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Wednesday January 30th, 2019
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BME590

HW 1

1 Linear algebra refresher

a The inner product

$$\mathbf{u^T v} = \begin{bmatrix}1 & 5 & 1 & 4\end{bmatrix}\begin{bmatrix}0\\1\\5\\1\end{bmatrix} = 1*0+5*1+1*5+4*1 = 14$$

b The outer product

$$\mathbf{uv^T} = \begin{bmatrix}1\\5\\1\\4\end{bmatrix}\begin{bmatrix}0 & 1 & 5 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 5 & 1\\0 & 5 & 25 & 5\\0 & 1 & 5 & 1\\0 & 4 & 20 & 4\end{bmatrix}$$

c The Hadamard product

$$\mathbf{u} \circ \mathbf{v} = \begin{bmatrix}1\\5\\1\\4\end{bmatrix} \circ \begin{bmatrix}0\\1\\5\\1\end{bmatrix} = \begin{bmatrix}0\\5\\5\\4\end{bmatrix}$$

d The complex inner product

$$\mathbf{a^H b} = \begin{bmatrix}1 & e^{\frac{-i\pi}{4}} & e^{\frac{-i\pi}{2}} & e^{\frac{-i\pi}{4}}\end{bmatrix}\begin{bmatrix}0\\e^{i\pi}\\5e^{i\pi}\\e^{-i\pi}\end{bmatrix} = 0 + e^{\frac{3i\pi}{4}} + 5e^{\frac{i\pi}{2}} + e^{\frac{-5i\pi}{4}} = e^{\frac{3i\pi}{4}} + 5e^{\frac{i\pi}{2}} + e^{\frac{-5i\pi}{4}}$$

e The complex inner product

$$\mathbf{a^H a} = \begin{bmatrix}1 & e^{\frac{-i\pi}{4}} & e^{\frac{-i\pi}{2}} & e^{\frac{-i\pi}{4}}\end{bmatrix}\begin{bmatrix}1\\e^{\frac{i\pi}{4}}\\e^{\frac{i\pi}{2}}\\e^{\frac{i\pi}{4}}\end{bmatrix} = 1+1+1+1 = 4$$

$$\mathbf{b^H b} = \begin{bmatrix}0 & e^{-i\pi} & 5e^{-i\pi} & e^{i\pi}\end{bmatrix}\begin{bmatrix}0\\e^{i\pi}\\5e^{i\pi}\\e^{-i\pi}\end{bmatrix} = 0+1+25+1 = 27$$

f The complex Hadamard product

$$\mathbf{a} \circ \mathbf{a}^* = \begin{bmatrix}1\\e^{\frac{i\pi}{4}}\\e^{\frac{i\pi}{2}}\\e^{\frac{i\pi}{4}}\end{bmatrix} \circ \begin{bmatrix}1\\e^{\frac{-i\pi}{4}}\\e^{\frac{-i\pi}{2}}\\e^{\frac{-i\pi}{4}}\end{bmatrix} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

g The complex Hadamard product

$$\mathbf{b} \circ \mathbf{b}^* = \begin{bmatrix} 0 \\ e^{i\pi} \\ 5e^{i\pi} \\ e^{-i\pi} \end{bmatrix} \circ \begin{bmatrix} 0 \\ e^{-i\pi} \\ 5e^{-i\pi} \\ e^{i\pi} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 25 \\ 1 \end{bmatrix}$$

h Check

$$|\mathbf{a^H b}| \leq ||\mathbf{a}||_\mathbf{2} ||\mathbf{b}||_\mathbf{2}$$
$$\left| e^{\frac{3i\pi}{4}} + 5e^{\frac{i\pi}{2}} + e^{\frac{-5i\pi}{4}} \right| \leq 2 \cdot 3\sqrt{3}$$

This checks out. The left side has a maximum value of 7 whereas the left side is greater than 7.

2 Convolutions in 1D

a The discrete convolution

$$\mathbf{u} * \mathbf{v} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & -5 \end{bmatrix}$$

b The convolution matrix

$$\mathbf{w} = \mathbf{U}\mathbf{v}$$
$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

This convolution is a discrete derivative (ignoring the edges).

c The deconvolution matrix

$$\mathbf{v} = \mathbf{D}\mathbf{w}$$
$$\mathbf{D} = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

This deconvolution is a discrete integral.

d The $\mathbf{DD^T}$ matrix

This matrix represents a double integral and is invertible. It must be invertible because it is square and nondegenerate. Its inverse represents a double derivative.

3 Fourier transforms

a $U(x) = \mathrm{rect}(2x)$

$F\left[U(x)\right] = 0.5 * \mathrm{sinc}\left(\frac{\omega}{4}\right)$

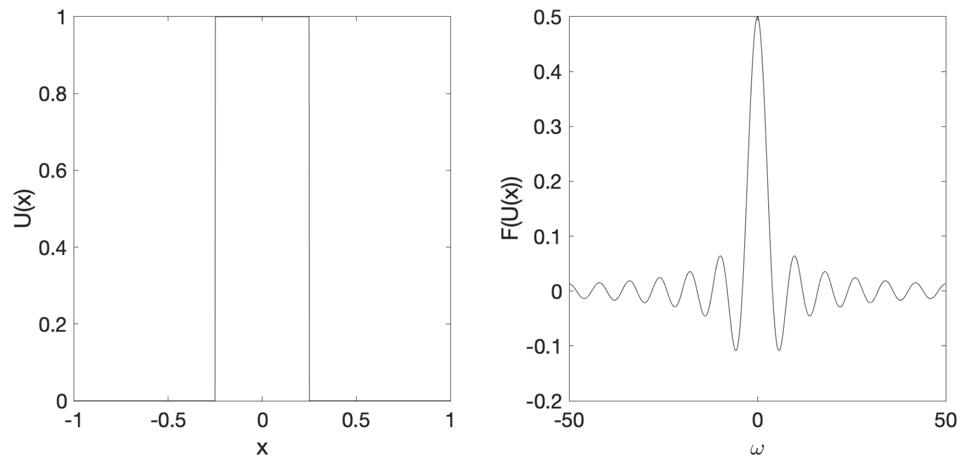

Figure 1: Function and its fourier transform

b $U(x) = e^{\frac{-x^2}{4}}$

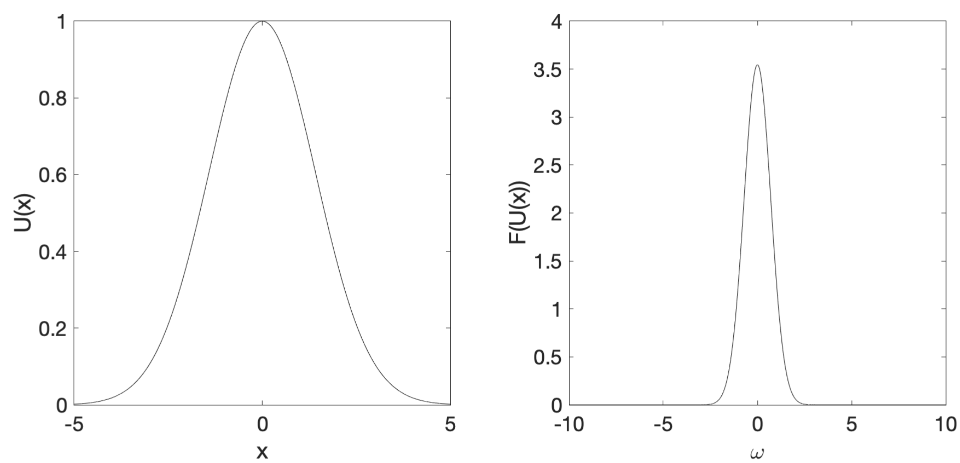

Figure 2: Function and its fourier transform

c $U(x) = \delta(x-3)$

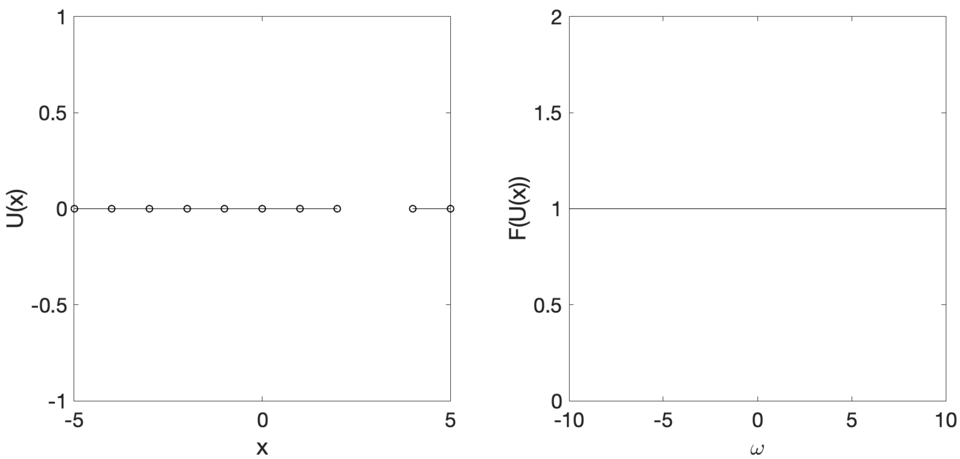

Figure 3: Function and its fourier transform

d $U(x) = \mathrm{ReLU}(x)$

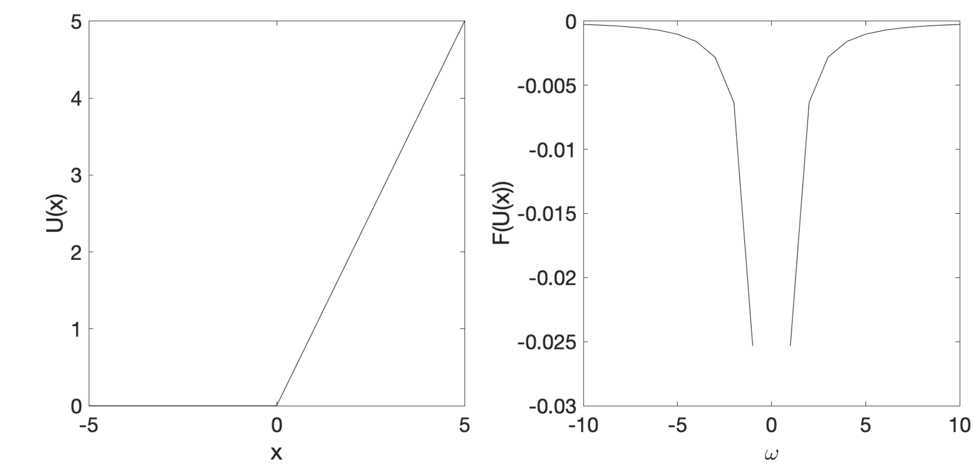

Figure 4: Function and its fourier transform

e The convolution theorem The convolution theorem can be restated and proved accordingly:

$$F[u(x) * v(x)] = F[u(x)]F[v(x)]$$

$$V(\omega) \equiv F[v(x)]$$
$$U(\omega) \equiv F[u(x)]$$

$$\begin{aligned} v(x) &= F^{-1}[V(\omega)] \\ &= \int_{-\infty}^{\infty} V(\omega) e^{2\pi i \omega x} d\omega \end{aligned}$$

$$\begin{aligned} u * v &\equiv \int_{-\infty}^{\infty} u(x')v(x-x')dx' \\ &= \int_{-\infty}^{\infty} u(x') \left[\int_{-\infty}^{\infty} V(\omega) e^{2\pi i \omega (x-x')} d\omega \right] dx' \\ &= \int_{-\infty}^{\infty} V(\omega) \left[\int_{-\infty}^{\infty} u(x') e^{-2\pi i \omega x'} dx' \right] e^{2\pi i \omega x} d\omega \\ &= \int_{-\infty}^{\infty} V(\omega) U(\omega) e^{2\pi i \omega x} d\omega \\ &= F^{-1}[V(\omega)U(\omega)] \\ F[u * v] &= V(\omega)U(\omega) \end{aligned}$$

$\blacksquare$

4 Quick and conceptual

a The Nyquist Criterion

The Nyquist criterion states that the sampling frequency must be at minimum twice the maximum frequency of the phenomenon being sampled.

A comb of pitch $= 10\mu$m is insufficient and would lead to an aliased image.

A comb of pitch $= 4\mu$m is below the maximum (5μm) and would be acceptable.

b $f(x) = x^2 \sin(x)$

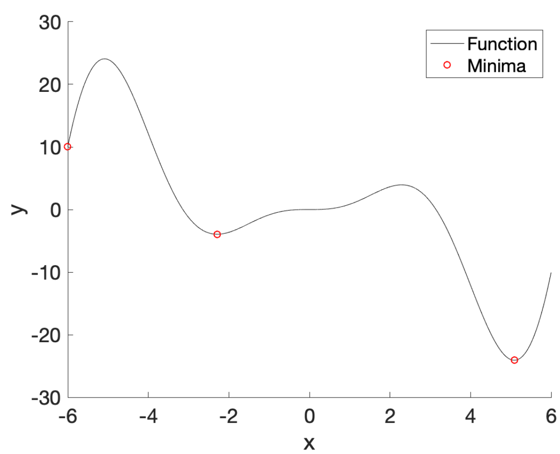

Figure 5: Function with minima labeled

Because of the presence of multiple minima, we are not always guaranteed to locate the global minimum. We would need to pick $x > 2.289$ to find the global minimum; anything less would result in one of the other two being determined as the global minimum.

c $f(x) = \text{sign}(\mathbf{w^T x})$

After performing the inner product, we can expand f: $f(x) = \text{sign}(3 + 2x_1 + x_2)$. The inside of the sign function is a linear function which will map a 2D hyperplane in 3D. The separation of the two opposite signs sits at the intersection of this hyperplane and the $x_1 - x_2$ plane. Because the intersection of two planes (in this case) is a line, we can conclude that the two regions of f are separated by a line.

This conclusion is reinforced by the plot below:

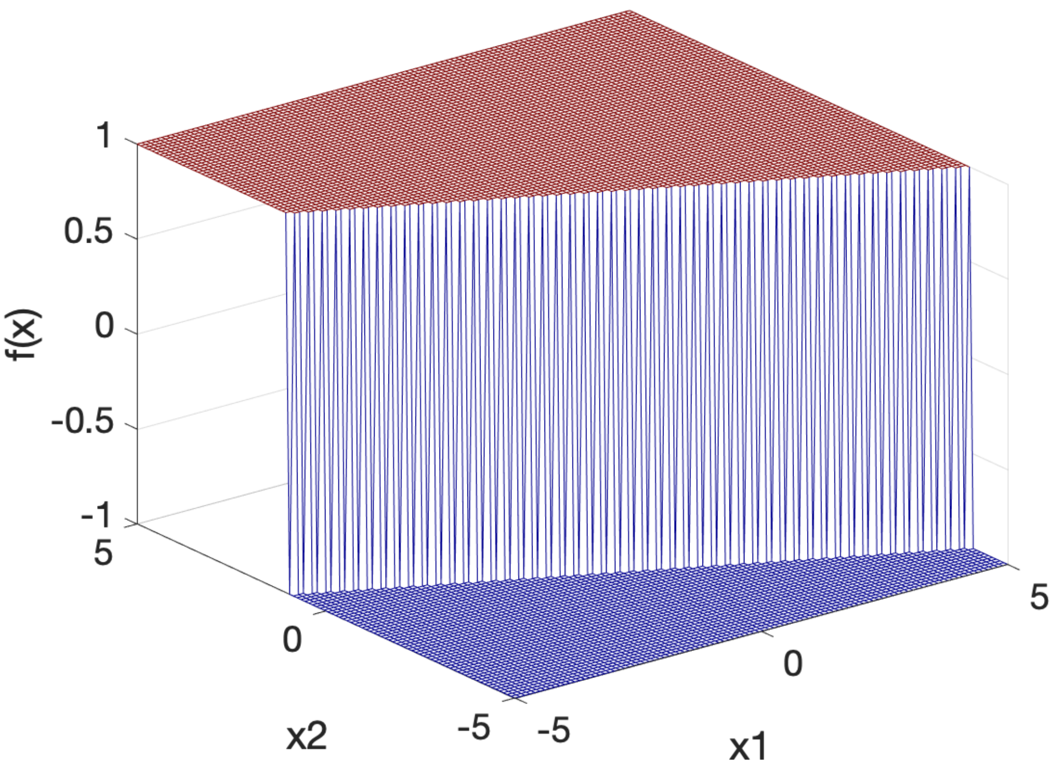

Figure 6: Mesh of f(x) generated in MATLAB