Splitability in minor-closed classes to graphs of bounded maximum degree

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Abstract

It is easy to see that every planar graph is a minor of another planar graph of maximum degree 3. Georgakopoulos proved that every finite K_5-minor free graph is a minor of another K_5-minor-free graph of maximum degree 22, and inquired if this is smallest possible.
This motivates the following generalization: Let C be a minor-closed class. What is the minimum k such that any graph in C is a minor of a graph in C of maximum degree k? Denote the minimum by $\Delta(C)$ and set it to be ∞ if no such k exists.
We explore the value of $\Delta(C)$ for various minor closed classes, and eventually prove that a minor-closed class C excludes an apex graph if and only if there exists a proper minor-closed superclass C' of C with $\Delta(C') = 3$ if and only if there exists a proper minor-closed superclass C' of C with finite $\Delta(C')$. This complements a list of 5 other characterizations of the minor-closed classes excluding an apex graph by Dujmovic, Morin and Wood.
Furthermore, we extend and simplify Markov and Shi's result that not every graph of treewidth $\leq$ k has a degree 3 expansion of treewidth $\leq k$. Finally, we simplify Georgakopoulos' proof on the existence of a countable universal graph of $Forb(K_5)$.

Contents

1 Introduction

It is easy to see that every planar graph is a minor of another planar graph of maximum degree 3. In [6], Georgakopoulos proved that every K_5-minor free graph is a minor of another K_5-minor-free graph of maximum degree 22, but did not explore if this is smallest possible.
This motivates the following question [6]: Let C be a minor-closed class. What is the minimum k such that any graph in C is a minor of a graph in C of maximum degree k? Denote the minimum by $\Delta(C)$ and set it to be ∞ if no such k exists. This is a very general, yet elegant definition.
In this thesis, we explore properties of this parameter, for example, given a minor-closed class C, change its obstruction set "a little bit" to obtain C'. How does $\Delta(C)$ relate to $\Delta(C')$? What if C' is just a minor-closed superclass of C? We also find its specific value for a number of minor-closed classes, such as the class of graphs of genus $\leq k$, the K_5-free graphs, graphs of treewidth $\leq k$, the apex graphs, e.t.c. In this process, we find relations between our results and the literature. For example, our main result is that a minor-closed class excludes an apex graph as a minor iff has a proper minor-closed superclass with finite Δ iff it has a proper minor-closed superclass of $\Delta = 3$. This complements a list of 5 other characterizations of the minor-closed classes excluding an apex graph by Dujmovic, Morin and Wood [3].

2 Definitions and Preliminaries

Graph theory has the unusual phenomenon that while graphs are technically duplets of sets, we tend to think of them not as sets but visually. Furthermore, when we refer to e.g the clique of size 3 G, we don't discuss if $G = (\{1,2,3\}, \{(1,2), (2,3), (3,1)\})$ or $G = (\{4,5,6\}, \{(4,5), (5,6), (6,4)\})$, really we only care that it belongs to the equivalence class of graphs isomorphic to $(\{1,2,3\}, \{(1,2), (2,3), (3,1)\})$. As a byproduct, well understood definitions are oftentimes hand-wavy and not technically rigorous.
The aim in this section is to introduce, in a rigorous manner from the ground up, notions needed during this thesis or at least to clarify what is left to common sense.

As a byproduct, the introduction section is quite large; the reader may skip it and refer to it as needed.

2.1 Basics

All graphs are simple and undirected. All graphs are finite unless stated otherwise. Though the focus of this thesis is on finite graphs, some results on infinite graphs are also presented. All infinite graphs are countable. The reader may also refer to Diestel [2], the standard reference book.

Definition 1. A *pair* is a set of cardinality 2.

Definition 2. A *graph* is an ordered pair $G = (V, E)$, where V is a finite set and E is a set of pairs of V. We call the elements of V the *vertices* of G and the elements of E the *edges* of G. For each edge $e = \{v, u\} \in E$, we call the vertices v and u *ends* of e and say that the vertices v and u are *connected or adjacent or neighbors* in G. The *order* of G is $|V|$.

In an abuse of notation, we write uv or (u, v) rather than $\{u, v\}$ for edges.

Definition 3. An *infinite graph* is defined in the same manner as a finite graph, the only difference being that V must be infinite. Similarly, a *countable graph* has vertex set V countable.

Definition 4. For subgraph H_1 of graph $G = (V, E)$, we say that H_1 and $v \in V$ are *connected or adjacent or neighbors* in G if there is $u \in H_1$, with u, v adjacent in G. For subgraphs H_1, H_2 of graph $G = (V, E)$, we say that H_1 and H_2 are *connected or adjacent or neighbors* in G if there are $u \in H_1$, $v \in H_2$ with u, v adjacent in G.

Definition 5. Graph $H = (V_H, E_H)$ is a *subgraph* of graph $G = (V_G, E_G)$, denoted $H \subseteq G$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. H is an *induced subgraph* of G if $V_H \subseteq V_G$ and E_H is E_G limited precisely to pairs with both ends in H. The induced subgraph of G with vertex set $S \subseteq V_G$ is denoted $G[S]$.

Definition 6. For subgraphs S_1, S_2 of a graph G, an S_1, S_2 *edge* is an edge with one endpoint on S_1 and one endpoint on S_2. We say that S_1, S_2 *touch or are adjecent or neighbors* if there is an S_1, S_2 edge in G.

Definition 7. Graph $H = (V_H, E_H)$ is *isomorphic to* graph $G = (V_G, E_G)$, denoted $H \cong G$ if there is a 1-1 and onto function $f : V(G) \to V(H)$ such that $(u, v) \in E(G) \iff (f(u), f(v)) \in E(H)$. We may call G a *relabelling* of H.

Definition 8. Let $G = (V, E)$ and let $v \in V$. The *degree of v in G $d_G(v)$* is the number of edges with it as an endpoint, $|\{(v, u) : (v, u) \in E\}|$.

We define a few basic graphs.

The trivial or single vertex graph is the graph of 1 vertex, $(\{v\}, \{\})$. In rigorous terms:

Definition 9. A trivial or single vertex graph is any graph belonging to the graph isomorphism class of $(\{1\}, \{\})$.

A path is a non-empty graph $P = (V, E)$ of the form $V = \{v_0, v_1, ..., v_k\}$ $E = \{(v_0, v_1), (v_1, v_2), ..., (v_{k-1}, v_k)\}$. Rigorously:

Definition 10. A *path graph P of length $n \geq 0$* is any graph belonging to the graph isomorphism class of the graph with vertex set $\{1, 2, ..., n, n+1\}$ and edge set $\{(1, 2), (2, 3), ..., (n, n+1)\}$. A path graph of length 0 is defined to be a single-vertex graph and is called *trivial.* A *path graph* is a graph belonging to the graph isomorphism class of the path graph of length n for some n.

Some additional notation for paths is of use. Let P be path with edge set $(v_1, v_2), ..., (v_{k-1}, v_k)$. We often denote P as $v_1 v_2 ... v_k$ or as $(v_1, v_2), (v_2, v_3), ...$ Other notation follows.

Definition 11. Let P be path $v_1 v_2 ... v_k$. v_1 and v_k are its *endpoints* or *ends*. $int(P) := v_2, ..., v_{k-1}$ are its *internal vertices*. $Pv_i := v_1 v_2 ... v_i$. $v_i P := v_i v_{i+1} ... v_k$. $Pv_i := v_1 v_2 ... v_i$. $v_i P v_j := v_i v_{i+1} ... v_{j-1} v_j$.

Definition 12. A *cycle* is any graph belonging to the graph isomorphism class of the graph with vertex set $\{1, 2, ..., n\}$ and edge set $\{(1, 2), (2, 3), ..., (n - 1, n), (n, 1)\}$ for some n.

Definition 13. A *clique* is any graph belonging to the graph isomorphism class of the graph with vertex set $V = \{1, 2, ..., n\}$ for some n and edge set all pairs of V. The *size of the clique* is n.

Given graph G, rather than say G has a clique subgraph K, we say K is a clique of G. The same goes for the other named graphs.

Definition 14. For non zero natural numbers N, M, the $N \times M$ *grid graph* is the graph with vertex set $1, 2, ..., N \times 1, 2, ..., M$ and edge set $\{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}$. See figure 1.

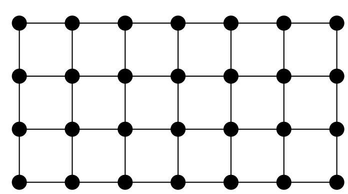

Figure 1: The 4×7 grid graph.

2.2 Graph operations

When defining graphs, it is often easier to do so using graph operators. Just like with number operations, a graph operator is a function $\otimes$ that takes two graphs as input and outputs a graph. Given two graphs G_1, G_2 we usually write $G_1 \otimes G_2$ to denote $\otimes(G_1, G_2)$. A few definitions follow.

Definition 15. Given two graphs $G = (V_G, E_G)$, $H = (V_H, E_H)$, define the *graph union* $G \cup H$ as $(V_G \cup V_H, E_G \cup E_H)$ and the *graph intersection* $G \cap H$ as $(V_G \cap V_H, E_G \cap E_H)$. If $G_V \cap G_H = \emptyset$, then G and H are *disjoint*.

Definition 16. If U is a set of vertices, we define $G - U$ as $G[V_G \setminus U]$. In an abuse of notation, if U is the single-vertex graph v we write $G - v$ rather than $G - \{v\}$ and if G' is a graph, $G - G'$ rather than $G - V(G')$.
If F is a set of pairs of vertices of G, we define $G - F$ to be the graph

$(V(G), E(G) \setminus F)$, and $G + F$ to be $(V(G), E(G) \cup F)$. In an abuse of notation, $G - e := G - \{e\}$ and $G + e := G + \{e\}$. To *join vertex u to vertex v* in G means to add (u, v) to G. To *join subgraph S_1 to subgraph S_2* of G means to join (u, v) in G for all $u \in S_1, v \in S_2$.

Definition 17. Given graphs $G_1,\ G_2$ we define the *disjoint union or graph sum or graph addition of G_1 and G_2*, denoted $G_1 + G_2$, to be $G_1 \cup G_2'$ where G_2' is a graph isomorphic to G_2 so that $G_1 \cap G_2' = \emptyset$.

Notice the similarity to the disjoint union of sets. Indeed, we could have very easily defined the disjoint union of graphs using it.
By "the subgraph S of G_2 in $G_1 + G_2$" it is obvious what we mean, but as the goal of this section is rigor: We changed the labels of G_2 while defining $G_1 + G_2$. Let f be the isomorphism in the above definition, and let $S \subseteq G_2$. By *the subgraph S of G_2 in $G_1 + G_2$* we mean the subgraph induced by $f(V_S)$ in $G_1 + G_2$. The same is said for vertices v of G_2.

Definition 18. Given graph G, *adding a vertex* is defined as the graph sum of G and the single vertex graph.

Definition 19. Given a graph $G = (V, E)$, to *identify* or *glue* vertices u and v of G means to replace all instances of u and v in V and E with a new element $w \notin V$. Remove any loops or parallel edges.

Definition 20. Take graphs $G_1,\ G_2$, and let $S \subseteq G_1$ be isomorphic to $S' \subseteq G_2$, let f be the isomorphism. The *identification of G_1 and G_2 over S and S'* is $G_1 + G_2$, whereby we identify in $G_1 + G_2$ the vertex $v \in G_1$ with $f(v) \in G_2$. See figure 2.2.

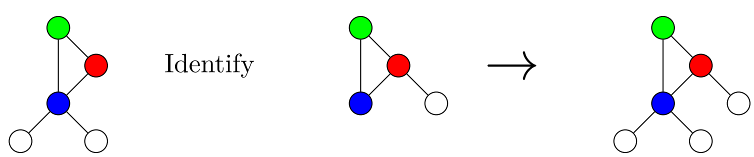

Figure 2: Intuitively, one may picture the identification of two graphs over e.g isomorphic triangles as putting the vertices of one on top of the vertices of the other.

Definition 21. Given graphs $G,\ H$, their *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u, v) and (u', v') are adjacent if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Intuitively, for each vertex of H take a copy of G, and if two vertices in H are connected, connect the corresponding G copies by their identical vertices.

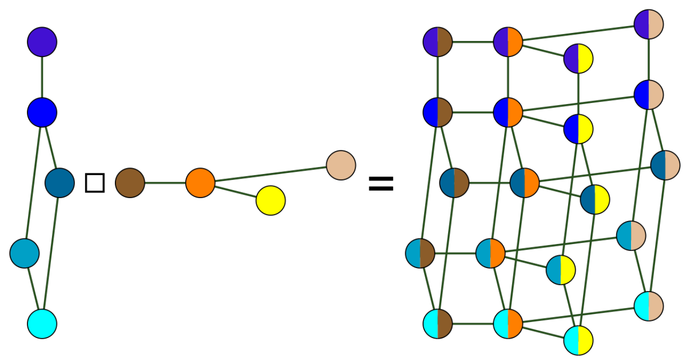

Figure 3: The Cartesian product of two graphs Courtesy: Wikipedia.

Definition 22. For fixed $u \in G$, we denote by by (u, H) the $G \square H$ limited to all vertices of the form (u, v) where v ranges over H. We call (u, H) the H-subgraph of $V(G) \times V(H)$ corresponding to u.

Definition 23. Given graphs G, H such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by taking $G \cup H$ and possibly removing a few edges of the clique. See figures 2.2, 4.

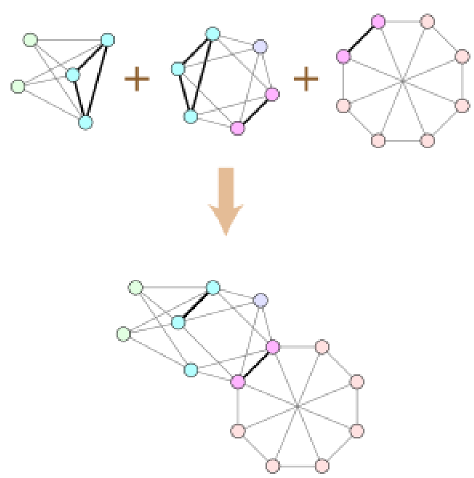

Figure 4: Two clique sums to create a single big graph. Notice how a few clique edges are removed. Courtesy: Wikipedia.

For the operation to be well defined, the edges to be removed must be declared. Still, we often make statements that stand regardless of the specific choice of removed edges. In this case, as happens often in graph theory, we omit mention of the edges to be removed. Similarly, we may omit mention of the cliques the two graphs are clique summed on.

Definition 24. The clique sum of G and H on clique $G \cap H$ of k vertices is called a *k-sum*. The clique sum of G and H on clique $G \cap H$ of $\leq k$ vertices is called a *$\leq k$-sum*.

Notice that 0-sums are well defined, and are the disjoint union. Now, we would like to clique-sum without caring about vertex labels.

Definition 25. Given graphs G, H and isomorphic clique subgraphs $S_G \subseteq G$, $S_H \subseteq H$, their *clique sum $G \oplus H$ over common cliques S_G and S_H* is defined by identifying G and H over S_H and S_G. We may denote this $G \oplus_{S_G, S_H} H$.

Similarly with the disjoint union, suppose some $G' \subseteq G$, we may make mention of G' as a subgraph of $G \oplus_{S_G, S_H} H$, ignoring the relabelling that occurred.

2.3 Treewidth

We now introduce the treewidth of a graph. While it is usually defined as the minimum necessary bag size of a tree-decomposition, I find its definition through clique-sums of smaller graphs, equivalently carefully selected unions of smaller graphs, to provide a better understanding of the notion, and understanding naturally is the primary goal when dealing with theory.

The following says that a graph has treewidth $\leq k$ if it can be built by the clique sum of graphs of order $\leq k+1$.

Definition 26. Let there be a natural number k. Let there be graph H_1 of order $\leq k+1$, and let graph H_2 be a graph of order $\leq k+1$. Let G_2 be $H_1 \oplus H_2$.Let there be a graph H_3 of order $\leq k+1$. Let G_3 be $G_2 \oplus H_3$. Let there be a graph H_4 of order $\leq k+1$. Let G_4 be $G_3 \oplus H_4$. Any graph G_i that can be built by this procedure is said to belong to *the class of graphs $TW_{\leq k}$ of graphs of treewidth $\leq k$*.

Definition 27. If a graphs G belongs to $TW_{\leq k}$ but not $TW_{\leq k-1}$, then it is said to be a graph of *treewidth k*.

The previous definition says that graphs of treewidth k are precisely the graphs which in order to be constructed as described above, it suffices and there need be some graphs H_i of order as large as $k+1$.
The reader may inquire why the $+1$ exists in the definition. It is a historical convention with no substantial meaning.

The classic notion of a tree-decomposition of a graph is directly related to a construction of it by clique-sums and vice-versa. Given a graph constructed by the clique sums of graphs H_i, we can find a tree-decomposition; simply take the vertices of the tree to be t_{H_i}, take the bag of t_{H_i} to be $V(H_i)$, and connect t_{H_i} and t_{H_j} in the tree decomposition if H_i was chosen for H_j to clique sum on. See [11] for a full and more detailed proof.

Definition 28. Let there be graph G constructed by the clique sum of graphs $H_1,\ H_2, ...,\ H_n$ as described in the definition of treewidth. We call $V(H_i)$ the *bags* of G, and denote them as B_{H_i} or $B(H_i)$. If minor bags are involved as well, we call them the *tree-decomposition bags* to avoid confusion.

The following says that a graph has treewidth $\leq k$ if it can be constructed by starting from a graph H_1 of order at most k and iteratively glueing graphs H_i of order at most k on top to build a bigger graph, each time selecting a previously added graph $H_j,\ j < i$ to glue on. While this is my definition of choice, I have funnily enough never seen another human or text mention it. We thus do not use the following alternative definition of treewidth in this text, but I still wished to include it.

Theorem 1. *Let there be a natural number k. Let there be graph H_1 of order $\leq k+1$, and let graph H_2 be a graph of order $\leq k+1$. Let G_2 be $H_1 \cup H_2$. Let there be a graph H_3 of order $\leq k+1$ such that $G_2 \cap H_3 \subseteq H_1$ or $G_2 \cap H_3 \subseteq H_2$. Let G_3 be $G_2 \cup H_3$. Let there be a graph H_4 of order $\leq k+1$ such that $G_3 \cap H_4 \subseteq H_1$ or H_2 or H_3, and so on. A graph G_i belongs to $TW_{\leq k}$ iff it can be built by this procedure.*

To shortly touch on this, indeed, if one can build a graph by the unions of smaller graphs as described above, one can also build it by clique sums of the same smaller graphs, with some extra edges so that the clique sum is well-defined, removed when no longer needed.
The mainstream definition of treewidth is not utilized in this text and is thus not presented.

Definition 29. Let there be graph F with vertex set $v_1, ..., v_n$. Let there be graph H_1. Let G_2 be $H_1 \cup H_2$. Let there be a graph H_3 such that $G_2 \cap H_3 \subseteq \bigcup H_i$ taken over all H_i such that $(v_i, v_3) \in E(F)$. Let G_3 be $G_2 \cup H_3$. Let there be a graph H_4 such that $G_3 \cap H_4 \subseteq \bigcup H_i$ taken over all H_i such $(v_i, v_4) \in E(F)$ and so on, n times. Any graph G_n that can be built in this manner by H_i of order $\leq k+1$ is said to have an $F-$*decomposition of width* k. We call $V(H_i)$ the *bags* of G, and denote them as B_{H_i} or $B(H_i)$. If minor bags are involved as well, we call them the *F-decomposition bags* to avoid confusion.

2.4 Minors, Topological Minors

Subgraphs capture the intuitive notion that a graph is inside another graph. One may however protest that given graphs G, and G', where G' is obtained from G by replacing some edge of G with a path of degree 2 nodes, G is inside G', because the path basically functions as an edge. Taking this idea a step further, given a graph G and G', where G' is obtained from G by replacing some node v of G with a connected graph adjacent to all nodes v was adjacent to, one may say G is inside G' because the connected graph can function as a big node.

It is helpful to define the operations of suppression and contraction before proceeding.

Definition 30. Given a graph G and a (possibly trivial) path $P = v_1 v_2 ... v_k$ of G of $d_G(v_i) = 2$ for all v_i, where l, the neighbor of $v_1 \in G \setminus P$, and r the neighbor of $v_k \in G \setminus P$ are distinct, the operation of *suppressing the path in* G, denoted $suppr_G(P)$ outputs a graph $G' = G - P + (l, r)$.

Given a graph G and a (possibly single-vertex) connected subgraph S of G, the operation of *contracting* S in G, denoted G/S, outputs a graph $G' = G - S +$ a new vertex v_S neighboring all vertices of $G - S$ that S did in G. Given a set of nodes U of G, the contraction of U is defined to be the contraction of $G[U]$.

Definition 31. Let G be a graph, and let S be a subgraph of G. Let S_2 be $suppr_S(P)$ for some path P of G (chosen so that the suppression is well-defined). Let S_2 be $suppr_{S_1}(P')$ for some path P' of S_1 and so on. If a graph G' is isomorphic to some S_i that can be constructed in this manner from G, then G *contains* G' *a topological minor*, denoted $G \geq_{tm} G'$.

Definition 32. Let G be a graph, let S be a subgraph of G and let H be a connected subgraph of S. Let S_2 be S/H. Let H' be a connected subgraph of S_2. Let S_3 be S_2/H'. If a graph G' is isomorphic to some S_i that can be constructed in this manner from G, then G *contains* G' *a minor*, denoted $G \geq_m G'$.

Observing that if a node that arose from a contraction is used in another contraction, we could have just done a single big contraction instead, one may verify that the following are equivalent:

Theorem 2. *The following are equivalent for two graphs* G, G'*:*

(1) $G \geq_m G'$

(2) For some subgraph R *of* G *there are pairwise disjoint subgraphs* $R_1, R_2, ..., R_{|V(G')|}$ *of* R *such that* $(((R/R_1)/R_2)/...)/R_{|V(G')|}$ *is isomorphic to* G'

(3) For some subgraph R *of* G *there are pairwise disjoint subgraphs* $R_1, R_2, ..., R_{|V(G')|}$ *of* R *and there is a bijection* $R_1 \leftrightarrow v_1$, $R_2 \leftrightarrow v_2$, $...$, $R_{|V(G')|} \leftrightarrow v_{|V(G')|}$, *where* $V(G') = \{v_1, ..., v_{|V(G')|}\}$, *such that* $(v_i, v_j) \in E(G')$ *iff* R_i, R_j *are adjacent.*

We work most with the third definition. Some terminology is of use.

Definition 33. A bijection $\mu(v_i) = R_i$ as in (3), is called a *model* of G' in G. We call R_i the *bag* or *branch* of v_i in G and also denote it $B(v_i)$ or G^{v_i}. For $H \subseteq G$, we denote with $\mu(H)$ or $B(H)$ or G^H the subgraph of G induced by the $\cup_{v \in V(H)} B(v)$.

As with edges removed after clique sums, when a statement holds for any choice of μ or μ is clear by context, we omit mention of μ.

Definition 34. Give a graph class C, we call C *closed under minors* or *minor-closed* if $G \in C$ and $G \geq_m G'$ implies $G' \in C$.

Definition 35. Give a graph class C, denote by *minor-closure*(C) its *minor closure*, i.e minor-closure(C)=$\{G : G \leq_m G'$ for some $G' \in C\}$

Definition 36. A graph G *forbids* a graph G' as a minor if $G \ngeq_m G'$.

Definition 37. By $Forb(G)$ we denote the class of graphs not containing G as a minor. It is easy to observe this class is closed under minors.

Definition 38. A minor-closed graph class C *does not contain* a graph G as a minor if $G \notin C$. A graph G is a *forbidden minor of* C or *excluded minor of* C or *in the obstruction set of* C if C forbids G as a minor and G is minimal in this regards, i.e $G' \in C$ for all other $G' \leq_m G$.

The following by Robertson and Seymour is one of the deepest results in all of graph theory. It was proved over a series of 20 papers amounting to 500 pages, over a period of 20 years.

Theorem 3 (The graph minor theorem [16])**.** *Every graph class C closed under minors can be characterized by a finite set of forbidden minors.*

2.5 Planar graphs, Graphs on Surfaces, Elements of topology

As in other subjects in graph theory, and especially in the one that proceeds, one may reason about concepts through visual intuition rather than rigor, and this is often what the community does in practise. Mohar's Topological graph theory [13]) provides for a more rigorous introduction to the topic, though he assumes some topological knowledge. For the topology fundamentals, we recommend Kinsey's topology of surfaces [8]. While this thesis is not focused on topology or bibliography, and thus many topological results are listed without proof, we still try to be as analytical and rigorous as possible.
The reader is probably already familiar with planar graphs. Some of the most deep results in minor theory mention graphs embeddedable on surfaces more complex than the plane or the sphere, such as the torus.

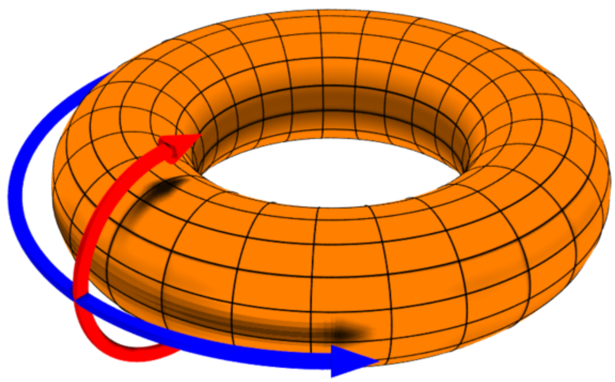

Figure 5: The torus. Courtesy: Wikipedia.

2.5.1 Elements of surfaces

Let (X, τ) be a topological space. Let an *element* of X be any $x \in X$. Some definitions apply more generally, but we only care about metrizable spaces, in fact only about surfaces, which we define shortly.

Definition 39. A *curve* or *arc* on X is the image $f([0,1])$ of a continuous function f from [0,1] to X. A curve is simple if f is 1-1. The curve *connects* $f(0)$ and $f(1)$, which are called the *ends* or *endpoints* of the curve, while $f((0,1))$ is its *interior*. For $a, b \in [0,1]$, a subset of the curve of the form $f([a,b])$ is called a *segment* of the curve, while a subset of the form $f([0,a])$ or $f([a,1])$ is called an *initial segment*. A *simple closed curve* is a curve such that f is 1-1 on (0,1) and $f(0) = f(1)$.

Notice that as the image of a continuous function on a compact set, a curve is compact.

Definition 40. A topological space (X, τ) is *path* or *arcwise* or *curve connected* if for every two points in it, there is a simple curve connecting them. A subset of X is called path-connected if the subspace induced by X under the subspace topology is path-connected. A maximal path-connected subset of X is called a *path-connected component* or *region*[REMOVE THIS?] of X.

A surface is a connected compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$. Intuitively, the reader may visualize them as 3 dimensional shapes, such as donuts, coffee mugs, spheres, chairs, e.t.c.

Definition 41. A topological space (X, τ) is called *Hausdorff* if for all distinct $x, y \in X$, there are disjoint U_x and U_y with $x \in U_x$, $y \in U_y$.

Hausdorff spaces have nice properties metric spaces do. It says we have enough open sets to separate points.

Definition 42. A topological space (X, τ) is called *locally homeomorphic to* (X', τ') if for all distinct $x \in X$, there is $\mathrm{O} \in \tau$ including x and homeomorphic to (X', τ') in the subspace topology.

Many subsets of $\mathbb{R}^2$ are homeomorphic to $\mathbb{R}^2$, such as any open ball of radius 1. Any of them could have been used in this definition.

Definition 43. Given a topological space (X, τ) an *open disc* is a subset of (X, τ) homeomorphic to the open ball of radius 1 of $\mathbb{R}^2$. A *closed disc* is a subset of (X, τ) homeomorphic to the closed ball of radius 1 of $\mathbb{R}^2$.

Surfaces have a few nice natural properties. For example:

Theorem 4. *A surface is a path-connected space. In fact, we could define them to be path-connected instead of connected without loss of generality.*

Theorem 5. *Every surface is a metrizable space.*

The reasoning is that a compact Hausdorff space is metrizable if it is locally metrizable, and surfaces are locally metrizable because they are locally homeomorphic to $\mathbb{R}^2$.

2.5.2 Graphs on Surfaces

A graph is *embeddable* on a surface if we can draw it on the surface so that edges do not intersect.

Definition 44. A graph G is *embeddable* on (X, τ) if there is a function f mapping vertices to elements of X, and edges to simple curves on X so that $f(v_1) \neq f(v_2)$ for $v_1 \neq v_2$, and curve $f(uv)$ connects $f(u)$ and $f(v)$, and has no intersection with the image of other vertices and only intersects other edges on its endpoints.
f is an *embeddeding* of G on X. The image of f, $f[(V(G) \cup E(G))]$, is called the *embedded graph*, and though it is technically not a graph, one may produce a graph from one in the obvious manner. For ease of notation, the embedded graph is also abusively denoted $f(G)$.

As the finite union of compact sets, any embedded graph is compact and therefore closed.

Definition 45. A *face* of an embedded graph G on (X, τ) is a region of $X \setminus G$ (equipped with the subspace topology of course).

Given a face of an embedded graph G, the boundary of the face is an embedded subgraph of G [prove it? nah]. If this subgraph is a cycle, it call it a facial cycle.

Definition 46. Let there be embeddable graph G, let f be an embedding, and let the boundary b of a face of $f(G)$ be a cycle, i.e let G limited to the vertices and edges of $f^{-1}(b)$ be a cycle. We call the boundary of b a *facial cycle*.

Definition 47. A graph embeddable on the plane $\mathbb{R}^2$ (with the standard topology always) is called *planar*. The embedded graph is called the *plane graph*.

Planar graphs are often introduced with arcs being polygonal. However, the two definitions are equivalent (see Mohar's Topological graph theory chapter 2.1 [13]).

Definition 48. A curve is *polygonal* if it is the union of a finite number of straight line segments. A *straight line segment* is a curve that is a subset of a line of $\mathbb{R}^2$.

Theorem 6. *A graph is embeddable on the plane if and only if it is embeddable on the plane with edges mapped to polygonal curves.*

For proofs on planar graphs, topological tools on $\mathbb{R}^2$ are useful. The Jordan Curve theorem is an intuitively obvious but infamously difficult to prove theorem. Naturally, we make use of it.

Theorem 7 (The Jordan Curve Theorem)**.** *Let C be a simple closed curve on $\mathbb{R}^2$. $\mathbb{R}^2 \setminus C$ has exactly two connected components, one being bounded and the other unbounded, with C as the boundary of both.*

The bounded component is called the *interior*, while the unbounded is called the *exterior*. The following extension exists.

Theorem 8 (The Jordan-Schoenflies Curve Theorem)**.** *For any two simple closed curves C_1, C_2, their interiors are homeomorphic and their exteriors are homeomorphic.*

A graph is embeddable on the plane if and only if it is embeddable on the sphere. The following theorem provides for a well-defined topology on the sphere that is useful for embeddings.

Theorem 9. *The unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \sqrt{x_1^2 + ... + x_n^2} = 1\}$ is a complete metric space when equipped with the metric, $d(x, y) := \arccos(x \cdot y)$ where $\cdot$ denotes the standard dot product.*

We need only consider the sphere S^2 on $\mathbb{R}^3$. The next theorem following from the definitions of homeomorphity and embeddability.

Theorem 10. *Let there be two homeomorphic surfaces Σ_1, Σ_2. Then a graph is embeddable on Σ_1 if and only if it is embeddable on Σ_2.*

Theorem 11. *The sphere minus an element is homeomorphic to the plane.*

Clearly any embedded graph on the sphere is not equal to the sphere. Thus

Corollary 1. *A graph can be embedded on the plane if and only if it can be embedded on the sphere.*

As mentioned, we wish to embed graphs on other surfaces as well. While intuitively we can visualize what a torus or a double-torus is, and therefore work with graphs embedded on it, it would be nice to also define those surfaces, starting from topology.

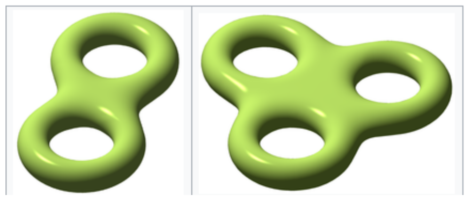

Figure 6: Surfaces of genus 2 and 3 respectively. The double and triple torus. Courtesy: Wikipedia.

2.5.3 Genus of surfaces and graphs, the classification theorem, handles and crosscaps, topological operations

Definition 49. A topological space (X, τ) is called *locally Euclidean of dimension n* if for every $x \in X$, x has an open neighborhood $U \in \tau$ homeomorphic to $\mathbb{R}^n$ (that is, the subspace topology of (X, τ) limited to U, (U, τ_U) has a homeomorphism $h : U \to \mathbb{R}^n$).

Intuitively, it is easy to define the torus; simply take the square [0,1]×[0,1], "glue together" the top side with the bottom side to obtain a hollow cylinder, then glue together the two opposing ends of the cylinder. One may do this with a piece of paper.

We want to formally define the intuitive notion of gluing topological sets together. This is done through the quotient topology.

Definition 50. Let $= (X, \tau)$ be a topological space. Let there be function $f : X \to Y$. The *biggest* or *finest continuous topology* induced by X and f on Y is (Y, τ') where $O' \in \tau'$ iff $f^{-1}(\mathrm{O}') \in \tau$.

Definition 51. Let $= (X, \tau)$ be a topological space. Let $\sim$ be an equivalence relation on X. The *quotient or identification set* $X/\!\sim$ is $\{[x] | x \in X\}$ where $[x]$ is the equivalence set of x under $\sim$. The function $f(x) = [x]$ is called the *identification or quotient* mapping.

The reader may notice that this space has sets as elements. This is of no importance; we could very well replace them with their representing element, and to avoid notational overencumbering we do.

One may visualize the identification set as X with equivalent points glued or contracted on each other. We now add a topology on the quotient set, because to work with notions such as continuity we need to have an underlying topological space. In the following we still work with general topology, but all spaces we work with will be metrizable, and I have found that thinking with metric distance functions often provides better understanding, so let me briefly mention the quotient metric as a side note. What should the metric d' of $X/\sim$ after gluing together some points of (X, d) be? Let x be a point in X, not glued to other points. Clearly its distance from $y \in X$ remains same if all other points of X of distances $\leq d(x, y)$ from x are also not glued. If however a glued point z exists in this ball, we must consider if using it allows us to reach y in a shorter fashion. Thus $d'(x, y)$ is something like $inf_{w \in [z]}(d(x, w) + d(w, y))$, in fact we should also consider other equivalence classes that one may utilize, possibly in succession. This only defines a pseudometric, as it may yield distinct elements of distance 0 (try [-1,1] with the Euclidean metric and [-1,0) contracted to the same equivalence set and (0,1] contracted). For specific metrizable topological sets and well chosen equivalence partitions, this does yield a metric, which induces the quotient topology.

Definition 52. Let $= (X, \tau)$ be a topological space. Let $\sim$ be an equivalence relation on X. $X/\sim$ equipped with the biggest topology making the identification mapping continuous is called the *quotient or identification topology* of X on $\sim$.

Definition 53. Let (X, τ) be a topological space. To *glue* x and $x' \in X$ means to take the quotient space on X defined by the equivalence relationship $x \sim x'$.

We can now properly define the topological space of the torus.

Definition 54. Let there be the metric space $[0, 1] \times [0, 1]$, equipped with the euclidean metric and take the topological space induced by the metric. For all $t \in [0, 1]$, glue $[0, t]$ with $[1, t]$. The resulting topological space is called a *cylinder*. The cylinder has two *opposing ends*, the sets $\{[t, 0] | t \in [0, 1]\}$ and $\{[t, 1] | t \in [0, 1]\}$.

Let there be the metric space $[0, 1] \times [0, 1]$, equipped with the euclidean metric and take the topological space induced by the metric. For all $t \in [0, 1]$, glue $[0, t]$ with $[1, t]$, and then for all $t \in [0, 1]$ glue $[t, 0]$ with $[t, 1]$ (the opposing ends). The resulting donut-shaped topological space is called the *torus*.

We now present a fundamental theorem in the topology of surfaces, the classification theorem, which says that any surface can be constructed by the sphere and a few simple operations. Some definitions are needed.

Definition 55. To *remove* a subset S of a topological space (X, τ) means to take the subspace topology induced by $X \setminus S$.

Much like with graphs, the disjoint union of sets expresses the idea of putting both sets separately together.

Definition 56. The *disjoint union* of two not necessarily disjoint sets A, B is the set $\{(x,1)|x \in A\} \cup \{(x,2)|x \in B\}$.

Definition 57. The *disjoint union topology* of two topological spaces A, B with bases $U_a,\ U_b$ is the disjoint union of A and B equipped with the base defined by the disjoint union of U_a and U_b.

It is interesting to notice that the following is equivalent: Let f be the natural map from $A \cup B$ to the disjoint union of A, B. We can define the disjoint union topology as the disjoint union of A, B equipped with the biggest topology making f continuous.
This was the case for the quotient topology as well. Thus it starts to become clear that the finest/biggest topology making f continuous is the one that conserves best the initial topological space in the image space.

Definition 58. Let there be a surface S. Let there be two subsets $C_1,\ C_2$ of S homeomorphic to an open ball of $\mathbb{R}^2$, and let the closure of C_1 and C_2 be disjoint. Remove C_1 and C_2 from S, take the disjoint union of the resulting topological space with a cylinder, and glue one end of the cylinder to the boundary of C_1 in the natural manner and the other end to the boundary of C_2. We then say we *added a handle* to S.

Definition 59. Let there be a surface S. Let there be a subset C of S homeomorphic to an open ball of $\mathbb{R}^2$. Remove C from S, and if $x,\ x' \in S \setminus C$ are on the boundary of C and diametrically opposite (on the circle homeomorphic to C of course), glue them. We then say we *added a crosscap* to S.

Adding a crosscap is homeomorphic to adding a mobius strip.

Theorem 12 (The classification theorem)**.** *Let S be a surface. S is homeomorphic to one of the following:*

1. *The sphere after adding $k \in \mathbb{Z}_{\geq 0}$ handles.*
2. *The sphere after adding $k \in \mathbb{Z}_{\geq 0}$ crosscaps.*

Definition 60. The *genus* of a connected orientable surface is the maximum amount of pair-wise disjoint simple closed curves that can be removed without rendering it disconnected. The *non-orientable genus* of a connected non-orientable surface is the maximum amount of pair-wise disjoint simple closed curves that can be removed without rendering it disconnected. [1]

Theorem 13. *The genus of an orientable surface is equal to the number of handles we need to add to construct it starting with a sphere. The non-orientable genus of a non-orientable surface is equal to the number of cross-caps we need to add to construct it starting with a sphere.*

[1] So if we add 10 handles to the sphere and then 1 cross-cap, this is a non-orientable surface. Can we really build the same surface by just adding cross-caps? Yes! We need 2 crosscaps for each handle

Thus, up to homeomorphism there is only one surface of orientable or non-orientable genus g, the surface of obtained from the sphere after adding g handles or g crosscaps.
Euler's theorem says that for an embedded graph in the plane, $n - m + f = 2$ where n is the number of vertices, m the edges, and f the distinct faces. This results extends to higher (non-orientable) genus surfaces.

Definition 61. Let S be a surface. Then for some possibly negative integer χ, called the *euler characteristic* of S, and for any embedded graph G on Σ such that every face is homeomorphic to an open ball in $\mathbb{R}^2$, $n - m + f = \chi$.

Theorem 14. *Let G be a graph embedded on Σ and not embeddable on a surface of lower genus. Then every face is homeomorphic to an open ball in $\mathbb{R}^2$*

Definition 62. The *genus of a graph* G is the smallest integer n such that G can be embedded on the surface of genus n. The *non-orientable genus of an graph* G is the smallest integer n such that G can be embedded on the non-orientable surface of genus n.

Definition 63. The *euler genus* of a surface with euler characteristic χ is $2-\chi$.

Theorem 15. *Let Σ be a surface built from the sphere after adding k handles. Then its euler genus is $2k$.*
Let Σ be a surface built from the sphere after adding k crosscaps. Then its euler genus is k.

In other words, the Euler genus of a non-orientable surface is its non-orientable genus, and the Euler genus of an orientable surface is double its genus. With this in mind, working with the euler genus instead of the regular genus and non-orientable genus is somewhat of an overcomplication for our purposes. In any case, The graph theory community seems to like not to concern itself with whether a surface is orientable or non-orientable and abolishing the established conventions is not a priority of this text.

Definition 64. The *euler genus* of a graph is the smallest integer n such that G can be embedded on the surface of euler genus n.

Euler's theorem implies that for any planar graph G of n vertices and m edges, $m \leq 3n-6$. This also generalizes to graphs embeddable on higher genus surfaces:

Theorem 16. *Let G be embeddable on Σ. Then $m \leq 3n - 6 + 3 eul\_genus(\Sigma)$.*

3 The graph class parameter Δ

One may easily observe that every planar graph is a minor of another planar graph of maximum degree 3. In [6], Georgakopoulos observed that every K_5-minor free graph is a minor of another K_5-minor-free graph of maximum degree 22, but did not find if this is smallest possible.
Given a minor-closed class C, define as $\Delta(C)$ the minimum k such that every graph in C is a minor of another graph in C of maximum degree $\leq k$. If there is no such k, define $\Delta(C)$ to be infinite.
Independently, Joret and Wood examined which minor-closed classes C have a minor closed superclass C', so that every graph G in C is a minor of a graph G' in C' of maximum degree 3. They called such a graph G' a degree-3 splitting of G [2]. As we will see, a minor-closed class has such a minor-closed superclass if and only if it excludes an apex graph as a minor.

The function Δ does not seem to have any clear general pattern at first glance. An easy observation to make is that if for some minor-closed class C we have $\Delta(C) \leq 2$, equivalently every graph in C is a minor of a graph in C of maximum degree ≤ 2, then C consists of the disjoint union of circles and paths, as any G' of $\Delta(G') \leq 2$ is isomorphic to the disjoint union of some paths and circles. We don't bother ourselves with such trivial classes; all results considered from now on shall be for C of $\Delta(C) \geq 3$, even if not explicitly stated.
For a more interesting result, one may conjecture that Δ is increasing with regard to the subset relationship, i.e $C \subseteq C' \implies \Delta(C) \leq \Delta(C')$. This is not the case; The class of stars [3] $\{K_{1,k} | k \in \mathbb{Z}_{\geq 0}\}$ has $\Delta = \infty$, because the only way to include a star as a minor is to use a bigger star. The planar graphs are a superset of the class of stars, yet they have $\Delta = +3$. The apex graphs in turn include the planar graphs, but as we will see they have $\Delta = \infty$.
We proceed to examine the properties of Δ and the precise value for a few minor-closed graph classes.

[2] They also added the inconsequential for our purposes restriction that there may not be contractions of triangle edges and of edges adjacent to degree 2 vertices

[3] Technically, this is not a minor-closed class. No matter; take the minor-closure of stars instead, which is almost same.

4 The Δ value of various minor-closed classes

In this section, we find the Δ value of a few minor-closed classes, such as K5-minor-free graphs.

4.1 Planar graphs, Graphs of Euler genus $\leq k$, Outerplanar graphs, Linklessly embeddable graphs

It is easy to conclude that every planar graph has a planar graph of maximum degree 3 by visual intuition alone. The following figure illustrates that.

Figure 7: By replacing each vertex of a plane graph with a circle on the boundary of an open ball around the vertex, we may create a plane graph of maximum degree 3 containing the first as a minor.

Let's write the actual proof! We remind that a planar graph has a function f mapping its vertices to points and its edges to curves on the plane. Note that an embedded graph is a compact subset of $\mathbb{R}^2$, being the finite union of compact sets, curves being compact as the continuous image of the compact set [0,1]. We remind that the initial segment of a curve $c([0,1])$ is a subset of the curve of the form $c([0,a])$ or $c([a,1])$. The following lemma says that with the right embedding, for each vertex one may find a closed ball centered on the vertex, only including the vertex and initial segments of the edges incident to the vertex (that is, edges only exit the ball once).

Lemma 1. *Let G be a planar graph. G has an embedding f with the following properties: For every embedded vertex $f(v)$, there is a closed ball centered on $f(v)$ such that*

- *The closed ball includes no other embedded vertices.*
- *The closed ball intersects only embedded edges incident to v.*
- *The closed ball intersects only an initial segment of those edges.*

Proof. Let f be any planar embedding of G. For a ball of $f(v)$ without other vertices inside, simply pick a ball with radius smaller than the minimum distance between $f(v)$ and other embedded vertices, $\min_v d(f(u), f(v))$, where $d()$ is the euclidean distance.
Moving to edges not incident to v, suppose towards contradiction that every closed ball around v intersects such an edge. Let E be the set of edges incident to v. We can thus pick a sequence a_n of $f(G \setminus E \setminus v)$ such that as n increases, the

distance from $f(v)$ decreases and tends to 0, e.g $a_n =$ some element of distance $\leq 1/n$. By definition, this sequence converges to $f(v)$. Furthermore, $f(G \setminus E \setminus v)$ is compact in $\mathbb{R}^2$ and thus closed, therefore $f(v) \in f(G \setminus E \setminus v)$, a contradiction to the definition of embeddings.
Moving to edges incident to v, pick some ε such that $B_\varepsilon(v)$ intersects from $f(G)$ only $f(v)$ and those edges. Simply erase the inside of the ball (except v of course) and reconnect v with its edges by a straight line segment going from $f(v)$ to where the embedded edge last exits $B_\varepsilon(v)$, erasing it before that point (to explain where to connect it in rigorous terms, let $e([0,1])$ where $e : [0,1] \to \mathbb{R}^2$ be such an embedded edge, with $e(0)$ being v. Let x be $\sup_y[e(y) \in B_\varepsilon(v)]$. Connect v to $e(x)$). It is simple geometry this remains an embedding satisfying the lemma. $\square$

For every embedding, we thus found an embedding very similar to it with all these nice properties. The reader may inquire whether these properties hold without changing the original embedding, in other words, if they are true for all embeddings. The answer is actually negative! There are graphs such that the final property does not hold.

For example: Let there be function

$$q(x) = \begin{cases} x\sin(1/x), & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0 \end{cases}$$

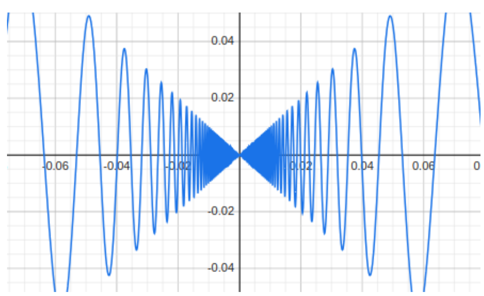

Figure 8: Function $x\sin(1/x)$. Our intuition can be false in topology, even on $\mathbb{R}^2$.

.

Notice that q is a continuous function on $[0,1]$, i.e a curve. Let there be some planar graph G with some embedding such that $q(0)$ and $q(1)$ are embedded vertices u_1 and u_2 of G, and $q([0,1])$ is an embedded edge. For some $r_0 > 0$, all circles of radius less that r_0 intersect the edge at least twice. (Indeed, its distance from the origin is $x\sqrt{1+\sin^2(1/x)}$. The reader may verify the rest by setting values of the form $1/k\pi$ for very large k.) Now, let there be an embedded vertex v of distance less than r_0 to u_1. There is no ball of u_1 satisfying both properties 1 and 3 of the lemma for this embedding.

Theorem 17. *Let PLANAR be the class of planar graphs. Δ(PLANAR)=3.*

Proof. Let there be planar graph G. Take the embedding of lemma 1, and take the balls small enough that they do not intersect and let v be a vertex of degree ≥ 3. Erase everything inside the closed ball of v, then let $p_1, ..., p_k$ be the points where the boundary of the closed ball last intersected the edges of v $e_1, ..., e_k$, the p_i ordered in a counterclockwise manner starting from some point of the boundary of the ball. Add the p_i back as embedded vertices v_i. Then, connect p_i with p_{i+1} by a curve running along the perimeter of the boundary and also connect p_k with p_1 in the same manner (of course these are well defined curves. Take the polar coordinate formula, mapping the angle to points on the circle.). Notice that all such vertices are of degree at most 3, and that their contraction yields the original graph. Doing this for every vertex of degree ≥ 3, we create an embedded graph of maximum degree 3 including G as a minor. $\square$

Much the same holds for graphs embeddable on a surface of euler genus k, equivalently graphs of euler genus $\leq k$. The fact that every graph of euler genus k is included as minor in a graph of euler genus k and maximum degree 3 is visualized in much the same manner and the proof is almost identical. We simply have to work with the open discs provided by the definition of a surface instead of open balls. We present them without proofs.
Note than for a point x of a surface, and any ball of x, there exists an open disc inside the ball. To see this, let D be an open disc of x homeomorphic to the open ball of $\mathbb{R}^2$ by homeomorphism f, take an open ball O of x, map it by f to $\mathbb{R}^2$. $f[O]$ is an open set (by homeomorphism) and thus it has inside an open ball centered on x. Map this open ball back to the surface by f^{-1}. Thus, for any ball $B_\varepsilon(x) \subseteq D$, we have found a subset D' of $B_\varepsilon(x)$, mapped by f to an open ball of $\mathbb{R}^2$. Limiting f to D', it is easy to see that we still have a homeomorphism.

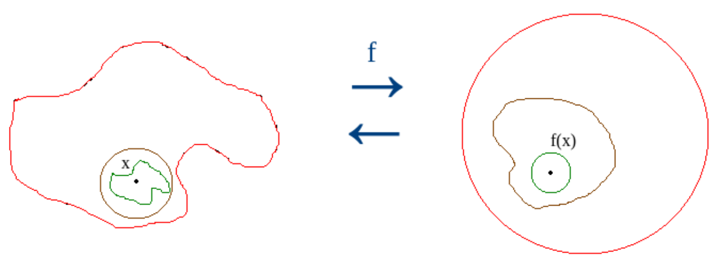

Figure 9: Reasoning about open discs through their homeomorphism to the open ball.

Lemma 2. *Let G be a graph with embedding f on some surface. For every embedded vertex $f(v)$, there is an open ball centered on $f(v)$ and an open disc inside the ball including no other embedded vertices, and only embedded edges incident to v. Furthermore, let $g : [0,1] \to \mathbb{R}^2$ be one such embedded edge.*

If $g(0) = f(v)$ the open disc only contains a subset of the form $g([0, \varepsilon])$. If $g(1) = f(v)$ the open disc only contains a subset of the form $g([1 - \varepsilon, 1])$.

Theorem 18. *Let $EUL\_GENUS_{\leq k}$ be the class of graphs of euler genus $\leq k$. $\Delta(EUL\_GENUS_{\leq k})$=3.*

Definition 65. Given graph G, we call the graph $G' \geq_m G$ of maximum degree 3 as in the proof that $\Delta(PLANAR) = 3$ the *fattening* or *ballooning* of G, and denote it $Bl(G)$. The circle we replace vertex $v \in G$ with we denote by $Bl(v)$. This is also the model function showing $G' \geq_m G$.

The outerplanar graphs are closely related to planar graphs. One expects that the same methods apply, and indeed this is the case. Let OUTERPLANAR be the class of outerplanar graphs.

Theorem 19. $\Delta(OUTERPLANAR) = 3$

The proof is summed up in the following figure.

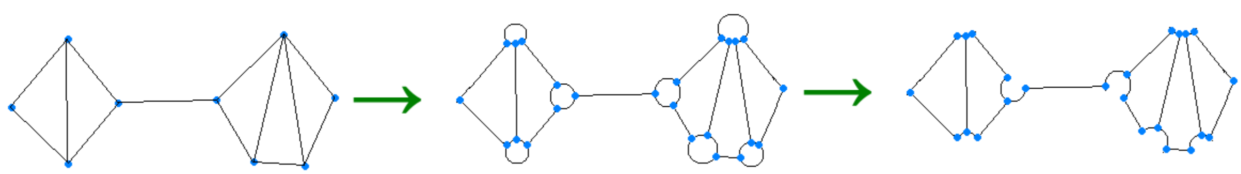

Figure 10: A picture is a thousand words. It unfortunately is not also a proof.

Proof. Let there be an outerplanar graph G. There is a common face f of $\mathbb{R}^2 \setminus G$ on which all vertices lie. So for a small enough ε a closed ball $B_v(\varepsilon)$ around a vertex v intersects with f. More specifically, its boundary intersects f. To prove this, observe that for ε small enough, there is a point $p \in f$ with $d(v, p) > \varepsilon$, and a simple curve $c : [0, 1] \to \mathbb{R}^2$ connecting v and p and having interior in f. The function d_v mapping a point of $\mathbb{R}^2$ to the distance from point v is continuous, therefore $d_v \circ c$ is continuous, and by the mean value theorem for all $\varepsilon' \in (0, \varepsilon)$ there is a point on the interior of the curve with distance ε' from v. Let $p_{\varepsilon'}$ be such a point. Even more specifically, since f is open, we may take an open ball of f around $p_{\varepsilon'}$, and by geometry notice that its entire intersection with the boundary of $B_v(\varepsilon)$ is in f.
We create from G a graph $G' := Bl(G)$ as in the proof of $\Delta(PLANAR) = 3$. Clearly $G' \geq_m G$ by contracting $Bl(v)$ for each v. Notice that this still holds if we remove any 1 edge from each $Bl(v)$.

Since the edges of $Bl(v)$ cover the circle $Bl(v)$ was embedded on, at least one such edge e must intersect the boundary of f. We remove it. Both the ball bounding circle $Bl(v)$ and f are faces, i.e maximal connected sets of $\mathbb{R}^2 \setminus G$, with an intersecting boundary, so $G' \setminus e$ now has a face= the interior of $e \cup$ f $\cup$ the ball bounding $Bl(v)$. This face intersects all vertices of $Bl(v)$. Doing this for all $Bl(v)$, we acquire an outerplanar graph of maximum degree 3 containing G as a minor. □

4.1.1 Linklessly Embeddable graphs

With all the above positive results in mind, one may thus conjecture that the linklessly embeddable graphs, a well-known three dimensional analogue of the planar graphs consisting of all graphs that have a linkless or flat embedding on 3D-space, also has a low Δ. This is not the case. As we will see, the linklessly embeddable graphs have $\Delta = \infty$.

The facts proved in this section, while not at all trivial in a topological sense, were for the most part visually obvious. We try to find the Δ value of various minor-closed classes, and in doing so, we move on to less obvious results.

4.2 K_5-minor-free and forb($K_{3,3}$) graphs

4.2.1 K_5-minor-free graphs

In [6], Georgakopoulos proved the existence of a countably infinite K_5-minor-free universal graph. As a corollary of his results, he obtained that every finite K_5-minor-free graph is a minor of another finite K_5-minor-free graph of maximum degree ≤ 22. A natural question to ask is if this number can be lowered. Let forb(K_5) be the class of K_5-minor-free graphs. We prove that $\Delta(\text{forb}(K_5))$=3. The following theorem by Wagner is essential.

Theorem 20 (Wagner [17])**.** *A graph G excludes K_5 as a minor if and only if it can be constructed by the ≤ 3-clique-sums of planar graphs and the Wagner graph $W[8]$.*

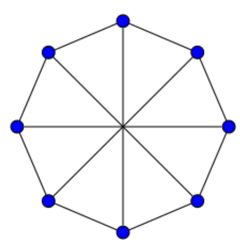

Figure 11: The Wagner graph $W[8]$, also known as the 8-wheel. Courtesy: Wikipedia.

We do not use the following observation, but it is nice to notice that 4-clique-sums do not add any extra graph creating power (Indeed, take Whitney's theorem that up to isomorphism, K_4 can be embedded in only one "manner" in the

plane. Then notice that anything we add by 4-sums we could have added by at most 4 3-sums, one for each face of the K_4). Thus a nice way to reformulate this theorem is that K_5-minor-free graphs are precisely the clique-sum closure of planar graphs and $W[8]$.
When I read a proof, I usually end up reading it a few times over while asking myself what the main mechanisms are that make the proven theorem provable. Would it not be nice if mathematicians separated them as lemmas? The following two lemmas are the main mechanisms used in the proof that $\Delta(\text{forb}(K_5))$=3.

Lemma 3. *Let C be a graph class closed under n-clique-sums such that the graph product $K_n \square P_2$ is in C. Then $K_n \square T$ is in C for any tree T of more than 1 vertex.*

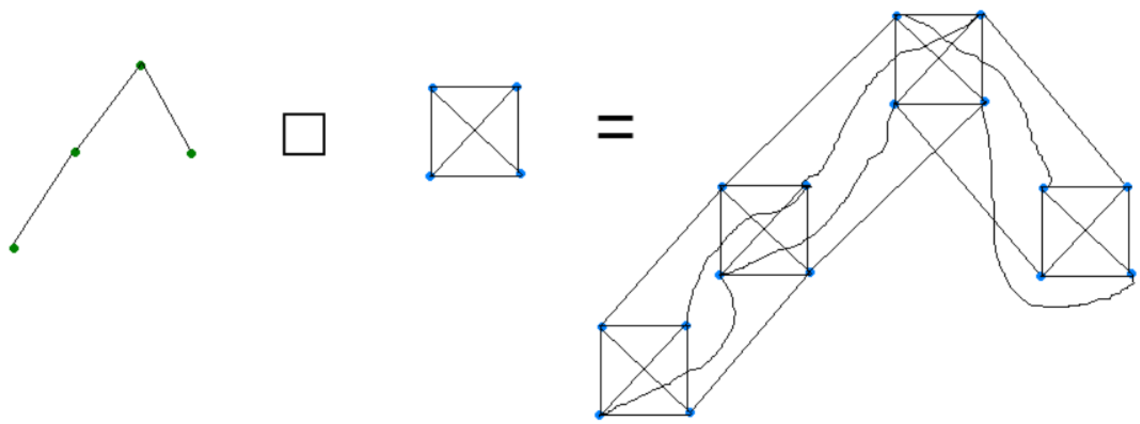

Figure 12: The cartesian product of a tree and a 4-clique, visualized.

The proof is conceptually very simple; imagine $K_n \square T$ as a tree where instead of vertices we have cliques. Much like we can create any tree by adding each of its edges one by one starting from the root in a DFS or BFS manner, we can create $K_n \square T$ by adding each of its n-cliques in the same order.

Proof. Let there be graph $K_n \square T$ some tree T. We have that $V(K_n \square T)$=$(V(T) \times \{1, ..., n\})$ and $((t_1, v_1), (t_2, v_2)) \in E(K_n \square T) \iff t_1 = t_2$ or (t_1 neighbors t_2 in T and $v_1 = v_2$).
The result is by induction of the number of vertices of T. If T is the edge graph, then the result holds trivially. Now let $K_n \square T$ for all T of some fixed number of vertices n. Let there be T' of $n+1$ vertices. This is constructed by some T of n vertices after adding a vertex t_2 to T and joining it to the correct vertex t_1. We have $K_n \square T \in C$. Clique sum either of the cliques of $K_n \square P_2$ to the clique of $K_n \square T$ corresponding to t_1, i.e to the subgraph of $K_n \square T$ induced by $\{(t_1, i) | i \in \{1, ..., n\}\}$. The resulting graph is (isomorphic to) $K_n \square T'$: Relabel

the new n vertices as $(t_2, 1), ..., (t_2, n)$ and notice that (t_2, i) neighbors (t, j) iff $(t_2 = t)$ or t_2 neighbors t in T' and $i = j$). $\square$

We remind $G_1 \oplus_{K_1,K_2} G_2$ is the clique sum of G_1 and G_2 over isomoprhic cliques $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$.

Lemma 4. *Let P_1, P_2 be some graphs. Let $P = P_1 \oplus_{K_1,K_2} P_2$. Let there be graph $P_1' \geq_m P_1$, let μ_1 be the model, such that $\mu(K_1)$ has a clique K_1' with one node in each branch and let there be similar graph P_2'. Then $P_1' \oplus_{K_1',K_2'} P_2' \geq_m P$.*

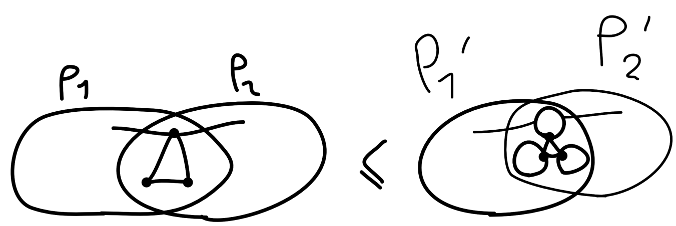

Figure 13: Example for size 3 cliques of graphs P_1 and P_2. To the right the triangle K_1' is depicted, one vertex in each branch of K.

Proof. Let μ_1, μ_2 be the model functions mapping connected components of P_i' to P_i. We define the branches of $P' := P_1' \oplus_{K_1',K_2'} P_2'$, i.e the model function μ from connected components of P' to vertices in P. Let vertex v of $P \notin$ the common clique, let it only $\in P_i$. Then $\mu(v) := \mu_i(v)$. Let $v \in$ the common clique. Then $\mu(v) := \mu_1(v) \cup \mu_2(v)$.

If $v \in P$, $v \notin$ the common clique, let it only $\in P_i$, then $(u, v) \in G \implies (u, v) \in P_i \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch.

If $v \in$ the common clique K_1 of P', then $(u, v) \in P \implies (u, v) \in$ one of the P_i containing $K_1 \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch. $\square$

We now move on to the proof that $\Delta(\text{forb}(K_5))$=3. Our previous result for planar graphs is of use. It suffices to consider clique sums that do not remove edges. Furthermore, we divert our attention mostly to the case of 3-sums. The reader may fill in the rest easily.

Before diving in, let us explain the proof conceptually. We decompose the K_5 minor free graph, to the clique sum of planar graphs, and we replace each planar graph with a bigger planar graph of maximum degree 3 containing it as a minor. We add a few extra triangles so that clique sums between big planar graphs are still possible. The triangles are placed so that the clique sum of the big planar graphs contains the clique sum of the original planars as a minor. By adding enough such triangles, we never need reuse a triangle, keeping the maximum degree low. My approach bloats the graphs quite a bit; it is not my intention to present the most economical approach in vertex or edge number.

Theorem 21. *Δ(forb(K_5))=3.*

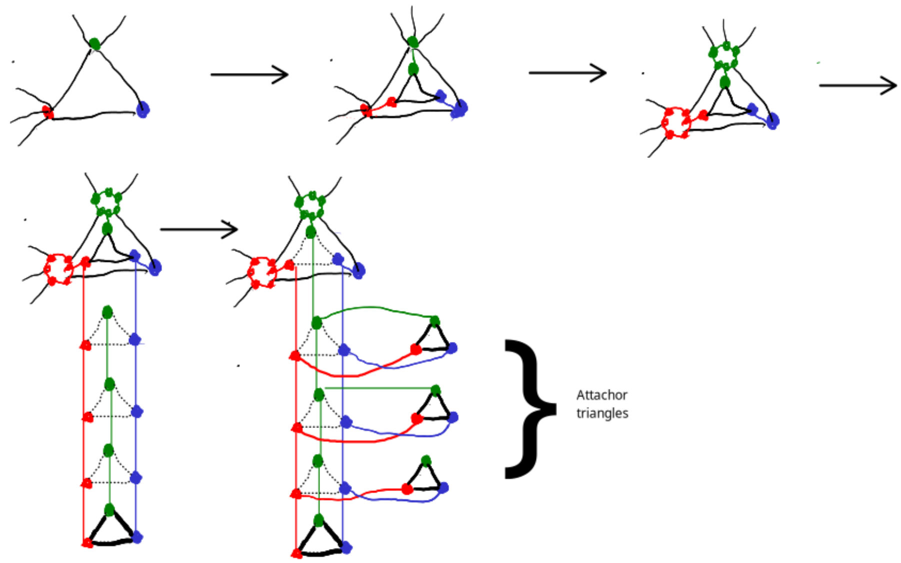

Figure 14: A triangle of G_i modified step by step. G_i (1st graph), G_i', G_i'', G_i''' (3rd, 4th and 5th graphs) are pictured. By contracting along same-colored segments, we regain the original graph. By clique summing on attachor triangles, we keep the maximum degree low. Delete dotted edges after you're done.

Let G be a K_5-minor-free graph. We construct the K_5-minor-free graph of maximum degree 3 containing G step by step, because it makes the construction easier to understand and better motivated.

Let there be K_5-minor-free graph G. Let $G_1, ..., G_k$ be its $\leq$ 3-clique-sum decomposition into planar graphs and Wagner graphs, clique summed in this order. We can assume all embedded triangles abc of (planar graphs) G_i have either an empty interior or an empty exterior; for let this not be the case, then by the definitions of planarity and the Jordan curve theorem, the triangle is a separator, and thus it can be further decomposed into the 3-clique-sum of smaller planar graphs. By the Jordan-Schoenflies Curve Theorem, this region is homeomorphic either to the interior or the exterior of a circle C of radius 1 on $\mathbb{R}^2$. One may then add a new triangle $a'b'c'$ to G, a joined to a', b joined to b', c joined to c', and embed it in the empty face. See image 4.2.1. [4]

[4]Visually, adding the triangle of course looks obvious, but for illustration purposes and since it's nice not to have gaps in our understanding, let's explain it. Let H be the homeomorphism function, and w.l.g. let the empty face be homeomorphic to the interior of C. One may embed the triangle by e.g taking a circle of half radius to C and same centre, noting the point p_a where the line segment from $H(a)$ to the centre of C intersects the smaller circle, let points p_b and p_c be defined in the same manner, and letting the embedded triangle be the embedded vertices $H^{-1}(p_a)$, $H^{-1}(p_b)$, $H^{-1}(p_c)$, and the embedded edges of the triangle

Do this for all triangles of G_i to obtain graph H_i. See figure 4.2.1

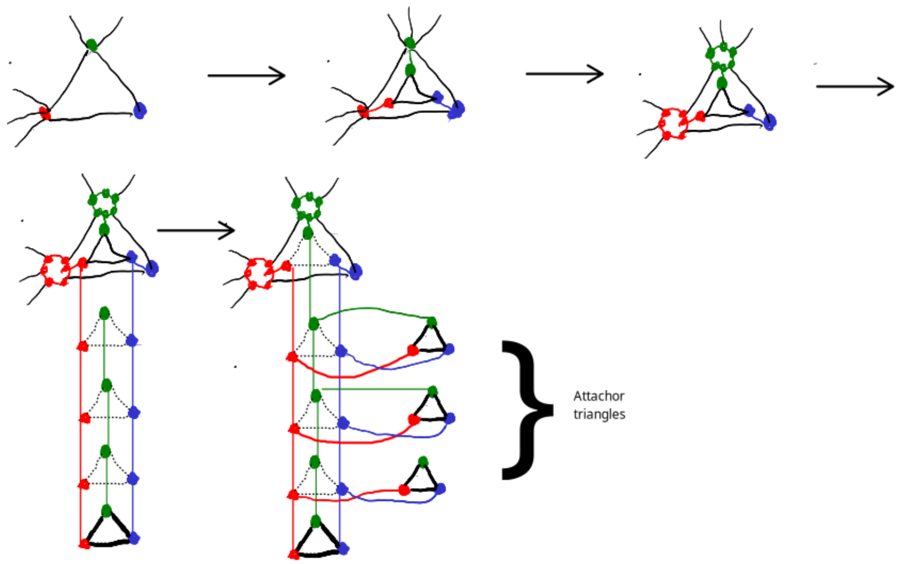

Figure 15: A triangle of G_i modified step by step. G_i H_i, G_i', G_i'', G_i''' are pictured in order. By contracting along same-colored segments, we regain the original graph. By clique summing on attachor triangles, we keep the maximum degree low. Delete dotted edges after you're done.

We call a triangle added in this manner on the empty face bounding abc a *representor triangle* of abc, and denote it $a'b'c'$. Now let there be planar graph $G_i' \geq_m H_i$ of maximum degree 3 created by H_i by replacing each vertex v with $Bl(v)$ as in the proof that $\Delta(PLANARS = 3)$, but leaving the vertices of representor triangles as is. This way, we can keep doing 3-sums. For every edge uv of G, call the unique $Bl(u) - Bl(v)$ edge the representor edge of uv. For every vertex u of G, add an additional vertex u' to G' and embed it on the circle $Bl(u)$ is embedded on, on the interior of an edge and let that u' be the representor of u. Naturally, replace that edge xw u' is on with the edges xu' and $u'w$, embedded on the circle.

Theorem 22. *($G_1' \oplus ... \oplus G_k' \geq_m G_1 \oplus ... \oplus G_k$), where if G_i and G_{i+1} were clique summed on common cliques abc and def, G_i' and G_{i+1}' were clique summed on common cliques $a'b'c'$ and $d'e'f'$. See image 4.2.1. (Analogously, if G_i and G_{i+1} were clique summed on a common 1-clique or 2-clique, G_{i+1}' were clique summed on the representors of those cliques).*

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be the the reverse under H of the 3 arcs of the small circle. Similar arguments apply if the empty face of abc is homeomorphic to the exterior of C.

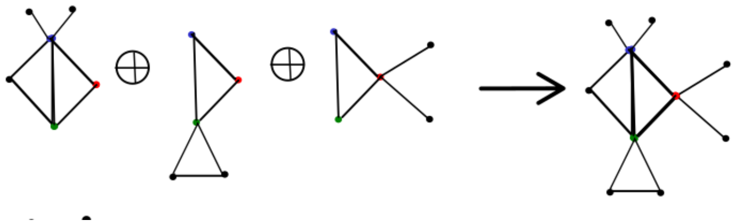


Figure 16: The clique sum of 3 planar graphs, leading to a graph of max degree >3.

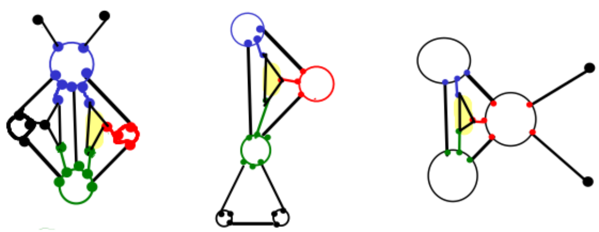


Figure 17: The graphs G_i' are clique summed over the shaded triangles now.

We discuss only 3-sums from now on. 2 and 1 sums are completely analogous.

Proof. Notice that $G_i' \geq_m G_i$ by contracting each $Bl(v)$ to get back v and for each representor triangle $x'y'z'$ contracting x' to x, y' to y, z' to z. Therefore, let μ_i be the model function of $G_i' \geq_m G_i$, $x' \in \mu_i(x)$, $y' \in \mu_i(y)$, $z' \in \mu_i(z)$, and $G_1' \oplus G_2' \geq_m G_1 \oplus G_2$ by lemma 4. Furthermore, representor triangles in $G_1' \oplus G_2'$ continue to have a vertex in each branch of the triangle they model. $(G_1' \oplus G_2') \oplus G_3' \geq_m (G_1 \oplus G_2) \oplus G_3$ by lemma 4. Furthermore, representor triangles continue to have a vertex in each branch of the triangle they model, and so on. The result follows inductively. $\square$

In this manner, we obtain a graph $G' = (G_1' \oplus ... \oplus G_k')$ containing G as a minor, with all non-representor vertices having degree 3 or less. However, if an unbounded amount of clique sums occur on a specific representor, we could still get a G' of unbounded degree.
Utilizing clique sums, we make some additional modifications to G_i'. See figure **??**.

Let $a'b'c'$ be a representor triangle in G_i'. Let there be graph $K_3 \square P_k$ with vertex set $(\{1,2,...,k\} \times \{1,2,3\})$. We call the clique corresponding to the *nth* vertex of P_k, i.e for fixed $n \in \{1,2,...,k\}$ we call the clique of $K_3 \square P_k$ induced by the vertices (p,k) with $p = n$ the *nth clique* of $K_3 \square P_k$. Clique sum the 1st K_3 of a $K_3 \square P_k$ graph to a representor triangle $a'b'c'$ to obtain G_i''. We call the nth clique of a $K_3 \square P_k$ in G_i'' added in this manner to representor triangle $a'b'c'$ the nth copy of $a'b'c'$ (with this terminology, $a'b'c'$ is the 1st copy of

$a'b'c'$). By lemma 10, the graph remains K_5-minor-free. Make the analogous modifications for 2 and 1 sums. Again, we discuss only of 3-sums - the reader may verify 2 and 1 sums have completely analogous proofs.

Theorem 23. *($G''_1 \oplus ... \oplus G''_k \geq_m G'_1 \oplus ... \oplus G'_k$), where if G'_i and G'_{i+1} were clique summed on common cliques $a'b'c'$ and $d'e'f'$, G''_i and G''_{i+1} were clique summed on the ith copy of $a'b'c'$ and $d'e'f'$. See image 4.2.1 and 4.2.1 again and then 4.2.1.*

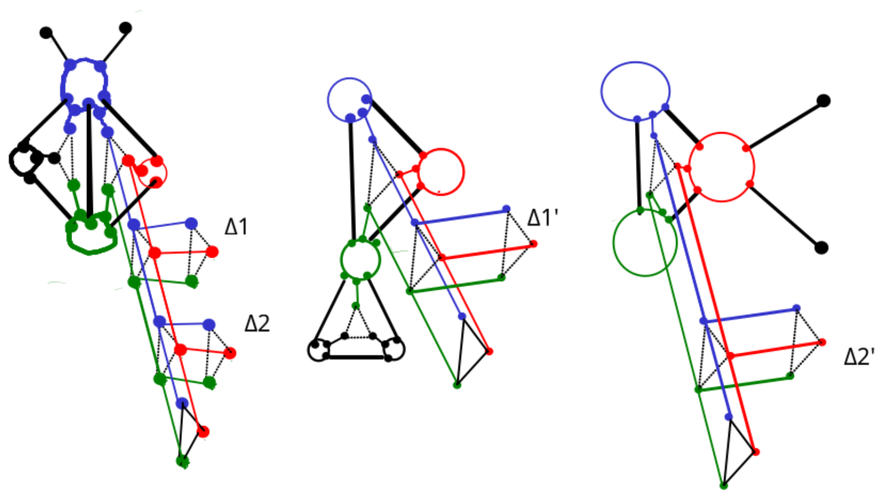


Figure 18: The graphs G''_i are clique summed over the attachor triangles now. $\Delta 1$ is summed to $\Delta 1'$ and $\Delta 2$ to $\Delta 2'$. By contracting along same colored components, we obtain G. Remove dotted edges after the last sum. This yields a graph of maximum degree 3.

Proof. Notice that $G''_i \geq G'_i$. This is done by contracting the first vertex of all copies of representor triangle $a'b'c'$ of G''_i, i.e the path of the $K_3 \square P_k$ induced by the vertices (p, k) with $k = 1$. Then by contracting the second vertex of all copies of representor triangle $a'b'c'$, and the the third. Do this for all representor triangles. Notice that every copy has 1 vertex in each branch of the $a'b'c'$ model. By lemma 4, the result then follows inductively as in the previous proof. $\square$

Notice that $G'' := (G''_1 \oplus ... \oplus G''_k)$ has maximum degree 6. Naturally we still call triangles in G'' copies if they came from a copy of G''_i for some i. Vertices that don't belong to a representor copy have maximum degree 3 still. Unused copies have degree 4. At most, we have two copies of representor triangles clique summed on each other for a degree of 6. This can be reduced to 4 as well. Notice that the last copy of each representor remains unused.

Claim 1. Let xyz be a copy of a representor triangle of G'' except the kth copy. $G'' \geq_m G'$ still holds after removing edges xy, yz, zx of G'' and doing this for all such xyz.

Proof. Let xyz be some representor. The model function showing $G'' \geq_m G'$ contracts the first vertex of each xyz copy together, the second vertex of each copy together, and the third vertex of each copy together (regaining xyz). It suffices that one copy retain its edges, because the rest of the edges are redundant once the contraction is finished. □

Now non-copies have degree at most 3, and copies have at most 4. Can the maximum degree be reduced to 3? The answer is positive. We further modify the clique sums.

Definition 66. Let there be a path graph $u_1 u_2 ... u_k$, and for each u_i, add a vertex v_i, and join it to u_i. The resulting graph is called the *comb graph* of length k or k-comb graph. The subpath $u_1 u_2 ... u_k$ is called the *spine* of the comb graph and u_i is the *ith* spine vertex. The v_i are the *teeth* of the comb.

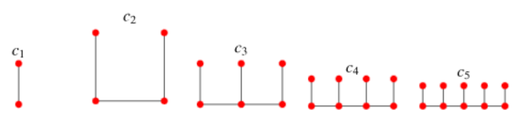


Figure 19: The 1, 2, 3, 4 and 5 comb graphs. Courtesy: Wolframalpha

Let $a'b'c'$ be a representor triangle in G'_i. We clique sum to $a'b'c'$ the first spine clique of $K_3 \square T$ where T is the k comb. We call the spine cliques of $K_3 \square T$ the copies of $a'b'c'$ and the teeth clique the attachors. Do this for all representor triangles to obtain G'''_i.

Theorem 24. *$G'''_1 \oplus ... \oplus G'''_k \geq_m G'_1 \oplus ... \oplus G'_k$, where if G'_i and G'_{i+1} were clique summed on common cliques $a'b'c'$ and $d'e'f'$, G'''_i and G'''_{i+1} were clique summed on the attachor of the ith copy of $a'b'c'$ and $d'e'f'$. This still holds after removing all edges of $(G'''_1 \oplus ... \oplus G'''_k)$ from Δ to Δ, where Δ ranges over any copy of representor triangles and any attachor except the attachor of the copy numbered k.*

Proof. Notice that $G'''_i \geq G'_i$. This is seen by contracting each attachor to its copy to obtain G''_i. Attachors of copies of $a'b'c'$ still have one vertex in each branch of the $a'b'c'$ model. $G''' := G'''_1 \oplus ... \oplus G'''_k \geq_m G'$ then follows inductively from lemma 4 as before.
Furthermore, notice that in G''' as all copies and attachors of a representor triangle $a'b'c'$ are contracted regaining $a'b'c'$, it suffices that one copy or attachor retain its edges to get $a'b'c'$ from the contraction. The other edges are unneeded. The attachor of the copy k of $a'b'c'$ fills this role. □

Notice that G''' after removing the aforementioned edges has maximum degree 3.

Corollary 2. $\Delta(forb(K_5)) = 3$.

4.2.2 $K_{3,3}$-minor-free graphs, a first lower bound and an afterthought

In this section, we will show that $\Delta(\text{forb}(K_{3,3}))$=4, that is, for every forb($K_{3,3}$) graph there is a forb($K_{3,3}$) graph of maximum degree 4 including it as a minor, but not all forb($K_{3,3}$) graphs have a forb($K_{3,3}$) graph of maximum degree 3 including the first a minor. This is the first example of a graph class with a bounded Δ value different than 3.

Just like with K_5-minor free graphs, Wagner discovered the following.

Theorem 25 (Wagner [17])**.** *A graph G excludes $K_{3,3}$ as a minor if and only if it can be constructed by the ≤ 2-clique-sums of planar graphs and K_5.*

Naturally, the proof that $\Delta(\text{forb}(K_{3,3}))$=4 repeats many of the arguments of the previous subsection. Let's center our attention at the proof that $\Delta(\text{forb}(K_{3,3}))\neq$ 3, our first lower bound.

Fact 1. Let G_1, G_2 be two planar graphs. Then, their $\leq$ 2-sum over some edge or vertex remains planar.

One may observe this using Wagner's characterization of planar graphs, and the fact that the clique sums of two graphs cannot have higher Hadwinger number greater than both the first graph and the second.
This implies that to create a non-planar graph by clique summing planar graphs and K_5 graphs, one must use a K_5 at some point, which has vertices of degree 4. Now, observe that with the exception of a trivial 2-sum which only removes an edge, (we remind that one may use clique sums to remove any edge of a graph without adding any vertices), $\leq$ 2-sums cannot reduce the degree of a vertex. We arrive at the following conclusion which we now prove:

Theorem 26. *If G is non-planar $K_{3,3}$-minor-free graph, then $\Delta(G) \geq 4$.*

Definition 67. Let $G = G_1 \oplus G_2$, and let G be equal to G_1 after removing ≥ 0 edges. In other words, the clique sum did not add any vertices. We call such a clique sum *trivial*.

Proof. Let $G = G_1 \oplus ... \oplus G_k$ be a series of 2-sums of planar graphs and K_5 graphs, creating a non-planar graph. By the above, at least 1 K_5 was used in the construction of G. Now, observe that:

- 1-sums cannot reduce the degree of vertex.
- We can assume that no trivial 2-sums occur; rather than remove an edge by a trivial clique sum, we can remove it after the last clique sum that utilizes it to create the same graph.
- If $(G_1 \oplus ... \oplus G_{i-1}) \oplus G_i$ is a 2-sum over common edge uv, we can assume that the degree of u and v in $(G_1 \oplus ... \oplus G_{i-1})$ and G_i is greater than 1; neither vertex only neighbors the other. If not, let v have degree 1 in G_i, we can replace this 2-sum $(G_1 \oplus ... \oplus G_{i-1}) \oplus G_i$ on uv with a 1-sum $(G_1 \oplus ... \oplus G_{i-1}) \oplus (G_i \setminus v)$ on u, and if the edge uv was removed during the

2-sum operation, we add after the 1 sum a trivial 2 sum after to remove it.

Thus, G may be built by $\leq$ 2-sums of planar graphs and K_5, no 2-sum being trivial or occurring over an edge with a vertex of degree ≤ 1, and at least 1 K_5 must have been used during its construction. But notice that using these ingredients, once a graph G_i has been clique summed during the building of G, none of its vertices can have their degree lowered in G. Therefore, the vertices of the K_5 graph must have degree ≥ 4. $\square$

Now, let there be non-planar $K_{3,3}$-minor-free graph G. For a $K_{3,3}$-minor-free G' to include G as a minor, G' must also be non-planar of course. Therefore, it has $\Delta(G') \geq 4$. This proves that $\Delta(K_{3,3} - MINOR - FREE) \geq 4$.

As for the proof that every $K_{3,3}$-minor-free graph is a minor of a $K_{3,3}$-minor-free of maximum degree 4, the same arguments as for K_5-minor-free graphs apply. A proof sketch is given.

Theorem 27. $\Delta(forb(K_{3,3})) = 4$

Proof Sketch. Let G be a $K_{3,3}$-minor-free graph built by the clique-sum $G_1 \oplus ... \oplus G_k$. Let G'_i be the fattening $Bl(G_i)$ if G_i is a planar graph and let it remain K_5 if G_i is K_5. For every uv edge in planar graph G_i, clique sum to the unique $Bl(u) - Bl(v)$ edge in G'_i the first torso K_2 of the graph $K_2 \square T$ where T is the k-comb. Do this for all uv to obtain G''_i. If G_i is a K_5 graph, clique sum $K_2 \square T$ on every edge to obtain G''_i instead. $G_1 \oplus ... \oplus G_k \leq_m G''_1 \oplus ... \oplus G''_k$ where if G_i is ≤ 2 clique summed to G_{i+1} on common cliques uv and wz, G''_i is ≤ 2 clique summed to G''_{i+1} on the attachor of the ith copy of the representors of uv and wz. Let $G'' := G''_1 \oplus ... \oplus G''_k$ and notice that G is still included as a minor if we remove all edges corresponding to copies or attachors except the kth attachor (i.e all edges uv where uv is a copy or attachor)). Observe that after removing those edges, G'' has maximum degree at most 4, the 4 because of the G_i isomorphic to K_5. $\square$

Remark 1. There is something quite interesting to notice here. For a minor-closed class C, one way to reformulate the definition of $\Delta(C)$ is to define $\Delta(C)$ as the minimum k so that C =minor-closure$\{G \in C | \Delta(G) \leq k\}$. For classes C of $\Delta(C) = k > 3$, one may ask what minor-closure$\{G \in C | \Delta(G) \leq 3\}$ is, or more generally, for any k' smaller than k what minor-closure$\{G \in C | \Delta(G) \leq k'\}$ is. For $K_{3,3}$-minor-free graphs the answer is easy; minor-closure$\{G \in \text{forb}(K_5) \mid \Delta(G) \leq 3\}$= the planar graphs, as every such G is built by the 2-sum of planar graphs and subgraphs of K_5, which are also planar.
Repeating this question with other minor-closed graph classes of high Δ, we may find elegant and natural graph classes, just as we did with $K_{3,3}$-minor-free graphs, and even undiscovered ones. As a foreshadowing, let $\text{TW}_{\leq k}$ be the class of graphs of treewidth k or less. $\{G \in TW_{\leq k} \mid \Delta(G) \leq 3\}$ lies strictly between $TW_{\leq k-1}$ and $TW_{\leq k}$. Could it be formulated as a variation of treewidth, like simple treewidth?

4.2.3 K_n-minor free graphs for $n \geq 6$, $K_{n,n}$-minor-free graphs for $n \geq 4$.

The lack of structural theorems and characterizations for K_6-minor-free graphs makes them particularly hard to work with. Specific results giving some information that come to mind are [1] and [9] and of course the proof of Jorgersen's conjecture for large graphs [7], which aren't very helpful. It is thus nice that we are able to prove that the class of K_6-minor free graphs, has $\Delta(forb(K_6)) = \infty$. In fact, the following is a corollary of the main theorem of this thesis:

Theorem 28. *$\Delta(forb(K_n)) = \infty$, for all $n \geq 6$. $\Delta(forb(K_{n,n})) = \infty$, for all $n \geq 4$.*

4.3 Graphs of pathwidth $\leq k$, Graphs of treewidth $\leq k$

Definition 68. Given a graph G, an *expansion* of G is any graph $G' \geq_m G$.

In [12], Markov and Shi showed that every graph of treewidth $\leq$ k has a degree 3 expansion of treewidth $\leq k+1$, and that the $+1$ is necessary for $k \geq 19$, i.e, $\Delta(TW_k) > 3$ for $k \geq 19$. We extend and simplify their results; let TW_k be the class of graphs of treewidth $\leq k$, and PW_k be the class of graphs of pathwidth $\leq k$. We show that $\Delta(PW_k) = \Delta(TW_k) = k$ for all k. Our proof that $\Delta(TW_k) \geq k$ is notionally simpler in comparison.

We remind that a graph has treewidth $\leq k$ iff it can be constructed by the clique sum of graphs of $\leq k+1$ vertices. A graph has pathwidth $\leq k$ iff it can be constructed by the clique sum of graphs $G_1,\ G_2,\ ...$, each graph clique summed to the previous in the sequence, i.e. $(V(G_1) \cup ... \cup V(G_i)) \cap V(G_{i+1})$=$(V(G_i) \cap V(G_{i+1})$.
The following proposition is key. It is proved in the same manner that one proves that the $n \times n$ grid has treewidth $\leq k$.

Proposition 1. *$K_n \square P_2 \in PW_n$, where P_2 is the 2-vertex path.*

Proof. Let G_1 be a K_n graph, let $V(G_1) = \{1, 2, ..., n\}$ and clique sum it with a K_{n+1} graph G_2, let its nodes be $\{1, 2, ..., n, 1'\}$. Afterwards, we clique sum G_2 with a K_{n+1}, its nodes being $\{1', 2, ..., n, 2'\}$, then the node set will be $\{1', 2', 3, ..., n, 3'\}$ and so on n times. In the final graph, $\{1, 2, ..., n\}$ and $\{1', 2', ..., n'\}$ are cliques, with (i, i') connected for all $i \in \{1, 2, ..., n\}$. $\square$

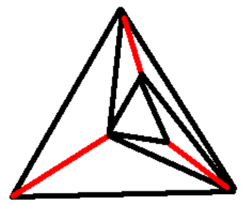


Figure 20: Creating a $K_3 \Box P_2$. We start from the exterior triangle xyz, and create the interior triangle $x'y'z'$ by clique-sums, one vertex at a time. The red edges are xx', yy', zz'

.

Instantly, we have as a corollary that $K_n \Box P_i \in PW_n$ for all paths P_i of length i, and by lemma 10 that $K_n \Box T \in TW_n$ for any graph T. Let's first observe that every graph in PW_n has a degree 3 splitting in PW_{n+1}:

4.3.1 Pathwidth $\leq n$

Let there be graph G of pathwidth $\leq n$, constructed by graphs $G_1, ..., G_k$ clique summed in this order. To observe that every graph in PW_n has a degree 3 splitting in PW_{n+1}, simply replace graph G_i with the following graph G'_i: Take $G_i \Box P_{|E(G_i)|+2}$, and let $P_{|E(G_i)|+2}$ have vertex set $p_1, p_2, ...$ and G_i vertex set $u_1, u_2, ...$ Let $e_1, ...$ be the edges of G_i. Delete all edges except e_1 in the G_i corresponding to p_2, delete all edges except e_2 in the G_i corresponding to p_3 and so on. Use the leftmost and rightmost cliques to perform the clique-sums: Add to the G_i corresponding to p_1 the clique G_i was summed on with G_{i-1} and to the G_i corresponding to $p_{|E(G_i)|+2}$ the clique G_i it was summed on with G_{i+1}. This completes the construction of graph G'_i of pathwidth $\leq n+1$ (G_1 and G_k are replaced with $G_1 \Box P_{|E(G_1)|+1}$ and $G_1 \Box P_{|E(G_k)|+1}$ of course). G' is defined as $G'_1 \oplus G'_2 ... \oplus G'_k$, G'_i clique summed on G'_{i+1} on their rightmost G_i and leftmost G_{i+1} copy of course. After clique summing G'_i with G'_{i+1}, remove the edges of the clique. It is easy to see that $G' \geq_m G$ with maximum degree 3.
We move on to the proof that $\Delta(PW_n) = n$. This is seperated in a lower and upper bound result. We first prove $\Delta(PW_n) \leq n$.

Proposition 2. *$\Delta(PW_n) \leq n$.*

Proof. Let there be pathwidth $\leq n$ graph $G = G_1 \oplus G_2 \oplus ... \oplus G_k$, clique summed in this order. It suffices to consider only the case where all the G_i are isomorphic to the $n+1$-clique. All other G in PW_n are subgraphs of such a graph. It also suffices to prove this for connected G.

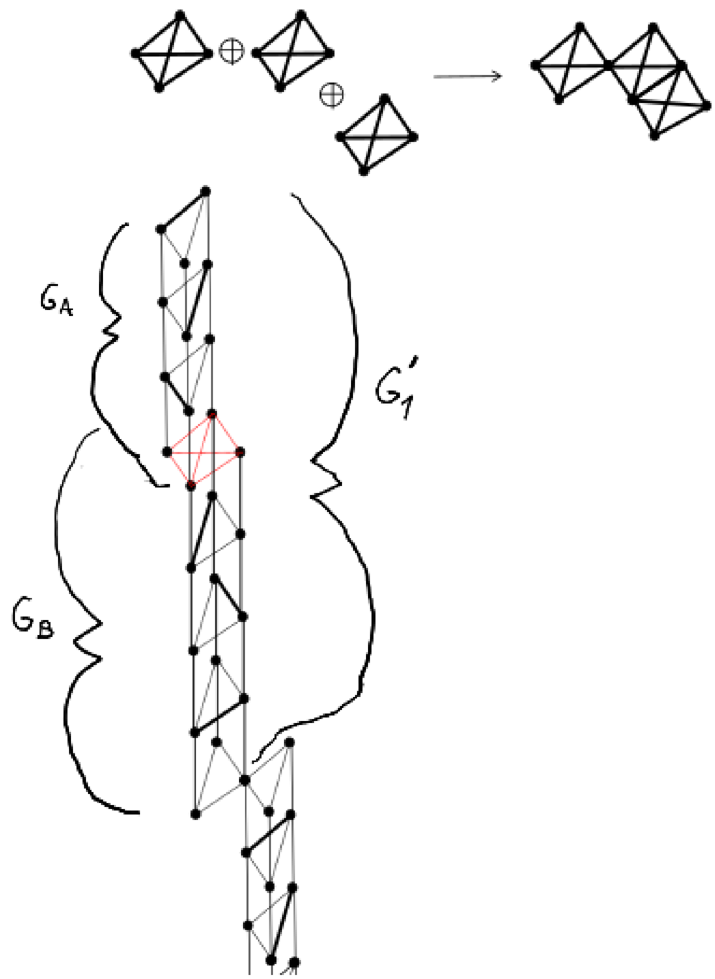


Figure 21: The clique sum of 3 4-cliques to create G and part of the corresponding G' below it. G_1' appears fully. The bold edge is the edge we do not remove in each triangle. It is easy to see that if we contract G_A downwards, and G_B upwards, we regain G_1.

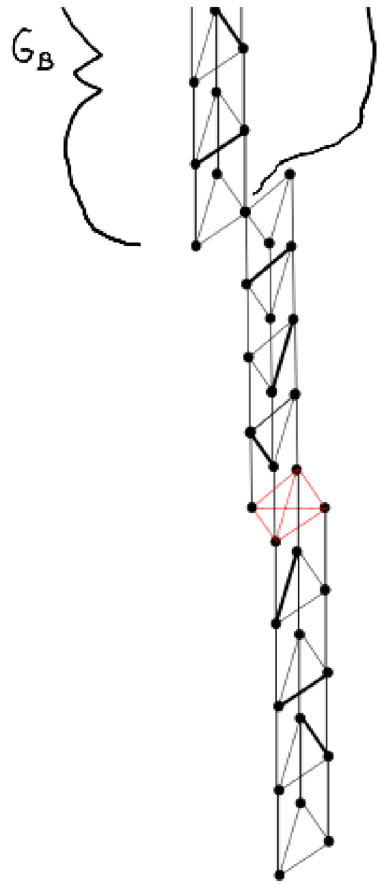


Figure 22: The rest of G'_2 in G' for illustration purposes.

Let $v \in G_1$ be some vertex. Similarly with above, we define the following graph G'_1. See figure 4.3.1: Let $E = e_1, ...$ be the edge set of $G_1 \setminus v$. Let there be graph $G_A = (G_1 \setminus v)\Box P_{|E|+1}$, where $P_{|E|+1} = p_1 p_2 ...$ is the path graph of $|E|+1$ vertices, and $V(G_1 \setminus v) = \{u_1, u_2, ...\}$. Now remove all edges of $(G_1 \setminus v, p_1)$ except e_1, all edges of $(G_1 \setminus v, p_2)$ except e_2, and so on, and remove all edges of $(G_1 \setminus v, P_{|E|+1})$.
We wish to add v, and to do that we have to drop another vertex. Let $v' \neq v$ be some vertex in G_1 and not in G_2 (it is safe to assume such a vertex exists w.l.g.). Do the same in $G_1 \setminus v'$, i.e define $G_B = (G_1 \setminus v')\Box P_{|E'|+1}$, where E' is the edge set of $G_1 \setminus v'$, and remove edges as before; remove all edges of $(G_1 \setminus v', p_1)$ except e'_1, all edges of $(G_1 \setminus v', p_2)$ except e'_2, and so on, and remove all edges of $(G_1 \setminus v, P_{|E'|+1})$, only this time keep the edges of the clique G_1 was clique-summed on to G_2 with (We shall use them for a clique sum. After the sum occurs, we shall remove those edges too).

Now take the disjoint union of G_A and G_B ($(G_1 \setminus v)\Box P_{|E|+1}$ and $(G_1 \setminus$

$v')\Box P_{|E'|+1})$ and identify same named vertices from $(G_1 \setminus v, P_{|E|+1})$ and from $(G_1 \setminus v', P_1)$ to obtain G'_1.
This is a graph of width n and maximum degree n (if we forget about the edges needed for the clique sum, which will be removed anyway), and by contracting in G'_1 the subgraphs $(u_1,P_{|E|+1})$ and $(u_1,P_{|E'|+1})$ together into 1 vertex, $(u_2,P_{|E|+1})$ and $(u_2,P_{|E'|+1})$ together into 1 vertex, and so on, and $(v',P_{|E|+1})$ into 1 vertex and $(v,P_{|E'|+1})$ into 1 vertex, we obtain G_1.
Do the same for the other G_i, only unlike before have $P_{|E|+2}$ instead of $P_{|E|+1}$, and have a clique on $(G_i \setminus v, p_1)$ of G_A and $(G_i \setminus v, p_{|E'|+2})$ of G_B (for the sums). Clique sum G'_i with G'_{i+1} in the obvious manner, removing the edges of the cliques after the clique sum. It is simple to observe that $G'_1 \oplus G'_2 \oplus ...$ has maximum degree n, is of pathwidth $\leq n$, and contains G as a minor by contracting as above.

□

We now move on to the second lower bound of this text. We need a graph G of pathwidth at most n such that any graph of pathwidth at most n containing it as a minor has maximum degree $\geq n$. This graph is the following:

Let there be a K_n clique with vertex set $\{1, 2, ..., n\}$. n-sum to it 1000 $n+1$-cliques, let the ith be $\{1, 2, ..., n, i\}$. This completes the construction of G.

Proposition 3. *There is no graph G' of pathwidth at most n containing G as a minor with $\Delta(G') < n$.*

The following well-known lemma (see e.g Diestel [2]) is of use:

Lemma 5. *Let G contain an n-clique, let G' contain G as a minor, and let there be a tree-decomposition of G'. Then there is some bag of the tree-decomposition which contains a vertex from each minor branch of the n-clique.*

Path-decompositions being tree-decompositions, this theorem applies here as well. We now prove proposition 3.

Proof. Let there be graph $G' \in PW_k$ containing G as a minor, and let G' be created by the clique sums $G'_1 \oplus G'_2 \oplus ....$ By proposition 5, for any of the 3 $(n+1)$-cliques of G there is a G'_i such that G'_i contains a vertex of each minor branch of the $(n+1)$-cliques. Let G'_i, G'_j, G'_k be these graphs, $i' \leq j' \leq k'$. Now, all graphs between G'_i and G'_k need to have a vertex from each branch of the central K_n clique. Therefore, the extra node of G'_j cannot be split. For let this be the case, let it be split into u and u', this edge does not fit anywhere. □

We move on to TW_k. The reader will notice that arguments are naturally similar.

4.3.2 Graphs of treewidth $\leq n$

We begin with the lower bound. In [12], Markov and Shi showed that there is a graph G of treewidth n and no degree 3 expansion of treewidth n. The example graph G we use is very similar in comparison and we now define it; let there be an $n+1$-clique graph with vertex set $\{1, 2, ..., n+1\}$, called the *central clique*. For every n-subclique with vertex set $\{1, ..., i-1, i+1, ..., n\}$, add a vertex labeled i' and join it to the subclique, call this $n+1$-clique $K_{n+1}^{(i)}$. This completes the construction of graph G. Markov's and Shi's example was the same, but they also removed all edges with both ends in the central clique of G. The following is both an extension and a notional simplification of their result.

Proposition 4. $\Delta(TW_k) \geq k$

Proof. Let $G' \geq_m G$ as a minor with model function μ. By lemma 5, for any tree-decomposition of G', if there is an $n+1$ clique in G, there is some bag of the tree-decomposition which contains a vertex from each minor branch of the $n+1$ clique. Call this a *model carrier* of that $n+1$-clique.
Let there be a width n tree-decomposition of G'. Notice that any tree decomposition vertex t adjacent to the centre clique bag carrier t_c must *drop* a centre clique bag node, i.e, for some $i \in \{1, ..., n\}$, $\mu(i) \cap V_{t_c}$ is not empty but $\mu(i) \cap V_t$ is, for there cannot be $n+1$ (possibly trivial) distinct paths from one bag to the other, as their intersection is a separator. Therefore there is a single centre clique model carrier. In fact this holds for all $n+1$ clique model carriers.
As every bag adjacent to the centre model bag must drop a vertex, the first internal vertex $t_{i'}$ on the path from the central bag carrier to the $K_n^{(i)}$ model carrier drops the bag vertex of i. Thus no vertex whose path to t_c uses $t_{i'}$ may have a vertex of the minor branch of i. All such vertices induce a subtree of the tree-decomposition, with $K_n^{(i)}$ in it. Lacking vertices from the model of i, for $j \neq i$ no other $K_n^{(j)}$ model carrier is included in this subtree.

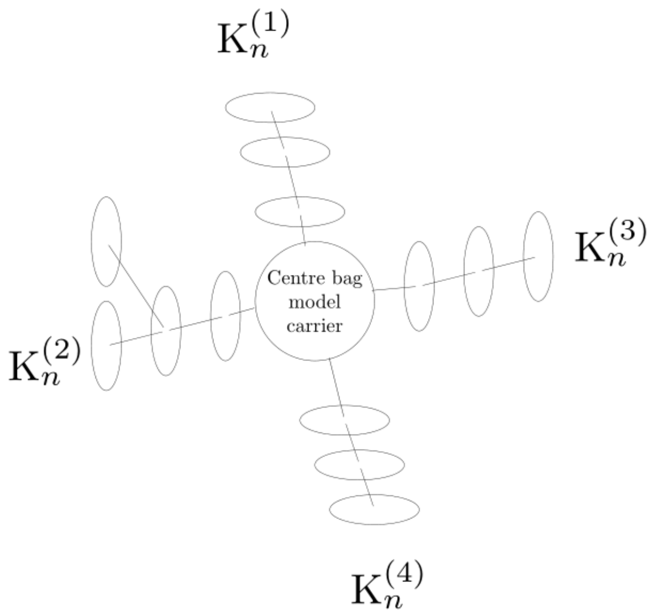


Figure 23: Example tree-decomposition of G' for $n = 4$. The centre bag model carrier separates the $K_n^{(i)}$.

Let v_i be both in the model carrier of $K_n^{(i)}$ and in the minor branch of i'. For G' to include G as a minor, there must be a path from v_i to all n nodes of the central bag carrier, except the one in the model of i. This path is internally disjoint to other such paths from a similar node v_j of a $K_n^{(j)}$ carrier, $j \neq i$. A vertex in the centre bag model carrier and the model of i thus receives n internally disjoint paths from each of the n $K_n^{(j)}$ model carriers, where $i \neq j$ Thus, each vertex of the central bag model carrier has degree $\geq n$. $\square$

We move on to the other direction. We have used the following ideas many times already, so we over them quickly.

Proposition 5. $\Delta(TW_n) \leq n$.

Let G be a graph produced by the clique sum of graphs G_1, G_2,..., G_k, in this order. It suffices to assume that the G_i are isomorphic $n+1$-cliques, as G made from such G_i includes all other graphs in TW_k as a subgraph.
Just like with previous classes, let there be some G_i with n-clique K, and let there be graph $T \square K_n$ where T is the $k+1$-comb graph, and K_n has vertex set $\{u_1, ..., u_n\}$. Call the subclique of $T \square K_n$ corresponding to the first spine vertex the first spine clique, and the subclique of $T \square K_n$ corresponding to the first hair vertex the first hair clique. n-sum G_i and $T \square K_n$ by identifying K and the first spine clique. Do this for all n cliques of size n of G_i to obtain G_i'.
Call the ith spine clique of the $T \square K_n$ attached to K the ith copy of K, and

the corresponding hair clique the ith attachor and call the entire $T\Box K_n$ the comb representor of K. Also for any clique of G_i, call a clique of size n of G_i containing it a representor clique.

Obviously $G'_i \geq_m G_i$. It is not hard to observe that in G'_i, if we remove all edges of a comb representor with both endpoints in the same copy or attachor, but leave the last attachor (numbered $k+1$) intact, we still contain G_i as a minor; simply contract the vertices of the comb representor corresponding to vertex v_1 of $T\Box K_n$, then contract the vertices corresponding to v_2, and so on for all v_i. We reobtain the original clique.
We now proceed to the clique sums.

Proposition 6. *$G_1 \oplus ... \oplus G_k \leq_m G'_1 \oplus ... \oplus G'_k$, where if G_{i+1} was m-summed to the G_j subgraph of $G_1 \oplus ... \oplus G_k$, on isomorphic cliques K and K', then G_{i+1}' was m summed to the G'_j subgraph of $(G'_1 \oplus ... \oplus G'_i)$ on the following isomorphic cliques: The ith attachor of the clique representors of K and K'.*

To obtain G as a minor of $G' := G'_1 \oplus ... \oplus G'_k$, for each G'_i, go to the G'_i subgraph of G', and for each n clique K of size $n+1$ of G_i, contract the vertices of the comb representor of K corresponding to vertex v_1 (we remind, the clique K_n of $T\Box K_n$ has vertex set $v_1, v_2, ...$), then contract the vertices corresponding to v_2, and so on for all v_j. It is easy to observe that doing this for all G'_i subgraphs of G', we obtain G.
Furthermore, if we remove all edges of a comb representor with both endpoints in the same copy or attachor but leave the last attachor (numbered $k+1$) intact, we still contain G as a minor by the same contractions. Remove those edges from all comb representors to obtain G''.
We have observed that $G'' \geq_m G$. Furthermore, $\Delta(G'') = n$, as the original vertices of the G_i in G'' and the last clique attachor of each comb has degree n, while other vertices of G'' have degree at most n. This completes the proof of the proposition.
By the two results of this subsection, we have that $\Delta(TW_n) = n$.

4.4 Apex graphs

We now consider the case of Apex graphs. As it turns out, apex graphs is the first example of a graph class C with $\Delta = \infty$.

We don't go too much into details in the proof, as it is a corollary of our main theorem.

5 Minor closure of class containing all pyramids

A natural question to ask is if Δ is increasing with respect to the subset relationship. This is not the case; STARS $\subseteq$ the class of planar graphs $\subseteq$ the class of apex graphs (where STARS is minor closure of the class of stars), but their Δ value is ∞, 3 and ∞ respectively. We do however have the following: Let $\mathcal{A}$ be the class of apex graphs.

Theorem 29. *If a proper minor closed class $C \supseteq \mathcal{A}$, then $\Delta(C) = \infty$.*

Formulated otherwise:

Theorem 30. *If for a minor closed class $C \supseteq \mathcal{A}$ it holds that $\Delta(C) = k \in \mathbb{N}$, then C contains all graphs.*

For non zero natural numbers N, M, the $N \times M$ *grid graph* is the graph with vertex set $\{1, 2, ..., N\} \times \{1, 2, ..., M\}$ and edge set $\{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}$. See figure 1.
The N*-pyramid* is the graph created by taking a $N \times N$ grid, adding a vertex, and joining it to all vertices of the grid.

Clearly a pyramid is an apex graph. As we now show, to prove Theorem 30, it suffices to prove the following: If a graph contains a large enough pyramid as a minor by a graph of $\Delta(G) \leq c$, then it contains an arbitrarily large clique.

Theorem 31. *For every $n,\ c \in \mathbb{N}$, there exists N such that if $\Delta(G) \leq c$, and G contains the N-pyramid as a minor, then G contains K_n as a minor.*

We prove Theorems 30 and 31 are equivalent.

Proof. If C includes all apex graphs as a minor with graphs of $\Delta(G) \leq k$ for some k, then it includes all N-pyramids with graphs of $\Delta(G) \leq k$, and then it includes all cliques. $\square$

We thus now only focus on Theorem 31. Let H be a subgraph of graph G. An H–path in G is a path of G internally disjoint from H with endpoints in H. To prove 31, the high level idea is to prove that if $\Delta(G) \leq c$ and $G \geq_m$ a large enough N-pyramid, then $G \geq_m$ an $N \times N$ grid H with many H-paths, their endpoints positioned to our liking (Lemma 7). It is well-known that a large enough grid H with $\binom{t}{2}$ H-paths with endpoints far apart from each other contains a K_t clique: See lemma 8.

Lemma 6. *For every $n, c \in \mathbb{N}$, there is N such that if $\Delta(G) \leq c$ and G contains the N-pyramid as a minor, then G also contains as a minor the $N \times N$ grid, call it H, with n pairwise edge-disjoint H-paths with discreet endpoints. Furthermore, there exists $s \in \mathbb{N}$ such that for any subgraph S of the grid H of order $|S| \geq s$, we can find in a minor of G S-paths with the same properties instead of H-paths.*

Lemma 7. *For every $n, c \in \mathbb{N}$, there is N and s such that if $\Delta(G) \leq c$ and G contains the N-pyramid as a minor, then G also contains as a minor the $N \times N$ grid, call it H, with n pairwise edge-disjoint S-paths with discreet endpoints, where S is any subgraph of H of more than s vertices.*

Lemma 8. *[14] If G is a wall with pairwise disjoint G-paths $P_1, ..., P_{\binom{n}{2}}$ where $n > 1$, there exists $d \geq 0$ such that if any 2 G-path endpoints $p \in P_i$, $p' \in P_j$ have $d(p, p') \geq d$, then $G \geq_m K_n$.*

A *wall* is an $(n \times 2n)$ grid, where ordering edges from top to bottom for each vertical path, we remove from the first vertical path the even ordered edges, from the second vertical path the odd ordered edges, from the third the even ordered edges and so on. Finally we remove degree 1 edges and then arbitrarily subdivide edges.

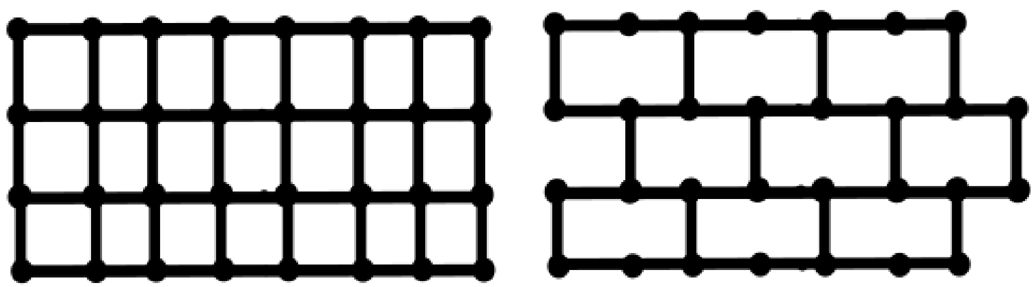


Figure 24: A 4×8 grid and a corresponding wall

Importantly, notice that lemma 8 also holds for $(n \times 2n)$ grids.
We are now ready to prove theorem 31.

Proof of Theorem 31. Fix some n and c. We want to prove that for some large enough $N = N(c)$, if a graph G has maximum degree at most c, it will contain K_n as a minor if it contains the N-pyramind as a minor. By lemma 7, for some large enough N, G will contain as a minor the $N \times N$ grid, call it H, with $\binom{n}{2}$ pairwise disjoint H-paths with discreet endpoints. Select some $(N/2 \times N)$ subgrid H' of the grid, and have the endpoints be in a subgraph S of H' such that for all $u_1 \neq u_2 \in S$, $d_{H'}(u_1, u_2) \geq d$. By lemma 8, $G \geq_m K_n$. $\square$

We present a few corollaries before proving lemma 1.

Corollary 3. *If C is a proper minor-closed superclass of the apex graphs, then $\Delta(C) = \infty$.*

The linklessly embeddable graphs are a well known 3-dimensional equivalent of the planar graphs. It is reasonable to ask if, like with planar graphs, one may by some geometric argument replace each node of a linklessly embeddable graph G

by some other structure to extend $\Delta(PLANARS) = 3$ to linklessly embeddable graphs. As the apex graphs are a subclass of the linklessly embeddable graphs, the answer is negative.

Corollary 4. *Let $\mathcal{L}$ be the class of linklessly-embeddable graphs. $\Delta(\mathcal{L})=\infty$.*

Corollary 5. *Let C be a class containing all apex graphs as minors. For some k, let f be any function mapping a graph to a graph containing in as a minor with maximum degree k. Then $f[C]$ contains all graphs as minors.*

Now follows the proof of lemma 7.

Proof. Let there be integer n. We would like to prove that if a graph G of $\Delta(G) \leq c$ contains a big enough pyramid as a minor, let it be a $N(n,c)$-pyramid, let $S(n,c)$ be a big enough subgraph of its grid, it contains the $N(n,c) \times N(n,c)$ grid with n pairwise edge-disjoint S-paths with discreet endpoints (N and S to be specified later).
So let a be the apex vertex of the $N(n,c)$-pyramid and X its grid and let μ be the model correspondence function mapping vertices of the pyramid to connected components of G. In G contract $\mu(v)$ for all grid vertices $v \in X$ to obtain X. We will use a to find n jumps, with endpoints in $S \subseteq X$. We remove edges until $\mu(a)$ is a tree, and it has precisely one $\mu(a) - X$ edge towards each vertex of S and 0 to $X \setminus S$.
Of course 2 vertices of $\mu(a)$ neighboring S along with the path of $\mu(a)$ between them form an S-path, but S-paths being internally disjoint, using it could make us lose many other S-paths. How should we proceed?
We may assume all subtrees in $\mu(a)$ have a vertex neighboring S. If not, we remove them. We may also assume all vertices of $\mu(a)$ that only neighbor $\mu(a)$ have degree ≥ 3. If they have degree 1 we delete them, and if they have degree 2 we dissolve them. We then take a maximal path $P = u_1, u_2...$ in $\mu(a)$. Call the u_i neighboring X good vertices, and the rest bad. Bad u_i vertices can be contracted into good vertices; since they must have degree>2 each must neighbor a subtree (which does not intersect P or other such subtrees, else there would be a cycle), which must include a vertex neighboring S. Remove all other vertices of the subtree except the path connecting u_i to the vertex neighboring S, then contract this path. Path P now has only good vertices, every two of which form the internal vertices of an S-path. How large is P? Notice that at the time we pick it, $\mu(a)$ still has maximum degree $\leq c$ and as it neighbors every vertex of S, $\mu(a)$ still has more than $\frac{N^2}{c}$ vertices. Fixing c and letting N and thus $|V(\mu(a))|$ grow larger and larger, the diameter of $\mu(a)$ must also increase, and thus the length of its maximum path. Pick s large enough for $\mu(a)$ to have diametre at least $2n$. Pick N large enough X can fit S. $\square$

Remark 2. Nowhere in this lemma did we use the fact that X is a grid. Indeed, rather than just pyramids, it holds for any infinite family of finite graphs as long as they all have a vertex connected to all other vertices.

6 A superclass of $\Delta = 3$ for any class excluding an apex graph

Definition 69. A graph class is proper if it does not include all graphs.

We have proved that any proper minor-closed class including all apex graphs must have $\Delta = \infty$, and any attempts to relax this fact to smaller classes while working on this thesis had failed. On the other hand, given a minor-closed class C excluding a planar graph, we have inspected that it is contained in a superclass C' of finite $\Delta(C')$, in fact of $\Delta(C') = 3$. We suspect the following.

Theorem 32. *Let C be a minor-closed class excluding an apex graph as a minor. There exists a proper minor-closed class $C' \supseteq C$ with $\Delta(C') = 3$.*

In [3] Dujmović, Morin and Wood proved that the following are equivalent for a proper minor-closed graph class C.

1. C forbids an apex graph as a minor.
2. C has bounded local treewidth.
3. C has linear local treewidth.
4. Every graph in C has bounded layered treewidth.
5. Every graph in G admits layered separations of bounded width.
6. For some k, every graph in C can be constructed by the clique-sum of strongly k-almost embeddable graphs.

Theorem 30 in combination with theorem 32, complements this result by adding the following characterization:

Theorem 33. *A proper minor-closed class C excludes an apex graph as a minor if and only if it has a minor-closed superclass C' with $\Delta(C') = 3$.*

The class C' of theorem 33 also excludes an apex graph. Furthermore, by theorem 30 one may replace $\Delta(C') = 3$ with $\Delta(C') \leq k$ for any finite k. Therefore, theorem 33 can be reformulated as:

Theorem 34. *A proper minor-closed class C excludes an apex graph as a minor if and only if it has a minor-closed superclass C' excluding an apex graph as a minor and with finite $\Delta(C')$.*

We prove the equivalence of theorem 33 with condition 6 above. Condition 6 is a corollary of a (rather strong) strengthening [4] of the graph minor structure theorem of Robertson and Seymour [15]. The theorem of Robertson and Seymour says that much like K_5-minor-free graphs can be built by clique-summing planar graphs and the Wagner graph, so can the K_n-minor-free graphs be built by clique summing graphs from a correctly selected family, the family of *k-almost-embeddable* graphs.

Theorem 35 (The graph minor structure theorem)**.** *Let there be a graph H, and let $G \in$ the H-minor-free graphs. Then G can be constructed from the clique-sum of k-almost embeddable graphs, where $k = k(H)$.*
Furthermore, it suffices to use graphs almost embeddable on surfaces that H does not embed on (of genus k or possibly less).

As an instant corollary, the graph minor structure theorem also holds for minor-closed graph families excluding more than 1 graph as a minor.
Now let us define what a k-almost embeddable graph is. Rather than take a planar graph to clique-sum, we take a graph embeddable on some surface of euler genus at most k, we embed it, and then choose up to k faces, to which we add potentially non-embeddable layers of "depth" $\leq k$. Finally we add k apex vertices.

Let's start by defining the non-embeddable layers of an almost embeddable graph, called *vortices*.

Definition 70. Let there be a graph G embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle [5] of G. Let there be graph G', and add [6] G' to G. Let there be a C-decomposition of G' with bags $B_{v_1}, ..., B_{v_n}$. Pick a distinct node u_i from each bag B_{v_i}, and in $G' + G$ identify v_i and u_i for all i to obtain a new graph G''. *Adding a vortex* G' to G over $v_1, ..., v_n$ and $u_1, ..., u_n$ is defined to be this sequence of operations. If the C-decomposition of G' has width k, then the vortex has *depth* k. We call G' a *vortex* of G''.

The reader may picture the vortex added inside the face. Since we usually do not care about the specific choice of G', we simply say we add a vortex to G on C. We now proceed to define a k-almost embeddable graph.

Definition 71. Let there be a graph G. Let G be embeddable on a surface of Euler genus $\leq k$. For some embedding, choose up to k pairwise disjoint facial cycles of G. Add to each of them a vortex of depth up to k, to obtain G'. Finally, add up to k vertices to G' to obtain G'', called the *apex vertices of* G'', and join them to any vertex in G'' (including other apex vertices). G'' is called a *k-almost embeddable* graph. We call G the *embedded part* of G'' and call G'' *almost embeddable* on the surface G was embedded on.

Reminding the minor structure theorem, for any H, all H-minor-free graphs can be constructed from the clique sum of k-almost embeddable graphs, where $k = k(H)$. For excluded minors H belonging to a more specific family of graphs, there exist more specific results than the graph minor structure theorem; for apex graphs it is mentioned above. If H is restricted to the planar graphs, then a $G \in \text{forb}(H)$ can be constructed from the clique-sum of graphs of $\leq k$ vertices, where $k = k(H)$ (in other words, $\textit{treewidth}(G) < k$). One could go on.

[5] A facial cycle is a cycle which is the boundary of a face of the embedded graph G.
[6] We remind we have defined the addition two graphs to be their disjoint union.

As already mentioned, on the other hand Dvořák and Thomas proved a strengthening of the graph minor structure theorem in the general case.

Definition 72. Given graph H and surface Σ, let $\alpha(H, \Sigma)$ be the minimum number of vertices one need remove from H to make it embeddable on Σ.

Theorem 36 (The graph minor structure theorem strengthened [4])**.** *The graph minor structure theorem holds even if we only use graphs almost-embedded on surface Σ such that every triangle of their embedded part is the boundary of a face homeomorphic to an open ball of $\mathbb{R}^2$, and all but $\alpha(H, \Sigma)$-1 of their apex vertices neighbor only other apex vertices and vortices.*

If H is an apex graph, then $\alpha(H, \Sigma) = 1$ of course. Condition 6 of theorem 32 follows:

Definition 73. A *strongly k-almost embeddable* is a k-almost embeddable graph where also all apex vertices neighbor only other apex vertices and vortex vertices.

Corollary 6. *Let there be an apex graph H, and let $G \in$ the H-minor-free graphs. Then G can be constructed from the clique-sum of strongly k-almost embeddable graphs, where $k = k(H)$.*

As implied by theorem 32, the converse also holds; if there is k such that every graph in some class can be constructed from the clique-sum of strongly k-almost embeddable graphs, then it excludes some apex graph.

The strengthened graph minor structure theorem has an important implication; We need only clique-sum almost embeddable graphs whose embedded part has no K_4 subgraph, or is trivially a K_4 graph.

Corollary 7. *Let there be connected graph $G \neq K_4$ embedded on some surface such that every triangle is the boundary of an open disc. Then G has no 4-cliques.*

Proof. Let there be a K_4 with vertex set $abcd$ in the graph G with embedding f. As G is connected and not a K_4, there must be a vertex v adjacent to some vertex of $abcd$, let it be adjacent to a. $f(a)$ has an open disc containing it and an initial segment of each edge incident to it. Without loss of generality, let the incident edges be clockwise around a in the order ab, ac, ad, av. Any face a participates in must include two clockwise adjacent edges in its boundary . Therefore, there is no face including only adb in its boundary. ☐

Naturally, the minor structure theorem would not be very interesting if it turned out that for some k we can create all graphs using k-almost embeddable ones. The following is a well known fact.

Theorem 37. *Let there be $k \in \mathbb{Z}_{\geq 0}$. Let C be the the class of all graphs that can be constructed by clique-summing k-almost embeddable graphs. Then $minor - closure(C)$ is proper.* [7]

This theorem holds for strongly k-almost embeddable graphs, as they are a subset of k-almost embeddable graphs [8].

In Jim Geelen's publicly available *Introduction to Graph Minors* course lectures, adding a vortex had a simpler definition, which is useful to us;

Definition 74. Let there be a graph G embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle of G. Add a K_k clique to G, and identify its first vertex to v_1. Add another K_k clique, and identify its first vertex to v_2 and so on. The clique identified with v_i is called the *vortex clique of* v_i. Now, join the clique of v_1 to the clique of v_2, join the clique of v_2 to the clique of v_3 and so on. Also join the clique of v_1 to the clique of v_n.
We call this sequence of operations as *adding a simple vortex of depth k*. The subgraph induced by the added cliques (i.e the union of the vortex clique of v_i over all i) is a *simple vortex.*[remove next?] The circle induced by the ith vertex of all simple vortex cliques is the *ith layer* of the simple vortex. We always have C be the 1st layer of the simple vortex.

Clearly this definition is different. The reader may notice that a simple vortex of depth k is a vortex of depth $2k + 1$ (the +1 needed because decompositions have that pointless -1 in their definition). Now, a k-depth vortex need not be isomorphic to any simple vortex, for example take a vortex which has a vertex neighboring all vertices of the facial cycle (this is possible if the vertex is in all branches of the cycle decomposition). However, any k-depth vortex is a *minor* of a $(k + 1)$-depth simple vortex:

Proposition 7. *Let there be embedded graph G on some surface, with facial cycle $C = v_1, ..., v_n$ and add vortex V of depth k on C to obtain G'. Alternatively, add to G a simple vortex sV of depth $k + 1$ to obtain G''. sV contains V as a minor.*

[7] Indeed, for fixed k none of the operations involved in constructing a k-almost embeddable graph can create an arbitrarily large clique minor; By Euler's formula for high genus (theorem 16), a graph G embedded on a surface of euler genus k must have at most $m \leq 3n - 6 + 3k$ where n are the vertices and m the edges of the graph, therefore too large a clique will not be embeddable on the surface. Graphs embeddable on a specific surface being closed under minors, G can't have too large a clique minor either for specific k. Similarly, adding k apex graphs can increase the Hadwinger number by at most k, and the clique sum of graphs G_1 and G_2 cannot create any larger clique minor either. For adding a vortex of depth k cannot create an arbitarily large minor, and more on the minor structure theorem, we refer the interested reader to Jim Geelen's graph minor recorded lectures, lecture 3 [5].

[8] This is significantly useful for our purposes, as opposed to the other characterizations of the class of apex graphs in theorem 32, such as layered treewidth, where the minor closure of graphs of layered treewidth k contains all graphs, even for $k = 3$. Indeed, the 3- dimensional $n \times n \times 2$ grid graph has layered TW 3 and a K_n minor, take a row from the first level and a column from the second to be each branch.

Proof. Let B_{v_i} be the bags of the C-decomposition of V of width k. We slowly contract and remove nodes from sV to prove it contains a V minor. In sV, for all $v_i \in C$, remove vertices from the simple vortex clique of v_i until it has as many vertices as B_{v_i} does. Let's now specify the model function μ. If $u \in B_{v_1}$ and $\in$ no other vortex bag, pick $\mu(u) = u'$ where u' is a vertex belonging to the simple vortex clique of v_1. If $u \in B_{v_1}$ also belongs to other bags, $B_{v_{n-j}}, ..., B_{v_n}, B_{v_1}, ..., B_{v_i}$, pick an unused by μ vertex from the simple vortex cliques of $v_{n-j}, ..., v_i$, and let the path P they define be modeled to u, i.e $\mu(P) = u$. Repeat this process for vertices of B_{v_2} not in B_{v_1} and so on. We never run out of unoccupied vertices in a simple vortex clique. If we do, let the simple vortex clique of v_i be such a clique, then B_{v_i} has more than $k+1$ vertices (a contradiction), as by construction of μ every occupied vertex of the simple vortex clique of v_i corresponds to exactly one vertex of B_{v_i}. It suffices to prove that if u and u' are adjacent in V then $\mu(u)$ and $\mu(u')$ are in sV. u neighbors u' in $V \implies$ they share a bag $B_{v_i} \implies$ (by construction) the simple-vortex clique of v_i has a vertex which μ corresponds to u and a vertex which μ corresponds to $u' \implies \mu(u)$ and $\mu(u')$ neighbor. ☐

Corollary 8. *Let there be graphs G' and G as above. $G' \geq_m G$.*

Proof. For vertices u of G' that are in the vortex V, let model function μ showing $G' \geq_m G$ be same as before, but making sure to set $\mu(v_i) = v_i$ for $v_i \in C$. If u is not in the vortex, once again set $\mu(u) = u$. Let there be vertex $v \notin$ a vortex. $(v, u) \in E(G) \implies (v, u) \in E(G') \implies (\mu(u), \mu(v)) \in E(G')$. ☐

We are now ready to prove theorem 33. By theorem 6 any minor closed class C excluding an apex graph can for some k be built by the clique sum of strongly k-almost embeddable graphs G. We will show that any such graph G, is the minor of a graph G' built by the clique sum of strongly $f(k^2 + k)$-almost embeddable graphs with $\Delta(G') = 3$. Taking the graph class of all such G', and taking its minor closure, we obtain a proper minor-closed graph class C' of $\Delta(C') = 3$ which contains C.

Rather than instantly give the final construction, it is more natural to see it produced step by step, adding more ingredients in each step. For each intermediate step we prove a few facts which we reuse in the next steps.

Let $C_1(k)$ be the class of graphs of genus $\leq k$, embeddable so each triangle bounds an open disc.

Let $C_2(k)$ be the class of graphs that can be obtained by adding at most k vortices of depth at most k to a graph of $C_1(k)$ (the graph of $C_1(k)$ embedded so that each triangle bounds an open disc of course).

Let $C_3(k)$ be the class of graphs that can be obtained by adding at most k apex vertices to a graph of $C_2(k)$, where the apex vertices may neighbor only other apex vertices and vortex vertices, i.e the class of strongly k-almost embeddable graphs.

Definition 75. Denote by $\oplus[C]$ the clique sum closure of class C. Denote by $\oplus^{\leq n}[C]$ the $\leq n$-sum closure of class C.

It is easy to see that much like planar graphs, $\Delta(C_1(k)) = 3$. We will add as few ingredients as possible; we will show that $\Delta(\oplus[C_1(k)]) = 3$. We will then show that $\oplus[C_2(k)]$ has a proper minor-closed superclass of $\Delta = 3$. We will then do the same for $\oplus[C_2(k)]$.

Proof. By [6], if a (finite) graph G is embedded on a surface, for any $v \in G$ there is an open disc D_v containing from G only v and an initial segment of edges incident to v [9]. Take the discs small enough that their boundaries do not intersect. Erase everything inside the closed disc D_v of v, then let $p_1, ..., p_k$ be the points where the boundary of the closed disc intersected the edges of v $e_1, ..., e_k$, ordered in a counterclockwise manner. Add the p_i back as embedded vertices v_i. Then, connect p_i with p_{i+1} by a curve running along the perimeter of the cycle. Call the resulting graph G'. Notice that $\Delta(G') \leq 3$ and $G' \geq_m G$, the model function is $\mu(v)$= all vertices of G' embedded on $D(v)$. $\square$

Definition 76. Given graph G, we call the graph $G' \geq_m G$ of maximum degree 3 as in the above proof the *fattening* or *ballooning* of G, and denote it $Bl(G)$. The cycle we replace vertex $v \in G$ with we denote by $Bl(v)$. [10]

We will prove that any graph G built by the clique sum of graphs of $C_1(k)$ is a minor of a G' built by the clique sum of graphs of $C_1(k)$ and $\Delta(G') = 3$. Let us develop a toolset to present theorem 38.

Definition 77. B is a *base* for C under $\leq n$-sums if $\oplus^{\leq n}[B] = C$. B is a *base* for C under clique sums if $\oplus[B] = C$.

Definition 78. Let $G' \geq_m G$, with model function μ. For clique $K \in G$, let its vertex set be $\{u_1, ...\}$, let there be isomorphic clique $K' \in G'$ with vertex set $\{u'_1, ...\}$ such that $u'_i \in \mu(u_i)$. We call K' a *representor clique* of K under μ.

Notice that clique representation is *transitive under minors*: If $G \leq_m G' \leq_m G''$ and K is a clique of G represented under μ by K' in G' and K' is represented under μ' in G'' by K'', then K is represented under $\mu \circ \mu'$ by K''. Also notice the following.

Proposition 8. *Let $G \leq_m G'$. If $K' \in G'$ is a representor clique of $K \in G$ under μ, we may remove from G' all $\mu(u) - \mu(v)$ edges, except the edges of K', for all distinct pairs $u, v \in K$ and still contain G as a minor under μ.* $\square$

[9] We may have to change the embedding a bit. Importantly, facial cycles remain same, and more generally the subgraphs induced by the boundary of faces remain same.

[10] This is also the model function showing $G' \geq_m G$

Almost entirely, in the following we want to restrict ourselves to a unique specific representor for each clique. This motivates the following definition.

Definition 79. Let $G \leq_m G'$ under μ. Correspond to some cliques in G a representor of theirs in G'. Call any such correspondence function from cliques in G to representor cliques in G' a *representation*. Call any 1-1 correspondence function a *1-1 representation* and if all cliques are represented call it *total*. Call the image of the correspondence function the set of *selected representors*.

We now give theorem 38.

Theorem 38. *Let there be a minor-closed class C closed under n-sums, such that $P_2 \square K_n \in C$. Let B be a base for C under $\leq n$-sums. For every graph G in B, let there be graph G' in C with*

- $G' \geq_m G$.
- *Every maximal clique in G has a representor clique in G'.*
- $\Delta(G') \leq d$.

Then $\Delta(C) \leq d$.

This theorem is a specialization of a more general theorem. For a maximal clique of a graph G, call its representor clique in $G' \geq_m G$ a *max representor clique*.

Definition 80. Given graphs $G,\ H$, their *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u, v) and (u', v') are adjacent if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Intuitively, for each vertex of H take a copy of G, and if two vertices in H are connected, connect the corresponding G copies by their identical vertices.

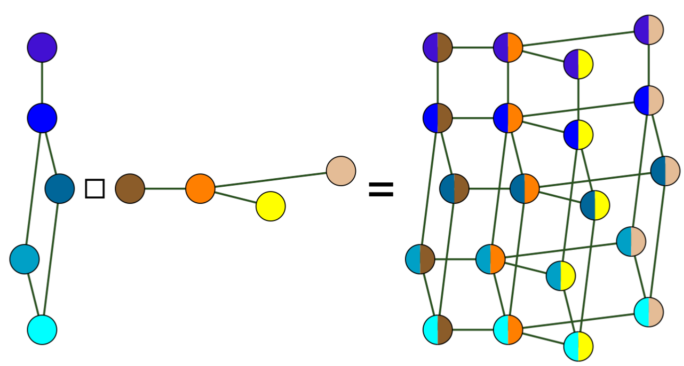


Figure 25: The Cartesian product of two graphs Courtesy: Wikipedia.

Definition 81. For $u \in G$, we call $G \square H$ limited to all vertices of the form (u, v) where $v \in H$, the H-subgraph of $V(G) \times V(H)$ corresponding to u.

Theorem 39. *Let there be a minor-closed class C closed under n-sums, such that $P_2 \square K_n \in C$. Let B be a base for C under $\leq n$-sums. For every graph G in B, let there be graph G' in C with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in G has a selected representor clique in G'.*
- *Every vertex v of G' of degree greater than d has degree at most $d - s$ if we remove for every selected max representor clique K it is in the edges of $G'[K]$, where s is the number of selected max representor cliques v is in.*

Then $\Delta(C) \leq d$.

This theorem is also a specialization of an even more general theorem! A *degree k expansion* of G is a graph $G' \geq_m G$ with $\Delta(G') = k$.

Theorem 40. *Let there be a class C' closed under n-sums, such that $P_2 \square K_n \in C'$. Let B be a base for minor-closed class C under $\leq n$-sums. For every graph G in B, let there be graph G' in C' with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in G has a selected representor clique in G'.*
- *Every vertex v of G' of degree greater than d has degree at most $d - s$, if we remove for every selected max representor clique K it is in the edges of $G'[K]$, where s is the number of selected max representor cliques v is in.*

Then every graph in C has an expansion of degree $\leq d$ in C'.

We remind one notation we use for clique sums: Given graphs G, H such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by the operation $G \cup H$. If $G \cap H = K$, denote this clique sum by $G \oplus_K H$.

Lemma 9. *Let $G = ((G_1 \oplus_{K_1} G_2) \oplus_{K_2} G_3) \oplus_{K_3} \ldots$ Let $G'_i \geq_m G_i$ be graphs with model function μ_i such that for every clique K of G_i, G'_i has a representor clique K'. Then $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3} \ldots =: G' \geq_m G$.* [11]

[11] $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3} \ldots$ is well-defined. If G_{i+1} is clique summed on $((G_1 \oplus_{K_1} G_2) \oplus \ldots \oplus_{K_{i-1}} G_i)$ on common clique K_i, then K_i must $\subseteq$ some graph G_j, $j < i$. $K_i \in G_j \implies K'_i \in G'_j \implies K'_i \in ((G'_1 \oplus_{K'_1} G'_2) \oplus \ldots \oplus_{K'_{i-1}} G'_i)$

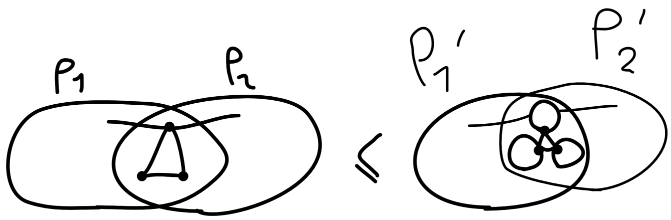


Figure 26: Example where K is a triangle. Graph $P_1 \oplus_K P_2$ is a minor of $P_1' \oplus_{K'} P_2'$

Proof. Call any K_j a *common clique*. We define the branches of G', i.e the model function μ from vertices in G to connected components of G': $\mu(v) := \bigcup_i \mu_i(v)$, where $\mu_i(v) = \emptyset$ if $v \notin G_i$.

If $v \in G$, $v \notin$ any common clique, let it only $\in G_i$, then $(u, v) \in G \implies (u, v) \in G_i \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch.

If $v \in$ some common clique K of G', then $(u, v) \in G \implies (u, v) \in$ one of the G_i containing $K \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch. $\square$

Lemma 10. *Let C be a graph class closed under n-clique-sums such that the graph product $K_n \square P_2$ is in C. Then $K_n \square T$ is in C for any tree T of more than 1 vertex.*

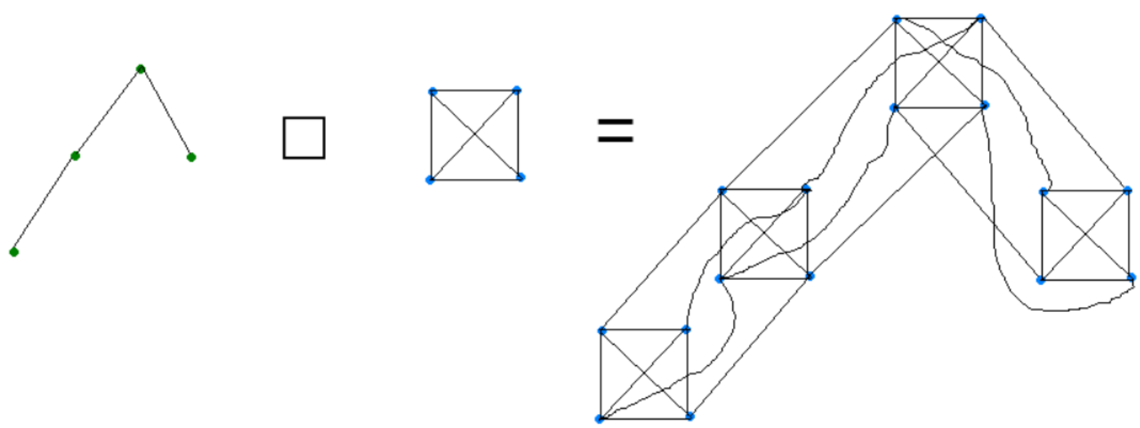


Figure 27: The cartesian product of a tree and a 4-clique, visualized.

The proof is conceptually very simple; imagine $K_n \square T$ as a tree where instead of vertices we have cliques. Much like we can create any tree by adding each of its edges one by one starting from the root in a DFS or BFS manner, we can create $K_n \square T$ by adding each of its n-cliques in the same order.

Proof. Let there be graph $K_n \square T$ some tree T. We have that $V(K_n \square T) = (V(T) \times \{1, ..., n\})$ and $((t_1, v_1), (t_2, v_2)) \in E(K_n \square T) \iff t_1 = t_2$ or (t_1 neighbors t_2 in T and $v_1 = v_2$).
The result is by induction of the number of vertices of T. If T is the edge graph, then the result holds trivially. Now let $K_n \square T$ for all T of some fixed number of vertices n. Let there be T' of $n+1$ vertices. This is constructed by some T of n vertices after adding a vertex t_2 to T and joining it to the correct vertex t_1. We have $K_n \square T \in C$. Clique sum either of the cliques of $K_n \square P_2$ to the clique of $K_n \square T$ corresponding to t_1, i.e to the subgraph of $K_n \square T$ induced by $\{(t_1, i) | i \in \{1, ..., n\}\}$. The resulting graph is (isomorphic to) $K_n \square T'$: Relabel the new n vertices as $(t_2, 1), ..., (t_2, n)$ and notice that (t_2, i) neighbors (t, j) iff $(t_2 = t)$ or t_2 neighbors t in T' and $i = j$). $\square$

We proceed with the proof of theorem 40.

Proof. Let there be graph G of C built by the clique sum of base graphs $G_1 \oplus_{K_1} ... \oplus_{K_k} G_k$. Suppose there exist graphs $G'_i \in C'$ with the aforementioned conditions, where μ_i is the model function for $G'_i \geq_m G_i$. Notice that since every maximal clique in G_i has a selected representor in G'_i, every clique in G_i has a representor in G'_i. By lemma 9, $(G'_1 \oplus_{K'_1} G'_2 \oplus_{K'_2} ... \oplus_{K'_k} G'_k) =: G' \geq_m G$, where $K'_i \in G'_{i+1}$ is a representor of K_i under μ_{i+1} and a representor of G_j under μ_j, G_j being the graph of G that G_i was clique summed while building G.
The common cliques K'_i of G' could have an arbitrarily large degree, so we make some adjustments. As $P_2 \square K_n \in C'$ and C' is closed under n-sums, by lemma 10 $T \square K_n \in C'$ where T is the $k+1$ comb graph. We remind we call the subclique of $T \square K_n$ corresponding to the ith spine vertex of the comb the ith spine clique, and the subclique of $T \square K_n$ corresponding to the *ith* hair vertex the ith hair clique. Furthermore, we call the sub-comb of $T \square K_n$ corresponding to ith vertex of K_n the ith comb running along $T \square K_n$.
To each selected max representor clique K' of G'_i, let K' have l vertices, l-sum a $P_2 \square K_l$, where P_2 is the path of two vertices. Call the l-clique of $P_2 \square K_l$ not used in the clique sum the *copy* of K'. To the copy of K', l-sum the first spine clique of a $T \square K_l$, to obtain $G''_i \in C'$. Call the $T \square K_l$ clique summed to the copy of K' its *representor comb*. $G''_i \geq_m G'_i$ of course, and let model function μ'_i showing that be $\mu'_i(v) = v$ if v is not in a max representor clique and if $v \in$ some max representor clique K, let v be the jth vertex of K, then let $\mu'_i(v)$ be the jth subcomb of the representor comb of K and the jth vertex of K.
By construction of μ'_i, if K' is a selected max representor clique of G'_i, all spine and hair cliques of the representor comb of K' in G''_i are representors of K' under μ'_i. We may use lemma 9 again; $(G''_1 \oplus G''_2 \oplus ... \oplus G''_k) =: G'' \geq_m G'$, where if during the construction of G' graph G'_i was clique summed on the subgraph G'_j on their common clique K'_i, then G''_i is clique summed on G''_j using the ith hair clique of the representor comb of K'_i in G'_i and the ith hair clique of the

representor comb of K_i' in G_j'.
Notice that lemma 9 gives us a specific model function μ' showing $G'' \geq_m G'$: The bag $\mu'(v)$ is the union of all $\mu_i'(v)$, if $v \in G_i$. By our choice of μ_i', we conclude that if v is in a selected max clique of G' under μ, let v be its jth vertex, then μ' puts in $\mu'(v)$ vertex v of G'' as well as the entire jth subcomb of its representor comb. Thus, by proposition 8, $G'' \geq_m G'$ even if for every selected max representor we remove edges with both endpoints in the representor, and for its representor comb we remove all edges with both endpoints on the same spine or hair clique, except from one such clique. Let G''' be G'' where we do just that, retaining only the edges of the last hair clique of every comb representor.
It suffices to prove that $\Delta(G''') \leq d$. As it turns out, we will need one more small change to do that. Let $v \in G'''$. We have the following cases.

- v does not belong to any representor comb or selected max clique of G'''. In this case, v also $\in G'$ and its degree remained unchanged during all of the above. $d_{G'''}(v) = d_G(v) \leq d$.

- v belongs to what was a selected max-clique representor K' in G'. If it has 1 vertex, then by construction $d_{G'''}(v) = 1$. For every selected max representor clique K' it was in, we removed the edges of $G'[K']$ and connected v to a copy of K', and made no other changes to the edges of v. By the conditions of the theorem, $d_{G'''}(v) \leq (d-s)+s = d$. Notice that $d_{G'''}(v) \leq d_{G'}(v)$, as the removal of each $G'[K']$ reduces the degree of v by 1 at least, so we need only consider v of $d_{G'}(v) > d$.

- v belongs to the spine clique of a comb representor. $d_{G'''}(v)$ is at most 3; It is incident precisely to an edge with endpoint the previous spine clique, the next spine clique if it has one, and its hair clique.

- v belongs to the hair clique of a comb representor. If the hair clique was not used in a clique sum and it is not the last hair clique, by construction $d_{G'''}(v)$=1. If it was used in a clique sum, by construction note that no hair clique is used in more than 1 clique sum, $d_{G'''}(v)$=2. If it is the last hair clique, let it have l vertices, then by construction v has degree l.

We now make changes to lower the degree of vertices of the last hair clique of a representor comb to 3, obtaining the intended claim. Let K be a last hair clique, let its edge set be $e_1, ..., e_m$. Let there be graph $P_m \square K$, where P_m is the path of m nodes. Let the K corresponding to the ith path vertex of $P_m \square K$ be called its ith clique, and the subpath corresponding to the ith clique vertex be the ith subpath running along $P_m \square K$. Clique sum to K the first clique of a $P_m \square K$. Then remove from the ith clique all edges with both endpoints in the clique except e_i. It is easy to see that all vertices of a $P_m \square K$ added in this manner have max degree 3, and by contracting the ith subpath running along the $P_m \square K$ we get G'''. Doing this for all hair cliques yields a graph G'''' with the required properties. $\square$

Using the previous lemmas, we can prove that $\Delta(\oplus[C_1(k)]) = 3$ fairly quickly.

Proposition 9. $\Delta(\oplus[C_1(k)]) = 3$.

Proof. We use theorem 38. The base B of $\oplus[C_1(k)]$ is of course $C_1(k)$. Let there be graph $G \in B$. We can assume that every triangle has an empty interior or exterior, else it is a separator and we can further decompose G to the clique sum of other base graphs. Let it be the interior, the other cases are analogous. On the open disc that has as boundary a triangle of G with vertex set abc, add a new triangle $a'b'c'$ embedded there, and connect a to a', b to b', c to c'. Let G' be the ballooning [12] $Bl(G)$, except we have not ballooned the vertices of any of the new triangles. Notice that $\Delta(G') = 3$. $G' \geq_m G$ by contracting each $Bl(v)$ back into v, and for each new triangle, $a'b'c'$ to a' to a, b' to b, c' to c. $a'b'c'$ in G' is a representor of abc in G. Let μ_1 be this model function. Each 2-clique $uv \in G$ has as representor the by construction unique $Bl(u) - Bl(v)$ edge of G'. By theorem 38, we have $\Delta(\oplus[C_1(k)]) = 3$. ☐

We now add the next ingredient, vortices. We will use theorem 40 to show that $\oplus[C_2(k)]$ has a degree 3 expansion in $C' = \oplus[C_2(2k)]$. [13]. In other words, for every $G \in \oplus[C_2(k)]$, there is $G' \in \oplus[C_2(2k)]$ with $G' \geq_m G$ and $\Delta(G') = 3$. Putting all those G' in a set, and taking the minor closure of the set, we obtain a minor-closed superclass of $\oplus[C_2(k)]$ of $\Delta = 3$ which is proper by theorem 37.

Proposition 10. $\Delta(\oplus[C_2(k)])$ *has a proper minor-closed superclass of* Δ *=3.*

Once again, the base is $C_2(k)$. Let there be graph G in $C_2(k)$, with embedded part $Emb(G)$ and at most k vortices of depth at most k added to pairwise disjoint facial cycles $C_1, ...C_k$.

Let G' be G with every vortex of depth d replaced by a simple vortex of depth $d+1$, as in proposition 7 and corollary 8. Use the model function defined there, call it μ_{sv}. Observe that there is a representation R_{sv} under μ_{sv}; if a clique K of G is in $Emb(G)$ trivially $R_{sv}(K) = K$. If a clique K of G is not in $Emb(G)$, it is in a vortex. In this case, let its facial cycle be $C = v_1v_2...$, then there must be a vortex bag B_{v_i} it is in. By construction of μ_{sv}, every vertex of B_{v_i} contains in its model in G' a distinct vertex of the simple vortex clique of v_i. But every vertex in the simple vortex clique of v_i is adjacent. $R_{sv}(K)$ is those simple vortex vertices.

As clique representation is transitive under minors, it suffices to find for every G' a graph $G'' \geq_m G'$ of $\oplus[C_2(2k+1)]$ such that there is a representation under some model function μ satisfying the conditions of theorem 40. Then, there will be such a representation for $G'' \geq_m G$ under $\mu \circ \mu_{sv}$.

Add triangles and repeat the same fattening procedure as before on $Emb(G)$ to obtain $Emb(G)'$. This time, rather than add 1 extra triangle $a'b'c'$ to the empty face of triangle abc of $Emb(G)$, we add two triangles $a'b'c'$ and $a''b''c''$,

[12]We remind a ballooning or fattening of G means to replace each vertex v with a cycle C embedded on the boundary of an open disc around the vertex, the vertices of C connected in a clockwise manner and each vertex of C adjacent to a single neighbor of v.

[13]In fact, we can show that $\Delta(\oplus[C_2(k)])$=3

$a'b'c'$ embedded on the empty face bounded by abc, $a''b''c''$ on the empty face bounded by $a'b'c'$, a joined to a', a' joined to a'' and so on. Both new triangles are not fattened. Call $a'b'c'$ and $a''b''c''$ the first and second *copies* of abc respectively. Fortunately, after fattening facial cycles are (almost) retained:

Definition 82. For $v \in Emb(G)$, let D_v be the disc on the boundary of which the cycle $Bl(v)$ was embedded on. Let $Bl(v \to u)$ or $Bl(u \leftarrow v)$ be the vertex of $Bl(v)$ incident to the unique $Bl(v) - Bl(u)$ edge of $Emb(G)'$.

If $C = u_1...u_n$, where $n > 3$ is a facial cycle in $Emb(G)$, then there is a facial cycle C'' in $Emb(G)'$, first with 1 or 2 vertices from $Bl(u_1)$, then with vertices from $Bl(u_2)$, and so on: Start from the vertex $Bl(u_1 \to u_2)$. Follow the $Bl(u_1)-Bl(u_2)$ edge to $Bl(u_2 \to u_1)$. If $d_{emb(G)}(u_2) > 2$, there is an edge $Bl(u_1 \leftarrow u_2) - Bl(u_2 \to u_3)$ in $Bl(u_2)$. Follow along it. Then take the $Bl(u_2 \to u_3)$ edge and so on. Call C'' the *corresponding* facial cycle of C. For triangles of $Emb(G)$ call their second copy in $Emb(G)'$ the corresponding facial cycle.
If to construct G' a simple vortex of depth k was added to a facial cycle of $Emb(G)$, add to the corresponding facial cycle of $Emb(G)'$ a simple vortex of depth k to obtain G''.
We prove G'' fulfils the conditions of theorem 40.

- To prove that $G'' \geq_m G'$, let μ_2 be the model function showing that, for v in the embedded part of G'' let $\mu_2(v) = \mu_1(v)$, where $\mu_1(v)$ is the model function of the proof that $\Delta(\oplus[C_1(k)]) = 3$, modified by putting a'' in the same bag as a' and a for triangles $abc \in G'$ of course. For $v \in$ a vortex, let the facial cycle be $C = v_1v_2...$ and let v belong to the simple vortex clique of v_i, let v be the ith vertex of the clique. Let C'' be the corresponding facial cycle and notice C'' of G'' is also in $Emb(G'') = Emb(G)'$. If $C = v_1v_2v_3$, then $C'' = v_1''v_2''v_3''$ and let $\mu_2(v)$ be the ith vertex of the simple vortex clique of v_i''. Else, set $\mu_2(v)$ to be the ith vertices of the vortex cliques of $Bl(v_{i-1} \leftarrow v_i)$ and $Bl(v_i \to v_{i+1})$. It is easy to observe that the contraction in G'' of each minor bag $\mu(v)$ yields G'.

- We find a representation R_2 under μ_2 so each maximal clique K is represented. For a cliques K of $Emb(G)$, set $R_2(K) = R_1(K)$, where for triangles K we use their first copy in G'' to represent them.
  With regard to simple vortex cliques K of G', let the simple vortex be of depth l and added on the facial cycle $C = u_1u_2...u_n$. There are precisely n maximal cliques of $2l$ vertices; the simple vortex clique of $u_i \cup$ the simple vortex clique of u_{i+1}, for $i \in \{1, ..., n\}$, where $u_{n+1} = u_1$. Its selected representor $R(K)$ in G'' is the simple vortex clique of $Bl(u_i \to u_{i+1}) \cup$ the simple vortex clique of $Bl(u_i \leftarrow u_{i+1})$.

- We prove the third condition. If $v \in G''$, is not in a vortex, then by construction it has max degree 3 unless if it is in the first copy $a'b'c'$ of a triangle abc. In this case it is a selected representor of abc, and it represents no other cliques. For the condition to be satisfied it must have at most $3-1$ edges adjacent to it, after removing the edges of $a'b'c'$, which is the case. If v is in a vortex, notice that all edges of the vortex have both endpoints in a selected max clique representor, and v belongs to exactly 2 selected representors. After removing the edges of the selected cliques, $d(v) = 1$ if v is on the facial cycle, and $d(v) = 0$ otherwise, satisfying the condition.

Therefore, every $G \in \oplus[C_2(k)]$ has a degree 3 expansion in $G' \in \oplus[C_2(2k)]$. Taking the minor closure of all such G', we obtain a proper minor-closed class of Δ 3 containing $\oplus[C_2(k)]$.

We now add the final ingredient, apex vertices only neighboring other apex vertices and vortex vertices. We will prove that $\oplus[C_3(k)]$, i.e the clique sum closure of strongly k-almost embeddable graphs has a proper minor-closed superclass of $\Delta = 3$. By theorem **??**, we thus obtain the the right direction of theorem 33.

Proposition 11. *$\oplus[C_3(k)]$ has a proper minor closed superclass of $\Delta = 3$.*

Let $G \in C_3(k)$. We will find an expansion of G in $C_3(k^2 + k)$, satisfying the conditions of theorem 40. Naturally, the base B is once again $C_3(k)$ and C' is $\oplus[C_3(k^2 + k)]$. It suffices to consider only G whose apex vertices neighbor all other apex vertices and all vortex vertices. All other graphs in $C_3(k)$ are subgraphs of such graphs and if $G_1 \subseteq G_2 \leq_m G'$ where $G_2 \leq_m G'$ has a representation under μ satisfying the conditions of theorem 40, so does $G_1 \leq_m G'$.
Let C be a facial cycle of $Emb(G)$. Let G' be G where instead of adding a vortex of depth k, we add a simple vortex of depth $k+1$ to C, and then connect all apex vertices to it. As in the previous proposition, $G' \geq_m G$ under a model function μ_{sv}, and there is a total representation r under μ_{sv}: If K is a clique not intersecting the apex vertices, $r(K) = R_{sv}(K)$ as we have already explained in the previous proposition. If K intersects only apex vertices, then trivially $r(K) = \mu_{sv}(K) = K$. If K intersects apex and the simple vortex's vertices, let the subcliques comprised by those vertices be K_a and K_{sv} respectively, then $r(K_a) = K_a$, and $r(K_{sv}) = R_{sv}(K_{sv})$.
Therefore it suffices to prove theorem 40 for G' in the place of G. We now construct the expansion G''' of G' with the desired properties; let G'' be defined exactly as in the previous proposition (fatten $emb(G)$ as in the previous proposition, adding two copies to the empty face of each triangle), apex vertices neighboring all vortex vertices and all other apex vertices.We still have to lower the degree of apex vertices.

Definition 83. Define the cycle induced by the ith vertex of all simple vortex cliques of a simple vortex to be the *ith layer* of the simple vortex. We always have C be the 1st layer of the simple vortex.

We replace each simple vortex of depth $k+1$ of G'' with a simple vortex of depth $2k+1$. Apex vertices no longer neighbor all vortex vertices; instead, give some ordering to the apex vertices, the ith apex vertex neighbors a single vertex of the $k + 1 + i$th layer of the first clique of the simple vortex. Finally, for each apex vertex a, add to G'' a path of $a_1 a_2 ... a_{k+1}$, identify a with a_1, remove the edge between a and its ith vortex neighbor and have the ith vortex neighbor be adjacent to a_{i+1} instead. Call this the *representor path* of a. This completes the construction of G'''. Notice that, treating the vertices of path representors as apex vertices, $G''' \in C_3(k(k+1))$ It now suffices to prove the three conditions of theorem 40.

- $G''' \geq_m G'$: For the ith apex vertex v of G', let $\mu_3(v)$ be the ith apex vertex of G''' together with its representor path, together with the $(k+1+i)$th layer of all simple vortices. Otherwise, let $\mu_3(v)$ be $\mu_2(v)$ as in the previous proposition.

- Let $R_3(K)$ be the representation. By the previous proposition, we have that every maximal clique K not having apex vertices has a representation $R_2(K)$. Let $R_3(K) = R_2(K)$ in this case. If K is the set of all apex vertices of G', then $R_3(K) = K$. If K=$K_a \cup K_{sv}$ is a set of apex vertices and simple vortex vertices of G', which by construction and maximality of K must consist precisely of all apex vertices and the simple vortex cliques of two consecutive facial cycle vertices, let them be c_i and c_{i+1}, then $R(K)$ is the two simple vortex cliques of c_i and c_{i+1} in G''.

- If $v \in G'''$ is an original apex vertex, then it belongs to a single max selected representor, that of all apex vertices. It has degree 1 excluding edges from that clique. If it does not, but still belongs to a path representor of an apex vertex, then it has degree 3 and belongs to no representor clique. If v is not an apex vertex, the same as in the previous proposition holds.

This completes the proof of the right direction of theorem 33.

7 Existence of countably infinite K_5-universal graph.

Given a class of infinite countable graphs C, a universal graph G is a graph such that $G >_m G'$ for all $G' \in C$. In [6], Georgakopoulos proved that there is a universal K_5-minor-free graph. The following is a simplification of this result.

Theorem 41. *There is a universal K_5 minor free graph.*

In [6], Georgakopoulos proved the existence of a countably infinite K_5 universal graph with regard to the minor relationship. I later reproved his result with a somewhat simpler construction. This section is a sketch of this proof.

For the remainder of this proof, we may assume without loss of generality that clique sum operations do not remove edges of the clique.

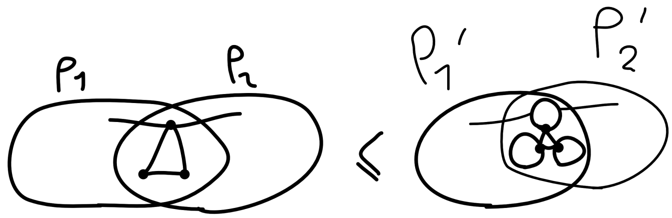
Let $\mathrm{K_5 f}$ be an infinite $\mathrm{K_5}$ free graph. By the paper of Thomas and Kritz [10], there exists a sequence $\{G_\alpha\}_{\alpha \leq \text{countable } \lambda}$ such that $G_{a+1} = G_a \oplus^3 \ P_a$ where P_a is planar (or w[8]) and $G_\lambda = K_5 f$ and $G_a = \liminf_{\beta < a} G_\beta$. Let $\{P_\alpha\}_{\alpha \leq \lambda}$ be the corresponding planar graphs (or w[8]). Let $P_{N(0)}, P_{N(1)}, ...$ be some enumeration of them. We print P_0, then dovetail the enumeration and print $P_{N(i)}$ once the ≤ 3 nodes it was clique-summed on during the construction of $K_5 f$ have already been printed (don't print already printed $P_{N(i)}$). Seeing clique sums as a union of graphs, it is easily seen that an ordering $\{P_\alpha\}_{\alpha \leq \omega}$ arises such that $G_0 = P_0$, $G_{a+1} = G_a \oplus^3 P_{a+1}$ and $G_\omega = K_5 f$. More generally,

Theorem 42. *Let a countable graph be k-summable over some Γ for some finite k, let the corresponding sequence be $\{G_\alpha\}_{\alpha \leq countable\ \lambda}$. Then there also exists such a sequence of the form $\{G_\alpha\}_{\alpha \leq \omega}$*

In the case clique sums remove edges this still holds. Break $\{G_\alpha\}_{\alpha \leq \text{countable } \lambda}$ in two sequences, one not removing and the other only removing edges.

So let $\mathrm{K_5 f} = ((P_1 \oplus_{\Delta_1} P_2) \oplus_{\Delta_2} P_3) \oplus_{\Delta_3}$ ..., for a class of countable planars P_i (or w[8]).

Lemma 11. *Let $G = ((P_1 \oplus_{\Delta_1} P_2) \oplus_{\Delta_2} P_3) \oplus_{\Delta_3} \ldots$ for arbitrary countable graphs P_i and cliques Δ_i, where for some $k \in \mathbb{N}$ all Δ_i are of size at most k . Let $P_i' > P_i$ be graphs such that for every clique Δ of P_i of size $\leq k$, ${P_i'}^{\Delta}$ has a clique Δ' with one node in each branch. Then $((P_1' \oplus_{\Delta_1'} P_2') \oplus_{\Delta_2'} P_3') \oplus_{\Delta_3'} ... =: G' >_m G$.*

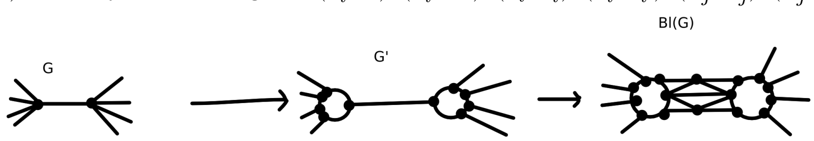


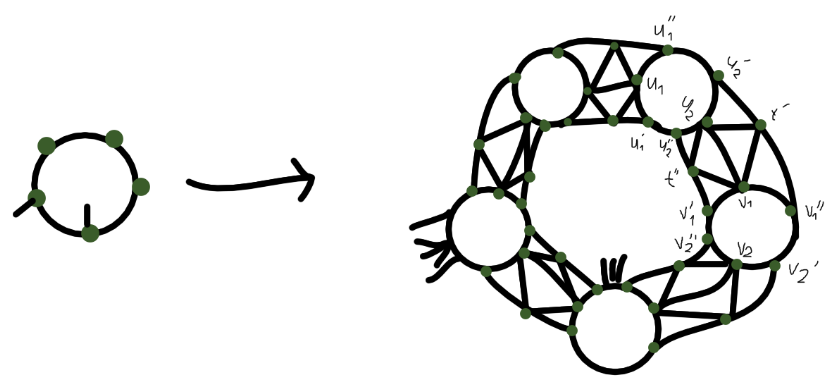
Proof. We define the branches of G' forming G. Let $v \notin$ any common clique, let it only $\in P_i$. Then $G'^v := {P_i'}^v$. Let $v \in$ some common clique Δ. Then ${G'}^v := \bigcup_{P_i \supseteq \Delta} {P_i'}^v$.

If $v \in G$, $v \notin$ any common clique, let it only $\in P_i$, then $(u,v) \in G \implies (u,v) \in P_i \implies {P_i'}^u, {P_i'}^v$ touch $\implies {G'}^u, {G'}^v$ touch.

If $v \in$ some common clique Δ, then $(u,v) \in G \implies (u,v) \in$ one of the planar P_i containing $\Delta \implies {P_i'}^u, {P_i'}^v$ touch $\implies {G'}^u, {G'}^v$ touch. $\square$

We now begin to construct the universal K_5-minor free graph. For a countable locally finite planar graph G, we inflate the nodes of G to obtain G': Take a generous embedding of G, and for every node v, take an open ball containing only v and its edges, delete the inside of the ball, and put a new vertex on the $\deg(v)$ points the edges of v first intersect the boundary, let these nodes be $v_1, v_2, \ldots$. Connect them in clockwise order around the boundary, with edges embedded on the boundary. Clearly G' remains planar and $G' > G$ by contracting the v_i. We inflate edges of G' to obtain $Bl(G)$. For every edge $(v_i, u_j), u \neq v$, notice there can only be one such edge for each vertex, add a node before and after v_i in the boundary, let them be v_i', v_i'', repeat for u_j then connect v_i' with u_j'' and v_i'' with u_j'. Then subdivide (v_i', u_j''), (v_i'', u_j') to add a new node to each, let it be t', t'' and connect the new nodes to v_i and u_j. $Bl(G)$ remains planar and $Bl(G) > G'$ by contracting the $(v_i', t'), (v_i'', t''), (v_i, v_i'), (v_i', v_i''), (u_j, u_j'), (u_j', u_j'')$.





Let $Bl(U_p)$ be any universal planar graph U_p inflated as above.

Claim 2. Let P be planar. $Bl(U_p)^P$ has a triangle Δ' with one vertex in each branch of $Bl(U_p)^\Delta$, for all $\Delta \in P$.

Proof. Let $\Delta = xyz \in P$. Pick a subpath of each of the three branch sets of U_p^Δ to form a minimal K_3 minor of P, let them be P_x, P_y, P_z. The subpaths can be chosen so that the minimal K_3 minor contains no node or edge of U_p^P

embedded on one of its two sides, w.l.g let it be the interior. Notice that the inner circle C_{in} of the fattened K_3 minimal minor thus contains no node or edge of $Bl(U_p)^P$. It is thus easy to see that $Bl(U_p)^P \setminus C_{in} > U_p^P > P$. Let uv be the $P_x - P_y$ edge of the K_3 minimal minor in U_p^P. By construction of $Bl(U_p)^P$, there is an edge (u_i, v_j) between $Bl(U_p)^u$ and $Bl(U_p)^v$ and they both neighbor an inner circle node t''. By reallocating C_{in} to $Bl(U_p)^{P_z}$, we have the desired triangle. □

We now define the universal K_5-free graph U_{K_5f}. Let $Bl(U_p)[1] := Bl(U_p)$. Let $Bl(U_p)[i+1]$ be $Bl(U_p)[i]$ clique summed with $Bl(U_p)$ or W[8] over all possible clique pairs. $\mathrm{U}_{K_5f} := \bigcup_{i=1}^{\infty} Bl(U_p)[i]$.

Theorem 43. *U_{K_5f} is a universal K_5-free graph.*

Proof. Let K_5f be any K_5-free graph, K_5f $= ((P_1 \oplus P_2) \oplus P_3) \oplus$ ... Notice that $Bl(U_p)$ has the properties of P_i' of lemma 1. It follows that, let $P_i' := Bl(U_p)$ for all i, K_5f ' $= ((P_1' \oplus P_2') \oplus P_3') \oplus$ ... for suitably selected cliques contains K_5f as a minor. But by definition of U_{K_5f}, K_5f ' is contained in it as a subgraph. □

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