Lower bounds on the sizes of integer programs without additional variables

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Presentation structure

1. Simple facts on integer linear programming and combinatorial optimization
2. Initial question and definitions
   1. How many inequalities needed to formulate a specific combinatorial problem as an ILP?
   2. No "extra" variables allowed.
3. Main technique and results
   1. Hiding sets
   2. Exponential lower bounds for many problems.
4. Later results on the topic

Solving combinatorial optimization problems by linear programming

Typical combinatorial optimization problem

- Finite ground set E.
- Feasible solution: Any set $F \subseteq E$ with some property.
- Vector $c \in \mathbb{R}^E$.
- Objective value of feasible solution F is $\sum_{e \in F} c_e$.

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Typical solution approach

- Make ordering for $e_1, e_2, \ldots$ of E
- Identify each F with its characteristic vector $\chi(F) \in \{0,1\}^E$
- $(\chi(F))_i = 1 \iff e_i \in F$
- Vector $c \in \mathbb{R}^E$.
- Objective value of feasible solution F is $c^T \chi(F)$.

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Need algebraic description of set $X := \{\chi(F) : F \subseteq E \text{ feasible}\}$.

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Traveling salesman problem:

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- Uses exponentially many (in n) linear inequalities.
- Nevertheless, computationally efficient (both in theory and practise). The separation problem associated with these inequalities can be solved efficiently.

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Do all formulations of the form $STSP_n = \{x \in \mathbb{Z}^E : Ax \leq b\}$ need exponential size?

Question answered for number of linear inequalities.

Motivators

- Pure mathematical curiosity
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We extend this question to many combinatorial problems.

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Integer programming

Given a language $L \subseteq \{0,1\}^* \in$ NP, there is a system $Ax + By \leq b$ such that

$$\{x \in \{0,1\}^k \ : \ x \in L\} = \{x \in \{0,1\}^k \ : \ \exists y \in \{0,1\}^m \ Ax + By \leq b\}$$

where the number of extra variables m and the number of inequalities is polynomially bounded by the size of the input x.

Definition

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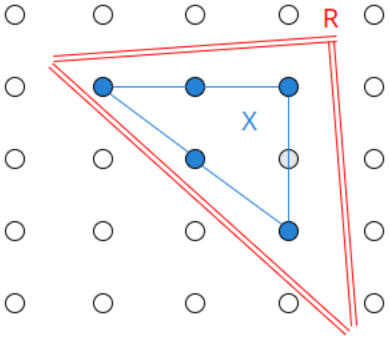

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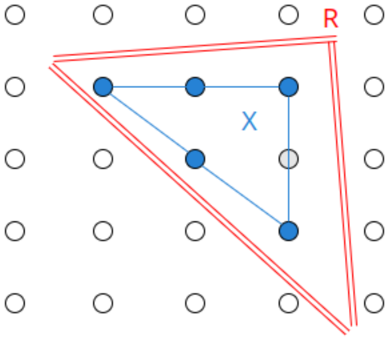

So they have the same set of integer solutions.
If $R = \{x \in \mathbb{R}^d | Ax \leq b\}$, then conv($X$)= $\{x \in \mathbb{Z}^d | Ax \leq b\}$, an ILP problem.

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With this notation, the initial question asks for the asymptotic behavior of $\text{rc}(\text{STSP}_n)$, the traveling salesman problem.

Previous work

- No reference that deals with a similar quantity except for a paper by Jeroslow.
- For a set $X \subseteq \{0,1\}^d$ of binary vectors, Jeroslow introduces the term index of X (short: ind(X)), defined as the smallest number of inequalities needed to separate X from the remaining points in $\{0,1\}^d$.
- Thus, relaxation complexity can be seen as a natural extension of the index with respect to general subsets of $\mathbb{Z}^d$.

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How to find RC(X)?

Lower bound framework

We introduce a simple framework to achieve that. Assume from now on X is polyhedral, that is, conv(X) is a polyhedron.

Definition

Let $X \subseteq \mathbb{Z}^d$. A set $H \subseteq \text{aff}(X) \cap \mathbb{Z}^d \setminus conv(X)$ is called a hiding set for X if for any two distinct points a, $b \in H$ we have that $conv\{a, b\} \cap conv(X) \neq \emptyset$.

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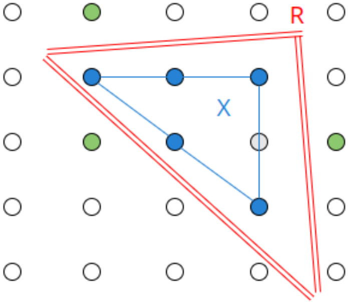

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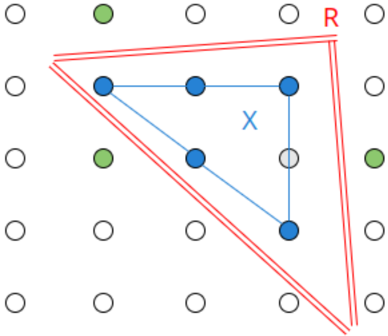

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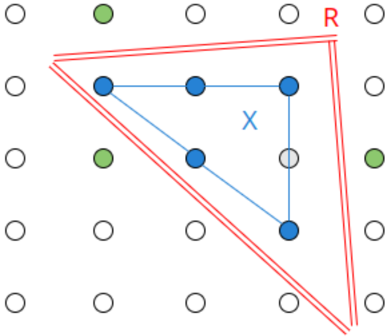

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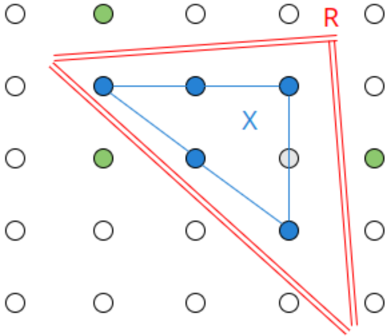

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Suppose $h_1, h_2 \in H$ violate $\langle a, x \rangle \leq \beta$. Then, all of conv($h_1$, h_2) does as well. But $conv\{a, b\} \cap conv(X) \neq \emptyset$, and $conv(X) \subseteq R$. □

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- In fact, we first prove this for the directed version, ATSP. We have an arc set A instead of edge set E.

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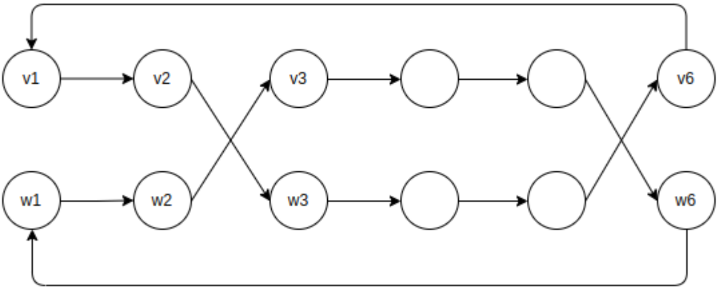
Let us keep only some edges to define the following subgraph (example for N=6).



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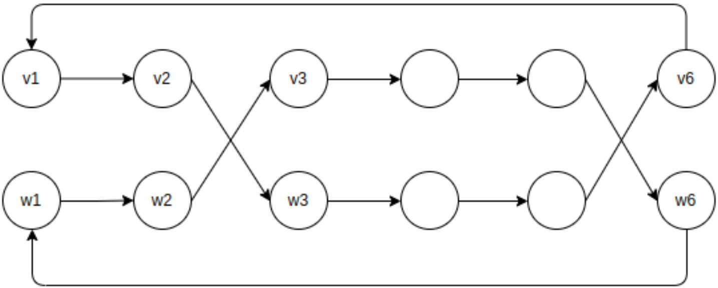
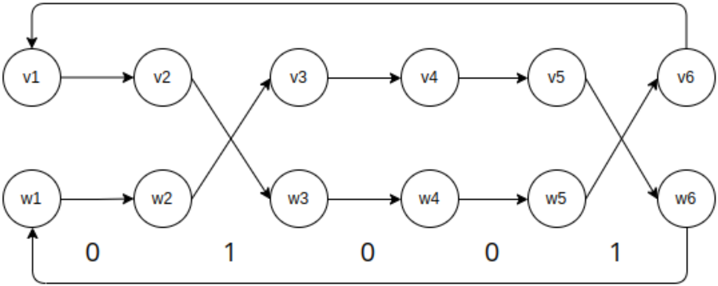


Not the only graph we care about, so to make notation easier, rather than $\{(v_6, v_1), (w_6, w_1)\} \cup \{(v_1, v_2), (w_1, w_2)\} \cup \{(v_2, w_3), (w_2, v_3)\} \cup \{(v_3, v_4), (w_3, w_4)\} \cup \{(v_4, v_5), (w_4, w_5)\} \cup \{(v_5, w_6), (w_5, v_6)\}$ represent this graph with "01001".



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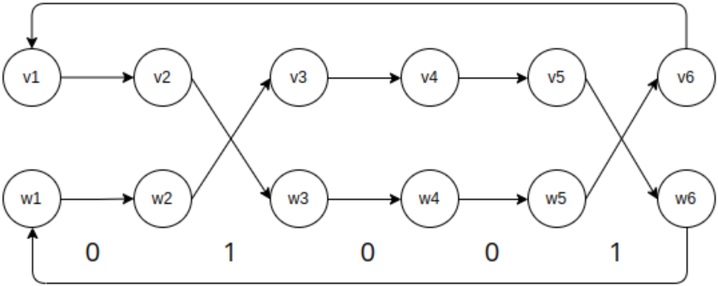

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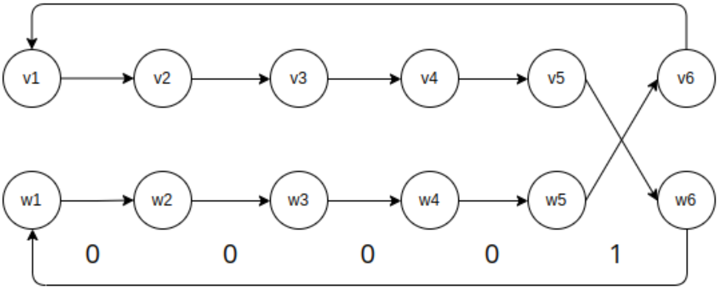

Definition

Let there be a binary vector $b \in \{0,1\}^N$. Let us define an arc set as a function of b: $E_b := \{(v_{N+1}, v_1), (w_{N+1}, w_1)\} \cup \bigcup_{i:b_i=0}\{(v_i, v_{i+1}), (w_i, w_{i+1})\} \cup \bigcup_{i:b_i=1}\{(v_i, w_{i+1}), (w_i, v_{i+1})\}$

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Note E_b is a directed Hamiltonian cycle on node set V if and only if $\sum_{i=1}^N b_i$ is odd. Define our hiding set as the other E_bs.
$H_N := \{\chi(E_b) : b \in \{0,1\}^N, \sum_{i=1}^N b_i \text{ is even}\}$

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conv$(\chi(E_b), \chi(E_b))\cap$conv$(ATSP_{2(N+1)}) \neq \emptyset$:

Let $b, b' \in \{0,1\}^N$ be distinct with b and b' having an even number of ones. Let j be an index with $b_j \neq b'_j$. Flip b_j, b'_j to get c, c'. c, c' have an odd number of ones, hence $\chi(E_c)$ and $\chi(E_{c'})$ are both contained in ATSP$_2(N+1)$. Clearly $\chi(E_b) + \chi(E_{b'}) = \chi(E_c) + \chi(E_{c'})$. □

STSP and ASTP

The asymptotic growth of rc(ATSP$_n$) and rc(STSP$_n$) is $2^{\Theta(n)}$.

Proof.

Clearly, $|H_N| = 2^{\Theta(n)}$. Note both ATSP and ASTP have formulations of size $2^{O(n)}$. rc(ATSP$_n$)$= 2^{\Theta(n)}$ follows instantly.
For STSP$_n$, replace all directed arcs with undirected edges, H_N is still a hiding set. $\square$

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  H_N from before for undirected graphs is still a hiding set:

Connected sets

Remember, the polytope $\{x \in [0,1]^{E_n} : x(\delta(S)) \geq 1, \ \forall \emptyset \neq S \subset V_n\}$ is a relaxation for CONN$_n$.

Proof.

$H_N \subseteq aff(CONN_n) \cap \mathbb{Z}^d \setminus conv(CONN_n)$:

- H_N only has values in $\mathbb{Z}^{E_n}$
- $\chi(\delta(S) \geq 1$ for all S for any element of conv(X), and not for all S for elements of H_N.
- It holds that aff(CONN$_n$)=$\mathbb{R}^{E_n}$.

Let $a, b \in H_N$.
$\emptyset \neq conv\{a,b\} \cap conv(STSP_n) \subseteq conv\{a,b\} \cap conv(CONN_n)$. □

Spanning trees and arborescences

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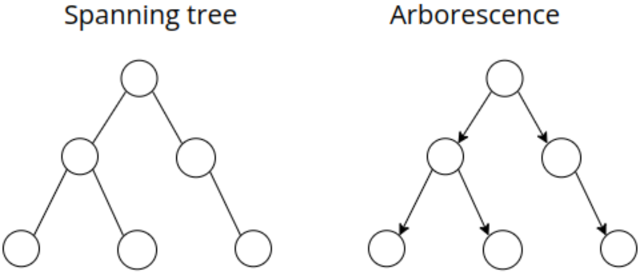

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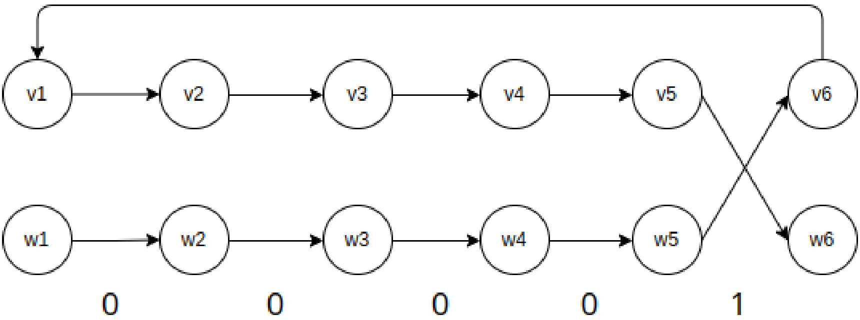
Towards a lower bound

Let us modify the definition of E_b by removing arc (w_{N+1}, w_1). Then, if $b \in \{0,1\}^N$ with b having an even number of ones, we have that E_b is a node-disjoint union of a cycle and a path and hence not an arborescence.

We will obtain that the modified set H_N is a hiding set for ARB_n.

Spanning trees and arborescences



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conv(ARB_N) $\cap$ conv($\chi(E_b), \chi(E_{b'})) \neq \emptyset$:

- Choosing b, b' with even number of ones, and flipping a bit they differ on, we still have $\chi(E_b) + \chi(E_{b'}) = \chi(E_c) + \chi(E_{c'})$, where E_c and $E_{c'}$ are hamilton paths, and thus arborescences, therefore conv(ARB_N) $\cap$ conv($\chi(E_b), \chi(E_{b'})) \neq \emptyset$.

$\square$

Spanning trees and arborescences

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□

By undirecting all arcs, H_N also yields a hiding set for SPT_n. We deduce a lower bound of $|H_N| = 2^{\Omega(n)}$for both rc(ARB$_n$) and rc(SPT$_n$).

Branchings and forests

Let BRANCH_n ($\mathrm{FORESTS}_n$) be the set of characteristic vectors of branchings (forests) in the complete directed (undirected) graph.

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So, if we could encode FORESTS_n in $2^{\mathcal{O}(n)}$ inequalities, we could do the same with SPT_n. Similarly,
$\text{ARB}_n = \text{BRANCH}_n \cap \{x \in \mathbb{R}^{A_n} : \sum_{e \in \mathbb{R}^{E_n}} x_e = n - 1\}$.

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$\mathsf{H}_{2,n} := \{(x,x)^T \in \{0,1\}^{2\times n} : x \in \{0,1\}^n\}$

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Proof.

$H_{2,n} \subseteq \mathrm{aff}(\mathrm{DIFF}_{2xn}) \cap \mathbb{Z}^d \setminus \mathrm{conv}(\mathrm{DIFF}_{m,n})$:

- $H_{2,n}$ only has values in $\mathbb{Z}^{2\times n}$
- conv($\mathrm{DIFF}_{m,n}$)) has no common element with $H_{2,n}$, clearly.
- It holds that aff(DIFF_{2xn})=$\mathbb{R}^{2xn}$.

Let $(x,x)^T, (y,y)^T \in H_{2,N}$.
$\frac{(x,x)^T+(y,y)^T}{2} = \frac{(x,y)^T+(y,x)^T}{2}$ $\in$CONV($\mathrm{DIFF}_{2,n}$). □

Permutahedron

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The asymptotic growth of rc(PERM$_n$) is $2^{\Theta(n)}$

Preliminary

Set $S := \{i \in [n] : i \text{ is odd}\}$. $|S| = \lfloor \frac{n}{2} \rfloor$. Select an integer vector $x \in \mathbb{Z}^n$ with $\{x_i : i \text{ is odd}(= S)\} = \{1, ..., \lfloor \frac{n}{2} \rfloor - 1\}$ and $\lfloor \frac{n}{2} \rfloor$ - 1 occurring twice, and $\{x_i : i \notin S\} = \{\lfloor \frac{n}{2} \rfloor + 2, ..., n\}$ and $\lfloor \frac{n}{2} \rfloor + 2$ occurring twice. Such a vector is not contained in conv(PERM$_n$).

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Preliminary

$$\sum_{i \in S} x_i = 1 + 2 + ... + (|S|-1) + (|S|-1) < |S|(|S|+1)/2$$

Note

- The only violated inequality is this one.
- $x \in \text{aff}(\text{Perm}_n)$ as it satisfies all implied equalities.
- We used only the cardinality of S so this holds for any S so that $|S| = \lfloor \frac{n}{2} \rfloor$

$H := \{x^S : S \subseteq [n], |S| = \lfloor \frac{n}{2} \rfloor - 1\}$ is a hiding set for PERM_n, where x^S is defined as follows: Select an integer vector $x \in \mathbb{Z}^n$ with $\{x_i : i \in S\} = \{1, ..., \lfloor \frac{n}{2} \rfloor - 1\}$ and $\lfloor \frac{n}{2} \rfloor$ - 1 occurring twice, and $\{x_i : i \notin S\} = \{\lfloor \frac{n}{2} \rfloor + 2, ..., n\}$ and $\lfloor \frac{n}{2} \rfloor + 2$ occurring twice.

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Proof.

$H \subseteq$ aff(PERM$_n$) $\cap \mathbb{Z}^n \setminus$ conv(PERM$_n$):

- H only has values in $\mathbb{Z}^n$.
- conv(PERM$_n$)) has no common element with H, as demonstrated.
- $H \subseteq$ aff(PERM$_n$), as demonstrated.

Let $S_1, S_2 \in H$. We will show that $x := \frac{1}{2} \cdot (x^{S_1} + x^{S_2}) \in$conv(PERM$_n$) holds. Since x satisfies all constraints that are satisfied by both x^{S_1} and x^{S_2} , it suffices to show that $\sum_{i \in S_1} x_i \geq |S_1|(|S_1| + 1)/2$, $\sum_{i \in S_1} x_i \geq |S_2|(|S_2| + 1)/2$ holds.

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$$\begin{aligned}
\sum_{i \in S_1} x_1 &= \frac{1}{2} \sum_{i \in S_1} x_i^{S_1} + \frac{1}{2} \sum_{i \in S_1} x_i^{S_2} \\
&= \frac{1}{2}(\frac{m(m+1)}{2} - 1) + \frac{1}{2} \frac{m(m+1)}{2} + 2) \\
&\geq \frac{m(m+1)}{2}
\end{aligned}$$

□

Rational relaxations

- What happens if we restrict ourselves to rational values in our description?
- We don't lose too much.

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Theorem

Let $X \subseteq \mathbb{Z}^d$ be finite and $rc_Q(X)$ be the smallest number of facets of any rational relaxation for X. Then, $rc_Q(X) \leq rc(X) + dim(X) + 1$.

If $X \subseteq \{0,1\}^d$ only, the coefficients of this rational relaxation R can be polynomially bounded in size by d.

Open questions

Questions

- Is it true that $rc(X) \geq dim(X) + 1$ holds for all polyhedral (or at least finite) sets $X \subseteq \mathbb{R}^d$? (holds for $X \subset \{0,1\}^d$).
- Is there any polyhedral (or even finite) set $X \subset \mathbb{Z}^d$ such that $rc(X) < rc_Q(X)$?

Later results

Weltge PHD thesis:

- In dimension two, rc(X) and rc_Q(X) coincide and are computable.
- $dim(X) \geq k! \implies rc(X) \geq k$

Later results

On the Size of Integer Programs with Bounded Coefficients or Sparse Constraints (2018)

- A simple algorithm to compute the maximum hiding set for $X \subset S$ where $|S| < |\mathbb{N}|$ is given.
- Given a nonempty set $X \subset \{0,1\}^n$, build a graph $G = (V, E)$ with $V = \{0,1\}^n \; X$. The edge set E of G is defined as $E = \{\{x,y\} : x, y \in V, conv(x,y) \cap conv(X) \neq \emptyset\}$.
- Any clique in G is a hiding set by definition. By computing the size of a maximum clique in G, we get the maximum size of any hiding set $H \subseteq 0, 1^n$ for X.

Later results

Complexity of linear relaxations in integer programming (2020)

Main results: (X assumed a finite lattice-convex set)

1. rc(X) = rc$_Q(X)$ when:
   1. X is at most four-dimensional,
   2. X represents every residue class in $(\mathbb{Z}/2\mathbb{Z})^d$
   3. the convex hull of X contains an interior integer point,
   4. the lattice-width of X is above a certain threshold.
2. rc(X) can be algorithmically computed when
   1. X is at most three-dimensional, or
   2. X satisfies one of the last three conditions above.
3. An improved lower bound on rc(X) in terms of the dimension of X is obtained. $rc(X) > log(dim(X)) - loglog(dim(X))$.

Later results

Strong IP formulations need large coefficients (2021)

Generalization of hiding sets.

- A set $H \subseteq (aff(X) \cap Z^n)\ X$ is called a hiding set for X if conv{x1, x2} ∩ conv(X) $\neq \emptyset$ for each pair of distinct points $x1, x2 \in H$.
- A set $H \subseteq (aff(X) \cap F)\ X$ is called an F-hiding set for X if conv{x1, x2} ∩ conv(X) $\neq \emptyset$ for each pair of distinct points $x1, x2 \in H$, where $F \subseteq \mathbb{R}^n$.
- Let $X \subseteq \mathbb{Z}^n$ be polyhedral, let $F \subseteq \mathbb{R}^n$, and let H be an F-hiding set for X. Then, every family of inequalities separating $F\cap$conv(X) and $F\setminus$ conv(X) has size at least $|H|$.

Later results

Computational Aspects of Relaxation Complexity: Possibilities and Limitations (2021)

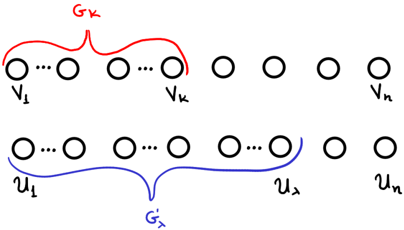
- Tight and computable upper bounds on $\mathrm{rc}_{\mathbb{Q}}(X)$.
- $\mathrm{rc}(X)$ can be computed in polynomial time if X is 2-dimensional.
- Explicit formula for $\mathrm{rc}(X)$ for specific classes of sets X.
- First practically applicable approach to compute $\mathrm{rc}(X)$ for specific classes of sets X.

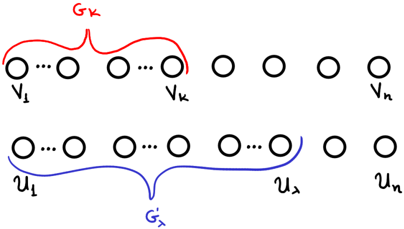
Later results

Efficient MIP Techniques for Computing the Relaxation Complexity (2022)

- Techniques to efficiently compute a numerically more robust variant of the relaxation complexity, $rc_\varepsilon(X)$, where conv$(X) \cap \mathbb{Z}^d$ $=\{x \in \mathbb{Z}^d | Ax \leq b + \varepsilon\}$ instead of $\{x \in \mathbb{Z}^d | Ax \leq b\}$.

Ευχαριστούμε!





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