Maximum degree in minor-closed graph classes

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Abstract

It is easy to see that every planar graph is a minor of another planar graph of maximum degree 3. Georgakopoulos proved that every finite K_5-minor free graph is a minor of another K_5-minor-free graph of maximum degree 22, and inquired if this is smallest possible.

This motivates the following generalization: Let C be a minor-closed class. What is the minimum k such that any graph in C is a minor of a graph in C of maximum degree k? Denote the minimum by $\Delta(C)$ and set it to be ∞ if no such k exists.

A graph class is proper if it does not contain all graphs. We prove that a minor-closed class C excludes an apex graph if and only if there exists a proper minor-closed superclass C' of C with $\Delta(C') = 3$. *This complements a list of 5 other characterizations of the minor-closed classes excluding an apex graph by Dujmovic, Morin and Wood.

*David wants me to remove this. Shouldn't we make clear how this contributes/relates to existing research?

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Contents

1 Introduction

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All graphs in this text are finite, simple and undirected.

1.1 The graph class parameter Δ

One may easily observe that every planar graph is a minor of another planar graph of maximum degree 3. Figure 1.1 illustrates that [1].

Figure 1: By replacing each vertex of a plane graph with a cycle on the boundary of an open ball around the vertex, we may create a plane graph of maximum degree 3 containing the first as a minor.

Georgakopoulos [5] observed that every K_5-minor free graph is a minor of another K_5-minor-free graph of maximum degree 22. This motivated the following definition:

Definition 1. Given a minor-closed class C, define as $\Delta(C)$ the minimum k such that every graph in C is a minor of another graph in C of maximum degree $\leq k$. If there is no such k, define $\Delta(C)$ to be infinite.

The function Δ does not seem to have any clear general pattern at first glance. One may observe that if for some minor-closed class C we have $\Delta(C) \leq 2$, then any graph in C consists of the disjoint union of cycles and paths. Therefore the interesting case is when $\Delta(C) \geq 3$; we may thus assume that* for all C, $\Delta(C) \geq 3$.

*David asked me to remove this. Does this mean start considering the other cases as well? We would have to change theorem 3 to Δ(C) =<3 for example.

Georgakopoulos [5] asked if every proper minor-closed class C has a proper minor-closed superclass C' of finite $\Delta(C')$. As we will see using criteria 6 of theorem 3, a minor-closed class has such a minor-closed superclass if and only if it excludes an apex graph as a minor. This complements theorem 3 by adding a previously unknown condition:

Theorem 1. *A proper minor-closed class C excludes an apex graph if and only if it has a minor-closed superclass C' with $\Delta(C') = 3$ if and only if it has a minor-closed superclass C' with $\Delta(C') \leq k$ for some k.*

The class C' of theorem 1 also excludes an apex graph.

[1] In fact, some topological caution is warranted. The text in its current form does not aim to cover this in detail.

1.2 The graph minor structure theorem

In one of the deepest results in all of graph theory, Robertson and Seymour showed that every graph class C closed under minors can be characterized by a finite set of excluded minors [10]. It was proved over a series of 20 papers amounting to 500 pages, over a period of 20 years. Another major result is the Graph structure theorem [9]. It says that just like all K_5 minor-free graphs can be constructed by the clique sum of planar graphs and the Wagner graph, so can the graphs in the class of graphs without an H-minor $Forb(H)$ be constructed by the clique sum of the so called k-almost embeddable graphs.

Theorem 2 (Graph Minor Structure Theorem)**.** *Let there be a graph H, and let $G \in Forb(H)$. Then G can be constructed from the clique-sum of k-almost embeddable graphs, where $k = k(H)$.*
Furthermore, it suffices to use graphs almost embeddable on surfaces that H does not embed on.

In 2016, Dvořák and Thomas found a strengthening of the graph minor structure theorem in the general case [3], which we go over later in this text. The following corollary resulted from the strengthening:

Corollary 1. *Let there be an apex graph H, and let $G \in Forb(H)$. Then G can be constructed from the clique-sum of strongly k-almost embeddable graphs, where $k = k(H)$.*

Using corollary 1 among other work, Dujmović, Morin and Wood [2] proved theorem 3.

Theorem 3. *The following are equivalent for a proper minor-closed graph class C.*

Remove unused *C excludes an apex graph.*

Remove unused *C has bounded local treewidth.*

Remove unused *C has linear local treewidth.*

Remove unused *Every graph in C has bounded layered treewidth.*

Remove unused *Every graph in G admits layered separations of bounded width.*

Remove unused *For some k, every graph in C can be constructed by the clique-sum of strongly k-almost embeddable graphs.*

David: Put AMST after definition of almost embeddable. Orestis: But isn't in good to establish how our research is related to the existing literature? I am describing how we arrive at our paper starting from the start of minor theory for people less familiar with the background.

2 Preliminaries

For a classic reference book, we refer the reader to e.g [1].

2.1 Basics

For subgraph H_1 of graph $G = (V, E)$, we say that H_1 and $v \in V$ are *adjacent or neighbors* in G if there is $u \in H_1$, with u, v adjacent in G. For subgraphs H_1, H_2 of graph $G = (V, E)$, we say that H_1 and H_2 are *adjacent or neighbors or touch* in G if there are $u \in H_1$, $v \in H_2$ with u, v adjacent in G.

Given two graphs $G = (V_G, E_G),\ H = (V_H, E_H)$, define the *graph union* $G \cup H$ as $(V_G \cup V_H, E_G \cup E_H)$ and the *graph intersection* $G \cap H$ as $(V_G \cap V_H, E_G \cap E_H)$. If $G_V \cap G_H = \varnothing$, then G and H are *disjoint*.

If U is a set of vertices, we define $G - U$ as $G[V_G \setminus U]$. In an abuse of notation, if U is the single-vertex graph v we write $G - v$ rather than $G - \{v\}$ and if G' is a graph, $G - G'$ rather than $G - V(G')$.

If F is a set of pairs of vertices of G, we define $G - F$ to be the graph $(V(G), E(G) \setminus F)$, and $G + F$ to be $(V(G), E(G) \cup F)$. In an abuse of notation, $G - e := G - \{e\}$ and $G + e := G + \{e\}$. To *join vertex u to vertex v* in G means to add (u, v) to G. To *join subgraph S_1 to subgraph S_2* of G means to join (u, v) in G for all $u \in S_1, v \in S_2$.

Given graphs $G_1,\ G_2$ we define the *disjoint union or graph sum or graph addition of G_1 and G_2*, denoted $G_1 + G_2$, to be $G_1 \cup G_2'$ where G_2' is a graph isomorphic to G_2 so that $G_1 \cap G_2' = \varnothing$.

2.2 Minors

Given a graph G and a (possibly single-vertex) connected subgraph S of G, the operation of *contracting* S in G, denoted G/S, outputs a graph $G' = G - S +$ a new vertex v_S neighboring all vertices of $G - S$ that S did in G. Given a set of nodes U of G, the contraction of U is defined to be the contraction of $G[U]$.

Let G and G' be graphs. Assume that for some subgraph R of G there are pairwise disjoint subgraphs $R_1, R_2, ..., R_{|V(G')|}$ of R and there is a bijection $R_1 \leftrightarrow v_1,\ R_2 \leftrightarrow v_2,\ ...,\ R_{|V(G')|} \leftrightarrow v_{|V(G')|}$, where $V(G') = \{v_1, ..., v_{|V(G')|}\}$, such that $(v_i, v_j) \in E(G')$ iff R_i, R_j are adjacent. Then G *contains G' a minor*, denoted $G \geq_m G'$.

A bijection $\mu(v_i) = R_i$ as above, is called a *model* of G' in G. We call R_i the *bag* or *branch* of v_i in G and also denote it $B(v_i)$ or $\mu(v_i)$. For $H \subseteq G$, we denote with $\mu(H)$ the subgraph of G induced by the $\cup_{v \in V(H)} \mu(v)$.

Given a graph class C, denote by *minor-closure*(C) its minor closure, i.e minor-closure(C)=$\{G : G \leq_m G'$ for some $G' \in C\}$.

2.3 Clique-sums

Definition 2. Given graphs G, H with same sized subcliques $K_G \subseteq G$, $K_H \subseteq H$, their *clique sum* $G \oplus H$ over K_G and K_H is defined as the graph obtained by identifying each vertex of K_G with a unique vertex of K_H. Afterwards, we may remove some edges from the clique.

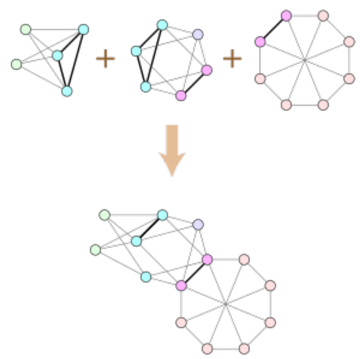

Figure 2: Two clique sums to create a single big graph Courtesy: Wikipedia.

When clear by context or unimportant, we omit mention of the edges to be removed.

Definition 3. The clique sum of G and H on a clique of k vertices is called a *k-sum.* The clique sum of G and H on a clique of $\leq k$ vertices is called a *$\leq k$-sum.*

0-sums are defined to be the disjoint union.

2.4 Apex graphs

A graph G is *apex* if there is a vertex whose removal makes G planar, or if G is planar. Given a graph class C, a graph is apex-C if it is in C or if there is a vertex whose removal makes G belong to C.

3 Looking for cliques in a class of low Δ containing all apex graphs

A natural question to ask is if Δ is increasing with respect to the subset relationship. This is not the case; STARS $\subseteq$ the class of planar graphs $\subseteq$ the class of apex graphs (where STARS is minor closure of the class of stars), but their Δ value is ∞, 3 and ∞ respectively. We do however have the following: Let $\mathcal{A}$ be the class of apex graphs.

Theorem 4. *If a proper minor closed class $C \supseteq \mathcal{A}$, then $\Delta(C) = \infty$.*

Formulated otherwise:

Theorem 5. *If for a minor closed class $C \supseteq \mathcal{A}$ it holds that $\Delta(C) = k \in \mathbb{N}$, then C contains all graphs.*

For non zero natural numbers N, M, the $N \times M$ *grid graph* is the graph with vertex set $\{1, 2, ..., N\} \times \{1, 2, ..., M\}$ and edge set $\{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}$. See figure 3.

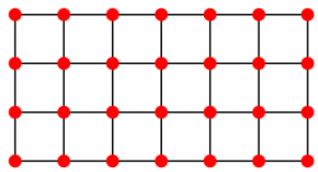

Figure 3: The 4×7 grid graph. Courtesy: Wolfram

The N*-pyramid* is the graph created by taking a $N \times N$ grid, adding a vertex, and joining it to all vertices of the grid.

Clearly a pyramid is an apex graph. As we now show, to prove Theorem 5 it suffices to prove the following: If a graph contains a large enough pyramid as a minor by a graph of $\Delta(G) \leq c$, then it contains an arbitrarily large clique.

have a look at this Cref command

Theorem 6. *For every n, $c \in \mathbb{N}$, there exists N such that if $\Delta(G) \leq c$, and G contains the N-pyramid as a minor, then G contains K_n as a minor.*

Orestis: Wait a second. Can't we just set c to 3?

We prove Theorems 5 and 6 are equivalent.

Proof. If C includes all apex graphs as a minor with graphs of $\Delta(G) \leq k$ for some k, then it includes all N-pyramids with graphs of $\Delta(G) \leq k$, and then it includes all cliques . $\square$

Let's give the right implication for now.

We thus now only focus on Theorem 6. Let H be a subgraph of graph G. An H–path in G is a path of G internally disjoint from H with endpoints in H. To prove 6, the high level idea is to prove that if $\Delta(G) \leq c$ and $G \geq_m$ a large enough N-pyramid, then $G \geq_m$ an $N \times N$ grid H with many H-paths, their endpoints positioned to our liking (Lemma 2). It is well-known that a large enough grid H with $\binom{t}{2}$ H-paths with endpoints far apart from each other contains a K_t clique: See lemma 3.

I replaced 'jump' here by H–path, which is the standard term; please use this term throughout.

Lemma 1. *For every $n, c \in \mathbb{N}$, there is N such that if $\Delta(G) \leq c$ and G contains the N-pyramid as a minor, then G also contains as a minor the $N \times N$ grid, call it H, with n pairwise edge-disjoint H-paths with discreet endpoints. Furthermore, there exists $s \in \mathbb{N}$ such that for any subgraph S of the grid H of order $|S| \geq s$, we can find in a minor of G S-paths with the same properties instead of H-paths.*

Lemma 2. *For every $n, c \in \mathbb{N}$, there is N and s such that if $\Delta(G) \leq c$ and G contains the N-pyramid as a minor, then G also contains as a minor the $N \times N$ grid, call it H, with n pairwise edge-disjoint S-paths with discreet endpoints, where S is any subgraph of H of more than s vertices.*

Maybe use this form?

Lemma 3. *[8] If G is a wall with pairwise disjoint G-paths $P_1, ..., P_{\binom{n}{2}}$ where $n > 1$, there exists $d \geq 0$ such that if any 2 G-path endpoints $p \in P_i$, $p' \in P_j$ have $d(p, p') \geq d$, then $G \geq_m K_n$.*

A *wall* is an $(n \times 2n)$ grid, where ordering edges from top to bottom for each vertical path, we remove from the first vertical path the even ordered edges, from the second vertical path the odd ordered edges, from the third the even ordered edges and so on. Finally we remove degree 1 edges and then arbitrarily subdivide edges. Importantly, notice that Lemma 3 also holds for $(n \times 2n)$ grids. We are now ready to prove Theorem 6.

Proof of Theorem 6. Fix some n and c. We want to prove that for some large enough $N = N(c)$, if a graph G has maximum degree at most c, it will contain K_n as a minor if it contains the N-pyramind as a minor. By lemma 2, for some large enough N, G will contain as a minor the $N \times N$ grid, call it H, with $\binom{n}{2}$ pairwise disjoint H-paths with discreet endpoints. Select some $(N/2 \times N)$ subgrid H' of the grid, and have the endpoints be in a subgraph S of H' such that for all $u_1 \neq u_2 \in S$, $d_{H'}(u_1, u_2) \geq d$. By lemma 3, $G \geq_m K_n$. $\square$

We present a few corollaries before proving lemma 1.

Corollary 2. *If C is a proper minor-closed superclass of the apex graphs, then $\Delta(C) = \infty$.*

The linklessly embeddable graphs are a well known 3-dimensional equivalent of the planar graphs. It is reasonable to ask if, like with planar graphs, one may by some geometric argument replace each node of a linklessly embeddable graph G by some other structure to extend $\Delta(PLANARS) = 3$ to linklessly embeddable graphs. As the apex graphs are a subclass of the linklessly embeddable graphs, the answer is negative.

Corollary 3. *Let $\mathcal{L}$ be the class of linklessly-embeddable graphs. $\Delta(\mathcal{L})=\infty$.*

Corollary 4. *Let C be a class containing all apex graphs as minors. For some k, let f be any function mapping a graph to a graph containing in as a minor with maximum degree k. Then $f[C]$ contains all graphs as minors.*

Now follows the proof of lemma 2.

Proof. Let there be integer n. We would like to prove that if a graph G of $\Delta(G) \leq c$ contains a big enough pyramid as a minor, let it be a $N(n, c)$-pyramid, let $S(n, c)$ be a big enough subgraph of its grid, it contains the $N(n, c) \times N(n, c)$ grid with n pairwise edge-disjoint S-paths with discreet endpoints (N and S to be specified later).

So let a be the apex vertex of the $N(n,c)$-pyramid and X its grid and let μ be the model correspondence function mapping vertices of the pyramid to connected components of G. In G contract $\mu(v)$ for all grid vertices $v \in X$ to obtain X. We will use a to find n jumps, with endpoints in $S \subseteq X$. We remove edges until $\mu(a)$ is a tree, and it has precisely one $\mu(a) - X$ edge towards each vertex of S and 0 to $X \setminus S$.
Of course 2 vertices of $\mu(a)$ neighboring S along with the path of $\mu(a)$ between them form an S-path, but S-paths being internally disjoint, using it could make us lose many other S-paths. How should we proceed?
We may assume all subtrees in $\mu(a)$ have a vertex neighboring S. If not, we remove them. We may also assume all vertices of $\mu(a)$ that only neighbor $\mu(a)$ have degree ≥ 3. If they have degree 1 we delete them, and if they have degree 2 we dissolve them. We then take a maximal path $P = u_1, u_2...$ in $\mu(a)$. Call the u_i neighboring X good vertices, and the rest bad. Bad u_i vertices can be contracted into good vertices; since they must have degree>2 each must neighbor a subtree (which does not intersect P or other such subtrees, else there would be a cycle), which must include a vertex neighboring S. Remove all other vertices of the subtree except the path connecting u_i to the vertex neighboring S, then contract this path. Path P now has only good vertices, every two of which form the internal vertices of an S-path. How large is P? Notice that at the time we pick it, $\mu(a)$ still has maximum degree $\leq c$ and as it neighbors every vertex of S, $\mu(a)$ still has more than $\frac{N^2}{c}$ vertices. Fixing c and letting N and thus $|V(\mu(a))|$ grow larger and larger, the diameter of $\mu(a)$ must also increase, and thus the length of its maximum path. Pick s large enough for $\mu(a)$ to have diametre at least $2n$. Pick N large enough X can fit S. □

Maybe we can remove N altogether from the initial statement

Remark 1. Nowhere in this lemma did we use the fact that X is a grid. Indeed, rather than just pyramids, it holds for any infinite family of finite graphs as long as they all have a vertex connected to all other vertices.

4 A superclass of $\Delta = 3$ for any class excluding an apex graph

As we said, we will prove that every proper minor-closed class C excluding an apex graph has a proper minor-closed superclass C' of $\Delta(C') = 3$, using condition 6 of Theorem 3.

4.1 Embeddings on Surfaces

Much like graphs can be embedded on the plane, they can be embedded on topological surfaces. A surface is a connected compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$. Mohar's Topological graph theory [7] provides for a rigorous introduction to the topic. For the topology fundamentals, we recommend Kinsey's topology of surfaces [6].

A graph is *embeddable* on a surface if we can draw it on the surface so that edges do not intersect.

Definition 4. A graph G is *embeddable* on topological space (X, τ) if there is a function f mapping vertices to elements of X, and edges to simple curves on X so that $f(v_1) \neq f(v_2)$ for $v_1 \neq v_2$, and curve $f(uv)$ connects $f(u)$ and $f(v)$, and has no intersection with the image of other vertices and only intersects other edges on its endpoints.
f is an *embeddeding* of G on X. The image of f, $f[(V(G) \cup E(G))]$, is called the *embedded graph*, and though it is technically not a graph, one may produce a graph from one in the obvious manner. For ease of notation, the embedded graph is also abusively denoted $f(G)$.

4.2 k-almost embeddable graphs, vortices, simple vortices, the graph minor structure theorem strengthened

Let us define what a k-almost embeddable graph is. Roughly speaking, we take a graph embeddable on some surface of Euler genus at most k, we embed it, and then choose up to k faces, to which we add potentially non-embeddable layers of "depth" $\leq k$. Finally we add k apex vertices.

Let's start by defining the non-embeddable layers of an almost embeddable graph, called *vortices*.

Definition 5. Let there be graph F with vertex set $v_1, ..., v_n$. Let there be graph H_1. Let G_2 be $H_1 \cup H_2$. Let there be a graph H_3 such that $G_2 \cap H_3 \subseteq \bigcup_i H_i$ taken over all $i < 3$ such that $(v_i, v_3) \in E(F)$. Let G_3 be $G_2 \cup H_3$. Let there be a graph H_4 such that $G_3 \cap H_4 \subseteq \bigcup_i H_i$ taken over all $i < 4$ such $(v_i, v_4) \in E(F)$ and so on, n times. Any graph G_n that can be built in this manner by H_i of at most $k+1$ vertices is said to have an $F-$*decomposition of width* k. We call $V(H_i)$ the *bags* of G, and denote them as B_{H_i} or $B(H_i)$. If minor bags are involved as well, we call them the *F-decomposition bags* to avoid confusion.

Definition 6. Let there be a graph G embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle [2] of G. Let there be graph G', and add [3] G' to G. Let there be a C-decomposition of G' with bags $B_{v_1}, ..., B_{v_n}$. Pick a distinct node u_i from each bag B_{v_i}, and in $G' + G$ identify v_i and u_i for all i to obtain a new graph G''. *Adding a vortex* G' to G over $v_1, ..., v_n$ and $u_1, ..., u_n$ is defined to be this sequence of operations. If the C-decomposition of G' has width k, then the vortex has *depth* k. We call G' a *vortex* of G''.

[ADD IMAGE HERE?]
The reader may picture the vortex added inside the face. Since we usually do not care about the specific choice of G', we simply say we add a vortex to G on C. We now proceed to define a k-almost embeddable graph.

[2] A facial cycle is a cycle which is the boundary of a face of the embedded graph G.
[3] We remind we have defined the addition two graphs to be their disjoint union.

Definition 7. Let there be a graph G. Let G be embeddable on a surface of Euler genus $\leq k$. For some embedding, choose up to k pairwise disjoint facial cycles of G. Add to each of them a vortex of depth up to k, to obtain G'. Finally, add up to k vertices to G' to obtain G'', called the *apex vertices of* G'', and join them to any vertex in G'' (including other apex vertices). G'' is called a *k-almost embeddable* graph. We call G the *embedded part* of G'' and call G'' *almost embeddable* on the surface G was embedded on.

Reminding the minor structure theorem, for any H, all H-minor-free graphs can be constructed from the clique sum of k-almost embeddable graphs, where $k = k(H)$. For excluded minors H belonging to a more specific family of graphs, there exist more specific results than the graph minor structure theorem; for apex graphs it is mentioned in Theorem 3. If H is restricted to the planar graphs, then a $G \in \mathrm{forb}(H)$ can be constructed from the clique-sum of graphs of $\leq k$ vertices, where $k = k(H)$ (in other words, $treewidth(G) < k$). One could go on.

As already mentioned, on the other hand Dvořák and Thomas proved a strengthening of the graph minor structure theorem in the general case.

Definition 8. Given graph H and surface Σ, let $\alpha(H, \Sigma)$ be the minimum number of vertices one need remove from H to make it embeddable on Σ.

Theorem 7 (The graph minor structure theorem strengthened [3])**.** *The graph minor structure theorem holds even if we only use graphs almost-embedded on surface Σ such that every triangle of their embedded part is the boundary of a face homeomorphic to an open ball of $\mathbb{R}^2$, and all but $\alpha(H, \Sigma)$-1 of their apex vertices neighbor only other apex vertices and vortices.*

Definition 9. A *strongly k-almost embeddable* is a k-almost embeddable graph where also all apex vertices neighbor only other apex vertices and vortex vertices.

If H is an apex graph, then $\alpha(H, \Sigma) = 1$ of course. Thus any $G \in Forb(H)$ can be constructed by the clique-sum of strongly k-almost embeddable graphs. Corollary 1 follows. As implied by theorem 3, the converse also holds; if there is k such that every graph in some class can be constructed from the clique-sum of strongly k-almost embeddable graphs, then it excludes some apex graph.

The strengthened graph minor structure theorem has an important implication; We need only clique-sum almost embeddable graphs whose embedded part has no K_4 subgraph, or is trivially a K_4 graph.

Corollary 5. *Let there be connected graph $G \neq K_4$ embedded on some surface such that every triangle is the boundary of an open disc. Then G has no 4-cliques.*

Proof. Let there be a K_4 with vertex set *abcd* in the graph G with embedding f. As G is connected and not a K_4, there must be a vertex v adjacent to some vertex of $abcd$, let it be adjacent to a. $f(a)$ has an open disc containing it and an initial segment of each edge incident to it. Without loss of generality, let the incident edges be clockwise around a in the order ab, ac, ad, av. Any face a participates in must contain two clockwise adjacent edges in its boundary . Therefore, there is no face including only adb in its boundary. □

Naturally, the minor structure theorem would not be very interesting if it turned out that for some k we can create all graphs using k-almost embeddable ones. The following is a well known fact.

Theorem 8. *Let there be $k \in \mathbb{Z}_{\geq 0}$. Let C be the the class of all graphs that can be constructed by clique-summing k-almost embeddable graphs. Then $minor - closure(C)$ is proper.* [4]

This theorem holds for strongly k-almost embeddable graphs, as they are a subset of k-almost embeddable graphs [5].

In Jim Geelen's publicly available *Introduction to Graph Minors* course lectures, adding a vortex had a simpler definition, which is useful to us;

Definition 10. Let there be a graph G embedded on a surface. Let $C = v_1, v_2, ..., v_n$ be a facial cycle of G. Add a K_k clique to G, and identify its first vertex to v_1. Add another K_k clique, and identify its first vertex to v_2 and so on. The clique identified with v_i is called the *vortex clique of* v_i. Now, join the clique of v_1 to the clique of v_2, join the clique of v_2 to the clique of v_3 and so on. Also join the clique of v_1 to the clique of v_n.
We call this sequence of operations as *adding a simple vortex of depth* k. The subgraph induced by the added cliques (i.e the union of the vortex clique of v_i over all i) is a *simple vortex.*[remove next?] The cycle induced by the ith vertex of all simple vortex cliques is the *ith layer* of the simple vortex. We always have C be the 1st layer of the simple vortex. [IMAGE]

Clearly this definition is different. The reader may notice that a simple vortex of depth k is a vortex of depth $2k + 1$ (the +1 needed because decompositions have that pointless -1 in their definition). Now, a k-depth vortex need not be

[4]Indeed, for fixed k none of the operations involved in constructing a k-almost embeddable graph can create an arbitrarily large clique minor; By Euler's formula for high genus , a graph G embedded on a surface of euler genus k must have at most $m \leq 3n - 6 + 3k$ where n are the vertices and m the edges of the graph, therefore too large a clique will not be embeddable on the surface. Graphs embeddable on a specific surface being closed under minors, G can't have too large a clique minor either for specific k. Similarly, adding k apex graphs can increase the Hadwinger number by at most k, and the clique sum of graphs G_1 and G_2 cannot create any larger clique minor either. For adding a vortex of depth k cannot create an arbitarily large minor, and more on the minor structure theorem, we refer the interested reader to Jim Geelen's graph minor recorded lectures, lecture 3 [4].

[5]This is significantly useful for our purposes, as opposed to the other characterizations of the class of apex graphs in theorem 3, such as layered treewidth, where the minor closure of graphs of layered treewidth k contains all graphs, even for $k = 3$. Indeed, the 3- dimensional $n \times n \times 2$ grid graph has layered TW 3 and a K_n minor, take a row from the first level and a column from the second to be each branch.

isomorphic to any simple vortex, for example take a vortex which has a vertex neighboring all vertices of the facial cycle (this is possible if the vertex is in all branches of the cycle decomposition). However, any k-depth vortex is a *minor* of a $(k+1)$-depth simple vortex:

Proposition 1. *Let there be embedded graph G on some surface, with facial cycle $C = v_1, ..., v_n$ and add vortex V of depth k on C to obtain G'. Alternatively, add to G a simple vortex sV of depth $k+1$ to obtain G''. sV contains V as a minor.*

Proof. Let B_{v_i} be the bags of the C-decomposition of V of width k. We slowly contract and remove nodes from sV to prove it contains a V minor. In sV, for all $v_i \in C$, remove vertices from the simple vortex clique of v_i until it has as many vertices as B_{v_i} does. Let's now specify the model function μ. If $u \in B_{v_1}$ and $\in$ no other vortex bag, pick $\mu(u) = u'$ where u' is a vertex belonging to the simple vortex clique of v_1. If $u \in B_{v_1}$ also belongs to other bags, $B_{v_{n-j}}, ..., B_{v_n}, B_{v_1}, ..., B_{v_i}$, pick an unused by μ vertex from the simple vortex cliques of $v_{n-j}, ..., v_i$, and let the path P they define be modeled to u, i.e $\mu(P) = u$. Repeat this process for vertices of B_{v_2} not in B_{v_1} and so on. We never run out of unoccupied vertices in a simple vortex clique. If we do, let the simple vortex clique of v_i be such a clique, then B_{v_i} has more than $k+1$ vertices (a contradiction), as by construction of μ every occupied vertex of the simple vortex clique of v_i corresponds to exactly one vertex of B_{v_i}. It suffices to prove that if u and u' are adjacent in V then $\mu(u)$ and $\mu(u')$ are in sV. u neighbors u' in $V \implies$ they share a bag $B_{v_i} \implies$ (by construction) the simple-vortex clique of v_i has a vertex which μ corresponds to u and a vertex which μ corresponds to $u' \implies \mu(u)$ and $\mu(u')$ neighbor. $\square$

Corollary 6. *Let there be graphs G' and G as above. $G' \geq_m G$.*

Proof. For vertices u of G' that are in the vortex V, let model function μ showing $G' \geq_m G$ be same as before, but making sure to set $\mu(v_i) = v_i$ for $v_i \in C$. If u is not in the vortex, once again set $\mu(u) = u$. Let there be vertex $v \notin$ a vortex. $(v, u) \in E(G) \implies (v, u) \in E(G') \implies (\mu(u), \mu(v)) \in E(G')$. $\square$

4.3 Right direction of main theorem

We are now ready to prove theorem 1. By theorem 1 any minor closed class C excluding an apex graph can for some k be built by the clique sum of strongly k-almost embeddable graphs G. We will show that any such graph G, is the minor of a graph G' built by the clique sum of strongly $f(k^2+k)$-almost embeddable graphs with $\Delta(G') = 3$. Taking the graph class of all such G', and taking its minor closure, we obtain a proper minor-closed graph class C' of $\Delta(C') = 3$ which contains C.

Rather than instantly give the final construction, it is more natural to see it produced step by step, adding more ingredients in each step. For each intermediate step we prove a few facts which we reuse in the next steps.

Let $C_1(k)$ be the class of graphs of genus $\leq k$, embeddable so each triangle bounds an open disc.

Let $C_2(k)$ be the class of graphs that can be obtained by adding at most k vortices of depth at most k to a graph of $C_1(k)$ (the graph of $C_1(k)$ embedded so that each triangle bounds an open disc of course).

Let $C_3(k)$ be the class of graphs that can be obtained by adding at most k apex vertices to a graph of $C_2(k)$, where the apex vertices may neighbor only other apex vertices and vortex vertices, i.e the class of strongly k-almost embeddable graphs.

Definition 11. Denote by $\oplus[C]$ the clique sum closure of class C. Denote by $\oplus^{\leq n}[C]$ the $\leq n$-sum closure of class C.

It is easy to see that much like planar graphs, $\Delta(C_1(k)) = 3$. We will add as few ingredients as possible; we will show that $\Delta(\oplus[C_1(k)]) = 3$. We will then show that $\oplus[C_2(k)]$ has a proper minor-closed superclass of $\Delta = 3$. We will then do the same for $\oplus[C_2(k)]$.

Orestis: Wait, is this minor closed?

We start by showing $\Delta(C_1(k)) = 3$. See figure 1.1.

Proof. By [5], if a (finite) graph G is embedded on a surface, for any $v \in G$ there is an open disc D_v containing from G only v and an initial segment of edges incident to v [6]. Take the discs small enough that their boundaries do not intersect. Erase everything inside the closed disc D_v of v, then let $p_1, ..., p_k$ be the points where the boundary of the closed disc intersected the edges of v $e_1, ..., e_k$, ordered in a counterclockwise manner. Add the p_i back as embedded vertices v_i. Then, connect p_i with p_{i+1} by a curve running along the perimeter of the cycle. Call the resulting graph G'. Notice that $\Delta(G') \leq 3$ and $G' \geq_m G$, the model function is $\mu(v)$= all vertices of G' embedded on $D(v)$. □

Definition 12. Given graph G, we call the graph $G' \geq_m G$ of maximum degree 3 as in the above proof the *fattening* or *ballooning* of G, and denote it $Bl(G)$. The cycle we replace vertex $v \in G$ with we denote by $Bl(v)$. [7]

We will prove that any graph G built by the clique sum of graphs of $C_1(k)$ is a minor of a G' built by the clique sum of graphs of $C_1(k)$ and $\Delta(G') = 3$. Let us develop a toolset to present theorem 9.

Definition 13. B is a *base* for C under $\leq n$-sums if $\oplus^{\leq n}[B] = C$. B is a *base* for C under clique sums if $\oplus[B] = C$.

[6] We may have to change the embedding a bit. Importantly, facial cycles remain same, and more generally the subgraphs induced by the boundary of faces remain same.

[7] This is also the model function showing $G' \geq_m G$

Definition 14. Let $G' \geq_m G$, with model function μ. For clique $K \in G$, let its vertex set be $\{u_1, ...\}$, let there be isomorphic clique $K' \in G'$ with vertex set $\{u'_1, ...\}$ such that $u'_i \in \mu(u_i)$. We call K' a *representor clique* of K under μ.

Notice that clique representation is *transitive under minors*: If $G \leq_m G' \leq_m G''$ and K is a clique of G represented under μ by K' in G' and K' is represented under μ' in G'' by K'', then K is represented under $\mu \circ \mu'$ by K''. Also notice the following.

Proposition 2. *Let $G \leq_m G'$. If $K' \in G'$ is a representor clique of $K \in G$ under μ, we may remove from G' all $\mu(u) - \mu(v)$ edges, except the edges of K', for all distinct pairs $u, v \in K$ and still contain G as a minor under μ.* $\square$

Almost entirely, in the following we want to restrict ourselves to a unique specific representor for each clique. This motivates the following definition.

Definition 15. Let $G \leq_m G'$ under μ. Correspond to some cliques in G a representor of theirs in G'. Call any such correspondence function from cliques in G to representor cliques in G' a *representation*. Call any 1-1 correspondence function a *1-1 representation* and if all cliques are represented call it *total*. Call the image of the correspondence function the set of *selected representors*.

We now give theorem 9.

Theorem 9. *Let there be a minor-closed class C closed under n-sums, such that $P_2 \square K_n \in C$. Let B be a base for C under $\leq$ n-sums. For every graph G in B, let there be graph G' in C with*

- $G' \geq_m G$.
- *Every maximal clique in G has a representor clique in G'.*
- $\Delta(G') \leq d$.

Then $\Delta(C) \leq d$.

This theorem is a specialization of a more general theorem. For a maximal clique of a graph G, call its representor clique in $G' \geq_m G$ a *max representor clique*.

Definition 16. Given graphs G, H, their *Cartesian product* $G \square H$ is the graph with vertex set $V(G) \times V(H)$ where two vertices (u, v) and (u', v') are adjacent if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Intuitively, for each vertex of H take a copy of G, and if two vertices in H are connected, connect the corresponding G copies by their identical vertices.

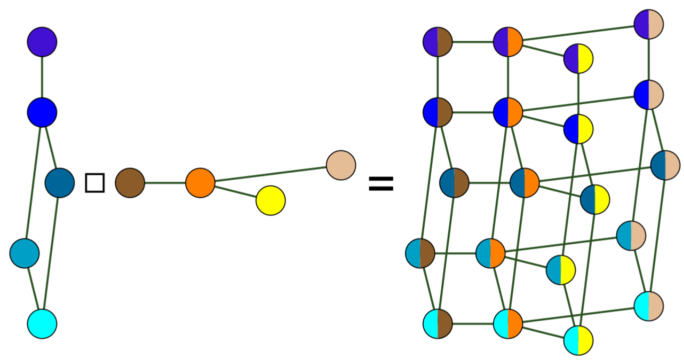

Figure 4: The Cartesian product of two graphs Courtesy: Wikipedia.

Definition 17. For $u \in G$, we call $G \Box H$ limited to all vertices of the form (u, v) where $v \in H$, the H-subgraph of $V(G) \times V(H)$ corresponding to u.

Theorem 10. *Let there be a minor-closed class C closed under n-sums, such that $P_2 \Box K_n \in C$. Let B be a base for C under $\leq n$-sums. For every graph G in B, let there be graph G' in C with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in G has a selected representor clique in G'.*
- *Every vertex v of G' of degree greater than d has degree at most $d - s$ if we remove for every selected max representor clique K it is in the edges of $G'[K]$, where s is the number of selected max representor cliques v is in.*

Then $\Delta(C) \leq d$.

This theorem is also a specialization of an even more general theorem! A *degree k expansion* of G is a graph $G' \geq_m G$ with $\Delta(G') = k$.

Theorem 11. *Let there be a class C' closed under n-sums, such that $P_2 \Box K_n \in C'$. Let B be a base for minor-closed class C under $\leq n$-sums. For every graph G in B, let there be graph G' in C' with*

- *$G' \geq_m G$ and there is a representation so that*
- *Every maximal clique in G has a selected representor clique in G'.*
- *Every vertex v of G' of degree greater than d has degree at most $d - s$, if we remove for every selected max representor clique K it is in the edges of $G'[K]$, where s is the number of selected max representor cliques v is in.*

Then every graph in C has an expansion of degree $\leq d$ in C'.

We remind one notation we use for clique sums: Given graphs G, H such that $G \cap H$ is a clique, their *clique sum* $G \oplus H$ is defined by the operation $G \cup H$. If $G \cap H{=}K$, denote this clique sum by $G \oplus_K H$.

Lemma 4. *Let* $G = ((G_1 \oplus_{K_1} G_2) \oplus_{K_2} G_3) \oplus_{K_3} \ldots$ *Let* $G'_i \geq_m G_i$ *be graphs with model function* μ_i *such that for every clique* K *of* G_i*,* G'_i *has a representor clique* K'*. Then* $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3} \ldots =: G' \geq_m G$*.* [8]

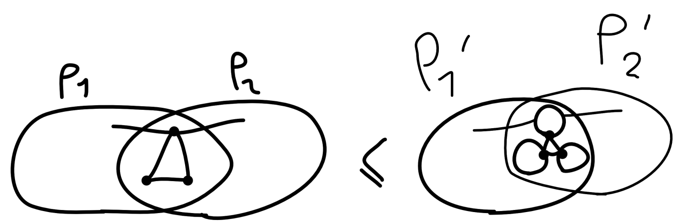

Figure 5: [IMPROVE THIS IMAGE] Example where K is a triangle. Graph $P_1 \oplus_K P_2$ is a minor of $P'_1 \oplus_{K'} P'_2$

Proof. Call any K_j a *common clique*. We define the branches of G', i.e the model function μ from vertices in G to connected components of G': $\mu(v) := \bigcup_i \mu_i(v)$, where $\mu_i(v) = \varnothing$ if $v \notin G_i$.

If $v \in G$, $v \notin$ any common clique, let it only $\in G_i$, then $(u, v) \in G \implies (u, v) \in G_i \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch.

If $v \in$ some common clique K of G', then $(u, v) \in G \implies (u, v) \in$ one of the G_i containing $K \implies \mu_i(u), \mu_i(v)$ touch $\implies \mu(u), \mu(v)$ touch. $\square$

Lemma 5. *Let* C *be a graph class closed under* n*-clique-sums such that the graph product* $K_n \square P_2$ *is in* C*. Then* $K_n \square T$ *is in* C *for any tree* T *of more than 1 vertex.*

[8] $((G'_1 \oplus_{K'_1} G'_2) \oplus_{K'_2} G'_3) \oplus_{K'_3} \ldots$ is well-defined. If G_{i+1} is clique summed on $((G_1 \oplus_{K_1} G_2) \oplus \ldots \oplus_{K_{i-1}} G_i)$ on common clique K_i, then K_i must $\subseteq$ some graph G_j, $j < i$. $K_i \in G_j \implies K'_i \in G'_j \implies K'_i \in ((G'_1 \oplus_{K'_1} G'_2) \oplus \ldots \oplus_{K'_{i-1}} G'_i)$

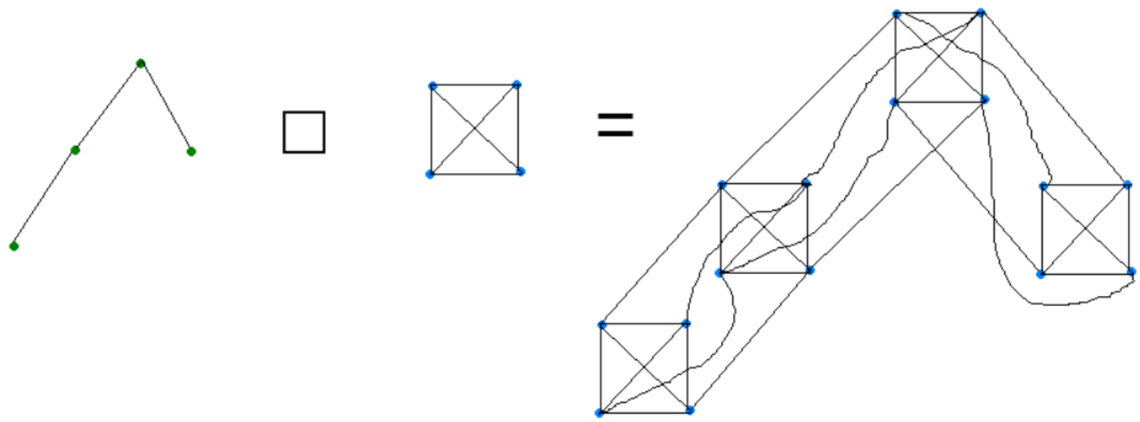

Figure 6: The cartesian product of a tree and a 4-clique, visualized.

The proof is conceptually very simple; imagine $K_n \square T$ as a tree where instead of vertices we have cliques. Much like we can create any tree by adding each of its edges one by one starting from the root in a DFS or BFS manner, we can create $K_n \square T$ by adding each of its n-cliques in the same order.

Proof. Let there be graph $K_n \square T$ some tree T. We have that $V(K_n \square T) = (V(T) \times \{1, ..., n\})$ and $((t_1, v_1), (t_2, v_2)) \in E(K_n \square T) \iff t_1 = t_2$ or (t_1 neighbors t_2 in T and $v_1 = v_2$).
The result is by induction of the number of vertices of T. If T is the edge graph, then the result holds trivially. Now let $K_n \square T$ for all T of some fixed number of vertices n. Let there be T' of $n+1$ vertices. This is constructed by some T of n vertices after adding a vertex t_2 to T and joining it to the correct vertex t_1. We have $K_n \square T \in C$. Clique sum either of the cliques of $K_n \square P_2$ to the clique of $K_n \square T$ corresponding to t_1, i.e to the subgraph of $K_n \square T$ induced by $\{(t_1, i) | i \in \{1, ..., n\}\}$. The resulting graph is (isomorphic to) $K_n \square T'$: Relabel the new n vertices as $(t_2, 1), ..., (t_2, n)$ and notice that (t_2, i) neighbors (t, j) iff ($t_2 = t$) or t_2 neighbors t in T' and $i = j$). $\square$

We proceed with the proof of theorem 11.

Proof. Let there be graph G of C built by the clique sum of base graphs $G_1 \oplus_{K_1} ... \oplus_{K_k} G_k$. Suppose there exist graphs $G'_i \in C'$ with the aforementioned conditions, where μ_i is the model function for $G'_i \geq_m G_i$. Notice that since every maximal clique in G_i has a selected representor in G'_i, every clique in G_i has a representor in G'_i. By lemma 4, $(G'_1 \oplus_{K'_1} G'_2 \oplus_{K'_2} ... \oplus_{K'_k} G'_k) =: G' \geq_m G$, where $K'_i \in G'_{i+1}$ is a representor of K_i under μ_{i+1} and a representor of G_j under μ_j, G_j being the graph of G that G_i was clique summed while building G. [IS THIS OVERCOMPLICATED?]

The common cliques K_i' of G' could have an arbitrarily large degree, so we make some adjustments. As $P_2 \Box K_n \in C'$ and C' is closed under n-sums, by lemma 5 $T \Box K_n \in C'$ where T is the $k+1$ comb graph. We remind we call the subclique of $T \Box K_n$ corresponding to the ith spine vertex of the comb the ith spine clique, and the subclique of $T \Box K_n$ corresponding to the ith hair vertex the ith hair clique. Furthermore, we call the sub-comb of $T \Box K_n$ corresponding to ith vertex of K_n the ith comb running along $T \Box K_n$.
To each selected max representor clique K' of G_i', let K' have l vertices, l-sum a $P_2 \Box K_l$, where P_2 is the path of two vertices. Call the l-clique of $P_2 \Box K_l$ not used in the clique sum the *copy* of K'. To the copy of K', l-sum the first spine clique of a $T \Box K_l$, to obtain $G_i'' \in C'$. Call the $T \Box K_l$ clique summed to the copy of K' its *representor comb*. $G_i'' \geq_m G_i'$ of course, and let model function μ_i' showing that be $\mu_i'(v) = v$ if v is not in a max representor clique and if $v \in$ some max representor clique K, let v be the jth vertex of K, then let $\mu_i'(v)$ be the jth subcomb of the representor comb of K and the jth vertex of K [IS THIS OVERCOMPLICATED TO SAY?].
By construction of μ_i', if K' is a selected max representor clique of G_i', all spine and hair cliques of the representor comb of K' in G_i'' are representors of K' under μ_i'. We may use lemma 4 again; $(G_1'' \oplus G_2'' \oplus ... \oplus G_k'') =: G'' \geq_m G'$, where if during the construction of G' graph G_i' was clique summed on the subgraph G_j' on their common clique K_i', then G_i'' is clique summed on G_j'' using the ith hair clique of the representor comb of K_i' in G_i' and the ith hair clique of the representor comb of K_i' in G_j'.
Notice that lemma 4 gives us a specific model function μ' showing $G'' \geq_m G'$: The bag $\mu'(v)$ is the union of all $\mu_i'(v)$, if $v \in G_i$. By our choice of μ_i', we conclude that if v is in a selected max clique of G' under μ, let v be its jth vertex, then μ' puts in $\mu'(v)$ vertex v of G'' as well as the entire jth subcomb of its representor comb. Thus, by proposition 2, $G'' \geq_m G'$ even if for every selected max representor we remove edges with both endpoints in the representor, and for its representor comb we remove all edges with both endpoints on the same spine or hair clique, except from one such clique. Let G''' be G'' where we do just that, retaining only the edges of the last hair clique of every comb representor.
It suffices to prove that $\Delta(G''') \leq d$. As it turns out, we will need one more small change to do that. Let $v \in G'''$. We have the following cases.

- v does not belong to any representor comb or selected max clique of G'''. In this case, v also $\in G'$ and its degree remained unchanged during all of the above. $d_{G'''}(v) = d_G(v) \leq d$.

- v belongs to what was a selected max-clique representor K' in G'. If it has 1 vertex, then by construction $d_{G'''}(v) = 1$. For every selected max representor clique K' it was in, we removed the edges of $G'[K']$ and connected v to a copy of K', and made no other changes to the edges of v. By the conditions of the theorem, $d_{G'''}(v) \leq (d-s)+s = d$. Notice that $d_{G'''}(v) \leq d_{G'}(v)$, as the removal of each $G'[K']$ reduces the degree of v by 1 at least, so we need only consider v of $d_{G'}(v) > d$.

- v belongs to the spine clique of a comb representor. $d_{G'''}(v)$ is at most 3; It is incident precisely to an edge with endpoint the previous spine clique, the next spine clique if it has one, and its hair clique.

- v belongs to the hair clique of a comb representor. If the hair clique was not used in a clique sum and it is not the last hair clique, by construction $d_{G'''}(v)$=1. If it was used in a clique sum, by construction note that no hair clique is used in more than 1 clique sum, $d_{G'''}(v)$=2. If it is the last hair clique, let it have l vertices, then by construction v has degree l.

We now make changes to lower the degree of vertices of the last hair clique of a representor comb to 3, obtaining the intended claim. Let K be a last hair clique, let its edge set be $e_1, ..., e_m$. Let there be graph $P_m \square K$, where P_m is the path of m nodes. Let the K corresponding to the ith path vertex of $P_m \square K$ be called its ith clique, and the subpath corresponding to the ith clique vertex be the ith subpath running along $P_m \square K$. Clique sum to K the first clique of a $P_m \square K$. Then remove from the ith clique all edges with both endpoints in the clique except e_i. It is easy to see that all vertices of a $P_m \square K$ added in this manner have max degree 3, and by contracting the ith subpath running along the $P_m \square K$ we get G'''. Doing this for all hair cliques yields a graph G'''' with the required properties. $\square$

Using the previous lemmas, we can prove that $\Delta(\oplus[C_1(k)]) = 3$ fairly quickly.

Proposition 3. $\Delta(\oplus[C_1(k)]) = 3$.

Proof. We use theorem 9. The base B of $\oplus[C_1(k)]$ is of course $C_1(k)$. Let there be graph $G \in B$. We can assume that every triangle has an empty interior or exterior, else it is a separator and we can further decompose G to the clique sum of other base graphs. Let it be the interior, the other cases are analogous. On the open disc that has as boundary a triangle of G with vertex set abc, add a new triangle $a'b'c'$ embedded there, and connect a to a', b to b', c to c'. Let G' be the ballooning [9] $Bl(G)$, except we have not ballooned the vertices of any of the new triangles. Notice that $\Delta(G') = 3$. $G' \geq_m G$ by contracting each $Bl(v)$ back into v, and for each new triangle, $a'b'c'$ to a' to a, b' to b, c' to c. $a'b'c'$ in G' is a representor of abc in G. Let μ_1 be this model function. Each 2-clique $uv \in G$ has as representor the by construction unique $Bl(u) - Bl(v)$ edge of G'. By theorem 9, we have $\Delta(\oplus[C_1(k)]) = 3$. $\square$

We now add the next ingredient, vortices. We will use theorem 11 to show that $\oplus[C_2(k)]$ has a degree 3 expansion in $C' = \oplus[C_2(2k)]$. [10]. In other words, for every $G \in \oplus[C_2(k)]$, there is $G' \in \oplus[C_2(2k)]$ with $G' \geq_m G$ and $\Delta(G') = 3$. Putting all those G' in a set, and taking the minor closure of the set, we obtain a minor-closed superclass of $\oplus[C_2(k)]$ of $\Delta = 3$ which is proper by theorem 8.

[9]We remind a ballooning or fattening of G means to replace each vertex v with a cycle C embedded on the boundary of an open disc around the vertex, the vertices of C connected in a clockwise manner and each vertex of C adjacent to a single neighbor of v.

[10]In fact, we can show that $\Delta(\oplus[C_2(k)])$=3

Proposition 4. $\Delta(\oplus[C_2(k)])$ *has a proper minor-closed superclass of* Δ *=3.*

Once again, the base is $C_2(k)$. Let there be graph G in $C_2(k)$, with embedded part $Emb(G)$ and at most k vortices of depth at most k added to pairwise disjoint facial cycles $C_1, ...C_k$.
Let G' be G with every vortex of depth d replaced by a simple vortex of depth $d+1$, as in proposition 1 and corollary 6. Use the model function defined there, call it μ_{sv}. Observe that there is a representation R_{sv} under μ_{sv}; if a clique K of G is in $Emb(G)$ trivially $R_{sv}(K) = K$. If a clique K of G is not in $Emb(G)$, it is in a vortex. In this case, let its facial cycle be $C = v_1v_2...$, then there must be a vortex bag B_{v_i} it is in. By construction of μ_{sv}, every vertex of B_{v_i} contains in its model in G' a distinct vertex of the simple vortex clique of v_i. But every vertex in the simple vortex clique of v_i is adjacent. $R_{sv}(K)$ is those simple vortex vertices.
As clique representation is transitive under minors, it suffices to find for every G' a graph $G'' \geq_m G'$ of $\oplus[C_2(2k+1)]$ such that there is a representation under some model function μ satisfying the conditions of theorem 11. Then, there will be such a representation for $G'' \geq_m G$ under $\mu \circ \mu_{sv}$.
Add triangles and repeat the same fattening procedure as before on $Emb(G)$ to obtain $Emb(G)'$. This time, rather than add 1 extra triangle $a'b'c'$ to the empty face of triangle abc of $Emb(G)$, we add two triangles $a'b'c'$ and $a''b''c''$, $a'b'c'$ embedded on the empty face bounded by abc, $a''b''c''$ on the empty face bounded by $a'b'c'$, a joined to a', a' joined to a'' and so on. Both new triangles are not fattened. Call $a'b'c'$ and $a''b''c''$ the first and second *copies* of abc respectively. Fortunately, after fattening facial cycles are (almost) retained:

Definition 18. For $v \in Emb(G)$, let D_v be the disc on the boundary of which the cycle $Bl(v)$ was embedded on. Let $Bl(v \to u)$ or $Bl(u \leftarrow v)$ be the vertex of $Bl(v)$ incident to the unique $Bl(v) - Bl(u)$ edge of $Emb(G)'$.

If $C = u_1...u_n$, where $n > 3$ is a facial cycle in $Emb(G)$, then there is a facial cycle C'' in $Emb(G)'$, first with 1 or 2 vertices from $Bl(u_1)$, then with vertices from $Bl(u_2)$, and so on: Start from the vertex $Bl(u_1 \to u_2)$. Follow the $Bl(u_1)-Bl(u_2)$ edge to $Bl(u_2 \to u_1)$. If $d_{emb(G)}(u_2) > 2$, there is an edge $Bl(u_1 \leftarrow u_2) - Bl(u_2 \to u_3)$ in $Bl(u_2)$. Follow along it. Then take the $Bl(u_2 \to u_3)$ edge and so on. Call C'' the *corresponding* facial cycle of C. For triangles of $Emb(G)$ call their second copy in $Emb(G)'$ the corresponding facial cycle.
If to construct G' a simple vortex of depth k was added to a facial cycle of $Emb(G)$, add to the corresponding facial cycle of $Emb(G)'$ a simple vortex of depth k to obtain G''.
We prove G'' fulfils the conditions of theorem 11.

- To prove that $G'' \geq_m G'$, let μ_2 be the model function showing that, for v in the embedded part of G'' let $\mu_2(v) = \mu_1(v)$, where $\mu_1(v)$ is the model function of the proof that $\Delta(\oplus[C_1(k)]) = 3$, modified by putting a'' in the same bag as a' and a for triangles $abc \in G'$ of course. For $v \in$ a vortex, let the facial cycle be $C = v_1v_2...$ and let v belong to the simple vortex clique of v_i, let v be the ith vertex of the clique. Let C'' be the corresponding facial cycle and notice C'' of G'' is also in $Emb(G'') = Emb(G)'$. If $C = v_1v_2v_3$, then $C'' = v_1''v_2''v_3''$ and let $\mu_2(v)$ be the ith vertex of the simple vortex clique of v_i''. Else, set $\mu_2(v)$ to be the ith vertices of the vortex cliques of $Bl(v_{i-1} \leftarrow v_i)$ and $Bl(v_i \rightarrow v_{i+1})$. It is easy to observe that the contraction in G'' of each minor bag $\mu(v)$ yields G'.

- We find a representation R_2 under μ_2 so each maximal clique K is represented. For a cliques K of $Emb(G)$, set $R_2(K) = R_1(K)$, where for triangles K we use their first copy in G'' to represent them.
  With regard to simple vortex cliques K of G', let the simple vortex be of depth l and added on the facial cycle $C = u_1u_2...u_n$. There are precisely n maximal cliques of $2l$ vertices; the simple vortex clique of u_i $\cup$ the simple vortex clique of u_{i+1}, for $i \in \{1, ..., n\}$, where $u_{n+1} = u_1$. Its selected representor $R(K)$ in G'' is the simple vortex clique of $Bl(u_i \rightarrow u_{i+1})\cup$ the simple vortex clique of $Bl(u_i \leftarrow u_{i+1})$.

- We prove the third condition. If $v \in G''$, is not in a vortex, then by construction it has max degree 3 unless if it is in the first copy $a'b'c'$ of a triangle abc. In this case it is a selected representor of abc, and it represents no other cliques. For the condition to be satisfied it must have at most $3-1$ edges adjacent to it, after removing the edges of $a'b'c'$, which is the case. If v is in a vortex, notice that all edges of the vortex have both endpoints in a selected max clique representor, and v belongs to exactly 2 selected representors. After removing the edges of the selected cliques, $d(v) = 1$ if v is on the facial cycle, and $d(v) = 0$ otherwise, satisfying the condition.

Therefore, every $G \in \oplus[C_2(k)]$ has a degree 3 expansion in $G' \in \oplus[C_2(2k)]$. Taking the minor closure of all such G', we obtain a proper minor-closed class of Δ 3 containing $\oplus[C_2(k)]$.

We now add the final ingredient, apex vertices only neighboring other apex vertices and vortex vertices. We will prove that $\oplus[C_3(k)]$, i.e the clique sum closure of strongly k-almost embeddable graphs has a proper minor-closed superclass of $\Delta = 3$. By theorem 3, we thus obtain the the right direction of theorem 1.

Proposition 5. *$\oplus[C_3(k)]$ has a proper minor closed superclass of $\Delta = 3$.*

Let $G \in C_3(k)$. We will find an expansion of G in $C_3(k^2 + k)$, satisfying the conditions of theorem 11. Naturally, the base B is once again $C_3(k)$ and C' is $\oplus[C_3(k^2 + k)]$. It suffices to consider only G whose apex vertices neighbor all other apex vertices and all vortex vertices. All other graphs in $C_3(k)$ are subgraphs of such graphs and if $G_1 \subseteq G_2 \leq_m G'$ where $G_2 \leq_m G'$ has a representation under μ satisfying the conditions of theorem 11, so does $G_1 \leq_m G'$.
Let C be a facial cycle of $Emb(G)$. Let G' be G where instead of adding a vortex of depth k, we add a simple vortex of depth $k+1$ to C, and then connect

all apex vertices to it. As in the previous proposition, $G' \geq_m G$ under a model function μ_{sv}, and there is a total representation r under μ_{sv}: If K is a clique not intersecting the apex vertices, $r(K) = R_{sv}(K)$ as we have already explained in the previous proposition. If K intersects only apex vertices, then trivially $r(K) = \mu_{sv}(K) = K$. If K intersects apex and the simple vortex's vertices, let the subcliques comprised by those vertices be K_a and K_{sv} respectively, then $r(K_a) = K_a$, and $r(K_{sv}) = R_{sv}(K_{sv})$.
Therefore it suffices to prove theorem 11 for G' in the place of G. We now construct the expansion G''' of G' with the desired properties; let G'' be defined exactly as in the previous proposition (fatten $emb(G)$ as in the previous proposition, adding two copies to the empty face of each triangle), apex vertices neighboring all vortex vertices and all other apex vertices.We still have to lower the degree of apex vertices.

Definition 19. Define the cycle induced by the ith vertex of all simple vortex cliques of a simple vortex to be the *ith layer* of the simple vortex. We always have C be the 1st layer of the simple vortex.

We replace each simple vortex of depth $k+1$ of G'' with a simple vortex of depth $2k+1$. Apex vertices no longer neighbor all vortex vertices; instead, give some ordering to the apex vertices, the ith apex vertex neighbors a single vertex of the $k+1+i$th layer of the first clique of the simple vortex. Finally, for each apex vertex a, add to G'' a path of $a_1a_2...a_{k+1}$, identify a with a_1, remove the edge between a and its ith vortex neighbor and have the ith vortex neighbor be adjacent to a_{i+1} instead. Call this the *representor path* of a. This completes the construction of G'''. Notice that, treating the vertices of path representors as apex vertices, $G''' \in C_3(k(k+1))$ It now suffices to prove the three conditions of theorem 11.

- $G''' \geq_m G'$: For the ith apex vertex v of G', let $\mu_3(v)$ be the ith apex vertex of G''' together with its representor path, together with the $(k+1+i)$th layer of all simple vortices. Otherwise, let $\mu_3(v)$ be $\mu_2(v)$ as in the previous proposition.

- Let $R_3(K)$ be the representation. By the previous proposition, we have that every maximal clique K not having apex vertices has a representation $R_2(K)$. Let $R_3(K) = R_2(K)$ in this case. If K is the set of all apex vertices of G', then $R_3(K) = K$. If K=$K_a \cup K_{sv}$ is a set of apex vertices and simple vortex vertices of G', which by construction and maximality of K must consist precisely of all apex vertices and the simple vortex cliques of two consecutive facial cycle vertices, let them be c_i and c_{i+1}, then $R(K)$ is the two simple vortex cliques of c_i and c_{i+1} in G''.

- If $v \in G'''$ is an original apex vertex, then it belongs to a single max selected representor, that of all apex vertices. It has degree 1 excluding edges from that clique. If it does not, but still belongs to a path representor of an apex vertex, then it has degree 3 and belongs to no representor clique. If v is not an apex vertex, the same as in the previous proposition holds.

This completes the proof of the right direction of theorem 1.

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