Master's thesis presentation

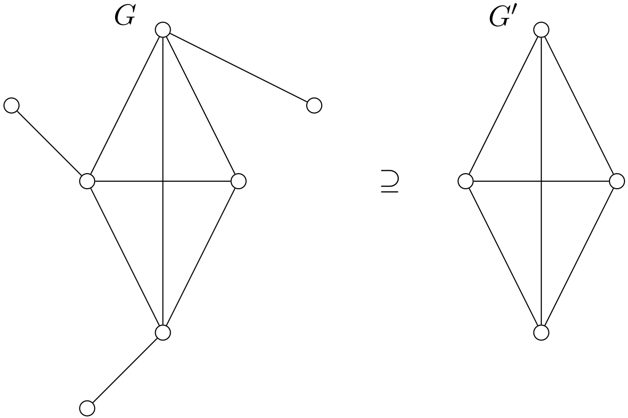
Orestis Milolidakis

Minors

Let's explain what a minor is just in case.

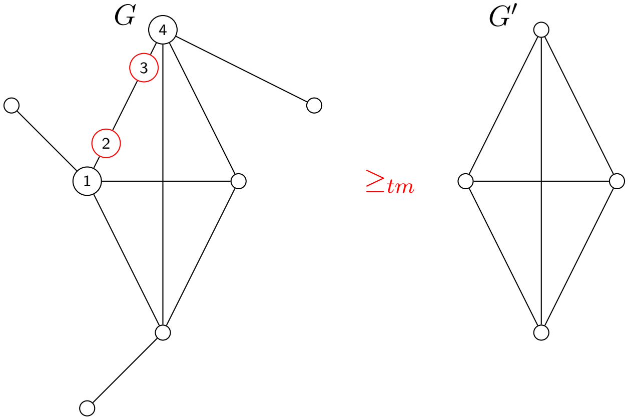
Minors

So we all know what a subgraph is. It captures the notion of containment.



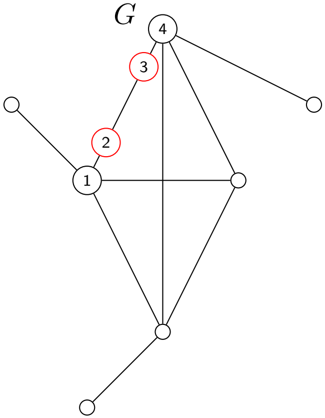
Minors

Now let's add a few vertices between two neighboring vertices.

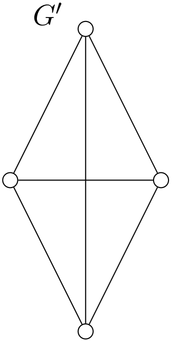


Minors

One might say, look, path 1234 functions like a big edge. We should still say G contains G'. This is called a topological minor.

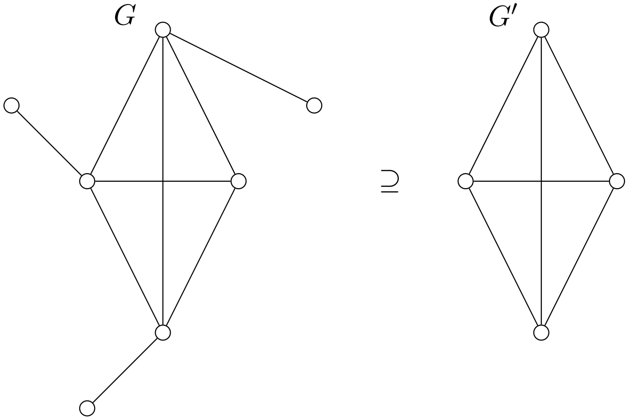


$\geq_{tm}$



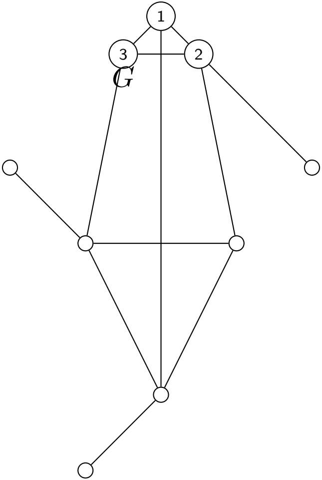
Minors

Someone could take this logic a step further, and argue, let's take G,

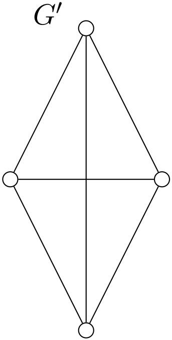


Minors

And let's replace a vertex with a connected component, that neighbors all vertices v neighbored.

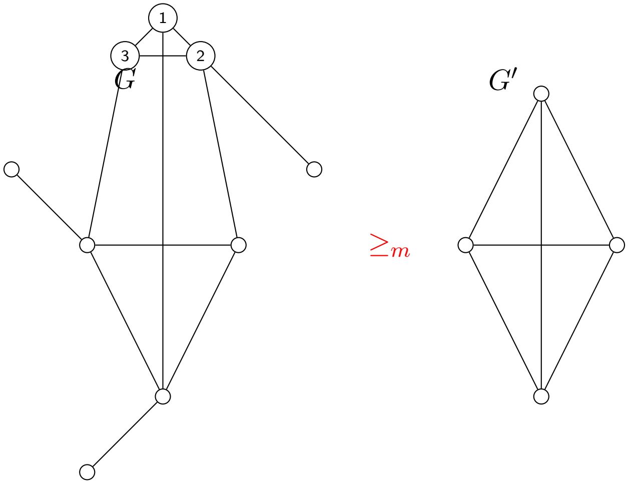


$\geq_m$



Minors

Then it still contains G', because 123 can be treated like a big node. We say G contains G' as a minor if we can find a number of connected components which if turned into nodes, we get G'.



Every planar graph is a minor of a planar graph of maximum degree ≤ 3.
Can you see why?

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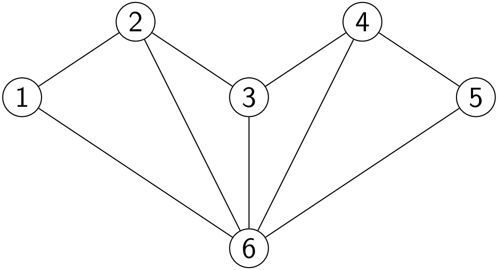

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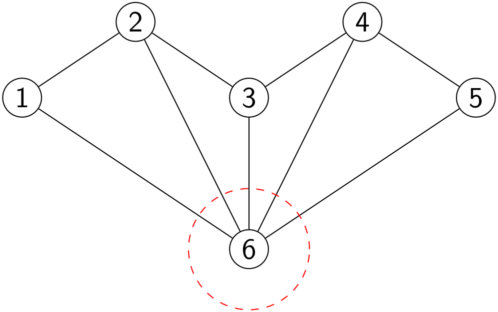

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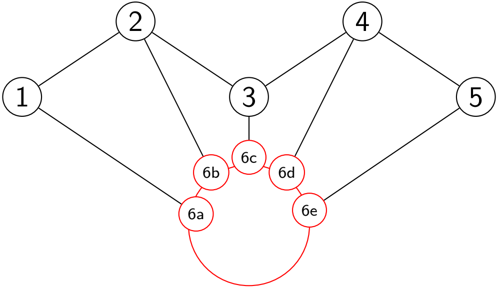

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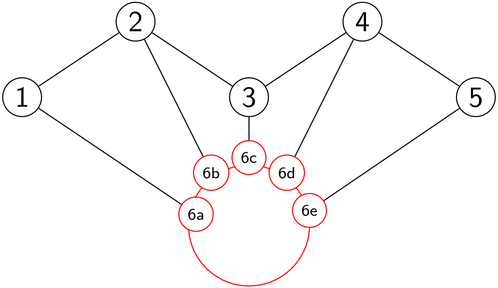

Obviously contracting the red cycle gives back the original graph.

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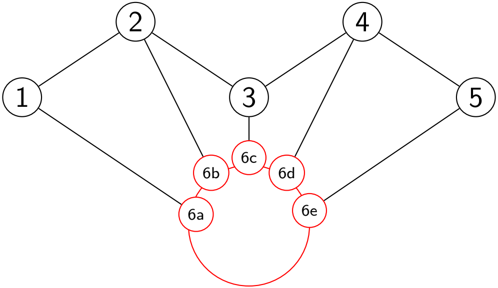

Obviously contracting the red cycle gives back the original graph. Do this for all vertices of degree ≥ 3. In this way we get a graph of maximum degree 3.

Central notion of thesis

Motivating question

- $\forall G \quad$ planar $\quad \exists G' \quad$ planar $\quad$ with $G \leq_m G'$ and $\Delta(G') \leq 3$.

- $\Delta()$ is not well defined yet, in the sense that there could be no such k.
- We care only for $\mathcal{C}$ minor closed (we don't really lose anything from this requirement).
- A minor of a graph with maximum degree 2 also has maximum degree 2, so the classes for which $\Delta(C) \leq 2$ are trivial, and we don't take such cases into account in this presentation.

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- $\forall G$ planar $\exists G'$ planar with $G \leq_m G'$ and $\Delta(G') \leq 3$.

What if we changed "planar" with some other graph class?

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In 2021 Georgakopoulos showed that every graph in Forb(K_5) is a minor of another graph in Forb(K_5) with maximum degree at most 22.

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- $\forall G \in$ Forb(K_5) $\exists G' \in$ Forb(K_5) with $G \leq_m G'$ and $\Delta(G') \leq 22$

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He asked if this is smallest possible.

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•$\forall G \in$	$\mathcal{C}$	$\exists G' \in$	$\mathcal{C}$	with $G \leq_m G'$ and $\Delta(G') \leq k$

This motivates the following definition:
Let C be some graph class.

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This motivates the following definition:

Let C be some graph class. Let $\Delta(C)$ be the minimum such k.

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Let $\Delta(C)$ be the minimum such k.

This is an elegant parameter, yet it is very general. We are interested in it and want to see how it behaves. This is the central notion of this thesis.

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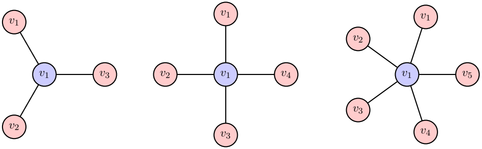

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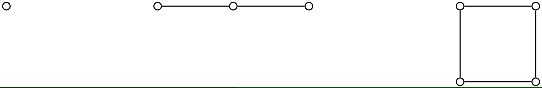

Frame Title

So we have

- $\Delta(PLANARS) = 3$
- $\Delta(Forb(K_5)) = 3$
- $\Delta(Forb(K_{3,3})) = 4$
- $\Delta(TW_{\leq k}) = k$ for $k \in \mathbb{N}$.

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Can we get values greater than 3?

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In fact we can get any possible value.

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Let's give a feeling of how these proofs might go before showing how this relates to other research.

$\Delta(Forb(K_5)) = 3$

How do we go about this?

Proof that $\Delta(Forb(K5)) = 3$.

□

$\Delta(Forb(K_5)) = 3$

Wagner (1937)

$G \in Forb(K_5) \iff G$ can be constructed by the clique sums of planar graphs and $W[8]$.

Proof that $\Delta(Forb(K5)) = 3$.

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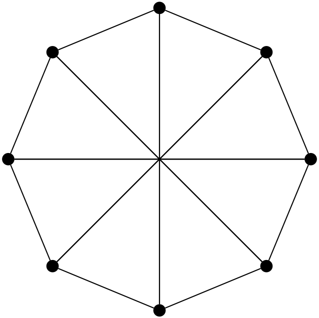

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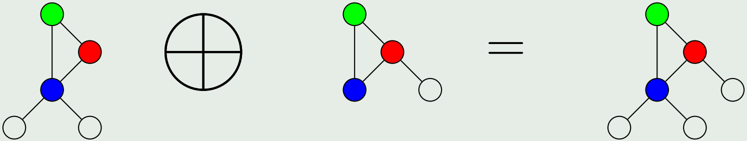
Clique sums

To clique sum two graphs means to pick a clique from each graph (same sized) and to identify their vertices in some 1-1 manner of our choosing.



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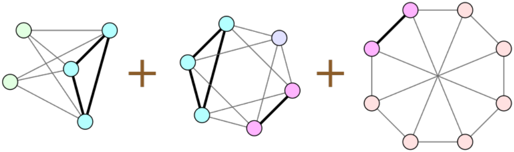

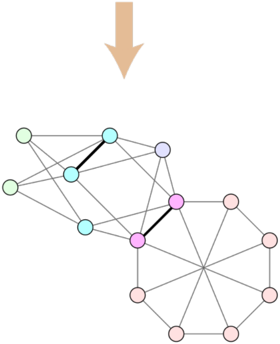

May also remove a edges from those cliques.

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The trick is to use Wagner's theorem combined with the splitting of planar graphs to planar graphs of maximum degree 3.

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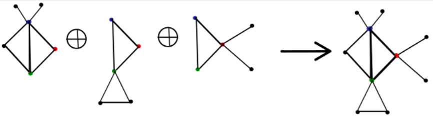

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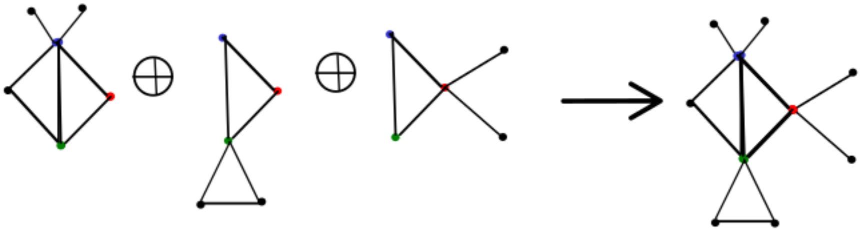

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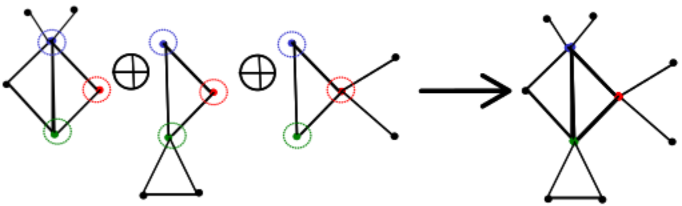

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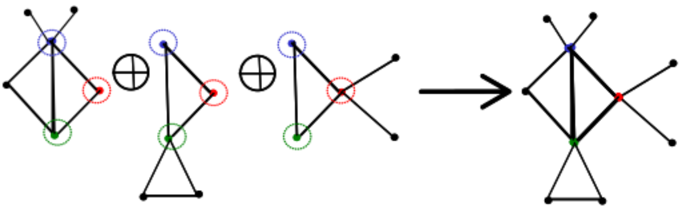

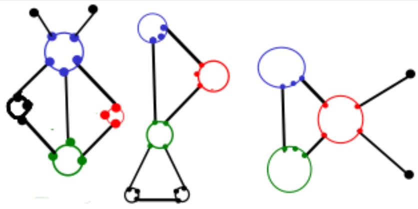

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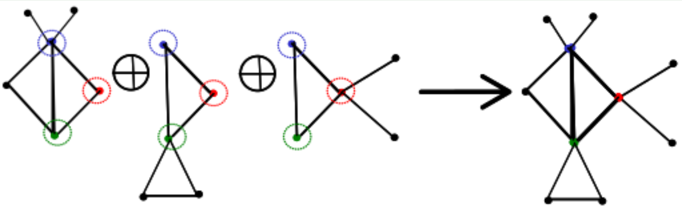

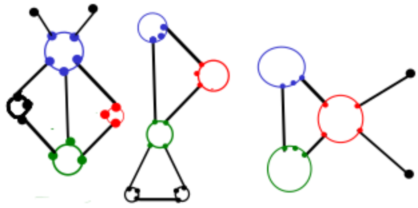

Would like to just identify same-colored the circles as if in clique sums.

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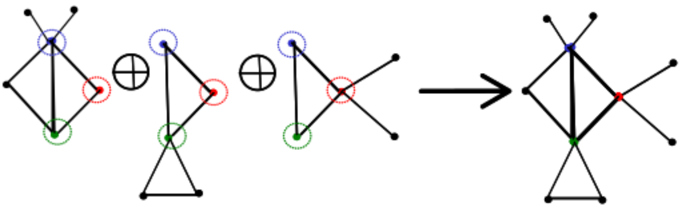

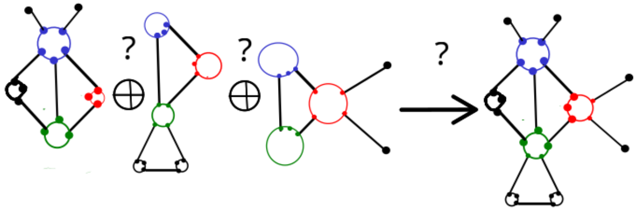

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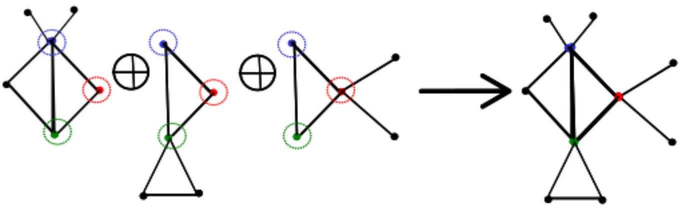

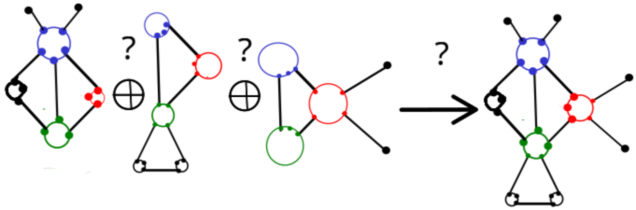

Then, we could just contract same-colored vertices to get G.

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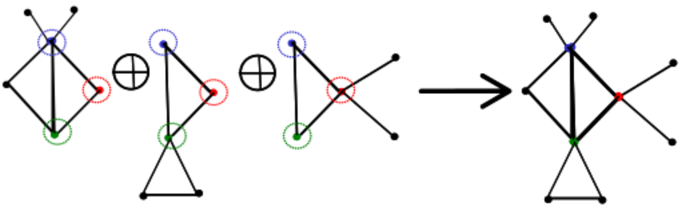

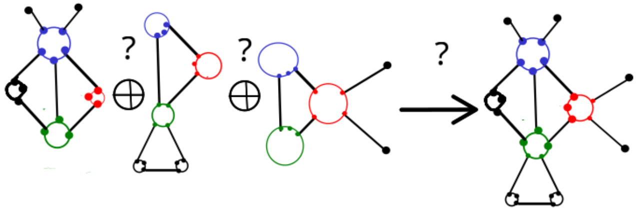

Since those aren't clique sums, not guaranteed to be in $Forb(K_5)$!

Observation: Wagner's theorem can be limited to planar graphs where every triangle has an empty exterior or an empty interior.

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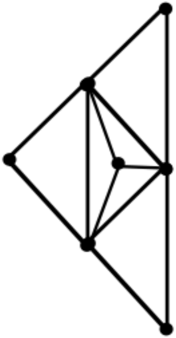

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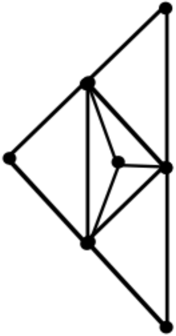

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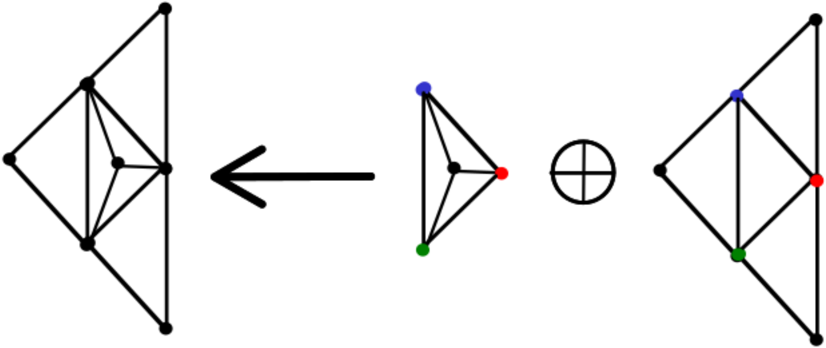

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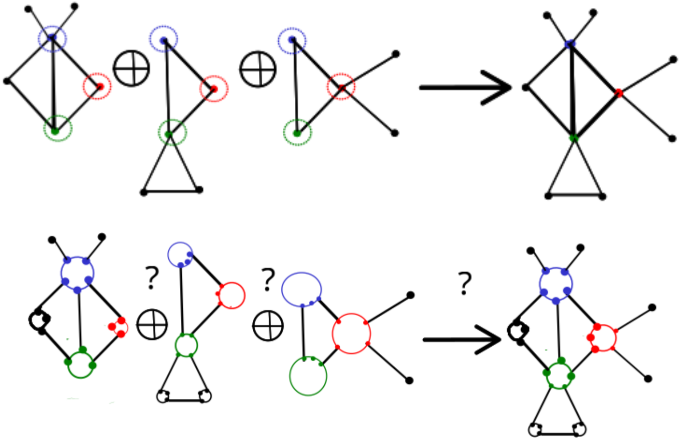

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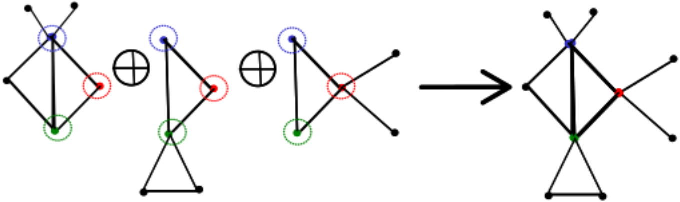
Here is what we do instead. For ease, assume interiors of triangle empty.



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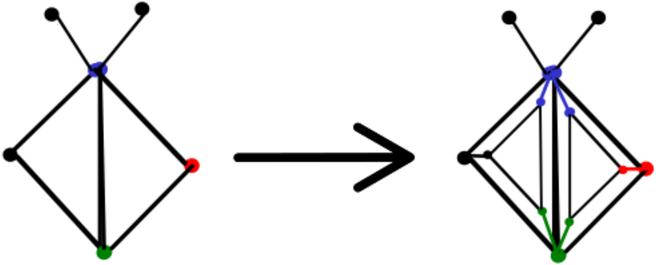

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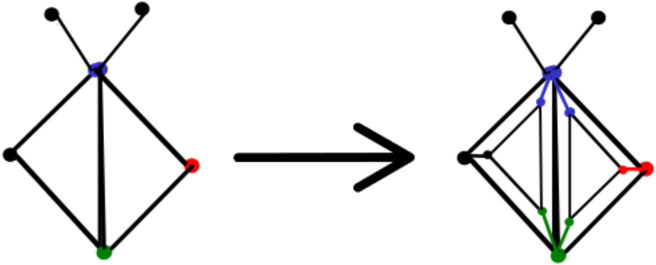

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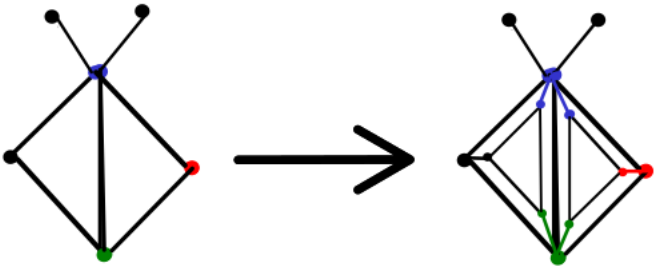

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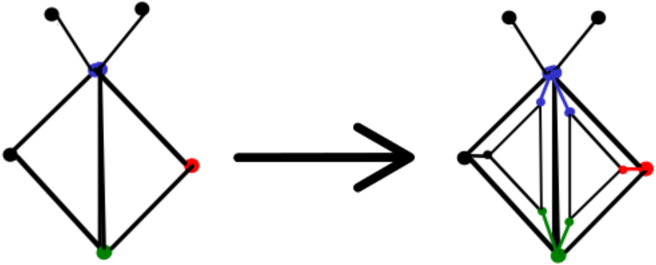

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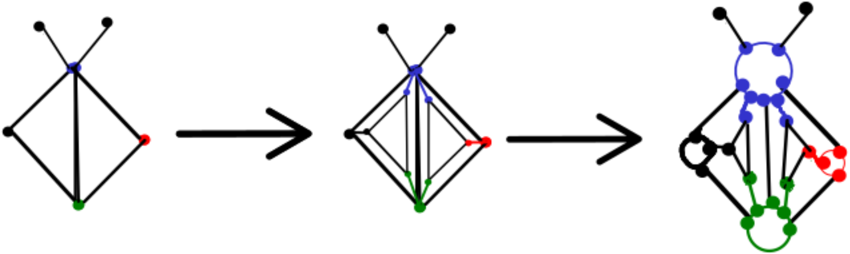

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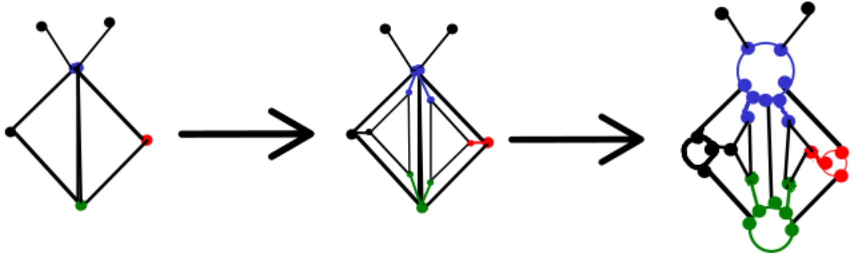

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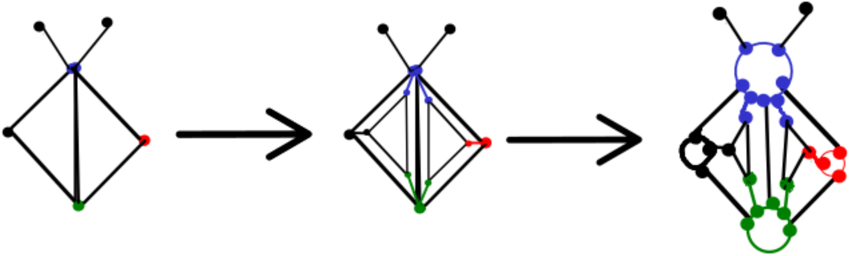
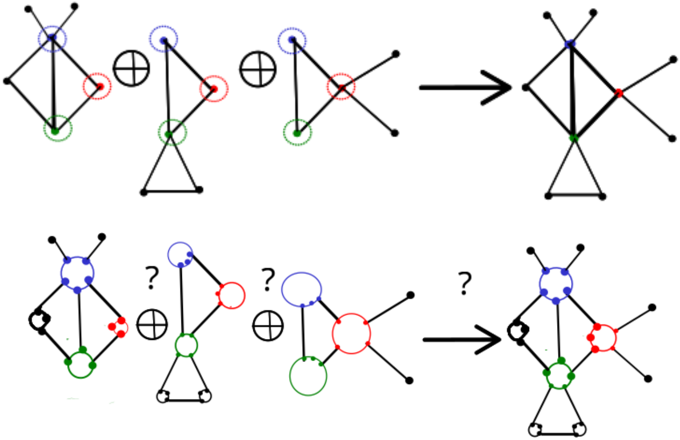


By contracting the same colored cycles of the third graph we get the second graph, and by contracting the same colored vertices of the second graph we get the first graph. Also, if a triangle was clique summed on the blue/green/triangle of the third, after these contractions it will remained clique summed on the red/blue/green triangle.



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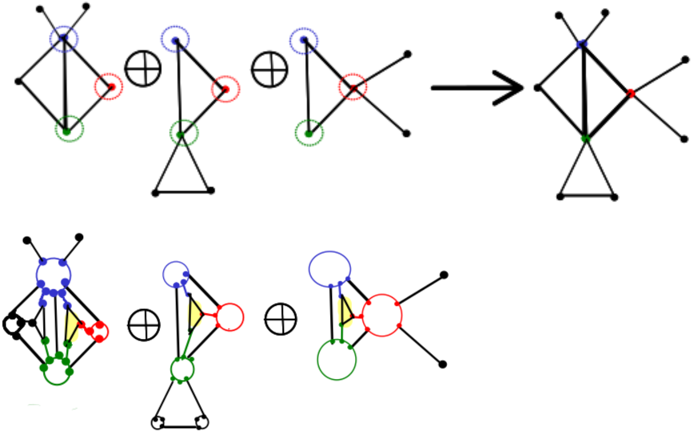

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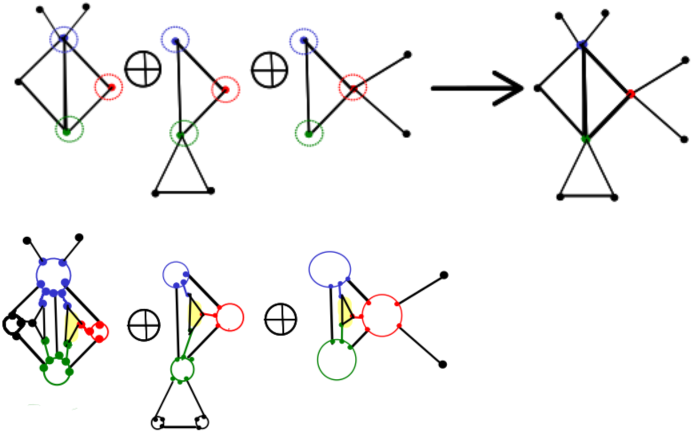

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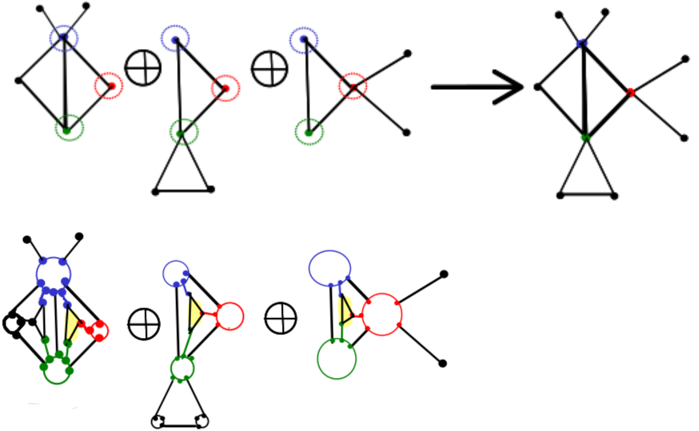
Now we have a triangle to clique sum to! After clique sum on the highlighted triangle, contract the same colored vertices together, then the cycles to get the original graph.



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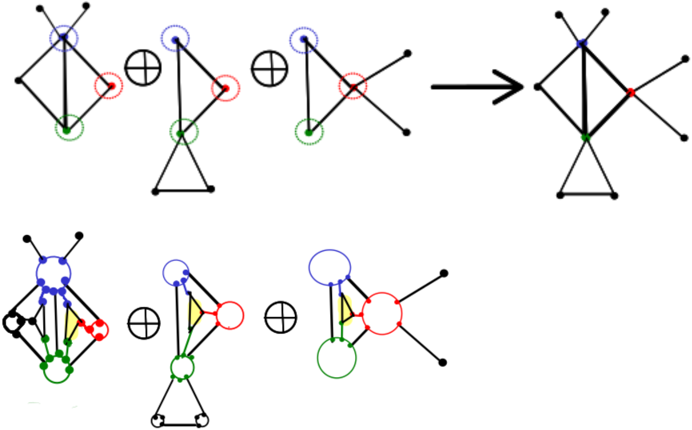
Good news: Non-extra triangle vertices now have max degree 3!
Bad news: Extra triangles can still have unbounded degree.



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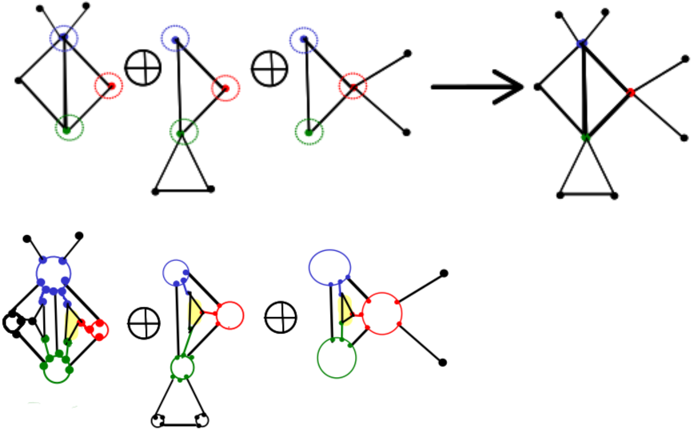
Solution: Instead of a single extra triangle, we have a chain of extra triangles.



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A final adjustment

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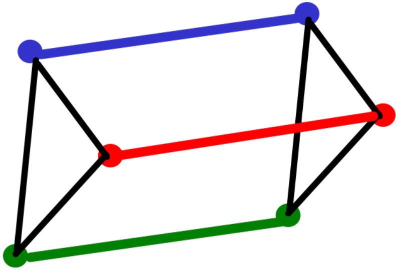

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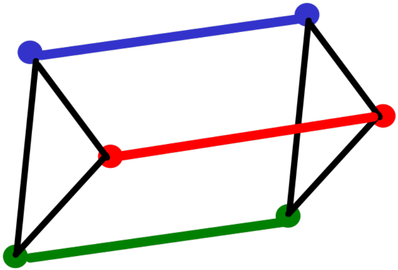
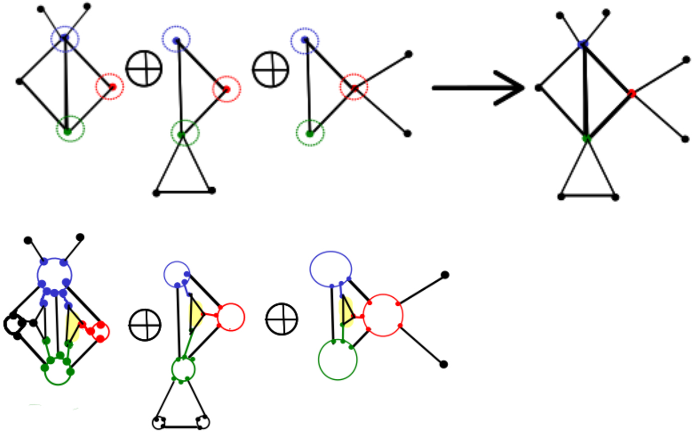


But this means we can start clique summing it with Wagner's theorem.



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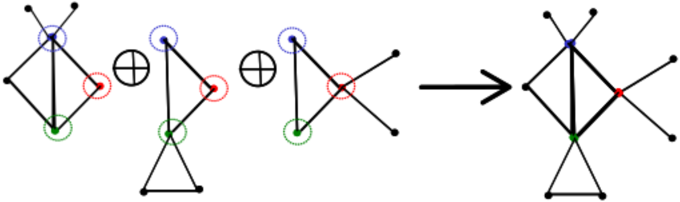
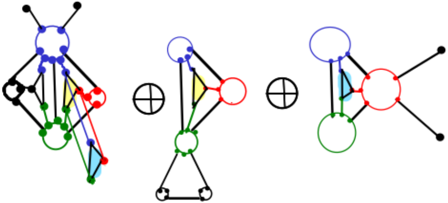
Clique sum the first graph to the triangle copy, and the second graph to the second triangle copy.





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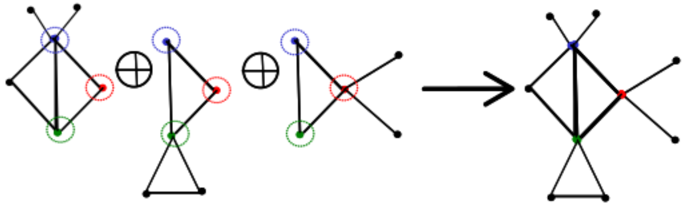
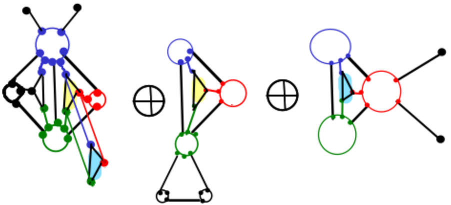
Clique sum the first graph to the triangle copy, and the second graph to the second triangle copy. Each extra triangle participates in one clique sum now.





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Small sidenote

It's nice to see how our results tie to the literature. In 2009 Markov and Shi proved that $\Delta(TW_{\leq k}) > 3$ for $k \geq 19$. As mentioned earlier, we have extended this by showing $\Delta(TW_{\leq k}) = k$ for all $k \geq 3$.

The structure of Δ

Is $\Delta(C)$ increasing?

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$STARS \subseteq PLANAR \subseteq APEX$ A graph is apex if it is planar or if it becomes planar by the removal of a vertex.

The structure of Δ

Is $\Delta(C)$ increasing?

i.e if $C \supseteq C'$ then $\Delta(C) \leq \Delta(C')$? This is not the case.

$$\begin{array}{ccccc} STARS & \subseteq & PLANAR & \subseteq & APEX \\ \downarrow & & \downarrow & & \downarrow \\ \infty & & 3 & & \infty \end{array}$$

The structure of Δ

What about a relaxation at least?

The structure of Δ

At the very least, we would like the following:

The structure of Δ

Conjecture

For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

The structure of Δ

Conjecture

For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

Surely this looks like it should hold right? There must be *some* class including C and not having $\Delta()$ infinite.

The structure of Δ

Conjecture

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Theorem

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For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

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For a proper minor-closed $C \supseteq APEX$, $\Delta(C) = \infty$.

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For a proper minor-closed $C \supseteq APEX$, $\Delta(C) = \infty$.

Which is a nice result right? And it is kind of unexpected.

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For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

Theorem

For a proper minor-closed $C \supseteq APEX$, $\Delta(C) = \infty$.

This has a few corollaries.

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For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

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For a proper minor-closed $C \supseteq APEX$, $\Delta(C) = \infty$.

Corollaries

fForb(K_6) includes all apex graphs. o it has $\Delta()$ infinite. A class usually hard to work with.

- $\Delta(Forb(K_n)) = \infty, \ n \geq 6$
- $\Delta(Forb(K_{n,n})) = \infty, \ n \geq 4$
- $\Delta(LINKLESSLYEMBEDDABLE)) = \infty$

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For every minor-closed C there exists a proper minor closed class C' that includes C with $\Delta(C')$ finite.

Theorem

For a proper minor-closed $C \supseteq APEX$, $\Delta(C) = \infty$.

Corollaries

fForb(K_6) includes all apex graphs. So it has $\Delta()$ infinite. A class usually hard to work with.

- $\Delta(Forb(K_n)) = \infty,\ n \geq 6$
- $\Delta(Forb(K_{n,n})) = \infty,\ n \geq 4$
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The linklessly embeddable graphs are a well-known 3D analogue to planar graphs.

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You would think that we would find some trick with them like we did with planar graphs.

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But they also include all apex graphs.

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So it's nice to find such general results.

And let's tie this a bit to known results and see how it connects and let's get into our major contribution. . .

Graph minor theorem

If C is a proper minor-closed class, it can be characterized by a finite list of excluded minors.

. .

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If C is a proper minor-closed class, it can be characterized by a finite list of excluded minors.

Proved by Seymour and Robertson after 30 years of work, 500 pages of pappers, 20 or so papers, completely seminal, may God allow us to pray to Robertson and Seymour rather than him, blah blah . . .

Another significant result they proved during this is the graph structure theorem . . .

Wagner (1937)

$G \in Forb(K_5) \iff G$ can be constructed by the clique sums of planar graphs and $W[8]$.

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Robertson and Seymour said "look it would be really nice if we could prove this for some class excluding an arbitrary minor. . . .

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Graph structure theorem

$G \in Forb(H) \implies G$ can be constructed by the clique sums of k-almost embeddable graphs, where H is some arbitrary graph.

We will get into $k-$almost embeddable graphs if we have the time . . .

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The Graph structure theorem has many variations. For more non-arbitary more specific H we can get more specific results. E.g if H is planar, G can be constructed by the clique sum of graphs of at most k vertices for some constant k. .

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. In 2016 Dvorak and Thomas proved such a result.

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Graph structure theorem

$G \in Forb(H) \iff G$ can be constructed by the clique sums of strongly k-almost embeddable graphs, where $H \in$ APEX.

Graph structure theorem specialized

$G \in Forb(\mathsf{H}) \iff G$ can be constructed by the clique sums of strongly k-almost embeddable graphs, where $H \in$ APEX.

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Using this result, Wood, Morin and Djukovic proved the following result

Graph structure theorem specialized

$G \in Forb(\mathsf{H}) \iff G$ can be constructed by the clique sums of strongly k-almost embeddable graphs, where $H \in$ APEX.

Theorem

The following are equivalent for a proper minor-closed class C.

- C excludes an apex graph as a minor
- $\forall G \in C$, G can be constructed by the clique sums of strongly k-almost embeddable graphs.
- [...]
- [...]
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So that's a new result.

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It's not obvious why this should be the case right? The cutoff is too nice.

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We are talking about $\Delta(C)$, but in the definition of $\Delta()$ we could use another parameter, like degeneracy, called it $dg(C)$, and have explored $dg(C)$ instead. And it turns out every class has a superclass of $dg(C) = 3$.

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- $\exists$ proper minor-closed $C' \supseteq C$ of $\Delta(C') = 3$

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And one direction of this addition we get from our previous result

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$\uparrow$ direction:

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$\uparrow$ direction: Let there be a proper minor-closed $C' \supseteq C$ with $\Delta(C') = 3$. Then, C must exclude an apex graph, because let's say it includes all apex graphs, then C' will also include all apex graphs, and by our fact $\Delta(C') = \infty$, a contradiction.

Theorem

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How might we prove the fact?

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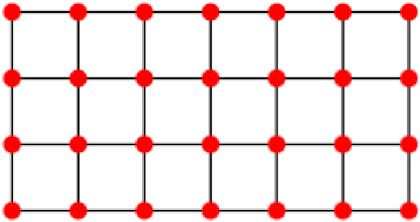

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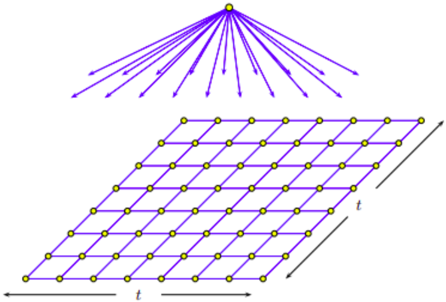

A pyramid is just a grid with an additional apex vertex.

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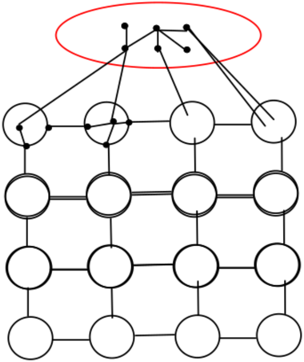
We include an inflation of each pyramid as a minor.



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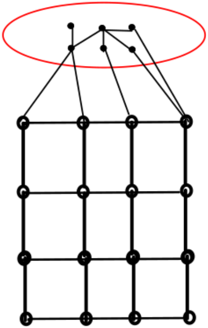
This is in C. The red guy must neighbor all grid vertices.



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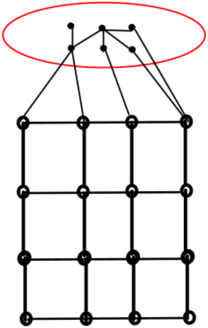
We have in C arbitrarily large pyramids.



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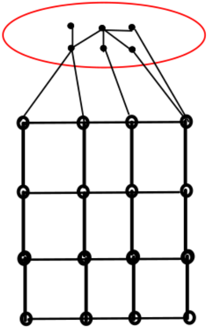
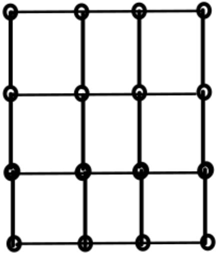
The idea is to show that we have arbitrarily large cliques, and being minor closed this means we include all graphs.

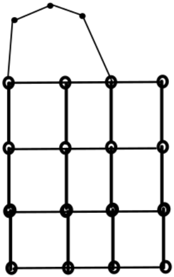


There is a well known structural theorem.

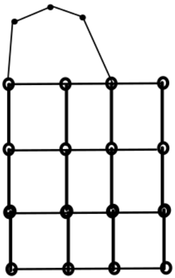
It says, take a grid H with many H-paths



An H-path is a path with endpoints in H but not intersecting H otherwise.



Fact: If we have a grid H with n choose 2 vertex disjoint H-paths, we will include K_n as a minor if the endpoints are far apart enough from each other.



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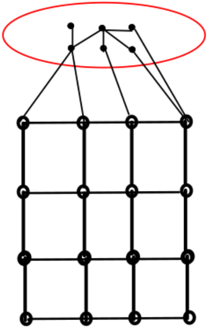

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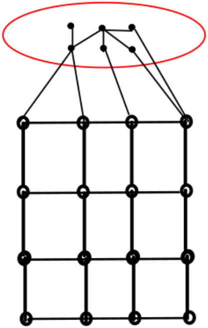
Showing we have n choose 2 vertex-disjoint H-paths far apart from each other suffices.



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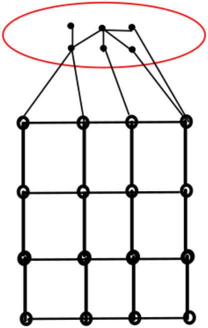
We can find H-paths using the inflated apex vertex.



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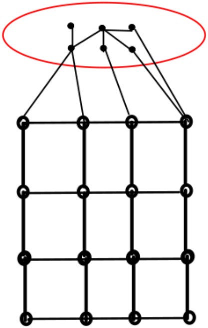
Problem: Other H-paths may not be vertex disjoint to this one.



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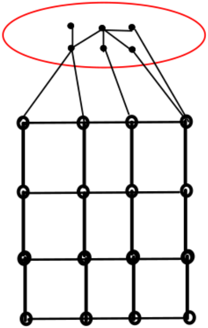
Solution: We can assume the red part to be a tree



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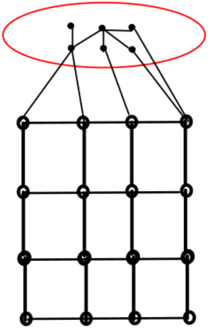
We can assume all its leaves to have an edge towards the grid.



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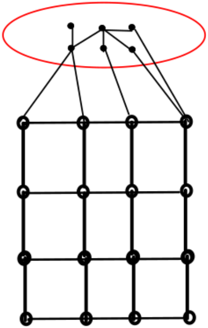

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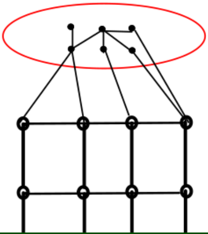
Recall the vertices of the tree have maximum degree k. We include all pyramids as minors. So let's take some super large pyramid. The inflated apex vertex must neighbor all grid vertices, and having bounded max degree, as the grid gets larger and larger the inflated apex vertex must become larger and larger.



One last thing

One last thing

Here is a last thing I would like to say.

- $\Delta(\leq_m[A])$=3
- It is a minor-closed class by construction
- It is a subclass of $(Forb(K_{3,3})$.

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Recall $\Delta(Forb(K_{3,3})) = 4$.

- $\Delta(\leq_m[A])$=3
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- It is a subclass of $(Forb(K_{3,3})$.

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So every graph G in $(Forb(K_{3,3})$ is a minor of another graph G' in $(Forb(K_{3,3})$ of maximum degree 4.

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One last thing

Let's take $(Forb(K_{3,3})$ and let's restrict it into a minor-closed class of $\Delta() = 3$. How?

- $\Delta(\leq_m[A])$=3
- It is a minor-closed class by construction
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One last thing

One last thing

Take all graphs of $(Forb(K_{3,3})$ of $\Delta(G) \leq 3$. Let this be some class, call it A, and take $\leq_m [A]$.

What can we say about $\leq_m [A]$?

- $\Delta(\leq_m[A])$=3
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- It is a subclass of $(Forb(K_{3,3})$.

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- $\Delta(\leq_m[A])$=3
- It is a minor-closed class by construction
- It is a subclass of $(Forb(K_{3,3})$.

Every graph in $\leq_m[A]$ is a minor of a graph of maximum degree 3 in $\leq_m [A]$. So it has

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- $\Delta(\leq_m[A])$=3
- It is a minor-closed class by construction
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It was nice to notice that $\leq_m [A]$ was exactly the planar graphs.

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With treewidth now

We created a minor-closed class in a kind of natural manner, and we got a natural answer as a result.

- $TW_{\leq k-1} \subset \leq_m [A] \subset TW_{\leq k}$
- $TW_{\leq k-1} \subset \ STW_{\leq k} \subset TW_{\leq k}$

With treewidth now

So let's do this again, but with treewidth.

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With treewidth now

Recall $TW_{\leq k}$ the class of treewidth $\leq k$ has $\Delta(TW_{\leq k}) = k$.

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Let's take all graphs of $TW_{\leq k}$ of $\Delta(G) \leq 3$. Let this be some class, call it A, and take $\leq_m [A]$.

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Markov and Shi showed that every graph of treewidth $k-1$ is a minor of a graph of treewidth k and maximum degree 3.

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We have a minor closed class between those guys, we have defined it in a natural way, that seems to give us natural graph classes. And we know that there are many interesting variations of treewidth.

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Is $\leq_m [A]$ also a natural variation of treewidth?

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Thank you!