Linear Algebra Study Guide

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Chapter 1

Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be consistent if it has either one solution or infinitely many solutions; a system is inconsistent if it has no solution.

Given an example System of Equations:

$$\begin{aligned} x_1 - 2x_2 + x3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned}$$

The matrix of coefficients is:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

A system of linear equations can be solved using elementary row operations. Elementary row operations are:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Interchange) Interchange two rows.

3. (Scaling) Multiply all entries in a row by a nonzero constant.

Row operations can be applied to any matrix and are reversible.

1.2 Row Reduction and Echelon Forms

Definition 1.2.1 (Echelon Form) *A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:*

1. *All nonzero rows are above any rows of all zeros.*

2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*

3. *All entries in a column below a leading entry are zeros.*

   If a matrix in echelon form satisfies the following additional conditions, then it is in reduced row echelon form.

4. *The leading entry in each nonzero row is 1.*

5. *Each leading 1 is the only nonzero entry in its column.*

Theorem 1.2.1 (Uniqueness of the Reduced Echelon Form) *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Definition 1.2.2 (Pivot Position) *A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A .*

Definition 1.2.3 (Pivot Column) *A pivot column is a column of A that contains a pivot position.*

Theorem 1.2.2 (Existence and Uniqueness Theorem) *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot columnthat is, if and only if an echelon form of the augmented matrix has no row of the form:*

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ With } b \text{ non-zero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Using Row Reduction to Solve a Linear System:

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Write the system of equations corresponding to the matrix obtained in step 3.
4. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

1.3 Vector Equations

A matrix with only one column is called a column vector, or simply a vector.

Algebraic Properties of $\mathbb{R}^n$:

For all $\vec{u}$, $\vec{v}$, $\vec{w}$ in $\mathbb{R}^n$ and all scalars c and d:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
3. $\vec{u}+0=0+\vec{u}=0$
4. $\vec{u}+(-\vec{u})=-\vec{u}+\vec{u}=0$, where $-\vec{u}$ denotes $-1\vec{u}$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
6. $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
7. $c(d\vec{u})=(cd)\vec{u}$
8. $1\vec{u}=\vec{u}$

A vector equation

$$x_1\vec{a_1}+x_2\vec{a_2}+\cdots+x_n\vec{a_n}=\vec{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c}\vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & \vec{b}\end{array}\right]$$

Definition 1.3.1 (span) *If $\vec{v_1},\ldots,\vec{v_p}$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\vec{v_1},\ldots,\vec{v_p}$ is denoted by $Span\{\vec{v_1},\ldots,\vec{v_p}\}$ and is called the subset of $\mathbb{R}^n$ spanned by $\vec{v_1},\ldots,\vec{v_p}$. That is $Span\{\vec{v_1},\ldots,\vec{v_p}\}$ is the collection of all vectors that can be written in the form $c_1\vec{v_1}+c_2\vec{v_2}+\cdots+c_p\vec{v_p}$ with $c_1,\ldots,c_p$ scalars.*

1.4 The Matrix Equation $A\vec{x} = \vec{b}$

Theorem 1.4.1 *If A is an $m \times n$ matrix, with colums $\vec{a_1}, \ldots, \vec{a_n}$, and if $\vec{b}$ is in $\mathbb{R}^m$, the matrix equation*

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \vec{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \ldots & \vec{a_n} & | & \vec{b} \end{bmatrix}$$

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A .

Theorem 1.4.2 *Let A be an $m \times n$ matrix.Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.*

1. *For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution*
2. *Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A*
3. *The columns of A span $\mathbb{R}^m$*
4. *A has a pivot position in every row*

Theorem 1.4.3 *If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:*

1. $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
2. $A(c\vec{u}) = c(A\vec{u})$

1.5 Solution Sets of Linear Systems

The homogeneous equation $A\vec{x} = \vec{0}$ has a non trivial solution if and only if the equation has at least one free variable.

Theorem 1.5.1 *Suppose the equation $A\vec{x} = \vec{b}$ is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then the solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v_h}$, where $\vec{v_h}$ is any solution to the homogeneous equation $A\vec{x} = \vec{0}$.*

Writing a solution set (of a consistent system) in parametric vector form:

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\vec{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\vec{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

1.6 Linear Independence

Definition 1.6.1 (Linearly Independent) *An indexed set of vectors $\{\vec{v_1}, \ldots, \vec{v_p}\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation*

$$x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_p\vec{v_p} = 0$$

has only the trivial solution. The set $\{\vec{v_1}, \ldots, \vec{v_p}\}$ is said to be linearly dependent if it is not linearly independent.

The columns of a matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

A set of two vectors $\{\vec{v_1}, \vec{v_2}\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 1.6.1 (Characterization of Linearly Dependent Sets) *An indexed set $S = \{\vec{v_1}, \ldots, \vec{v_p}\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.*

Theorem 1.6.2 *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $S = \{\vec{v_1}, \ldots, \vec{v_p}\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.*

Theorem 1.6.3 *If a set $S = \{\vec{v_1}, \ldots, \vec{v_p}\}$ in $\mathbb{R}^n$, contains the zero vector, then the set is linearly dependent.*

1.7 Introduction to Linear Transformations

Definition 1.7.1 (Linear Transformation) *A transformation of mapping T is linear if:*

1. *$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T;*

2. *$T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in the domain of T;*

Every matrix transformation is a linear transformation.
All Linear transformations can be written as matrix multiplication.

1.8 The Matrix of a Linear Transformation

Theorem 1.8.1 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that:*

$$T(\vec{x}) = A\vec{x} \text{ For all } \vec{x} \text{ in } \mathbb{R}^n$$

Chapter 2

Matrix Algebra

2.1 Matrix Operations

Theorem 2.1.1 *Let A, B, C be matrices of the same size and let r and s be scalars:*

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Theorem 2.1.2 *Let A be a $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.*

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
   For any scalar r
5. $I_m A = A = AI_n$

Theorem 2.1.3 *Let A and B denote matrices whose sizes are appropriate for the following sums and products.*

1. $(A^T)^T = A$

2. $(A+B)^T = A^T + B^T$

3. For any scalar r,
$(rA)^T = rA^T$

4. $(AB)^T = B^T A^T$

2.2 Matrix Inverse

Theorem 2.2.1 *If A is an invertible $n \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$*

1. $(AB)^{-1} = B^{-1}A^{-1}$

2. $(A^T)^{-1} = (A^{-1})^T$

The product of an $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

Theorem 2.2.2 *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}*

2.3 Subspaces of $\mathbb{R}^n$

Definition 2.3.1 (Subspace) *A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has 3 properties:*

1. *The zero vector is in H*

2. *For each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u} + \vec{v}$ is in H*

3. *For each $\vec{u}$ and each scalar c in H, the vector $c\vec{u}$ is in H*

Definition 2.3.2 (Column Space) *The column space of a matrix A is the set Col A of all linear combinations of the columns of A.*

Definition 2.3.3 (Null Space) *The null space of a matrix A is the set Nul A of all solutions of the homogeneous equation $A\vec{x} = \vec{0}$*

Definition 2.3.4 (Basis) *A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H*

2.4 Dimension and Rank

Definition 2.4.1 (dimension) *The dimension of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be zero.*

Definition 2.4.2 (rank) *The rank of a matrix A, denoted by rank A, is the dimension of the column space of A .*

Theorem 2.4.1 (Rank Theorem) *If a matrix A has n columns, then rank $A + dimNulA = n$*

Theorem 2.4.2 (Basis Theorem) *Let H be a p-dimensional subspace of $\mathbb{R}^n$: Any linearly independent set of exactly p elements in H is automatically a basis for H: Also, any set of p elements of H that spans H is automatically a basis for H.*

Chapter 3

Determinants

3.1 Introduction to Determinants

Definition 3.1.1 (determinant) *For $n \geq 2$ the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum on n terms of the form $\pm a_{1j} det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A.*

$$det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det A_{1j}$$

Theorem 3.1.1 *If A is a triangular matrix, then $det A$ is the product of the entries on the main diagonal of A.*

3.2 Properties of Determinants

Theorem 3.2.1 (Row Operations) *Let A be a square matrix:*

1. *If a multiple of one row of A is added to another row to produce a matrix B, then $det B = det A$.*
2. *If two rows of A are interchanged to produce B, then $det B = -det A$.*
3. *If one row of A is multiplied by k to produce B , then $det B = k \times det A$.*

Theorem 3.2.2 *A square matrix A is invertible if and only if $det A \neq 0$.*

Theorem 3.2.3 *If A is an $n \times n$ matrix, then $det A^T = det A$.*

Theorem 3.2.4 (Multiplicative Property) *If A and B are $n \times n$ matrices, then $det AB = (det A)(det B)$*

Chapter 4

Eigenvalues and Eigenvectors

4.1 Eigenvalues and Eigenvectors

Definition 4.1.1 (Eigenvector and Eigenvalue) *An eigenvector of an $n \times n$ matrix A is a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$; such an $\vec{x}$ is called an eigenvector corresponding to λ.*

4.2 The Characteristic equation

Theorem 4.2.1 (The invertible Matrix theorem) *Let A be an $n \times n$ matrix. Then A is invertible if and only if:*

1. *The number 0 is not an eigenvalue of A.*
2. *The determinant of A is not 0.*

The scalar equation $det(A - \lambda I) = 0$ is called the characteristic equation of A.

4.3 Diagonalization

Theorem 4.3.1 (The Diagonalization Theorem) *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*