Parametrized hardness results for game theory

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Presentation structure

1. Games, definitions, theorems
2. Fixed parameter intractability results
   1. k-UNIFORM NASH
   2. MAXIMUM PAYOFF FOR THE COLUMN PLAYER
   3. k-MINIMAL NASH SUPPORT

Games

- N players $i = 1..., N$
- For every player i, a set of available strategies S^i
- For every player i, a utility function $u_i : S^1$ x ... x $S^n \to \mathcal{R}$

All players simoultaneously make a decision $s^i \in S^i$ and receive payments $u_i(s^1, .., s^n)$. Players seek to maximize their utility.

2-player Games

It suffices to show our results for 2-player games. 2-player games can be represented in bimatrix form.

2	0	0	0
1	0	0	0
2	-1	4	2

Table: $Matrix A$

3	2	-2	3
2	0	0	0
2	0	2	2

Table: $Matrix B$

	t_1	t2	t3	t4
s1	2	0	0	0
s2	1	0	0	0
s3	2	-1	4	2

Table: Matrix A

	t_1	t_2	t_3	t_4
s_1	3	2	-2	3
s_2	2	0	0	0
s_3	2	0	2	2

Table: Matrix B

- $N = 2$
- Player I also called the row player selects a strategy s_i
- Player II also called the column player selects a strategy t_j
- Player I is paid $u_1 = A(i, j)$, Player II is paid $u_2 = B(i, j)$

A tuple (s_i, t_j) is sometimes called a strategy profile.

	t_1	t2	t3	t4
s1	2	0	0	0
s2	1	0	0	0
s3	2	-1	4	2

Table: Matrix A

	t_1	t_2	t_3	t_4
s_1	3	2	-2	3
s_2	2	0	0	0
s_3	2	0	2	2

Table: Matrix B

Strategy s_i is an optimal response to t_j if $v^I(s_i, t_j) \geq v^I(s, t_j)$ for every alternative strategy $s \in S^I$.
Strategy t_j is an optimal response to s_i if $v^{II}(s_i, t_j) \geq v^{II}(s, t_j)$ for every alternative strategy $s \in S^I$.
For example here s_3 is an optimal response to t_3 and vice versa.

	t_1	t2	t3	t4
s1	2	0	0	0
s2	1	0	0	0
s3	2	-1	4	2

Table: Matrix A

	t_1	t_2	t_3	t_4
s_1	3	2	-2	3
s_2	2	0	0	0
s_3	2	0	2	2

Table: Matrix B

If s_i is an optimal response to t_j and vice versa, we call the profile (s_i, t_j) a nash equilibrium.

	t_1	t2	t3	t4
s1	2	0	0	0
s2	1	0	0	0
s3	2	-1	4	2

Table: Matrix A

	t_1	t_2	t_3	t_4
s_1	3	2	-2	3
s_2	2	0	0	0
s_3	2	0	2	2

Table: Matrix B

Mixed extension

Definition

A mixed strategy of a player is a probability distribution on the set of his available choices

- If $S = (s_1, s_2, ..., s_n)$ is the set of available strategies of a player then a mixed strategy is a vector in the form $p = (p_1, ..., p_n)$, where $p_i \geq 0$ for $i = 1, ..., n$, and $p_1 + ... + p_n = 1$.
- Any mixed strategy of the form $p = (p_1, ..., p_i, ..., p_n) = (0, ..., 1, ..., 0)$ is called a pure strategy, and corresponds to playing strategy s_i.
- Let $p = (p_1, ..., p_n)$ be a mixed strategy of I and $q = (q_1, ..., q_m)$ be a mixed strategy of II.

$$u^I(p, q) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u^I(s_i, t_i)$$

$$u^{II}(p, q) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i q_j u^{II}(s_i, t_i)$$

Mixed extension

Definitions for optimal response and nash equilibrium are same.

Definition

If $S = (s_1, s_2, ..., s_n)$ is the set of available strategies of a player then the support supp(p) of mixed strategy p are all $s_i \in S$ so that $p(s_i) > 0$.

Theorem

Strategy x is an optimal response to y iff all $x_i \in supp(x)$ are optimal responses to y. That is x is a mix of optimal responses.

Κάποιοι τύποι παιγνίων

Win-lose game

- $N = 2$
- $h^I, h^{II} : (S^I \text{x} S^{II}) \to \{0, 1\}$

Imitation game

- $S^I = S^{II}$
- I rewarded 1 if he plays same strategy as II, 0 else.
- Payment matrices are of the form (I, M), I identity matrix, M some diagonal matrix.

Symmetric Imitation game

- Imitation game where M is symmetric.

Κάποιοι ορισμοί

k-uniform strategy

Mixed strategy choosing between k different pure strategies with chance $1/k$

Uniform nash equilibrium

A strategy profile (x, x) that is a nash equilibrium and x is uniform.

Κάποιοι ορισμοί

ISWLG class

Collection of

- win-lose imitation games (I_{nxn}, M_{nxn}), with
- M symmetric
- $m_{jj} = 0, j \in \{1, ..., n\}$ (Diagonal of M is 0)

ISWLG(1) class

Collection of ISWLG games where one column has all ones except the diagonal.

Next steps

Theorem

k-UNIFORM NASH is W[2]-complete.

k-UNIFORM NASH problem

- Instance: An game in ISWLG, $G = (I, M)$
- Paremeter: k
- Question: Is there a uniform nash equilibrium (x, x) with |support(x)|=k ?

Corollary

The existence of a uniform nash equilibrium (x, x) with |support(x)|=k is W[2]-complete for any 2-player game.

It suffices to show this for ISWLG(1).

Max clique problem

Does G have a maximal clique of size k? Parameterized by k
W[2]-complete.

s-Max clique

Does G have a maximal clique of size k? Also there is a vertex neighboring all vertices in G. Parameterized by k. Also W[2]-complete.

We will use s-Max clique to prove k-Uniform nash is W[2]-complete.

ISWLG games: Corresponding graph

Corresponding graph of game

Notice that every game in ISWLG (I, M) can be encoded as an undirected simple graph G, where M is the adjacency matrix of G. This mapping is reversible.

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Pure strategies $=$ Vertices
II wins $=$ Edge

Lemma

A maximal clique of G corresponds to a uniform nash equilibrium (The reverse isn't necessary).

Lemma

Let G be the graph representation of a game $\mathcal{G} = (I, M)$ in ISWLG and G_x be a maximal clique of size k in graph G. Then the mixed strategy profile (x, x) constitutes a uniform Nash equilibrium of the game $\mathcal{G}$ where x is defined as follows:

$$x_i = \begin{cases} \textit{1/k if } i \textit{ is a vertex of } G_x, \\ 0, \ \textit{otherwise} \end{cases}$$

Proof.

I plays a best response to II: In a k-uniform nash equilibrium, player I earns $1/k$ by choosing a pure strategy in II's support, and 0 otherwise. So he plays a mix of best response pure strategies.
II plays a best response to I: Columns in supp(x) give exactly $(p-1)/p$. Because the payoff is 1 everywhere except the diagonal. Any pure strategy e_i of II from outside supp(x) will correspond to a vertex i not neighboring all vertices of G_x. So column i has at least one 0, and gives payment at most $(p-1)/p$. □

k-UNIFORM NASH problem is in W[2]

Reminder:

k-UNIFORM NASH problem

- Instance: An game in ISWLG(1), $G = (I, M)$
- Paremeter: k
- Question: Is there a uniform nash equilibrium (x, x) with |support(x)|=k ?

We will now prove that k-UNIFORM NASH is W[2]-complete. We first show inclusion by reduction to s-Max clique.

Proof.

Given an instance of a game $\mathcal{G} = (I, M)$ in ISWLG(1), and parameter k, the reduction simply creates $\mathcal{G}$'s corresponding graph G, with parameter k. This reduction is polynomial in time.
$\mathcal{G}$ has a k-uniform nash equilibrium (x, x) iff G_x is a maximal clique of size k.

k-UNIFORM NASH problem is in W[2]

Proof.

$\mathcal{G}$ has a k-uniform nash equilibrium (x, x) iff G_x is a maximal clique of size k.

$\Longleftarrow$) Already proven

$\Longrightarrow$) Let m be the vertex neighboring all vertices of G.

1. $m \in \text{supp}(x)$. Because a vertex neighboring all vertices in supp(x) not in x itself gives value 1. (all 1s except diagonal, which can't happen since I doesn't play m.), while the pures in supp(x) give $(k-1)/k$.
2. G_x is a clique. Since m gives value $(k-1)/k$, all strategies must be all ones except the diagonal to give the same value. So they form a clique.
3. G_x is maximal. Similarly with 1, a vertex neighboring all vertices in supp(x) not in x itself gives value 1.

$\square$

k-UNIFORM NASH problem is W[2]-hard

To prove k-UNIFORM NASH problem is W[2]-hard, we just take the opposite reduction.

Proof.

Given an instance of a graph G, with parameter k, the reduction simply creates the corresponding $\mathcal{G} = (I, M)$ with parameter k . This reduction is polynomial in time.
Already proven. □

Another hardness result

MAXIMUM PAYOFF FOR THE COLUMN PLAYER

- Instance: An imitation game $\mathcal{G} = (I, M)$
- Paremeter: k
- Question: Does $\mathcal{G}$ have a uniform Nash equilibrium (x, x) such that the payoff for the column player is at least k?

We prove MPCP is W[2]-hard by reduction from s-MAX CLIQUE.

MAXIMUM PAYOFF FOR THE COLUMN PLAYER

Proof.

Given an instance of a graph G, with parameter k, the reduction creates the corresponding imitation game $\mathcal{G} = (I, M)$ with parameter $k' = 2k - 2$, where now

$$m_{ij} = \begin{cases} 2k \text{ (rather than 1) if } ij \in E \\ 0, \text{ otherwise} \end{cases}$$

There is a maximal clique of size k in G iff $\mathcal{G}$ has a uniform nash (x, x) of payment $\geq 2k - 2$.

Let m be the vertex neighboring all vertices of G. Remember that in the previous reduction, we proved that $m \in supp(x)$, then that G_x is a clique, then that it is maximal. These claims still hold. For a clique of size k, the payoff is $2k(k-1)/k = 2(k-1) = 2k - 2$. It follows that G has a maximal clique of size k if and only if $\mathcal{G}$ has a nash equilibrium is which the column player receivers at least k'.

□

Yet another hardness result

k-MINIMAL NASH SUPPORT

- Instance: A zero-sum game $\mathcal{G} = (A, -A)$.
- Paremeter: k
- Question: Does $\mathcal{G}$ have a Nash equilibrium (x, y) such that $\max\{|supp(x)|, |supp(y)|\} \leq k$

We prove that k-MINIMAL NASH SUPPORT is W[2]-hard by reduction from set cover, a known W[2]-hard problem.

SET COVER

- Instance: A family $S = S_1, ..., S_r$ of r subsets of the set $N = \{1, ..., n\}$ that covers N, that is $\bigcup_{i=1,...,r} S_i = N$
- Paremeter: A positive integer $k(\leq r)$
- Question: Does S have a subset of size at most k that covers N?

We reduce set cover to k-minimal nash support.

Reduction

Let (N, S, k) be an instance of set cover. We make an instance of k-minimal nash support. It suffices to give A, r rows, $n+1$ columns.
For $i \in \{1, ..., r\}$, $j \in \{1, ..., n\}$

$$a_{ij} = \begin{cases} 1 \text{ if } j \in S_i \\ 0, \text{ if } j \notin S_i \end{cases}$$

For $i \in \{1, ..., r\}$, $j = n+1$:

$$a_{ij} = 1/k$$

	1	2	3	...	t	...	...	$n+1$
s_1					0			$1/k$
s_2					0			$1/k$
$\vdots$					$\vdots$			$1/k$
s_k					0			$1/k$
s_{k+1}								$1/k$
$\vdots$								$1/k$
s_r								$1/k$

set cover to k-minimal nash support

The rows in supp(x) correspond to sets. The columns to elements of N.

Proof.

$\Longrightarrow$) There is a k-cover. So there are k rows so that the game limited to these rows has at least a "1" in every column except column $n+1$. Let I play these rows uniformly and let II play column $n+1$. Both players respond optimally: No matter what I does, he will receive $1/k$ and any other pure strategy of II yields $\leq \frac{1}{k} \cdot (-1)$.

$\Longleftarrow$) There is a nash equilibrium (x, y), $\max\{|supp(x)|, |supp(y)|\} \leq k$. The corresponding sets of the rows in supp(x) form the desired set cover. This follows from the fact that in the matrix $A := A$ limited to the rows of supp(x), every column has at least one "1". Every column has at least one "1" for if all were "0", then all the columns in supp(y) must be all 0 as best responses to x. Since S covers N, every column has a "1" somehwere. I has incentive to change to that line. □

Thank you!