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**DEPARTMENT OF INFORMATICS AND TELECOMMUNICATIONS**

**MASTER'S PROGRAM "ALGORITHMS, LOGIC AND DISCRETE MATHEMATICS"**


**MSc THESIS**

# Splittability within minor-closed classes to graphs of low maximum degree.


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**ΣΧΟΛΗ ΘΕΤΙΚΩΝ ΕΠΙΣΤΗΜΩΝ**
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# Διασπάσεις εντός κλάσσεων κλειστών υπό ελάσσονα προς γραφήματα χαμηλού μέγιστου βαθμού

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## ABSTRACT


It is easy to see that every planar graph is a minor of another planar graph of maximum degree 3. Georgakopoulos proved that every finite $K_5$-minor free graph is a minor of another $K_5$-minor-free graph of maximum degree 22, and inquired if this is smallest possible. This motivates the following generalization: Let $C$ be a minor-closed class. What is the minimum $k$ such that any graph in $C$ is a minor of a graph in $C$ of maximum degree $k$? Denote the minimum by $\Delta(C)$ and set it to be $\infty$ if no such $k$ exists.

We explore the value of $\Delta(C)$ for various minor closed classes, and eventually prove that a minor-closed class $C$ excludes an apex graph if and only if there exists a proper minor-closed superclass $C'$ of $C$ with $\Delta(C') = 3$ if and only if there exists a proper minor-closed superclass $C'$ of $C$ with finite $\Delta(C')$. This complements a list of 5 other characterizations of the minor-closed classes excluding an apex graph by Dujmovic, Morin and Wood.

Furthermore, we extend and simplify Markov and Shi's result that not every graph of treewidth $\leq$ k has a degree 3 expansion of treewidth $\leq k$. Finally, we simplify Georgakopoulos' proof on the existence of a countable universal graph of $Forb(K_5)$.



**SUBJECT AREA**: Structural Graph Theory

**KEYWORDS**: minor-closed classes, splittings, maximum degree, graph minor structure theorem

# ΠΕΡΙΛΗΨΗ

Είναι εύκολο να δει κανείς ότι κάθε επίπεδο γράφημα είναι έλασσον ενός επιπέδου γραφήματος μέγιστου βαθμού 3. Ο Γεωργακόπουλος απέδειξε ότι κάθε γράφημα που εξαιρεί το $\mathrm{K}_5$ ως έλασσον είναι έλασσον ενός άλλου γραφήματος που εξαιρεί το $\mathrm{K}_5$ ως έλασσον μέγιστου βαθμού 22, και ρώτησε αν αυτός είναι ο ελάχιστος δυνατόν.
Αυτό παρακινεί την εξής γενίκευση. Έστω $C$ μία κλάση κλειστή υπό ελάσσονα. Ποιο είναι το ελάχιστο $k$ έτσι ώστε οποιοδήποτε γράφημα της $C$ είναι έλασον ενός γραφήματος της $C$ μέγιστου βαθμού $k$; Συμβολίζουμε το ελάχιστο με $\Delta(C)$ και θέτουμε την τιμή του σε $\infty$ εάν δεν υπάρχει τέτοιο $k$.
Εξερευνούμε την τιμή του $\Delta(C)$ για ποικίλες κλάσεις κλειστές υπό ελάσσονα και τελικά αποδεικνύουμε ότι μια κλάση κλειστή υπό ελάσσονα $C$ αποκλείει ένα απόγειο γράφημα ως έλασσον εάν και μόνο εάν υπάρχει μια κλειστή υπό ελάσσονα υπερκλάση $C'$ της $C$ με $\Delta(C') = 3$ εάν και μόνο εάν υπ άρχει κλειστή υπό ελάσσονα υπερκλάση $C'$ με πεπερασμένο $\Delta(C')$. Αυτό επαυξάνει μια λίστα με 5 άλλους χαρακτηρισμούς των κλάσεων κλειστών υπό ελάσσονα που αποκλείουν ένα απόγειο γράφημα από τους Dujmovic, Morin και Wood.
Επιπλέον, επεκτείνουμε και απλοποιούμε το αποτέλεσμα των Markov και Shi ότι δεν έχει κάθε γράφημα δενδροπλάτους $\leq$ k διάσπαση μέγιστου βαθμού 3 και δενδροπλάτους $\leq k$.
Τέλος, απλοποιούμε την απόδειξη του Γεωργακόπουλου για την ύπαρξη ενός αριθμήσιμα άπειρου καθολικού γραφήματος για την $Forb(K_5)$.



**ΘΕΜΑΤΙΚΗ ΠΕΡΙΟΧΗ**: Δομική θεωρία γραφημάτων

**ΛΕΞΕΙΣ ΚΛΕΙΔΙΑ**: κλάσσεις κλειστές υπό ελάσσονα, διασπάσεις, μέγιστος βαθμός, δομικό θεώρημα ελάσσονων γραφημάτων

*Στον Γιάννη και πλειότερο στον Φίλιππο.*
*Γιατί ήταν ό,τι δεν ήταν οι υπόλοιποι.*

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# 1. INTRODUCTION

The organization of this text is as follows: Chapter 1 contains the introduction and lists our results. Chapter 2 contains a minimal preliminaries section, complemented by an extended appendix. Chapter 3 contains an overview of proofs and techniques, followed by an extensive section with the full proofs in chapter 4; if a topic is covered by both chapters 3 and 4, preferring chapter 3 is *highly* recommended. Chapters 5 and 6 contains the two directions of our main proof.

## 1.1 The graph class parameter $\Delta$

One may observe that every planar graph is a minor of another planar graph of maximum degree 3. Can the reader see why? Figure 3.1 gives a simple explanation.

In 2021, Georgakopoulos observed that every $K_5$-minor free graph is a minor of another $K_5$-minor-free graph of maximum degree 22, but did not find if this is smallest possible. A graph $G'$ including $G$ as a minor is a *splitting* of $G$.
This motivates the following question [1]: Let $C$ be a minor-closed class. What is the minimum $k$ such that any graph in $C$ is a minor of a graph in $C$ of maximum degree $k$? Denote the minimum by $\Delta(C)$ and set it to be $\infty$ if no such $k$ exists. This is a general, yet elegant definition. We are thus interested in it and this text is devoted in exploring its properties. All results, established mostly through my work, are original.

As it turns out, it is easy to show that $\Delta(Forb(K_5))$ also is equal to 3. One may then ask if there is a class whose $\Delta$ does not fall down to 3. Note that there are classes of $\Delta(C) \leq 2$, but all of them consist of disjoint unions of circles and paths, and we don't care for such trivial classes. Similarly, we don't consider finite classes.

The answer is negative, $\Delta(Forb(K_{3,3}))$ being equal to 4. In fact, for any $k \geq 3$, the minor-closed class $TW_{\leq k}$ of graphs of treewidth $\leq k$ has $\Delta(TW_{\leq k}) = k$. As implied in the definition of $\Delta(C)$, there are also classes $C$ for which no $k$ exists so that every graph in $C$ is a minor of a graph in $C$ of maximum degree $k$. The class of stars [1] $\{K_{1,k} | k \in \mathbb{Z}_{\geq 0}\}$ has $\Delta = \infty$, because the only way to include a star as a minor is to use a bigger star.

Of more interest to me are structural question that might arise. Let $C$ be a minor-closed class, and change an excluded minor "a little bit" to obtain another class $C'$. Given $\Delta(C)$, can we say something about $\Delta(C')$? If there is an elegant way to approach this question, it evades me. What if $C'$ is just any superclass of $C$? Is $\Delta$ an increasing function

[1] Technically, this is not a minor-closed class. No matter; take the minor-closure of stars instead, which is almost same.

perhaps, i.e $C \subseteq C' \implies \Delta(C) \leq \Delta(C')$? Not the case; the planar graphs are a superset of the class of stars. The apex graphs in turn include the planar graphs, but as we will see they have $\Delta = \infty$, so it is not decreasing either. The function $\Delta$ does not seem to have any clear general pattern at first glance.

Georgakopoulos conjectured that at least every proper minor-closed class $C$ has a proper minor-closed superclass $C$' of finite $\Delta(C')$. We proved this conjecture to be wrong in fact;

**Theorem 1.** *If a proper minor-closed graph class $C \supseteq$ the apex graphs, then $\Delta(C) = \infty$.*

Now, we may ask if there is a strict subclass of the apex graphs with the property that all classes above it have $\Delta(C) = \infty$. As far as smaller classes are concerned, we do already have that such a class would have to include all planar graphs; By a known theorem, if minor-closed $C$ excludes a planar graph, it is a subclass of $TW_{\leq k}$ for some $k$. So such a class must include all planar graphs, but not all apex graphs. Can we make the ”floor” of the planar graphs and the ”ceiling” of apex graphs collapse on each other? As it turns out, apex graphs are the cutoff.

**Theorem 2.** *For a proper minor-closed class $C$, the following are equivalent:*

1. *$C$ excludes an apex graph;*
2. *there is a minor-closed superclass $C' \supseteq C$ such that $\Delta(C')$ is finite.*
3. *there is a minor-closed superclass $C' \supseteq C$ such that $\Delta(C') = 3$;*

Note that apex-minor-free graphs arise in a variety of settings. In particular, for a number of graph parameters $f$, a minor-closed class $C$ has bounded $f$ if and only if some apex graph is not in $C$ (see [2, 3, 4, 5] for examples).

The above statement still holds if, for some fixed constant $k \geq 3$, instead of $\Delta(C') = 3$ we demand $\Delta(C') = k$ or if we instead demand $\Delta(C') \leq k$. It still holds if for any of the equivalent cases, we further demand that $C'$ also excludes an apex graph as a minor.

### 1.2 Other results

Theorem 2 isn't the only result that has ties to the bibliography: Markov and Shi [6] proved that for every graph $G$ there is a graph $G'$ with maximum degree 3 such that $G$ is a minor of $G'$ and $TW(G') \leq TW(G) + 1$. Moreover, this treewidth bound is best possible for $TW(G) \geq 18$.

In particular, for $k \geq 18$, Markov and Shi [6] constructed a graph $G$ of treewidth $k$ such that if $G$ is a minor of a graph $G'$ with maximum degree 3, then $TW(G') \geq k+1$. In our terminology, for $k \geq 18$, $\Delta(TW_{\leq k}) \geq 4$. The aforementioned result that for $k \geq 3$, $\Delta(TW_{\leq k}) = k$ extends this. As it turned out the construction used for my proof is similar to theirs. The proof in this text could be considered notionally simpler.

Let's mention other results of this thesis. The linklessly embeddable graphs $\mathcal{L}$ are a well studied 3-dimensional analogue of the planar graphs [7]. It is reasonable to ask if, like with planar graphs, one may by some geometric argument replace each node of a linklessly embeddable graph $G$ by some bounded-degree graph to show that $\Delta(\mathcal{L}) = 3$ or at least finite. But since the linklessly embeddable graphs are a superclass of the apex graphs, by theorem 1 the answer is negative:

**Corollary 1.** *$\Delta(\mathcal{L})=\infty$.*

Likewise, by theorem 1 we have the following.

**Corollary 2.** *$\Delta(Forb(K_n)) = \infty$ for $n \geq 6$, $\Delta(Forb(K_{n,n})) = \infty$ for $n \geq 4$.*

Finally I simplify Georgakopoulos' proof that there is a universal graph for the class of countably infinite $K_5$-minor-free graphs [1]. A universal graph for a class of infinite graphs $C$ is a graph in $C$ that includes all graphs in $C$ as minors, and it is interesting in the sense that it serves as a representative for the entire class. Universal graphs and related problems have been studied in the literature [8], [9].

Other results in this text is that the class of outerplanar and series-parallel graphs have $\Delta = 3$, and that for $k \geq 3$ the class of graphs of pathwidth at most $k$, $PW_{\leq k}$, has $\Delta(PW_{\leq k}) = k$.

# 2. DEFINITIONS AND PRELIMINARIES

Originally, the aim in this section was to collect and introduce, in a rigorous manner from the ground up, all notions needed during this thesis or at least to clarify what is left to common sense or used as a black box. As a byproduct, it was quite large and for this reason it has been moved to the appendix, which the reader may check as needed. A minimal version is here instead.

## 2.1 Preliminaries

All graphs are simple and undirected. All graphs are finite unless stated otherwise. Though the focus of this thesis is on finite graphs, a result on infinite graphs is also presented. All infinite graphs are countable. The reader may also refer to Diestel [10], the standard reference book.

### 2.1.1 Basics

If $F$ is a set of pairs of vertices of $G$, we define $G-F$ to be the graph $(V(G), E(G)\setminus F)$, and $G+F$ to be $(V(G), E(G)\cup F)$. In an abuse of notation, $G-e := G-\{e\}$ and $G+e := G+\{e\}$. To *join vertex $u$ to vertex $v$* in $G$ means to add $(u,v)$ to $G$. To *join subgraph $S_1$ to subgraph $S_2$* of $G$ means to join $(u,v)$ in $G$ for all $u \in S_1, v \in S_2$.

Given graphs $G_1,\ G_2$ we define the *disjoint union of $G_1$ and $G_2$* , denoted $G_1 + G_2$, to be $G_1 \cup G'_2$ where $G'_2$ is a graph isomorphic to $G_2$ so that $V(G_1) \cap V(G'_2) = \emptyset$.

For subgraphs $S_1, S_2$ of a graph $G$, an *$S_1, S_2$ edge* is an edge with one endpoint on $S_1$ and one endpoint on $S_2$. We say that *$S_1, S_2$ are adjacent or neighbors* if there is an $S_1, S_2$ edge in $G$.

### 2.1.2 Minors

Given a graph $G$ and a (possibly single-vertex) connected subgraph $S$ of $G$, the *contraction $G/S$* is the graph obtained from $G - V(S)$ by adding a new vertex $v_S$ adjacent to every neighbour of $S$ in $V(G)\setminus V(S)$. We say $G/S$ is obtained from $G$ by *contracting* $S$. Given a set of vertices $U$ of $G$ such that $G[U]$ is connected, the contraction of $U$ is defined to be the contraction of $G[U]$.

Let $G$ and $G'$ be graphs. Assume that for some subgraph $R$ of $G$ there are pairwise disjoint subgraphs $R_1, R_2, ..., R_{|V(G')|}$ of $R$ and there is a bijection $R_1 \leftrightarrow v_1$, $R_2 \leftrightarrow v_2$, ..., $R_{|V(G')|} \leftrightarrow v_{|V(G')|}$, where $V(G') = \{v_1, ..., v_{|V(G')|}\}$, such that $(v_i, v_j) \in$

$E(G')$ iff $R_i, R_j$ are adjacent. Then $G$ *contains* $G'$ *a minor*, denoted $G \geq_m G'$. $G$ is called an *expansion* or *splitting* of $G'$.

A bijection $\mu(v_i) = R_i$ as above, is called a *model* of $G'$ in $G$. We call $R_i$ the *bag* or *branch* of $v_i$ in $G$ and also denote it $B(v_i)$ or $\mu(v_i)$. For $H \subseteq G$, we denote with $\mu(H)$ the subgraph of $G$ induced by the $\cup_{v \in V(H)} \mu(v)$.

Given a graph class $C$, denote by *minor-closure*$(C)$ the set $\{G : G \leq_m G'$ for some $G' \in C\}$.

By the famous Robertson-Seymour theorem, every class closed under minors can be characterized by a finite set of forbidden minors. If the excluded minors of $G$ are $H_1, H_2, ...$, we may denote $C$ by $Forb(H_1, H_2, ...)$.

### 2.1.3 Apex graphs

A graph is *apex* if it is planar or becomes planar after the removal of a single vertex. Given a graph class $C$, a graph is apex-$C$ if it is in $C$ or if there is a vertex whose removal makes $G$ belong to $C$.

### 2.1.4 Graphs on Surfaces

The reader is probably already familiar with planar graphs. Some of the most deep results in minor theory mention graphs embeddable on surfaces more complex than the plane or the sphere, such as the torus. See the index for an exhaustive list of definitions and discussions.

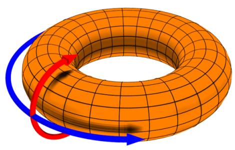

**Figure 2.1: The torus. Courtesy: Wikipedia.**

Much like graphs can be embedded on the plane, they can be embedded on topological surfaces. A surface is a connected compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$. Mohar and Thomassen's Topological graph theory [11] provides for a rigorous introduction to the topic.

A graph is *embeddable* on a surface if we can draw it on the surface so that edges do not intersect:

**Definition 1.** A graph $G$ is *embeddable* on surface $X$ if there is a function $f$ mapping vertices to elements of $X$, and edges to simple curves on $X$ so that $f(v_1) \neq f(v_2)$ for $v_1 \neq v_2$, and curve $f(uv)$ connects $f(u)$ and $f(v)$, and has no intersection with the image of other vertices and only intersects other edges on its endpoints.
$f$ is an *embeddeding* of $G$ on $X$. The image of $f$, $f[(V(G) \cup E(G))]$, is called the *embedded graph*, and though it is technically not a graph, one may produce a graph from one in the obvious manner. For ease of notation, the embedded graph is also abusively denoted $f(G)$.

A graph embeddable on $\mathbb{R}^2$ is called *planar*. A graph embedded on $\mathbb{R}^2$ is called *plane*.

**Definition 2.** A *face* of an embedded graph $G$ on $X$ is a region of $X \setminus G$.

Given a face of an embedded graph $G$, the boundary of the face is an embedded subgraph of $G$. If this subgraph is a cycle, call it a facial cycle.

**Definition 3.** Let $G$ be an embeddable graph , let $f$ be an embedding, and let the boundary $b$ of a face of $f(G)$ be a cycle, i.e let $G$ limited to the vertices and edges of $f^{-1}(b)$ be a cycle. We call the boundary of $b$ a *facial cycle*.

# 3. OVERVIEW OF PROOFS AND TECHNIQUES, OPEN PROBLEMS

The goal of this subsection is to present sketches of proofs of this text in an easily readable manner. Chapter 4 of this text containing the full proofs is a bit more bulky and pedantic than we would like; thus, if the reader reads this overview and does not really look into chapter 4, it will have done its job well.

## 3.1 Classes with a geometric interpretation

We proceed to find the $\Delta$ value of a few minor-closed classes. We start with graphs that have geometric interpretations; the planar graphs, the graphs of euler genus $\leq k$ for fixed $k \in \mathbb{N}$, the outerplanar graphs, and the series-parallel graphs, all of which have $\Delta = 3$. Interestingly, all of them admit the same approach: Replace each vertex with a cycle. The following image shows that planar graphs have $\Delta = 3$.

**Figure 3.1: By replacing each vertex of a plane graph with a circle on the boundary of an open ball around the vertex, we may create a plane graph of maximum degree 3 containing the first as a minor.**

Given $k$, much the same can be said for the class of graphs embeddable on a surface of euler genus $k$ [1], showing this class has $\Delta = 3$ as well. The proof for outerplanar graphs is summed up in the following figure.

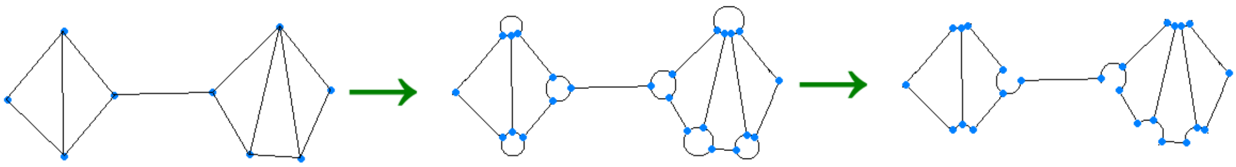

**Figure 3.2: For $G$ an outerplanar graph with common face $f$ for all vertices, do as with planar graphs to obtain $G'$, then remove the edge intersecting $f$.**

The proof that the class of series-parallel graphs has $\Delta = 3$ is omitted. It would not be hard to prove for the interested reader.

[1] This notion doesn't become important until chapter 6, so we don't spend more time on it here. See 14 for a definition. Further information is on the appendix.

### 3.2 Classes closed under clique-sums

We move on to graph classes closed under clique-sums; $Forb(K_5)$, $Forb(K_{3,3})$ and, let $k \geq 3$ be fixed, $TW_{\leq k}$. Surprisingly, they also admit a unified approach.

**Definition 4.** Given graphs $G,\ H$ such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by taking $G \cup H$ and possibly removing a few edges of the clique. See figure3.3.

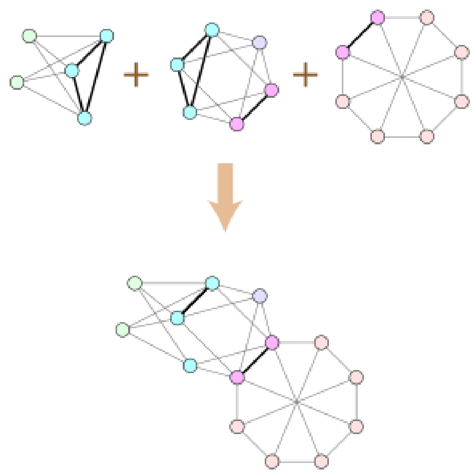

**Figure 3.3: Two clique sums to create a single big graph. Notice how a few clique edges are removed. Courtesy: Wikipedia.**

**Definition 5.** The clique sum of $G$ and $H$ on clique $G \cap H$ of $k$ vertices is called a $k$*-sum*. The clique sum of $G$ and $H$ on clique $G \cap H$ of $\leq k$ vertices is called a $\leq k$*-sum*.

Notice that 0-sums are well defined, and are the disjoint union. Now, we would like to clique-sum without caring about vertex labels.

**Theorem 3** (Wagner [12])**.** *A graph $G$ excludes $K_5$ as a minor if and only if it can be constructed by the $\leq 3$-clique-sums of planar graphs and the Wagner graph $W[8]$. See figure 3.4.*

**Theorem 4** (Wagner [12])**.** *A graph $G$ excludes $K_{3,3}$ as a minor if and only if it can be constructed by the $\leq 2$-clique-sums of planar graphs and $K_5$.*

**Definition 6.** A graph is said to have treewidth $\leq k$ iff it can be constructed by the clique sum of graphs of $\leq k+1$ vertices.

**Definition 7.** A graph is said to have treewidth = $k$ iff it has treewidth $\leq k$, but it doesn't have treewidth $\leq k-1$.

The aforementioned definition of treewidth is somewhat unorthodox, but I find it to be most intuitive. Isn't it much simpler to conceptualize than tree-decompositions?

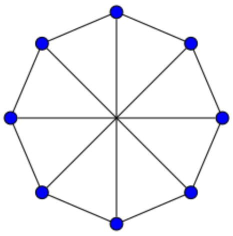

**Figure 3.4: The Wagner graph $W[8]$, also known as the 8-wheel. Courtesy: Wikipedia.**

We develop the toolset for the key lemma that will allow us to approach all 3 problems in a unified fashion. Our reward will be that the proofs giving a (strict as it turns out) upper bound to their $\Delta$ value will be quite easy and short.

**Definition 8.** Denote by $\oplus[C]$ the clique sum closure of class $C$ [2]. Denote by $\oplus^{\leq n}[C]$ the $\leq n$-sum closure of class $C$.

**Definition 9.** $B$ is a *base* for $C$ under $\leq n$-sums if $\oplus^{\leq n}[B] = C$. $B$ is a *base* for $C$ under clique sums if $\oplus[B] = C$.

**Definition 10.** Let $G' \geq_m G$, with model function $\mu$. For clique $K \in G$, let its vertex set be $\{u_1, ...\}$, let $K' \in G'$ be isomorphic clique with vertex set $\{u'_1, ...\}$ such that $u'_i \in \mu(u_i)$. We call $K'$ a *representor clique of $K$ under $\mu$*.

Notice that clique representation is *transitive under minors*: If $G \leq_m G' \leq_m G''$ and $K$ is a clique of $G$ represented under $\mu$ by $K'$ in $G'$ and $K'$ is represented under $\mu'$ in $G''$ by $K''$, then $K$ is represented under $\mu \circ \mu'$ by $K''$.

**Definition 11.** Given graphs $G,\ H$, their *Cartesian product* $G\Box H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices $(u, v)$ and $(u', v')$ are adjacent if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Intuitively, for each vertex of $H$ take a copy of $G$, and if two vertices in $H$ are connected, connect the corresponding $G$ copies by their identical vertices.

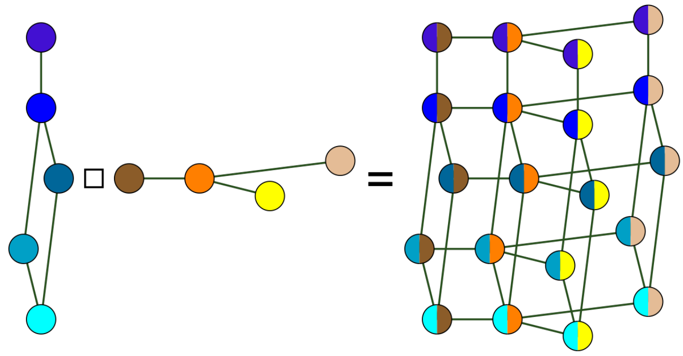

**Figure 3.5: The Cartesian product of two graphs Courtesy: Wikipedia.**

[2]So $G \in C \implies G \in \oplus[C]$ and $G,\ G' \in \oplus[C] \implies G \oplus G' \in \oplus[C]$.

**Definition 12.** For fixed $u \in G$, we denote by by $(u, H)$ the $G \square H$ limited to all vertices of the form $(u, v)$ where $v$ ranges over $H$. We call $(u, H)$ the $H$-subgraph of $V(G) \times V(H)$ corresponding to $u$.

We may now give the key lemma we will be using. $P_2$ is the path of 2 vertices.

**Lemma 1.** *Let $d \geq 3$. Let $C$ be a minor-closed class closed under $n$-sums, such that $P_2 \square K_n \in C$. Let $B$ be a base for $C$ under $\leq n$-sums. For every graph G in $B$, let $G'$ in $C$ be a graph with*

- *$G' \geq_m G$.*
- *Every maximal clique in $G$ has a representor clique in $G'$.*
- *$\Delta(G') \leq d$.*

*Then $\Delta(C) \leq d$.*

We give a short overview of why this lemma holds in the end of the section. The next chapter with the full proofs does not use the lemma, but they are much easier with it, so I recommend focusing on this chapter.

**Proposition 1.** $\Delta(forb(K_5)) = 3$.

Given planar graph $G$, we call the graph $G' \geq_m G$ of maximum degree 3 as in the proof that the class of planar graphs has $\Delta = 3$ the *ballooning* of $G$, and denote it $Bl(G)$. The cycle we replace vertex $v \in G$ with we denote by $Bl(v)$.

*Proof of proposition 1.* We use lemma 1, where $C$ is of course $Forb(K_5)$ and $n = 3$. The base $B$ is the Wagner graph along with the class $B'$ of planar graphs $G$ such that all embedded triangles $abc$ of $G$ have either an empty interior or an empty exterior. $B'$ is enough to construct all planar graphs with clique-sums; for let $G$ be a planar graph embeddable so some triangle $abc$ has neither empty interior nor exterior, then by the definitions of planarity and the Jordan curve theorem, the triangle is a separator, and thus it can be further decomposed into the 3-clique-sum of smaller planar graphs. Therefore by Wagner's theorem, the clique sum closure of $B$ gives $Forb(K_5)$.

We now find for every $G \in B$ a $G' \in C$ as in lemma 1. The Wagner graph has maximum degree 3 so it already is of the desired form (the corresponding $G' \in C$ being again the Wagner graph). For $G \in B'$, $G' \in C$ will be as follows. Let $abc$ be a triangle in $G$ of empty interior or exterior. Add a new triangle $a'b'c'$ to $G$, $a$ joined to $a'$, $b$ joined to $b'$, $c$ joined to $c'$, and embed it in the empty face. See image 3.6. Do this for all triangles of $G$. Now balloon $G$, but leave the vertices of the new triangles as is. This completes the construction of $G'$. It is clearly still planar and has maximum degree at most 3. By contracting as explained in the image, we regain the original graph, $a'b'c'$ being a representor of $abc$. For any potential maximal 2-cliques $uv$ of $G$, the unique $Bl(u) - Bl(v)$ edge is a representor. □

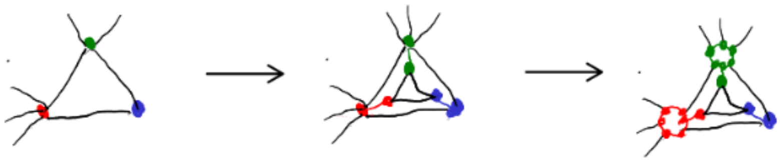

**Figure 3.6: A triangle of $G$ modified step by step. By contracting along same-colored segments, we regain the original graph.**

**Proposition 2.** $\Delta(forb(K_{3,3})) \leq 4$

*Proof.* We use lemma lemma 1 where $n = 2$. The base $B$ is the $K_5$ graph along with the class $B'$ of planar graphs $G$ such that all embedded triangles $abc$ of $G$ have either an empty interior or an empty exterior. We now find for every $G \in B$ a $G' \in C$ as in lemma 1. If $G$ is $K_5$, then $G'$ is also $K_5$. If $G$ is a planar graph, then $G'$ is $Bl(G)$. The reader may verify the rest. □

**Proposition 3.** $\Delta(TW_{\leq k}) \leq k$

We use lemma 1 where $n = k$. The base $B$ is the graphs of at most $k + 1$ vertices. We should first prove that $P_2 \square K_n \in C$.

**Proposition 4.** $K_n \square P_2 \in TW_{\leq n}$.

*Proof.* See figure 3.7. Let $G_1$ be a $K_n$ graph, let $V(G_1) = \{1, 2, ..., n\}$ and clique sum it with a $K_{n+1}$ graph $G_2$, let its nodes be $\{1, 2, ..., n, 1'\}$. Afterwards, we clique sum $G_2$ with a $K_{n+1}$, its nodes being $\{1', 2, ..., n, 2'\}$, then the node set will be $\{1', 2', 3, ..., n, 3'\}$ and so on $n$ times. In the final graph, $\{1, 2, ..., n\}$ and $\{1', 2', ..., n'\}$ are cliques, with $(i, i')$ connected for all $i \in \{1, 2, ..., n\}$. Remove surplus edges as needed (note one can use clique-sums to remove edges without adding any new vertex). □

In fact, we proved that $K_n \square P_2 \in PW_{\leq n}$ where $PW_{\leq n}$ is the class of graphs of path-width $n$ or less.

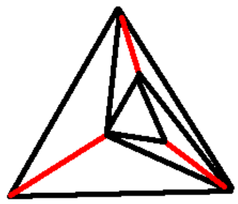

**Figure 3.7: Creating a $K_3 \square P_2$. We start from the outermost triangle, call it $xyz$, and create the innermost triangle $x'y'z'$ by clique-sums, one vertex at a time. The red edges are $xx'$, $yy'$, $zz'$**

**.**

*Proof of proposition 3.* As mentioned take lemma 1 for $n = k$ and $B$ the graphs of at most $k+1$ vertices. We are in fact already done. For $G \in B$, $G'$ is again $G$. □

We move on to lower bounds. We prove that $\Delta(forb(K_{3,3})) \neq 3$. As it cannot be 2 or 1 either, this combined with $\Delta(forb(K_{3,3})) \leq 4$ implies that $\Delta(forb(K_{3,3})) = 4$.

**Proposition 5.** $\Delta(forb(K_{3,3})) \geq 4$

*Proof.* To give the idea in brief, by Wagner any graph in $forb(K_{3,3})$ is constructed by the $\leq 2$ sum of planar graphs and $K_5$. Now, observe by geometric intuition that the $\leq 2$ sum of planar graphs remains planar, therefore to create a non-planar graph of $forb(K_{3,3})$ using Wagner's theorem a $K_5$ must be used at some point. Also observe the only way to reduce the degree of a vertex with a $\leq 2$ sum is to use a 2-sum that does not add vertices and removes a single edge, call this a *trivial* 2-sum. But rather than remove an edge by a trivial clique sum, we can remove it after the last (non-trivial) clique sum that utilizes the edge. Therefore any $G \in forb(K_{3,3})$ can be constructed by a series of clique-sums where no trivial 2-sum occurs. Therefore, a non-planar $G \in forb(K_{3,3})$ must have degree 4 or more as the $K_5$ that was added while building it this way cannot have the degrees of its vertices reduced. Of course, if $G \in forb(K_{3,3})$ is non-planar and $G' \in forb(K_{3,3})$ includes $G$ as a minor, then $G'$ must also be non-planar, and thus $\Delta(G') \geq 4$. The reader may also refer to section 4.2.2 for this proof, which I find to be of satisfactory quality. □

Similarly, we would like to prove that for $n \geq 3$, $\Delta(TW_{\leq n}) \geq n$.

**Proposition 6.** $\Delta(TW_{\leq n}) \geq n$

To prove this, we want a graph $G \in TW_{\leq n}$ so that any graph $G' \in TW_{\leq n}$ including it as a minor must have $\Delta(G') \geq n$. In [13], Markov and Shi showed that there is a graph $G$ of treewidth $n$ and no degree $3$ expansion of treewidth $n$.

Let there be an $n+1$-clique graph with vertex set $\{1, 2, ..., n+1\}$, called the *central clique* $K_c$. For $i \in \{1, 2, ..., n+1\}$ add a vertex labeled $i'$ and join [3] it to the subclique $\{1, ..., i-1, i+1, ..., n, n+1\}$. Call the $n+1$-clique with vertex set $\{1, ..., i-1, i', i+1, ..., n, n+1\}$ by the name $K^{(i)}$. This completes the construction of graph $G$. It is clear that it is in $TW_{\leq n}$ as each vertex we joined can be added by a clique sum. Markov's and Shi's example was the same, but they also removed all edges with both ends in the central clique of $G$.

We use the following known lemma. In case the reader is not familiar with the notion, the definition of tree-decompositions can be found in section 4.3.

**Lemma 2.** *Let $G$ contain a clique $K$, let $G'$ contain $G$ as a minor, and let $(X, T)$ be a tree-decomposition of $G'$. Then there is some vertex $t \in T$ such that its bag $B(t)$ contains for each $v \in V(K)$ a vertex from $\mu(v)$.*

Call any such $t$ a *model carrier* of $K$, and denote it $t_K$. What follows is both an extension and a notional simplification of Markov's and Shi's result.

*Proof of proposition 6.* Let $G$ be the graph constructed above. Let $G' \geq_m G$ as a minor with model function $\mu$, where $G' \in TW_{\leq n}$. We will show $\Delta(G') \geq n$.

To do this, we will show that any tree decomposition $(X, T)$ of $G'$ looks like fig. 3.8; that is, removing the centre clique model carrier seperates the tree such that for all $i, j$, $t_{K^{(i)}}$ and $t_{K^{(j)}}$ do not share a connected component. This will imply that any vertex $v$ in $B(t_{K_c})$ must have $d(v) \geq n$, by the following argument (recall $V(K_c)$={1,...,n+1}):

Let $v_i$ be both in $B(t_{K^{(i)}})$ (the bag of the model carrier of $K^{(i)}$) and in $\mu(i')$ (the minor branch of $i'$). For $G'$ to include $G$ as a minor, there must be a path from $v_i$ to all vertices of $B(t_{K_c})$, except the one vertex of $B(t_{K_c})$ also in $\mu(i)$ (observe this path intersects $B(t_{K_c})$ only at its endpoint, as each vertex of $B(t_{K_c})$ belongs to a different minor branch). A vertex in $B(t_{K_c}) \cap \mu(i)$ thus receives $n$ internally disjoint paths, 1 from each of the $n$ $K^{(j)}$ model carriers, where $i \neq j$ (they are internally disjoint as by a known theorem removing $B(t_{K_c})$ from $G'$ separates all the $v_j$ from each other). Thus, each vertex of $B(t_{K_c})$ has degree $\geq n$.

To see that any tree decomposition has the form of fig. 3.8, assume towards contradiction that a connected component of $T \setminus t_{K_c}$ has model carriers for both $K^{(1)}$ and $K^{(2)}$

[3]We remind that to join a vertex to a graph $H$ means to connect it to all vertices in $H$

w.l.g. Now let $v_i$ where $i \neq 1$ be the vertex $\in \mu(i) \cap B(t_{K^{(1)}})$, and let $v_1$ be the vertex $\in \mu(1) \cap B(t_{K^{(2)}})$. Let $u_i$ be the vertex $\in \mu(i)$ and $B(t_{K_c})$. Since $G' \geq_m G$, for each $i \in \{1, ..., n+1\}$ there is a (possibly trivial) path from $v_i$ to $u_i$, all of them pairwise vertex disjoint. So we have $n+1$ pairwise vertex disjoint such paths. This cannot be, as by known tree decomposition properties, there must be a separator of size at most $n$ eliminating all paths from $B(t_{K_c})$ to $B(t_{K^{(1)}}) \cup B(t_{K^{(2)}})$ (namely let $t$ be the first vertex in the $t_{K_c} - t_{K^{(1)}}$ path, then $B(t_{K_c}) \cap B(t)$ separates $B(t_{K_c})$ from the bag of any other vertex $t'$ in the connected component of $t$ in $T \setminus T_{K_c}$). □

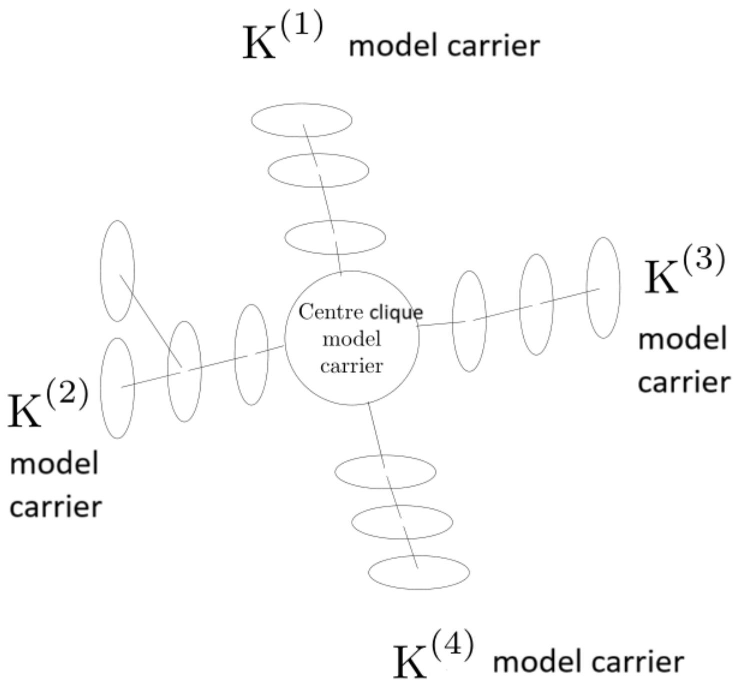


**Figure 3.8: Example tree-decomposition of $G'$ for $n = 3$. The centre bag model carrier separates the model carriers of $K^{(i)}$.**

The same proof with slightly different arguments and reinterpreted notation can be found in the extended version.

We finish this section with a high level overview of the proof of lemma 1:

For $H$ in a minor-closed class $C$ which has the required properties of lemma 1, $H$ can be constructed by the clique-sum $G_1 \oplus G_2 \oplus ...$ where $G_i \in B$. We want to find $H' \in C$ that includes $H$ as a minor and has $\Delta(H') \leq d$. The idea is to use the $G_i'$ provided by the lemma instead to build $H'$; If $G_1$ and $G_2$ were clique summed over common clique $K$, we use its maximal representor $K'$ in $G_1'$ and $G_2'$ and clique sum them. This way, $H' := G_1' \oplus G_2' \oplus ...$ is a well defined clique-sum. One may check that by contracting each graph $G_i'$ that $H'$ comprises of back into $G_i$, we obtain $H$. Finally, the vertices of $H$ that did not participate in a clique sum have degree at most $d$. All that remains is to deal with the potentially high degree of the common cliques, as they may participate in an unbounded number of clique sums. The next lemma given without proof follows easily by iterated clique-summing:

**Lemma 3.** *Let $C$ be a graph class closed under $n$-clique-sums such that the graph product $K_n \Box P_2$ is in $C$. Then $K_n \Box T$ is in $C$ for any tree $T$ of more than 1 vertex.*

Notice lemma 1 satisfies the requirements for lemma 3. To deal with the potentially high degree of the common cliques, *before* clique summing $G_1'$ on $G_2'$ on common clique $K'$, we first clique sum on $K'$ the graph $K' \Box T$ for some big enough comb $T$. See figure 3.10 for an example with a 2-comb.

**Definition 13.** Let $u_1 u_2 ... u_k$ be a path graph, and for each $u_i$, add a vertex $v_i$, and join it to $u_i$. The resulting graph is called the *comb graph* of length $k$ or $k$-comb graph. The subpath $u_1 u_2 ... u_k$ is called the *spine* of the comb graph and $u_i$ is the $ith$ spine vertex. The $v_i$ are the *teeth* of the comb. See figure 3.9.

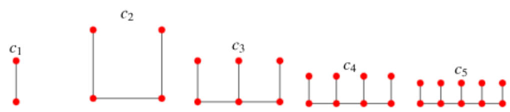


**Figure 3.9: The 1, 2, 3, 4 and 5 comb graphs. Courtesy: Wolframalpha**

Figure 3.10 explains how we use the newly clique summed graph. It is simple to contract the new graph back into $G_1'$. Since we remove clique edges, we keep an additional tooth clique not to be used in clique-sums ($\Delta 3$ in the figure), which will help us regain our clique's edges once the contractions happen; to keep the maximum degree of this tooth clique low, we further break it up into a path of cliques $K \Box P$, each clique retaining a single edge of $K$, see figure 3.11. Once this is done, the maximum degree will be $d$. This completes the description of how we build the splitting of $H$.

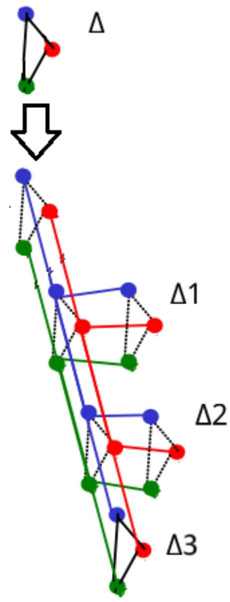


**Figure 3.10: A representor clique $\triangle$, where a comb of cliques is attached. Tooth clique $\triangle 1$ is used in place of $\triangle$ for a clique sum, and then $\triangle 2$ is used in place of $\triangle$ for a second clique sum. Extend the comb if more clique sums are needed. By contracting along same colored components, we reobtain $\triangle$ and it is as if we had clique summed everything on $\triangle$. Remove dotted edges after the comb of cliques in no longer needed for sums. This yields a graph of low maximum degree. The last clique $\triangle 3$ is not used in a clique sum, but stays as is so that the edges of $\triangle$ are reobtained when contracting along same colored components.**

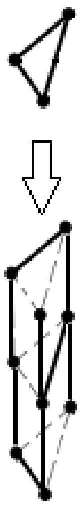

**Figure 3.11: Example with 3-clique. A 3-clique $K$ is replaced with a clique path $K \square P$, where the dashed edges are removed. This keeps the maximum degree down to 3 no matter how big the clique, and one may simply contract upwards to regain the clique.**

The full proof of this lemma (of a more general form actually) is in section 6, though it may be somewhat bulky.

Overviews of the main proof (chapters 5 and 6) are not included. Chapter 5 is decently written, and the interested reader may look into it.

### 3.3 Open problems

This section quickly touches on some potential open problems.

Recall if $C$ is a class, apex-$C$ is the class of graphs $G$ that are in $C$ or that have a vertex $v$ so that $G \setminus v$ is in $C$. One may observe that except the strict superclasses of the class of apex graphs, the two examples we have given of a minor-closed class $C$ of $\Delta(C) = \infty$, are the stars [4] and the apex graphs. Now, both are of the form apex-$C$, where $C$ is a minor-closed class; for the apex graphs $C$ is of course the planar graphs, and for the stars $C$ is the class of the disjoint union of single-vertex graphs. Furthermore, I recently proved the following, not included in this text.

**Proposition 7.** *If $C$ is a minor-closed proper class, $\Delta(apex - C) = \infty$.*

This hints the following is an approachable problem.

**Problem 1.** What minor-closed classes $C$ have finite $\Delta(C)$?

One could conjecture it is all the classes $C$ not of the form apex-$C'$ for some minor-closed class $C'$. This is not true, as the class of stars union the class of paths has $\Delta = \infty$ without being of this form. Naturally, this counterexample feel unethical, so one could formulate a conjecture similar to the above that eliminates such pathological counterexamples.

On another note, when I proved that $Forb(K_{3,3}) = 4$, I made an interesting observation. For a minor-closed class $C$, one way to reformulate the definition of $\Delta(C)$ is to define $\Delta(C)$ as the minimum $k$ so that $C$ =minor-closure$(\{G \in C|\Delta(G) \leq k\})$. For classes $C$ of $\Delta(C) = k > 3$, one may then ask what minor-closure$(\{G \in C|\Delta(G) \leq 3\})$ is, or more generally, for any $k'$ smaller than $k$ what minor-closure$(\{G \in C|\Delta(G) \leq k'\})$ is. For $K_{3,3}$-minor-free graphs the interesting question is when $k' = 3$ and the answer is easy; minor-closure$(\{G \in$ forb($K_{3,3}$) | $\Delta(G) \leq 3\})$= the planar graphs, as every non-planar $G \in Forb(K_{3,3})$ cannot have maximum degree 3 as we have seen in proposition 5.

We asked this question for $Forb(K_{3,3})$ and got a natural graph class as a response. By repeating this question with other minor-closed graph classes of high $\Delta$ we could again find elegant and natural graph classes, or we might even find undiscovered ones. Let TW$_{\leq k}$ be the class of graphs of treewidth $k$ or less. $C := minor - closure(\{G \in TW_{\leq k}$ | $\Delta(G) \leq 3\})$ is a treewidth-like minor-closed class. Could it be formulated as a natural variation of treewidth? Notice $TW_{\leq k-1} \subset C \subset TW_{\leq k}$.

[4] Again, this is technically not a minor-closure class; think of its minor-closure when we speak of it

Simple treewidth is an interesting minor-closed variation of treewidth with geometric applications, also with the property that $TW_{\leq k-1} \subset STW_{\leq k} \subset TW_{\leq k}$, where $STW_{\leq k}$ is the class of graphs with simple treewidth at most $k$. Could $C = STW_{\leq k}$? The answer to this conjecture is negative, as $\Delta(STW_{\leq k}) \geq k$ (it has the same construction $G$ showing $\Delta(STW_{\leq k}) \geq k$ as proposition 6). The question remains.

**Problem 2.** Does $C$ have a natural description with treewidth like properties?

On another topic, given minor-closed class $C$, with $\Delta(C) = k$, which we know the excluded minors of, it is not way too difficult to find the excluded minors of minor-closure$\{G \in C | \Delta(G) \leq k'\}$ where $k' < k$. To see this, one may try describing the excluded minors of minor-closure$(\{G \in TW_{\leq k} | \Delta(G) \leq 3\})$ as a function of the minors of $TW_{\leq k}$, by finding minor-minimal constructions (such as the graph $G$ of proposition 6) showing that $\Delta(TW_{\leq k}) \geq k' + 1$.

**Problem 3.** Can we express a minor-closed class $C'$ whose excluded minors we do not know as minor-closure$\{G \in C | \Delta(G) \leq k'\}$, where $C$ is a minor-closed class whose excluded minors we do know?

If we could, it is very possible we could find the excluded minors of $C'$. If the answer to the question is that we cannot, another question would be to find other functions $f$ from the set of minor closed graph classes to the set of minor closed graph classes, with the property that if we know the excluded minors of $C$ we can find the excluded minors of $f(C)$; once we have enough such functions, we might have enough expressive power to achieve what problem 3 requests.

Here is a final toy question I found interesting. Let APEX be the class of apex graphs.

**Problem 4.** Notice $minor - closure(\{G \in APEX \mid \Delta(G) \leq 3\}) \subset minor - closure(\{G \in APEX \mid \Delta(G) \leq 4\}) \subseteq minor - closure(\{G \in APEX \mid \Delta(G) \leq 5\}) \subseteq ... \subseteq minor - closure(\{G \in APEX \mid \Delta(G) \leq 1000\}) \subseteq ....$ Do its classes admit an interesting description?

It is easy to observe this hierarchy does not collapse, and it has an unbounded number of proper inclusions. I still do not know if its classes admit an interesting description or if they relate to each other in a meaningful manner.

# 4. THE $\Delta$ VALUE OF VARIOUS MINOR-CLOSED CLASSES

In this section, we find the $\Delta$ value of a few minor-closed classes, such as K5-minor-free graphs. This is the extended version of chapter 3, the overview. I would *strongly* advise the reader to look into the overview first, as the present section is more pedantic and extensive than we would like.

## 4.1 Planar graphs, Graphs of Euler genus $\leq k$, Outerplanar graphs, Linklessly embeddable graphs

### 4.1.1 Planar graphs

It is easy to conclude that every planar graph has a planar graph of maximum degree 3 by visual intuition alone. The following figure illustrates that.

**Figure 4.1: By replacing each vertex of a plane graph with a circle on the boundary of an open ball around the vertex, we may create a plane graph of maximum degree 3 containing the first as a minor.**

Let's write the actual proof! We remind that a planar graph has a function $f$ mapping its vertices to points and its edges to curves on the plane. Note that an embedded graph is a compact subset of $\mathbb{R}^2$, being the finite union of compact sets, curves being compact as the continuous image of the compact set [0,1]. We remind that the initial segment of a curve $c([0,1])$ is a subset of the curve of the form $c([0,a])$ or $c([a,1])$. The following lemma says that with the right embedding, for each vertex one may find a closed ball centered on the vertex, only including the vertex and initial segments of the edges incident to the vertex (that is, edges only exit the ball once).

**Lemma 4.** *Let $G$ be a planar graph. $G$ has an embedding $f$ with the following properties: For every embedded vertex $f(v)$, there is a closed ball centered on $f(v)$ such that*

- *The closed ball includes no other embedded vertices.*
- *The closed ball intersects only embedded edges incident to $v$.*
- *The closed ball intersects only an initial segment of those edges.*

*Proof.* Let $f$ be any planar embedding of $G$. For a ball of $f(v)$ without other vertices inside, simply pick a ball with radius smaller than the minimum distance between $f(v)$

and other embedded vertices, $\min_v d(f(u), f(v))$, where $d()$ is the euclidean distance. Moving to edges not incident to $v$, suppose towards contradiction that every closed ball around $v$ intersects such an edge. Let $E$ be the set of edges incident to $v$. We can thus pick a sequence $a_n$ of $f(G \setminus E \setminus v)$ such that as $n$ increases, the distance from $f(v)$ decreases and tends to 0, e.g $a_n =$ some element of distance $\leq 1/n$. By definition, this sequence converges to $f(v)$. Furthermore, $f(G \setminus E \setminus v)$ is compact in $\mathbb{R}^2$ and thus closed, therefore $f(v) \in f(G \setminus E \setminus v)$, a contradiction to the definition of embeddings. Moving to edges incident to $v$, pick some $\varepsilon$ such that $B_\varepsilon(v)$ intersects from $f(G)$ only $f(v)$ and those edges. Simply erase the inside of the ball (except $v$ of course) and reconnect $v$ with its edges by a straight line segment going from $f(v)$ to where the embedded edge last exits $B_\varepsilon(v)$, erasing it before that point (to explain where to connect it in rigorous terms, let $e([0, 1])$ where $e : [0, 1] \to \mathbb{R}^2$ be such an embedded edge, with $e(0)$ being $v$. Let $x$ be $\sup_y [e(y) \in B_\varepsilon(v)]$. Connect $v$ to $e(x)$). It is simple geometry this remains an embedding satisfying the lemma. $\square$

For every embedding, we thus found an embedding very similar to it with all these nice properties. The reader may inquire whether these properties hold without changing the original embedding, in other words, if they are true for all embeddings. The answer is actually negative! There are graphs such that the final property does not hold.

For example: Let there be function

$$q(x) = \begin{cases} x \sin(1/x), & \text{if } x \in (0, 1] \\ 0, & \text{if } x = 0 \end{cases}$$

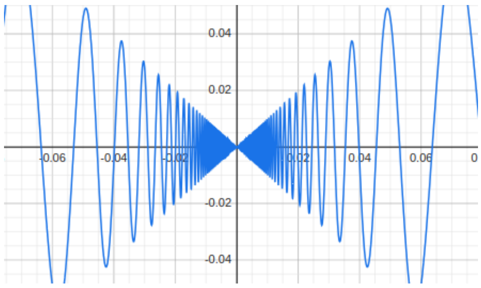


**Figure 4.2: Function $x \sin(1/x)$. Our intuition can be false in topology, even on $\mathbb{R}^2$.**

.
Notice that $q$ is a continuous function on $[0, 1]$, i.e a curve. Let $G$ be some planar graph with some embedding such that $q(0)$ and $q(1)$ are embedded vertices $u_1$ and $u_2$ of $G$, and $q([0, 1])$ is an embedded edge. For some $r_0 > 0$, all circles of radius less that $r_0$ intersect the edge at least twice. (Indeed, its distance from the origin is $x\sqrt{1 + \sin^2(1/x)}$. The reader may verify the rest by setting values of the form $1/k\pi$ for very large $k$.) Now, let $v$ be an embedded vertex of distance less than $r_0$ to $u_1$. There is no ball of $u_1$ satisfying both properties 1 and 3 of the lemma for this embedding.

**Theorem 5.** *Let PLANAR be the class of planar graphs. Δ(PLANAR)=3.*

*Proof.* Let $G$ be a planar graph. Take the embedding of lemma 4, and take the balls small enough that they do not intersect and let $v$ be a vertex of degree $\geq 3$. Erase everything inside the closed ball of $v$, then let $p_1, ..., p_k$ be the points where the boundary of the closed ball last intersected the edges of $v$ $e_1, ..., e_k$, the $p_i$ ordered in a counterclockwise manner starting from some point of the boundary of the ball. Add the $p_i$ back as embedded vertices $v_i$. Then, connect $p_i$ with $p_{i+1}$ by a curve running along the perimeter of the boundary and also connect $p_k$ with $p_1$ in the same manner (of course these are well defined curves. Take the polar coordinate formula, mapping the angle to points on the circle.). Notice that all such vertices are of degree at most 3, and that their contraction yields the original graph. Doing this for every vertex of degree $\geq 3$, we create an embedded graph of maximum degree 3 including $G$ as a minor. □

Much the same holds for graphs embeddable on a surface of euler genus $k$, equivalently graphs of euler genus $\leq k$.

### 4.1.2 Graphs of Euler genus $\leq k$

**Definition 14.** Let $\Sigma$ be a surface built from the sphere after adding $k$ handles. Then its euler genus is $2k$.
Let $\Sigma$ be a surface built from the sphere after adding $k$ crosscaps. Then its euler genus is $k$.

The classification theorem of closed surfaces states that any connected closed surface is homeomorphic to a surface as in the above definition, where $k \geq 0$.

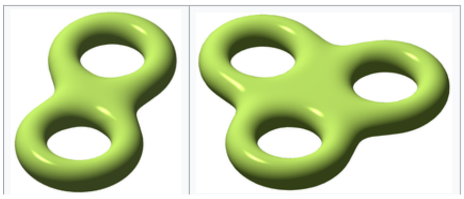

**Figure 4.3: Surfaces where we have added 2 or 3 handles respectively. The double and triple torus. Courtesy: Wikipedia.**

**Definition 15.** The *euler genus* of a graph is the smallest integer $n$ such that $G$ can be embedded on the surface of euler genus $n$.

We may abusively call a graph of Euler genus $n$ a graph of genus $n$; in this text we always refer to the Euler Genus.

The fact that every graph of euler genus $k$ is included as minor in a graph of euler genus $k$ and maximum degree 3 is visualized in much the same manner and the proof is almost identical. We simply have to work with the open discs provided by the definition of a

surface instead of open balls. We present them without proofs.
Note than for a point $x$ of a surface, and any ball of $x$, there exists an open disc inside the ball. To see this, let $D$ be an open disc of $x$ homeomorphic to the open ball of $\mathbb{R}^2$ by homeomorphism $f$, take an open ball $O$ of $x$, map it by $f$ to $\mathbb{R}^2$. $f[O]$ is an open set (by homeomorphism) and thus it has inside an open ball centered on $x$. Map this open ball back to the surface by $f^{-1}$. Thus, for any ball $B_\varepsilon(x) \subseteq D$, we have found a subset $D'$ of $B_\varepsilon(x)$, mapped by $f$ to an open ball of $\mathbb{R}^2$. Limiting $f$ to $D'$, it is easy to see that we still have a homeomorphism.

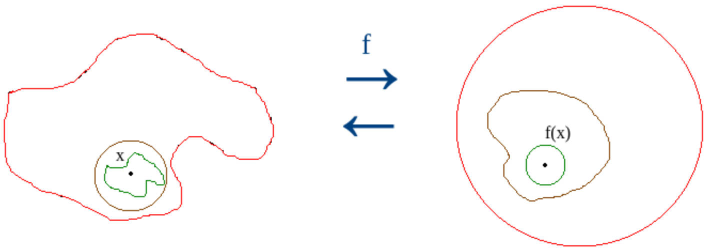


**Figure 4.4: Reasoning about open discs through their homeomorphism to the open ball.**

**Lemma 5.** *Let $G$ be a graph with embedding $f$ on some surface. For every embedded vertex $f(v)$, there is an open ball centered on $f(v)$ and an open disc inside the ball including no other embedded vertices, and only embedded edges incident to $v$. Furthermore, let $g : [0, 1] \to \mathbb{R}^2$ be one such embedded edge. If $g(0) = f(v)$ the open disc only contains a subset of the form $g([0, \varepsilon])$. If $g(1) = f(v)$ the open disc only contains a subset of the form $g([1 - \varepsilon, 1])$.*

**Theorem 6.** *Let EUL_GENUS$_{\leq k}$ be the class of graphs of euler genus $\leq k$. Δ(EUL_GENUS$_{\leq k}$)=3.*

**Definition 16.** Given graph $G$, we call the graph $G' \geq_m G$ of maximum degree 3 as in the proof that $\Delta(PLANAR) = 3$ the *fattening* or *ballooning* of $G$, and denote it $Bl(G)$. The circle we replace vertex $v \in G$ with we denote by $Bl(v)$. This is also the model function showing $G' \geq_m G$.

### 4.1.3 Outerplanar graphs

The outerplanar graphs are closely related to planar graphs. One expects that the same methods apply, and indeed this is the case. Let OUTERPLANAR be the class of outerplanar graphs.

**Theorem 7.** $\Delta(OUTERPLANAR) = 3$

The proof is summed up in the following figure.

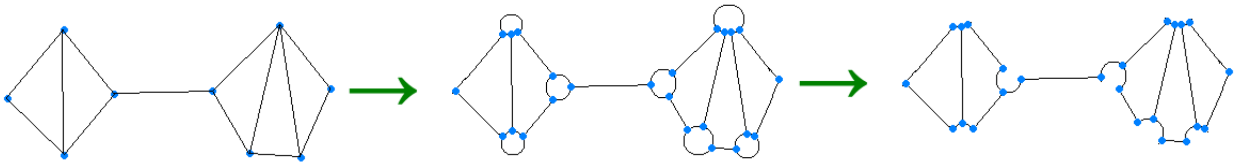

**Figure 4.5: A picture is a thousand words. It unfortunately is not also a proof.**

*Proof.* Let $G$ be an outerplanar graph. There is a common face $f$ of $\mathbb{R}^2 \setminus G$ on which all vertices lie. So for a small enough $\varepsilon$ a closed ball $B_v(\varepsilon)$ around a vertex $v$ intersects with $f$. More specifically, its boundary intersects $f$. To prove this, observe that for $\varepsilon$ small enough, there is a point $p \in f$ with $d(v,p) > \varepsilon$, and a simple curve $c : [0,1] \to \mathbb{R}^2$ connecting $v$ and $p$ and having interior in $f$. The function $d_v$ mapping a point of $\mathbb{R}^2$ to the distance from point $v$ is continuous, therefore $d_v \circ c$ is continuous, and by the mean value theorem for all $\varepsilon' \in (0,\varepsilon)$ there is a point on the interior of the curve with distance $\varepsilon'$ from $v$. Let $p_{\varepsilon'}$ be such a point. Even more specifically, since $f$ is open, we may take an open ball of $f$ around $p_{\varepsilon'}$, and by geometry notice that its entire intersection with the boundary of $B_v(\varepsilon)$ is in $f$.
We create from $G$ a graph $G' := Bl(G)$ as in the proof of $\Delta(PLANAR) = 3$. Clearly $G' \geq_m G$ by contracting $Bl(v)$ for each $v$. Notice that this still holds if we remove any 1 edge from each $Bl(v)$.

Since the edges of $Bl(v)$ cover the circle $Bl(v)$ was embedded on, at least one such edge $e$ must intersect the boundary of $f$. We remove it. Both the ball bounding circle $Bl(v)$ and $f$ are faces, i.e maximal connected sets of $\mathbb{R}^2 \setminus G$, with an intersecting boundary, so $G' \setminus e$ now has a face= the interior of $e \cup$ f $\cup$ the ball bounding $Bl(v)$. This face intersects all vertices of $Bl(v)$. Doing this for all $Bl(v)$, we acquire an outerplanar graph of maximum degree 3 containing $G$ as a minor. □

### 4.1.4 Linklessly Embeddable graphs

With all the above positive results in mind, one may thus conjecture that the linklessly embeddable graphs, a well-known three dimensional analogue of the planar graphs consisting of all graphs that have a linkless or flat embedding on 3D-space, also has a low $\Delta$. This is not the case. As we will see, the linklessly embeddable graphs have $\Delta = \infty$.

The facts proved in this section, while not at all trivial in a topological sense, were for the most part visually obvious. We try to find the $\Delta$ value of various minor-closed classes, and in doing so, we move on to less obvious results.

## 4.2 Forb($K_5$) and forb($K_{3,3}$) graphs

Chapter 3 includes an overview of this section; reading it first is *strongly* recommended.

### 4.2.1 $K_5$-minor-free graphs

In [1], Georgakopoulos proved the existence of a countably infinite $K_5$-minor-free universal graph. As a corollary of his results, he obtained that every finite $K_5$-minor-free graph is a minor of another finite $K_5$-minor-free graph of maximum degree $\leq 22$. A natural question to ask is if this number can be lowered. Let forb($K_5$) be the class of $K_5$-minor-free graphs. We prove that Δ(forb($K_5$))=3.

**Definition 17.** Given graphs $G,\ H$ and isomorphic clique subgraphs $S_G \subseteq G$, $S_H \subseteq H$, their *clique sum* $G \oplus H$ *over common cliques* $S_G$ *and* $S_H$ is defined by identifying $G$ and $H$ over $S_H$ and $S_G$. We may denote this $G \oplus_{S_G, S_H} H$.

Theorem 3 by Wagner is essential.

We do not use the following observation, but it is nice to notice that for theorem 3 4-clique-sums would not add any extra graph creating power (Indeed, take Whitney's theorem that up to isomorphism, $K_4$ can be embedded in only one "manner" in the plane. Then notice that anything we add by 4-sums we could have added by at most 4 3-sums, one for each face of the $K_4$). Thus a nice way to reformulate this theorem is that $K_5$-minor-free graphs are precisely the clique-sum closure of planar graphs and $W[8]$.

The following two lemmas are the main mechanisms used in the proof that Δ(forb($K_5$))=3.

One is lemma 3.

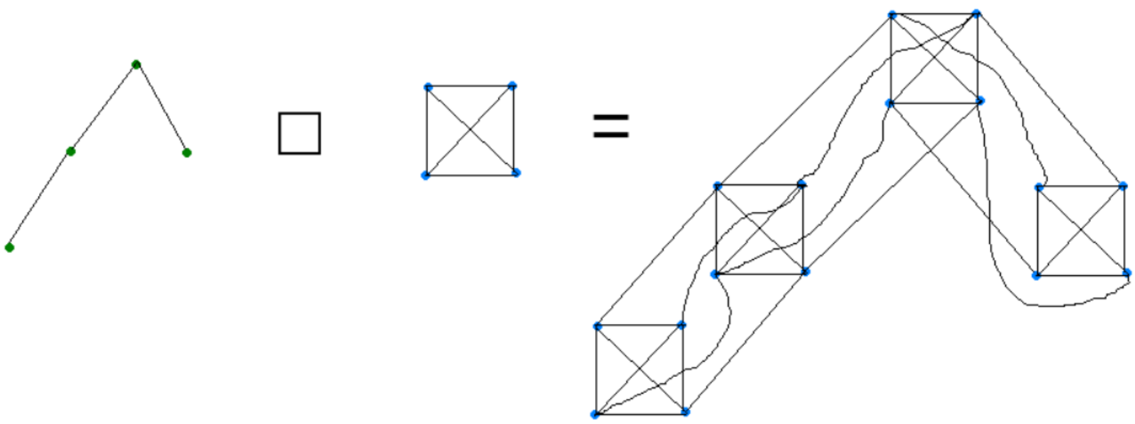

**Figure 4.6: The cartesian product of a tree and a 4-clique, visualized.**

Its proof is conceptually very simple; imagine $K_n \square T$ as a tree where instead of vertices we have cliques. Much like we can create any tree by adding each of its edges one by one starting from the root in a DFS or BFS manner, we can create $K_n \square T$ by adding each of its $n$-cliques in the same order.

*Proof.* Let there be graph $K_n \square T$ for some tree $T$. We have that $V(K_n \square T)$=($V(T) \times \{1, ..., n\}$) and $((t_1, v_1), (t_2, v_2)) \in E(K_n \square T) \iff t_1 = t_2$ or ($t_1$ neighbors $t_2$ in $T$ and $v_1 = v_2$).
The result is by induction of the number of vertices of $T$. If $T$ is the edge graph, then the result holds trivially. Now let $K_n \square T$ for all $T$ of some fixed number of vertices $n$. Let there be $T'$ of $n + 1$ vertices. This is constructed by some $T$ of $n$ vertices after adding a vertex $t_2$ to $T$ and joining it to the correct vertex $t_1$. We have $K_n \square T \in C$. Clique sum either of the cliques of $K_n \square P_2$ to the clique of $K_n \square T$ corresponding to $t_1$, i.e to the subgraph of $K_n \square T$ induced by $\{(t_1, i) | i \in \{1, ..., n\}\}$. The resulting graph is (isomorphic to) $K_n \square T'$: Relabel the new $n$ vertices as $(t_2, 1), ..., (t_2, n)$ and notice that $(t_2, i)$ neighbors $(t, j)$ iff $(t_2 = t)$ or $t_2$ neighbors $t$ in $T'$ and $i = j$). □

We remind $G_1 \oplus_{K_1, K_2} G_2$ is the clique sum of $G_1$ and $G_2$ over isomoprhic cliques $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$.

**Lemma 6.** *Let $P_1, P_2$ be some graphs. Let $P = P_1 \oplus_{K_1, K_2} P_2$. Let there be graph $P_1' \geq_m P_1$, let $\mu_1$ be the model, such that $\mu(K_1)$ has a clique $K_1'$ with one node in each branch and let there be similar graph $P_2'$. Then $P_1' \oplus_{K_1', K_2'} P_2' \geq_m P$.*

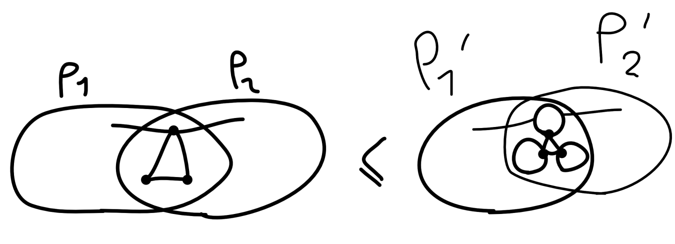


**Figure 4.7: Example for size 3 cliques of graphs $P_1$ and $P_2$. To the right the triangle $K_1'$ is depicted, one vertex in each branch of $K$.**

*Proof.* Let $\mu_1$, $\mu_2$ be the model functions mapping connected components of $P_i'$ to $P_i$. We define the branches of $P' := P_1' \oplus_{K_1', K_2'} P_2'$, i.e the model function $\mu$ from connected components of $P'$ to vertices in $P$. Let vertex $v$ of $P \notin$ the common clique, let it only $\in P_i$. Then $\mu(v) := \mu_i(v)$. Let $v \in$ the common clique. Then $\mu(v) := \mu_1(v) \cup \mu_2(v)$.

If $v \in P$, $v \notin$ the common clique, let it only $\in P_i$, then $(u, v) \in G \implies (u, v) \in P_i \implies \mu_i(u), \mu_i(v)$ are neighbors $\implies \mu(u), \mu(v)$ are neighbors .

If $v \in$ the common clique $K_1$ of $P'$, then $(u, v) \in P \implies (u, v) \in$ one of the $P_i$ containing $K_1 \implies \mu_i(u), \mu_i(v)$ neighbor $\implies \mu(u), \mu(v)$ neighbor. □

We now move on to the proof that Δ(forb($K_5$))=3. Our previous result for planar graphs is of use. It suffices to consider clique sums that do not remove edges. Furthermore, we divert our attention mostly to the case of 3-sums. The reader may fill in the rest easily.

Before diving in, let us explain the proof conceptually. We decompose the $K_5$ minor free graph, to the clique sum of planar graphs, and we replace each planar graph with a bigger planar graph of maximum degree 3 containing it as a minor. We add a few extra triangles so that clique sums between big planar graphs are still possible. The triangles are placed so that the clique sum of the big planar graphs contains the clique sum of the original planars as a minor. By adding enough such triangles, we never need reuse a triangle, keeping the maximum degree low. My approach bloats the graphs quite a bit; it is not my intention to present the most economical approach in vertex or edge number.

**Theorem 8.** *Δ(forb($K_5$))=3.*

Let $G$ be a $K_5$-minor-free graph. We construct the $K_5$-minor-free graph of maximum degree 3 containing $G$ step by step, because it makes the construction easier to understand and better motivated.

Let $G$ be a $K_5$-minor-free graph. Let $G_1, ..., G_k$ be its $\leq 3$-clique-sum decomposition into planar graphs and Wagner graphs, clique summed in this order. We can assume all embedded triangles $abc$ of (planar graphs) $G_i$ have either an empty interior or an empty exterior; for let this not be the case, then by the definitions of planarity and the Jordan curve theorem, the triangle is a separator, and thus it can be further decomposed into the 3-clique-sum of smaller planar graphs. By the Jordan-Schoenflies Curve Theorem, this region is homeomorphic either to the interior or the exterior of a circle $C$ of radius 1 on $\mathbb{R}^2$. One may then add a new triangle $a'b'c'$ to $G$, $a$ joined to $a'$, $b$ joined to $b'$, $c$ joined to $c'$, and embed it in the empty face. [1]

Do this for all triangles of $G_i$ to obtain graph $H_i$. See figure 4.8

[1]Visually, adding the triangle of course looks obvious, but for illustration purposes and since it's nice not to have gaps in our understanding, let's explain it. Let $H$ be the homeomorphism function, and w.l.g. let the empty face be homeomorphic to the interior of $C$. One may embed the triangle by e.g taking a circle of half radius to $C$ and same centre, noting the point $p_a$ where the line segment from $H(a)$ to the centre of $C$ intersects the smaller circle, let points $p_b$ and $p_c$ be defined in the same manner, and letting the embedded triangle be the embedded vertices $H^{-1}(p_a)$, $H^{-1}(p_b)$, $H^{-1}(p_c)$, and the embedded edges of the triangle be the the reverse under $H$ of the 3 arcs of the small circle. Similar arguments apply if the empty face of $abc$ is homeomorphic to the exterior of $C$.

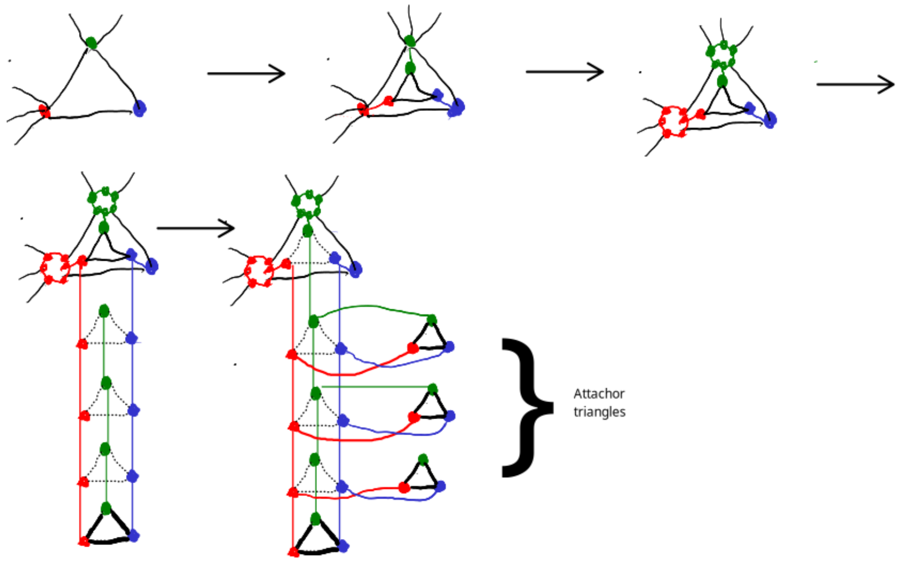


**Figure 4.8: A triangle of $G_i$ modified step by step. $G_i$ $H_i$, $G_i'$, $G_i''$, $G_i'''$ are pictured in order. By contracting along same-colored segments, we regain the original graph. By clique summing on attachor triangles, we keep the maximum degree low. Delete dotted edges after you're done.**

We call a triangle added in this manner on the empty face bounding $abc$ a *representor triangle* of $abc$, and denote it $a'b'c'$. Now let $G_i' \geq_m H_i$ be planar graph of maximum degree 3 created by $H_i$ by replacing each vertex $v$ with $Bl(v)$ as in the proof that $\Delta(PLANARS = 3)$, but leaving the vertices of representor triangles as is. This way, we can keep doing 3-sums. For every edge $uv$ of $G$, call the unique $Bl(u) - Bl(v)$ edge the representor edge of $uv$. For every vertex $u$ of $G$, add an additional vertex $u'$ to $G'$ and embed it on the circle $Bl(u)$ is embedded on, on the interior of an edge and let that $u'$ be the representor of $u$. Naturally, replace that edge $xw$ $u'$ is on with the edges $xu'$ and $u'w$, embedded on the circle.

**Theorem 9.** *$(G_1' \oplus ... \oplus G_k' \geq_m G_1 \oplus ... \oplus G_k)$, where if $G_i$ and $G_{i+1}$ were clique summed on common cliques $abc$ and $def$, $G_i'$ and $G_{i+1}'$ were clique summed on common cliques $a'b'c'$ and $d'e'f'$. See image 4.10. (Analogously, if $G_i$ and $G_{i+1}$ were clique summed on a common 1-clique or 2-clique, $G_{i+1}'$ were clique summed on the representors of those cliques).*

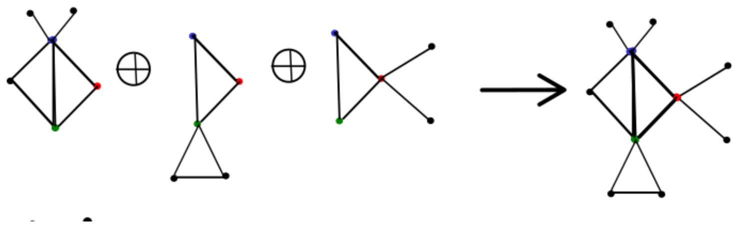

**Figure 4.9: The clique sum of 3 planar graphs, leading to a graph of max degree >3.**

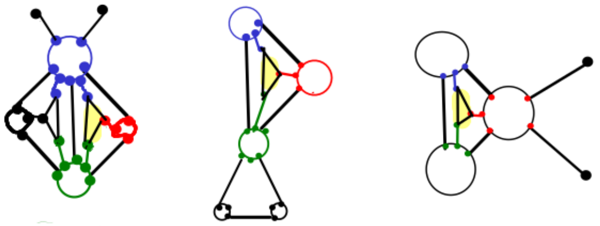

**Figure 4.10: The graphs $G_i'$ are clique summed over the shaded triangles now.**

We discuss only 3-sums from now on. 2 and 1 sums are completely analogous.

*Proof.* Notice that $G_i' \geq_m G_i$ by contracting each $Bl(v)$ to get back $v$ and for each representor triangle $x'y'z'$ contracting $x'$ to $x$, $y'$ to $y$, $z'$ to $z$. Therefore, let $\mu_i$ be the model function of $G_i' \geq_m G_i$, $x' \in \mu_i(x)$, $y' \in \mu_i(y)$, $z' \in \mu_i(z)$, and $G_1' \oplus G_2' \geq_m G_1 \oplus G_2$ by lemma 6. Furthermore, representor triangles in $G_1' \oplus G_2'$ continue to have a vertex in each branch of the triangle they model. $(G_1' \oplus G_2') \oplus G_3' \geq_m (G_1 \oplus G_2) \oplus G_3$ by lemma 6. Furthermore, representor triangles continue to have a vertex in each branch of the triangle they model, and so on. The result follows inductively. □

In this manner, we obtain a graph $G' = (G_1' \oplus ... \oplus G_k')$ containing $G$ as a minor, with all non-representor vertices having degree 3 or less. However, if an unbounded amount of clique sums occur on a specific representor, we could still get a $G'$ of unbounded degree. Utilizing clique sums, we make some additional modifications to $G_i'$. See figure 4.11.

Let $a'b'c'$ be a representor triangle in $G_i'$. Let $K_3 \square P_k$ be graph with vertex set $(\{1, 2, ..., k\} \times \{1, 2, 3\})$. We call the clique corresponding to the $nth$ vertex of $P_k$, i.e for fixed $n \in \{1, 2, ..., k\}$ we call the clique of $K_3 \square P_k$ induced by the vertices $(p, k)$ with $p = n$ the *nth clique* of $K_3 \square P_k$. Clique sum the 1st $K_3$ of a $K_3 \square P_k$ graph to a representor triangle $a'b'c'$ to obtain $G_i''$. We call the $n$th clique of a $K_3 \square P_k$ in $G_i''$ added in this manner to representor triangle $a'b'c'$ the $n$th copy of $a'b'c'$ (with this terminology, $a'b'c'$ is the 1st copy of $a'b'c'$). By lemma 3, the graph remains $K_5$-minor-free. Make the analogous modifications for 2 and 1 sums. Again, we discuss only of 3-sums - the reader may verify 2 and 1 sums have completely analogous proofs.

**Theorem 10.** *$(G_1'' \oplus ... \oplus G_k'' \geq_m G_1' \oplus ... \oplus G_k')$, where if $G_i'$ and $G_{i+1}'$ were clique summed on common cliques $a'b'c'$ and $d'e'f'$, $G_i''$ and $G_{i+1}''$ were clique summed on the $ith$ copy of $a'b'c'$ and $d'e'f'$. See images 4.9 and 4.10 again and then 4.11.*

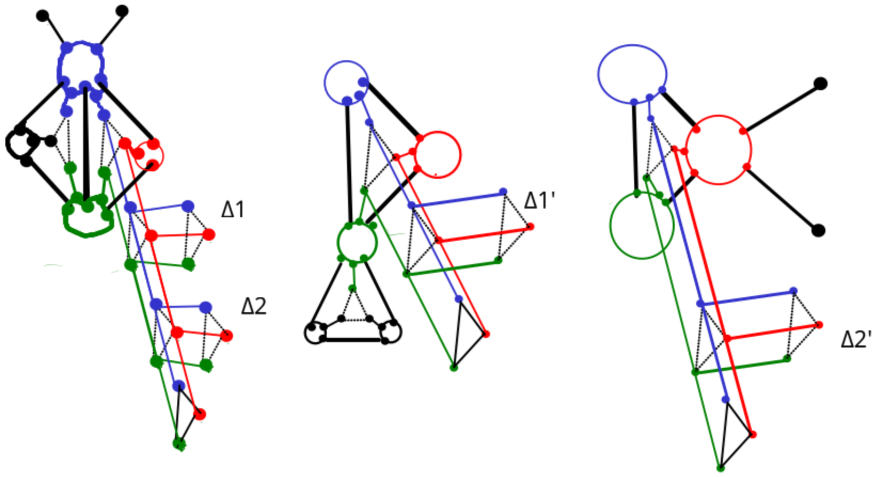


**Figure 4.11: The graphs $G_i''$ are clique summed over the attachor triangles now. $\Delta 1$ is summed to $\Delta 1'$ and $\Delta 2$ to $\Delta 2'$. By contracting along same colored components, we obtain $G$. Remove dotted edges after the last sum. This yields a graph of maximum degree 3.**

*Proof.* Notice that $G_i'' \geq G_i'$. This is done by contracting the first vertex of all copies of representor triangle $a'b'c'$ of $G_i''$, i.e the path of the $K_3 \square P_k$ induced by the vertices $(p, k)$ with $k = 1$. Then by contracting the second vertex of all copies of representor triangle $a'b'c'$, and the the third. Do this for all representor triangles. Notice that every copy has 1 vertex in each branch of the $a'b'c'$ model. By lemma 6, the result then follows inductively as in the previous proof. □

Notice that $G'' := (G_1'' \oplus ... \oplus G_k'')$ has maximum degree 6. Naturally we still call triangles in $G''$ copies if they came from a copy of $G_i''$ for some $i$. Vertices that don't belong to a representor copy have maximum degree 3 still. Unused copies have degree 4. At most, we have two copies of representor triangles clique summed on each other for a degree of 6. This can be reduced to 4 as well. Notice that the last copy of each representor remains unused.

*Claim* 1. Let $xyz$ be a copy of a representor triangle of $G''$ except the $k$th copy. $G'' \geq_m G'$ still holds after removing edges $xy,\ yz,\ zx$ of $G''$ and doing this for all such $xyz$.

*Proof.* Let $xyz$ be some representor. The model function showing $G'' \geq_m G'$ contracts the first vertex of each $xyz$ copy together, the second vertex of each copy together, and the third vertex of each copy together (regaining $xyz$). It suffices that one copy retain its edges, because the rest of the edges are redundant once the contraction is finished. □

Now non-copies have degree at most 3, and copies have at most 4. Can the maximum degree be reduced to 3? The answer is positive. We further modify the clique sums.

Let $a'b'c'$ be a representor triangle in $G_i'$. We clique sum to $a'b'c'$ the first spine clique of $K_3 \square T$ where $T$ is the $k$ comb. We call the spine cliques of $K_3 \square T$ the copies of $a'b'c'$ and the teeth clique the attachors. Do this for all representor triangles to obtain $G_i'''$.

**Theorem 11.** *$G_1''' \oplus ... \oplus G_k''' \geq_m G_1' \oplus ... \oplus G_k'$, where if $G_i'$ and $G_{i+1}'$ were clique summed on common cliques $a'b'c'$ and $d'e'f'$, $G_i'''$ and $G_{i+1}'''$ were clique summed on the attachor of the $ith$ copy of $a'b'c'$ and $d'e'f'$. This still holds after removing all edges of $(G_1''' \oplus ... \oplus G_k''')$ from $\Delta$ to $\Delta$, where $\Delta$ ranges over any copy of representor triangles and any attachor except the attachor of the copy numbered $k$.*

*Proof.* Notice that $G_i''' \geq G_i'$. This is seen by contracting each attachor to its copy to obtain $G_i''$. Attachors of copies of $a'b'c'$ still have one vertex in each branch of the $a'b'c'$ model. $G''' := G_1''' \oplus ... \oplus G_k''' \geq_m G'$ then follows inductively from lemma 6 as before. Furthermore, notice that in $G'''$ as all copies and attachors of a representor triangle $a'b'c'$ are contracted regaining $a'b'c'$, it suffices that one copy or attachor retain its edges to get $a'b'c'$ from the contraction. The other edges are unneeded. The attachor of the copy $k$ of $a'b'c'$ fills this role. ☐

Notice that $G'''$ after removing the aforementioned edges has maximum degree 3.

**Corollary 3.** *$\Delta(forb(K_5)) = 3$.*

### 4.2.2 $K_{3,3}$-minor-free graphs, a first lower bound and an afterthought

In this section, we will show that Δ(forb($K_{3,3}$))=4, that is, for every forb($K_{3,3}$) graph there is a forb($K_{3,3}$) graph of maximum degree 4 including it as a minor, but not all forb($K_{3,3}$) graphs have a forb($K_{3,3}$) graph of maximum degree 3 including the first a minor. This is the first example of a graph class with a bounded $\Delta$ value different than 3.

Just like with $K_5$-minor free graphs, Wagner discovered theorem 4, which is of use. Naturally, the proof that Δ(forb($K_{3,3}$))=4 repeats many of the arguments of the previous subsection. Let's center our attention at the proof that Δ(forb($K_{3,3}$))$\neq 3$, our first lower bound.

*Fact* 1. Let $G_1$, $G_2$ be two planar graphs. Then, their $\leq 2$-sum over some edge or vertex remains planar.

One may observe this by geometric intuition or by using Wagner's characterization of planar graphs, and the fact that the clique sums of two graphs cannot have higher Hadwinger number greater than both the first graph and the second.
This implies that to create a non-planar graph by clique summing planar graphs and $K_5$ graphs, one must use a $K_5$ at some point, which has vertices of degree 4. Now, observe that with the exception of a trivial 2-sum which only removes an edge, (we remind that one may use clique sums to remove any edge of a graph without adding any vertices), $\leq 2$-sums cannot reduce the degree of a vertex. We arrive at the following conclusion which we now prove:

**Theorem 12.** *If $G$ is non-planar $K_{3,3}$-minor-free graph, then $\Delta(G) \geq 4$.*

**Definition 18.** Let $G = G_1 \oplus G_2$, such that $V(G) = V(G_1)$ or $V(G) = V(G_2)$. In other words, the clique sum did not add any vertices. We call such a clique sum *trivial*.

*Proof.* Let $G = G_1 \oplus ... \oplus G_k$ be a series of 2-sums of planar graphs and $K_5$ graphs, creating a non-planar graph. By the above, at least 1 $K_5$ was used in the construction of $G$. Now, observe that:

- 1-sums cannot reduce the degree of vertex.
- We can assume that no trivial 2-sums occur; rather than remove an edge by a trivial clique sum, we can remove it after the last clique sum that utilizes the edge.
- If $(G_1 \oplus ... \oplus G_{i-1}) \oplus G_i$ is a 2-sum over common edge $uv$, we can assume that the degree of $u$ and $v$ in $(G_1 \oplus ... \oplus G_{i-1})$ and in $G_i$ is greater than 1; If not, let w.l.g $v$ have degree 1 in $G_i$, we can replace the 2-sum $(G_1 \oplus ... \oplus G_{i-1}) \oplus G_i$ on $uv$ with a 1-sum $(G_1 \oplus ... \oplus G_{i-1}) \oplus (G_i \setminus v)$ on $u$, and if the edge $uv$ was removed during the 2-sum operation, we add after the 1 sum a trivial 2 sum to remove it.

Thus, $G$ may be built by $\leq 2$-sums of planar graphs and $K_5$, no 2-sum being trivial or occurring over an edge with a vertex of degree $\leq 1$, and at least 1 $K_5$ must have been used during its construction. But notice that using these ingredients, once a graph $G_i$ has been clique summed during the building of $G$, none of its vertices can have their degree lowered in $G$. Therefore, the vertices of the $K_5$ graph must have degree $\geq 4$. □

Now, let there be non-planar $K_{3,3}$-minor-free graph $G$. For a $K_{3,3}$-minor-free $G'$ to include $G$ as a minor, $G'$ must also be non-planar of course. Therefore, it has $\Delta(G') \geq 4$. This proves that $\Delta(K_{3,3} - MINOR - FREE) \geq 4$.

As for the proof that every $K_{3,3}$-minor-free graph is a minor of a $K_{3,3}$-minor-free of maximum degree 4, the same arguments as for $K_5$-minor-free graphs apply. A proof sketch is given.

**Theorem 13.** $\Delta(forb(K_{3,3})) = 4$

*Proof Sketch.* Let $G$ be a $K_{3,3}$-minor-free graph built by the clique-sum $G_1 \oplus ... \oplus G_k$. Let $G_i'$ be the fattening $Bl(G_i)$ if $G_i$ is a planar graph and let it remain $K_5$ if $G_i$ is $K_5$. For every $uv$ edge in planar graph $G_i$, clique sum to the unique $Bl(u) - Bl(v)$ edge in $G_i'$ the first torso $K_2$ of the graph $K_2 \square T$ where $T$ is the $k$-comb. Do this for all $uv$ to obtain $G_i''$. If $G_i$ is a $K_5$ graph, clique sum $K_2 \square T$ on every edge to obtain $G_i''$ instead. $G_1 \oplus ... \oplus G_k \leq_m G_1'' \oplus ... \oplus G_k''$ where if $G_i$ is $\leq 2$ clique summed to $G_{i+1}$ on common cliques $uv$ and $wz$, $G_i''$ is $\leq 2$ clique summed to $G_{i+1}''$ on the attachor of the $ith$ copy of the representors of $uv$ and $wz$. Let $G'' := G_1'' \oplus ... \oplus G_k''$ and notice that $G$ is still included as a minor if we remove all edges corresponding to copies or attachors except the $k$th attachor (i.e all edges $uv$ where $uv$ is a copy or attachor)). Observe that after removing those edges, $G''$ has maximum degree at most 4, the 4 because of the $G_i$ isomorphic to $K_5$. □

*Remark* 1. There is something quite interesting to notice here. For a minor-closed class $C$, one way to reformulate the definition of $\Delta(C)$ is to define $\Delta(C)$ as the minimum $k$ so that $C$ =minor-closure$\{G \in C|\Delta(G) \leq k\}$. For classes $C$ of $\Delta(C) = k > 3$, one may ask what minor-closure$\{G \in C|\Delta(G) \leq 3\}$ is, or more generally, for any $k'$ smaller than $k$ what minor-closure$\{G \in C|\Delta(G) \leq k'\}$ is. For $K_{3,3}$-minor-free graphs the answer is easy; minor-closure$\{G \in$ forb($K_{3,3}$) | $\Delta(G) \leq 3\}$= the planar graphs, as every such $G$ is built by the 2-sum of planar graphs and subgraphs of $K_5$, which are also planar.
Repeating this question with other minor-closed graph classes of high $\Delta$, we may find elegant and natural graph classes, just as we did with $K_{3,3}$-minor-free graphs, and even undiscovered ones. As a foreshadowing, let TW$_{\leq k}$ be the class of graphs of treewidth $k$ or less. $\{G \in TW_{\leq k}$ | $\Delta(G) \leq 3\}$ lies strictly between $TW_{\leq k-1}$ and $TW_{\leq k}$. Could it be formulated as a variation of treewidth, like simple treewidth?

#### 4.2.3 $K_n$-minor free graphs for $n \geq 6$, $K_{n,n}$-minor-free graphs for $n \geq 4$.

The lack of structural theorems and characterizations for $K_6$-minor-free graphs makes them particularly hard to work with. Specific results giving some information that come to mind are [14] and [15] and of course the proof of Jorgersen's conjecture for large graphs [16], which aren't very helpful. It is thus nice that we are able to prove that the class of $K_6$-minor free graphs, has $\Delta(forb(K_6)) = \infty$. In fact, the following is a corollary of the main theorem of this thesis:

**Theorem 14.** *$\Delta(forb(K_n)) = \infty$, for all $n \geq 6$. $\Delta(forb(K_{n,n})) = \infty$, for all $n \geq 4$.*

### 4.3 Graphs of pathwidth $\leq k$, Graphs of treewidth $\leq k$

Chapter 3 includes an overview of the results on treewidth; reading it first is *strongly* recommended.

We have already defined treewidth through clique sums in section 3.2.

**Definition 19.** A graph has pathwidth $\leq k$ iff it can be constructed by the clique sum of graphs $G_1$, $G_2$, ..., each graph clique summed to the previous in the sequence, i.e. $(V(G_1) \cup ... \cup V(G_i)) \cap V(G_{i+1})$=$(V(G_i) \cap V(G_{i+1})$.

**Definition 20.** A graph is said to have treewidth = $k$ iff it has treewidth $\leq k$, but it doesn't have treewidth $\leq k-1$. Similarly for pathwidth.

Notice that in treewidth, by definition of clique sums each new graph we add as we build $G$ can be thought to be added to a single previous graph, i.e for all $i$ there is $j < i+1$ such that $(V(G_1) \cup ... \cup V(G_i)) \cap V(G_{i+1})$=$(V(G_j) \cap V(G_{i+1})$.

**Definition 21.** Let there be a graph $G$ constructible by the clique-sum of graphs $G_1, \ldots, G_k$, in this order. A corresponding *tree decomposition* is the tuple $(X, T)$ with set $X = \{V(G_1), V(G_2), \ldots\}$ and $T$ a tree with $V(T) = \{V(G_1), V(G_2), \ldots\}$. To define $E(T)$, for each $i \geq 1$ pick a single arbitrary $j < i+1$ such that $(V(G_1) \cup \ldots \cup V(G_i)) \cap V(G_{i+1})$=$(V(G_j) \cap V(G_{i+1})$, and have $V(G_i)$ be adjacent to $V(G_j)$. A *path decomposition* is a tree decomposition where $T$ is a path. The *width* of the decomposition is the size of the largest bag -1.

We call $V(G_i)$ the *bags* of $G$, and given $t \in T$ denote the corresponding vertex set $V_t$. Every tree decomposition has a corresponding clique sum sequence and vice versa [2].

**Definition 22.** Given a graph $G$, an *expansion* or *splitting* of $G$ is any graph $G' \geq_m G$.

In [13], Markov and Shi showed that every graph of treewidth $\leq$ k has a degree 3 expansion of treewidth $\leq k+1$, and that the +1 is necessary for $k \geq 19$, i.e, $\Delta(TW_{\leq k}) > 3$ for $k \geq 19$. We extend and simplify their results; let $TW_{\leq k}$ be the class of graphs of treewidth $\leq k$, and $PW_{\leq k}$ be the class of graphs of pathwidth $\leq k$. We show that $\Delta(PW_{\leq k}) = \Delta(TW_{\leq k}) = k$ for all $k$. Our proof that $\Delta(TW_{\leq k}) \geq k$ is notionally simpler in comparison.

We remind that a graph has treewidth $\leq k$ iff it can be constructed by the clique sum of graphs of $\leq k+1$ vertices. A graph has pathwidth $\leq k$ iff it can be constructed by the clique sum of graphs $G_1$, $G_2$, ..., each graph clique summed to the previous in the sequence, i.e. $(V(G_1) \cup \ldots \cup V(G_i)) \cap V(G_{i+1})$=$(V(G_i) \cap V(G_{i+1})$.
Proposition 4 is key, the statement still holding and the proof being same for $PW_{\leq n}$. It is proved in the same manner that one proves that the $n \times n$ grid has treewidth $\leq k$.

Instantly, we have as a corollary that $K_n \square P_i \in PW_{\leq n}$ for all paths $P_i$ of length $i$, and by lemma 3 that $K_n \square T \in TW_{\leq n}$ for any graph $T$. Let's first observe that every graph in $PW_{\leq n}$ has a degree 3 splitting in $PW_{n+1}$:

### 4.3.1 Pathwidth $\leq n$

Let there be graph $G$ of pathwidth $\leq n$, constructed by graphs $G_1, \ldots, G_k$ clique summed in this order. To observe that every graph in $PW_{\leq n}$ has a degree 3 splitting in $PW_{n+1}$, simply replace graph $G_i$ with the following graph $G_i'$: Take $G_i \square P_{|E(G_i)|+2}$, and let $P_{|E(G_i)|+2}$ have vertex set $p_1, p_2, \ldots$ and $G_i$ vertex set $u_1, u_2, \ldots$. Let $e_1, \ldots$ be the edges of $G_i$. Delete all edges except $e_1$ in the $G_i$ corresponding to $p_2$, delete all edges except $e_2$ in the $G_i$ corresponding to $p_3$ and so on. Use the leftmost and rightmost cliques to perform the clique-sums: Add to the $G_i$ corresponding to $p_1$ the clique $G_i$ was summed on

[2]Simply take the vertices of the tree to be $t_{H_i}$, take the bag of $t_{H_i}$ to be $V(H_i)$, and connect $t_{H_i}$ and $t_{H_j}$ in the tree decomposition if $H_i$ was chosen for $H_j$ to clique sum on. See [17] for a full and more detailed proof.

with $G_{i-1}$ and to the $G_i$ corresponding to $p_{|E(G_i)|+2}$ the clique $G_i$ it was summed on with $G_{i+1}$. This completes the construction of graph $G_i'$ of pathwidth $\leq n+1$ ($G_1$ and $G_k$ are replaced with $G_1 \square P_{|E(G_1)|+1}$ and $G_1 \square P_{|E(G_k)|+1}$ of course). $G'$ is defined as $G_1' \oplus G_2' ... \oplus G_k'$, $G_i'$ clique summed on $G_{i+1}'$ on their rightmost $G_i$ and leftmost $G_{i+1}$ copy of course. After clique summing $G_i'$ with $G_{i+1}'$, remove the edges of the clique. It is easy to see that $G' \geq_m G$ with maximum degree 3.
We move on to the proof that $\Delta(PW_{\leq n}) = n$. This is seperated in a lower and upper bound result. We first prove $\Delta(PW_{\leq n}) \leq n$.

**Proposition 8.** *$\Delta(PW_{\leq n}) \leq n$.*

*Proof.* Let there be pathwidth $\leq n$ graph $G = G_1 \oplus G_2 \oplus ... \oplus G_k$, clique summed in this order. It suffices to consider only the case where all the $G_i$ are isomorphic to the $n+1$-clique. All other $G$ in $PW_{\leq n}$ are subgraphs of such a graph. It also suffices to prove this for connected $G$.

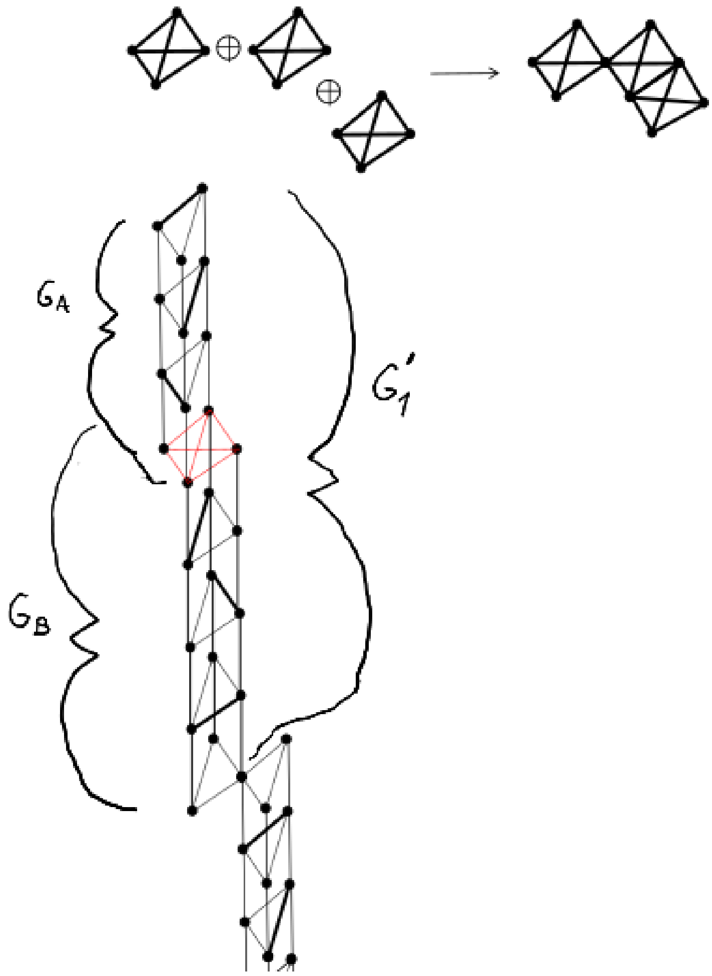


**Figure 4.12: The clique sum of 3 4-cliques to create $G$ and part of the corresponding $G'$ below it. $G_1'$ appears fully. The bold edge is the edge we do not remove in each triangle. It is easy to see that if we contract $G_A$ downwards, and $G_B$ upwards, we regain $G_1$.**

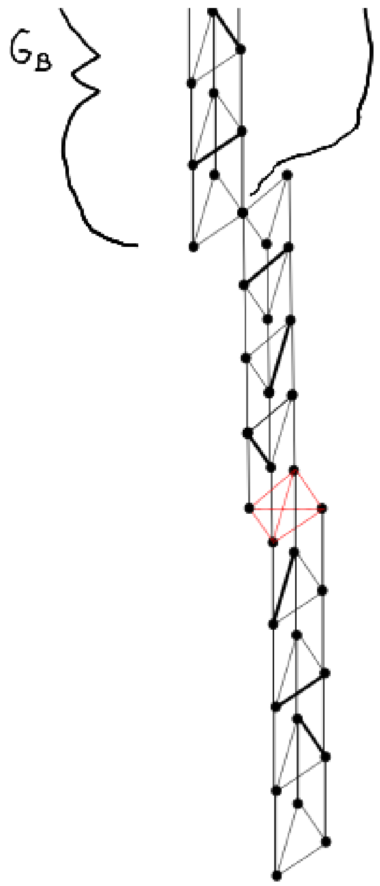


**Figure 4.13: The rest of $G_2'$ in $G'$ for illustration purposes.**

Let $v \in G_1$ be some vertex. Similarly with above, we define the following graph $G_1'$. See figure 4.12: Let $E = e_1, ...$ be the edge set of $G_1 \setminus v$. Let there be graph $G_A = (G_1 \setminus v)\Box P_{|E|+1}$, where $P_{|E|+1} = p_1 p_2 ...$ is the path graph of $|E|+1$ vertices, and $V(G_1 \setminus v) = \{u_1, u_2, ...\}$. Now remove all edges of $(G_1 \setminus v, p_1)$ except $e_1$, all edges of $(G_1 \setminus v, p_2)$ except $e_2$, and so on, and remove all edges of $(G_1 \setminus v, P_{|E|+1})$.
We wish to add $v$, and to do that we have to drop another vertex. Let $v' \neq v$ be some vertex in $G_1$ and not in $G_2$ (it is safe to assume such a vertex exists w.l.g.). Do the same in $G_1 \setminus v'$, i.e define $G_B = (G_1 \setminus v')\Box P_{|E'|+1}$, where $E'$ is the edge set of $G_1 \setminus v'$, and remove edges as before; remove all edges of $(G_1 \setminus v', p_1)$ except $e_1'$, all edges of $(G_1 \setminus v', p_2)$ except $e_2'$, and so on, and remove all edges of $(G_1 \setminus v, P_{|E'|+1})$, only this time keep the edges of the clique $G_1$ was clique-summed on to $G_2$ with (We shall use them for a clique sum. After the sum occurs, we shall remove those edges too).

Now take the disjoint union of $G_A$ and $G_B$ ($(G_1 \setminus v)\Box P_{|E|+1}$ and $(G_1 \setminus v')\Box P_{|E'|+1}$) and identify same named vertices from $(G_1 \setminus v, P_{|E|+1})$ and from $(G_1 \setminus v', P_1)$ to obtain $G_1'$.
This is a graph of width $n$ and maximum degree $n$ (if we forget about the edges needed for the clique sum, which will be removed anyway), and by contracting in $G_1'$ the subgraphs

($u_1$,$P_{|E|+1}$) and ($u_1$,$P_{|E'|+1}$) together into 1 vertex, ($u_2$,$P_{|E|+1}$) and ($u_2$,$P_{|E'|+1}$) together into 1 vertex, and so on, and ($v'$,$P_{|E|+1}$) into 1 vertex and ($v$,$P_{|E'|+1}$) into 1 vertex, we obtain $G_1$.
Do the same for the other $G_i$, only unlike before have $P_{|E|+2}$ instead of $P_{|E|+1}$, and have a clique on ($G_i \setminus v, p_1$) of $G_A$ and ($G_i \setminus v, p_{|E'|+2}$) of $G_B$ (for the sums). Clique sum $G'_i$ with $G'_{i+1}$ in the obvious manner, removing the edges of the cliques after the clique sum. It is simple to observe that $G'_1 \oplus G'_2 \oplus$ ... has maximum degree $n$, is of pathwidth $\leq n$, and contains $G$ as a minor by contracting as above.

□

We now move on to the second lower bound of this text. We need a graph $G$ of pathwidth at most $n$ such that any graph of pathwidth at most $n$ containing it as a minor has maximum degree $\geq n$. This graph is the following:

Let there be a $K_n$ clique with vertex set $\{1, 2, ..., n\}$. $n$-sum to it 1000 $n + 1$-cliques, let the $i$th be $\{1, 2, ..., n, i\}$. This completes the construction of $G$.

**Proposition 9.** *There is no graph $G'$ of pathwidth at most $n$ containing $G$ as a minor with $\Delta(G') < n$.*

The following well-known lemma (see e.g Diestel [10]) is of use:

**Lemma 7.** *Let $G$ contain an $n$-clique, let $G'$ contain $G$ as a minor, and let there be a tree-decomposition of $G'$. Then there is some bag of the tree-decomposition which contains a vertex from each minor branch of the $n$-clique.*

Path-decompositions being tree-decompositions, this theorem applies here as well. We now prove proposition 9.

*Proof.* Let there be graph $G' \in PW_k$ containing $G$ as a minor, and let $G'$ be created by the clique sums $G'_1 \oplus G'_2 \oplus$ .... By proposition 7, for any of the 3 ($n + 1$)-cliques of $G$ there is a $G'_i$ such that $G'_i$ contains a vertex of each minor branch of the $(n+1)$-cliques. Let $G'_i$, $G'_j$, $G'_k$ be these graphs, $i' \leq j' \leq k'$. Now, all graphs between $G'_i$ and $G'_k$ need to have a vertex from each branch of the central $K_n$ clique. Therefore, the extra node of $G'_j$ cannot be split. For let this be the case, let it be split into $u$ and $u'$, this edge does not fit anywhere. □

We move on to $TW_{\leq k}$. The reader will notice that arguments are naturally similar.

#### 4.3.2 Graphs of treewidth $\leq n$

We begin with the lower bound. In [13], Markov and Shi showed that there is a graph $G$ of treewidth $n$ and no degree $3$ expansion of treewidth $n$. The example graph $G$ we use is very similar in comparison and we now define it; let there be an $n+1$-clique graph with vertex set $\{1, 2, ..., n+1\}$, called the *central clique*. For every $n$-subclique with vertex set $\{1, ..., i-1, i+1, ..., n, n+1\}$, add a vertex labeled $i'$ and join it to the subclique, call this $n+1$-clique by the name $K_{n+1}^{(i)}$. This completes the construction of graph $G$. Markov's and Shi's example was the same, but they also removed all edges with both ends in the central clique of $G$. The following is both an extension and a notional simplification of their result.

**Proposition 10.** $\Delta(TW_{\leq n}) \geq n$

This is a slightly different proof to the one presented in the overview. Notation is a bit different here; For tree decomposition $(X, T)$, model carriers denote tree-decomposition bags rather than tree decomposition vertices $t \in V(T)$. Also, the bag of $t$ is denoted $V_t$ instead of $B(t)$.

*Proof.* Let $G' \geq_m G$ as a minor with model function $\mu$, where $G' \in TW_{\leq n}$. By lemma 7, for any tree-decomposition of $G'$, if there is an $n+1$ clique in $G$, there is some bag of the tree-decomposition which contains a vertex from each minor branch of the $n+1$ clique. Call this a *model carrier* of that $n+1$-clique.
Let there be a width $n$ tree-decomposition of $G'$. Notice that any tree decomposition vertex $t$ adjacent to the centre clique bag carrier $t_c$ must *drop* a centre clique bag node, i.e, for some $i \in \{1, ..., n\}$, $\mu(i) \cap V_{t_c}$ is not empty but $\mu(i) \cap V_t$ is, for there cannot be $n+1$ (possibly trivial) distinct paths from one bag to the other, as their intersection is a separator. Therefore there is a single centre clique model carrier. In fact this holds for all $n+1$ clique model carriers.
As every bag adjacent to the centre model bag must drop a vertex, the first internal vertex $t_{i'}$ on the path from the central bag carrier to the $K_n^{(i)}$ model carrier drops the bag vertex of $i$. Thus no vertex whose path to $t_c$ uses $t_{i'}$ may have a vertex of the minor branch of $i$. All such vertices induce a subtree of the tree-decomposition, with $K_n^{(i)}$ in it. Lacking vertices from the model of $i$, for $j \neq i$ no other $K_n^{(j)}$ model carrier is included in this subtree.

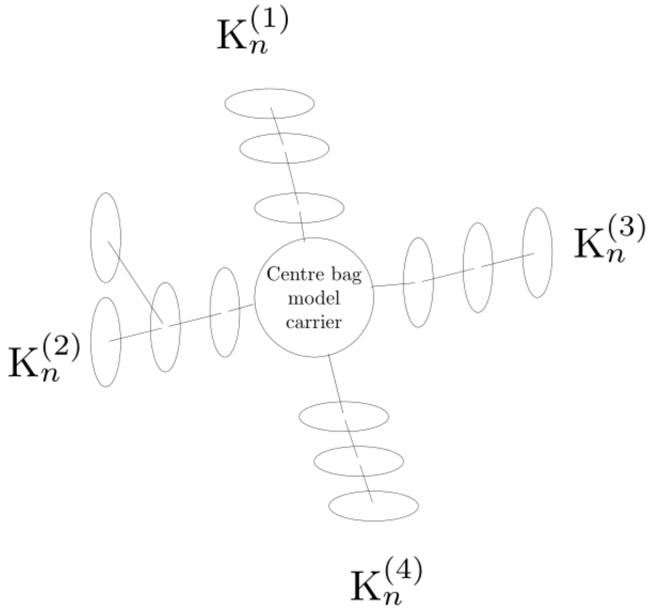


**Figure 4.14: Example tree-decomposition of $G'$ for $n = 3$. The centre bag model carrier separates the $K_n^{(i)}$.**

Let $v_i$ be both in the model carrier of $K_n^{(i)}$ and in the minor branch of $i'$. For $G'$ to include $G$ as a minor, there must be a path from $v_i$ to all $n$ nodes of the central bag carrier, except the one in the model of $i$. This path is internally disjoint to other such paths from a similar node $v_j$ of a $K_n^{(j)}$ carrier, $j \neq i$. A vertex in the centre bag model carrier and the model of $i$ thus receives $n$ internally disjoint paths from each of the $n$ $K_n^{(j)}$ model carriers, where $i \neq j$ Thus, each vertex of the central bag model carrier has degree $\geq n$. $\square$

We move on to the other direction. We have used the following ideas many times already, so we over them quickly.

**Proposition 11.** *$\Delta(TW_{\leq n}) \leq n$.*

Let $G$ be a graph produced by the clique sum of graphs $G_1$, $G_2$,..., $G_k$, in this order. It suffices to assume that the $G_i$ are isomorphic $n+1$-cliques, as $G$ made from such $G_i$ includes all other graphs in $TW_{\leq k}$ as a subgraph.
Just like with previous classes, let there be some $G_i$ with $n$-clique $K$, and let there be graph $T \Box K_n$ where $T$ is the $k+1$-comb graph, and $K_n$ has vertex set $\{u_1, ..., u_n\}$. Call the subclique of $T \Box K_n$ corresponding to the first spine vertex the first spine clique, and the subclique of $T \Box K_n$ corresponding to the first hair vertex the first hair clique. $n$-sum $G_i$ and $T \Box K_n$ by identifying $K$ and the first spine clique. Do this for all $n$ cliques of size $n$ of $G_i$ to obtain $G'_i$.
Call the $i$th spine clique of the $T \Box K_n$ attached to $K$ the $i$th copy of $K$, and the corresponding hair clique the $i$th attachor and call the entire $T \Box K_n$ the comb representor of

$K$. Also for any clique of $G_i$, call a clique of size $n$ of $G_i$ containing it a representor clique.

Obviously $G_i' \geq_m G_i$. It is not hard to observe that in $G_i'$, if we remove all edges of a comb representor with both endpoints in the same copy or attachor, but leave the last attachor (numbered $k+1$) intact, we still contain $G_i$ as a minor; simply contract the vertices of the comb representor corresponding to vertex $v_1$ of $T\Box K_n$, then contract the vertices corresponding to $v_2$, and so on for all $v_i$. We reobtain the original clique.
We now proceed to the clique sums.

**Proposition 12.** *$G_1 \oplus ... \oplus G_k \leq_m G_1' \oplus ... \oplus G_k'$, where if $G_{i+1}$ was $m$-summed to the $G_j$ subgraph of $G_1 \oplus ... \oplus G_k$, on isomorphic cliques $K$ and $K'$, then $G_{i+1}$' was $m$ summed to the $G_j'$ subgraph of $(G_1' \oplus ... \oplus G_i')$ on the following isomorphic cliques: The $i$th attachor of the clique representors of $K$ and $K'$.*

To obtain $G$ as a minor of $G' := G_1' \oplus ... \oplus G_k'$, for each $G_i'$, go to the $G_i'$ subgraph of $G'$, and for each $n$ clique $K$ of size $n+1$ of $G_i$, contract the vertices of the comb representor of $K$ corresponding to vertex $v_1$ (we remind, the clique $K_n$ of $T\Box K_n$ has vertex set $v_1, v_2, ...$), then contract the vertices corresponding to $v_2$, and so on for all $v_j$. It is easy to observe that doing this for all $G_i'$ subgraphs of $G'$, we obtain $G$.
Furthermore, if we remove all edges of a comb representor with both endpoints in the same copy or attachor but leave the last attachor (numbered $k+1$) intact, we still contain $G$ as a minor by the same contractions. Remove those edges from all comb representors to obtain $G''$.
We have observed that $G'' \geq_m G$. Furthermore, $\Delta(G'') = n$, as the original vertices of the $G_i$ in $G''$ and the last clique attachor of each comb has degree $n$, while other vertices of $G''$ have degree at most $n$. This completes the proof of the proposition.
By the two results of this subsection, we have that $\Delta(TW_{\leq n}) = n$.

# 5. MAIN THEOREM PART 1: MINOR CLOSURE OF CLASS CONTAINING ALL PYRAMIDS

A natural question to ask is if $\Delta$ is increasing with respect to the subset relationship. This is not the case; STARS $\subsetneq$ the class of planar graphs $\subsetneq$ the class of apex graphs (where STARS is minor closure of the class of stars), but their $\Delta$ value is $\infty$, 3 and $\infty$ respectively. We do however have the following: Let $\mathcal{A}$ be the class of apex graphs.

**Theorem 15.** *If a proper minor closed class $C \supseteq \mathcal{A}$, then $\Delta(C) = \infty$.*

This gives one direction of theorem 2, our main theorem. Formulated otherwise:

**Theorem 16.** *If for a minor closed class $C \supseteq \mathcal{A}$ it holds that $\Delta(C) = k \in \mathbb{N}$, then $C$ contains all graphs.*

For non zero natural numbers $N, M$, the $N \times M$ *grid graph* is the graph with vertex set $\{1, 2, ..., N\} \times \{1, 2, ..., M\}$ and edge set $\{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}$. See figure A.1.
The $N$*-pyramid* is the graph created by taking a $N \times N$ grid, adding a vertex, and joining it to all vertices of the grid.

Clearly a pyramid is an apex graph. As we now show, to prove Theorem 16, it suffices to prove the following: If a graph contains a large enough pyramid as a minor by a graph of $\Delta(G) \leq c$, then it contains an arbitrarily large clique.

**Theorem 17.** *For every $n,\ c \in \mathbb{N}$, there exists $N$ such that if $\Delta(G) \leq c$, and $G$ contains the $N$-pyramid as a minor, then $G$ contains $K_n$ as a minor.*

We prove theorem 17 implies theorem 16.

*Proof.* If $C$ includes all apex graphs as a minor with graphs of $\Delta(G) \leq k$ for some $k$, then it includes all $N$-pyramids with graphs of $\Delta(G) \leq k$, and then it includes all cliques. □

We thus now only focus on Theorem 17. Let $H$ be a subgraph of graph $G$. An $H$–path in $G$ is a path of $G$ internally disjoint from $H$ with endpoints in $H$. To prove theorem 17, the high level idea is to prove that if $\Delta(G) \leq c$ and $G \geq_m$ a large enough $N$-pyramid, then $G \geq_m$ an $N \times N$ grid $H$ with many $H$-paths, their endpoints positioned to our liking (Lemma 8). It is well-known that a large enough grid $H$ with $\binom{t}{2}$ $H$-paths with endpoints far apart from each other contains a $K_t$ clique: See lemma 9.

**Lemma 8.** *For every $n, c \in \mathbb{N}$, there is $N$ and $s$ such that if $\Delta(G) \leq c$ and $G$ contains the $N$-pyramid as a minor, then $G$ also contains as a minor the $N \times N$ grid, call it $H$, with $n$ pairwise edge-disjoint $S$-paths with discreet endpoints, where $S$ is any subgraph of $H$ of more than $s$ vertices.*

Let $G$ be a graph and $u, v \in V(G)$. The distance $d(u, v)$ is the length (number of edges) of the shortest path between them.

**Lemma 9.** *[18] If $G$ is a wall with pairwise disjoint $G$-paths $P_1, ..., P_{\binom{n}{2}}$ where $n > 1$, there exists $d \geq 0$ such that if any 2 $G$-path endpoints $p \in P_i$, $p' \in P_j$ have $d(p, p') \geq d$, then $G \geq_m K_n$.*

A *wall* is an $(n \times 2n)$ grid, where ordering edges from top to bottom for each vertical path, we remove from the first vertical path the even ordered edges, from the second vertical path the odd ordered edges, from the third the even ordered edges and so on. Finally we remove degree 1 edges and then arbitrarily subdivide edges.

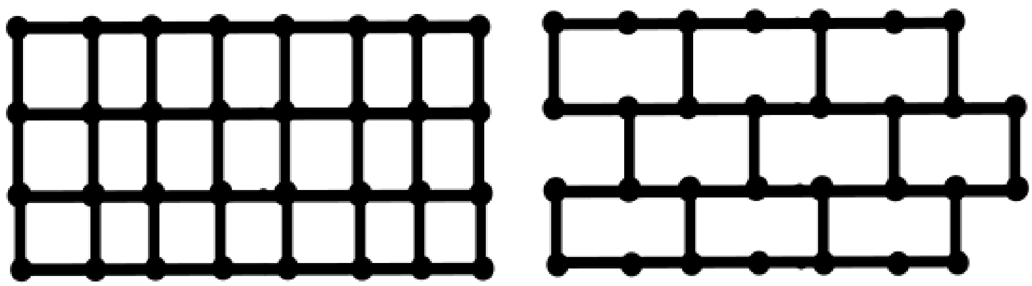

**Figure 5.1: A $4 \times 8$ grid and a corresponding wall**

Importantly, notice that lemma 9 also holds for $(n \times 2n)$ grids.
We are now ready to prove theorem 17.

*Proof of Theorem 17.* Fix some $n$ and $c$. We want to prove that for some large enough $N = N(c)$, if a graph $G$ has maximum degree at most $c$, it will contain $K_n$ as a minor if it contains the $N$-pyramind as a minor. By lemma 8, for some large enough $N$, $G$ will contain as a minor the $N \times N$ grid, call it $H$, with $\binom{n}{2}$ pairwise disjoint $H$-paths with discreet endpoints. Select some $(N/2 \times N)$ subgrid $H'$ of the grid, and have the endpoints be in a subgraph $S$ of $H'$ such that for all $u_1 \neq u_2 \in S$, $d_{H'}(u_1, u_2) \geq d$. By lemma 9, $G \geq_m K_n$. $\square$

We present a few corollaries before proving lemma 1.

**Corollary 4.** *If $C$ is a proper minor-closed superclass of the apex graphs, then Δ($C$) = ∞.*

The linklessly embeddable graphs are a well known 3-dimensional equivalent of the planar graphs. It is reasonable to ask if, like with planar graphs, one may by some geometric argument replace each node of a linklessly embeddable graph $G$ by some other structure to extend $\Delta(PLANARS) = 3$ to linklessly embeddable graphs. As the apex graphs are a subclass of the linklessly embeddable graphs, the answer is negative.

**Corollary 5.** *Let $\mathcal{L}$ be the class of linklessly-embeddable graphs. $\Delta$($\mathcal{L}$)=$\infty$.*

**Corollary 6.** *Let $C$ be a class containing all apex graphs as minors. For some $k$, let $f$ be any function mapping a graph to a graph containing it as a minor with maximum degree $k$. Then $f[C]$ contains all graphs as minors.*

Now follows the proof of lemma 8.

*Proof.* Let there be integer $n$. We would like to prove that if a graph $G$ of $\Delta(G) \leq c$ contains a big enough pyramid as a minor, let it be a $N(n,c)$-pyramid, let $S(n,c)$ be a big enough subgraph of its grid, it contains the $N(n,c) \times N(n,c)$ grid with $n$ pairwise edge-disjoint $S$-paths with discreet endpoints ($N$ and $S$ to be specified later).
So let $a$ be the apex vertex of the $N(n,c)$-pyramid and $X$ its grid and let $\mu$ be the model function mapping vertices of the pyramid to connected components of $G$. In $G$ contract $\mu(v)$ for all grid vertices $v \in X$ to obtain $X$. We will use $a$ to find $n$ jumps, with endpoints in $S \subseteq X$. We remove edges until $\mu(a)$ is a tree, and it has precisely one $\mu(a) - X$ edge towards each vertex of $S$ and 0 to $X \setminus S$.
Of course 2 vertices of $\mu(a)$ neighboring $S$ along with the path of $\mu(a)$ between them form an $S$-path, but $S$-paths being internally disjoint, using it could make us lose many other $S$-paths. How should we proceed?
We may assume all subtrees in $\mu(a)$ have a vertex neighboring $S$. If not, we remove them. We may also assume all vertices of $\mu(a)$ that only neighbor $\mu(a)$ have degree $\geq$ 3. If they have degree 1 we delete them, and if they have degree 2 we dissolve them. We then take a maximal path $P = u_1, u_2 ...$ in $\mu(a)$. Call the $u_i$ neighboring $X$ good vertices, and the rest bad. Bad $u_i$ vertices can be contracted into good vertices; since they must have degree>2 each must neighbor a subtree (which does not intersect $P$ or other such subtrees, else there would be a cycle), which must include a vertex neighboring $S$. Remove all other vertices of the subtree except the path connecting $u_i$ to the vertex neighboring $S$, then contract this path. Path $P$ now has only good vertices, every two of which form the internal vertices of an $S$-path. How large is $P$? Notice that at the time we pick it, $\mu(a)$ still has maximum degree $\leq c$ and as it neighbors every vertex of $S$, $\mu(a)$ still has more than $\frac{N^2}{c}$ vertices. Fixing $c$ and letting $N$ and thus $|V(\mu(a))|$ grow larger and larger, the diameter of $\mu(a)$ must also increase, and thus the length of its maximum path. Pick $s$ large enough for $\mu(a)$ to have diametre at least $2n$. Pick $N$ large enough $X$ can fit $S$. □

*Remark* 2. Nowhere in this lemma did we use the fact that $X$ is a grid. Indeed, rather than just pyramids, it holds for any infinite family of finite graphs as long as they all have a vertex connected to all other vertices.

# 6. MAIN THEOREM PART 2: A SUPERCLASS OF Δ = 3 FOR ANY CLASS EXCLUDING AN APEX GRAPH

**Definition 23.** A graph class is proper if it does not include all graphs.

We have proved that any proper minor-closed class including all apex graphs must have $\Delta = \infty$, and any attempts to relax this fact to smaller classes while working on this thesis had failed. On the other hand, given a minor-closed class $C$ excluding a planar graph, we have inspected that it is contained in a superclass $C'$ of finite $\Delta(C')$, in fact of $\Delta(C') = 3$. We suspect the following.

**Theorem 18.** *Let $C$ be a minor-closed class excluding an apex graph as a minor. There exists a proper minor-closed class $C' \supseteq C$ with $\Delta(C') = 3$.*

In [2] Dujmović, Morin and Wood proved that the following are equivalent for a proper minor-closed graph class $C$.

1. $C$ forbids an apex graph as a minor.
2. $C$ has bounded local treewidth.
3. $C$ has linear local treewidth.
4. Every graph in $C$ has bounded layered treewidth.
5. Every graph in G admits layered separations of bounded width.
6. For some $k$, every graph in $C$ can be constructed by the clique-sum of strongly $k$-almost embeddable graphs.

Theorem 16 in combination with theorem 18, complements this result by adding the following characterization:

**Theorem 19.** *A proper minor-closed class $C$ excludes an apex graph as a minor if and only if it has a minor-closed superclass $C'$ with $\Delta(C') = 3$.*

The class $C'$ of theorem 19 will by construction also exclude an apex graph. Furthermore, by theorem 16 one may replace $\Delta(C') = 3$ with $\Delta(C') \leq k$ for any finite $k$. Therefore, theorem 19 can be reformulated as:

**Theorem 20.** *A proper minor-closed class $C$ excludes an apex graph as a minor if and only if it has a minor-closed superclass $C'$ excluding an apex graph as a minor and with finite $\Delta(C')$.*

Theorems 19 and 20 give theorem 2.

We prove the equivalence of theorem 19 with condition 6 above. Condition 6 is a corollary of a strengthening [19] of the graph minor structure theorem of Robertson and Seymour [20]. The theorem of Robertson and Seymour says that much like $K_5$-minor-free graphs can be built by clique-summing planar graphs and the Wagner graph, so can the $K_n$-minor-free graphs be built by clique summing graphs from a correctly selected family, the family of $k$-*almost-embeddable* graphs.

**Theorem 21** (The graph minor structure theorem)**.** *Let there be a graph $H$, and let $G \in$ the $H$-minor-free graphs. Then $G$ can be constructed from the clique-sum of $k$-almost embeddable graphs, where $k = k(H)$.*
*Furthermore, it suffices to use graphs almost embeddable on surfaces that $H$ does not embed on (of genus $k$ or possibly less).*

As an instant corollary, the graph minor structure theorem also holds for minor-closed graph families excluding more than 1 graph as a minor.
Now let us define what a $k$-almost embeddable graph is. Rather than take a planar graph to clique-sum, we take a graph embeddable on some surface of euler genus at most $k$, we embed it, and then choose up to $k$ faces, to which we add potentially non-embeddable layers of "depth" $\leq k$. Finally we add $k$ apex vertices.

Let's start by defining the non-embeddable layers of an almost embeddable graph, called *vortices*.

**Definition 24.** Let there be a graph $G$ embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle [1] of $G$. Let there be graph $G'$, and add [2] $G'$ to $G$. Let there be a $C$-decomposition of $G'$ with bags $B_{v_1}, ..., B_{v_n}$. Pick a distinct node $u_i$ from each bag $B_{v_i}$, and in $G' + G$ identify $v_i$ and $u_i$ for all $i$ to obtain a new graph $G''$. *Adding a vortex* $G'$ to $G$ over $v_1, ..., v_n$ and $u_1, ..., u_n$ is defined to be this sequence of operations. If the $C$-decomposition of $G'$ has width $k$, then the vortex has *depth* $k$. We call $G'$ a *vortex* of $G''$.

The reader may picture the vortex added inside the face. Since we usually do not care about the specific choice of $G'$, we simply say we add a vortex to $G$ on $C$. We now proceed to define a $k$-almost embeddable graph.

**Definition 25.** Let there be a graph $G$. Let $G$ be embeddable on a surface of Euler genus $\leq k$. For some embedding, choose up to $k$ pairwise disjoint facial cycles of $G$. Add to each of them a vortex of depth up to $k$, to obtain $G'$. Finally, add up to $k$ vertices to $G'$ to obtain $G''$, called the *apex vertices of* $G''$, and join them to any vertex in $G''$ (including other apex vertices). $G''$ is called a $k$-*almost embeddable* graph. We call $G$ the

[1] A facial cycle is a cycle which is the boundary of a face of the embedded graph $G$.

[2] We remind we have defined the addition two graphs to be their disjoint union.

*embedded part* of $G''$ and call $G''$ *almost embeddable* on the surface $G$ was embedded on.

Reminding the minor structure theorem, for any $H$, all $H$-minor-free graphs can be constructed from the clique sum of $k$-almost embeddable graphs, where $k = k(H)$. For excluded minors $H$ belonging to a more specific family of graphs, there exist more specific results than the graph minor structure theorem; for apex graphs it is mentioned above. If $H$ is restricted to the planar graphs, then a $G \in$ forb($H$) can be constructed from the clique-sum of graphs of $\leq k$ vertices, where $k = k(H)$ (in other words, $treewidth(G) < k$). One could go on.

As already mentioned, on the other hand Dvořák and Thomas proved a strengthening of the graph minor structure theorem in the general case.

**Definition 26.** Given graph $H$ and surface $\Sigma$, let $\alpha(H, \Sigma)$ be the minimum number of vertices one need remove from $H$ to make it embeddable on $\Sigma$.

**Theorem 22** (The graph minor structure theorem strengthened [19])**.** *The graph minor structure theorem holds even if we only use graphs almost-embedded on surface $\Sigma$ such that every triangle of their embedded part is the boundary of a face homeomorphic to an open ball of $\mathbb{R}^2$, and all but $\alpha(H, \Sigma)$-1 of their apex vertices neighbor only other apex vertices and vortices.*

If $H$ is an apex graph, then $\alpha(H, \Sigma) = 1$ of course. Condition 6 of theorem 18 follows:

**Definition 27.** A *strongly $k$-almost embeddable* is a $k$-almost embeddable graph where also all apex vertices neighbor only other apex vertices and vortex vertices.

**Corollary 7.** *Let there be an apex graph $H$, and let $G$ $\in$ the $H$-minor-free graphs. Then $G$ can be constructed from the clique-sum of strongly $k$-almost embeddable graphs, where $k = k(H)$.*

As implied by theorem 18, the converse also holds; if there is $k$ such that every graph in some class can be constructed from the clique-sum of strongly $k$-almost embeddable graphs, then it excludes some apex graph.

The strengthened graph minor structure theorem has an important implication; We need only clique-sum almost embeddable graphs whose embedded part has no $K_4$ subgraph, or is trivially a $K_4$ graph.

**Corollary 8.** *Let there be connected graph $G \neq K_4$ embedded on some surface such that every triangle is the boundary of an open disc. Then $G$ has no $4$-cliques.*

*Proof.* Let there be a $K_4$ with vertex set $abcd$ in the graph $G$ with embedding $f$. As $G$ is connected and not a $K_4$, there must be a vertex $v$ adjacent to some vertex of $abcd$, let it be adjacent to $a$. $f(a)$ has an open disc containing it and an initial segment of each edge incident to it. Without loss of generality, let the incident edges be clockwise around $a$ in the order $ab,\ ac,\ ad,\ av$. Any face $a$ participates in must include two clockwise adjacent edges in its boundary . Therefore, there is no face including only $adb$ in its boundary. □

Naturally, the minor structure theorem would not be very interesting if it turned out that for some $k$ we can create all graphs using $k$-almost embeddable ones. The following is a well known fact.

**Theorem 23.** *Let there be $k \in \mathbb{Z}_{\geq 0}$. Let $C$ be the the class of all graphs that can be constructed by clique-summing $k$-almost embeddable graphs. Then $minor - closure(C)$ is proper.* [3]

This theorem holds for strongly $k$-almost embeddable graphs, as they are a subset of $k$-almost embeddable graphs [4].

In Jim Geelen's publicly available *Introduction to Graph Minors* course lectures, adding a vortex had a simpler definition, which is useful to us;

**Definition 28.** Let there be a graph $G$ embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle of $G$. Add a $K_k$ clique to $G$, and identify its first vertex to $v_1$. Add another $K_k$ clique, and identify its first vertex to $v_2$ and so on. The clique identified with $v_i$ is called the *vortex clique of* $v_i$. Now, join the clique of $v_1$ to the clique of $v_2$, join the clique of $v_2$ to the clique of $v_3$ and so on. Also join the clique of $v_1$ to the clique of $v_n$.
We call this sequence of operations as *adding a simple vortex of depth* $k$. The subgraph induced by the added cliques (i.e the union of the vortex clique of $v_i$ over all $i$) is a *simple vortex*. The circle induced by the $i$th vertex of all simple vortex cliques is the *ith layer* of the simple vortex. We always have $C$ be the 1st layer of the simple vortex.

Clearly this definition is different. The reader may notice that a simple vortex of depth $k$ is a vortex of depth $2k + 1$ (the +1 needed because decompositions have that pointless -1 in their definition). Now, a $k$-depth vortex need not be isomorphic to any simple vortex, for example take a vortex which has a vertex neighboring all vertices of the facial cycle (this is possible if the vertex is in all branches of the cycle decomposition). However, any $k$-depth vortex is a *minor* of a $(k + 1)$-depth simple vortex:

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[3] Indeed, for fixed $k$ none of the operations involved in constructing a $k$-almost embeddable graph can create an arbitrarily large clique minor; By Euler's formula for high genus (theorem 44), a graph $G$ embedded on a surface of euler genus $k$ must have at most $m \leq 3n - 6 + 3k$ where $n$ are the vertices and $m$ the edges of the graph, therefore too large a clique will not be embeddable on the surface. Graphs embeddable on a specific surface being closed under minors, $G$ can't have too large a clique minor either for specific $k$. Similarly, adding $k$ apex graphs can increase the Hadwinger number by at most $k$, and the clique sum of graphs $G_1$ and $G_2$ cannot create any larger clique minor either. For adding a vortex of depth $k$ cannot create an arbitarily large minor, and more on the minor structure theorem, we refer the interested reader to Jim Geelen's graph minor recorded lectures, lecture 3 [21].

[4] This is significantly useful for our purposes, as opposed to the other characterizations of the class of apex graphs in theorem 18, such as layered treewidth, where the minor closure of graphs of layered treewidth $k$ contains all graphs, even for $k = 3$. Indeed, the 3- dimensional $n \times n \times 2$ grid graph has layered TW 3 and a $K_n$ minor, take a row from the first level and a column from the second to be each branch.

**Proposition 13.** *Let there be embedded graph $G$ on some surface, with facial cycle $C = v_1, ..., v_n$ and add vortex $V$ of depth $k$ on $C$ to obtain $G'$. Alternatively, add to $G$ a simple vortex $sV$ of depth $k+1$ to obtain $G''$. $sV$ contains $V$ as a minor.*

*Proof.* Let $B_{v_i}$ be the bags of the C-decomposition of $V$ of width $k$. We slowly contract and remove nodes from $sV$ to prove it contains a $V$ minor. In $sV$, for all $v_i \in C$, remove vertices from the simple vortex clique of $v_i$ until it has as many vertices as $B_{v_i}$ does. Let's now specify the model function $\mu$. If $u \in B_{v_1}$ and $\in$ no other vortex bag, pick $\mu(u) = u'$ where $u'$ is a vertex belonging to the simple vortex clique of $v_1$. If $u \in B_{v_1}$ also belongs to other bags, $B_{v_{n-j}}, ..., B_{v_n}, B_{v_1}, ..., B_{v_i}$, pick an unused by $\mu$ vertex from the simple vortex cliques of $v_{n-j}, ..., v_i$, and let the path $P$ they define be modeled to $u$, i.e $\mu(P) = u$. Repeat this process for vertices of $B_{v_2}$ not in $B_{v_1}$ and so on. We never run out of unoccupied vertices in a simple vortex clique. If we do, let the simple vortex clique of $v_i$ be such a clique, then $B_{v_i}$ has more than $k+1$ vertices (a contradiction), as by construction of $\mu$ every occupied vertex of the simple vortex clique of $v_i$ corresponds to exactly one vertex of $B_{v_i}$. It suffices to prove that if $u$ and $u'$ are adjacent in $V$ then $\mu(u)$ and $\mu(u')$ are in $sV$. $u$ neighbors $u'$ in $V \implies$ they share a bag $B_{v_i} \implies$ (by construction) the simple-vortex clique of $v_i$ has a vertex which $\mu$ corresponds to $u$ and a vertex which $\mu$ corresponds to $u' \implies \mu(u)$ and $\mu(u')$ neighbor. ☐

**Corollary 9.** *Let there be graphs $G'$ and $G$ as above. $G' \geq_m G$.*

*Proof.* For vertices $u$ of $G'$ that are in the vortex $V$, let model function $\mu$ showing $G' \geq_m G$ be same as before, but making sure to set $\mu(v_i) = v_i$ for $v_i \in C$. If $u$ is not in the vortex, once again set $\mu(u) = u$. Let there be vertex $v \notin$ a vortex. $(v, u) \in E(G) \implies (v, u) \in E(G') \implies (\mu(u), \mu(v)) \in E(G')$. ☐

We are now ready to prove theorem 19. By theorem 7 any minor closed class $C$ excluding an apex graph can for some $k$ be built by the clique sum of strongly $k$-almost embeddable graphs $G$. We will show that any such graph $G$, is the minor of a graph $G'$ built by the clique sum of strongly $f(k^2+k)$-almost embeddable graphs with $\Delta(G') = 3$. Taking the graph class of all such $G'$, and taking its minor closure, we obtain a proper minor-closed graph class $C'$ of $\Delta(C') = 3$ which contains $C$.

Rather than instantly give the final construction, it is more natural to see it produced step by step, adding more ingredients in each step. For each intermediate step we prove a few facts which we reuse in the next steps. If $C$ is not a minor-closed class, set $\Delta(C)$ to be $\Delta(MINOR - CLOSURE(C))$.

Let $C_1(k)$ be the class of graphs of genus $\leq k$, embeddable so each triangle bounds an open disc.

Let $C_2(k)$ be the class of graphs that can be obtained by adding at most $k$ vortices of depth at most $k$ to a graph of $C_1(k)$ (the graph of $C_1(k)$ embedded so that each triangle bounds an open disc of course).

Let $C_3(k)$ be the class of graphs that can be obtained by adding at most $k$ apex vertices to a graph of $C_2(k)$, where the apex vertices may neighbor only other apex vertices and vortex vertices, i.e the class of strongly $k$-almost embeddable graphs. It is easy to see that much like planar graphs, $\Delta(C_1(k)) = 3$. We will add as few ingredients as possible; we will show that $\Delta(\oplus[C_1(k)]) = 3$. We will then show that $\oplus[C_2(k)]$ has a proper minor-closed superclass of $\Delta = 3$. We will then do the same for $\oplus[C_2(k)]$.

*Proof.* By [1], if a (finite) graph $G$ is embedded on a surface, for any $v \in G$ there is an open disc $D_v$ containing from $G$ only $v$ and an initial segment of edges incident to $v$ [5]. Take the discs small enough that their boundaries do not intersect. Erase everything inside the closed disc $D_v$ of $v$, then let $p_1, ..., p_k$ be the points where the boundary of the closed disc intersected the edges of $v$ $e_1, ..., e_k$, ordered in a counterclockwise manner. Add the $p_i$ back as embedded vertices $v_i$. Then, connect $p_i$ with $p_{i+1}$ by a curve running along the perimeter of the cycle. Call the resulting graph $G'$. Notice that $\Delta(G') \leq 3$ and $G' \geq_m G$, the model function is $\mu(v)$= all vertices of $G'$ embedded on $D(v)$. □

**Definition 29.** Given graph $G$, we call the graph $G' \geq_m G$ of maximum degree 3 as in the above proof the *fattening* or *ballooning* of $G$, and denote it $Bl(G)$. The cycle we replace vertex $v \in G$ with we denote by $Bl(v)$. [6]

We will prove that any graph $G$ built by the clique sum of graphs of $C_1(k)$ is a minor of a $G'$ built by the clique sum of graphs of $C_1(k)$ and $\Delta(G') = 3$. We will use theorem 1. Also notice the following.

**Proposition 14.** *Let $G \leq_m G'$. If $K' \in G'$ is a representor clique of $K \in G$ under $\mu$, we may remove from $G'$ all $\mu(u) - \mu(v)$ edges, except the edges of $K'$, for all distinct pairs $u, v \in K$ and still contain $G$ as a minor under $\mu$.* □

Almost entirely, in the following we want to restrict ourselves to a unique specific representor for each clique. This motivates the following definition.

**Definition 30.** Let $G \leq_m G'$ under $\mu$. Correspond to some cliques in $G$ a representor of theirs in $G'$. Call any such correspondence function from cliques in $G$ to representor cliques in $G'$ a *representation*. Call any 1-1 correspondence function a *1-1 representation* and if all cliques are represented call it *total*. Call the image of the correspondence function the set of *selected representors*.

[5] We may have to change the embedding a bit. Importantly, facial cycles remain same, and more generally the subgraphs induced by the boundary of faces remain same.

[6] This is also the model function showing $G' \geq_m G$

We have already given theorem 1. This theorem is a specialization of a slightly more general theorem. For a maximal clique of a graph $G$, call its representor clique in $G' \geq_m G$ a *max representor clique*.

**Theorem 24.** *Let $d \geq 3$. Let there be a minor-closed class $C$ closed under $n$-sums, such that $P_2 \square K_n \in C$. Let $B$ be a base for $C$ under $\leq n$-sums. For every graph $G$ in $B$, let there be graph $G'$ in $C$ with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in $G$ has a selected representor clique in $G'$.*
- *Every vertex $v$ of $G'$ of degree greater than $d$ has degree at most $d-s$ if we remove for every selected max representor clique $K$ it is in the edges of $G'[K]$, where $s$ is the number of selected max representor cliques $v$ is in.*

*Then $\Delta(C) \leq d$.*

This theorem is also a specialization of an even more general theorem! A *degree $k$ expansion* or *splitting* of $G$ is a graph $G' \geq_m G$ with $\Delta(G') = k$.

**Theorem 25.** *Let $d \geq 3$. Let there be a class $C'$ closed under $n$-sums, such that $P_2 \square K_n \in C'$. Let $B$ be a base for minor-closed class $C$ under $\leq n$-sums. For every graph $G$ in $B$, let there be graph $G'$ in $C'$ with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in $G$ has a selected representor clique in $G'$.*
- *Every vertex $v$ of $G'$ of degree greater than $d$ has degree at most $d-s$, if we remove for every selected max representor clique $K$ it is in the edges of $G'[K]$, where $s$ is the number of selected max representor cliques $v$ is in.*

*Then every graph in $C$ has an expansion of degree $\leq d$ in $C'$.*

We remind one notation we use for clique sums: Given graphs $G,\ H$ such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by the operation $G \cup H$. If $G \cap H$=$K$, denote this clique sum by $G \oplus_K H$.

**Lemma 10.** *Let $G = ((G_1 \oplus_{K_1} G_2) \oplus_{K_2} G_3) \oplus_{K_3}$ ... Let $G'_i \geq_m G_i$ be graphs with model function $\mu_i$ such that for every clique $K$ of $G_i$, $G'_i$ has a representor clique $K'$. Then $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3} ... =: G' \geq_m G$.* [7]

[7] $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3}$... is well-defined. If $G_{i+1}$ is clique summed on $((G_1 \oplus_{K_1} G_2) \oplus \ldots \oplus_{K_{i-1}} G_i)$ on common clique $K_i$, then $K_i$ must $\subseteq$ some graph $G_j$, $j < i$. $K_i \in G_j \implies K'_i \in G'_j \implies K'_i \in ((G'_1 \oplus_{K'_1} G'_2) \oplus \ldots \oplus_{K'_{i-1}} G'_i)$

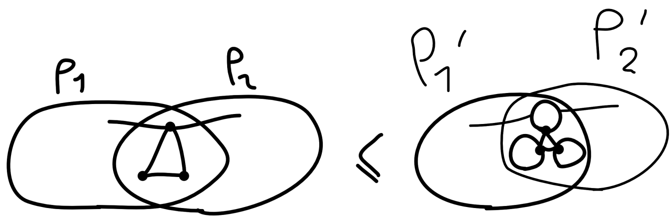



**Figure 6.1: Example where $K$ is a triangle. Graph $P_1 \oplus_K P_2$ is a minor of $P_1' \oplus_{K'} P_2'$**

*Proof.* Call any $K_j$ a *common clique*. We define the branches of $G'$, i.e the model function $\mu$ from vertices in $G$ to connected components of $G'$: $\mu(v) := \bigcup_i \mu_i(v)$, where $\mu_i(v) = \emptyset$ if $v \notin G_i$.

If $v \in G$, $v \notin$ any common clique, let it only $\in G_i$, then $(u, v) \in G \implies (u, v) \in G_i \implies \mu_i(u), \mu_i(v)$ neighbor $\implies \mu(u), \mu(v)$ neighbor.

If $v \in$ some common clique $K$ of $G'$, then $(u, v) \in G \implies (u, v) \in$ one of the $G_i$ containing $K \implies \mu_i(u), \mu_i(v)$ neighbor $\implies \mu(u), \mu(v)$ neighbor. $\square$

The proof is conceptually very simple; imagine $K_n \square T$ as a tree where instead of vertices we have cliques. Much like we can create any tree by adding each of its edges one by one starting from the root in a DFS or BFS manner, we can create $K_n \square T$ by adding each of its $n$-cliques in the same order.

*Proof.* Let there be graph $K_n \square T$ some tree $T$. We have that $V(K_n \square T)$=($V(T) \times \{1, ..., n\}$) and $((t_1, v_1), (t_2, v_2)) \in E(K_n \square T) \iff t_1 = t_2$ or ($t_1$ neighbors $t_2$ in $T$ and $v_1 = v_2$).
The result is by induction of the number of vertices of $T$. If $T$ is the edge graph, then the result holds trivially. Now let $K_n \square T$ for all $T$ of some fixed number of vertices $n$. Let there be $T'$ of $n + 1$ vertices. This is constructed by some $T$ of $n$ vertices after adding a vertex $t_2$ to $T$ and joining it to the correct vertex $t_1$. We have $K_n \square T \in C$. Clique sum either of the cliques of $K_n \square P_2$ to the clique of $K_n \square T$ corresponding to $t_1$, i.e to the subgraph of $K_n \square T$ induced by $\{(t_1, i) | i \in \{1, ..., n\}\}$. The resulting graph is (isomorphic to) $K_n \square T'$: Relabel the new $n$ vertices as $(t_2, 1), ..., (t_2, n)$ and notice that $(t_2, i)$ neighbors $(t, j)$ iff $(t_2 = t)$ or $t_2$ neighbors $t$ in $T'$ and $i = j$). $\square$

We proceed with the proof of theorem 25.

*Proof.* Let there be graph $G$ of $C$ built by the clique sum of base graphs $G_1 \oplus_{K_1} ... \oplus_{K_k} G_k$. Suppose there exist graphs $G_i' \in C'$ with the aforementioned conditions, where $\mu_i$ is the model function for $G_i' \geq_m G_i$. Notice that since every maximal clique in $G_i$ has a selected representor in $G_i'$, every clique in $G_i$ has a representor in $G_i'$. By lemma 10, $(G_1' \oplus_{K_1'} G_2' \oplus_{K_2'} ... \oplus_{K_k'} G_k') =: G' \geq_m G$, where $K_i' \in G_{i+1}'$ is a representor of

$K_i$ under $\mu_{i+1}$ and a representor of $G_j$ under $\mu_j$, $G_j$ being the graph of $G$ that $G_i$ was clique summed while building $G$.
The common cliques $K_i'$ of $G'$ could have an arbitrarily large degree, so we make some adjustments. As $P_2 \square K_n \in C'$ and $C'$ is closed under $n$-sums, by lemma 3 $T \square K_n \in C'$ where $T$ is the $k+1$ comb graph. We remind we call the subclique of $T \square K_n$ corresponding to the $i$th spine vertex of the comb the $i$th spine clique, and the subclique of $T \square K_n$ corresponding to the $ith$ hair vertex the $i$th hair clique. Furthermore, we call the sub-comb of $T \square K_n$ corresponding to $i$th vertex of $K_n$ the $i$th comb running along $T \square K_n$.
To each selected max representor clique $K'$ of $G_i'$, let $K'$ have $l$ vertices, $l$-sum a $P_2 \square K_l$, where $P_2$ is the path of two vertices. Call the $l$-clique of $P_2 \square K_l$ not used in the clique sum the *copy* of $K'$. To the copy of $K'$, $l$-sum the first spine clique of a $T \square K_l$, to obtain $G_i'' \in C'$. Call the $T \square K_l$ clique summed to the copy of $K'$ its *representor comb*. $G_i'' \geq_m G_i'$ of course, and let model function $\mu_i'$ showing that be $\mu_i'(v) = v$ if $v$ is not in a max representor clique and if $v \in$ some max representor clique $K$, let $v$ be the $j$th vertex of $K$, then let $\mu_i'(v)$ be the $j$th subcomb of the representor comb of $K$ and the $j$th vertex of $K$.
By construction of $\mu_i'$, if $K'$ is a selected max representor clique of $G_i'$, all spine and hair cliques of the representor comb of $K'$ in $G_i''$ are representors of $K'$ under $\mu_i'$. We may use lemma 10 again; $(G_1'' \oplus G_2'' \oplus ... \oplus G_k'') =: G'' \geq_m G'$, where if during the construction of $G'$ graph $G_i'$ was clique summed on the subgraph $G_j'$ on their common clique $K_i'$, then $G_i''$ is clique summed on $G_j''$ using the $i$th hair clique of the representor comb of $K_i'$ in $G_i'$ and the $i$th hair clique of the representor comb of $K_i'$ in $G_j'$.
Notice that lemma 10 gives us a specific model function $\mu'$ showing $G'' \geq_m G'$: The bag $\mu'(v)$ is the union of all $\mu_i'(v)$, if $v \in G_i$. By our choice of $\mu_i'$, we conclude that if $v$ is in a selected max clique of $G'$ under $\mu$, let $v$ be its $j$th vertex, then $\mu'$ puts in $\mu'(v)$ vertex $v$ of $G''$ as well as the entire $j$th subcomb of its representor comb. Thus, by proposition 14, $G'' \geq_m G'$ even if for every selected max representor we remove edges with both endpoints in the representor, and for its representor comb we remove all edges with both endpoints on the same spine or hair clique, except from one such clique. Let $G'''$ be $G''$ where we do just that, retaining only the edges of the last hair clique of every comb representor.
It suffices to prove that $\Delta(G''') \leq d$. As it turns out, we will need one more small change to do that. Let $v \in G'''$. We have the following cases.

- $v$ does not belong to any representor comb or selected max clique of $G'''$. In this case, $v$ also $\in G'$ and its degree remained unchanged during all of the above. $d_{G'''}(v) = d_G(v) \leq d$.

- $v$ belongs to what was a selected max-clique representor $K'$ in $G'$. If it has 1 vertex, then by construction $d_{G'''}(v) = 1$. For every selected max representor clique $K'$ it was in, we removed the edges of $G'[K']$ and connected $v$ to a copy of $K'$, and made no other changes to the edges of $v$. By the conditions of the theorem,

$d_{G'''}(v) \leq (d-s)+s=d$. Notice that $d_{G'''}(v) \leq d_{G'}(v)$, as the removal of each $G'[K']$ reduces the degree of $v$ by 1 at least, so we need only consider $v$ of $d_{G'}(v) > d$.

- $v$ belongs to the spine clique of a comb representor. $d_{G'''}(v)$ is at most 3; It is incident precisely to an edge with endpoint the previous spine clique, the next spine clique if it has one, and its hair clique.
- $v$ belongs to the hair clique of a comb representor. If the hair clique was not used in a clique sum and it is not the last hair clique, by construction $d_{G'''}(v)$=1. If it was used in a clique sum, by construction note that no hair clique is used in more than 1 clique sum, $d_{G'''}(v)$=2. If it is the last hair clique, let it have $l$ vertices, then by construction $v$ has degree $l$.

We now make changes to lower the degree of vertices of the last hair clique of a representor comb to 3, obtaining the intended claim. Let $K$ be a last hair clique, let its edge set be $e_1, ..., e_m$. Let there be graph $P_m \square K$, where $P_m$ is the path of $m$ nodes. Let the $K$ corresponding to the $i$th path vertex of $P_m \square K$ be called its $i$th clique, and the subpath corresponding to the $i$th clique vertex be the $i$th subpath running along $P_m \square K$. Clique sum to $K$ the first clique of a $P_m \square K$. Then remove from the $i$th clique all edges with both endpoints in the clique except $e_i$. It is easy to see that all vertices of a $P_m \square K$ added in this manner have max degree 3, and by contracting the $i$th subpath running along the $P_m \square K$ we get $G'''$. Doing this for all hair cliques yields a graph $G''''$ with the required properties. □

Using the previous lemmas, we can prove that $\Delta(\oplus[C_1(k)]) = 3$ fairly quickly.

**Proposition 15.** $\Delta(\oplus[C_1(k)]) = 3.$

*Proof.* We use theorem 1. The base $B$ of $\oplus[C_1(k)]$ is of course $C_1(k)$. Let there be graph $G \in B$. We can assume that every triangle has an empty interior or exterior, else it is a separator and we can further decompose $G$ to the clique sum of other base graphs. Let it be the interior, the other cases are analogous. On the open disc that has as boundary a triangle of $G$ with vertex set $abc$, add a new triangle $a'b'c'$ embedded there, and connect $a$ to $a'$, $b$ to $b'$, $c$ to $c'$. Let $G'$ be the ballooning [8] $Bl(G)$, except we have not ballooned the vertices of any of the new triangles. Notice that $\Delta(G') = 3$. $G' \geq_m G$ by contracting each $Bl(v)$ back into $v$, and for each new triangle, $a'b'c'$ to $a'$ to $a$, $b'$ to $b$, $c'$ to $c$. $a'b'c'$ in $G'$ is a representor of $abc$ in $G$. Let $\mu_1$ be this model function. Each 2-clique $uv \in G$ has as representor the by construction unique $Bl(u) - Bl(v)$ edge of $G'$. By theorem 1, we have $\Delta(\oplus[C_1(k)]) = 3$. □

[8] We remind a ballooning or fattening of $G$ means to replace each vertex $v$ with a cycle $C$ embedded on the boundary of an open disc around the vertex, the vertices of $C$ connected in a clockwise manner and each vertex of $C$ adjacent to a single neighbor of $v$.

We now add the next ingredient, vortices. We will use theorem 25 to show that $\oplus[C_2(k)]$ has a degree 3 expansion in $C' = \oplus[C_2(2k)]$. [9]. In other words, for every $G \in \oplus[C_2(k)]$, there is $G' \in \oplus[C_2(2k)]$ with $G' \geq_m G$ and $\Delta(G') = 3$. Putting all those $G'$ in a set, and taking the minor closure of the set, we obtain a minor-closed superclass of $\oplus[C_2(k)]$ of $\Delta = 3$ which is proper by theorem 23.

**Proposition 16.** *$\Delta(\oplus[C_2(k)])$ has a proper minor-closed superclass of $\Delta$ =3.*

Once again, the base is $C_2(k)$. Let there be graph $G$ in $C_2(k)$, with embedded part $Emb(G)$ and at most $k$ vortices of depth at most $k$ added to pairwise disjoint facial cycles $C_1, ...C_k$.
Let $G'$ be $G$ with every vortex of depth $d$ replaced by a simple vortex of depth $d+1$, as in proposition 13 and corollary 9. Use the model function defined there, call it $\mu_{sv}$. Observe that there is a representation $R_{sv}$ under $\mu_{sv}$; if a clique $K$ of $G$ is in $Emb(G)$ trivially $R_{sv}(K) = K$. If a clique $K$ of $G$ is not in $Emb(G)$, it is in a vortex. In this case, let its facial cycle be $C = v_1 v_2...$, then there must be a vortex bag $B_{v_i}$ it is in. By construction of $\mu_{sv}$, every vertex of $B_{v_i}$ contains in its model in $G'$ a distinct vertex of the simple vortex clique of $v_i$. But every vertex in the simple vortex clique of $v_i$ is adjacent. $R_{sv}(K)$ is those simple vortex vertices.
As clique representation is transitive under minors, it suffices to find for every $G'$ a graph $G'' \geq_m G'$ of $\oplus[C_2(2k+1)]$ such that there is a representation under some model function $\mu$ satisfying the conditions of theorem 25. Then, there will be such a representation for $G'' \geq_m G$ under $\mu \circ \mu_{sv}$.
Add triangles and repeat the same fattening procedure as before on $Emb(G)$ to obtain $Emb(G)'$. This time, rather than add 1 extra triangle $a'b'c'$ to the empty face of triangle $abc$ of $Emb(G)$, we add two triangles $a'b'c'$ and $a''b''c''$, $a'b'c'$ embedded on the empty face bounded by $abc$, $a''b''c''$ on the empty face bounded by $a'b'c'$, $a$ joined to $a'$, $a'$ joined to $a''$ and so on. Both new triangles are not fattened. Call $a'b'c'$ and $a''b''c''$ the first and second *copies* of $abc$ respectively. Fortunately, after fattening facial cycles are (almost) retained:

**Definition 31.** For $v \in Emb(G)$, let $D_v$ be the disc on the boundary of which the cycle $Bl(v)$ was embedded on. Let $Bl(v \to u)$ or $Bl(u \leftarrow v)$ be the vertex of $Bl(v)$ incident to the unique $Bl(v) - Bl(u)$ edge of $Emb(G)'$.

If $C = u_1...u_n$, where $n > 3$ is a facial cycle in $Emb(G)$, then there is a facial cycle $C''$ in $Emb(G)'$, first with 1 or 2 vertices from $Bl(u_1)$, then with vertices from $Bl(u_2)$, and so on: Start from the vertex $Bl(u_1 \to u_2)$. Follow the $Bl(u_1) - Bl(u_2)$ edge to $Bl(u_2 \to u_1)$. If $d_{emb(G)}(u_2) > 2$, there is an edge $Bl(u_1 \leftarrow u_2) - Bl(u_2 \to u_3)$ in $Bl(u_2)$. Follow along it. Then take the $Bl(u_2 \to u_3)$ edge and so on. Call $C''$ the *corresponding* facial cycle of $C$. For triangles of $Emb(G)$ call their second copy in $Emb(G)'$ the corresponding facial cycle.
If to construct $G'$ a simple vortex of depth $k$ was added to a facial cycle of $Emb(G)$, add

[9] In fact, we can show that $\Delta(\oplus[C_2(k)])$=3

to the corresponding facial cycle of $Emb(G)'$ a simple vortex of depth $k$ to obtain $G''$. We prove $G''$ fulfils the conditions of theorem 25.

- To prove that $G'' \geq_m G'$, let $\mu_2$ be the model function showing that, for $v$ in the embedded part of $G''$ let $\mu_2(v) = \mu_1(v)$, where $\mu_1(v)$ is the model function of the proof that $\Delta(\oplus[C_1(k)]) = 3$, modified by putting $a''$ in the same bag as $a'$ and $a$ for triangles $abc \in G'$ of course. For $v \in$ a vortex, let the facial cycle be $C = v_1 v_2 ...$ and let $v$ belong to the simple vortex clique of $v_i$, let $v$ be the $i$th vertex of the clique. Let $C''$ be the corresponding facial cycle and notice $C''$ of $G''$ is also in $Emb(G'') = Emb(G)'$. If $C = v_1 v_2 v_3$, then $C'' = v_1'' v_2'' v_3''$ and let $\mu_2(v)$ be the $i$th vertex of the simple vortex clique of $v_i''$. Else, set $\mu_2(v)$ to be the $i$th vertices of the vortex cliques of $Bl(v_{i-1} \leftarrow v_i)$ and $Bl(v_i \rightarrow v_{i+1})$. It is easy to observe that the contraction in $G''$ of each minor bag $\mu(v)$ yields $G'$.

- We find a representation $R_2$ under $\mu_2$ so each maximal clique $K$ is represented. For a cliques $K$ of $Emb(G)$, set $R_2(K) = R_1(K)$, where for triangles $K$ we use their first copy in $G''$ to represent them.
  With regard to simple vortex cliques $K$ of $G'$, let the simple vortex be of depth $l$ and added on the facial cycle $C = u_1 u_2 ... u_n$. There are precisely $n$ maximal cliques of $2l$ vertices; the simple vortex clique of $u_i \cup$ the simple vortex clique of $u_{i+1}$, for $i \in \{1, ..., n\}$, where $u_{n+1} = u_1$. Its selected representor $R(K)$ in $G''$ is the simple vortex clique of $Bl(u_i \rightarrow u_{i+1}) \cup$ the simple vortex clique of $Bl(u_i \leftarrow u_{i+1})$.

- We prove the third condition. If $v \in G''$, is not in a vortex, then by construction it has max degree 3 unless if it is in the first copy $a'b'c'$ of a triangle $abc$. In this case it is a selected representor of $abc$, and it represents no other cliques. For the condition to be satisfied it must have at most $3-1$ edges adjacent to it, after removing the edges of $a'b'c'$, which is the case. If $v$ is in a vortex, notice that all edges of the vortex have both endpoints in a selected max clique representor, and $v$ belongs to exactly 2 selected representors. After removing the edges of the selected cliques, $d(v) = 1$ if $v$ is on the facial cycle, and $d(v) = 0$ otherwise, satisfying the condition.

Therefore, every $G \in \oplus[C_2(k)]$ has a degree 3 expansion in $G' \in \oplus[C_2(2k)]$. Taking the minor closure of all such $G'$, we obtain a proper minor-closed class of $\Delta$ 3 containing $\oplus[C_2(k)]$.

We now add the final ingredient, apex vertices only neighboring other apex vertices and vortex vertices. We will prove that $\oplus[C_3(k)]$, i.e the clique sum closure of strongly $k$-almost embeddable graphs has a proper minor-closed superclass of $\Delta = 3$. By characterization 6 of the minor-closed classes excluding an apex graph, we thus obtain the right direction of theorem 19.

**Proposition 17.** *$\oplus[C_3(k)]$ has a proper minor closed superclass of $\Delta = 3$.*

Let $G \in C_3(k)$. We will find an expansion of $G$ in $C_3(k^2+k)$, satisfying the conditions of theorem 25. Naturally, the base $B$ is once again $C_3(k)$ and $C'$ is $\oplus[C_3(k^2+k)]$. It suffices to consider only $G$ whose apex vertices neighbor all other apex vertices and all vortex vertices. All other graphs in $C_3(k)$ are subgraphs of such graphs and if $G_1 \subseteq G_2 \leq_m G'$ where $G_2 \leq_m G'$ has a representation under $\mu$ satisfying the conditions of theorem 25, so does $G_1 \leq_m G'$.
Let $C$ be a facial cycle of $Emb(G)$. Let $G'$ be $G$ where instead of adding a vortex of depth $k$, we add a simple vortex of depth $k+1$ to $C$, and then connect all apex vertices to it. As in the previous proposition, $G' \geq_m G$ under a model function $\mu_{sv}$, and there is a total representation $r$ under $\mu_{sv}$: If $K$ is a clique not intersecting the apex vertices, $r(K) = R_{sv}(K)$ as we have already explained in the previous proposition. If $K$ intersects only apex vertices, then trivially $r(K) = \mu_{sv}(K) = K$. If $K$ intersects apex and the simple vortex's vertices, let the subcliques comprised by those vertices be $K_a$ and $K_{sv}$ respectively, then $r(K_a) = K_a$, and $r(K_{sv}) = R_{sv}(K_{sv})$.
Therefore it suffices to prove theorem 25 for $G'$ in the place of $G$. We now construct the expansion $G'''$ of $G'$ with the desired properties; let $G''$ be defined exactly as in the previous proposition (fatten $emb(G)$ as in the previous proposition, adding two copies to the empty face of each triangle), apex vertices neighboring all vortex vertices and all other apex vertices.We still have to lower the degree of apex vertices.

**Definition 32.** Define the cycle induced by the $i$th vertex of all simple vortex cliques of a simple vortex to be the *ith layer* of the simple vortex. We always have $C$ be the 1st layer of the simple vortex.

We replace each simple vortex of depth $k+1$ of $G''$ with a simple vortex of depth $2k+1$. Apex vertices no longer neighbor all vortex vertices; instead, give some ordering to the apex vertices, the $i$th apex vertex neighbors a single vertex of the $k+1+i$th layer of the first clique of the simple vortex. Finally, for each apex vertex $a$, add to $G''$ a path of $a_1 a_2 ... a_{k+1}$, identify $a$ with $a_1$, remove the edge between $a$ and its $i$th vortex neighbor and have the $i$th vortex neighbor be adjacent to $a_{i+1}$ instead. Call this the *representor path* of $a$. This completes the construction of $G'''$. Notice that, treating the vertices of path representors as apex vertices, $G''' \in C_3(k(k+1))$ It now suffices to prove the three conditions of theorem 25.

- $G''' \geq_m G'$: For the $i$th apex vertex $v$ of $G'$, let $\mu_3(v)$ be the $i$th apex vertex of $G'''$ together with its representor path, together with the $(k+1+i)$th layer of all simple vortices. Otherwise, let $\mu_3(v)$ be $\mu_2(v)$ as in the previous proposition.
- Let $R_3(K)$ be the representation. By the previous proposition, we have that every maximal clique $K$ not having apex vertices has a representation $R_2(K)$. Let $R_3(K) = R_2(K)$ in this case. If $K$ is the set of all apex vertices of $G'$, then $R_3(K) = K$. If $K$=$K_a \cup K_{sv}$ is a set of apex vertices and simple vortex vertices of $G'$, which by construction and maximality of $K$ must consist precisely of all apex vertices and the simple vortex cliques of two consecutive facial cycle vertices, let

them be $c_i$ and $c_{i+1}$, then $R(K)$ is the two simple vortex cliques of $c_i$ and $c_{i+1}$ in $G''$.

- If $v \in G'''$ is an original apex vertex, then it belongs to a single max selected representor, that of all apex vertices. It has degree 1 excluding edges from that clique. If it does not, but still belongs to a path representor of an apex vertex, then it has degree 3 and belongs to no representor clique. If $v$ is not an apex vertex, the same as in the previous proposition holds.

This completes the proof of the right direction of theorem 19.

## 7. EXISTENCE OF COUNTABLY INFINITE $K_5$-UNIVERSAL GRAPH.

Given a class of infinite countable graphs $C$, a universal graph $G$ is a graph such that $G >_m G'$ for all $G' \in C$. In [1], Georgakopoulos proved that there is a universal $K_5$-minor-free graph. The following is a simplification of this result.

**Theorem 26.** *There is a universal $K_5$ minor free graph.*

In [1], Georgakopoulos proved the existence of a countably infinite $K_5$ universal graph with regard to the minor relationship. I later reproved his result with a simpler construction. This section is a quickly written sketch of this proof. In this chapter, if $G' \geq_m G$ with model $\mu$ and $S \subseteq G$, we denote $\mu(S)$ by $G'^S$ instead.

For the remainder of this proof, we may assume without loss of generality that clique sum operations do not remove edges of the clique.

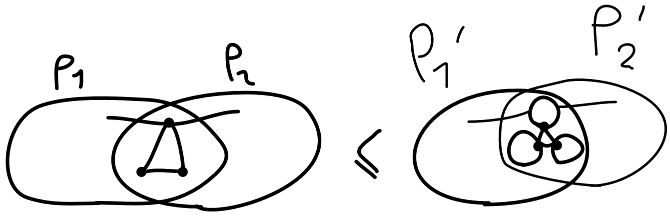
Let K$_5$f be an infinite K$_5$ free graph. By the paper of Thomas and Kritz [22], there exists a sequence $\{G_\alpha\}_{\alpha \leq \text{countable } \lambda}$ such that $G_{a+1} = G_a \oplus^3 P_a$ where $P_a$ is planar (or w[8]) and $G_\lambda = K_5 f$ and $G_a = \liminf_{\beta<a} G_\beta$. Let $\{P_\alpha\}_{\alpha\leq\lambda}$ be the corresponding planar graphs (or w[8]). Let $P_{N(0)}, P_{N(1)}, \ldots$ be some enumeration of them. We print $P_0$, then dovetail the enumeration and print $P_{N(i)}$ once the $\leq 3$ nodes it was clique-summed on during the construction of $K_5 f$ have already been printed (don't print already printed $P_{N(i)}$). Seeing clique sums as a union of graphs, it is easily seen that an ordering $\{P_\alpha\}_{\alpha\leq\omega}$ arises such that $G_0 = P_0$, $G_{a+1} = G_a \oplus^3 P_{a+1}$ and $G_\omega = K_5 f$. More generally,

**Theorem 27.** *Let a countable graph be $k$-summable over some $\Gamma$ for some finite $k$, let the corresponding sequence be $\{G_\alpha\}_{\alpha \leq \text{countable } \lambda}$. Then there also exists such a sequence of the form $\{G_\alpha\}_{\alpha\leq\omega}$*

In the case clique sums remove edges this still holds. Break $\{G_\alpha\}_{\alpha \leq \text{countable } \lambda}$ in two sequences, one not removing and the other only removing edges.

So let K$_5$f = $((P_1 \oplus_{\Delta_1} P_2) \oplus_{\Delta_2} P_3) \oplus_{\Delta_3}$ ..., for a class of countable planars $P_i$ (or w[8]).

**Lemma 11.** *Let $G = ((P_1 \oplus_{\Delta_1} P_2) \oplus_{\Delta_2} P_3) \oplus_{\Delta_3} \ldots$ for arbitrary countable graphs $P_i$ and cliques $\Delta_i$, where for some $k \in \mathbb{N}$ all $\Delta_i$ are of size at most $k$ . Let $P_i' > P_i$ be graphs such that for every clique $\Delta$ of $P_i$ of size $\leq k$, ${P_i'}^\Delta$ has a clique $\Delta'$ with one node in each branch. Then $((P_1' \oplus_{\Delta_1'} P_2') \oplus_{\Delta_2'} P_3') \oplus_{\Delta_3'} \ldots =: G' >_m G$.*

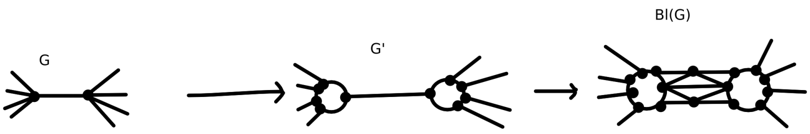


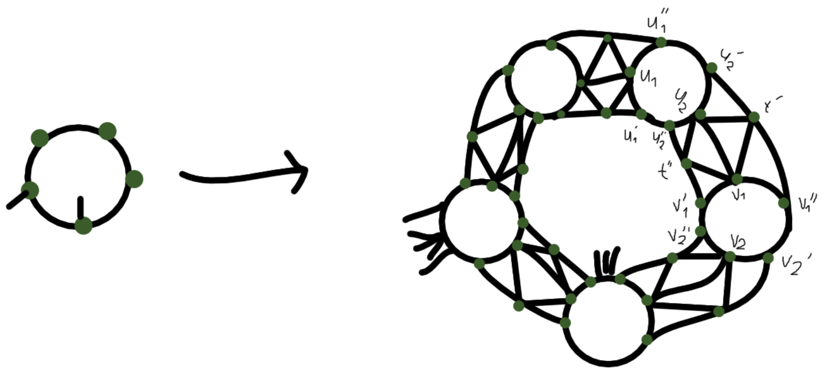

*Proof.* We define the branches of $G'$ forming $G$. Let $v \notin$ any common clique, let it only $\in P_i$. Then $G'^v := {P_i'}^v$. Let $v \in$ some common clique $\Delta$. Then ${G'}^v := \bigcup_{P_i \supseteq \Delta} {P_i'}^v$.

If $v \in G$, $v \notin$ any common clique, let it only $\in P_i$, then $(u,v) \in G \implies (u,v) \in P_i \implies {P_i'}^u, {P_i'}^v$ neighbor $\implies {G'}^u, {G'}^v$ neighbor.

If $v \in$ some common clique $\Delta$, then $(u,v) \in G \implies (u,v) \in$ one of the planar $P_i$ containing $\Delta \implies {P_i'}^u, {P_i'}^v$ neighbor $\implies {G'}^u, {G'}^v$ neighbor. $\square$

We now begin to construct the universal $K_5$-minor free graph. For a countable locally finite planar graph $G$, we inflate the nodes of $G$ to obtain $G'$: Take a generous embedding of $G$, and for every node $v$, take an open ball containing only $v$ and its edges, delete the inside of the ball, and put a new vertex on the deg($v$) points the edges of $v$ first intersect the boundary, let these nodes be $v_1, v_2, \ldots$. Connect them in clockwise order around the boundary, with edges embedded on the boundary. Clearly $G'$ remains planar and $G' > G$ by contracting the $v_i$. We inflate edges of $G'$ to obtain $Bl(G)$. For every edge $(v_i,u_j), u \neq v$, notice there can only be one such edge for each vertex, add a node before and after $v_i$ in the boundary, let them be $v_i', v_i''$, repeat for $u_j$ then connect $v_i'$ with $u_j''$ and $v_i''$ with $u_j'$. Then subdivide ($v_i'$, $u_j''$), ($v_i''$,$u_j'$) to add a new node to each, let it be $t', t''$ and connect the new nodes to $v_i$ and $u_j$. $Bl(G)$ remains planar and $Bl(G) > G'$ by contracting the $(v_i', t'), (v_i'', t''), (v_i, v_i'), (v_i', v_i''), (u_j, u_j'), (u_j', u_j'')$.





Let $Bl(U_p)$ be any universal planar graph $U_p$ inflated as above.

*Claim* 2. Let $P$ be planar. $Bl(U_p)^P$ has a triangle $\Delta'$ with one vertex in each branch of $Bl(U_p)^\Delta$, for all $\Delta \in P$.

*Proof.* Let $\Delta = xyz \in P$. Pick a subpath of each of the three branch sets of $U_p^\Delta$ to form a minimal K$_3$ minor of $P$, let them be $P_x, P_y, P_z$. The subpaths can be chosen so that the minimal K$_3$ minor contains no node or edge of $U_p^P$ embedded on one of its two sides, w.l.g let it be the interior. Notice that the inner circle $C_{in}$ of the fattened K$_3$ minimal minor thus contains no node or edge of $Bl(U_p)^P$. It is thus easy to see that $Bl(U_p)^P \setminus C_{in} > U_p^P > P$. Let $uv$ be the $P_x - P_y$ edge of the K$_3$ minimal minor in $U_p^P$. By construction of $Bl(U_p)^P$, there is an edge $(u_i, v_j)$ between $Bl(U_p)^u$ and $Bl(U_p)^v$ and they both neighbor an inner circle node $t''$. By reallocating $C_{in}$ to $Bl(U_p)^{P_z}$, we have the desired triangle. □

We now define the universal K$_5$-free graph U$_{K_5 f}$. Let $Bl(U_p)[1] := Bl(U_p)$. Let $Bl(U_p)[i+1]$ be $Bl(U_p)[i]$ clique summed with $Bl(U_p)$ or W[8] over all possible clique pairs. U$_{K_5 f} := \bigcup_{i=1}^{\infty} Bl(U_p)[i]$.

**Theorem 28.** *U$_{K_5 f}$ is a universal K$_5$-free graph.*

*Proof.* Let K$_5$f be any K$_5$-free graph, K$_5$f = $((P_1 \oplus P_2) \oplus P_3) \oplus$ ... Notice that $Bl(U_p)$ has the properties of $P_i'$ of lemma 1. It follows that, let $P_i' := Bl(U_p)$ for all $i$, K$_5$f ' = $((P_1' \oplus P_2') \oplus P_3') \oplus$ ... for suitably selected cliques contains K$_5$f as a minor. But by definition of $U_{K_5 f}$, K$_5$f ' is contained in it as a subgraph. □

# ABBREVIATIONS - ACRONYMS

| | |
|---|---|
| AI | Artificial Intelligence |
| SPARQL | SPARQL Protocol and RDF Query Language |
| OWL | Web Ontology Language |
| OGC | Open Geospatial Consortium |

# APPENDIX A. BASIC DEFINITIONS

Graph theory has the unusual phenomenon that while graphs are technically duplets of sets, we tend to think of them not as sets but visually. Furthermore, when we refer to e.g the clique of size 3 $G$, we don't discuss if $G =(\{1,2,3\},\{(1,2)\,,(2,3),(3,1)\})$ or $G =(\{4,5,6\},\{(4,5),(5,6),(6,4)\})$, really we only care that it belongs to the equivalence class of graphs isomorphic to $(\{1,2,3\},\{(1,2),(2,3)\,,(3,1)\})$. As a byproduct, well understood definitions are oftentimes hand-wavy and not technically rigorous.
The aim in this section is to introduce, in a rigorous manner from the ground up, notions needed during this thesis or at least to clarify what is left to common sense.

As a byproduct, the introduction section is quite large; the reader may skip it and refer to it as needed.

## A.1 Basics

All graphs are simple and undirected. All graphs are finite unless stated otherwise. Though the focus of this thesis is on finite graphs, some results on infinite graphs are also presented. All infinite graphs are countable. The reader may also refer to Diestel [10], the standard reference book.

**Definition 33.** A *pair* is a set of cardinality 2.

**Definition 34.** A *graph* is an ordered pair $G = (V,E)$, where $V$ is a finite set and $E$ is a set of pairs of $V$. We call the elements of $V$ the *vertices* of $G$ and the elements of $E$ the *edges* of $G$. For each edge $e = \{v,u\} \in E$, we call the vertices $v$ and $u$ *ends* of e and say that the vertices $v$ and $u$ are *connected or adjacent or neighbors* in $G$. The *order* of $G$ is $|V|$.

In an abuse of notation, we write $uv$ or $(u,v)$ rather than $\{u,v\}$ for edges.

**Definition 35.** An *infinite graph* is defined in the same manner as a finite graph, the only difference being that $V$ must be infinite. Similarly, a *countable graph* has vertex set $V$ countable.

**Definition 36.** For subgraph $H_1$ of graph $G = (V,E)$, we say that $H_1$ and $v \in V$ are *connected or adjacent or neighbors* in $G$ if there is $u \in H_1$, with $u,v$ adjacent in $G$. For subgraphs $H_1$, $H_2$ of graph $G = (V,E)$, we say that $H_1$ and $H_2$ are *connected or adjacent or neighbors* in $G$ if there are $u \in H_1$, $v \in H_2$ with $u,v$ adjacent in $G$.

**Definition 37.** Graph $H = (V_H,E_H)$ is a *subgraph* of graph $G = (V_G,E_G)$, denoted $H \subseteq G$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. $H$ is an *induced subgraph* of $G$ if $V_H \subseteq V_G$ and $E_H$ is $E_G$ limited precisely to pairs with both ends in $H$. The induced subgraph of $G$ with vertex set $S \subseteq V_G$ is denoted $G[S]$.

**Definition 38.** For subgraphs $S_1, S_2$ of a graph $G$, an $S_1, S_2$ *edge* is an edge with one endpoint on $S_1$ and one endpoint on $S_2$. We say that $S_1, S_2$ *touch or are adjecent or neighbors* if there is an $S_1, S_2$ edge in $G$.

**Definition 39.** Graph $H = (V_H, E_H)$ is *isomorphic to* graph $G = (V_G, E_G)$, denoted $H \cong G$ if there is a 1-1 and onto function $f : V(G) \to V(H)$ such that $(u, v) \in E(G) \iff (f(u), f(v)) \in E(H)$. We may call $G$ a *relabelling* of $H$.

**Definition 40.** Let $G = (V, E)$ and let $v \in V$. The *degree of $v$ in $G$ $d_G(v)$* is the number of edges with it as an endpoint, $|\{(v, u) : (v, u) \in E\}|$.

We define a few basic graphs.

The trivial or single vertex graph is the graph of 1 vertex, ($\{v\}, \{\}$). In rigorous terms:

**Definition 41.** A trivial or single vertex graph is any graph belonging to the graph isomorphism class of ($\{1\}, \{\}$).

A path is a non-empty graph $P = (V, E)$ of the form $V = \{v_0, v_1, ..., v_k\}$ $E = \{(v_0, v_1), (v_1, v_2), ..., (v_{k-1}, v_k)\}$. Rigorously:

**Definition 42.** A *path graph $P$ of length $n \geq 0$* is any graph belonging to the graph isomorphism class of the graph with vertex set $\{1, 2, ..., n, n+1\}$ and edge set $\{(1, 2), (2, 3), ..., (n, n+1)\}$. A path graph of length 0 is defined to be a single-vertex graph and is called *trivial*. A *path graph* is a graph belonging to the graph isomorphism class of the path graph of length $n$ for some $n$.

Some additional notation for paths is of use. Let $P$ be path with edge set $(v_1, v_2), ..., (v_{k-1}, v_k)$. We often denote $P$ as $v_1v_2...v_k$ or as $(v_1, v_2), (v_2, v_3), ....$ Other notation follows.

**Definition 43.** Let $P$ be path $v_1v_2...v_k$. $v_1$ and $v_k$ are its *endpoints* or *ends*. $int(P) := v_2, ..., v_{k-1}$ are its *internal vertices*. $Pv_i := v_1v_2...v_i$. $v_iP := v_iv_{i+1}...v_k$. $Pv_i := v_1v_2...v_i$. $v_iPv_j := v_iv_{i+1}...v_{j-1}v_j$.

**Definition 44.** A *cycle* is any graph belonging to the graph isomorphism class of the graph with vertex set $\{1, 2, ..., n\}$ and edge set $\{(1, 2), (2, 3), ..., (n-1, n), (n, 1)\}$ for some $n$.

**Definition 45.** A *clique* is any graph belonging to the graph isomorphism class of the graph with vertex set $V = \{1, 2, ..., n\}$ for some $n$ and edge set all pairs of $V$. The *size of the clique* is $n$.

Given graph $G$, rather than say $G$ has a clique subgraph $K$, we say $K$ is a clique of $G$. The same goes for the other named graphs.

**Definition 46.** For non zero natural numbers $N, M$, the $N \times M$ *grid graph* is the graph with vertex set $1, 2, ..., N \times 1, 2, ..., M$ and edge set $\{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}$. See figure A.1.

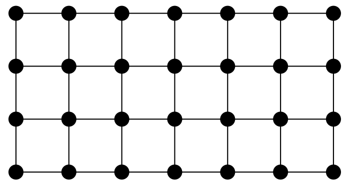


**Figure A.1: The $4 \times 7$ grid graph.**

## A.2 Graph operations

When defining graphs, it is often easier to do so using graph operators. Just like with number operations, a graph operator is a function $\otimes$ that takes two graphs as input and outputs a graph. Given two graphs $G_1, G_2$ we usually write $G_1 \otimes G_2$ to denote $\otimes(G_1, G_2)$. A few definitions follow.

**Definition 47.** Given two graphs $G = (V_G, E_G),\ H = (V_H, E_H)$, define the *graph union* $G \cup H$ as $(V_G \cup V_H, E_G \cup E_H)$ and the *graph intersection* $G \cap H$ as $(V_G \cap V_H, E_G \cap E_H)$. If $G_V \cap G_H = \emptyset$, then $G$ and $H$ are *disjoint*.

**Definition 48.** If $U$ is a set of vertices, we define $G - U$ as $G[V_G \setminus U]$. In an abuse of notation, if $U$ is the single-vertex graph $v$ we write $G - v$ rather than $G - \{v\}$ and if $G'$ is a graph, $G - G'$ rather than $G - V(G')$.
If $F$ is a set of pairs of vertices of $G$, we define $G - F$ to be the graph $(V(G), E(G) \setminus F)$, and $G + F$ to be $(V(G), E(G) \cup F)$. In an abuse of notation, $G - e := G - \{e\}$ and $G + e := G + \{e\}$. To *join vertex $u$ to vertex $v$* in $G$ means to add $(u, v)$ to $G$. To *join subgraph $S_1$ to subgraph $S_2$* of $G$ means to join $(u, v)$ in $G$ for all $u \in S_1, v \in S_2$.

**Definition 49.** Given graphs $G_1,\ G_2$ we define the *disjoint union or graph sum or graph addition of $G_1$ and $G_2$*, denoted $G_1 + G_2$, to be $G_1 \cup G_2'$ where $G_2'$ is a graph isomorphic to $G_2$ so that $G_1 \cap G_2' = \emptyset$.

Notice the similarity to the disjoint union of sets. Indeed, we could have very easily defined the disjoint union of graphs using it.
By "the subgraph $S$ of $G_2$ in $G_1 + G_2$" it is obvious what we mean, but as the goal of this section is rigor: We changed the labels of $G_2$ while defining $G_1 + G_2$. Let $f$ be the isomorphism in the above definition, and let $S \subseteq G_2$. By *the subgraph $S$ of $G_2$ in $G_1 + G_2$* we mean the subgraph induced by $f(V_S)$ in $G_1 + G_2$. The same is said for vertices $v$ of $G_2$.

**Definition 50.** Given graph $G$, *adding a vertex* is defined as the graph sum of $G$ and the single vertex graph.

**Definition 51.** Given a graph $G = (V, E)$, to *identify* or *glue* vertices $u$ and $v$ of $G$ means to replace all instances of $u$ and $v$ in $V$ and $E$ with a new element $w \notin V$. Remove any loops or parallel edges.

**Definition 52.** Take graphs $G_1, G_2$, and let $S \subseteq G_1$ be isomorphic to $S' \subseteq G_2$, let $f$ be the isomorphism. The *identification of* $G_1$ *and* $G_2$ *over* $S$ *and* $S'$ is $G_1 + G_2$, whereby we identify in $G_1 + G_2$ the vertex $v \in G_1$ with $f(v) \in G_2$. See figure A.2.

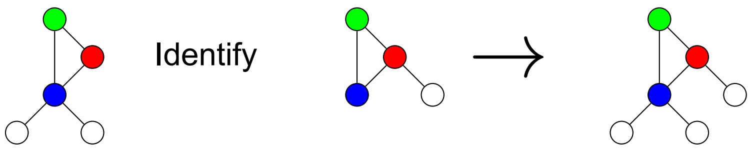



**Figure A.2: Intuitively, one may picture the identification of two graphs over e.g isomorphic triangles as putting the vertices of one on top of the vertices of the other.**

**Definition 53.** Given graphs $G, H$, their *Cartesian product* $G\square H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices $(u, v)$ and $(u', v')$ are adjacent if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Intuitively, for each vertex of $H$ take a copy of $G$, and if two vertices in $H$ are connected, connect the corresponding $G$ copies by their identical vertices.

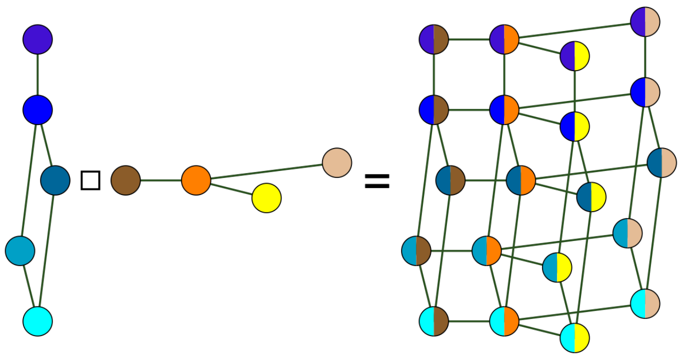


**Figure A.3: The Cartesian product of two graphs Courtesy: Wikipedia.**

**Definition 54.** For fixed $u \in G$, we denote by by $(u, H)$ the $G\square H$ limited to all vertices of the form $(u, v)$ where $v$ ranges over $H$. We call $(u, H)$ the $H$-subgraph of $V(G) \times V(H)$ corresponding to $u$.

**Definition 55.** Given graphs $G, H$ such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by taking $G \cup H$ and possibly removing a few edges of the clique. See figures A.2, A.4.

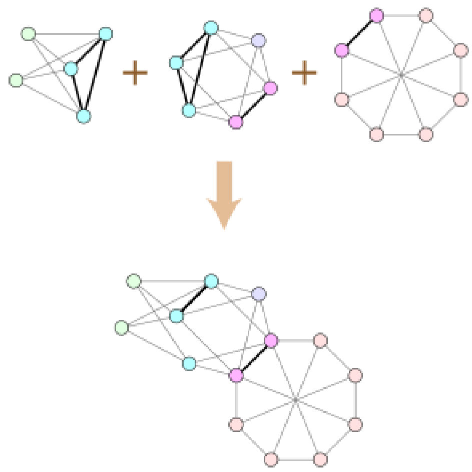


**Figure A.4: Two clique sums to create a single big graph. Notice how a few clique edges are removed. Courtesy: Wikipedia.**

For the operation to be well defined, the edges to be removed must be declared. Still, we often make statements that stand regardless of the specific choice of removed edges. In this case, as happens often in graph theory, we omit mention of the edges to be removed. Similarly, we may omit mention of the cliques the two graphs are clique summed on.

**Definition 56.** The clique sum of $G$ and $H$ on clique $G \cap H$ of $k$ vertices is called a *$k$-sum*. The clique sum of $G$ and $H$ on clique $G \cap H$ of $\leq k$ vertices is called a *$\leq k$-sum*.

Notice that 0-sums are well defined, and are the disjoint union. Now, we would like to clique-sum without caring about vertex labels.

**Definition 57.** Given graphs $G,\ H$ and isomorphic clique subgraphs $S_G \subseteq G$, $S_H \subseteq H$, their *clique sum $G \oplus H$ over common cliques $S_G$ and $S_H$* is defined by identifying $G$ and $H$ over $S_H$ and $S_G$. We may denote this $G \oplus_{S_G, S_H} H$.

Similarly with the disjoint union, suppose some $G' \subseteq G$, we may make mention of $G'$ as a subgraph of $G \oplus_{S_G, S_H} H$, ignoring the relabelling that occurred.

## A.3 Treewidth

We now introduce the treewidth of a graph. While it is usually defined as the minimum necessary bag size of a tree-decomposition, I find its definition through clique-sums of smaller graphs, equivalently carefully selected unions of smaller graphs, to provide a better understanding of the notion, and understanding naturally is the primary goal when dealing with theory.

The following says that a graph has treewidth $\leq k$ if it can be built by the clique sum of graphs of order $\leq k + 1$.

**Definition 58.** Let there be a natural number $k$. Let there be graph $H_1$ of order $\leq k+1$ , and let graph $H_2$ be a graph of order $\leq k+1$. Let $G_2$ be $H_1 \oplus H_2$.Let there be a graph $H_3$ of order $\leq k+1$. Let $G_3$ be $G_2 \oplus H_3$. Let there be a graph $H_4$ of order $\leq k+1$. Let $G_4$ be $G_3 \oplus H_4$. Any graph $G_i$ that can be built by this procedure is said to belong to *the class of graphs $TW_{\leq k}$ of graphs of treewidth $\leq k$.*

**Definition 59.** If a graphs $G$ belongs to $TW_{\leq k}$ but not $TW_{\leq k-1}$, then it is said to be a graph of *treewidth* $k$.

The previous definition says that graphs of treewidth $k$ are precisely the graphs which in order to be constructed as described above, it suffices and there need be some graphs $H_i$ of order as large as $k+1$.
The reader may inquire why the +1 exists in the definition. It is a historical convention with no substantial meaning.

The classic notion of a tree-decomposition of a graph is directly related to a construction of it by clique-sums and vice-versa. Given a graph constructed by the clique sums of graphs $H_i$, we can find a tree-decomposition; simply take the vertices of the tree to be $t_{H_i}$, take the bag of $t_{H_i}$ to be $V(H_i)$, and connect $t_{H_i}$ and $t_{H_j}$ in the tree decomposition if $H_i$ was chosen for $H_j$ to clique sum on. See [17] for a full and more detailed proof.

**Definition 60.** Let there be graph $G$ constructed by the clique sum of graphs $H_1,\ H_2, ...,\ H_n$ as described in the definition of treewidth. We call $V(H_i)$ the *bags* of $G$, and denote them as $B_{H_i}$ or $B(H_i)$. If minor bags are involved as well, we call them the *tree-decomposition bags* to avoid confusion.

The following says that a graph has treewidth $\leq k$ if it can be constructed by starting from a graph $H_1$ of order at most $k$ and iteratively glueing graphs $H_i$ of order at most $k$ on top to build a bigger graph, each time selecting a previously added graph $H_j$, $j < i$ to glue on. While this is my definition of choice, I have funnily enough never seen another human or text mention it. We thus do not use the following alternative definition of treewidth in this text, but I still wished to include it.

**Theorem 29.** *Let there be a natural number $k$. Let there be graph $H_1$ of order $\leq k+1$ , and let graph $H_2$ be a graph of order $\leq k+1$. Let $G_2$ be $H_1 \cup H_2$. Let there be a graph $H_3$ of order $\leq k+1$ such that $G_2 \cap H_3 \subseteq H_1$ or $G_2 \cap H_3 \subseteq H_2$. Let $G_3$ be $G_2 \cup H_3$. Let there be a graph $H_4$ of order $\leq k+1$ such that $G_3 \cap H_4 \subseteq H_1$ or $H_2$ or $H_3$, and so on. A graph $G_i$ belongs to $TW_{\leq k}$ iff it can be built by this procedure.*

To shortly touch on this, indeed, if one can build a graph by the unions of smaller graphs as described above, one can also build it by clique sums of the same smaller graphs, with some extra edges so that the clique sum is well-defined, removed when no longer needed. The mainstream definition of treewidth is not utilized in this text and is thus not presented.

**Definition 61.** Let there be graph $F$ with vertex set $v_1, ..., v_n$. Let there be graph $H_1$. Let $G_2$ be $H_1 \cup H_2$. Let there be a graph $H_3$ such that $G_2 \cap H_3 \subseteq \bigcup H_i$ taken over all $H_i$ such that $(v_i, v_3) \in E(F)$. Let $G_3$ be $G_2 \cup H_3$. Let there be a graph $H_4$ such that $G_3 \cap H_4 \subseteq \bigcup H_i$ taken over all $H_i$ such $(v_i, v_4) \in E(F)$ and so on, $n$ times. Any graph $G_n$ that can be built in this manner by $H_i$ of order $\leq k+1$ is said to have an $F - decomposition$ *of width* $k$. We call $V(H_i)$ the *bags* of $G$, and denote them as $B_{H_i}$ or $B(H_i)$. If minor bags are involved as well, we call them the *F-decomposition bags* to avoid confusion.

## A.4 Minors, Topological Minors

Subgraphs capture the intuitive notion that a graph is inside another graph. One may however protest that given graphs $G$, and $G'$, where $G'$ is obtained from $G$ by replacing some edge of $G$ with a path of degree 2 nodes, $G$ is inside $G'$, because the path basically functions as an edge. Taking this idea a step further, given a graph $G$ and $G'$, where $G'$ is obtained from $G$ by replacing some node $v$ of $G$ with a connected graph adjacent to all nodes $v$ was adjacent to, one may say $G$ is inside $G'$ because the connected graph can function as a big node.

It is helpful to define the operations of suppression and contraction before proceeding.

**Definition 62.** Given a graph $G$ and a (possibly trivial) path $P = v_1 v_2 ... v_k$ of $G$ of $d_G(v_i) = 2$ for all $v_i$, where $l$, the neighbor of $v_1 \in G \setminus P$, and $r$ the neighbor of $v_k \in G \setminus P$ are distinct, the operation of *suppressing the path in* $G$, denoted $suppr_G(P)$ outputs a graph $G' = G - P + (l, r)$.

Given a graph $G$ and a (possibly single-vertex) connected subgraph $S$ of $G$, the operation of *contracting* $S$ in $G$, denoted $G/S$, outputs a graph $G' = G - S +$ a new vertex $v_S$ neighboring all vertices of $G - S$ that $S$ did in $G$. Given a set of nodes $U$ of $G$, the contraction of $U$ is defined to be the contraction of $G[U]$.

**Definition 63.** Let $G$ be a graph, and let $S$ be a subgraph of $G$. Let $S_2$ be $suppr_S(P)$ for some path $P$ of $G$ (chosen so that the suppression is well-defined). Let $S_2$ be $suppr_{S_1}(P')$ for some path $P'$ of $S_1$ and so on. If a graph $G'$ is isomorphic to some $S_i$ that can be constructed in this manner from $G$, then $G$ *contains* $G'$ *a topological minor*, denoted $G \geq_{tm} G'$.

**Definition 64.** Let $G$ be a graph, let $S$ be a subgraph of $G$ and let $H$ be a connected subgraph of $S$. Let $S_2$ be $S/H$. Let $H'$ be a connected subgraph of $S_2$. Let $S_3$ be $S_2/H'$. If a graph $G'$ is isomorphic to some $S_i$ that can be constructed in this manner from $G$, then $G$ *contains* $G'$ *a minor*, denoted $G \geq_m G'$.

Observing that if a node that arose from a contraction is used in another contraction, we could have just done a single big contraction instead, one may verify that the following are equivalent:

**Theorem 30.** *The following are equivalent for two graphs* $G$, $G'$*:*

*(1)* $G \geq_m G'$

*(2) For some subgraph* $R$ *of* $G$ *there are pairwise disjoint subgraphs* $R_1, R_2, ..., R_{|V(G')|}$ *of* $R$ *such that* $(((R/R_1)/R_2)/...)/R_{|V(G')|}$ *is isomorphic to* $G'$

*(3) For some subgraph* $R$ *of* $G$ *there are pairwise disjoint subgraphs* $R_1, R_2, ..., R_{|V(G')|}$ *of* $R$ *and there is a bijection* $R_1 \leftrightarrow v_1$*,* $R_2 \leftrightarrow v_2$*, ...,* $R_{|V(G')|} \leftrightarrow v_{|V(G')|}$*, where* $V(G') = \{v_1, ..., v_{|V(G')|}\}$*, such that* $(v_i, v_j) \in E(G')$ *iff* $R_i, R_j$ *are adjacent.*

We work most with the third definition. Some terminology is of use.

**Definition 65.** A bijection $\mu(v_i) = R_i$ as in (3), is called a *model* of $G'$ in $G$. We call $R_i$ the *bag* or *branch* of $v_i$ in $G$ and also denote it $B(v_i)$ or $G^{v_i}$. For $H \subseteq G$, we denote with $\mu(H)$ or $B(H)$ or $G^H$ the subgraph of $G$ induced by the $\cup_{v \in V(H)} B(v)$.

As with edges removed after clique sums, when a statement holds for any choice of $\mu$ or $\mu$ is clear by context, we omit mention of $\mu$.

**Definition 66.** Give a graph class $C$, we call $C$ *closed under minors* or *minor-closed* if $G \in C$ and $G \geq_m G'$ implies $G' \in C$.

**Definition 67.** Give a graph class $C$, denote by *minor-closure*$(C)$ its *minor closure*, i.e minor-closure($C$)=$\{G : G \leq_m G'$ for some $G' \in C\}$

**Definition 68.** A graph $G$ *forbids* a graph $G'$ as a minor if $G \not\geq_m G'$.

**Definition 69.** By $Forb(G)$ we denote the class of graphs not containing $G$ as a minor. It is easy to observe this class is closed under minors.

**Definition 70.** A minor-closed graph class $C$ *does not contain* a graph $G$ as a minor if $G \notin C$. A graph $G$ is a *forbidden minor of* $C$ or *excluded minor of* $C$ or *in the obstruction set of* $C$ if $C$ forbids $G$ as a minor and $G$ is minimal in this regards, i.e $G' \in C$ for all other $G' \leq_m G$.

The following by Robertson and Seymour is one of the deepest results in all of graph theory. It was proved over a series of 20 papers amounting to 500 pages, over a period of 20 years.

**Theorem 31** (The graph minor theorem [23])**.** *Every graph class* $C$ *closed under minors can be characterized by a finite set of forbidden minors.*

# APPENDIX B. TOPOLOGY FUNDAMENTALS

As in other subjects in graph theory, and especially in the one that proceeds, one may reason about concepts through visual intuition rather than rigor, and this is often what the community does in practise. Mohar's Topological graph theory [11]) provides for a more rigorous introduction to the topic, though he assumes some topological knowledge. For the topology fundamentals, we recommend Kinsey's topology of surfaces [24]. While this thesis is not focused on topology or bibliography, and thus many topological results are listed without proof, we still try to be as analytical and rigorous as possible.
The reader is probably already familiar with planar graphs. Some of the most deep results in minor theory mention graphs embeddedable on surfaces more complex than the plane or the sphere, such as the torus.

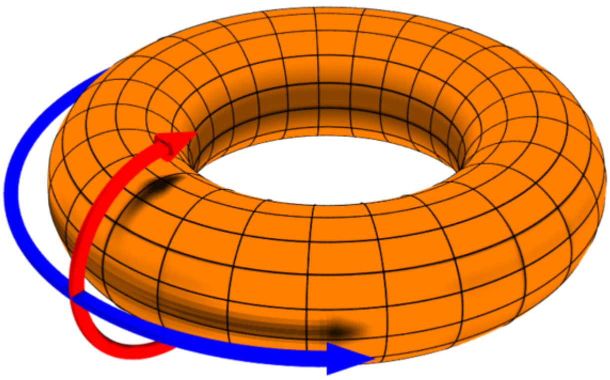


**Figure B.1: The torus. Courtesy: Wikipedia.**

## B.1 Elements of surfaces

Let $(X, \tau)$ be a topological space. Let an *element* of $X$ be any $x \in X$. Some definitions apply more generally, but we only care about metrizable spaces, in fact only about surfaces, which we define shortly.

**Definition 71.** A *curve* or *arc* on $X$ is the image $f([0,1])$ of a continuous function $f$ from [0,1] to $X$. A curve is simple if $f$ is 1-1. The curve *connects* $f(0)$ and $f(1)$, which are called the *ends* or *endpoints* of the curve, while $f((0,1))$ is its *interior*. For $a, b \in [0,1]$, a subset of the curve of the form $f([a,b])$ is called a *segment* of the curve, while a subset of the form $f([0,a])$ or $f([a,1])$ is called an *initial segment*. A *simple closed curve* is a curve such that $f$ is 1-1 on (0,1) and $f(0) = f(1)$.

Notice that as the image of a continuous function on a compact set, a curve is compact.

**Definition 72.** A topological space $(X, \tau)$ is *path* or *arcwise* or *curve connected* if for every two points in it, there is a simple curve connecting them. A subset of $X$ is called path-connected if the subspace induced by $X$ under the subspace topology is path-connected. A maximal path-connected subset of X is called a *path-connected component* or *region* of $X$.

A surface is a connected compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$. Intuitively, the reader may visualize them as 3 dimensional shapes, such as donuts, coffee mugs, spheres, chairs, e.t.c.

**Definition 73.** A topological space $(X, \tau)$ is called *Hausdorff* if for all distinct $x, y \in X$, there are disjoint $U_x$ and $U_y$ with $x \in U_x$, $y \in U_y$.

Hausdorff spaces have nice properties metric spaces do. It says we have enough open sets to separate points.

**Definition 74.** A topological space $(X, \tau)$ is called *locally homeomorphic to* $(X', \tau')$ if for all distinct $x \in X$, there is $\mathrm{O} \in \tau$ including $x$ and homeomorphic to $(X', \tau')$ in the subspace topology.

Many subsets of $\mathbb{R}^2$ are homeomorphic to $\mathbb{R}^2$, such as any open ball of radius 1. Any of them could have been used in this definition.

**Definition 75.** Given a topological space $(X, \tau)$ an *open disc* is a subset of $(X, \tau)$ homeomorphic to the open ball of radius 1 of $\mathbb{R}^2$. A *closed disc* is a subset of $(X, \tau)$ homeomorphic to the closed ball of radius 1 of $\mathbb{R}^2$.

Surfaces have a few nice natural properties. For example:

**Theorem 32.** *A surface is a path-connected space. In fact, we could define them to be path-connected instead of connected without loss of generality.*

**Theorem 33.** *Every surface is a metrizable space.*

The reasoning is that a compact Hausdorff space is metrizable if it is locally metrizable, and surfaces are locally metrizable because they are locally homeomorphic to $\mathbb{R}^2$.

## B.2 Graphs on Surfaces

A graph is *embeddable* on a surface if we can draw it on the surface so that edges do not intersect.

**Definition 76.** A graph $G$ is *embeddable* on $(X, \tau)$ if there is a function $f$ mapping vertices to elements of $X$, and edges to simple curves on $X$ so that $f(v_1) \neq f(v_2)$ for $v_1 \neq v_2$, and curve $f(uv)$ connects $f(u)$ and $f(v)$, and has no intersection with the image of other vertices and only intersects other edges on its endpoints.
$f$ is an *embeddeding* of $G$ on $X$. The image of $f$, $f[(V(G) \cup E(G))]$, is called the *embedded graph*, and though it is technically not a graph, one may produce a graph from one in the obvious manner. For ease of notation, the embedded graph is also abusively denoted $f(G)$.

As the finite union of compact sets, any embedded graph is compact and therefore closed.

**Definition 77.** A *face* of an embedded graph $G$ on $(X, \tau)$ is a region of $X \setminus G$ (equipped with the subspace topology of course).

Given a face of an embedded graph $G$, the boundary of the face is an embedded subgraph of $G$. If this subgraph is a cycle, it call it a facial cycle.

**Definition 78.** Let there be embeddable graph $G$, let $f$ be an embedding, and let the boundary $b$ of a face of $f(G)$ be a cycle, i.e let $G$ limited to the vertices and edges of $f^{-1}(b)$ be a cycle. We call the boundary of $b$ a *facial cycle*.

**Definition 79.** A graph embeddable on the plane $\mathbb{R}^2$ (with the standard topology always) is called *planar*. The embedded graph is called the *plane graph*.

Planar graphs are often introduced with arcs being polygonal. However, the two definitions are equivalent (see Mohar's Topological graph theory chapter 2.1 [11]).

**Definition 80.** A curve is *polygonal* if it is the union of a finite number of straight line segments. A *straight line segment* is a curve that is a subset of a line of $\mathbb{R}^2$.

**Theorem 34.** *A graph is embeddable on the plane if and only if it is embeddable on the plane with edges mapped to polygonal curves.*

For proofs on planar graphs, topological tools on $\mathbb{R}^2$ are useful. The Jordan Curve theorem is an intuitively obvious but infamously difficult to prove theorem. Naturally, we make use of it.

**Theorem 35** (The Jordan Curve Theorem)**.** *Let $C$ be a simple closed curve on $\mathbb{R}^2$. $\mathbb{R}^2 \setminus C$ has exactly two connected components, one being bounded and the other unbounded, with $C$ as the boundary of both.*

The bounded component is called the *interior*, while the unbounded is called the *exterior*. The following extension exists.

**Theorem 36** (The Jordan-Schoenflies Curve Theorem)**.** *For any two simple closed curves $C_1,\ C_2$, their interiors are homeomorphic and their exteriors are homeomorphic.*

A graph is embeddable on the plane if and only if it is embeddable on the sphere. The following theorem provides for a well-defined topology on the sphere that is useful for embeddings.

**Theorem 37.** *The unit sphere $S^{n-1}:= \{x \in \mathbb{R}^n : \sqrt{x_1^2 + ... + x_n^2} = 1\}$ is a complete metric space when equipped with the metric, $d(x, y) :=$ arccos$(x \cdot y)$ where $\cdot$ denotes the standard dot product.*

We need only consider the sphere $S^2$ on $\mathbb{R}^3$. The next theorem following from the definitions of homeomorphity and embeddability.

**Theorem 38.** *Let there be two homeomorphic surfaces $\Sigma_1$, $\Sigma_2$. Then a graph is embeddable on $\Sigma_1$ if and only if it is embeddable on $\Sigma_2$.*

**Theorem 39.** *The sphere minus an element is homeomorphic to the plane.*

Clearly any embedded graph on the sphere is not equal to the sphere. Thus

**Corollary 10.** *A graph can be embedded on the plane if and only if it can be embedded on the sphere.*

As mentioned, we wish to embed graphs on other surfaces as well. While intuitively we can visualize what a torus or a double-torus is, and therefore work with graphs embedded on it, it would be nice to also define those surfaces, starting from topology.

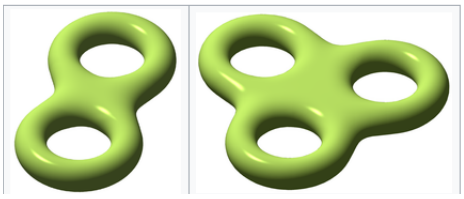


**Figure B.2: Surfaces of genus 2 and 3 respectively. The double and triple torus. Courtesy: Wikipedia.**

### B.3 Genus of surfaces and graphs, the classification theorem, handles and cross-caps, topological operations

**Definition 81.** A topological space $(X, \tau)$ is called *locally Euclidean of dimension $n$* if for every $x \in X$, $x$ has an open neighborhood $U \in \tau$ homeomorphic to $\mathbb{R}^n$ (that is, the subspace topology of $(X, \tau)$ limited to $U$, $(U, \tau_U)$ has a homeomorphism $h : U \to \mathbb{R}^n$).

Intuitively, it is easy to define the torus; simply take the square [0,1]×[0,1], "glue together" the top side with the bottom side to obtain a hollow cylinder, then glue together the two opposing ends of the cylinder. One may do this with a piece of paper.

We want to formally define the intuitive notion of gluing topological sets together. This is done through the quotient topology.

**Definition 82.** Let $= (X, \tau)$ be a topological space. Let there be function $f : X \to Y$. The *biggest* or *finest continuous topology* induced by $X$ and $f$ on $Y$ is $(Y, \tau')$ where $O' \in \tau'$ iff $f^{-1}(\mathrm{O}') \in \tau$.

**Definition 83.** Let $= (X, \tau)$ be a topological space. Let $\sim$ be an equivalence relation on $X$. The *quotient or identification set* $X/\sim$ is $\{[x] | x \in X\}$ where $[x]$ is the equivalence set of $x$ under $\sim$. The function $f(x) = [x]$ is called the *identification or quotient* mapping.

The reader may notice that this space has sets as elements. This is of no importance; we could very well replace them with their representing element, and to avoid notational overencumbering we do.

One may visualize the identification set as $X$ with equivalent points glued or contracted on each other. We now add a topology on the quotient set, because to work with notions such as continuity we need to have an underlying topological space. In the following we still work with general topology, but all spaces we work with will be metrizable, and I have found that thinking with metric distance functions often provides better understanding, so let me briefly mention the quotient metric as a side note. What should the metric $d'$ of $X/\sim$ after gluing together some points of $(X, d)$ be? Let $x$ be a point in $X$, not glued to other points. Clearly its distance from $y \in X$ remains same if all other points of $X$ of distances $\leq d(x, y)$ from $x$ are also not glued. If however a glued point $z$ exists in this ball, we must consider if using it allows us to reach $y$ in a shorter fashion. Thus $d'(x, y)$ is something like $inf_{w \in [z]}(d(x, w) + d(w, y))$, in fact we should also consider other equivalence classes that one may utilize, possibly in succession. This only defines a pseudometric, as it may yield distinct elements of distance 0 (try [-1,1] with the Euclidean metric and [-1,0) contracted to the same equivalence set and (0,1] contracted). For specific metrizable topological sets and well chosen equivalence partitions, this does yield a metric, which induces the quotient topology.

**Definition 84.** Let $= (X, \tau)$ be a topological space. Let $\sim$ be an equivalence relation on $X$. $X/\sim$ equipped with the biggest topology making the identification mapping continuous is called the *quotient or identification topology* of $X$ on $\sim$.

**Definition 85.** Let $(X, \tau)$ be a topological space. To *glue* $x$ and $x' \in X$ means to take the quotient space on $X$ defined by the equivalence relationship $x \sim x'$.

We can now properly define the topological space of the torus.

**Definition 86.** Let there be the metric space $[0,1] \times [0,1]$, equipped with the euclidean metric and take the topological space induced by the metric. For all $t \in [0,1]$, glue $[0,t]$ with $[1,t]$. The resulting topological space is called a *cylinder*. The cylinder has two *opposing ends*, the sets $\{[t,0] | t \in [0,1]\}$ and $\{[t,1] | t \in [0,1]\}$.

Let there be the metric space $[0,1] \times [0,1]$, equipped with the euclidean metric and take the topological space induced by the metric. For all $t \in [0,1]$, glue $[0,t]$ with $[1,t]$, and then for all $t \in [0,1]$ glue $[t,0]$ with $[t,1]$ (the opposing ends). The resulting donut-shaped topological space is called the *torus*.

We now present a fundamental theorem in the topology of surfaces, the classification theorem, which says that any surface can be constructed by the sphere and a few simple operations. Some definitions are needed.

**Definition 87.** To *remove* a subset $S$ of a topological space $(X, \tau)$ means to take the subspace topology induced by $X \setminus S$.

Much like with graphs, the disjoint union of sets expresses the idea of putting both sets separately together.

**Definition 88.** The *disjoint union* of two not necessarily disjoint sets $A, B$ is the set $\{(x,1) | x \in A\} \cup \{(x,2) | x \in B\}$.

**Definition 89.** The *disjoint union topology* of two topological spaces $A, B$ with bases $U_a,\ U_b$ is the disjoint union of $A$ and $B$ equipped with the base defined by the disjoint union of $U_a$ and $U_b$.

It is interesting to notice that the following is equivalent: Let $f$ be the natural map from $A \cup B$ to the disjoint union of $A, B$. We can define the disjoint union topology as the disjoint union of $A, B$ equipped with the biggest topology making $f$ continuous.
This was the case for the quotient topology as well. Thus it starts to become clear that the finest/biggest topology making $f$ continuous is the one that conserves best the initial topological space in the image space.

**Definition 90.** Let there be a surface $S$. Let there be two subsets $C_1,\ C_2$ of $S$ homeomorphic to an open ball of $\mathbb{R}^2$, and let the closure of $C_1$ and $C_2$ be disjoint. Remove $C_1$ and $C_2$ from $S$, take the disjoint union of the resulting topological space with a cylinder, and glue one end of the cylinder to the boundary of $C_1$ in the natural manner and the other end to the boundary of $C_2$. We then say we *added a handle* to $S$.

**Definition 91.** Let there be a surface $S$. Let there be a subset $C$ of $S$ homeomorphic to an open ball of $\mathbb{R}^2$. Remove $C$ from $S$, and if $x,\ x' \in S \setminus C$ are on the boundary of $C$ and diametrically opposite (on the circle homeomorphic to $C$ of course), glue them. We then say we *added a crosscap* to $S$.

Adding a crosscap is homeomorphic to adding a mobius strip.

**Theorem 40** (The classification theorem)**.** *Let $S$ be a compact surface. $S$ is homeomorphic to one of the following:*

1. *The sphere after adding $k \in \mathbb{Z}_{\geq 0}$ handles.*
2. *The sphere after adding $k \in \mathbb{Z}_{\geq 0}$ crosscaps.*

**Definition 92.** The *genus* of a connected orientable surface is the maximum amount of pair-wise disjoint simple closed curves that can be removed without rendering it disconnected. The *non-orientable genus* of a connected non-orientable surface is the maximum amount of pair-wise disjoint simple closed curves that can be removed without rendering it disconnected. [1]

**Theorem 41.** *The genus of an orientable surface is equal to the number of handles we need to add to construct it starting with a sphere. The non-orientable genus of a non-orientable surface is equal to the number of cross-caps we need to add to construct it starting with a sphere.*

Thus, up to homeomorphism there is only one surface of orientable or non-orientable genus $g$, the surface of obtained from the sphere after adding $g$ handles or $g$ crosscaps. Euler's theorem says that for an embedded graph in the plane, $n - m + f = 2$ where $n$ is the number of vertices, $m$ the edges, and $f$ the distinct faces. This results extends to higher (non-orientable) genus surfaces.

**Definition 93.** Let $S$ be a surface. Then for some possibly negative integer $\chi$, called the *euler characteristic* of $S$, and for any embedded graph $G$ on $\Sigma$ such that every face is homeomorphic to an open ball in $\mathbb{R}^2$, $n - m + f = \chi$.

**Theorem 42.** *Let $G$ be a graph embedded on $\Sigma$ and not embeddable on a surface of lower genus. Then every face is homeomorphic to an open ball in $\mathbb{R}^2$*

**Definition 94.** The *genus of a graph $G$* is the smallest integer $n$ such that $G$ can be embedded on the surface of genus $n$. The *non-orientable genus of an graph $G$* is the smallest integer $n$ such that $G$ can be embedded on the non-orientable surface of genus $n$.

**Definition 95.** The *euler genus* of a surface with euler characteristic $\chi$ is $2 - \chi$.

**Theorem 43.** *Let $\Sigma$ be a surface built from the sphere after adding $k$ handles. Then its euler genus is $2k$.*
*Let $\Sigma$ be a surface built from the sphere after adding $k$ crosscaps. Then its euler genus is $k$.*

[1] So if we add 10 handles to the sphere and then 1 cross-cap, this is a non-orientable surface. Can we really build the same surface by just adding cross-caps? Yes! We need 2 crosscaps for each handle

In other words, the Euler genus of a non-orientable surface is its non-orientable genus, and the Euler genus of an orientable surface is double its genus. With this in mind, working with the euler genus instead of the regular genus and non-orientable genus is somewhat of an overcomplication for our purposes. In any case, The graph theory community seems to like not to concern itself with whether a surface is orientable or non-orientable and abolishing the established conventions is not a priority of this text.

**Definition 96.** The *euler genus* of a graph is the smallest integer $n$ such that $G$ can be embedded on the surface of euler genus $n$.

Euler's theorem implies that for any planar graph $G$ of $n$ vertices and $m$ edges, $m \leq 3n - 6$. This also generalizes to graphs embeddable on higher genus surfaces:

**Theorem 44.** *Let $G$ be embeddable on $\Sigma$. Then $m \leq 3n - 6 + 3$eul_genus$(\Sigma)$.*

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