Linear Algebr

MAT223 Slides

Core Exercise 1

You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 km East and 1 km North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 km East and 2 km North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 km East and 64 km North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Core Exercise 2

You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 km East and 1 km North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 km East and 2 km North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation? Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Set. A *set* is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set *elements*.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a *subset* Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called *equal* if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using *set-builder notation*. That is, if X is a set, we can define the subset

$$Y = \{a \in X \ : \ \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$

$\mathbb{R} = \{\text{real numbers}\}.$

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$

3.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$.

(b) $1.5 \in \{1,2,3\}$.

(c) $4 \in \{1,2,3\}$.

(d) "b"$\in \{x : x$ is an English letter$\}$.

(e) "ò"$\in \{x : x$ is an English letter$\}$.

(f) $\{1,2\} \subseteq \{1,2,3\}$.

(g) For some $a \in \{1,2,3\}$, $a \geq 3$.

(h) For any $a \in \{1,2,3\}$, $a \geq 3$.

(i) $1 \subseteq \{1,2,3\}$.

(j) $\{1,2,3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$.

(k) $\{1,2,3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$.

Core Exercise 4

Write the following in set-builder notation

4.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

4.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

Unions & Intersections. Let X and Y be sets. The *union* of X and Y and the *intersection* of X and Y are defined as follows.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

5.1 $X \cup Y$

5.2 $X \cap Y$

5.3 $X \cup Y \cup Z$

5.4 $X \cap Y \cap Z$

Core Exercise 6

Draw the following subsets of $\mathbb{R}^2$.

6.1 $V = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

6.2 $H = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

6.3 $D = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

6.4 $N = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R} \right\}$.

6.5 $V \cup H$.

6.6 $V \cap H$.

6.7 Does $V \cup H = \mathbb{R}^2$?

Linear Combination. A *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ are called the *coefficients* of the linear combination.

Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

7.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

7.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

7.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Core Exercise 8

Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

8.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

8.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

8.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

Non-negative & Convex Linear Combinations.
Let $\vec w = \alpha_1\vec v_1 + \alpha_2\vec v_2 + \cdots + \alpha_n\vec v_n$. The vector $\vec w$ is called a ***non-negative*** linear combination of $\vec v_1, \vec v_2, \ldots, \vec v_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0.$$

The vector $\vec w$ is called a ***convex*** linear combination of $\vec v_1, \vec v_2, \ldots, \vec v_n$ if

$$\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1.$$

Let

$$\vec a = \begin{bmatrix}1\\1\end{bmatrix} \qquad \vec b = \begin{bmatrix}-1\\1\end{bmatrix} \qquad \vec c = \begin{bmatrix}0\\1\end{bmatrix} \qquad \vec d = \begin{bmatrix}0\\2\end{bmatrix} \qquad \vec e = \begin{bmatrix}-1\\-1\end{bmatrix}.$$

9.1 Out of $\vec a$, $\vec b$, $\vec c$, $\vec d$, and $\vec e$, which vectors are

(a) linear combinations of $\vec a$ and $\vec b$?

(b) non-negative linear combinations of $\vec a$ and $\vec b$?

(c) convex linear combinations of $\vec a$ and $\vec b$?

9.2 If possible, find two vectors $\vec u$ and $\vec v$ so that

(a) $\vec a$ and $\vec c$ are non-negative linear combinations of $\vec u$ and $\vec v$ but $\vec b$ is not.

(b) $\vec a$ and $\vec e$ are non-negative linear combinations of $\vec u$ and $\vec v$.

(c) $\vec a$ and $\vec b$ are non-negative linear combinations of $\vec u$ and $\vec v$ but $\vec d$ is not.

(d) $\vec a$, $\vec c$, and $\vec d$ are convex linear combinations of $\vec u$ and $\vec v$.

Otherwise, explain why it's not possible.

Core Exercise 10

Let L be the set of points $(x,y)\in\mathbb{R}^2$ such that $y=2x+1$.

10.1 Describe L using set-builder notation.

10.2 Draw L as a subset of $\mathbb{R}^2$.

10.3 Add the vectors $\vec a=\begin{bmatrix}-1\\-1\end{bmatrix}$, $\vec b=\begin{bmatrix}1\\3\end{bmatrix}$ and $\vec d=\vec b-\vec a$ to your drawing.

10.4 Is $\vec d\in L$? Explain.

10.5 For which $t\in\mathbb{R}$ is it true that $\vec a+t\vec d\in L$? Explain using your picture.

Vector Form of a Line. Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p} \text{ for some } t \in \mathbb{R}\}$, we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in *vector form*. The vector $\vec{d}$ is called a *direction vector* for ℓ.

Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

11.1 Write ℓ in vector form. Is vector form of ℓ unique?

11.2 Write L in vector form.

11.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

Core Exercise 12

Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

12.1 Do the lines A and B intersect? Justify your conclusion.

12.2 Do the lines A and C intersect? Justify your conclusion.

12.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

Vector Form of a Plane. A plane $\mathcal{P}$ is written in *vector form* if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t, s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called *direction vectors* for $\mathcal{P}$.

Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

13.1 Find two direction vectors for $\mathcal{P}$.

13.2 Write $\mathcal{P}$ in vector form.

13.3 Describe how vector form of a plane relates to linear combinations.

13.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

Core Exercise 14

Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

14.1 Find three points in $\mathcal{Q}$.

14.2 Find two direction vectors for $\mathcal{Q}$.

14.3 Write $\mathcal{Q}$ in vector form.

Span. The *span* of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec{0}\}$.

Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

15.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

15.2 Draw $\operatorname{span}\{\vec{v}_2\}$.

15.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

15.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$.

15.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Core Exercise 16

Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

16.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

16.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

16.3 Does the expression $\operatorname{span}(\ell_1)$ make sense? If so, what is it? How about $\operatorname{span}(\ell_2)$?

Set Addition. If A and B are sets of vectors, then the *set sum* of A and B, denoted $A+B$, is

$$A+B=\{\vec{x}\ :\ \vec{x}=\vec{a}+\vec{b} \text{ for some } \vec{a}\in A \text{ and } \vec{b}\in B\}.$$

Let $A=\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell=\operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

17.1 Draw A, B, and $A+B$ in the same picture.

17.2 Is $A+B$ the same as $B+A$?

17.3 Draw $\ell+A$.

17.4 Consider the line ℓ_2 given in vector form by $\vec{x}=t\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

Core Exercise 18

Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.

2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?

3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?

4. What is $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}$?

Linearly Dependent & Independent (Geometric). We say the vectors $\vec v_1, \vec v_2, \ldots, \vec v_n$ are *linearly dependent* if for at least one i,

$$\vec v_i \in \operatorname{span}\{\vec v_1, \vec v_2, \ldots, \vec v_{i-1}, \vec v_{i+1}, \ldots, \vec v_n\}.$$

Otherwise, they are called *linearly independent*.

Let $\vec u = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec v = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec w = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

19.1 Describe $\operatorname{span}\{\vec u, \vec v, \vec w\}$.

19.2 Is $\{\vec u, \vec v, \vec w\}$ linearly independent? Why or why not?

Let $X = \{\vec u, \vec v, \vec w\}$.

19.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

19.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

Trivial Linear Combination.
The linear combination $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1 = \cdots = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called ***non-trivial***.

Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

20.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

20.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

We now have an equivalent definition of linear dependence.

Linearly Dependent & Independent (Algebraic).
The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are *linearly dependent* if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

1.1 Explain how the geometric definition of linear dependence (original) implies this algebraic one (new).

1.2 Explain how this algebraic definition of linear dependence (new) implies the geometric one (original).

Since we have geometric def $\implies$ algebraic def, and algebraic def $\implies$ geometric def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as algebraic def $\iff$ geometric def).

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

2.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

2.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

2.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

2.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Norm. The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula $\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$.

Dot Product. If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

4.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.

(b) Compute $\vec{a} \cdot \vec{b}$.

(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

4.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.

4.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.

4.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

For a vector $\vec{v} \in \mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

always holds.

Distance. The *distance* between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

Unit Vector. A vector $\vec{v}$ is called a *unit vector* if $\|\vec{v}\| = 1$.

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

5.1 Find the distance between $\vec{u}$ and $\vec{v}$.

5.2 Find a unit vector in the direction of $\vec{u}$.

5.3 Does there exist a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

5.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

Orthogonal. Two vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

6.1 Find two vectors orthogonal to $\vec{a} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Can you find two such vectors that are not parallel?

6.2 Find two vectors orthogonal to $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$. Can you find two such vectors that are not parallel?

6.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\| = 5$ and $\|\vec{y}\| = 3$. What is the distance between $\vec{x}$ and $\vec{y}$?

7.1 Draw $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.

7.2 If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x} \cdot \vec{u}$?

7.3 Expand the dot product $\vec{u} \cdot \vec{x}$ to get an equation for A.

7.4 If possible, express A as a span.

Normal Vector. A *normal vector* to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{d}\} + \{\vec{p}\}.$$

8.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

8.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n} \cdot \vec{v}$? What about $\vec{n} \cdot (\vec{u} - \vec{p})$? Explain using a picture.

8.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n} \cdot (\vec{x} - \vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

8.4 Some textbooks would claim that ℓ_2 could be expressed in normal form as $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \vec{x} = 3$. How does this relate to the $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$ normal form? Where does the 3 come from?

Let $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

9.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

9.2 Describe X using an equation.

9.3 Describe X as a span.

Projection. Let $X \subseteq \mathbb{R}^n$ be a set. The ***projection*** of the vector $\vec{v} \in \mathbb{R}^n$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

Let $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\ell = \operatorname{span}\{\vec{a}\}$.

0.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.

0.2 Find $\operatorname{proj}_{\{\vec{b}\}} \vec{v}$, $\operatorname{proj}_{\{\vec{a},\vec{b}\}} \vec{v}$.

0.3 Find $\operatorname{proj}_\ell \vec{v}$. (Recall that a quadratic $at^2 + bt + c$ has a minimum at $t = -\dfrac{b}{2a}$).

0.4 Is $\vec{v} - \operatorname{proj}_\ell \vec{v}$ a normal vector for ℓ? Why or why not?

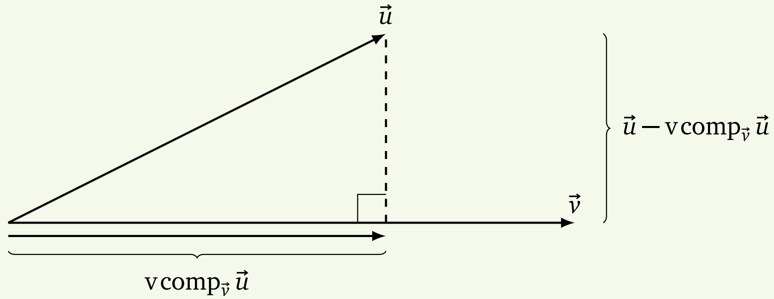
Let K be the line given in vector form by $\vec{x} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and let $\vec{c} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1.1 Make a sketch with $\vec{c}$, K, and $\operatorname{proj}_K \vec{c}$ (you don't need to compute $\operatorname{proj}_K \vec{c}$ exactly).

1.2 What should $(\vec{c} - \operatorname{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

1.3 Use your formula from the previous part to find $\operatorname{proj}_K \vec{c}$ *without* computing any distances.

Vector Components. Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\operatorname{vcomp}_{\vec{v}} \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \operatorname{vcomp}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.



Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

2.1 List two conditions that $\operatorname{vcomp}_{\vec{b}} \vec{a}$ must satisfy.

2.2 Find a formula for $\operatorname{vcomp}_{\vec{b}} \vec{a}$.

Let $\vec d = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\vec u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

3.1 Draw $\vec d$, $\vec u$, $\operatorname{span}\{\vec d\}$, and $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ in the same picture.

3.2 How do $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ and $\operatorname{vcomp}_{\vec d} \vec u$ relate?

3.3 Compute $\operatorname{proj}_{\operatorname{span}\{\vec d\}} \vec u$ and $\operatorname{vcomp}_{\vec d} \vec u$.

3.4 Compute $\operatorname{vcomp}_{\vec d} \vec u$. Is this the same as or different from $\operatorname{vcomp}_{\vec d} \vec u$? Explain.

Subspace. A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec u, \vec v \in V$ and all scalars k we have

(i) $\vec u + \vec v \in V$; and

(ii) $k\vec u \in V$.

Subspaces give a mathematically precise definition of a “flat space through the origin.”

For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

4.1 $A = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z} \right\}.$

4.2 $B = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$

4.3 $C = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.4 $D = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.5 $E = \left\{ \vec{x} \in \mathbb{R}^2 \ : \ \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.6 $F = \left\{ \vec{x} \in \mathbb{R}^2 \,:\, \vec{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}.$

4.7 $G = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$

4.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

Basis. A *basis* for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

Dimension. The *dimension* of a subspace V is the number of elements in a basis for V.

Let $\vec u = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec v = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec w = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec u, \vec v, \vec w\}$.

5.1 Describe V.

5.2 Is $\{\vec u, \vec v, \vec w\}$ a basis for V? Why or why not?

5.3 Give a basis for V.

5.4 Give another basis for V.

5.5 Is $\operatorname{span}\{\vec u, \vec v\}$ a basis for V? Why or why not?

5.6 What is the dimension of V?

Let $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 7 \\ 8 \\ 8 \end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

6.1 Give a basis for and the dimension of P.

6.2 Give a basis for and the dimension of Q.

6.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

6.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

7.1 Compute the product $A\vec{x}$.

7.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

7.3 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

7.4 Let $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

8.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

8.2 Can your method be rephrased in terms of a matrix equation? Explain.

Consider the system represented by

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{b}.$$

9.1 If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

9.2 If $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

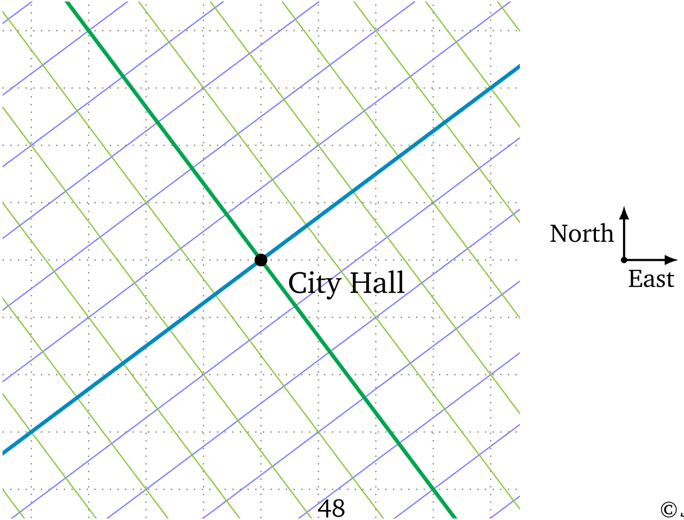
0.1 How many rows does M have?

0.2 Find such an M.

0.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

0.4 Find a matrix K so that non-zero solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

The mythical town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:


City Hall
North
East

Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \dfrac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

1.1 The points $A = (2, 1)$ and $B = (3, -1)$ are given in street coordinates. Find their east-north coordinates.

1.2 The points $X = (4, 3)$ and $Y = (1, 7)$ are given in east-north coordinates. Find their street coordinates.

1.3 Define $\vec{e}_1 = \begin{bmatrix} 1 \text{ east} \\ 0 \text{ north} \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \text{ east} \\ 1 \text{ north} \end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \operatorname{span}\{\vec{d}_1, \vec{d}_2\}$?

1.4 Notice that $Y = 5\vec{d}_1 + 5\vec{d}_2 = \vec{e}_1 + 7\vec{e}_2$. Is the point Y better represented by the pair $(5, 5)$ or by the pair $(1, 7)$? Explain.

Representation in a Basis.
Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for a subspace V and let $\vec{v} \in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

where $\alpha_1, \dots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$.
Conversely,

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}_{\mathcal{B}} = \alpha_1 \vec{b}_1 + \cdots + \alpha_n \vec{b}_n$$

is notation for the linear combination of $\vec{b}_1, \dots, \vec{b}_n$ with coefficients $\alpha_1, \dots, \alpha_n$.

Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ where $\vec{c}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

2.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$.

2.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$.

2.3 Let $\vec{v} = 2\vec{e}_1 + 2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

2.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}} = [\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

2.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}} = [\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

2.6 What is YX?

Orientation of a Basis. The ordered basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is *right-handed* or *positively oriented* if it can be continuously transformed to the standard basis (with $\vec{b}_i \mapsto \vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called *left-handed* or *negatively oriented*.

Let $\{\vec{e}_1, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

3.1 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ a linearly independent set?

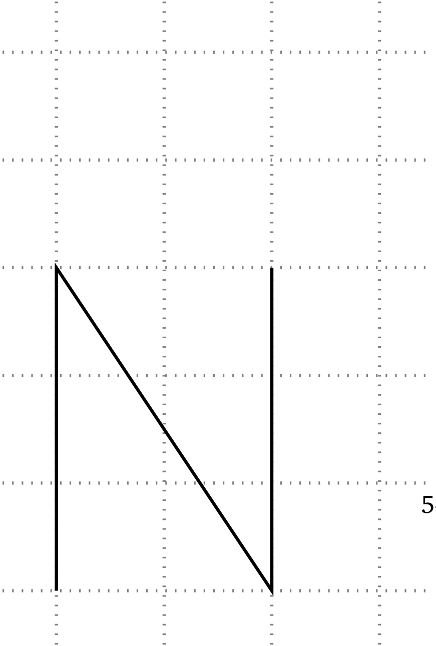
3.2 For which θ can $\{\vec{e}_1, \vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1, \vec{e}_2\}$ and remain linearly independent the whole time?

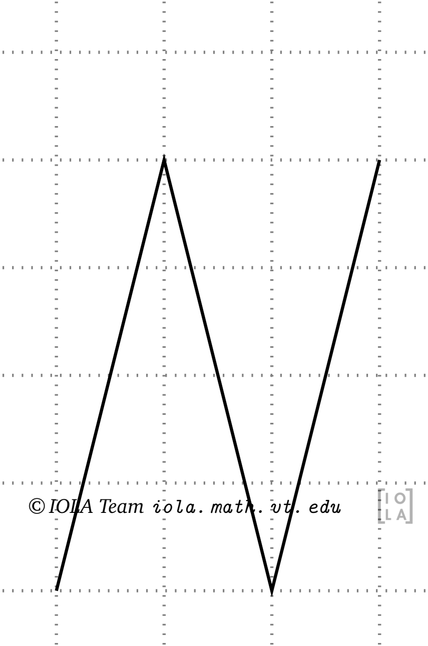
3.3 For which θ is $\{\vec{e}_1, \vec{u}_\theta\}$ right-handed? Left-handed?

3.4 For which θ is $\{\vec{u}_\theta, \vec{e}_1\}$ (in that order) right-handed? Left-handed?

3.5 Is $\{2\vec{e}_1, 3\vec{e}_2\}$ right-handed or left-handed? What about $\{2\vec{e}_1, -3\vec{e}_2\}$?

The citizens of Oronto want to erect a sign welcoming visitors to the city. They've commissioned letters to be built, but at the last council meeting, they decided they wanted italicised letters instead of regular ones. Can you help them?

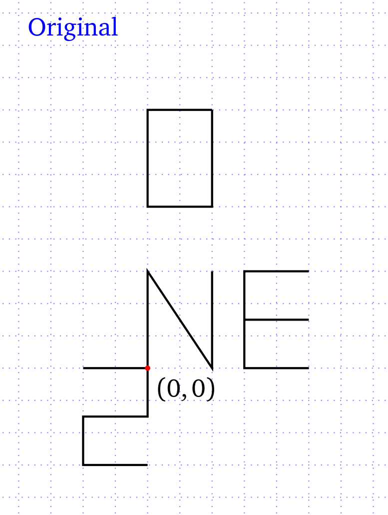


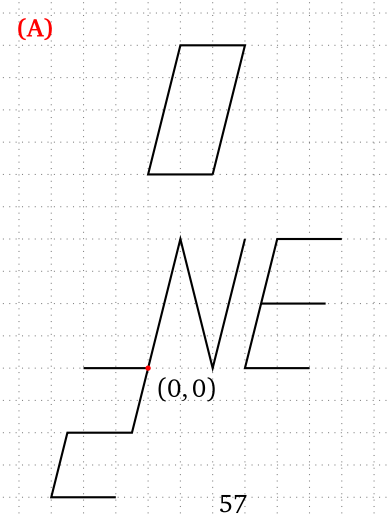


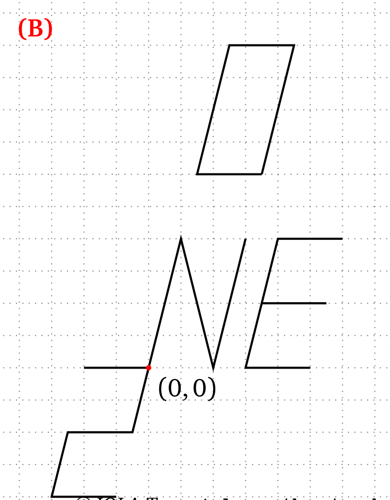
Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

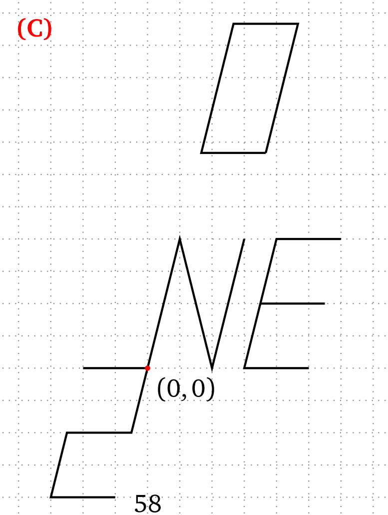
Some council members were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If other letters are placed around the "N," the council members argued over four different possible results for the transformed letters. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?


Original
(0, 0)

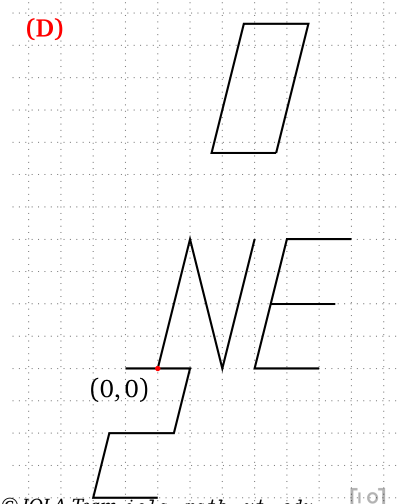

(A)
(0, 0)


(B)
(0, 0)

(C)



(D)



$\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

6.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$.

6.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

6.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

6.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

Linear Transformation. Let V and W be subspaces. A function $T : V \to W$ is called a *linear transformation* if

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \text{and} \qquad T(\alpha\vec{v}) = \alpha T(\vec{v})$$

for all vectors $\vec{u}, \vec{v} \in V$ and all scalars α.

7.1 Classify the following as linear transformations or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

(b) $W : \mathbb{R}^2 \to \mathbb{R}^2$ where $W\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 \\ y \end{bmatrix}$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$.

(d) $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix} x \\ y \end{bmatrix} = \mathrm{vcomp}_{\vec{u}}\begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Image of a Set. Let $L:\mathbb{R}^n\to\mathbb{R}^m$ be a transformation and let $X\subseteq\mathbb{R}^n$ be a set. The *image of the set X under L*, denoted $L(X)$, is the set

$$L(X)=\{\vec y\in\mathbb{R}^m : \vec y=L(\vec x)\text{ for some }\vec x\in X\}.$$

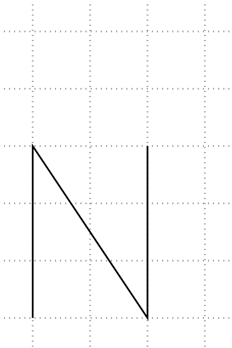
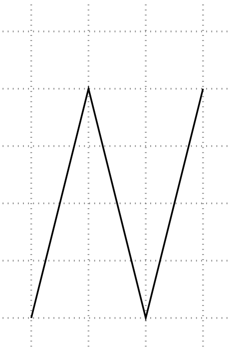
Let $S=\left\{\begin{bmatrix}x\\y\end{bmatrix} : 0\leq x\leq 1\text{ and }0\leq y\leq 1\right\}\subseteq\mathbb{R}^2$ be the filled-in unit square and let $C=\{\vec 0,\vec e_1,\vec e_2,\vec e_1+\vec e_2\}\subseteq\mathbb{R}^2$ be the corners of the unit square.

8.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

8.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).

8.3 Let $\ell=\{\text{all convex combinations of }\vec a\text{ and }\vec b\}$ be a line segment with endpoints $\vec a$ and $\vec b$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

8.4 Explain how images of sets relate to the *Italicizing N* task.

Suppose that the “N” on the left is written in regular 12-point font. Find a matrix A that will transform the “N” into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to italicizing the N as follows:

In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.

1. Do you think Pat and Jamie's approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?

2. Try Pat and Jamie's approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

Define $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ to be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation counter-clockwise by $90°$.

0.1 Find a matrix P so that $P\vec{x} = \mathcal{P}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

0.2 Find a matrix R so that $R\vec{x} = \mathcal{R}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

0.3 Write down matrices A and B for $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$.

0.4 How do the matrices A and B relate to the matrices P and R?

Range. The *range* (or *image*) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\operatorname{range}(T) = \{\vec{y} \in W \ : \ \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

Null Space. The *null space* (or *kernel*) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to the zero vector under T. That is,

$$\operatorname{null}(T) = \{\vec{x} \in V \ : \ T\vec{x} = \vec{0}\}.$$

Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

1.1 What is the range of $\mathcal{P}$?

1.2 What is the null space of $\mathcal{P}$?

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

2.1 Show that the null space of T is a subspace.

2.2 Show that the range of T is a subspace.

Induced Transformation.
Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[\mathcal{T}_M \vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $\vec{v} = \vec{e}_1 + \vec{e}_2 \in \mathbb{R}^2$, and let $\mathcal{T}_M$ be the transformation induced by M.

3.1 What is the difference between "$M\vec{v}$" and "$M[\vec{v}]_{\mathcal{E}}$"?

3.2 What is $[\mathcal{T}_M \vec{e}_1]_{\mathcal{E}}$?

3.3 Can you relate the columns of M to the range of $\mathcal{T}_M$?

Fundamental Subspaces. Associated with any matrix M are three fundamental subspaces: the *row space* of M, denoted $\operatorname{row}(M)$, is the span of the rows of M; the *column space* of M, denoted $\operatorname{col}(M)$, is the span of the columns of M; and the *null space* of M, denoted $\operatorname{null}(M)$, is the set of solutions to $M\vec{x}=\vec{0}$.

Consider $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$.

4.1 Describe the row space of A.

4.2 Describe the column space of A.

4.3 Is the row space of A the same as the column space of A?

4.4 Describe the set of all vectors orthogonal to the rows of A.

4.5 Describe the null space of A.

4.6 Describe the range and null space of T_A, the transformation induced by A.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \mathrm{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

5.1 How does the row space of B relate to the row space of C?

5.2 How does the null space of B relate to the null space of C?

5.3 Compute the null space of B.

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \mathrm{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

6.1 How does the column space of P relate to the column space of Q?

6.2 Describe the column space of P and the column space of Q.

Rank. For a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$, the ***rank*** of T, denoted $\operatorname{rank}(T)$, is the dimension of the range of T. For an $m\times n$ matrix M, the ***rank*** of M, denoted $\operatorname{rank}(M)$, is the dimension of the column space of M.

Let $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ be projection onto $\operatorname{span}\{\vec{u}\}$ where $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$, and let $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ be rotation counter-clockwise by 90°.

7.1 Describe $\operatorname{range}(\mathcal{P})$ and $\operatorname{range}(\mathcal{R})$.

7.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

7.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

7.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

8.1 Determine the rank of (a) $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$ (b) $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ (c) $\begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ (d) $\begin{bmatrix}3\\ 3\\ 2\end{bmatrix}$ (e) $\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

Consider the homogeneous system

$$\begin{array}{rrrl} x & +2y & +z & =0\\ x & +2y & +3z & =0\\ -x & -2y & +z & =0 \end{array} \tag{1}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix}1 & 2 & 1\\ 1 & 2 & 3\\ -1 & -2 & 1\end{bmatrix}$.

9.1 What is rank(A)?

9.2 Give the general solution to system (1).

9.3 Are the column vectors of A linearly independent?

9.4 Give a non-homogeneous system with the same coefficients as (1) that has

(a) infinitely many solutions

(b) no solutions.

0.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

0.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

ank-nullity Theorem

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

1.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

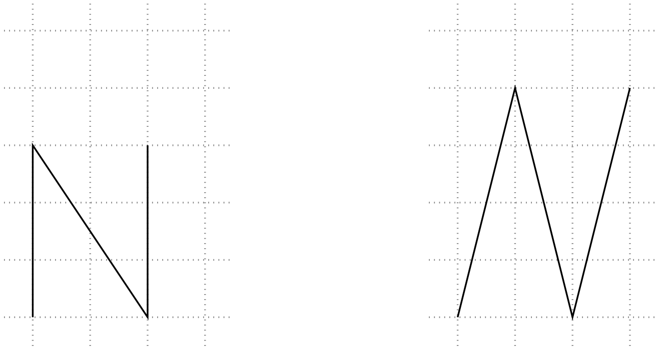
The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

2.1 What is the rank and nullity of A?

2.2 What is the rank and nullity of A^T?

“We’ve made a terrible mistake,” a council member says. “Can we go back to the regular N?”

Recall the original Italicising N task.



Suppose that the “N” on the left is written in regular 12-point font. Find a matrix A that will transform the “N” into the letter on the right which is written in an *italic* 16-point font.

Pat and Jamie explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

The Oronto city council has asked you to *unitalicise* the N. Your new task is to find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

4.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

4.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

Elementary Matrix.
A matrix is called an *elementary matrix* if it is an identity matrix with a single elementary row operation applied.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

4.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

4.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify

4.5 Without computing, what should E_2E_1 be? What about E_1E_2? Now compute and verify.

Matrix Inverse.
The *inverse* of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated A^{-1}.

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -3 & -6 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5.1 Which pairs of matrices above are inverses of each other?

$$B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

6.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

6.2 What is E_2E_1B?

6.3 Find B^{-1}.

6.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad C = [A|\vec{b}] \qquad A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$$

7.1 What is $A^{-1}A$?

7.2 What is rref(A)?

7.3 What is rref(C)? (Hint, there is no need to actually do row reduction!)

7.4 Solve the system $A\vec{x} = \vec{b}$.

8.1 For two square matrices X, Y, should $(XY)^{-1} = X^{-1}Y^{-1}$?

8.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ where $\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{E}}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{E}}$ and let $X = [\vec{b}_1|\vec{b}_2]$ be the matrix whose columns are $[\vec{b}_1]_{\mathcal{E}}$ and $[\vec{b}_2]_{\mathcal{E}}$.

9.1 Write down X.

9.2 Compute $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$.

9.3 Compute $X[\vec{e}_1]_{\mathcal{B}}$ and $X[\vec{e}_2]_{\mathcal{B}}$. What do you notice?

9.4 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$ and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be another basis for $\mathbb{R}^n$. Define the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ to be the matrix whose columns are the $\vec{b}_i$ vectors written in the standard basis. Notice that X converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_{\mathcal{B}} = [\vec{v}]_{\mathcal{E}}.$$

0.1 Should X^{-1} exist? Explain.

0.2 Consider the equation

$$X^{-1}[\vec{v}]_{?} = [\vec{v}]_{?}.$$

Can you fill in the "?" symbols so that the equation makes sense?

0.3 What is $[\vec{b}_1]_{\mathcal{B}}$? How about $[\vec{b}_2]_{\mathcal{B}}$? Can you generalize to $[\vec{b}_i]_{\mathcal{B}}$?

Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

1.1 Compute $[\vec{c}_1]_{\mathcal{C}}$ and $[\vec{c}_2]_{\mathcal{C}}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

1.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$.

1.3 Compute $[T\vec{c}_1]_{\mathcal{C}}$ and $[T\vec{c}_2]_{\mathcal{C}}$.

1.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

1.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

1.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

Similar Matrices. The matrices A and B are called ***similar matrices***, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

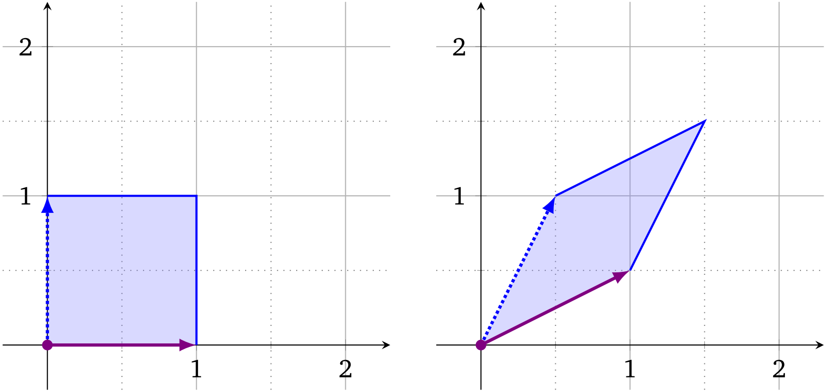
$$A = XBX^{-1}.$$

Unit n-cube. The *unit n-cube* is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \ldots, \alpha_n \in [0,1] \right\} = [0,1]^n.$$

The sides of the unit n-cube are always length 1 and its volume is always 1.

The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



2.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

2.2 Write down a matrix for T.

2.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

Determinant. The *determinant* of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$, denoted $\det(\mathcal{T})$ or $|\mathcal{T}|$, is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

We know the following about the linear transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

3.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.

3.2 Compute the area of $A(C_2)$.

3.3 Compute $\det(A)$.

Suppose R is a rotation counter-clockwise by 30°.

4.1 Draw C_2 and $R(C_2)$.

4.2 Compute the area of $R(C_2)$.

4.3 Compute $\det(R)$.

We know the following about the linear transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \quad \text{and} \quad F\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}.$$

5.1 What is $\det(F)$?

olume Theorem I

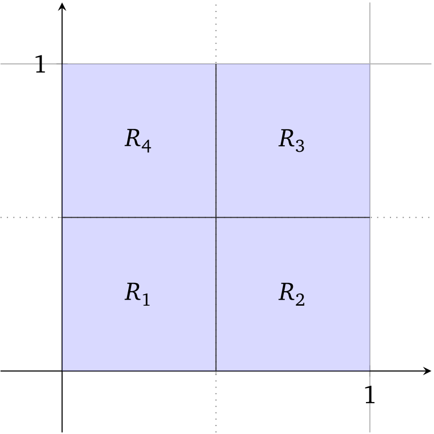
For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors of M.

olume Theorem II

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the row vectors of M.

6.1 Explain Volume Theorem I using the definition of determinant.

6.2 Based on Volume Theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix M?



Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. You know the following about the linear transformations M, T, and S.

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 2$$

$$S : \mathbb{R}^2 \to \mathbb{R}^2 \text{ has determinant } 3$$

7.1 Find the volumes (areas) of R_1, R_2, R_3, R_4, and R.

7.2 Compute the oriented volume of $M(R_1)$, $M(R_2)$, and $M(R)$.

7.3 Do you have enough information to compute the oriented volume of $T(R_2)$? What about the oriented volume of $T(R+\{\vec{e}_2\})$?

7.4 What is the oriented volume of $S \circ T(R)$? What is $\det(S \circ T)$?

- E_f is $I_{3\times 3}$ with the first two rows swapped.
- E_m is $I_{3\times 3}$ with the third row multiplied by 6.
- E_a is $I_{3\times 3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

8.1 What is $\det(E_f)$?

8.2 What is $\det(E_m)$?

8.3 What is $\det(E_a)$?

8.4 What is $\det(E_f E_m)$?

8.5 What is $\det(4I_{3\times 3})$?

8.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

9.1 What is $\det(U)$?

9.2 V is a square matrix and $\operatorname{rref}(V)$ has a row of zeros. What is $\det(V)$?

0.1 V is a square matrix whose columns are linearly dependent. What is $\det(V)$?

0.2 P is projection onto $\operatorname{span}\left\{\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$. What is $\det(P)$?

Suppose you know $\det(X) = 4$.

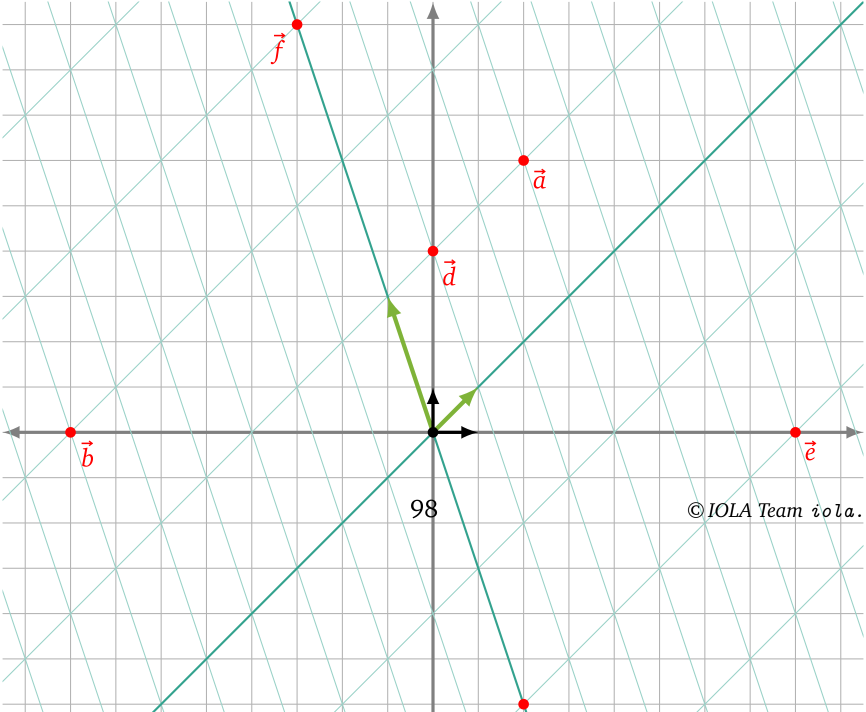
1.1 What is $\det(X^{-1})$?

1.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

1.3 Suppose Y is not invertible. What is $\det(Y)$?

The subway system of Oronto is laid out in a skewed grid. All tracks run parallel to one of the green lines shown. Compass directions are given by the black lines.

While studying the subway map, you decide to pick two bases to help: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$, and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.


$\vec{f}$
$\vec{a}$
$\vec{d}$
$\vec{b}$
$\vec{e}$

1. Write each point above in both the green and the black bases.

2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.

3. The city commission is considering renumbering all the stops along the $y = -3x$ direction. You deduce that the commission's proposal can be modeled by a linear transformation.

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

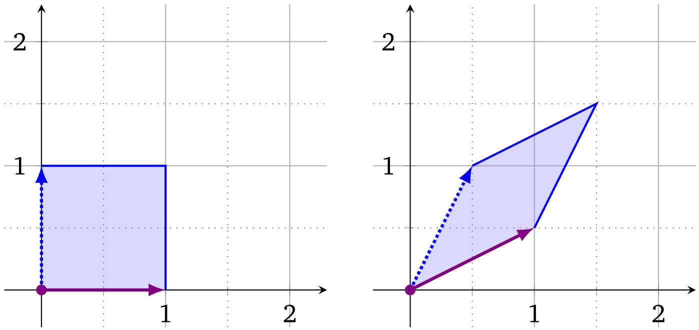
4. When working with the transformation T, which basis do you prefer vectors be represented in? What coordinate system would you propose the city commission use to describe their plans?

Eigenvector. Let X be a linear transformation or a matrix. An *eigenvector* for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the *eigenvalue* of X corresponding to the eigenvector $\vec{v}$.

The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



3.1 Give an eigenvector for T. What is the eigenvalue?

3.2 Can you give another?

For some matrix A,

$$A\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B=A-\frac{2}{3}I.$$

4.1 Give an eigenvector and a corresponding eigenvalue for A.

4.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

4.3 What is the dimension of $\operatorname{null}(B)$?

4.4 What is $\det(B)$?

Let $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ and $E_\lambda = C - \lambda I$.

5.1 For what values of λ does E_λ have a non-trivial null space?

5.2 What are the eigenvalues of C?

5.3 Find the eigenvectors of C.

Characteristic Polynomial.

For a matrix A, the *characteristic polynomial* of A is

$$\operatorname{char}(A) = \det(A - \lambda I).$$

Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

6.1 Compute $\operatorname{char}(D)$.

6.2 Find the eigenvalues of D.

Suppose $\operatorname{char}(E) = -\lambda(2-\lambda)(-3-\lambda)$ for some unknown 3×3 matrix E.

7.1 What are the eigenvalues of E?

7.2 Is E invertible?

7.3 What can you say about nullity(E), nullity($E-3I$), nullity($E+3I$)?

Consider

$$A=\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix}\qquad \vec{v}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}\qquad \vec{v}_2=\begin{bmatrix}1\\1\\-2\end{bmatrix}\qquad \vec{v}_3=\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

and notice that $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

8.1 Find the eigenvalues of T_A.

8.2 Find the characteristic polynomial of T_A.

8.3 Compute $T_A\vec{w}$ where $\vec{w}=2\vec{v}_1-\vec{v}_2$.

8.4 Compute $T_A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

8.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $T_A\vec{x}$ in the $\mathcal{V}$ basis.

Recall from Problem 88 that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Let T_A be the transformation induced by A and let $P = [\vec{v}_1|\vec{v}_2|\vec{v}_3]$ be the matrix with columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ (written in the standard basis).

9.1 Describe in words what P and P^{-1} do in terms of change-of-basis.

9.2 If you were asked to compute $T_A\vec{y}$ for some $\vec{y} \in \mathbb{R}^3$, which basis would you prefer to do your computations in? Explain.

9.3 Given a vector $\vec{y} \in \mathbb{R}^3$ written in the standard basis, is there a way to compute $T_A\vec{y}$ without using the matrix A? (You may use P and P^{-1}, just not A.) Explain.

9.4 Can you find a matrix D so that

$$PDP^{-1} = A?$$

9.5 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. Compute $T_A^{100}\vec{x}$. Express your answer in both the $\mathcal{V}$ basis and the standard basis.

Diagonalizable. A matrix is *diagonalizable* if it is similar to a diagonal matrix.

Let B be an $n \times n$ matrix and let T_B be the induced transformation. Suppose T_B has eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ which form a basis for $\mathbb{R}^n$, and let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues.

0.1 How do the eigenvalues and eigenvectors of B and T_B relate?

0.2 Is B diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

0.3 What if one of the eigenvalues of T_B is zero? Would B be diagonalizable?

0.4 What if the eigenvectors of T_B did not form a basis for $\mathbb{R}^n$. Would B be diagonalizable?

Eigenspace. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$. The *eigenspace* of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.
The *geometric multiplicity* of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The *algebraic multiplicity* of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Let $F = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $G = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

1.1 Is F diagonalizable? Why or why not?

1.2 Is G diagonalizable? Why or why not?

1.3 What are the geometric and algebraic multiplicities of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

1.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?