1 Degree of Connectedness

We would like to have a measure of how well connected is the graph. Viewed as a whole, a network with a lot of disconnected cliques should score lower than a loosely connected network. From the perspective of a single player, this measure can be expressed as the expected mean indirect trust, defined incrementally as follows:

$$Tr(A) = \sum_{B \in \mathcal{V} \setminus \{A\}} Tr_{A \to B} \text{ (Total indirect trust for player } A\text{)}$$

$$ETr\left(A\right) = \frac{Tr(A)}{|\mathcal{V} \setminus \{A\}|} = \frac{Tr(A)}{|\mathcal{V}| - 1} \text{ (Expected indirect trust for player } A\text{)}$$

$$EMTr = \frac{1}{|\mathcal{V}|} \sum_{A \in \mathcal{V}} ETr\left(A\right) \text{ (Expected mean indirect trust)}$$

2 Centrality

Another important measure is the network centrality. We propose here some different measures to that end.

2.1 Degree Centrality

One possible approach is the degree centrality [Freeman citation], that can be broken down as in-degree and out-degree centrality. We first define the node in-degree centrality.

$$C_{in}^{d}\left(A\right) = \sum_{B \in \mathcal{V} \setminus \{A\}} DTr_{B \to A}{}^{1} \text{ (Node in-degree centrality)}$$

Let $A^* = \underset{A \in \mathcal{V}}{\operatorname{argmax}} C_{in}^{d}\left(A\right)$. The network in-degree centrality is defined as:

$$C_{in}^{d} = \sum_{A \in \mathcal{V}} \left(C_{in}^{d}\left(A^*\right) - C_{in}^{d}\left(A\right) \right) \text{ (Network in-degree centrality)}$$

Similarly, for the out-degree centrality we have:

$$C_{out}^{d}\left(A\right) = \sum_{B \in \mathcal{V} \setminus \{A\}} DTr_{A \to B}{}^{1} \text{ (Node out-degree centrality)}$$

[1] Maybe indirect trust is more intuitive here than direct trust.

Let $A^* = \operatorname*{argmax}_{A \in \mathcal{V}} C^d_{out}(A)$. The network out-degree centrality is defined as:

$$C^d_{out} = \sum_{A \in \mathcal{V}} \left(C^d_{out}(A^*) - C^d_{out}(A) \right) \text{ (Network out-degree centrality)}$$

A problem of these centrality measures is that their unit is the currency used (i.e. Bitcoin) and thus may not always have an intuitive meaning. We would thus like to have a measure of *centalization* that has no units and can take values in the interval $[0, 1]$, with 0 corresponding to a network of no centralization (all nodes are equal, e.g. cycle) and 1 to a network with the maximum centralization possible (there is one central vital node for all, e.g. star). A centralization measure that achieves this target is proposed in [Freeman citation]. Here we use the following modified form. For a graph $\mathcal{G}$, the in- and out-centralization are defined as:

$$Cn^d_{in} = \frac{C^d_{in}}{\max C^d_{in}} \text{ (in-degree centralization)}$$
$$Cn^d_{out} = \frac{C^d_{out}}{\max C^d_{out}} \text{ (out-degree centralization)} ,$$

where $\max C^d_{in}$ is defined as the maximum in-degree centrality for any graph with the same number of nodes for which the maximum direct[2] trust is equal to the maximum direct[2] trust of $\mathcal{G}$; $\max C^d_{out}$ is defined equivalently.

2.2 maxFlow Centrality

An alternative measure of centrality for a player $A \in \mathcal{V}$ of a network $\mathcal{G}$ can be defined as the impact that the removal of A would have on the indirect trust between the rest of the players. More specifically, let $\mathcal{G}' = \mathcal{G} \setminus \{A\}$. Then it is:

$$C^{mF}(A) = \sum_{B,C \in \mathcal{V}'} \left(Tr_{\mathcal{G},B \to C} - Tr_{\mathcal{G}',B \to C} \right) \text{ (Node maxFlow centrality)} .$$

We can now follow the same steps as previously for the relevant network definitions. Let A^* be the player with the maximum maxFlow centrality:

$$A^* = \operatorname*{argmax}_{A \in \mathcal{V}} C^{mF}(A) .$$

[2] Maybe indirect trust is more intuitive here than direct trust.

Then the network maxFlow centrality is defined as follows:

$$C^{mF} = \sum_{A \in \mathcal{V}} \left(C^{mF}\left(A^{*}\right) - C^{mF}\left(A\right) \right)$$

and the centralization:

$$Cn^{mF} = \frac{C^{mF}}{\max C^{mF}} \text{ (Network maxFlow centralization) },$$

where $\max C^{mF}$ is the maximum centrality for any graph with the same number of nodes for which the maximum direct[3] trust is equal to the maximum direct[3] trust of $\mathcal{G}$.

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[3] Maybe indirect trust is more intuitive here than direct trust.

Here we can see several possible network health measures. A combination of them will probably do the job.

1. $n := |\mathcal{V}|$ — How many players
2. $e := |\mathcal{E}| - |\{A : DTr_{A \to A} > 0\}|$[4] — How many direct trust lines
3. $\frac{e}{n}$ — Mean direct trust lines per player
4. $DTr := \sum_{\substack{A,B \in \mathcal{V} \\ A \neq B}} DTr_{A \to B}$[4] — How much direct trust in total
5. $\mu := \frac{DTr}{n}$ — Mean direct trust per player
6. $Cap := \sum_{A \in \mathcal{V}} DTr_{A \to A}$ — Total capital
7. $\frac{Cap}{n}$ — Mean capital per player
8. $\frac{DTr - Cap}{n}$ — Mean direct trust minus capital per player
9. $\frac{1}{n} \sum_{A \in \mathcal{V}} \left(\sum_{\substack{B \in V \\ B \neq A}} DTr_{A \to B} - \mu \right)^2$[4] — Outgoing direct trust variance
10. $\frac{1}{n} \sum_{B \in \mathcal{V}} \left(\sum_{\substack{A \in V \\ A \neq B}} DTr_{A \to B} - \mu \right)^2$[4] — Incoming direct trust variance
11. $\frac{Cap}{DTr}$ — Total capital to total direct trust ratio

[4] Maybe it makes sense to include looped direct trusts (Capital) as well.