MAT1856/APM466 Assignment 1

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Fundamental Questions - 25 points

1.
   (a) Bonds allow governments to borrow money from individuals, creating an obligation to repay over time with interest, rather than causing immediate monetary distortions(by printing money) that could result in a higher inflation rate.

   (b) When a two-year bond offer a yield of 3%, and a five-year bond of 3.1%, yield curve with little difference between short-term and long-term rates would flatten in the long-term part.

   (c) Quantitative easing involves purchasing financial assets to stimulate economic activity, the US Fed resumed purchasing massive amounts of debt securities in early 2020.[1]

2. CAN 1.25 March 25; CAN 0.50 September 25;
   CAN 0.25 March 26; CAN 1.00 September 26;
   CAN 1.25 March 27; CAN 2.75 September 27;
   CAN 3.50 March 28; CAN 3.25 September 28;
   CAN 4.00 March 29; CAN 3.50 September 29;
   Reasoning: All of these bonds mature in either March or September with a regular intervals of six months (semi-annual maturities), which provides an even distribution of data points for bootstrapping. Also, their coupon rates are relatively close, excluding the outliers like CAN 9.0 June 25, making them more consistent and hence easier to compare.

3. We are basically creating a summary by reducing the number of dimensions in large datasets to principal components that retain most of the original information.[2] In this case, the eigenvectors represent the directions or patterns in the data along which the variation occurs, and the eigenvalues tell us how much of the total variation(i.e. the magnitude) in the stochastic processes is explained by each principal component.

Empirical Questions - 75 points

4.
   (a)
$$\text{price} = \sum_{t=1}^{T} \frac{\text{Coupon}}{\left(1+\frac{\text{ytm}}{2}\right)^t} + \frac{\text{FaceValue}}{\left(1+\frac{\text{ytm}}{2}\right)^T}$$

   By using this formula, the yield to maturity (YTM) for each of the 10 selected bonds was calculated for each date. An iterative numerical method (Newton's method) was used to solve the YTM equation. This ensures precision by solving the nonlinear equation for yield. The maturity values are evenly spaced by six months, so linear interpolation provides a good approximation without overcomplicating the curve.[3]

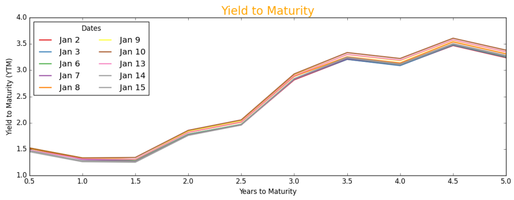

Figure 1: YTM curve plotted for 4a

(b) We split into two cases: for bonds that mature in 6 months (no more coupon payments, treated as zero-coupon bonds), we calculate its value directly by:

$$r(T) = -\frac{\log(\text{Price}/\text{Notional})}{T}$$

For the bond with maturity more than 6 months, I use bootstrapping under the equation:

$$P = p_1 \cdot e^{-r(t_1)\cdot t_1} + p_2 \cdot e^{-r(t_2)\cdot t_2}$$

where P is the dirty price, which equals accrued interest plus the clean price; t_1 and t_2 are current timing and the next timing; p_1 and p_2 stand for payment at t_1 and t_2; and $r(t_1)$ is the spot rate. We could use known information to calculate the unknown spot rate $r(t_2)$ at the next timing.

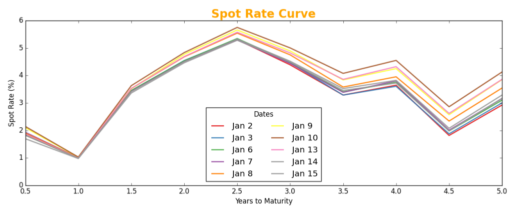

Figure 2: Spot Rate curve plotted for 4b

(c)

$$F_{1,1+n} = \left(\frac{(1+S_{1+n})^{2(1+n)}}{(1+S_1)^2} \right)^{\frac{1}{2n}} - 1$$

where[5]: S_1 is the spot rate for 1 year,S_{1+n} is the spot rate for $1+n$ years,2 represents the number of compounding periods per year for semi-annual compounding,n is the forward term in years (e.g., 2, 3, 4, or 5). Calculate $F_{1,3}, F_{1,4}, F_{1,5}$ based on the formula provided above, and return the list of forward rates $F_{1,2}, F_{1,3}, F_{1,4}, F_{1,5}$.

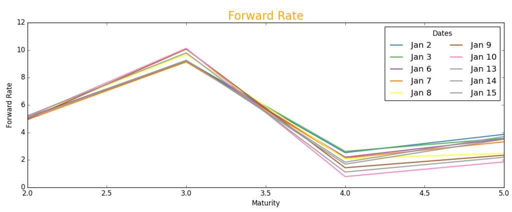

Figure 3: Forward Rate curve plotted for 4c

5.

$$A = \begin{pmatrix} 1.11484674\times 10^{-4} & 9.12271430\times 10^{-5} & 7.17022974\times 10^{-5} & 5.81492222\times 10^{-5} & 5.24709038\times 10^{-5} \\ 9.12271430\times 10^{-5} & 4.20506905\times 10^{-4} & 2.87211478\times 10^{-4} & 1.78288188\times 10^{-4} & 1.90984901\times 10^{-4} \\ 7.17022974\times 10^{-5} & 2.87211478\times 10^{-4} & 2.45337412\times 10^{-4} & 1.45699606\times 10^{-4} & 1.54446224\times 10^{-4} \\ 5.81492222\times 10^{-5} & 1.78288188\times 10^{-4} & 1.45699606\times 10^{-4} & 1.02538664\times 10^{-4} & 1.02136397\times 10^{-4} \\ 5.24709038\times 10^{-5} & 1.90984901\times 10^{-4} & 1.54446224\times 10^{-4} & 1.02136397\times 10^{-4} & 1.08212169\times 10^{-4} \end{pmatrix}$$

$$B = \begin{pmatrix} 0.00107593 & 0.00040861 & -0.00207116 & -0.00409468 \\ 0.00040861 & 0.00118082 & -0.00664452 & -0.00658925 \\ -0.00207116 & -0.00664452 & 0.10343898 & 0.05297293 \\ -0.00409468 & -0.00658925 & 0.05297293 & 0.0505417 \end{pmatrix}$$

Here, A is the covariance matrix for the time series of daily log-returns of yield, and B is the covariance matrix for the forward rates.

6. Eigenvalues for A: [8.44891656e-04 8.82171192e-05 4.01965304e-05 1.23362941e-05 2.43822297e-06], with corresponding eigenvectors:

$$\begin{pmatrix} [-0.18664191, -0.95559138, -0.17761245, 0.13442787, 0.04891702]^T \\ [-0.68468208, 0.25588825, -0.68109436, -0.01720793, -0.03931925]^T \\ [-0.51903145, 0.08695472, 0.54325097, 0.64795832, -0.08985336]^T \\ [-0.32841323, -0.11704666, 0.3375431, -0.61635776, -0.62017143]^T \end{pmatrix}$$

The largest eigenvalue here(8.44891656e-04) has around 85.5% in size among all eigenvalues, meaning that its corresponding eigenvector $[-0.18664191, -0.95559138, -0.17761245, 0.13442787, 0.04891702]^T$ represents the direction for the largest variance.

Eigenvalues for B: [0.13692609 0.01836272 0.00068708 0.00026154], with corresponding eigenvectors:

$$\begin{pmatrix} [0.02897599, -0.13587661, 0.92671328, 0.34914244]^T \\ [0.06709715, -0.11680601, -0.36612536, 0.92076411]^T \\ [-0.84729374, -0.52893852, -0.04265332, -0.02231697]^T \\ [-0.52607193, 0.82952875, 0.07304392, 0.17261218]^T \end{pmatrix}$$

The largest eigenvalue here(0.13692609) has around 87.6% in size among all eigenvalues, meaning that its corresponding eigenvector $[0.02897599, -0.13587661, 0.92671328, 0.34914244]^T$ represents the direction for the largest variance.

References and GitHub Link to Code

Github Link: https://github.com/Orggz/APM466

1. What did the Fed do in response to the COVID-19 crisis?
https://www.brookings.edu/articles/fed-response-to-covid19/
2. What is principal component analysis (PCA)?
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3. How to calculate Yield To Maturity with Python?
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4. Textbook Week1 "Bootstrapping" Section
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5. Hint Section from Assignment 1
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