Notes Taken from 18.100A 2020
Dr. Casey Rodriguez Lecture.

**Q:** When do two sets $A$ and $B$ have the same $\boxed{\text{size}}$?

**A (kind of):** When the elements of the two sets can be paired off.

**Eg:** $\{a, b, c\}$, $\{1, 2, 3\}$ (Theory of Cardinality)

$a \to 1$
$b \to 2$
$c \to 3$

## Functions:

**Def:** If $A$ and $B$ are sets, a function $f: A \to B$ is a mapping that assigns to each $x \in A$ a unique element $f(x) \in B$

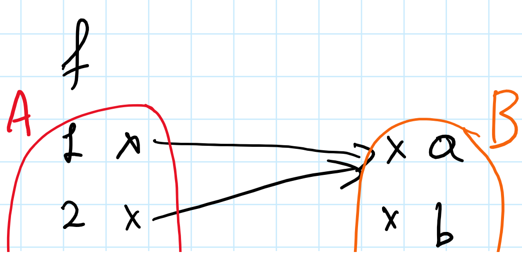
**Def:** Let $f: A \to B$

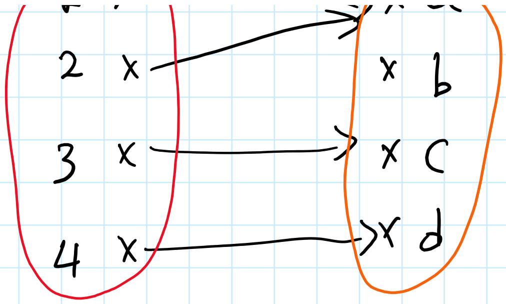
1) if $C \subset A$, we define

$$f(C) = \{ y \in B \mid \exists x \in C \text{ s.t. } y = f(x) \}$$
$$= \{ f(x) \mid x \in C \}$$

2) If $D \subset B$, we define $f^{-1}(D) = \{ x \in A \mid f(x) \in D \}$

**Eg:**





$f(\{1,2\}) = \{a\}$ $\qquad$ $f(\{1,3\}) = \{a, c\}$

$f^{-1}(\{a\}) = \{1,2\}$ $\qquad$ $f^{-1}(\{a, c, d\}) = \overset{=A}{\{1, 2, 3, 4\}}$

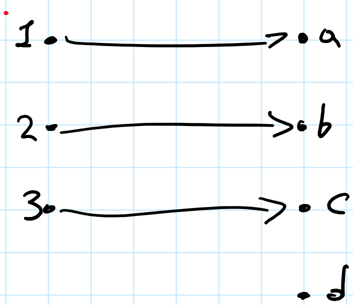
**Def:** Let $f: A \to B$

1) $f$ is <u>injective</u> or <u>1-1</u> if $\qquad$ <u>Equiv:</u> $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

2) $f$ is <u>surjective</u> or onto if $\qquad$ <u>Equiv:</u> $\forall y \in B\ \exists x \in A$ s.t. $f(x) = y$
$f(A) = B$

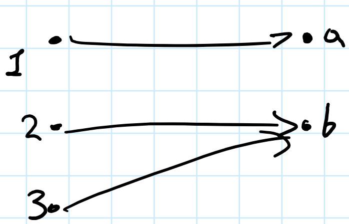
3) $f$ is <u>bijective</u> if $f$ is <u>1-1</u> and <u>onto</u>

<u>Eg:</u> 1.



<u>inj.</u> but not surjective

2.



<u>Surj.</u> but not injective

3. $a \to 1$

$b \to 2$

$c \to 3$

bijective

**Def:** If $f: A \to B$, $g: B \to C$,

1. $g \circ f: A \to C$ is defined by $(g \circ f)(x) = g(f(x))$.

2. If $f: A \to B$ is bijective then $f^{-1}: B \to A$ defined by: if $y \in B$ then $f^{-1}(y) \in A$ is the unique element in $A$ s.t. $f(f^{-1}(y)) = y$

## Cardinality

**Def:** Two sets $A$ and $B$ have the same cardinality if $\exists$ bijective function $f: A \to B$.

Notation: 1) if $A, B$ have the same card., we write $|A| = |B|$.

2) if $|A| = |\{1, 2, \ldots, n\}|$ we write $|A| = n$. ($A$ is finite)

3) if $\exists$ injective fn. $f: A \to B$ we write $|A| \leq |B|$

4) if $|A| \le |B|$ but $|A| \neq |B|$ we write $|A| < |B|$.

**Theorem "Cantor-Schroder-Bernstein Theorem":**

if $|A| \le |B|$ and $|B| \le |A| \implies |A| = |B|$

**Def:** If $|A| = |\mathbb{N}|$ then $A$ is <u>countably</u> <u>infinite</u>

If $A$ is finite or countably infinite we say $A$ is <u>countable</u>

Otherwise, we say $A$ is <u>uncountable</u>

**Theorem:** If $|A| = |B|$ then $|B| = |A|$

**Proof:** Suppose $|A| = |B|$ then $\exists$ bijective function $f: A \to B$

Then $f^{-1}: B \to A$ is a bijection so $|B| = |A|$. $\square$

**Theorem:** if $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$

**Proof:** Suppose $|A| = |B|$ and $|B| = |C|$. Then bijections from $f: A \to B$ and $g: B \to C$. Let $h: A \to C$ be the fn. $h(x) = (g \circ f)(x)$

We want to prove $h$ is a bijection.

We first show $h$ is 1-1. if $h(x_1) = h(x_2)$ then $x_1 = x_2$.

If $h(x_1) = h(x_2)$ then $g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ (Since $g$ is 1-1)

$\Rightarrow x_1 = x_2$ (Since $f$ is 1-1)

$\therefore$ $h$ is 1-1

Now we prove $h(A) = C$: $\forall z \in C \ \exists x \in A$ s.t. $h(x) = z$

Now we prove $h(A) = C$: $\forall z \in C \ \exists x \in A$ s.t. $h(x) = z$

Let $z \in C$. Since $g$ is surjective, $\exists y \in B$ s.t $g(y) = z$

Since $f$ is surjective, $\exists x \in A$ s.t. $f(x) = y$. Then $h(x) = g(f(x))$
$= g(y) = z$ $\square$

**Theorem:** 1) $|\{2n : n \in \mathbb{N}\}| = |\mathbb{N}|$

2) $|\{2n-1 : n \in \mathbb{N}\}| = |\mathbb{N}|$

**proof:** 1) We want to show $|\mathbb{N}| = |\{2n : n \in \mathbb{N}\}|$

Let $f : \mathbb{N} \longrightarrow \{2n : n \in \mathbb{N}\}$

$$f(n) = 2n, \quad n \in \mathbb{N}$$

We first show $f$ is 1-1 i.e. $f(n_1) = f(n_2)$ then $n_1 = n_2$

Suppose $f(n_1) = f(n_2) \implies 2n_1 = 2n_2 \implies n_1 = n_2$ $\therefore f$ is 1-1.

We now show $f$ is onto: i.e. $\forall m \in \{2k : k \in \mathbb{N}\} \ \exists n$ s.t. $f(n) = m$

let $m \in \{2k : k \in \mathbb{N}\}$. Then $\exists n \in \mathbb{N}$ s.t. $m = 2n$. Then $f(n) = 2n = m$.

$\therefore f$ is onto. $\square$

2) HW

**Theorem:** $|\mathbb{Z}| = |\mathbb{N}|$

**Proof:** HW

**Theorem:**

$|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|$

$r_j, s_k \in \mathbb{N}, \forall_{j,k} \; q_j \neq p_k$

**Remark:** Every $q \in \mathbb{Q}, q > 0$ can be written as $q = \dfrac{p_1^{r_1} \cdots p_N^{r_N}}{q_1^{s_1} \cdots q_M^{s_M}}$,

$$p_1^{2r_1} \cdots p_N^{2r_N} \cdot q_1^{2s_1 - 1} \cdots q_M^{2s_N - 1}$$

(HW: is a bijection)

**Theorem:**

$$|\mathbb{Q}| = |\mathbb{N}|$$

**Proof (sketch):** We have $|\{q \in \mathbb{Q} : q > 0\}| = |\{r \in \mathbb{Q} : r < 0\}|$

Since $f(q) := -q$ is a bijection from $\{q \in \mathbb{Q} : q > 0\}$

Since $\{r \in \mathbb{Q} : r < 0\}$. $\therefore |\{r \in \mathbb{Q} : r < 0\}| = |\mathbb{N}|$

Then $\exists$ bijections $f : \{q \in \mathbb{Q} : q > 0\} \longrightarrow \mathbb{N}$ and

$g : \{r \in \mathbb{Q} : r < 0\} \longrightarrow \mathbb{N}$. Define $h : \mathbb{Q} \longrightarrow \mathbb{Z}$

by
$$h(x) = \begin{cases} 0 & x = 0 \\ f(x) & x > 0 \\ -g(x) & x < 0 \end{cases}$$

Then $h$ is a bijection. So $|\mathbb{Q}| = |\mathbb{Z}| \Rightarrow |\mathbb{Q}| = |\mathbb{N}|$ $\square$

**Q:** Does there exist a set $A$ s.t. $|\mathbb{N}| < |A|$?

**Def:** If $A$ is a set, we define the power set of $A$:

$$\mathcal{P}(A) = \{B : B \subset A\}$$

Ex: $A = \emptyset$, $\mathcal{P}(A) = \{\emptyset\}$

$A = \{1\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}\}$

$A = \{1,2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Theorem: If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$

Theorem "Cantor":

If $A$ is a set then $|A| < |\mathcal{P}(A)|$.

Theorem: $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \ldots$

Remark: Informally, there are infinety of infinitudes.

Q: Does there exist set $A$ s.t. $|\mathbb{N}| < |A| < |\mathcal{P}(\mathbb{N})|$?

(Continicum Hypothesis)