Tree packing and covering

Diestel, Graph Theory, Chapter 2.4

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Presentation structure

1. Presentation of two classic theorems of the 60s.
    1. Tree covering theorem
    2. Tree packing theorem
2. We prove them through a recent result
    - Packing-covering theorem
3. Proof of packing-covering theorem

Tree packing and covering

Let $\mathsf{n}(G)$ and $\mathsf{m}(G)$ be the number of vertices and edges of a graph G.

Tree packing theorem (Nash-Williams 1961; Tutte 1961)

A multigraph contains k edge-disjoint spanning trees if and only if for every partition P of its vertex set it has at least $k(|P|-1)$ cross-edges.

Packing covering theorem (Nash-Williams 1964)

The edges of a connected multigraph $G=(V,E)$ can be covered by k trees if and only if $\mathsf{m}(S) \leq k(\mathsf{n}(S)-1)$ for every non-empty induced subgraph S of G.

Also stated as:

Tree covering theorem (Nash-Williams 1964)

The edges of a connected multigraph $G=(V,E)$ can be covered by k trees if and only if $\mathsf{m}(G[U]) \leq k(|U|-1)$ for every non-empty set $U \subseteq V$.

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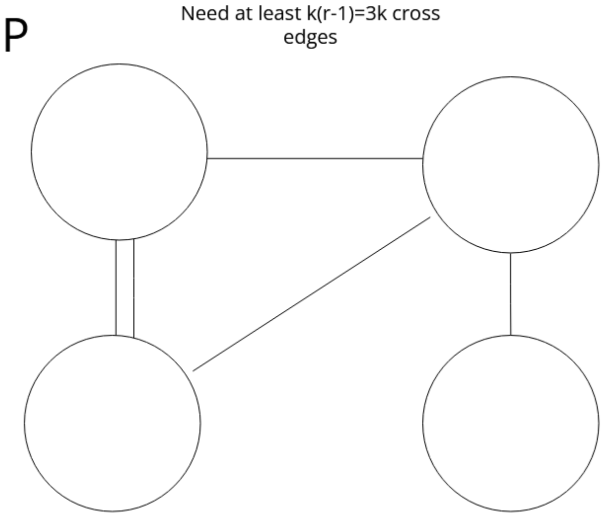


Tree packing and covering

About the tree packing theorem

- The tree packing theorem has applications in computer networks: The existence of k edge-disjoint spanning trees in a network graph implies not only the existence of k edge-disjoint $u - v$ paths for every pair of nodes u, v, which function as backup-paths for each other in case one fails or is overloaded, but also the ability to cheaply store the paths in the form of trees and retrieve the paths from them.
- The theorem also entails theoretical results: By the tree packing theorem, it can be proven that every $2k$ edge-connected multigraph has k edge-disjoint spanning trees.

About the tree covering theorem

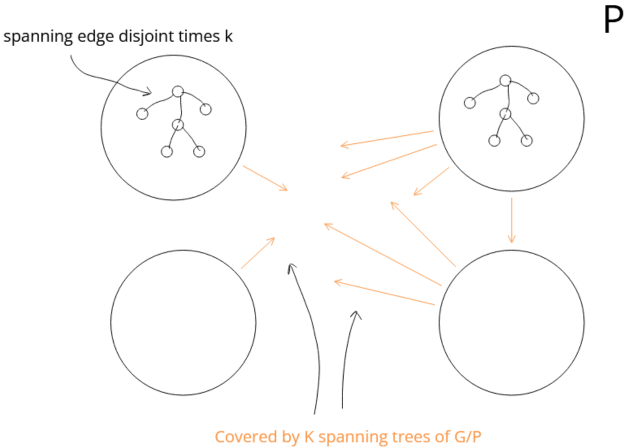
The tree covering theorem relates the arboricity of a graph, that is the number of trees needed to cover its edges, with its maximum local density.

Definition

Given a multigraph G, and a partition of its vertex set P, a contraction minor G/P is defined as the graph that is derived from G by contracting all vertices in the same partition class. Notice that since we are dealing with multigraphs, edges between the same two partition classes become parallel edges in G/P.

Packing-covering theorem (Bowler-Carmesin 2015)

For every connected multigraph $G = (V, E)$ and every $k \in N$ there is a partition P of V such that every $G[U]$ with $U \in P$ has k edge-disjoint spanning trees and the edges of G/P can be covered by k spanning trees.



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Notice the similarity between the packing-covering theorem and the tree packing and tree covering theorems.

Tree packing theorem (Nash-Williams 1961; Tutte 1961)

A multigraph G contains k edge-disjoint spanning trees if and only if for every partition P of its vertex set it has at least $k(|P| - 1)$ cross-edges.

Tree covering theorem (Nash-Williams 1964)

The edges of a connected multigraph $G = (V, E)$ can be covered by k trees if and only if $\mathrm{m}(G[U]) \leq k(|U| - 1)$ for every non-empty set $U \subseteq V$.

Packing-covering theorem (Bowler-Carmesin 2015)

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Proof of the tree packing and tree covering theorems

It may not come as a surprise that the packing-covering theorem derives the other two theorems in short fashion.
It is worth noting that the packing-covering theorem makes no structural assumptions for the given graph, unlike the other two theorems.

Next steps

The right direction of the equivalence in the proof of both the tree covering and tree packing is seen by contradiction, without the use of the packing-covering theorem. The left direction now follows.

Tree packing theorem (Nash-Williams 1961; Tutte 1961)

A multigraph G contains k edge-disjoint spanning trees if and only if for every partition P of its vertex set it has at least $k(|P|-1)$ cross-edges.

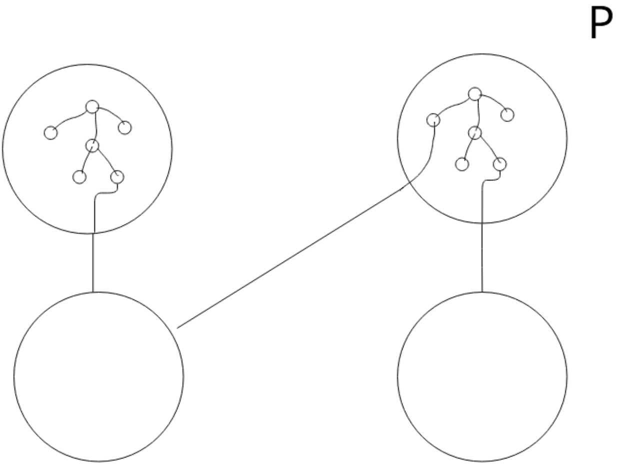
Proof of the tree packing theorem

Let it be given that the multigraph has for every partition P of $\mathsf{V}(G)$ at least $k(|P|-1)$ cross-edges.

By the packing-covering theorem, there is a partition P such that G/P has a collection $\mathcal{T}_1$ of k spanning trees covering its edges, each covering at most $|P|-1$. Since $\mathsf{m}(G/P) \geq k(|P|-1)$, we have that

$$\mathsf{m}(G/P) = k(|P|-1)$$

and the trees $\mathcal{T}_1$ are edge-disjoint. We now construct the k edge-disjoint spanning trees of G of the tree packing theorem by combining the trees $\mathcal{T}_1$ of G/P with the edge-disjoint spanning trees in $G[U]$ that are also given for P by the packing-covering theorem, let them be called $\mathcal{T}_2^U$: Combine any one of $\mathcal{T}_1$ with any $|P|$ trees of $\mathcal{T}_2^U$, one for each $U \in P$. $\square$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Proof of the tree packing and tree covering theorems

Definition

Given a partition P of $\mathsf{V}(G)$, a spanning tree T of G/P and $|P|$ spanning trees T^U, one for each $U \in P$, we define a combination of T and the trees T^U as the tree

$$\bigcup_U T^U \cup T'$$

where T' is defined from T by replacing vertices representing a partition class U from $\mathsf{V}(T)$ and from all edges $\in \mathsf{E}(T)$ with the corresponding vertices of G.

Tree covering theorem (Nash-Williams 1964)

The edges of a connected multigraph $G = (V, E)$ can be covered by k trees if and only if $\mathrm{m}(G[U]) \leq k(|U| - 1)$ for every non-empty set $U \subseteq V$.

Proof of the tree covering theorem

Given that every $U \subseteq V$ induces $\leq k(|U| - 1)$ edges in G, then for every $U \in P$, where P is the partition of $V(G)$ provided by the packing-covering theorem, the k edge-disjoint spanning trees of $G[U]$ provided by the theorem, each covering $|U| - 1$ different edges, in total cover $k(|U| - 1)$ edges, so $\mathrm{m}(G[U]) = k(|U| - 1)$ and the edges of $G[U]$ are partitioned by the k trees.
Combining them with the k spanning trees of G/P that cover its edges, also provided by the packing-covering theorem, we obtain k spanning trees of G covering its edges. $\square$

We now proceed with the proof of the main theorem.

Definition

We define a *chord* to be an edge of $G \setminus T$ where T is a spanning tree. The unique cycle created by adding some $e \notin T$ to T is called the *fundamental cycle* C_e.

Remark

Notice that adding some $e' \notin T$ to T and then removing some other $e \in C_{e'}$ from T results in a new tree T', as T' remains connected and has $n-1$ edges. When we create T' from T in this manner, we say we exchange e for e' or that e' replaces e.

Definition

Let $\mathcal{T} = (T_1, ..., T_k)$ be a family of spanning trees of G. We call $e_0, ..., e_n$ an *exchange chain* for $\mathcal{T}$ if $\forall e_i$ (except e_n of course) there is some tree (of $\mathcal{T}$) such that e_{i+1} can replace e_i. Furthermore, e_n lies in no tree of $\mathcal{T}$. We say the exchange chain is *started by* e_0.

We may denote the index j of the tree T_j such that e_{i+1} can replace e_i, as $j(i)$.

Let $\mathsf{E}(\mathcal{T})$ denote $\bigcup\{E(T) \mid T \in \mathcal{T}\}$.

Use of exchange chain

The general idea of the exchange chain is that, some details excluded, we exchange e_i for e_{i+1} in the tree T for which this is possible, producing a new family where T is different and all other trees are same. If done for all i, one after another, we produce a family $\mathcal{T}$' with $E(\mathcal{T}') = \mathsf{E}(\mathcal{T}) + e_n - e_0$ or even $E(\mathcal{T}') = \mathsf{E}(\mathcal{T}) + e_n$:

Property 1

If for a family $\mathcal{T}$, e_0 starts an exchange chain, and lies in more than 1 tree of $\mathcal{T}$, then we can construct a family $\mathcal{T}$', for which $E(\mathcal{T}') = \mathsf{E}(\mathcal{T}) + e$ for some edge e.

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Proof of property 1

Among all the exchange chains that start with e_0, pick one of minimal length, $e_0, ..., e_n$. Since the chain is minimal, it isn't of the form $e_0, ..., e_i, e_{i+1}, ..., e_l, ..., ...e_n$, where e_l can replace e_i on some tree. Because in this case, $e_0, e_i, e_l, ..., ...e_n$ is an exchange chain, contradicting the minimality.

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Now let's implement the exchanges. From $\mathcal{T} = \mathcal{T}^0$, construct $\mathcal{T}^1$, by replacing e_0 with e_1 in the corresponding tree T_j, and letting all other trees be same. Repeat this until $T^n = T'$ is constructed.

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Since every step removes e_i from a tree and adds e_{i+1} to another, $E(\mathcal{T}') = \mathsf{E}(\mathcal{T}) + e_n$.

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Since every step removes e_i from a tree and adds e_{i+1} to another, $E(\mathcal{T}') = \mathsf{E}(\mathcal{T}) + e_n$.

With the use of Property 1, we are now set to prove the packing-covering theorem.

Packing-covering theorem (Bowler-Carmesin 2015)

For every connected multigraph $G = (V, E)$ and every $k \in N$ there is a partition P of V such that every $G[U]$ with $U \in P$ has k edge-disjoint spanning trees and the edges of G/P can be covered by k spanning trees.

Proof

Choose a collection of k spanning trees of $G = (V, E)$ with the edges it covers $\mathsf{E}(\mathcal{T})$ maximal. Let D be the set of all edges that start an exchange chain for $\mathcal{T}$. An edge e not in $\mathsf{E}(\mathcal{T})$ is in D as it is a single-element exchange chain. Notice that by property 1, each edge of D lies in no more than 1 tree of $\mathcal{T}$ or $\mathsf{E}(\mathcal{T})$ would not be maximal. We will prove that the partition into the vertexes of each connected component of (V, D), let it be called P, is the partition with properties as described by the packing-covering theorem.

Proof (continued)

We start by proving that every class U_i of the partition has k edge-disjoint spanning trees. Let $U_i \in P$ be one of those vertex sets. For every $j = 1...k$ let S_j be the T_j limited to the nodes of U_i and the edges in D, that is, to make S_j take $T_j[U_i]$ and discard its edges not in D. As S_j may or may not have all edges of $T_j[U_i]$, it is a spanning forest of U_i. As each edge of D lies in no more than 1 T_j, all S_j are edge disjoint.

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Proof

We now show that S_j is also connected, that it is a tree. By definition of U_i, (V, D) is a supergraph of U_i, and by definition of S_j, U_i is a supergraph of S_j. It thus suffices to show that for every edge (u, u') in D, where $u,\ u' \in U_i$, there is a $u - u'$ path in S_j. If (u, u') in D is also in T_j, then it is also in S_j. If (u, u') in $D \notin T_j$, we can prove that the $u - u'$ path of T_j is in D, and as such is also in S_j: Since $(u, u') \in D$, there is an exchange chain of the form $(u, u') = e_0, e_1, \ldots$. But since $(u, u') \notin T_j$, adding (u, u') to T_j creates a circle, all of the edges of the $u - u'$ path of T_j are in. In other words, any edge e on the $u - u'$ path of T_j can be exchanged with e_0, and so $e, (u, u') = e_0, e_1, \ldots$ is also an exchange chain.

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Proof

So we have derived that every class U_i of the partition has k edge-disjoint spanning trees S_j. We now derive that the edges of G/P can be covered by k spanning trees. Contracting the partition classes U_i to form G/P, the k spanning trees of G, T_j turn into spanning trees of G/P and let's denote them T'_j. The edges of G/P are in $E \setminus D$, and since $E \setminus \mathrm{E}(\mathcal{T}) \subseteq D$, the edges of G/P are covered by the k T'_j. □

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Sketch Proof

1. Choose a collection of k spanning trees of $G = (V, E)$ with the edges it covers $\mathrm{E}(\mathcal{T})$ maximal. Let D be the set of all edges that start an exchange chain for $\mathcal{T}$. An edge e not in $\mathrm{E}(\mathcal{T})$ is in D as it is a single-element exchange chain.

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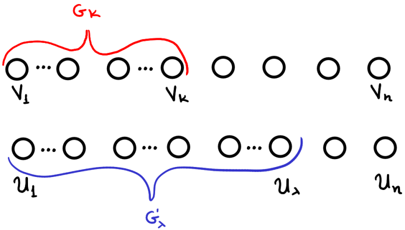
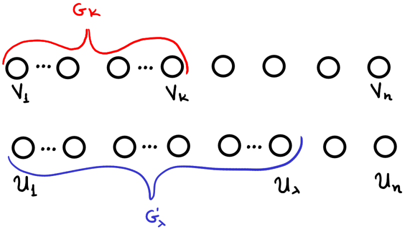
Thank you!


G_k
v_1
v_k
v_n
u_1
u_λ
u_n
G'_λ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