Lecture 1: Sets, Set Operations, and Mathematical Induction

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Basic Set Theory

Notes Taken from 18.100A 2020 Dr. Casey Rodriguez Lecture.

Sets

- A set is a collection of objects called elements or members of that set. The empty set ($\emptyset$) is the set with no elements.
- Let S be a set. Then:

- $a \in S$ "a is an element in S"
- $a \notin S$ "a is not an element in S"
- $\forall$ "for all"
- $:=$ "define"
- $\exists$ "There exists"
- $\exists!$ "There exists a unique"
- $\Rightarrow$ "implies"
- $\Longleftrightarrow$ "if and only if"

Definitions

1. A set is a subset of B, $A \subset B$, if every element of A is in B. Given $A \subset B$, if $a \in A \Rightarrow a \in B$.

2. Two sets A and B are equal, $A = B$, if $A \subset B$ and $B \subset A$

3. A set A is a proper subset of B, $A \subsetneq B$ if $A \subset B$ and $A \neq B$.

Set building notation

$$\{x \in A \mid P(x)\} \text{ or } \{x \mid P(x)\}$$

- This means "all $x \in A$ that satisfy property $P(x)$".

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- E.g. $\{x \mid x \text{ is an even number}\}$.

Key sets

1. Set of natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
2. Set of integers : $\mathbb{Z} = \{1, -1, 2, -2, \ldots\}$
3. Set of rational numbers: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$
4. The set of Real numbers: $\mathbb{R}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} (\subset \mathbb{C})$$

Defenitions

1. The Union of A and B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
2. The intersection of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
3. The set difference of A and B is the set $A \setminus B = \{x \in A \mid x \notin B\}$
4. The complement of A is the set $A^c = \{x \mid x \notin A\}$
5. A and B are disjoint if $A \cap B = \emptyset$

Theorem: "De Morgan's Laws"

If A, B, C are sets then:

1. $(B \cup C)^c = B^c \cap C^c$
2. $(B \cap C)^c = B^c \cup C^c$
3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
4. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof:

1. To prove: $(B \cup C)^c \subset B^c \cap C^c$ and $B^c \cap C^c \subset (B \cup C)^c$

- if $x \in (B \cup C)^c \Rightarrow x \notin B \cup C \Rightarrow x \notin B$ and $x \notin C$. Hence $x \in B^c$ and $x \in C^c \Rightarrow x \in B^c \cap C^c$. Thus $(B \cup C)^c \subset B^c \cap C^c$

- if $x \in B^c \cap C^c \Rightarrow x \in B^c$ and $x \in C^c \Rightarrow x \notin B$ and $x \notin C$. $\Rightarrow x \notin B \cup C \Rightarrow x \in (B \cup C)^c$ Hence $B^c \cap C^c \subset (B \cup C)^c$ $\square$

• if $x \in B^c \cap C^c \Rightarrow x \in B^c$ and $x \in C^c \Rightarrow x \notin B$ and $x \notin C$.
$\Rightarrow x \notin B \cup C \Rightarrow x \in (B \cup C)^c$ Hence $B^c \cap C^c \subset (B \cup C)^c$ $\square$

2. To prove: $(B \cap C)^c \subset B^c \cup C^c$ and $B^c \cup C^c \subset (B \cap C)^c$

• if $x \in (B \cap C)^c \Rightarrow x \notin B \cap C \Rightarrow x \notin B$ or $x \notin C \Rightarrow x \in B^c$ or $x \in C^c \Rightarrow x \in B^c \cup C^c$. Hence $(B \cap C)^c \subset B^c \cup C^c$

• if $x \in B^c \cup C^c \Rightarrow x \in B^c$ or $x \in C^c \Rightarrow x \notin B$ or $x \notin C$
$\Rightarrow x \notin B \cap C \Rightarrow x \in (B \cap C)^c$. Hence $B^c \cup C^c \subset (B \cap C)^c$ $\square$

3. $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$ and $(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$

• if $x \in A \setminus (B \cup C) \Rightarrow x \in A \mid x \notin B \cup C \Rightarrow x \in A \mid x \notin B$ and $x \notin C \Rightarrow x \in A \mid x \notin B$ and $x \in A \mid x \notin C$
$\Rightarrow x \in (A / B) \cap (A / B)$
$\therefore A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$

• if $x \in (A \setminus B) \cap (A \setminus C) \Rightarrow x \in A \mid x \notin B$ and $x \in A \mid x \notin C$
$\Rightarrow x \in A \mid x \notin B \cup C \Rightarrow x \in A \setminus (B \cup C)$
$\therefore (A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$ $\square$

4. To prove: $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$ and $(A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C)$

• if $x \in A \mid x \notin (B \cap C) \Rightarrow x \in A \mid x \notin B$ or $x \in A \mid x \notin C$
$\Rightarrow x \in (A / B) \cup (A \mid C)$
$\therefore A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$

• if $x \in (A \setminus B) \cup (A \setminus C) \Rightarrow x \in A \mid x \notin B$ or $x \in A \mid x \notin C$
$\Rightarrow x \in A \mid x \notin B \cap C \Rightarrow x \in A \setminus (B \cap C)$
$\therefore (A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C)$ $\square$

Mathematical Induction

Axiom: "The well-ordering property"

- The well-ordering property of $\mathbb{N}$ states that if $S \subset \mathbb{N}$ then $\exists$ an $x \in S$ s.t. $x \leq y$ $\forall y \in S$.

i.e. there is always a smallest element.

Theorem: Induction

Let $P(n)$ be a statement depending on $n \in \mathbb{N}$. Assume that:

1. (Base Case) $P(1)$ is True and
2. (Inductive step) if $P(m)$ is true then $P(m+1)$ is True.

Then, $P(n)$ is True $\forall n \in \mathbb{N}$.

Proof: Let $S = \{n \in \mathbb{N} \mid P(n)$ is not true$\}$

Wish to show: $S = \emptyset$

- Suppose that $S \neq \emptyset$. Then by WOP of $\mathbb{N}$, S has the least element $m \in S$. Since $P(1)$ is true, $m \neq 1$, i.e. $m > 1$.

Since m is a least element $\Rightarrow m-1 \notin S \Rightarrow P(m-1)$ is True.

this implies $P(m)$ is true $\Rightarrow m \notin S$ by assumption.

But then $m \in S$ and $m \notin S$. This is a contradiction.

Thus $S = \emptyset$ and $\therefore$ $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Remark: When we prove something by contradiction, we assume the conclusion we want is false, and then show that we will reach a false statement. Rules of logic thus imply that the initial statement must be false. Thus in this case, we assumed $S \neq \emptyset$ and derived a false statement.

Theorem:

$\forall$ $c \in \mathbb{R}$, $c \neq 1$ and $\forall n \in \mathbb{N}$,

$$1 + c + c^2 + \ldots + c^n = \frac{1 - c^{n+1}}{1 - c}$$

Proof (by induction):

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- $(n=1)$ $1+c = \frac{1-c^{1+1}}{1-c} \Rightarrow 1+c = \frac{1-c^2}{1-c} \Rightarrow 1+c = \frac{\cancel{(1-c)}(1+c)}{\cancel{1-c}}$

$\therefore 1+c = 1+c$ ✓

- Assume that eq. is true for $k \in \mathbb{N}$:

$$1+c+c^2+\dots+c^k = \frac{1-c^{k+1}}{1-c}$$

Thus

$$\begin{aligned} 1+c+c^2+\dots+c^k+c^{k+1} &= (1+c+c^2+\dots+c^k)+c^{k+1} \\ &= \frac{1-c^{k+1}}{1-c}+c^{k+1} \\ &= \frac{1-c^{k+1}+c^{k+1}(1-c)}{1-c} \\ &= \frac{1-c^{(k+1)+1}}{1-c} \quad \square \end{aligned}$$

Theorem

$\forall\, c \geq -1,\ (1+c)^n \geq 1+nc \quad \forall n \in \mathbb{N}.$

Proof (by induction):

$(n=1)$ $1+c \geq 1+1c$ ✓

Suppose that $(1+c)^m \geq 1+mc$

Then,

$$(1+c)^{m+1} = (1+c)^m(1+c)$$

By assumption,

$$\begin{aligned} &\geq (1+mc)(1+c) \\ &= 1+(m+1)c+mc^2 \\ &\geq 1+(m+1)c \end{aligned}$$

$\square$