg y$ , then by Proposition 1.4(ii),  $a \nleq y$ , whence  $a \notin \downarrow y$ . Since  $a \leq x$ ,  $a \in y$ . In conclusion,  $x \neq y$ .

Part (ii) in Proposition 1.5 is an example of a "representation theorem". Its role is to give the reader a concrete realization of every finite boolean algebra A as the boolean algebra of all subsets of the set of atoms of A, equipped with union, intersection and complement.

1.2. The geometry of finite boolean algebras in  $\mathbb{R}^n$ . As usual,  $\mathbb{R}$  denotes the real line,  $\mathbb{R}^2$  the cartesian plane, and  $\mathbb{R}^n$  the *n*-dimensional euclidean space. We let  $[0,1]^n$  denote the *unit n-cube* in  $\mathbb{R}^n$ . Then  $\{0,1\}^n$  is the set of vertices of  $[0,1]^n$ . The *standard basis vectors* of  $\mathbb{R}^n$  are denoted  $e_1,\ldots,e_n$ . Throughout this paper, such basic facts as the linear independence of the basis vectors of  $\mathbb{R}^n$  will be used tacitly.

**Definition 1.6.** The *product order* of  $\{0,1\}^n$  is defined by stipulating that  $x \leq y$  if and only if, whenever a coordinate  $x_i$  of x is 1 then so is the coordinate  $y_i$  of y.

Geometrically, x < y means that there is a path from x to y consisting of steps along consecutive edges of the unit cube  $[0,1]^n$ , where each step moves away from the origin.

A direct inspection yields:

**Proposition 1.7.** The product order makes  $\{0,1\}^n$  into a boolean algebra with bottom element 0 = the origin in  $\mathbb{R}^n$  and top element  $1 = (1, \ldots, 1)$ . For any  $x, y \in \{0,1\}^n$  the ith coordinate  $(\neg x)_i$  of  $\neg x$  equals  $1 - x_i$ . Further,  $(x \land y)_i = \min(x_i, y_i)$  and  $(x \lor y) = \max(x_i, y_i)$ . The atoms of  $\{0,1\}^n$  are the standard basis vectors  $e_1, \ldots, e_n$ , at  $(\{0,1\}^n) = \{e_1, \ldots, e_n\}$ .

**Theorem 1.8.** (Geometric representation of finite boolean algebras in  $\mathbb{R}^n$ ) With  $\{a_1, \ldots, a_n\} = \operatorname{at}(A)$  and  $\{e_1, \ldots, e_n\}$  the standard basis vectors in  $\mathbb{R}^n$ , let the function  $\iota \colon A \to \{0,1\}^n$  by defined by stipulating that for all  $x \in A$ ,

$$\iota(x) = \bigvee \{e_i \mid a_i \le x\} = \sum \{e_i \mid a_i \le x\} \in \mathbb{R}^n.$$
 (6)

Then  $\iota$  is an isomorphism of A onto  $\{0,1\}^n$ ,  $\iota: A \cong \{0,1\}^n$ .

*Proof.* By (5) and Proposition 1.5, the linear independence of the standard basis vectors ensures that  $\iota$  is one-one. To prove that every  $v \in \{0,1\}^n$  is in the range of  $\iota$ , let us write

$$v = e_{n_1} + \dots + e_{n_k} \tag{7}$$

for a unique subset  $\{e_{n_1}, \ldots, e_{n_k}\}$  of  $\{e_1, \ldots, e_n\}$ . Let  $\{a_{n_1}, \ldots, a_{n_k}\}$  be the corresponding set of atoms of A, and  $x_v = a_{n_1} \vee \cdots \vee a_{n_k}$ . By Proposition 1.5(i),  $\downarrow x_v = \{a_{n_1}, \ldots, a_{n_k}\}$ . By (6),  $\iota(x_v) = v$ , which shows that the function  $\iota$  is onto  $\{0,1\}^n$ 

In view of Proposition 1.4(vi), there remains to be proved that for all  $x, y \in A$   $x \le y \Leftrightarrow \iota(x) \le \iota(y)$ . By Definition 1.6 and (6), for all  $i = 1, \ldots n$  we have

$$e_i < \iota(x) \Leftrightarrow a_i < x$$
.

As a consequence,

 $\begin{array}{lll} x \leq y & \Leftrightarrow & \downarrow x \subseteq \downarrow y, & \text{by Proposition 1.5(ii)} \\ & \Leftrightarrow & \text{for all } \ a \in \operatorname{at}(A) \ \text{ such that } \ a \leq x \ \text{ we have } \ a \leq y \\ & \Leftrightarrow & \text{for all } \ e \in \operatorname{at}(\{0,1\}^n) \ \text{ such that } \ e \leq \iota(x) \ \text{ we have } \ e \leq \iota(y) \\ & \Leftrightarrow & \iota(x) \leq \iota(y). \end{array}$ 

For every (always column) vector  $v \in \mathbb{R}^n$  we let  $v^T$  denote its transpose. Thus, e.g., if v is the column vector  $\binom{x}{y}$  then  $v^T = (x, y)$ . For any vector w in  $\mathbb{R}^n$  we let  $v^T w$  denote the matrix multiplication of  $v^T$  and w, i.e., their scalar product in  $\mathbb{R}^n$ .

For any two (possibly infinite) sets X and Y, by a *one-one correspondence* between X and Y we mean an injective function  $f: X \to Y$  with f(X) = Y.

We use the notation

for the set of homomorphisms of A into the two-element boolean algebra  $\{0,1\}$ . It follows that any  $\eta \in \text{hom}(A)$  satisfies the identities

$$\eta(\neg x) = 1 - \eta(x), \ \eta(x \land y) = \min(\eta(x), \eta(y)), \ \eta(x \lor y) = \max(\eta(x), \eta(y)).$$

Intuitively, the homomorphisms of A into  $\{0,1\} = \{no, yes\}$  are the "possible outcomes" of the "events" (i.e., the elements) of A.

The most general example of a homomorphism of A into  $\{0,1\}$  is given by the following result, which in general fails when A is an infinite boolean algebra:

Corollary 1.9. (Geometric representation of hom(A)) For any A we have:

- (i) Let the function  $a \mapsto \eta_a$  send every atom a into the function  $\eta_a \colon A \to \{0,1\}$  such that for all  $x \in A$ ,  $\eta_a(x) = 1$  iff  $a \le x$ . Then  $a \mapsto \eta_a$  is a one-one correspondence between  $\operatorname{at}(A)$  and  $\operatorname{hom}(A)$ . The inverse correspondence sends any  $\eta \in \operatorname{hom}(A)$  to the only atom  $a_{\eta}$  of A such that  $\eta(a_{\eta}) = 1$ .
- (ii) Let  $\{e_1,\ldots,e_n\}=\operatorname{at}(\{0,1\}^n)$ . For each  $i=1,\ldots,n$  let the function  $\theta_{e_i}:\{0,1\}^n\to\{0,1\}$  be defined by stipulating that for every  $v\in\{0,1\}^n$

$$\theta_{e_i}(v) = 1$$
 iff  $e_i \le v$  in the product order of  $\{0, 1\}^n$ . (8)

Then the function  $e_i \mapsto \theta_{e_i}$ , (i = 1, ..., n) is a one-one correspondence between  $\operatorname{at}(\{0,1\}^n)$  and  $\operatorname{hom}(\{0,1\}^n)$ . For any atom e of  $\{0,1\}^n$ ,  $\theta_e$  is the only homomorphism of  $\operatorname{hom}(\{0,1\}^n)$  such that  $\theta_e(e) = 1$ . The inverse correspondence sends every  $\theta \in \operatorname{hom}(\{0,1\}^n)$  to the uniquely determined standard basis vector  $e_\theta \in \operatorname{at}(\{0,1\}^n) \subseteq \mathbb{R}^n$  such that  $\theta(e_\theta) = 1$ .

(iii) For every  $v \in \{0, 1\}^n$  and i = 1, ..., n,

$$\theta_{e_i}(v) = e_i^T v = ith \ coordinate \ of \ v.$$
 (9)

Proof. (i) For all  $x, y \in A$ ,  $\eta_a(x \wedge y) = 1$  iff  $x \wedge y \geq a$  iff  $x \geq a$  and  $y \geq a$  iff  $\eta_a(x) = 1$  and  $\eta_a(y) = 1$ . Thus  $\eta_a(x \wedge y) = \eta_a(x) \wedge \eta_a(y)$ . Similarly,  $\eta_a$  preserves the  $\vee$  operation. Preservation of the  $\neg$  operation follows from Proposition 1.4(ii). Therefore,  $\eta_a \in \text{hom}(A)$ . If a and b are distinct atoms of A then  $\eta_a(a) = 1$  and  $\eta_b(a) = 0$ , because  $a \ngeq b$ . Thus the function  $a \mapsto \eta_a$  is one-one. For any  $\eta \in \text{hom}(A)$  let  $B_{\eta} \subseteq \text{at}(A)$  be the set of atoms b such that  $\eta(b) = 1$ .  $B_{\eta}$  is nonempty, for otherwise (absurdum hypothesis),

$$1 = \eta(1) = \eta(\bigvee_{i=1}^{n} a_i) = \max\{\eta(a) \mid a \in \text{at}(A)\} = 0,$$

which is impossible. We have just proved that the function  $a \mapsto \eta_a$  is onto hom(A). Finally,  $B_{\eta}$  cannot contain two distinct atoms a, b. For otherwise, (absurdum hypothesis), Proposition 1.4(ii) yields  $b \leq \neg a$ , whence  $0 = 1 - \eta(a) = \eta(\neg a) \geq \eta(b) = 1$ , a contradiction. So precisely one atom  $a_{\eta}$  belongs to  $B_{\eta}$ . Evidently,  $\eta = \eta_{a_{\eta}}$ .

(ii) By Theorem 1.8, this is the special case of (i) for  $A = \{0,1\}^n$ .

(iii) For arbitrary  $v \in \{0,1\}^n$ , let  $e_{n_1}, \ldots, e_{n_k} \in \mathbb{R}^n$  be the basis vectors  $\leq v$ . By Proposition 1.5(i) and (7) we have  $v = e_{n_1} + \cdots + e_{n_k}$ . From (8) we obtain

$$\theta_{e_i}(v) = 1$$
 iff  $e_i \leq v$ , (8)  
iff  $e_i \in \{e_{n_1}, \dots, e_{n_k}\}$ , by Definition 1.6  
iff  $e_i^T(e_{n_1} + \dots + e_{n_k}) = 1$   
iff  $e_i^Tv = 1$ .

1.3. Events, atomic events, and possible worlds. Fix an integer n = 1, 2, ... and a set  $G = \{X_1, ..., X_n\}$ . Does there exist a largest boolean algebra containing G as a generating set?

Take a coin, toss it n times, and record the result (head=1 or tail=0) of each toss. Suppose for each i = 1, ..., n,  $X_i$  stands for the event "the result of the ith toss of my coin is head". With  $\neg, \land, \lor$  the usual connectives of boolean logic, let us consider the  $2^n$  boolean formulas (called miniterms)

$$X_1^{\beta_1} \wedge \dots, \wedge X_n^{\beta_n}$$
 with  $X_i^{\beta_i} = X_i$  if  $\beta_i = 1$ , and  $X_i^{\beta_i} = \neg X_i$  if  $\beta_i = 0$ . (10)

These miniterms record any possible outcome of your n tosses of a coin. Each miniterm stands for a sequence of n "independent" events, in the sense that the occurrence or non-occurrence of  $X_i$  does not interfere with the occurrence or non-occurrence of  $X_j$ ,  $(i \neq j)$ .

Let  $\mathsf{F}_n$  be the set of all boolean formulas in the variables  $X_1,\ldots,X_n$ , where two formulas are identified iff they are logically equivalent: thus for instance,  $\neg\neg X_1$  is identified with  $X_1, X_1 \wedge X_2$  is identified with  $X_2 \wedge X_1$ , and  $X_1 \vee (X_2 \wedge X_3)$  is identified with  $(X_1 \vee X_2) \wedge (X_1 \vee X_3)$ .

The result is the free boolean algebra on the free generating set  $G = \{X_1, \ldots, X_n\}$ . Readers who (like de Finetti) have little propensity for logic may adopt any of the following two alternative definitions of  $\mathsf{F}_n$ :

$$G$$
 generates  $\mathsf{F}_n$  and  $X_1^{\beta_1} \wedge \ldots, \wedge X_n^{\beta_n} \neq 0$  for all  $(\beta_1, \ldots, \beta_n) \in \{0,1\}^n$ ;

or, equivalently,

G generates  $\mathsf{F}_n$ , and for every boolean algebra A and function  $f\colon G\to A,\ f$  uniquely extends to a homomorphism of  $\mathsf{F}_n$  into A.

As expected,  $\mathsf{F}_n$  is the largest possible boolean algebra containing G as a generating set. For, if F is another boolean algebra generated by G, then by our last definition of  $\mathsf{F}_n$ , the identity function  $\epsilon\colon X_i\mapsto X_i$  extends to a homomorphism  $\tilde{\epsilon}$  of  $\mathsf{F}_n$  into F. Now,  $\tilde{\epsilon}$  is onto F, because the  $X_i$  generate F. We conclude that  $|F|\leq |\mathsf{F}_n|$ .

The miniterms of  $\mathsf{F}_n$  in (10) are the  $2^n$  atoms of  $\mathsf{F}_n$ . Intuitively, they are the "atomic events" of  $\mathsf{F}_n$ . As in Corollary 1.9, each miniterm  $t \in \mathsf{F}_n$  uniquely determines a "possible world" of  $\mathsf{F}_n$ , i.e., a homomorphism  $\mathsf{F}_n$  of into  $\{0,1\}$ , assigning 1 or 0 to any event  $e \in \mathsf{F}_n$ , according as e dominates the atom t or is disjoint from t. In Proposition 1.4(ii) it is shown that this alternative always occurs. Conversely, any  $\eta \in \mathsf{hom}(\mathsf{F}_n)$  uniquely determines the atom  $a_\eta$  given by the smallest element  $a \in \mathsf{F}_n$  such that  $\eta(a) = 1$ . The independence of the events  $X_i$  results in the largest set of possible worlds.

 $<sup>^3</sup>$ By assigning 0 or 1 to the variables in all possible ways and working in the two-element boolean algebra  $\{0,1\}$  one has a familiar mechanical procedure to check if two formulas are logically equivalent.

<sup>&</sup>lt;sup>4</sup>Kolmogorov calls each atom of  $F_n$  an "elementary event". For Boole, the atoms of  $F_n$  are its "constituents". De Finetti says that each atomic event is a "case". of the dual space of  $F_n$ . An

Now suppose n=3 and  $X_1,X_2,X_3$  stand for the following events, where "wins" is shorthand for "wins the next FIFA Club World Cup":

$$X_1 = \text{Brazil wins}, \quad X_2 = \text{Spain wins}, \quad X_3 = \text{France wins}.$$

From the rules of the FIFA World Cup it follows that not all  $2^3$  atomic events of  $\mathsf{F}_3$ , coded by the miniterms (10), can occur. For instance, the atomic event  $X_1 \wedge X_2 \wedge \neg X_3$  is impossible, and so is, a fortiori, the atomic event  $X_1 \wedge X_2 \wedge X_3$ . The impossible atomic events are precisely those stating that the number of winners is  $\geq 2$ . On the other hand,  $\neg X_1 \wedge \neg X_2 \wedge \neg X_3$  is possible. It follows that the boolean algebra A generated by the three events  $X_1, X_2, X_3$  is strictly smaller than the free three-generator algebra  $\mathsf{F}_3$ , the largest possible boolean three-generator algebra. We construct A by deleting from the set of atoms of  $\mathsf{F}_3$  those which code atomic events forbidden by the rules of the FIFA cup. A moment's reflection shows that the surviving four atomic events in A are as follows:

$$X_1 \wedge \neg X_2 \wedge \neg X_3$$
,  $\neg X_1 \wedge X_2 \wedge \neg X_3$ ,  $\neg X_1 \wedge \neg X_2 \wedge X_3$ ,  $\neg X_1 \wedge \neg X_2 \wedge \neg X_3$ .

As expected, the boolean algebra A has  $2^4 = 16$  elements/events, fewer than the  $2^8 = 256$  elements of  $F_3$ . Beyond  $X_1, X_2, X_3$  themselves, and the four atoms of A, examples of events of A include  $\neg X_1 \wedge X_2$  (which in A is the same as  $X_2$ ), the impossible event  $X_1 \wedge X_2$  (i.e., the zero element of A), the sure event  $X_1 \vee \neg X_1$  (i.e., the top element 1 of A), and a few others.

As in Proposition 1.5(i), each event of A is the disjunction (algebraically speaking, the supremum) of the atomic events it dominates. Thus, e.g., in A we have

$$\neg X_1 = (\neg X_1 \land \neg X_2 \land X_3) \lor (\neg X_1 \land X_2 \land \neg X_3) \lor (\neg X_1 \land \neg X_2 \land \neg X_3),$$

while an easy exercise shows that in  $F_3$  the element  $\neg X_1$  is the supremum of four atoms.

By Proposition 1.5(ii), up to isomorphism, every n-generator boolean algebra arises from a similar reduction procedure of the atoms of a free boolean algebra  $F_n$ .

# 1.4. States.

**Definition 1.10.** A state of a boolean algebra A is a function  $\sigma: A \to [0,1]$  with  $\sigma(1) = 1$ , having the additivity property: For all  $x, y \in A$  if  $x \wedge y = 0$  then  $\sigma(x \vee y) = \sigma(x) + \sigma(y)$ .

An easy verification shows that every homomorphism of A into  $\{0,1\}$  is a state.

infinite boolean algebra A need not have atoms. The set hom(A) conveniently replaces the set of atoms of A in any case.

# Historical/Terminological remark (for a second reading).

States of boolean algebras are also known as "finitely additive probability measures". Carathéodory extension theorem yields an affine homeomorphism of the space of states of every finite or infinite boolean algebra A onto the space of regular Borel probability measures on the Stone space of A. Hence our terminology is preferable, not only for its conciseness, but also to avoid confusion between the finite additivity of states at the algebraic level of A and the countable additivity of Borel probability measures at the topological level of the dual Stone space of A. The specific choice of the term "state" rests on the categorical equivalence  $\Gamma$ between MV-algebras and unital  $\ell$ -groups, whose "states", i.e., unit-preserving monotone homomorphisms, are deeply related to the "states" of C\*-algebraic quantum systems. For every unital  $\ell$ -group (G, u), setting  $A = \Gamma(G, u) = [0, u]$ , the restriction function  $\sigma \mapsto \sigma \upharpoonright [0, u]$  is an affine homeomorphism of the state space of (G, u) onto the state space of A. Since boolean algebras are precisely idempotent MV-algebras, every state of a boolean algebra B uniquely corresponds to the only state of the unital  $\ell$ -group (H, v) associated to B by  $\Gamma$ , and also determines a state of the C\*-algebra associated to (H, v).

The geometric representation of every state  $\sigma$  of a boolean algebra A is the object of the following corollary, where  $\sigma$  is identified with a suitable vector in euclidean space, and the value assigned by  $\sigma$  to an event is the scalar product of the vector representing the state and the vector representing the event.

A (possibly infinite) set X in euclidean space  $\mathbb{R}^n$  is *convex* if for any two points  $x, y \in X$  the segment joining x and y is contained in X.

For any set  $\{x_1, \ldots, x_l\} \subseteq \mathbb{R}^n$ , the convex hull  $\operatorname{conv}(x_1, \ldots, x_l)$  is the set of all convex combinations of  $x_1, \ldots, x_l$ , i.e., all points x of  $\mathbb{R}^n$  of the form

$$x = \lambda_1 x_1 + \dots + \lambda_n x_n, \quad 0 \le \lambda_i \in \mathbb{R}, \quad (i = 1, \dots, n), \quad \sum_{i=1}^n \lambda_i = 1.$$

An element  $z \in X$  is extremal (in X) if whenever  $z \in conv(a,b)$  for some  $a,b \in X$  then z=a or z=b.

Corollary 1.11. (Geometric representation of the states of A) Let  $\sigma$  be a state of the boolean algebra  $\{0,1\}^n$ . Let the closed convex set  $S \subseteq \mathbb{R}^n$  and the vector  $v_{\sigma} \in \mathbb{R}^n$  be respectively defined by

$$S = \operatorname{conv}(e_1, \dots, e_n) \quad and \quad v_{\sigma} = (\sigma(e_1), \dots, \sigma(e_n)). \tag{11}$$

Then

- (i)  $(v_{\sigma})_1 + \cdots + (v_{\sigma})_n = 1$ . Thus  $v_{\sigma} \in S$ .
- (ii) For all  $v = (v_1, ..., v_n) \in \{0, 1\}^n$ ,  $\sigma(v) = v_{\sigma}^T v$ .
- (iii)  $v_{\sigma}$  is the only vector in  $\mathbb{R}^n$  such that for all  $v \in \{0,1\}^n$ ,  $\sigma(v) = v_{\sigma}^T v$ . Therefore, any state is uniquely determined by the values it assigns to the atoms  $e_i$ .

*Proof.* (i) The top element 1 in the boolean algebra  $\{0,1\}^n$  is the sum  $e_1 + \cdots + e_n = e_1 \vee \cdots \vee e_n$ . Since  $e_i \wedge e_j = 0$  for  $i \neq j$  then 1 is the supremum of incompatible elements of  $\{0,1\}^n$ . By additivity,  $1 = \sigma(\bigvee_{i=1}^n e_i) = \sum_{i=1}^n \sigma(e_i)$ , whence  $v_{\sigma} \in S$ .

(ii) For each i = 1, ..., n let us agree to write  $0e_i$  = the origin in  $\mathbb{R}^n$  and  $1e_i = e_i$ . The vector v is a linear combination of the pairwise orthogonal vectors  $e_1, ..., e_n$  with uniquely determined coefficients 1 or 0 according as  $e_i \leq v$  or  $e_i \nleq v$ . Further,

v is the supremum of the  $e_i$  with the same coefficients. By Corollary 1.9(iii),  $v_i = \theta_{e_i}(v) \in \{0,1\}$ , whence

$$v = \sum_{i=1}^{n} \theta_{e_i}(v)e_i = \bigvee_{i=1}^{n} \theta_{e_i}(v)e_i.$$

By (11) and the additivity property of  $\sigma$  we can write

$$\sigma(v) = \sigma\left(\bigvee_{i=1}^{n} \theta_{e_i}(v)e_i\right) = \sum_{i=1}^{n} \sigma(\theta_{e_i}(v)e_i) = \sum_{i=1}^{n} \theta_{e_i}(v)\sigma(e_i)$$
$$= \sum_{i=1}^{n} v_i(v_\sigma)_i = v^T v_\sigma = v_\sigma^T v.$$

- (iii) Suppose  $w \in \mathbb{R}^n$  satisfies  $w^T v = \sigma(v)$  for all  $v \in \{0,1\}^n$ . Then by (i)-(ii), for all  $i = 1, \ldots n$ , we have  $(w^T v_\sigma^T)e_i = 0$  whence  $w_i = (v_\sigma)_i$  and  $w = v_\sigma$ .  $\square$
- 1.5. **De Finetti's "Fundamental Theorem of Probability".** In his "Investigation of the laws of thought", [1, Chapter XVI, 4, p. 246], Boole writes:

"the object of the theory of probabilities might be thus defined. Given the probabilities of any events, of whatever kind, to find the probability of some other event connected with them."

In his paper [3] de Finetti gave a criterion for the probability of the new event to exist. This is his consistency theorem 2.2, based on Definition 2.1. Furthermore, in [4, p. 13] and on page 112 of his book [5, 3.10.1], de Finetti showed that the set of possible probabilities of the new event (if nonempty) is a closed interval contained in [0,1].

He named his result "the Fundamental Theorem of Probability".

For our self-contained proof of this theorem we prepare the following notation and terminology, which will also be used in a later chapter: A (closed) hyperplane H in euclidean space  $\mathbb{R}^n$  is a subset of  $\mathbb{R}^n$  of the form  $\{x \in \mathbb{R}^n \mid l^Tx = w\}$  for some nonzero vector  $l \in \mathbb{R}^n$  and real number w. The sets  $\{x \in \mathbb{R}^n \mid l^Tx \geq w\}$  and  $\{x \in \mathbb{R}^n \mid l^Tx \leq w\}$  are the two (closed) half-spaces with boundary H.

**Theorem 1.12.** (De Finetti's "Fundamental Theorem of Probability") Let A be a boolean algebra with n atoms and state space S(A). Fix a finite subset  $E = \{h_1, \ldots, h_m\}$  of A along with a function  $\beta \colon E \to [0, 1]$ , and let

$$S_{\beta} = \{ \sigma \in S(A) \mid \sigma \supseteq \beta \} = be \text{ the set of states of } A \text{ extending } \beta.$$

Next, for any  $h \in A$  let

$$S_{\beta}(h) = \{ \sigma(h) \mid \sigma \in S_{\beta} \} \tag{12}$$

be the set of possible values assigned to h by all states extending  $\beta$ . Then  $S_{\beta}(h)$  is either empty, or is a closed interval contained in [0,1] possibly consisting of a single point.

*Proof.* In view of the representation Theorem 1.8, we can identify A with the boolean algebra  $\{0,1\}^n$ . Then E is a set of vertices  $h_j \in \{0,1\}^n$  of the n-cube  $[0,1]^n$ . For all  $\sigma \in S(A)$  and  $w \in \{0,1\}^n$ , the value  $\sigma(w)$  is given by the scalar product  $v_\sigma^T w$ , with  $v_\sigma = (\sigma(e_1), \ldots, \sigma(e_n))$  the vector associated to  $\sigma$  by the geometric representation in Corollary 1.11. In particular,  $v_\sigma$  lies in the closed convex set

 $S = \text{conv}(e_1, \dots, e_n)$ , with  $e_1, \dots, e_n$  the standard basis vectors of  $\mathbb{R}^n$ .

For every j = 1, ..., m, the set  $\{\sigma \in S(A) \mid \sigma(h_j) = \beta(h_j)\}$  of states of A extending the singleton function  $h_j \mapsto \beta(h_j)$  is the set of vectors

$$\{v_{\sigma} \in \mathsf{S} \mid \sigma(h_j) = \beta(h_j)\} = \{v_{\sigma} \in \mathsf{S} \mid v_{\sigma}^T h_j = \beta(h_j)\}.$$

This is the intersection of S with the hyperplane  $H_j \subseteq \mathbb{R}^n$  of all vectors  $w \in \mathbb{R}^n$  such that  $w^T h_j = \beta(h_j)$ . As a consequence, the set  $S_\beta \subseteq S$  of states of A extending  $\beta$  coincides with  $S \cap H_1 \cap \cdots \cap H_m$ . It follows that  $S_\beta$  is a closed convex (a fortioriconnected) subset of S. The set  $S_\beta(h)$  in (12) has the form  $\{v_\sigma^T h \in [0,1] \mid \beta \subseteq \sigma \in S\}$ . Thus  $S_\beta(h)$  is the range of the continuous [0,1]-valued function  $v_\sigma \mapsto v_\sigma^T h$ , where the vector  $v_\sigma$  ranges over the closed connected space  $S_\beta$ . Elementary topology shows that  $S_\beta(h)$  is closed and connected. In conclusion, if nonempty,  $S_\beta(h)$  is a closed interval in [0,1].

**Remarks 1.13.** Boole's remarks on the object of the theory of probabilities are taken up today in the PSAT (probabilistic satisfiability) problem and its optimization version. Here events  $h_1, \ldots, h_m$  are assigned probabilities  $p_1, \ldots, p_m$ , and one has to specify the possible values of the probability  $p_{m+1}$  of a new event  $h_{m+1}$ . For computational purposes, all events  $p_i$  are coded by boolean formulas, and all  $p_1, \ldots, p_m$ , are rational numbers.

Theorem 1.12 shows that the set of possible values of  $p_{m+1}$ , if nonempty, is a closed interval contained in [0,1]. To provide a fundamental necessary and sufficient condition for this set to be nonempty, it will take de Finetti's consistency ( $\sim$  Dutch book) theorem 2.2. PSAT is the problem of checking if this condition is valid. PSAT is a generalization of the satisfiability problem SAT for boolean formulas, and has the same computational complexity as SAT.

PSAT and its optimization version pertain to a vibrant research area in various domains, including defeasible reasoning, automated deduction, formal epistemology, and uncertainty management. For more information, also including de Finetti's contributions to Boole's problem and its modern reformulation, see P. Hansen, B. Jaumard, "Probabilistic Satisfiability", in: Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 5. (J. Kohlas, S. Moral, Editors). Springer, 2000, pp. 321-367.

# 2. De Finetti's Consistency Theorem

Lot T. Tery is an honest and experienced manager of a worldwide lottery. Each ticket reads:

I, the undersigned seller of this ticket, will buy it back paying the bearer one euro, if Spain wins the next FIFA world cup.

(Signed: the seller)

The price of each ticket is p (euro, to fix ideas).

If p is a fair price for you and you decide to buy N tickets, you now pay Np to Lot T. Tery. He, the signatory ticket seller, will pay you N if Spain wins.

On the other hand, if the ticket price seems too high to you, why not ask Lot T. Tery to  $buy\ M$  tickets from you? What better proof that the price p is right? His acclaimed honesty makes him willing to exchange his managerial role with you: he pays you pM for M tickets. You, the signatory seller of the ticket, will pay him M if Spain wins.

As de Finetti notes on page 309 of his paper [3], the price p set by any experienced manager like Lot T. Tery must satisfy the trivial inequalities  $0 \le p \le 1$ . Otherwise, you could bankrupt him, whatever the outcome of the next FIFA world cup. As a matter of fact, if p > 1, selling him Z tickets, with Z = one zillion, you will get

(p-1)Z if Spain wins and pZ otherwise. Lot T. Tery will get even worse if he sells Z tickets at a price p < 0.

Therefore, first of all, consistency requires that the ticket price be in the range [0,1]. Any inconsistency on Lot T. Tery's part can be punished by you, sending him to ruin, whatever the outcome of the FIFA cup.

What other conditions must the ticket price p meet, to keep you from bankrupting Lot T. Tery? Definition 2.1 provides the answer, even for the general case where tickets for multiple events and sold/bought. De Finetti's consistency theorem 2.2 gives a characterization of consistency.

2.1. Consistency. Following de Finetti, we define the fundamental notion of a consistent assignment of numbers in [0,1] to "events"  $h_1, \ldots h_m$  understood as elements of a finite boolean algebra A. The finiteness hypothesis will be dropped in Section 2.4.

With  $\{a_1, \ldots, a_n\}$  the set of atoms of A, let  $\{\eta_{a_1}, \ldots, \eta_{a_n}\} = \{\eta_1, \ldots, \eta_n\} = \text{hom}(A)$  be the set of homomorphisms of A into the two-element boolean algebra  $\{0,1\}$ , as in Corollary 1.9(i).

**Definition 2.1.** (De Finetti, [3, pp. 304-305]) Let  $E = \{h_1, \dots h_m\}$  be a subset of A and  $\beta \colon E \to [0,1]$  a function. Then  $\beta$  is said to be *inconsistent in* A if there exists a function  $s \colon E \to \mathbb{R}$  such that

$$\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \eta_i(h_j)) < 0 \text{ for every } i = 1, \dots, n.$$
 (13)

If  $\beta$  is not inconsistent in A we say it is consistent in A.

To better understand Definition 2.1 and its characterization in theorem 2.2, recalling de Finetti's second citation at the outset of Chapter 1, let us replace Lot T. Tery by a bookmaker named Bookie grappling with a clever bettor named Betty. Bookie posts "betting odds", or "betting rates"

$$p_1 = \beta(h_1), \dots, p_n = \beta(h_m) \in [0, 1]$$

on future events  $h_1, \ldots, h_m \in A$ . As in the example above,  $p_j = \beta(h_j)$  is the price of each ticket for a payoff of 1 euro in case  $h_i$  occurs, and 0 otherwise.<sup>5</sup>

If Bookie's price  $\beta(h_j)$  is deemed reasonable by Betty, she now places a stake  $s(h_j) \geq 0$ , pays  $\beta(h_j)s(h_j)$ , hoping to win  $s(h_j)$  if  $h_j$  occurs. It goes without saying that if  $h_j$  does not occur Betty's win will be zero.

On the other hand, if Betty finds  $\beta(h_j)$  excessive — as is almost always the case with bookmakers' odds in real life — she can place a stake  $s(h_j) < 0$ . So Bookie pays Betty  $\beta(h_j) |s(h_j)|$ , and wins  $|s(h_j)|$  if  $h_j$  occurs. In this case, Bookie/Betty's roles swap. This never happens in real life.

For definiteness, let us make the following stipulation:

Then for any positive or negative stake  $s(h_j)$ , the balance of this single bet on event  $h_j$  is given by  $s(h_j)(\beta(h_j) - \eta_i(h_j))$  in the "possible world"  $\eta_i \in \text{hom}(A)$ . 6 Since zero bets are possible, we may assume Betty is betting on all events  $h_1, \ldots, h_m$ . Bookie's balance is then

$$\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \eta_i(h_j)). \tag{15}$$

<sup>&</sup>lt;sup>5</sup>Real-life bookmakers prefer to post their odds as  $1/p_j \ge 1$  instead of  $p_j$  (always > 0), guess why.

<sup>&</sup>lt;sup>6</sup>It is understood that  $\eta_i(h_i)$  equals 1 or 0 according as  $h_i$  occurs or does not occur in  $\eta_i$ .

Bookie's book  $\beta$  is inconsistent in the sense of Definition 2.1 iff Betty can devise positive or negative stakes  $s(h_1), \ldots, s(h_m) \in \mathbb{R}$  that guarantee her a minimum profit of 1 —equivalently, one zillion— "regardless of the outcome of the events  $h_j$ ", i.e., in any "possible world"  $\eta_i \in \text{hom}(A)$ . Correspondingly, Bookie has a sure loss of at least one(zillion).

The totality of "possible outcomes", or "possible worlds", is made precise by the homomorphisms  $\eta_1, \ldots, \eta_n \in \text{hom}(A)$ . Definition 1.3 ensures that the laws of logic hold in each  $\eta_i$ .

To view panoramically all events  $h_j$ , (j = 1, ..., m) and all possible worlds  $\eta_i$ , (i = 1, ..., n), we prepare the following  $n \times m$  matrix M:

$$M = \begin{pmatrix} \beta(h_1) - \eta_1(h_1) & \beta(h_2) - \eta_1(h_2) & \dots & \beta(h_m) - \eta_1(h_m) \\ \beta(h_1) - \eta_2(h_1) & \beta(h_2) - \eta_2(h_2) & \dots & \beta(h_m) - \eta_2(h_m) \\ \dots & \dots & \dots & \dots \\ \beta(h_1) - \eta_n(h_1) & \beta(h_2) - \eta_n(h_2) & \dots & \beta(h_m) - \eta_n(h_m) \end{pmatrix}$$
(16)

Let  $s(h_1), \ldots, s(h_m) \in \mathbb{R}$  be Betty's stakes. Fix a row  $i = 1, \ldots, n$  and a column  $j = 1, \ldots, m$ .

If  $\eta_i(h_j) = 0$ , i.e., if  $h_j$  does not occur in the possible world  $\eta_i$ , then the term  $M_{i,j}$  in M coincides with  $\beta(h_j)$ . Recalling (14), Bookie's profit/loss is  $\beta(h_j)s(h_j) \in \mathbb{R}$ . On the other hand, if  $h_j$  occurs in the possible world  $\eta_i$  then  $M_{i,j} = \beta(h_j) - 1$  and Bookie's profit/loss is  $(\beta(h_j) - 1)s(h_j)$ . In either case, Bookie's profit/loss for Betty's stake  $s(h_j)$  on event  $h_j$  is  $(\beta(h_j) - \eta_i(h_j))s(h_j)$  in the possible world  $\eta_i$ .

Let now  $s \in \mathbb{R}^m$  be the (always column) vector whose coordinates are  $s_1 = s(h_1), \ldots, s_m = s(h_m)$ . Then the vector  $Ms \in \mathbb{R}^n$  gives the balance (15) in all possible worlds  $\eta_1, \ldots, \eta_n$ . Bookie's  $\beta$  is inconsistent iff Betty can devise a vector  $s \in \mathbb{R}^n$  (equivalently,  $s \in \mathbb{Q}^n$ ) such that all coordinates of Ms are < 0.

#### 2.2. The Consistency (alias Dutch book) Theorem.

**Theorem 2.2.** (De Finetti's consistency theorem, [3, pp. 309-313]) For any boolean algebra A with atoms  $a_1, \ldots, a_n$  and corresponding homomorphisms

$$\eta_1,\ldots,\eta_n$$

of A into  $\{0,1\}$ , let  $E = \{h_1, \ldots, h_m\} \subseteq A$  and  $\beta \colon E \to [0,1]$ . Then precisely one of the following conditions holds:

(i)  $\beta$  is inconsistent in A in the sense of Definition 2.1. Thus there is a function  $s \colon E \to \mathbb{R}$  such that, letting  $s_1 = s(h_1), \ldots, s_m = s(h_m)$ , every coordinate of the vector  $M(s_1, \ldots, s_m) \in \mathbb{R}^n$  is < 0. In symbols,

$$\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \eta_i(h_j)) < 0 \text{ for every } i = 1, \dots, n.$$

(ii)  $\beta$  is extendable to a state  $\sigma$  of A.

Stated otherwise,  $\beta$  is consistent in A iff it is extendable to a state of A.

*Proof.* In view of Theorem 1.8, by identifying A with the boolean algebra  $\{0,1\}^n$  of the vertices of the n-cube  $[0,1]^n$ , each atom  $a_i$  is geometrically realized as the ith standard basis vector  $e_i$  in  $\mathbb{R}^n$ . Furthermore, each element  $h_j$  is realized as a vertex of the n-cube  $[0,1]^n$ . Since the isomorphism  $\iota$  of Theorem 1.8 now coincides with identity, by Corollary 1.9(ii) we can write

$$\eta_i = \theta_i = \theta_{e_i} \in \text{hom}(\{0, 1\}^n) \text{ for each } i = 1, \dots, n.$$

$$\tag{17}$$

Proof that if condition (i) fails then condition (ii) holds. Suppose condition (i) fails. Gordan's theorem<sup>7</sup> gives a nonzero column vector  $(u_1, \ldots, u_n) \in \mathbb{R}^n$  with each coordinate  $u_i \geq 0$  and  $u^T M = 0$  = the origin in  $\mathbb{R}^m$ . Without loss of generality,  $u_1 + \cdots + u_n = 1$ . Summing up,

$$u^{T}M = 0, \quad 0 \le u_{i} \ (i = 1, ..., n), \quad u_{1} + \dots + u_{n} = 1.$$
 (18)

For each j = 1, ..., m the jth column  $C_j$  of the  $n \times m$  matrix M in (16) has the form  $C_j = C_{j,\text{left}} - C_{j,\text{right}}$ , where

$$C_{j,\text{left}} = \underbrace{(\beta(h_j), \dots, \beta(h_j))}_{i \text{ times}}.$$
 (19)

For each i = 1, ..., n, the *i*th term  $(C_{j,right})_i$  satisfies the identities

$$(C_{j,\text{right}})_i = \eta_i(h_j) = \theta_i(h_j) = e_i^T h_j = i\text{th coordinate of } h_j \in \{0,1\}^n.$$
 (20)

This follows from 1.8-1.9 and (17). By (18)-(20), the assumption  $u^T M = 0$ , (i.e.,  $u^T C_{j,\text{right}} = u^T C_{j,\text{left}}$  for each j = 1, ..., m) amounts to writing

$$\beta(h_j) = u^T C_{j,\text{right}} = u^T h_j, \text{ for each } j = 1, \dots, m.$$
 (21)

Let the state  $\sigma$  of the boolean algebra  $\{0,1\}^n$  be defined by

$$\sigma(e_i) = u_i, \ (i = 1, \dots, n).$$

By Corollary 1.11(iii) and (21),  $\sigma(h_j) = u^T h_j = \beta(h_j)$  for each j = 1, ..., m. This shows that  $\sigma$  extends  $\beta$ , whence condition (ii) holds.

Proof that conditions (i) and (ii) are incompatible. By way of contradiction, assume both conditions hold. As in Corollary 1.11, let the vector  $v_{\sigma} \in S$  be defined by  $v_{\sigma}^T w = \sigma(w)$  for all  $w \in \{0,1\}^n$ . In particular,  $v_{\sigma}^T h_j = \sigma(h_j)$  for each  $j = 1, \ldots, m$ . Since all coordinates of  $v_{\sigma}$  are  $\geq 0$  and their sum is equal to 1, then

$$v_{\sigma}^T C_{j,\text{left}} = \beta(h_j)$$
 for all  $j = 1, \dots, m$ .

Further, by (20),  $v_{\sigma}^T C_{j,\text{right}} = v_{\sigma}^T h_j = \sigma(h_j)$ . Since, by condition (ii),  $\sigma$  extends  $\beta$  then

$$v_{\sigma}^T C_{j,\text{left}} = v_{\sigma}^T C_{j,\text{right}}$$
 for all  $j = 1, \dots, m$ ,

whence  $v_{\sigma}^T M = 0$ . As in Gordan's theorem 2.10, conditions (i) and (ii) imply the contradiction

$$0 \neq \underbrace{v_{\sigma}^{T}}_{v_{\sigma} \in S} \underbrace{(M(s_{1}, \dots, s_{m}))}_{s_{\sigma} \in S} = (v_{\sigma}^{T}M)(s_{1}, \dots, s_{m}) = 0.$$

We have thus shown that conditions (i) and (ii) are incompatible.

Conclusion. If  $\beta$  is consistent then condition (i) fails, whence, by condition (ii),  $\beta$  is extendable to a state. Conversely, if condition (ii) holds then  $\beta$  cannot satisfy condition (i), i.e.,  $\beta$  is consistent.

2.3. Interlude: Logical consistency and de Finetti's consistency. This section requires some acquaintance with the syntax and semantics of boolean propositional logic: formulas, truth-valuations, logical equivalence and consistency. The first pages of any book on mathematical logic will provide all necessary background. Readers not interested in the relationships between logical consistency and de Finetti's consistency can skip this section on a first reading, as no results proved here will be used in the rest of this paper.

Let  $\mathsf{F}_n$  be the free boolean algebra over the free generating set  $\{X_1,\ldots,X_n\}$  introduced in Section 1.3. Elements of  $\mathsf{F}_n$  are boolean formulas  $\phi$  in the variables  $X_1,\ldots,X_n$  up to logical equivalence. The boolean operations naturally act on  $\mathsf{F}_n$ 

<sup>&</sup>lt;sup>7</sup>A self-contained proof of Gordan's theorem is given in 2.10.

as the boolean connectives act on formulas. Thus, e.g., let  $\phi$  be a formula and  $\tilde{\phi} \in \mathsf{F}_n$  the (infinite!) set of all formulas logically equivalent to  $\phi$ . Then for any formula  $\psi$  logically equivalent to  $\phi$  the equivalence class  $\widetilde{-\psi}$  of  $\neg \psi$  coincides with the complement  $\neg \tilde{\phi} \in \mathsf{F}_n$  of the equivalence class  $\tilde{\phi}$ . Any formula  $\psi$  logically equivalent to  $\phi$  is said to code the element  $\tilde{\phi}$  of  $\mathsf{F}_n$ .

For each  $i=1,\ldots,n,\ X_i$  is a special kind of a formula, known as a *variable*. Traditionally, the same notation is used for  $X_i$  and for the equivalence class  $\widetilde{X_i} \in \mathsf{F}_n$ . Likewise,  $\neg$ ,  $\wedge$ ,  $\vee$  denote both the connectives acting on formulas, and the operations of the boolean algebra  $\mathsf{F}_n$ .

A truth-valuation is a  $\{0,1\}$ -valued function v defined on all formulas  $\psi = \psi(X_1,\ldots,X_n)$ , having the following properties:

```
v(\neg \psi) = 1 - v(\psi), \ v(\psi \land \phi) = \min(v(\psi), v(\phi)), \ v(\psi \lor \phi) = \max(v(\psi), v(\phi)).
```

The non-ambiguity of the syntax ensures that v is uniquely determined by the values it assigns to the variables  $X_1, \ldots, X_n$ .

Let the formulas  $\phi_1, \ldots, \phi_m$  respectively code elements  $h_1, \ldots, h_m$  of  $\mathsf{F}_n$ . Then the set  $\{\phi_1, \ldots, \phi_m\}$  is said to be *logically consistent* if some truth-valuation assigns value 1 to each formula  $\phi_i$ . In equivalent algebraic terms,  $\eta(h_1) = \cdots = \eta(h_m) = 1$  for some  $\eta \in \mathsf{hom}(\mathsf{F}_n)$ .

**Corollary 2.3.** Let  $E = \{h_1, \ldots, h_m\}$  be a finite subset of the boolean algebra  $F_n$ . Let the function  $\beta \colon E \to \{0,1\}$  assign the constant value 1 to each element of E. Then the following conditions are equivalent:

- (i) E can be coded by a logically consistent set E' of boolean formulas.
- (ii)  $\beta$  is (de Finetti) consistent in the sense of Definition 2.1.
- (iii)  $\beta$  can be extended to a homomorphism of  $F_n$  into  $\{0,1\}$ .
- (iv)  $\beta$  can be extended to a state of  $F_n$ .

*Proof.* (ii) $\Leftrightarrow$ (iv) By Theorem 2.2.

- (i) $\Rightarrow$ (iii) By hypothesis, some truth-valuation v assigns 1 to all formulas in E'. Equivalently,  $\eta(h_1) = \cdots = \eta(h_m) = 1$  for some  $\eta \in \text{hom}(\mathsf{F}_n)$ . Thus,  $\eta$  extends  $\beta$ .
- (iii) $\Rightarrow$ (ii) Every homomorphism of  $\mathsf{F}_n$  into  $\{0,1\}$  is a state. Now apply Theorem 2.2
- (ii) $\Rightarrow$ (i) Assume (i) does not hold, with the intent of proving that  $\beta$  is inconsistent in the sense of Definition 2.1. To this purpose, for each  $j=1,\ldots,m$ , Betty places the stake  $s(h_j)=-1$ . As a result, she now receives m from Bookie. In any possible world  $\eta \in \text{hom}(\mathsf{F}_n)$  she will have to return 1 to Bookie for each event to which  $\eta$  assigns the value 1. However, at least one event  $h=h_{\eta}\in E$  will be evaluated 0 by  $\eta$ , because the assumed logical inconsistency of E' entails that no truth-valuation v assigns value 1 to all formulas of E'. Thus Betty has a net profit  $\geq 1$  in any possible world  $\eta$ . By Definition 2.1,  $\beta$  is inconsistent.

Using the fact that every boolean algebra A is the homomorphic image of some free boolean algebra, with some extra work Corollary 2.3 can be shown to hold for A

2.4. **De Finetti's Consistency Theorem for all boolean algebras.** Generalizing Theorem 2.2, in Theorem 2.7 we will show that our restriction to finite boolean algebras is inessential. So let us relax this restriction:

A henceforth stands for an arbitrary (finite or infinite) boolean algebra.

The set hom(A) of homomorphisms of A into the two-element boolean algebra  $\{0,1\}$  is no longer indexed by the atoms of A. Suffice to say that an infinite

boolean algebra A need not have atoms. Thus the definition of consistency gets the following form, which is equivalent to Definition 2.1 for finite boolean algebras, and encompasses the general case when the boolean algebra A is infinite:

**Definition 2.4.** Let  $E = \{h_1, \dots h_m\}$  be a subset of a boolean algebra A and  $\beta \colon E \to [0,1]$  a function. Then  $\beta$  is said to be *inconsistent in* A if there is a function  $s \colon E \to \mathbb{R}$  such that

$$\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \eta(h_j)) < 0 \text{ for every } \eta \in \text{hom}(A).$$
 (22)

If  $\beta$  is not inconsistent in A we say it is consistent in A.

The definition of a *state* of A is verbatim the same as Definition 1.10.

**Lemma 2.5.** Let B be a finite boolean algebra and B' a subalgebra of B.

- (i) The homomorphisms of B' into  $\{0,1\}$  are precisely the restrictions to B' of the homomorphisms of B into  $\{0,1\}$ .
- (ii) The states of B' are precisely the restrictions to B' of the states of B.

*Proof.* (i) Let  $\epsilon \in \text{hom}(B)$ . Trivially, the restriction  $\epsilon \upharpoonright B'$  of  $\epsilon$  to B' is a homomorphism of B' into  $\{0,1\}$ .

Conversely, let  $\eta \in \text{hom}(B')$ , with the intent of extending  $\eta$  to a some  $\eta^* \in \text{hom}(B)$ . By Corollary 1.9(i) we may write  $\eta = \eta_e$  for a unique atom e of B'. By Proposition 1.5, each atom a of B' is a nonzero element of B, and dominates a nonempty set [a] of atoms of B. If  $a \neq b \in \text{at}(B')$  then  $[a] \cap [b] = \emptyset$ . (For otherwise, some atom c of B is dominated by both a and b, whence c is dominated by  $a \wedge b = 0$ , which is impossible.) As a consequence, upon writing

 $c \approx d$  iff c and d are dominated by the same atom a of B',  $(c, d \in at(B))$ ,

we obtain an equivalence relation  $\approx$  over at(B). The function  $a \mapsto [a]$ ,  $a \in \operatorname{at}(B')$  maps at(B') onto the set of  $\approx$ -equivalence classes. Furthermore, for each  $x \in B'$  letting  $a_1, \ldots, a_m$  be the atoms of B' dominated by x, we have the identity

$$x = \bigvee \{ b \in \operatorname{at}(B) \mid b \in ([a_1] \vee \dots \vee [a_m]) \}. \tag{23}$$

Turning to our homomorphism  $\eta_e \in \text{hom}(B')$ , let us arbitrarily pick an atom  $e^* \in [e]$ , and let  $\eta_{e^*} \in \text{hom}(B)$  be the corresponding homomorphism. By a final application of Corollary 1.9, for every  $y \in B$ ,  $\eta_{e^*}(y) = 1$  iff y dominates  $e^*$ . An easy verification using (23) shows that the homomorphism  $\eta^* = \eta_{e^*}$  is an extension of  $\eta_e$ .

(ii) The restriction to B' of any state of B is a state of B'. Conversely, let  $\tau$  be a state of B'. With the notation of (i), for each  $a \in \operatorname{at}(B')$  arbitrarily pick an atom  $a^* \in [a] \subseteq \operatorname{at}(B)$ . Let the function  $\tau^{\circ} \colon \operatorname{at}(B) \to \{0,1\}$  be defined by

$$\tau^{\circ}(a^*) = \tau(a)$$
 for each  $a \in \operatorname{at}(B')$ , and  $\tau^{\circ}(b) = 0$  for all other atoms of B.

By Corollary 1.11(iii), any state of the finite boolean algebra B is uniquely determined by the values it gives to the atoms of B. It is now easy to verify that  $\tau^{\circ}$  uniquely determines a state  $\tau^{*}$  of B which extends  $\tau$ .

**Lemma 2.6.** Let B' be a finite subalgebra of an infinite boolean algebra A.

- (i) The states of B' are precisely the restrictions to B' of the states of A.
- (ii) The homomorphisms of B' into  $\{0,1\}$  are precisely the restrictions to B' of the homomorphisms of A into  $\{0,1\}$ .

*Proof.* (i) Let  $\sigma$  be a state of B'. By Lemma 2.5(ii), for every *finite* subalgebra B of A containing B',  $\sigma$  is extendable to a state of B. Let the set  $\tilde{S}_{\sigma,B} \subseteq [0,1]^A$  be defined by

$$\tilde{S}_{\sigma,B} = \{ f \colon A \to [0,1] \mid f \upharpoonright B \text{ is a state of } B \text{ extending } \sigma \}.$$

By definition of the product topology of  $[0,1]^A$ ,  $\tilde{S}_{\sigma,B}$  is a nonempty *closed* subset of the compact Hausdorff space  $[0,1]^A$ .

For any finite family of finite subalgebras  $B_1, \ldots, B_u$  of A containing B' the intersection I of the closed sets  $S_{\sigma,B_1}, \ldots, S_{\sigma,B_u}$  is nonempty. As a matter of fact, letting  $B_{u+1}$  be the subalgebra of A generated by the finite set  $B_1 \cup \cdots \cup B_u$ , it follows that  $B_{u+1}$  is finite (see Section 1.3), and the set I contains the nonempty set  $\tilde{S}_{\sigma,B_{u+1}}$ . Next let

$$I^* = \bigcap \{\tilde{S}_{\sigma,B} \mid B \text{ a finite subalgebra of } A \text{ containing } B'\}.$$

From the compactness of the Tychonoff cube  $[0,1]^A$  it follows that  $I^*$  is nonempty. Any element of  $I^*$  is a state of A extending  $\sigma$ .

Theorem 2.2 has the following generalization to every (finite or infinite) boolean algebra A:

**Theorem 2.7.** (De Finetti consistency theorem, general case) Let E be a finite subset of a boolean algebra A and  $\beta \colon E \to [0,1]$  a function. Then  $\beta$  is consistent in A iff  $\beta$  is extendable to a state of A.

*Proof.* Say  $E = \{h_1, \ldots, h_m\}$ . The subalgebra  $B_E$  of A generated by E is finite. (As noted in Section 1.3, the number of elements of  $B_E$  is  $\leq 2^m$ .)

 $(\Rightarrow)$  Let  $s: E \to \mathbb{R}$  be Betty's bet on the events in E. Let us agree to say that any  $\eta = \eta_s \in \text{hom}(B_E)$  such that

$$\sum_{l=1}^{m} s(h_l)(\beta(h_l) - \eta_s(h_l)) \ge 0$$
 (24)

is a witness of the s-consistency of  $\beta$  in  $B_E$ . Since, by assumption,  $\beta$  is consistent in A, for every bet  $t \colon E \to \mathbb{R}$  there is a witness  $\theta = \theta_t \in \text{hom}(A)$  of the t-consistency of  $\beta$  in A.<sup>8</sup> A fortiori, the restriction of  $\theta_t$  to  $B_E$  witnesses the t-consistency of  $\beta$  in  $B_E$ . Since t is any arbitrary bet on E,  $\beta$  is consistent in  $B_E$ . By Theorem 2.2,  $\beta$  is extendable to a state  $\tau$  of  $B_E$ . By Lemma 2.6(i),  $\tau$  is extendable to a state  $\rho$  of A. A fortiori,  $\beta \subseteq \tau$  is extendable to  $\rho$ .

( $\Leftarrow$ ) Let  $\sigma$  be a state of A extending  $\beta$ . The restriction  $\sigma \upharpoonright B_E$  is a state of  $B_E$  extending  $\beta$ . By Theorem 2.2,  $\beta$  is consistent in  $B_E$ . We have to prove that  $\beta$  is consistent in A. Arguing by way of contradiction, let us assume Betty can devise a bet  $s: E \to [0,1]$  such that

$$\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \eta(h_j)) < 0 \text{ for every } \eta \in \text{hom}(A).$$

By Lemma 2.6(ii),  $\sum_{j=1}^{m} s(h_j)(\beta(h_j) - \epsilon(h_j)) < 0$  for every  $\epsilon \in \text{hom}(B_E)$ . By Definition 2.4,  $\beta$  is inconsistent in  $B_E$ , a contradiction.

<sup>&</sup>lt;sup>8</sup>The definition of  $\theta_t \in \text{hom}(A)$  witnessing the *t*-consistency of  $\beta$  in A is precisely the same as (24), with t in place of s.

**Corollary 2.8.** Let A be a boolean algebra. Then A has a state. A function  $f: A \to [0,1]$  is a state of A iff f is finitely consistent, in the sense that every finite restriction of f is consistent in A.

- **Remarks 2.9.** (i) The proof of Lemma 2.6 rests on the compactness of the Tychonoff cube  $[0,1]^A$ . In ZF (Zermelo-Fraenkel) set-theory this is an equivalent reformulation of the Axiom of Choice. The dependence of de Finetti's consistency theorem 2.7 on this axiom is rarely made explicit in the vast literature on this theorem.
- (ii) From Lemmas 2.5-2.6 and Theorem 2.7 it follows that a function  $\beta \colon E \to [0,1]$  is consistent in a boolean algebra  $A \supseteq E$  iff it is consistent in the algebra  $B_E$  generated by E in A, iff it is consistent in any algebra  $A^*$  containing A as a subalgebra. Hence the consistency of  $\beta$  is largely indifferent to the chosen boolean algebra  $A \supseteq E$ . However, the specification of *some* ambient boolean algebra A for the set E of "events" and their operations considered in de Finetti's consistency theorem is necessary to give a meaning to expressions such as "in any possible case", or "in any possible world", which occur in the definition of consistency.
- (iii) De Finetti's consistency theorem has two directions. He proved the  $(\Rightarrow)$ -direction in [3, pp.309-312], and the  $(\Leftarrow)$ -direction in [3, p.313]. Nevertheless, purported "converses" of de Finetti consistency ( $\sim$  Dutch book) theorem exist in the literature.
- 2.5. A self-contained proof of Gordan's Theorem. In this section we give a self-contained proof of Gordan's theorem, a basic ingredient of the proof of de Finetti's consistency theorem.

**Theorem 2.10.** (Gordan's theorem) Let M be a real matrix with n rows and m columns. Then precisely one of the following conditions holds:

- (i) There is a (column) vector  $s \in \mathbb{R}^m$  such that every coordinate of the vector  $Ms \in \mathbb{R}^n$  is < 0.
- (ii) There is a nonzero vector  $u \in \mathbb{R}^n$  such that every coordinate of u is  $\geq 0$  and  $u^T M = 0$  = the origin in  $\mathbb{R}^m$ . (As above,  $u^T$  is the transpose of u.)

*Proof.* Conditions (i) and (ii) in the statement of Gordan's theorem 2.10 are incompatible. For, if both are assumed to hold we have the contradiction

$$0 \neq \underbrace{u^{T}}_{\geq 0 \dots \geq 0, \ u \neq 0} \underbrace{(Ms)}_{< 0 \dots < 0} = (u^{T}M)s = 0.$$
 (25)

There remains to be proved that if (i) fails then (ii) holds. Failure of (i) means that the range R of the linear operator  $M: \mathbb{R}^m \to \mathbb{R}^n$  is disjoint from the south-west open octant O given by

$$O = \{v \in \mathbb{R}^n \mid \text{ each coordinate of } v \text{ is } < 0\}.$$

R is a linear subspace of  $\mathbb{R}^n$  containing the origin. The hyperplane separation lemma 2.14  $^9$  then yields a nonzero column vector  $u \in \mathbb{R}^n$ , along with a hyperplane H such that

$$R \subseteq H = \{x \in \mathbb{R}^n \mid u^T x = 0\} \text{ and } O \subseteq \{x \in \mathbb{R}^n \mid u^T x < 0\}.$$

The latter inclusion implies that all coordinates of u are  $\geq 0$ . It is now easy to see that  $u^TM=0$ . For otherwise, (absurdum hypothesis), there is a vector  $q \in \mathbb{R}^m$  with  $(u^TM)q \neq 0$ . As a member of R, the vector Mq is orthogonal to  $u^T$ , whence  $u^T(Mq)=0$ , a contradiction. We have thus proved that if (i) fails then (ii) holds.

<sup>&</sup>lt;sup>9</sup>A self-contained proof is given below.

To complete the proof of Gordan's theorem, as well as of de Finetti's theorems 2.2 and 2.7, in this section a routine proof is given of the hyperplane separation lemma 2.14.  $^{10}$ 

### **Fact 2.11.** The closure cl X of a convex set $X \subseteq \mathbb{R}^n$ is convex.

Proof. If  $X = \emptyset$  we are done. If  $X \neq \emptyset$ , for any  $x, y \in \operatorname{cl} X$  we must prove that the segment  $[x,y] = \{(1-\lambda)x + \lambda y \mid \lambda \in [0,1]\}$  lies in  $\operatorname{cl} X$ . By definition of closure, X contains converging sequences  $x_n \to x$  and  $y_n \to y$ . Since X is convex, the interval  $[x_n,y_n]$  lies in X. The sequence of midpoints  $m_n$  of  $[x_n,y_n]$  converges to the midpoint m of [x,y]. As a limit of a convergent sequence of points of X, m lies in  $\operatorname{cl} X$ . More generally, for each  $\lambda \in [0,1]$  the sequence  $(1-\lambda)x_n + \lambda y_n$  converges to  $(1-\lambda)x + \lambda y$ . So  $(1-\lambda)x + \lambda y$  lies in  $\operatorname{cl} X$ . As  $\lambda$  ranges over [0,1], all points of [x,y] are obtained. They all lie in  $\operatorname{cl} X$ .

Let us recall that the scalar product of two (always column) vectors  $u, v \in \mathbb{R}^n$  is given by matrix multiplication  $u^T v$ , with  $u^T$  the transpose of u.

For  $w \in \mathbb{R}^n$  we let |w| denote its euclidean norm, (or "length")  $(w^T w)^{1/2}$ .

# **Fact 2.12.** Let $X \subseteq \mathbb{R}^n$ be a nonempty closed convex set.

- (i) X has a unique shortest vector, i.e., a vector  $x_*$  with  $|x_*| = \inf\{|x| \mid x \in X\}$ .
- (ii) Let  $t \in \mathbb{R}^n$ . Then there is a unique point u in X such that  $|u t| = \inf\{|x t| \mid x \in X\}$ .
- (iii) Assume  $0 \notin X$ . Let  $x_*$  be the shortest vector in X given by (i). Then for all  $y \in X$ ,  $x_*^T y > |x_*|^2/2$ .
  - (iv) Let  $x_*$  be the shortest vector in X. Assume  $0 \notin X$ . Then the hyperplane

$$H = \{ z \in \mathbb{R}^n \mid x_*^T z = |x_*|^2 / 2 \}$$

strongly separates the origin from X.<sup>11</sup> More generally, for any nonempty closed convex set  $Y \subseteq \mathbb{R}^n$ , point  $e \in \mathbb{R}^n \setminus Y$ , letting  $x_*$  be the unique point of Y closest to e, we have  $(x_* - e)^T(x - e) > |x_* - e|^2/2$  for each  $x \in Y$ . In other words, the hyperplane

$$K = \{ z \in \mathbb{R}^n \mid (x_* - e)^T (z - e) = |x_* - e|^2 / 2 \}$$

strongly separates e from K.

- Proof. (i) If the origin 0 belongs to X we have nothing to prove. Otherwise, let  $\xi$  be shorthand for  $\inf\{|x|\mid x\in X\}$ . Since X is nonempty, it is no loss of generality to assume that X is bounded. By definition of infimum there is a sequence  $x_n\in X$  with  $|x_n|-\xi<1/n$ . There is a convergent subsequence  $x_{n_i}$ . The point  $x_*=\lim x_{n_i}$  belongs to X, because X is closed. Since  $x\mapsto |x|$  is a continuous real-valued function, then  $\xi=\lim |x_{n_i}|=|\lim x_{n_i}|=|x_*|$ . To prove the uniqueness of  $x_*$  suppose (absurdum hypothesis)  $y\in X$  is different from  $x_*$  and has the same length as  $x_*$ . The triangle with vertices  $0,x_*,y$  is isosceles. The midpoint of the interval  $[x_*,y]$  has a distance  $<|x_*|$  from 0, which is impossible.
- (ii) The translated set  $X t = \{x t \mid x \in X\}$  is also nonempty closed and convex. Distances |a b| are preserved under translation. Now apply (i).
- (iii) Arguing by way of contradiction, let us assume that there exists  $y \in X$  such that

$$x_*^T y \le |x_*|^2 / 2. (26)$$

<sup>&</sup>lt;sup>10</sup>Some proofs of de Finetti's consistency theorem in the literature cite a "hyperplane separation theorem" without indicating the specific result needed for the proof.

<sup>&</sup>lt;sup>11</sup>i.e., the distance of the origin from H is > 0, the distance from H of every  $y \in X$  is > 0, and X and the origin are contained in opposite closed half-spaces with boundary H.

The closed interval  $[x_*, y] = \{(1 - \lambda)x_* + \lambda y \mid \lambda \in [0, 1]\}$  is contained in X. For all  $0 \le \lambda \le 1$  we then have:

$$|x_*|^2 \leq |(1-\lambda)x_* + \lambda y|^2, \text{ since } x^* \text{ is the shortest vector in } X$$

$$= (1-\lambda)^2 |x_*|^2 + \lambda^2 |y|^2 + 2(1-\lambda) \cdot \lambda x_*^T y$$

$$= |x_*|^2 + \lambda^2 |x_*|^2 - 2\lambda |x_*|^2 + \lambda^2 |y|^2 + (1-\lambda)\lambda |x_*|^2, \text{ by (26)}.$$

Therefore,

$$0 \leq \lambda^{2}|x_{*}|^{2} - 2\lambda|x_{*}|^{2} + \lambda^{2}|y|^{2} + \lambda|x_{*}|^{2} - \lambda^{2}|x_{*}|^{2}$$
$$= -\lambda|x_{*}|^{2} + \lambda^{2}|y|^{2}.$$

We then have  $|x_*|^2 \le \lambda |y|^2$  for each  $\lambda > 0$ . The only possibility is  $|x_*|^2 = 0$  whence  $x_*$  coincides with the origin of  $\mathbb{R}^n$ , a contradiction.

(iv) Immediate from (iii). 
$$\Box$$

**Fact 2.13.** (Supporting hyperplane theorem) Let X be a closed convex subset of  $\mathbb{R}^n$  having a boundary point b. Then there is a hyperplane H containing b, such that X is contained in one of the two closed half-spaces bounded by H. H is known as a supporting hyperplane for X at b.

*Proof.* By definition of boundary, b lies in the closed set X, and there is a sequence of points  $b_n \notin X$  with  $b_n \to b$ . For each  $b_n$ , let  $x_n \in X$  be the closest point to  $b_n$  as given by Fact 2.12(ii). Let  $u_n$  be the unit vector  $(x_n - b_n)/|x_n - b_n|$ . By Fact 2.12(iv),

$$\frac{|x_n - b_n|}{2} < u_n^T (x - b_n) \quad \text{for each } x \in X, \text{ in particular for } x = b. \tag{27}$$

The bounded sequence  $u_n$  has a convergent subsequence, say converging to the unit vector u. Then by (27),

$$\lim \frac{|x_n - b_n|}{2} \le \lim u_n^T (b - b_n) = u^T \lim (b - b_n) = 0,$$

whence

$$\lim \frac{|x_n - b_n|}{2} = 0. \tag{28}$$

From (27)-(28), for each  $x \in X$  we have

$$0 \le \lim u_n^T(x - b_n) = u^T(x - b)$$
 whence  $u^T b \le u^T x$ .

Thus the hyperplane  $H = \{x \in \mathbb{R}^n \mid u^T x = u^T b\}$  supports X at b.  $\square$ 

We are now ready to prove the specific hyperplane separation lemma needed for the proof of Gordan's theorem and, ultimately, for the proof of de Finetti's consistency theorem.

**Lemma 2.14.** (Hyperplane Separation Theorem) Let  $O \subseteq \mathbb{R}^n$  be the set of all vectors whose coordinates are < 0. We say that O is the south-west open octant. Let R be a linear subspace of  $\mathbb{R}^n$  disjoint from O. Then for some vector  $u \in \mathbb{R}^n$  the hyperplane  $H = \{x \in \mathbb{R}^n \mid u^T x = 0\}$  has the following (separation) property for O and R:

$$R \subseteq H$$
, (i.e.,  $u^T r = 0$  for all  $r \in R$ ).

and

$$u^T y < 0$$
 for all  $y \in O$ , whence  $H \cap O$  is empty.

*Proof.* The set  $R+O=\{r+o\in\mathbb{R}^n\mid r\in R,\ o\in O\}$  is convex and does not contain the origin. Its closure  $\operatorname{cl}(R+O)$  is convex, by Fact 2.11. Since R and O are disjoint, the origin is a boundary point of  $\operatorname{cl}(R+O)$ .

Fact 2.13 provides a supporting hyperplane H for  $\operatorname{cl}(R+O)$  at 0.

Since R is linear and  $0 \in R \cap H$  then R is contained in H,

$$R \subseteq H.$$
 (29)

By construction, O is contained in one of the two *open* half-spaces with boundary H. The other open half-space contains a vector u orthogonal to H such that  $u^Ty < 0$  for all  $y \in O$ . For any such u, by (29), we automatically have  $u^Tr = 0$  for all  $r \in R$ .

The proof of Gordan's theorem is now complete, and so is the proof of de Finetti's consistency theorem 2.2, as well as of its generalization 2.7 to all boolean algebras.

Remarks 2.15. The main effect of de Finetti's consistency theorem is summarized by de Finetti himself in his quote at the beginning of this paper: all the results of probability theory are nothing more than consequences of his definition of consistency. In particular, the traditional "axiom of additivity for the probability of incompatible events" is shown by Theorem 2.2 to be a consequence of de Finetti's definition 2.1 of consistency. In a nutshell:

consistency + incompatibility 
$$\Rightarrow$$
 additivity axiom (30)

#### 3. DE FINETTI'S EXCHANGEABILITY THEOREM

This chapter provides a self-contained proof of de Finetti's exchangeability theorem, a seminal result which he first proved in his 1930 paper [2] and then in [4]. We only use the language of boolean algebras, doing without notions such as probability space, random variable, expectation, conditional, moment, Radon measure, martingale, variously present in the literature on this theorem. In a final section, the original formulation of de Finetti's theorem will be easily recovered from our proof. The proof given here, although elementary, may discourage the reader unfamiliar with long combinatorial calculations. Since this chapter is independent of the rest of this paper, it can be skipped on a first reading.

3.1. Product states and exchangeable states. Let A be a (finite or infinite) boolean algebra. By definition of product topology, the set S(A) of states of A, equipped with the restriction of the product topology of  $[0,1]^A$  is a convex compact subspace of  $\mathbb{R}^A$ . Generalizing the definition of  $\mathsf{F}_n$  given in Section 1.3, let

 $\mathsf{F}_{\omega}$  be the free boolean algebra over the free generating set  $\{X_1, X_2, \dots\}$ .

Equivalently,  $\{X_1, X_2, \dots\}$  generates  $\mathsf{F}_{\omega}$ , and for every  $m = 1, 2, \dots$  and m-tuple  $(\beta_1, \dots, \beta_m) \in \{0, 1\}^m$ ,

$$X_1^{\beta_1} \wedge \dots, \wedge X_m^{\beta_m} \neq 0.$$
 <sup>12</sup> (31)

Any element  $t \in \mathsf{F}_{\omega}$  of the form (31) is said to be a miniterm of  $\mathsf{F}_{\omega}$ . Each  $X_i^{\beta_i}$  is called a *conjunct* of t. Since, as we have seen, the set  $\{X_1, \ldots, X_m\}$  freely generates the free boolean algebra  $\mathsf{F}_m \subseteq \mathsf{F}_{\omega}$ , it follows that t is also a miniterm of  $\mathsf{F}_m$ . We let

$$pos(t)$$
 (resp.,  $neg(t)$ )

<sup>&</sup>lt;sup>12</sup>As in Section 1.3,  $X_i^{\beta_i} = X_i$  if  $\beta_i = 1$ , and  $X_i^{\beta_i} = \neg X_i$  if  $\beta_i = 0$ .

denote the number of non-negated (resp., the number of negated) conjuncts of t. Thus, e.g.,  $pos(t) = \sum_{i=1}^{m} \beta_i$ .

Arbitrarily fix  $p \in [0, 1]$ . Let the function  $f_p$  assign to every miniterm  $t \in \mathsf{F}_{\omega}$  the value  $p^{\mathsf{pos}(t)}(1-p)^{\mathsf{neg}(t)}$ ,

$$f_n(t) = p^{\mathsf{pos}(t)} (1-p)^{\mathsf{neg}(t)}.$$
 (32)

In the particular case p=1, we have  $\operatorname{neg}(t)=0$ . We then set  $\operatorname{f}_1(t)=1$ . Likewise, we set  $\operatorname{f}_0(t)=1$ . Since  $\{X_1,X_2,\dots\}$  freely generates  $\operatorname{F}_{\omega}$ , the function  $\operatorname{f}_p$  is well defined. Upon writing every element  $a\in\operatorname{F}_{\omega}$  as a disjunction of miniterms of some free algebra  $\operatorname{F}_m$ , it follows that (the actual choice of m is immaterial, and) the function  $\operatorname{f}_p$  is extendable to a unique state  $\pi_p$  of  $\operatorname{F}_{\omega}$ ,

$$\pi_p(a) = \text{ unique extension of } f_p, \qquad (p \in [0, 1]).$$
 (33)

For instance,

$$\pi_p((X_1 \wedge X_2 \wedge X_3) \vee (X_1 \wedge X_2 \wedge \neg X_3)) = p^3(1-p)^0 + p^2(1-p) = p^2 = \pi_p(X_1 \wedge X_2),$$
  
in agreement with the identity  $(X_1 \wedge X_2 \wedge X_3) \vee (X_1 \wedge X_2 \wedge \neg X_3) = X_1 \wedge X_2.$ 

**Definition 3.1.** For every  $p \in [0,1]$  we say that  $\pi_p$  is a *product state* of  $\mathsf{F}_{\omega}$ . The restriction  $\pi_p \upharpoonright \mathsf{F}_m$  of  $\pi_p$  to  $\mathsf{F}_m$  is said to be a *product state* of  $\mathsf{F}_m$ .

An exchangeable state of  $F_{\omega}$  is a state  $\sigma \colon F_{\omega} \to [0,1]$  such that for every miniterm t the value  $\sigma(t)$  only depends on the pair of integers (pos(t), neg(t)).

The proof of the following proposition is immediate:

**Proposition 3.2.** (i) For any  $p \in [0,1]$  the product state  $\pi_p$  is an exchangeable state of  $F_{\omega}$ .

- (ii) Every convex combination of product states of  $F_{\omega}$  in the vector space  $\mathbb{R}^{F_{\omega}} \supseteq [0,1]^{F_{\omega}}$  is exchangeable.
- 3.2. The Exchangeability Theorem. In his 1930 paper [2] de Finetti vastly extended Proposition 3.2(ii) with his characterization of exchangeable states. In our boolean algebraic language, "de Finetti theorem" by antonomasia is as follows:

**Theorem 3.3.** Let  $\mathsf{F}_{\omega}$  be the free boolean algebra over the free generating set  $\{X_1, X_2, \ldots\}$ . Let  $\sigma$  be a state of  $\mathsf{F}_{\omega}$ . Then  $\sigma$  is exchangeable iff it lies in the closure of the set of convex combinations of product states of  $\mathsf{F}_{\omega}$  in the vector space  $\mathbb{R}^{\mathsf{F}_{\omega}}$  endowed with the restriction of the product topology.

*Proof.* Self-contained proofs of deep results – the target of this paper – can be long and challenging. As with the consistency theorem, the reader's patient study will be rewarded with knowledge of another far-reaching de Finetti theorem.

(⇒)-direction. Arbitrarily fix  $n=1,2,\ldots$  The restriction  $\sigma_n=\sigma\upharpoonright \mathsf{F}_n$  is an exchangeable state of  $\mathsf{F}_n$ , in the sense that for every miniterm t of  $\mathsf{F}_n$ , the value  $\sigma_n(t)$  only depends on the pair of integers ( $\mathsf{pos}(t), \mathsf{neg}(t)$ ). For all integers N>n and  $K=0,\ldots,N$  let  $\xi_{N,K}$  be the state of the free boolean algebra  $\mathsf{F}_N$  assigning the value  $1/\binom{N}{K}$  to each miniterm u of  $\mathsf{F}_N$  with  $\mathsf{pos}(u)=K$ , and assigning 0 to the remaining  $2^N-\binom{N}{K}$  miniterms of  $\mathsf{F}_N$ .

We can easily verify that  $\xi_{N,K}$  is extremal in the convex set of exchangeable states of  $\mathsf{F}_N$ . In other words,  $\xi_{N,K}$  cannot be expressed as a nontrivial convex combination of two distinct exchangeable states of  $\mathsf{F}_N$ . Furthermore,  $\mathsf{F}_N$  has no other extremal exchangeable states beyond  $\xi_{N,0},\ldots,\xi_{N,N}$ .

Since the restriction  $\sigma_N = \sigma \upharpoonright \mathsf{F}_N$  is an exchangeable state of  $\mathsf{F}_N$  there are real numbers

$$\lambda_{N,0}, \dots, \lambda_{N,N} \ge 0$$
 with  $\sum_{l=0}^{n} \lambda_l = 1$ 

such that  $\sigma_N$  agrees over  $\mathsf{F}_N$  with the convex combination  $\sum_{K=0}^N \lambda_{N,K} \cdot \xi_{N,K}$  in the finite-dimensional vector space  $\mathbb{R}^{\mathsf{F}_N}$ . Since the restriction function

$$\psi \in S(\mathsf{F}_N) \mapsto \psi \upharpoonright \mathsf{F}_n \in S(\mathsf{F}_n)$$

is linear, the state  $\sigma_n = \sigma_N \upharpoonright \mathsf{F}_n = \sigma \upharpoonright \mathsf{F}_n$  agrees over  $\mathsf{F}_n$  with the convex combination  $\sum_{K=0}^N \lambda_{N,K} \cdot \xi_{N,K} \upharpoonright \mathsf{F}_n$ . In particular,

$$\sigma(r) = \sum_{K=0}^{N} \lambda_{N,K} \cdot \xi_{N,K}(r), \quad \text{for each miniterm } r \in \mathsf{F}_n \text{ and } N > n. \tag{34}$$

For some *n*-tuple of bits  $(\beta_1, \ldots, \beta_n) \in \{0, 1\}^n$ , let the miniterm  $t \in \mathsf{F}_n$  be defined by

$$t = X_1^{\beta_1} \wedge \dots \wedge X_n^{\beta_n}.$$

Suppose the miniterm  $w = X_1^{\beta'_1} \wedge \cdots \wedge X_N^{\beta'_N} \in \mathsf{F}_N$  satisfies  $t \geq w$ . Since the set  $\{X_1, X_2, \dots, X_N\}$  freely generates  $\mathsf{F}_N$  then  $\beta_1 = \beta'_1, \dots, \beta_n = \beta'_n$ , whence in particular

$$pos(t) \le pos(w) \le N - neg(t) = N - n + pos(t).$$

For each  $K = pos(t), \ldots, N - n + pos(t)$ , we have

$$v \leq t$$
 for precisely  $\binom{N-n}{K-\mathsf{pos}(t)}$  miniterms  $v$  of  $\mathsf{F}_N$  with  $\mathsf{pos}(v) = K$ .

As we already know, for any such v,  $\xi_{N,K}(v)$  coincides with  $1/\binom{N}{K}$ . Since t equals the disjunction of the miniterms u of  $\mathsf{F}_N$  satisfying  $t \geq u$ , then

$$\xi_{N,K}(t) = \begin{cases} \frac{\binom{N-n}{K-\mathsf{pos}(t)}}{\binom{N}{K}} & \text{if } K = \mathsf{pos}(t), \dots, N-n+\mathsf{pos}(t) \\ 0 & \text{if } K = 0, \dots, \mathsf{pos}(t)-1, \ N-n+\mathsf{pos}(t)+1, \dots, N. \end{cases}$$

Thus by (34), for any miniterm t of  $F_n$  and N > n we can write

$$\sigma(t) = \sum_{K=\mathsf{pos}(t)}^{N-n+\mathsf{pos}(t)} \lambda_{N,K} \cdot \frac{\binom{N-n}{K-\mathsf{pos}(t)}}{\binom{N}{K}}. \tag{35}$$

Claim 1: For every  $n=1,2,\ldots$  and  $\epsilon>0$  there is N>n such that for every miniterm  $t\in \mathsf{F}_n$ , letting

k be shorthand for pos(t),

we have

$$\left| \sigma(t) - \sum_{K=0}^{N} \lambda_{N,K} \, \pi_{K/N}(t) \right| = \left| \sigma(t) - \sum_{K=0}^{N} \lambda_{N,K} (K/N)^k (1 - K/N)^{n-k} \right| < \epsilon.$$

By way of contradiction, suppose there are n and  $\epsilon > 0$  such that for every N > n there is a miniterm  $u \in \mathsf{F}_n$  satisfying  $\left| \sigma(u) - \sum_{K=0}^N \lambda_{N,K} \pi_{K/N}(u) \right| \geq \epsilon$ . Since  $\mathsf{F}_n$  is finite there is  $\epsilon > 0$  and n, together with a miniterm  $t \in \mathsf{F}_n$  such that for infinitely many N,  $\left| \sigma(t) - \sum_{K=0}^N \lambda_{N,K} \pi_{K/N}(t) \right| \geq \epsilon$ .

Case 1:  $pos(t) \notin \{0, n\}$ . By (35) we can write

$$\sigma(t) = \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \frac{\binom{N-n}{K-k}}{\binom{N}{K}} = \sum_{K=k}^{N} \lambda_{N,K} \cdot \frac{(N-n)!}{(K-k)!(N-n-(K-k))!} \cdot \frac{K!(N-K)!}{N!}$$

$$= \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \frac{(K-k+1)\cdots K}{(N-n+1)\cdots N} \cdot (N-n-(K-k)+1)\cdots (N-K)$$

$$= \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \frac{(K-k+1)\cdots K}{(N-n+1)\cdots (N-n+k)} \cdot \frac{(N-K-n+k+1)\cdots (N-K)}{(N-n+k+1)\cdots N}$$

$$= \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \prod_{i=1}^{k} \frac{K/N - (k-i)/N}{1 - (n-i)/N} \cdot \prod_{i=1}^{n-k} \frac{1 - K/N - (n-k-j)/N}{1 - (n-k-j)/N}.$$

Since  $k = pos(t) \notin \{0, n\}$  and  $\lambda_{N,K} \leq 1$ , then

$$0 = \lim_{N \to \infty} \sum_{K=0}^{k} \lambda_{N,K} (K/N)^{k} (1 - K/N)^{n-k}$$
$$= \lim_{N \to \infty} \sum_{K=N-n+k}^{N} \lambda_{N,K} (K/N)^{k} (1 - K/N)^{n-k}.$$

So our absurdum hypothesis equivalently states that there is  $\epsilon > 0$  together with a miniterm  $t \in \mathsf{F}_n$  such that

$$\left| \sigma(t) - \sum_{K-k}^{N-n+k} \lambda_{N,K} \pi_{K/N}(t) \right| \ge \epsilon$$

for infinitely many N. For i = 1, ..., k and j = 1, ..., n - k) let the rationals  $c_i$  and  $d_j$  be defined by

$$c_i = \left(1 - \frac{k - i}{K}\right) / \left(1 - \frac{n - i}{N}\right)$$

and

$$d_{j} = \left(1 - \frac{n-k-j}{N-K}\right) / \left(1 - \frac{n-k-j}{N}\right).$$

Then

$$\epsilon \leq \sum_{K=k}^{N-n+k} \lambda_{N,K} \left| \prod_{i=1}^{k} \frac{K/N - (k-i)/N}{1 - (n-i)/N} \cdot \prod_{j=1}^{n-k} \frac{1 - K/N - (n-k-j)/N}{1 - (n-k-j)/N} - \pi_{K/N}(t) \right| \\
\leq \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \left| \prod_{i=1}^{k} c_i \cdot K/N \cdot \prod_{j=1}^{n-k} d_j \cdot (1 - K/N) - (K/N)^k (1 - K/N)^{n-k} \right| \\
= \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot (K/N)^k \cdot (1 - K/N)^{n-k} \cdot \left| 1 - \prod_i c_i \prod_j d_j \right| \\
\leq \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot (K/N)^k \cdot (1 - K/N)^{n-k} \cdot \left( 1 - \left( 1 - \frac{k}{K} \right)^k \cdot \left( 1 - \frac{n-k}{N-K} \right)^{n-k} \right),$$

because for all  $i, j - 1 - \frac{k}{K} \le c_i \le 1$  and  $1 - \frac{n-k}{N-K} \le d_j \le 1$ .

Fix  $\eta > 0$ . For all sufficiently large N we have  $k < \lfloor \eta N \rfloor < \lceil N - \eta N \rceil < N - n + k$ , where  $\lfloor x \rfloor$  (resp.,  $\lceil x \rceil$ ) is the largest integer  $\leq x$  (resp., the smallest integer  $\geq x$ ). As a consequence,

$$\epsilon \leq \sum_{K=k}^{\lfloor \eta N \rfloor} \lambda_{N,K} \cdot (K/N)^k \cdot (1 - K/N)^{n-k} \cdot \left(1 - \left(1 - \frac{k}{K}\right)^k \cdot \left(1 - \frac{n-k}{N-K}\right)^{n-k}\right)$$

$$+ \sum_{K=\lfloor \eta N \rfloor + 1}^{\lceil N - \eta N \rceil} \lambda_{N,K} \cdot (K/N)^k \cdot (1 - K/N)^{n-k} \cdot \left(1 - \left(1 - \frac{k}{K}\right)^k \cdot \left(1 - \frac{n-k}{N-K}\right)^{n-k}\right)$$

$$+ \sum_{K=\lceil N - \eta N \rceil + 1}^{N-n+k} \lambda_{N,K} \cdot (K/N)^k \cdot (1 - K/N)^{n-k} \cdot \left(1 - \left(1 - \frac{k}{K}\right)^k \cdot \left(1 - \frac{n-k}{N-K}\right)^{n-k}\right)$$

$$\leq \sum_{K=k}^{\lfloor \eta N \rfloor} \lambda_{N,K} \cdot \left(\frac{\eta N}{N}\right)^k \cdot 1 \cdot 1$$

$$+ \sum_{K=\lfloor \eta N \rfloor + 1}^{\lceil N - \eta N \rceil} \lambda_{N,K} \cdot 1 \cdot 1 \cdot \left(1 - \left(1 - \frac{k}{\eta N}\right)^k \cdot \left(1 - \frac{n-k}{N-(N-\eta N)}\right)^{n-k}\right)$$

$$+ \sum_{K=\lceil N - \eta N \rceil + 1}^{N-n+k} \lambda_{N,K} \cdot 1 \cdot \left(1 - \frac{N - \eta N + 1}{N}\right)^{n-k} \cdot 1$$

$$\leq \sum_{K=k}^{\lfloor \eta N \rfloor} \lambda_{N,K} \cdot \left[\left(\frac{\eta N}{N}\right)^k\right] + \sum_{K=\lfloor \eta N \rfloor + 1}^{\lceil N - \eta N \rceil} \lambda_{N,K} \cdot \left[1 - \left(1 - \frac{k}{\eta N}\right)^k \cdot \left(1 - \frac{n-k}{\eta N}\right)^{n-k}\right]$$

$$+ \sum_{K=\lceil N - \eta N \rceil + 1}^{N-n+k} \lambda_{N,K} \cdot \left[\left(\eta - \frac{1}{N}\right)^{n-k}\right].$$

Since  $\epsilon$  is fixed we may choose  $\eta > 0$  so small that for all sufficiently large N,

$$\begin{split} \epsilon & \leq & \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \left( \left( \frac{\eta N}{N} \right)^k + \left( 1 - \left( 1 - \frac{k}{\eta N} \right)^k \cdot \left( 1 - \frac{n-k}{\eta N} \right)^{n-k} \right) + \left( \eta - \frac{1}{N} \right)^{n-k} \right) \\ & \leq & \sum_{K=k}^{N-n+k} \lambda_{N,K} \cdot \frac{\epsilon}{1000} \leq \frac{\epsilon}{1000}, \text{ which is impossible.} \end{split}$$

Case 2:  $pos(t) \in \{0, n\}$ . Then a routine simplification of the proof of Case 1 again yields a contradiction.

Having thus settled our claim, from  $\mathsf{F}_1 \subseteq \mathsf{F}_2 \subseteq \cdots$  it follows that for every  $\epsilon > 0$  and  $n = 1, 2, \ldots$ , there is N > n together with a convex combination of product states of  $\mathsf{F}_N$  which agrees with  $\sigma$  over all miniterms of  $\mathsf{F}_n$  up to an error  $< \epsilon$ . Every set  $\{a_1, \ldots, a_z\} \subseteq \mathsf{F}_\omega$  is contained in some finitely generated free boolean algebra  $\mathsf{F}_n$ , and hence each  $a_i$  is a disjunction of miniterms of  $\mathsf{F}_n$ . In conclusion,

For all  $\{a_1, \ldots, a_z\} \subseteq \mathsf{F}_{\omega}$  and  $\epsilon > 0$  there is a convex combination of product states of  $\mathsf{F}_{\omega}$  agreeing with  $\sigma$  over  $\{a_1, \ldots, a_z\}$  up to an error  $< \epsilon$ 

By definition of the product topology of the compact space  $[0,1]^{\mathsf{F}_{\omega}}$ , this amounts to saying that  $\sigma$  belongs to the closure of the set of convex combinations of product states of  $\mathsf{F}_{\omega}$  in the vector space  $\mathbb{R}^{\mathsf{F}_{\omega}}$  equipped with the product topology.

 $(\Leftarrow)$ -direction. Easy now.

3.3. De Finetti's formulation of the Exchangeability Theorem. Fix an exchangeable state  $\sigma$  of the boolean algebra  $\mathsf{F}_{\omega}$  freely generated by  $X_1, X_2, \ldots$  The proof of Theorem 3.3 yields integers  $0 < N_1 < N_2 < \ldots$ , and for each  $i = 1, 2, \ldots$  real numbers  $\lambda_{N_i,0}, \ldots, \lambda_{N_i,N_i} \geq 0$  summing up to 1, along with Borel probability measures  $\mu_i$  on [0,1] such that for any miniterm  $t \in \mathsf{F}_{\omega}$ 

$$\sigma(t) = \lim_{i \to \infty} \sum_{K=0}^{N_i} \lambda_{N_i,K} (K/N_i)^{\mathsf{pos}(t)} (1 - K/N_i)^{\mathsf{neg}(t)}$$

$$= \lim_{i \to \infty} \sum_{K=0}^{N_i} \lambda_{N_i,K} \pi_{K/N_i}(t)$$

$$= \lim_{i \to \infty} \int_{[0,1]} p^{\mathsf{pos}(t)} (1 - p)^{\mathsf{neg}(t)} d\mu_i(p).$$

Elementary measure theory will now yield the original formulation [2] of de Finetti's exchangeability theorem by letting P([0,1]) be the compact metric space of Borel probability measures on [0,1] equipped with the weak topology. The sequential compactness of P([0,1]) yields a subsequence of the  $\mu_i$  converging to some  $\mu \in P([0,1])$ . Thus for every miniterm  $t \in \mathsf{F}_{\omega}$ 

$$\sigma(t) = \int_{[0,1]} p^{\mathsf{pos}(t)} \, (1-p)^{\mathsf{neg}(t)} \, \mathrm{d}\mu(p) = \int_{[0,1]} \mathsf{f}_p(t) \, \mathrm{d}\mu(p) = \int_{[0,1]} \pi_p(t) \, \mathrm{d}\mu(p).$$

Intuitively: Any exchangeable sequence of  $\{yes, no\}$ -events  $X_1, X_2, \ldots$  is a "mixture" of sequences of independent and identically distributed (i.i.d.) Bernoulli random variables.

Remarks 3.4. De Finetti proved the existence of  $\mu$  in the Appendix of [2, pp.124-133] using characteristic functions, the tools available to him when he communicated this result in the 1928 International Congress of Mathematicians in Bologna. In all his subsequent papers on exchangeability he abandoned this formalism and used distribution functions, probability measures, moments, and laws of large numbers. In [2, Chapter 4, §31, p.121], the free generating set  $\{X_1, X_2, \ldots, X_n\}$  of  $\mathsf{F}_n$  is said to constitute a "class of equivalent events".

A comprehensive advanced account of the ramifications of exchangeability is given by D.J. Aldous in "Exchangeability and related topics". In: École d' Été de Probabilités de Saint-Flour XIII–1983. Lecture Notes in Mathematics, Vol. 1117, Springer, Berlin, (1985), pp. 1–198. Further extensions of de Finetti's theorem to quantum states have been applied in other research areas, including quantum information theory.

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<sup>&</sup>lt;sup>13</sup>i.e., a sort of a "weighted average" computed by an integral.

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