Definition 1 (Trust Reduction). *Let $A, B \in \mathcal{V}$ and x_i flow to $N^+(A)_i$ resulting from $maxFlow(A, B), u_i = DTr_{A \to N^+(A)_i, j-1}, u'_i = DTr_{A \to N^+(A)_i, j}$, $i \in [|N^+(A)|], j \in \mathbb{N}$.*

1. *The Trust Reduction on neighbour i, δ_i is defined as $\delta_i = u_i - u'_i$.*
2. *The Flow Reduction on neigbour i, Δ_i is defined as $\Delta_i = x_i - u'_i$.*

We will also use the standard notation for 1-norm and ∞-norm, that is:

1. $||\delta_i||_1 = \sum\limits_{i \in N^+(A)} \delta_i$
2. $||\delta_i||_\infty = \max\limits_{i \in N^+(A)} \delta_i.$

Definition 2 (Restricted Flow). *Let $A, B \in \mathcal{V}, i \in [|N^+(A)|]$.*

1. *Let $F_{A_i \to B}$ be the flow from A to $N^+(A)_i$ as calculated by the $maxFlow(A, B)$ (x'_i) when $u'_i = u_i$,*
   $u'_k = 0 \,\forall k \in [|N^+(A)|] \wedge k \neq i.$
2. *Let $S \subset N^+(A)$. Let $F_{A_S \to B}$ be the sum of flows from A to S as calculated by the $maxFlow(A, B)$ ($\sum\limits_{i=1}^{|S|} x'_i$) when $u'_C = u_C \forall C \in S, u'_D = 0 \,\forall D \in N^+(A) \setminus S.$*

Theorem 1 (Saturation theorem).
Let s source, $n = |N^+(s)|, x_i, i \in [n]$, flows to s's neighbours as calculated by the maxFlow algorithm, u'_i new direct trusts to the n neighbours and x'_i new flows to the neighbours as calculated by the maxFlow algorithm with the new direct trusts, u'_i. It holds that $\forall i \in [n], u'_i \leq x_i \Rightarrow x'_i = u'_i$.

Proof. $\forall i \in [n], x'_i > u'_i$ is impossible because a flow cannot be higher than its corresponding capacity. Thus $\forall i \in [n], x'_i \leq u'_i$. (1)
In the initial configuration of u_i and according to the flow problem setting, a combination of flows y_i such that $\forall i \in [n], y_i = u'_i$ is a valid, albeit not necessarily maximum, configuration with a flow $\sum\limits_{i=1}^{n} y_i$. Suppose that $\exists k \in [n] : x'_k < u'_k$ as calculated by the maxFlow algorithm with the new direct trusts, u'_i. Then for the new maxFlow F' it holds that $F' = \sum\limits_{i=1}^{n} x'_i < \sum\limits_{i=1}^{n} y_i$ since $x'_k < y_k$ and (1) which is impossible because the configuration $\forall i \in [n], x'_i = y_i$ is valid since $\forall i \in [n], y_i = u'_i$ and also has a higher flow, thus the maxFlow algorithm will prefer the configuration with the higher flow. Thus we deduce that $\forall i \in [n], x'_i = u'_i$. $\qquad \square$

Theorem 2 (Trust transfer theorem (flow terminology)).
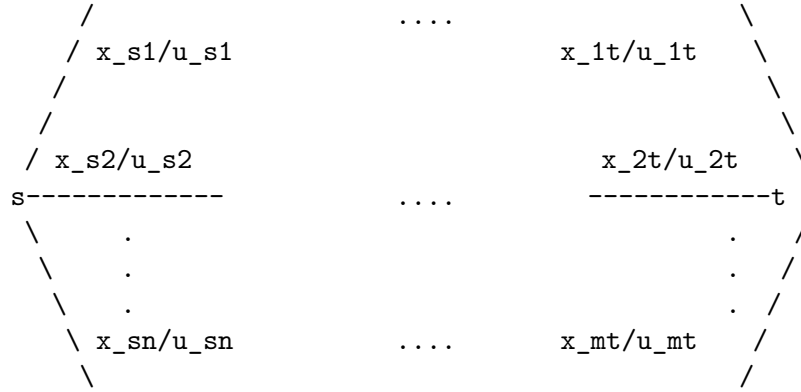*Let s source, t sink, $n = N^+(s)$*
$X = \{x_1, \ldots, x_n\}$ *outgoing flows from s,*
$U = \{u_1, \ldots, u_n\}$ *outgoing capacities from s,*
V the value to be transferred.
Nodes apart from s, t follow the conservative strategy.
Obviously $maxFlow = F = \sum\limits_{i=1}^{n} x_i$.

```
   /                    . . . .                        \
  / x_s1/u_s1                        x_1t/u_1t    \
 /                                                  \
/                                                    \
/ x_s2/u_s2                          x_2t/u_2t    \
s-------------          . . . .      -----------t
 \        .                                .    /
  \       .                                .   /
   \      .                                .  /
    \ x_sn/u_sn           . . . .    x_mt/u_mt   /
     \                                          /
```

We create a new graph where

1. $\sum\limits_{i=1}^{n} u_i' = F - V$
2. $\forall i \in [n]\, u_i' \le x_i$

It holds that $maxFlow' = F' = F - V$.

Proof. From theorem 1 we can see that $x_i' = u_i'$. It holds that $F' = \sum\limits_{i=1}^{n} x_i' = \sum\limits_{i=1}^{n} u_i' = F - V$. $\qquad\square$

Lemma 1 (Flow limit lemma).
It is impossible for the outgoing flow x_i from A to an out neighbour of A to be greater than $F_{A_i \to B}$. More formally, $x_i \le F_{A_i \to B}$.

Proof. Suppose a configuration where $\exists i : x_i > F_{A_i \to B}$. If we reduce the capacities $u_k, k \ne i$ the flow that passes from i in no case has to be reduced. Thus we can set $\forall k \ne i, u_k' = 0$ and $u_i' = u_i$. Then $\forall k \ne i, x_k' = 0, x_i' = x_i$ is a valid configuration and thus by definition $F_{A_i \to B} = x_i' = x_i > F_{A_i \to B}$, which is a contradiction. Thus $\forall i \in [|N^+(A)|], x_i \le F_{A_i \to B}$. $\qquad\square$

Theorem 3 (Trust Saving Theorem).

A configuration $U' : u_i' = F_{A_i \to B}$ for some $i \in [|N^+(A)|]$ can yield the same maxFlow with a configuration $U'' : u_i'' = u_i, \forall k \in [|N^+(A)|], k \neq i, u_k'' = u_k'$.

Proof. We know that $x_i \leq F_{A_i \to B}$ (lemma 1), thus we can see that any increase in u_i' beyond $F_{A_i \to B}$ will not influence x_i and subsequently will not incur any change on the rest of the flows. $\square$

Theorem 4 (Invariable trust reduction with naive algorithms).

Let A source, $n = |N^+(A)|$ and u_i' new direct trusts. If $\forall i \in [n], u_i' \leq x_i$, Trust Reduction $||\delta_i||_1$ is independent of $x_i, u_i' \forall$ valid configurations of x_i

Proof. Since $\forall i \in [n], u_i' \leq x_i$ it is (according to 1) $x_i' = u_i'$, thus $\delta_i = u_i - x_i'$. We know that $\sum\limits_{i=1}^{n} x_i' = F - V$, so we have $||\delta_i||_1 = \sum\limits_{i=1}^{n} \delta_i = \sum\limits_{i=1}^{n} (u_i - x_i') = \sum\limits_{i=1}^{n} u_i - F + V$ independent from x_i', u_i' $\square$

Here we show three naive algorithms for calculating new direct trusts so as to maintain invariable risk when paying a trusted party. To prove the correctness of the algorithms, it suffices to prove that $\forall i \in [n] u_i' \leq x_i$ and that $\sum\limits_{i=1}^{n} u_i' = F - V$ where $F = \sum\limits_{i=1}^{n} x_i$.

Algorithm 1: First-come, first-served trust transfer

    Input : x_i flows, $n = |N^+(s)|$, V value
    Output: u_i' capacities

1   $F \leftarrow \sum\limits_{i=1}^{n} x_i$

2   **if** $F < V$ **then**

3 | **return** $\perp$

4   $F_{cur} \leftarrow F$

5   **for** $i \leftarrow 1$ *to* n **do**

6 | $u_i' \leftarrow x_i$

7   $i \leftarrow 1$

8   **while** $F_{cur} > F - V$ **do**

9 | $reduce \leftarrow \min(x_i, F_{cur} - F + V)$

10 | $F_{cur} \leftarrow F_{cur} - reduce$

11 | $u_i' \leftarrow x_i - reduce$

12 | $i \leftarrow i + 1$

13 **return** $U' = \bigcup\limits_{k=1}^{n} \{u_k'\}$

Proof of correctness for algorithm 1. — We will show that $\forall i \in [n]\, u'_i \leq x_i$.

Let $i \in [n]$. In line 6 we can see that $u'_i = x_i$ and the only other occurence of u'_i is in line 11 where it is never increased ($reduce \geq 0$), thus we see that, when returned, $u'_i \leq x_i$.

— We will show that $\sum\limits_{i=1}^{n} u'_i = F - V$.

$F_{cur,0} = F$

If $F_{cur,i} \geq F - V$, then $F_{cur,i+1}$ does not exist because the *while* loop breaks after calculating $F_{cur,i}$.

Else $F_{cur,i+1} = F_{cur,i} - \min(x_{i+1}, F_{cur,i} - F + V)$.

If for some i, $\min(x_{i+1}, F_{cur,i} - F + V) = F_{cur,i} - F + V$, then $F_{cur,i+1} = F - V$, so if $F_{cur,i+1}$ exists, then $\forall k < i, F_{cur,k} = F_{cur,k-1} - x_k \Rightarrow F_{cur,i} = F - \sum\limits_{k=1}^{i} x_k$

Furthermore, if $F_{cur,i+1} = F - V$ then $u'_{i+1} = x_{i+1} - F_{cur,i} + F - V = x_i - F + \sum\limits_{k=1}^{i-1} x_k + F - V = \sum\limits_{k=1}^{i} x_k - V$, $\forall k \leq i, u'_k = 0$ and $\forall k > i + 1, u'_k = x_k$.

In total, we have $\sum\limits_{k=1}^{n} u'_k = \sum\limits_{k=1}^{i} x_k - V + \sum\limits_{k=i+1}^{n} x_k = \sum\limits_{k=1}^{n} x_k - V \Rightarrow \sum\limits_{k=1}^{n} u'_k = F - V$.

$\square$

Complexity of algorithm 1. First we will prove that on line 13 $i \leq n + 1$. Suppose that $i > n + 1$ on line 13. This means that $F_{cur,n}$ exists and $F_{cur,n} = F - \sum\limits_{i=1}^{n} x_i = 0 \leq F - V$ since, according to the condition on line 2, $F - V \geq 0$. This means however that the *while* loop on line 8 will break, thus $F_{cur,n+1}$ cannot exist and $i = n + 1$ on line 13, which is a contradiction, thus $i \leq n + 1$ on line 13. Since i is incremented by 1 on every iteration of the *while* loop (line 12), the complexity of the *while* loop is $O(n)$ in the worst case. The complexity of lines 2 - 4 and 7 is $O(1)$ and the complexity of lines 1, 5 - 6 and 13 is $O(n)$, thus the total complexity of algorithm 1 is $O(n)$. $\square$

Algorithm 2: Absolute equality trust transfer ($||\Delta_i||_\infty$ minimizer)

Input : x_i flows, $n = |N^+(s)|$, V value
Output: u_i' capacities

1 $F \leftarrow \sum\limits_{i=1}^{n} x_i$

2 **if** $F < V$ **then**

3 $\quad$ **return** $\perp$

4 **for** $i \leftarrow 1 \; to \; n$ **do**

5 $\quad$ $u_i' \leftarrow x_i$

6 $reduce \leftarrow \frac{V}{n}$

7 $reduction \leftarrow 0$

8 $empty \leftarrow 0$

9 $i \leftarrow 0$

10 **while** $reduction < V$ **do**

11 $\quad$ **if** $u_i' > 0$ **then**

12 $\quad\quad$ **if** $x_i < reduce$ **then**

13 $\quad\quad\quad$ $empty \leftarrow empty + 1$

14 $\quad\quad\quad$ **if** $empty < n$ **then**

15 $\quad\quad\quad\quad$ $reduce \leftarrow reduce + \frac{reduce - x_i}{n - empty}$

16 $\quad\quad\quad$ $reduction \leftarrow reduction + u_i'$

17 $\quad\quad\quad$ $u_i' \leftarrow 0$

18 $\quad\quad$ **else if** $x_i \geq reduce$ **then**

19 $\quad\quad\quad$ $reduction \leftarrow reduction + u_i' - (x_i - reduce)$

20 $\quad\quad\quad$ $u_i' \leftarrow x_i - reduce$

21 $\quad$ $i \leftarrow (i+1) mod \; n$

22 **return** $U' = \bigcup\limits_{k=1}^{n} \{u_k'\}$

We will start by showing some results useful for the following proofs. Let j be the number of iterations of the **while** loop for the rest of the proofs for algorithm 2 (think of i from line 21 without the $mod \; n$).

First we will show that $empty \leq n$. $empty$ is only modified on line 13 where it is incremented by 1. This happens only when $u_i' > 0$ (line 11), which is assigned the value 0 on line 17. We can see that the incrementation of $empty$ can happen at most n times because $|U'| = n$. Since $empty_0 = 0$, $empty \leq n$ at all times of the execution.

Next we will derive the recursive formulas for the various variables.
$empty_0 = 0$

$$empty_{j+1} = \begin{cases} empty_j, & u'_{(j+1) \bmod n} = 0 \\ empty_j + 1, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod n} < reduce_j \\ empty_j, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod n} \geq reduce_j \end{cases}$$

$$reduce_0 = \frac{V}{n}$$

$$reduce_{j+1} = \begin{cases} reduce_j, & u'_{(j+1) \bmod n} = 0 \\ reduce_j + \frac{reduce_j - x_{(j+1) \bmod n}}{n - empty_{j+1}}, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod n} < reduce_j \\ reduce_j, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod n} \geq reduce_j \end{cases}$$

$$reduction_0 = 0$$

$$reduction_{j+1} = \begin{cases} reduction_j, & u'_{(j+1) \bmod n} = 0 \\ reduction_j + u'_{(j+1) \bmod n}, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod} \\ reduction_j + u'_{(j+1) \bmod n} - x_{(j+1) \bmod n} + reduce_{j+1}, & u'_{(j+1) \bmod n} > 0 \wedge x_{(j+1) \bmod} \end{cases}$$

In the end, $r = reduce$ is such that $r = \frac{V - \sum\limits_{x \in S} x}{n - |S|}$ where $S = \{$flows y from s to $N^+(s)$ according to max

$y < r\}$. Also, $\sum\limits_{i=1}^{n} u'_i = \sum\limits_{i=1}^{n} \max(0, x_i - r)$. TOPROVE

Proof of correctness for algorithm 2. — We will show that $\forall i \in [n] \, u'_i \leq x_i$.

On line 5, $\forall i \in [n] \, u'_i = x_i$. Subsequently u'_i is modified on line 17, where it becomes equal to 0 and on line 20, where it is assigned $x_i - reduce$. It holds that $x_i - reduce \leq x_i$ because initially $reduce = \frac{V}{n} \geq 0$ and subsequently $reduce$ is modified only on line 15 where it is increased ($n > empty$ because of line 14 and $reduce > x_i$ because of line 12, thus $\frac{reduce - x_i}{n - empty} > 0$). We see that $\forall i \in [n], u'_i \leq x_i$.

— We will show that $\sum\limits_{i=1}^{n} u'_i = F - V$.

The variable $reduction$ keeps track of the total reduction that has happened and breaks the **while** loop when $reduction \geq V$. We will first show that $reduction = \sum\limits_{i=1}^{n} (x_i - u'_i)$ at all times and then we will prove that $reduction = V$ at the end of the execution. Thus we will have proven that $\sum\limits_{i=1}^{n} u'_i = \sum\limits_{i=1}^{n} x_i - V = F - V$.

- On line 5, $u'_i = x_i \Rightarrow \sum\limits_{i=1}^{n} (x_i - u'_i) = 0$ and $reduction = 0$.

  On line 17, u'_i is reduced to 0 thus $\sum\limits_{i=1}^{n} (x_i - u'_i)$ is increased by u'_i.

  Similarly, on line 16 $reduction$ is increased by u'_i, the same as the

increase in $\sum_{i=1}^{n} (x_i - u_i')$.

On line 20, u_i' is reduced by $u_i' - x_i + reduce$ thus $\sum_{i=1}^{n} (x_i - u_i')$ is increased by $u_i' - x_i + reduce$. On line 19, $reduction$ is increased by $u_i' - x_i + reduce$, which is equal to the increase in $\sum_{i=1}^{n} (x_i - u_i')$. We also have to note that neither u_i' nor $reduction$ is modified in any other way from line 10 and on, thus we conclude that $reduction = \sum_{i=1}^{n} (x_i - u_i')$ at all times.

- Suppose that $reduction_j > V$ on the line 22. Since $reduction_j$ exists, $reduction_{j-1} < V$. If $x_{j \bmod n} < reduce_{j-1}$ then $reduction_j = reduction_{j-1} + u_{j \bmod n}'$. Since $reduction_j > V$, $u_{j \bmod n}' > V - reduction_{j-1}$. TOCOMPLETE

$\square$

Complexity of algorithm 2. In the worst case scenario, each time we iterate over all capacities only the last non-zero capacity will become zero and every non-zero capacity must be recalculated. This means that every n steps exactly 1 capacity becomes zero and eventually all capacities (maybe except for one) become zero. Thus we need $O(n^2)$ steps in the worst case. $\square$

A variation of this algorithm using a Fibonacci heap with complexity $O(n)$ can be created, but that is part of further research.

Proof that algorithm 2 minimizes the $||\Delta_i||_\infty$ norm. Suppose that U' is the result of an execution of algorithm 2 that does not minimize the $||\Delta_i||_\infty$ norm. Suppose that W is a valid solution that minimizes the $||\Delta_i||_\infty$ norm. Let δ be the minimum value of this norm. There exists $i \in [n]$ such that $x_i - w_i = \delta$ and $u_i' < w_i$. Because both U' and W are valid solutions ($\sum_{i=1}^{n} u_i' = \sum_{i=1}^{n} w_i = F - V$), there must exist a set $S \subset U'$ such that $\forall u_j' \in S, u_j' > w_j$ TOCOMPLETE. $\square$

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Algorithm 3: Proportional equality trust transfer

Input : x_i flows, $n = |N^+(s)|$, V value
Output: u'_i capacities

1 $F \leftarrow \sum\limits_{i=1}^{n} x_i$

2 if $F < V$ **then**

3 | **return** $\perp$

4 for $i \leftarrow 1$ *to* n **do**

5 | $u'_i \leftarrow x_i - \frac{V}{F} x_i$

6 return $U' = \bigcup\limits_{k=1}^{n} \{u'_k\}$

Proof of correctness for algorithm 3. − We will show that $\forall i \in [n]\ u'_i \leq x_i$.

According to line 5, which is the only line where u'_i is changed, $u'_i = x_i - \frac{V}{F} x_i \leq x_i$ since $x_i, V, F > 0$ and $V \leq F$.

− We will show that $\sum\limits_{i=1}^{n} u'_i = F - V$.

With $F = \sum\limits_{i=1}^{n} x_i$, on line 6 it holds that $\sum\limits_{i=1}^{n} u'_i = \sum\limits_{i=1}^{n} (x_i - \frac{V}{F} x_i) = \sum\limits_{i=1}^{n} x_i - \frac{V}{F} \sum\limits_{i=1}^{n} x_i = F - V$.

$\square$

Complexity of algorithm 3. The complexity of lines 1, 4 - 5 and 6 is $O(n)$ and the complexity of lines 2 - 3 is $O(1)$, thus the total complexity of algorithm 3 is $O(n)$. $\square$

Naive algorithms result in $u'_i \leq x_i$, thus according to 4, $||\delta_i||_1$ is invariable for any of the possible solutions U', which is not necessarily the minimum (usually it will be the maximum). The following algorithms

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concentrate on minimizing two δ_i norms, $||\delta_i||_\infty$ and $||\delta_i||_1$.

Algorithm 4: $||\delta_i||_\infty$ minimizer

Input : $X = \{x_i\}$ flows, $n = |N^+(s)|$, V value, ϵ_1, ϵ_2
Output: u_i' capacities

1 **if** $\epsilon_1 < 0 \vee \epsilon_2 < 0$ **then**
2 | **return** $\bot$
3 $F \leftarrow \sum\limits_{i=1}^{n} x_i$
4 **if** $F < V$ **then**
5 | **return** $\bot$
6 $\delta_{max} \leftarrow \max\limits_{i \in [n]} \{u_i\}$
7 $\delta^* \leftarrow$ `BinSearch(`$0, \delta_{max}, F\text{-}V, n, X, \epsilon_1, \epsilon_2$`)`
8 **for** $i \leftarrow 1$ *to* n **do**
9 | $u_i' \leftarrow \max\left(u_i - \delta^*, 0\right)$
10 **return** $U' = \bigcup\limits_{k=1}^{n} \{u_k'\}$

Since trust should be considered as a continuous unit and binary search dissects the possible interval for the solution on each recursive call, inclusion of the ϵ-parameters in `BinSearch` is necessary for the algorithm to complete in a finite number of steps.

Algorithm 5: *

Input : *bot, top*, F', n, X, ϵ_1, ϵ_2
Output: δ^*

1 **function** `BinSearch` **if** $bot = top$ **then**
2 | **return** *bot*
3 **else**
4 | **for** $i \leftarrow 1$ *to* n **do**
5 | | $u_i' \leftarrow \max\left(0, u_i - \frac{top+bot}{2}\right)$
6 | **if** $maxFlow < F' - \epsilon_1$ **then**
7 | | **return** `BinSearch(`*bot*, $\frac{top+bot}{2}, F', n, X, \epsilon_1, \epsilon_2$`)`
8 | **else if** $maxFlow > F' + \epsilon_2$ **then**
9 | | **return** `BinSearch(`$\frac{top+bot}{2}$, *top*$, F', n, X.\epsilon_1, \epsilon_2$`)`
10 | **else**
11 | | **return** $\frac{top+bot}{2}$

Proof that $maxFlow(\delta)$ is strictly decreasing for $\delta : maxflow(\delta) < F$.
Let $maxFlow(\delta)$ be the $maxFlow$ with $\forall i \in [n], u_i' = max(0, u_i - \delta)$. We will prove that the function $maxFlow(\delta)$ is strictly decreasing for all

$\delta \leq \max\limits_{i\in[n]}\{u_i\}$ such that $maxFlow(\delta) < F$.

Supppose that $\exists \delta_1, \delta_2 : \delta_1 < \delta_2 \wedge maxFlow(\delta_1) \leq maxFlow(\delta_2) < F$. We will work with configurations of $x'_{i,j}$ such that $x'_{i,j} \leq x_i, j \in \{1,2\}$.

Let $S_j = \{i \in N^+(s) : i \in MinCut_j\}$. It holds that $S_1 \neq \emptyset$ because otherwise $MinCut_1 = MinCut_{\delta=0}$ which is a contradiction because then $maxFlow(\delta_1) = F$. Moreover, it holds that $S_1 \subseteq S_2$, since $\forall u'_{i,2} > 0, u'_{i,2} < u'_{i,1}$. Every node in the $MinCut_j$ is saturated, thus $\forall i \in S_1, x'_{i,j} = u'_{i,j}$. Thus $\sum\limits_{i\in S_1} x_{i,2} < \sum\limits_{i\in S_1} x_{i,1}$ and, since $maxFlow(\delta_1) \leq maxFlow(\delta_2)$, we conclude that for the same configurations, $\sum\limits_{i\in N^+(s)\setminus S_1} x_{i,2} > \sum\limits_{i\in N^+(s)\setminus S_1} x_{i,1}$.

However, since $x'_{i,j} \leq x_i, j \in \{1,2\}$, the configuration $[x''_{i,1} = x'_{i,2}, i \in N^+(s) \setminus S_1], [x''_{i,1} = x'_{i,1}, i \in S_1]$ is valid for $\delta = \delta_1$ and then $\sum\limits_{i\in S_1} x''_{i,1} + \sum\limits_{i\in N^+(s)\setminus S_1} x''_{i,1} = \sum\limits_{i\in S_1} x'_{i,1} + \sum\limits_{i\in N^+(s)\setminus S_1} x'_{i,2} > maxFlow(\delta_1)$, contradiction. Thus $maxFlow(\delta)$ is strictly decreasing. $\square$

We can see that if $V > 0, F' = F - V < F$ thus if $\delta \in (0, \max\limits_{i\in[n]}\{u_i\}]$: $maxFlow(\delta) = F' \Rightarrow \delta = \min ||\delta_i||_\infty : maxFlow(||\delta_i||_\infty) = F'$.

Proof of correctness for function 5. Supposing that $[F' - \epsilon_1, F' + \epsilon_2] \subset [maxFlow(top), maxFlow(bot)]$, or equivalently $maxFlow(top) \leq F' - \epsilon_1 \wedge maxFlow(bot) \geq F' + \epsilon_2$, we will prove that $maxFlow(\delta^*) \in [F' - \epsilon_1, F' + \epsilon_2]$.

First of all, we should note that if an invocation of `BinSearch` returns without calling `BinSearch` again (line 2 or 11), its return value will be equal to the return value of the initial invocation of `BinSearch`, as we can see on lines 7 and 9, where the return value of the invoked `BinSearch` is returned without any modification. The case where `BinSearch` is called again is analyzed next:

- If $maxFlow(\frac{top+bot}{2}) < F' - \epsilon_1 < F'$ (line 6) then, since $maxFlow(\delta)$ is strictly decreasing, $\delta^* \in [bot, \frac{top+bot}{2})$. As we see on line 7, the interval $(\frac{top+bot}{2}, top]$ is discarded when the next `BinSearch` is called. Since $F' + \epsilon_2 \leq maxFlow(bot)$, we have $[F' - \epsilon_1, F' + \epsilon_2] \subset [maxFlow(\frac{top+bot}{2}), maxFlow(bot)]$ and the length of the available interval is divided by 2.
- Similarly, if $maxFlow(\frac{top+bot}{2}) > F' + \epsilon_2 > F'$ (line 8) then $\delta^* \in (\frac{top+bot}{2}, top]$. According to line 9, the interval $[bot, \frac{top+bot}{2})$ is discarded when the next `BinSearch` is called. Since $F' - \epsilon_1 \geq maxFlow(top)$, we have $[F' - \epsilon_1, F' + \epsilon_2] \subset (maxFlow(top), maxFlow(\frac{top+bot}{2})]$ and the length of the available interval is divided by 2.

As we saw, $[F' - \epsilon_1, F' + \epsilon_2] \subset [maxFlow(top), maxFlow(bot)]$ in every recursive call and $top - bot$ is divided by 2 in every call. From topology we know that $A \subset B \Rightarrow |A| < |B|$, so the recursive calls cannot continue infinitely. $||[F' - \epsilon_1, F' + \epsilon_2]|| = \epsilon_1 + \epsilon_2$. Let bot_0, top_0 the input values given to the initial invocation of `BinSearch`, bot_j, top_j the input values given to the j-th recursive call of `BinSearch` and $len_j = ||[bot_j, top_j]|| = top_j - bot_j$. We have $\forall j > 0, len_j = top_j - bot_j = \frac{top_{j-1} - bot_{j-1}}{2} \Rightarrow \forall j > 0, len_j = \frac{top_0 - bot_0}{2^j}$. We understand that in the worst case $len_j = \epsilon_1 + \epsilon_2 \Rightarrow 2^j = \frac{top_0 - bot_0}{\epsilon_1 + \epsilon_2} \Rightarrow j = \log_2(\frac{top_0 - bot_0}{\epsilon_1 + \epsilon_2})$. Also, as we saw earlier, δ^* is always in the available interval, thus $maxFlow(\delta^*) \in [F' - \epsilon_1, F' + \epsilon_2]$. $\qquad\square$

Complexity of function 5. Lines 1 - 2 have complexity $O(1)$, lines 4 - 5 have complexity $O(n)$, lines 6 - 11 have complexity $O(maxFlow) + O(BinSearch)$. As we saw in the proof of correctness for function 5, we need at most $\log_2(\frac{top - bot}{\epsilon_1 + \epsilon_2})$ recursive calls of `BinSearch`. Thus the function 5 has worst-case complexity $O((maxFlow + n)\log_2(\frac{top - bot}{\epsilon_1 + \epsilon_2}))$. $\qquad\square$

Proof of correctness for algorithm 4. We will show that $maxFlow \in [F - V - \epsilon_1, F - V + \epsilon_2]$, with u_i' decided by algorithm 4.
Obviously $maxFlow(0) = F, maxFlow(\max_{i \in [n]}\{u_i\}) = 0$, thus $\delta^* \in \max_{i \in [n]}\{u_i\}$.
According to the proof of correctness for function 5, we can directly see that $maxFlow(\delta^*) \in [F - V - \epsilon_1, F - V + \epsilon_2]$, given that ϵ_1, ϵ_2 are chosen so that $F - V - \epsilon_1 \geq 0, F - V + \epsilon_2 \leq F$, so as to satisfy the condition $[F' - \epsilon_1, F' + \epsilon_2] \subset [maxFlow(top), maxFlow(bot)]$. $\qquad\square$

Complexity of algorithm 4. The complexity of lines 1 - 2 and 4 - 5 is $O(1)$, the complexity of lines 3, 6, 8 - 9 and 10 is $O(n)$ and the complexity of line 7 is $O(BinSearch) = O((maxFlow + n)\log_2(\frac{\delta_{max}}{\epsilon_1 + \epsilon_2}))$, thus the total complexity of algorithm 4 is $O((maxFlow + n)\log_2(\frac{\delta_{max}}{\epsilon_1 + \epsilon_2}))$. $\qquad\square$

However, we need to minimize $\sum_{i=1}^{n}(u_i - u_i') = ||\delta_i||_1$.