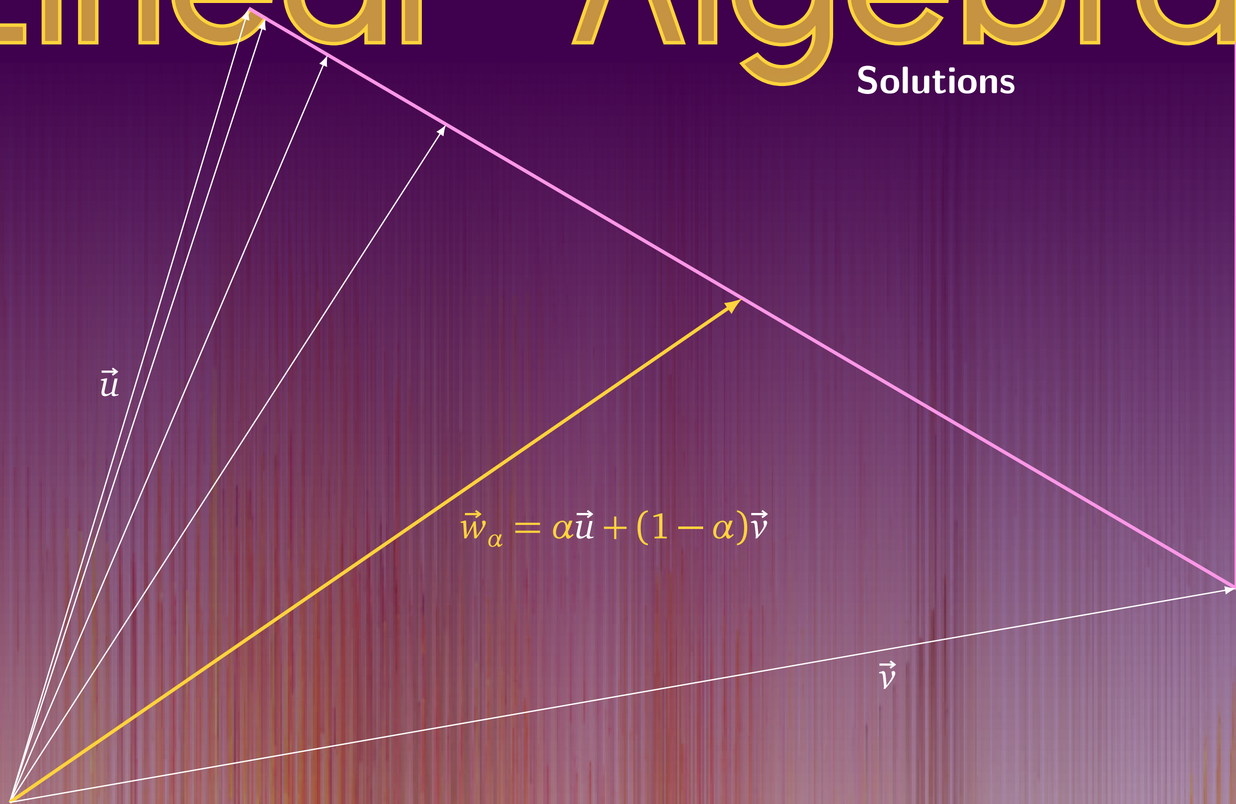
Linear Algebra

Solutions



Jason Siefken

The Magic Carpet Ride

1 You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

Hands-on experience with vectors as displacements.

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Use pre-existing knowledge of algebra to answer vector questions.

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 km East and 1 km North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 km East and 2 km North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 km East and 64 km North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

The Magic Carpet Ride, Hide and Seek

2 You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 km East and 1 km North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 km East and 2 km North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Address an existential question involving vectors: "Is it possible to find a linear combination that does. . . ?"

The goal of this problem is to

- Formalize geometric questions using the language of vectors.
- Find both geometric and algebraic arguments to support the same conclusion.
- Establish what a "negative multiple" of a vector should be.

Sets and Set Notation

DEFINITION

Set

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X : \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "$|$" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

3

Practice reading sets and set-builder notation.

The goal of this problem is to
- Become familiar with $\in$, $\subseteq$, and $=$ in the context of sets.
- Distinguish between $\in$ and $\subseteq$.
- Use quantifiers with sets.

3.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$. True

(b) $1.5 \in \{1,2,3\}$. False

(c) $4 \in \{1,2,3\}$. False

(d) "b"$\in \{x : x \text{ is an English letter}\}$. True

(e) "ò"$\in \{x : x \text{ is an English letter}\}$. False

(f) $\{1,2\} \subseteq \{1,2,3\}$. True

(g) For some $a \in \{1,2,3\}$, $a \geq 3$. True

(h) For any $a \in \{1,2,3\}$, $a \geq 3$. False

(i) $1 \subseteq \{1,2,3\}$. False

(j) $\{1,2,3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. False

(k) $\{1,2,3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$. True

---

4

Practice writing sets using set-builder notation.

The goal of this problem is to
- Express English descriptions using math notation.
- Recognize there is more than one correct way to write formal math.
- Preview vector form of a line.

Write the following in set-builder notation

4.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

$$\{x \in \mathbb{R} : x > \sqrt{2}\}.$$

4.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ with } a = 2b\right\} \text{ or } \left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} 2t \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : a = 2b\right\}.$$

Unions & Intersections (DEFINITION)

Let X and Y be sets. The ***union*** of X and Y and the ***intersection*** of X and Y are defined as follows.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

Apply the definition of ∪ and ∩.

5 Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

5.1 $X \cup Y$ $\{1,2,3,4,5\}$

5.2 $X \cap Y$ $\{2,3\}$

5.3 $X \cup Y \cup Z$ $\{1,2,3,4,5,6\}$

5.4 $X \cap Y \cap Z$ $\emptyset = \{\}$

Visualize sets of vectors.

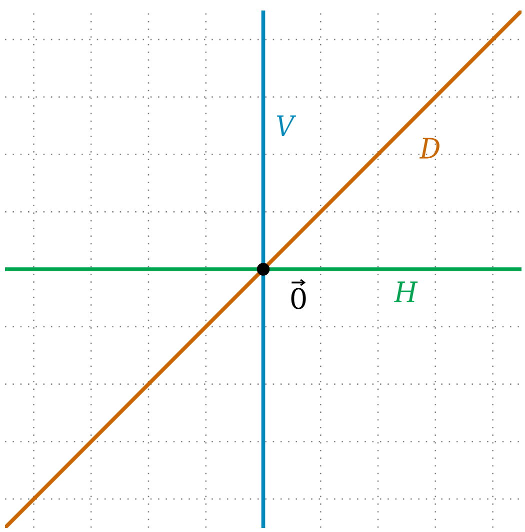
The goal of this problem is to
- Apply set-builder notation in the context of vectors.
- Distinguish between "for all" and "for some" in set builder notation.
- Practice unions and intersections.
- Practice thinking about set equality.

6 Draw the following subsets of $\mathbb{R}^2$.

6.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

6.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

6.3 $D = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.



6.4 $N = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R}\right\}$. $N = \{\}$.

6.5 $V \cup H$. $V \cup H$ looks like a "+" going through the origin.

6.6 $V \cap H$. $V \cap H = \{\vec{0}\}$ is just the origin.

6.7 Does $V \cup H = \mathbb{R}^2$?

No. $V \cup H$ does not contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ while $\mathbb{R}^2$ does contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Vector Combinations

Linear Combination (DEFINITION)

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

Practice linear combinations.

The goal of this problem is to
- Practice using the formal term *linear combination*.
- Foreshadow span.

7 Let $\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{w}=2\vec{v}_1+\vec{v}_2$.

7.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

$\vec{w}=\begin{bmatrix}3\\1\end{bmatrix}$; the coefficients are $(2,1)$.

7.2 Is $\begin{bmatrix}3\\3\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}3\\3\end{bmatrix}=3\vec{v}_1+0\vec{v}_2$.

7.3 Is $\begin{bmatrix}0\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\vec{0}=0\vec{v}_1+0\vec{v}_2$.

7.4 Is $\begin{bmatrix}4\\0\end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix}4\\0\end{bmatrix}=2\vec{v}_1+2\vec{v}_2$.

7.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

No. $\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{2}\vec{v}_1+\frac{1}{2}\vec{v}_2$ and $\begin{bmatrix}0\\1\end{bmatrix}=\frac{1}{2}\vec{v}_1-\frac{1}{2}\vec{v}_2$. Therefore

$$\begin{bmatrix}a\\b\end{bmatrix}=a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}=a(\tfrac{1}{2}\vec{v}_1+\tfrac{1}{2}\vec{v}_2)+b(\tfrac{1}{2}\vec{v}_1-\tfrac{1}{2}\vec{v}_2)=(\tfrac{a+b}{2})\vec{v}_1+(\tfrac{a-b}{2})\vec{v}_2.$$

Therefore any vector in $\mathbb{R}^2$ can be written as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

7.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Yes. All linear combinations of $\vec{v}_1$ have equal x and y coordinates, therefore $\vec{w}=\begin{bmatrix}2\\1\end{bmatrix}$ is not a linear combination of $\vec{v}_1$.

Practice formal writing.

8 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h}=\begin{bmatrix}3\\1\end{bmatrix}$ and the magic carpet could move in the direction $\vec{m}=\begin{bmatrix}1\\2\end{bmatrix}$.

8.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

Gauss's location can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

8.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

8.3 Rephrase the sentence "*$\mathbb{R}^2$ is the set of all linear combinations of $\vec{h}$ and $\vec{m}$*" using formal mathematical language.

$\mathbb{R}^2=\{\vec{v} : \vec{v}=t\vec{m}+s\vec{h} \text{ for some } t,s\in\mathbb{R}\}$.

Non-negative & Convex Linear Combinations

DEFINITION

Let $\vec{w}=\alpha_1\vec{v}_1+\alpha_2\vec{v}_2+\cdots+\alpha_n\vec{v}_n$. The vector $\vec{w}$ is called a ***non-negative*** linear combination of $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n$ if

$$\alpha_1,\alpha_2,\ldots,\alpha_n\geq 0.$$

The vector $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n$ if

$$\alpha_1,\alpha_2,\ldots,\alpha_n\geq 0 \qquad\text{and}\qquad \alpha_1+\alpha_2+\cdots+\alpha_n=1.$$

Geometric meaning of *non-negative* and *convex* linear combinations.

The goal of this problem is to
- Read and apply the definition of non-negative and convex linear combinations.
- Gain geometric intuition for non-negative and convex linear combinations.
- Learn how to describe line segments using convex linear combinations.

9 Let

$$\vec{a}=\begin{bmatrix}1\\1\end{bmatrix}\qquad \vec{b}=\begin{bmatrix}-1\\1\end{bmatrix}\qquad \vec{c}=\begin{bmatrix}0\\1\end{bmatrix}\qquad \vec{d}=\begin{bmatrix}0\\2\end{bmatrix}\qquad \vec{e}=\begin{bmatrix}-1\\-1\end{bmatrix}.$$

9.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$? All of them, since any vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$.

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$.

9.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

Let $\vec{u}=\vec{a}$ and $\vec{v}=\vec{c}$.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

Let $\vec{u}=\vec{a}$ and $\vec{v}=\vec{e}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

Impossible. If $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$, then every non-negative linear combination of $\vec{a}$ and $\vec{b}$ is also a non-negative linear combination of $\vec{u}$ and $\vec{v}$. And, we already concluded that $\vec{d}$ is a non-negative linear combination of $\vec{a}$ and $\vec{b}$.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Impossible. Convex linear combinations all lie on the same line segment, but $\vec{a}$, $\vec{c}$, and $\vec{d}$ are not collinear.

Otherwise, explain why it's not possible.

Lines and Planes

Link prior knowledge to new notation/concepts.

The goal of this problem is to

- Convert between $y=mx+b$ form of a line and the set-builder definition of the same line.
- Think about lines in terms of vectors rather than equations.

10 Let L be the set of points $(x,y)\in\mathbb{R}^2$ such that $y=2x+1$.

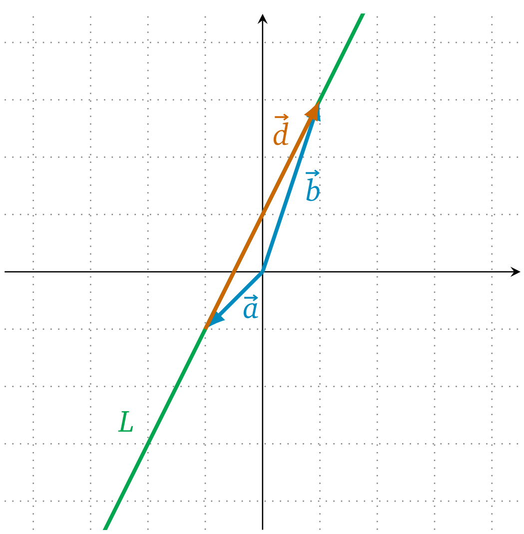
10.1 Describe L using set-builder notation.

$$\left\{\vec{v}\in\mathbb{R}^2 : \vec{v}=\begin{bmatrix}t\\2t+1\end{bmatrix} \text{ for some } t\in\mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2 : y=2x+1\right\} \text{ or } \left\{\begin{bmatrix}t\\2t+1\end{bmatrix}\in\mathbb{R}^2 : t\in\mathbb{R}\right\}$$

10.2 Draw L as a subset of $\mathbb{R}^2$.

10.3 Add the vectors $\vec{a}=\begin{bmatrix}-1\\-1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\3\end{bmatrix}$ and $\vec{d}=\vec{b}-\vec{a}$ to your drawing.



10.4 Is $\vec{d}\in L$? Explain.

No. $\vec{d}=\begin{bmatrix}2\\4\end{bmatrix}$ and so its entries don't satisfy $y=2x+1$.

10.5 For which $t\in\mathbb{R}$ is it true that $\vec{a}+t\vec{d}\in L$? Explain using your picture.

$\vec{a} + t\vec{d} \in L$ for any $t \in \mathbb{R}$. We can see this because if we start at the vector $\vec{a}$ and the displace by $t\vec{d}$, we will always be on the line L.

Vector Form of a Line

DEFINITION

Let ℓ be a line and let $\vec{d}$ and $\vec{p}$ be vectors. If $\ell = \{\vec{x} : \vec{x} = t\vec{d} + \vec{p}$ for some $t \in \mathbb{R}\}$, we say the vector equation

$$\vec{x} = t\vec{d} + \vec{p}$$

is ℓ expressed in ***vector form***. The vector $\vec{d}$ is called a ***direction vector*** for ℓ.

---

11 Let $\ell \subseteq \mathbb{R}^2$ be the line with equation $2x + y = 3$, and let $L \subseteq \mathbb{R}^3$ be the line with equations $2x + y = 3$ and $z = y$.

Practice with vector form.

The goal of this problem is to

- Express lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ in vector form.
- Produce direction vectors by subtracting two points on a line.
- Recognize vector form is not unique.

11.1 Write ℓ in vector form. Is vector form of ℓ unique?

$$\vec{x} = t\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$$

The vector form is not unique, as any non-zero scalar multiple of $\begin{bmatrix}1\\-2\end{bmatrix}$ can serve as a direction vector. Additionally, any other point on the line can be used in place of $\begin{bmatrix}0\\3\end{bmatrix}$. For example, $\vec{x} = t\begin{bmatrix}-4\\8\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}$ is another vector form of ℓ.

11.2 Write L in vector form. $\vec{x} = t\begin{bmatrix}1\\-2\\-2\end{bmatrix} + \begin{bmatrix}0\\3\\3\end{bmatrix}$. This is obtained by finding two points: one when $x = 0$ and one when $x = 1$ and subtracting them to find a direction vector for L.

11.3 Find another vector form for L where both "$\vec{d}$" and "$\vec{p}$" are different from before.

$$\vec{x} = t\begin{bmatrix}-3\\6\\6\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Again, any non-zero scalar multiple of the direction vector will work for $\vec{d}$, as will any other point on the line work for $\vec{p}$.

---

12 Let A, B, and C be given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B} \qquad \overbrace{\vec{x} = t\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}}^{C}.$$

Intersect lines in vector form.

The goal of this problem is to

- Practice computing the intersection between lines in vector form.
- Recognize "t" as a dummy variable as used in vector form and that, when comparing lines in vector form, "t" needs to be replaced with non-dummy variables.

12.1 Do the lines A and B intersect? Justify your conclusion.

Yes. $(0)\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} = (-1)\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}$.

To find the intersection, if there is one, we must solve the vector equation:

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

One solution is when $t = 0$ and $s = -1$.

12.2 Do the lines A and C intersect? Justify your conclusion.

No. The vector equation

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = s\begin{bmatrix}2\\-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix}$$

has no solutions. This is equivalent to saying that the following system of equations has no solutions:

$$\begin{aligned} t &= 2s+1 \\ 2t &= -s+1 \\ 3t+1 &= s+1 \end{aligned}$$

The third equation tells us that $s = 3t$, which when substituted into the first equation forces $t = -\frac{1}{5}$ and therefore $s = -\frac{3}{5}$. However, these two numbers don't satisfy the second equation.

12.3 Let $\vec{p} \neq \vec{q}$ and suppose X has vector form $\vec{x} = t\vec{d} + \vec{p}$ and Y has vector form $\vec{x} = t\vec{d} + \vec{q}$. Is it possible that X and Y intersect?

Yes. If $\vec{q} = \vec{p} + a\vec{d}$ for $a \neq 0$, then X and Y will actually be the same line, since in this case

$$\vec{x} = t\vec{d} + \vec{q} = t\vec{d} + (\vec{p} + a\vec{d}) = (t+a)\vec{d} + \vec{p}.$$

For example, the following two vector equations represent the same line.

$$\vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad \text{and} \qquad \vec{x} = t\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}7\\7\\7\end{bmatrix}.$$

DEFINITION

Vector Form of a Plane

A plane $\mathcal{P}$ is written in ***vector form*** if it is expressed as

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}$$

for some vectors $\vec{d}_1$ and $\vec{d}_2$ and point $\vec{p}$. That is, $\mathcal{P} = \{\vec{x} : \vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} \text{ for some } t,s \in \mathbb{R}\}$. The vectors $\vec{d}_1$ and $\vec{d}_2$ are called ***direction vectors*** for $\mathcal{P}$.

13 Recall the intersecting lines A and B given in vector form by

$$\overbrace{\vec{x} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}}^{A} \qquad \overbrace{\vec{x} = t\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}}^{B}.$$

Let $\mathcal{P}$ the plane that contains the lines A and B.

Apply vector form of a plane.

The goal of this problem is to
- Use direction vectors for lines given in vector form.
- Think about planes in terms of vectors rather than equations.
- Combine direction vectors in a plane to produce new direction vectors.

13.1 Find two direction vectors for $\mathcal{P}$.

Two possible answers are:

$$\vec{d}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \text{and} \qquad \vec{d}_2 = \begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

These are the two direction vectors we already know are in the plane—the ones from the two lines:

Note that neither of these is a multiple of the other, so they really are two unique direction vectors in $\mathcal{P}$.

13.2 Write $\mathcal{P}$ in vector form.

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p} = t\begin{bmatrix}1\\2\\3\end{bmatrix} + s\begin{bmatrix}-1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

We already have two direction vectors, so we just needed a point on the plane.

We used the point $\vec{p} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ that we already know is on line A.

13.3 Describe how vector form of a plane relates to linear combinations.

The vector form of a plane says that a vector $\vec{x}$ is on the plane exactly when it is equal to some linear combination of $\vec{d}_1$ and $\vec{d}_2$, plus $\vec{p}$.

Another way of saying the same thing is that the vector $\vec{x}$ is on the plane exactly when $\vec{x}-\vec{p}$ is equal to some linear combination of $\vec{d}_1$ and $\vec{d}_2$.

13.4 Write $\mathcal{P}$ in vector form using different direction vectors and a different point.

One possible answer:

$$\vec{x} = t\begin{bmatrix}-1\\-2\\-3\end{bmatrix} + s\begin{bmatrix}-7\\7\\7\end{bmatrix} + \begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

As with the equations of lines from before, we can use any non-zero scalar multiple of either direction vector and get the same plane. We also used the point $\vec{q} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$ that we already knew is on line B.

Connect vector form and scalar form of a plane.

The goal of this problem is to

- Produce direction vectors for a plane defined by an equation.
- Generalize the procedure for finding direction vectors that was used for lines.

14 Let $\mathcal{Q} \subseteq \mathbb{R}^3$ be a plane with equation $x + y + z = 1$.

14.1 Find three points in $\mathcal{Q}$.

There are many choices here, of course. Three natural ones are:

$$\vec{p}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad \vec{p}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \vec{p}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

14.2 Find two direction vectors for $\mathcal{Q}$.

Now that we have three points on the plane, we can use the direction vectors joining any two pairs of them. For example:

$$\vec{d}_1 = \vec{p}_1 - \vec{p}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{d}_2 = \vec{p}_1 - \vec{p}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

14.3 Write $\mathcal{Q}$ in vector form.

Using the point $\vec{p}_1$ from above, one possible answer is:

$$\vec{x} = t\vec{d}_1 + s\vec{d}_2 + \vec{p}_1 = t\begin{bmatrix}1\\-1\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\-1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Span

Span (DEFINITION)

The ***span*** of a set of vectors V is the set of all linear combinations of vectors in V. That is,

$$\operatorname{span} V = \{\vec{v} : \vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n \text{ for some } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in V \text{ and scalars } \alpha_1, \alpha_2, \ldots, \alpha_n\}.$$

Additionally, we define $\operatorname{span}\{\} = \{\vec{0}\}$.

Apply the definition of *span*.

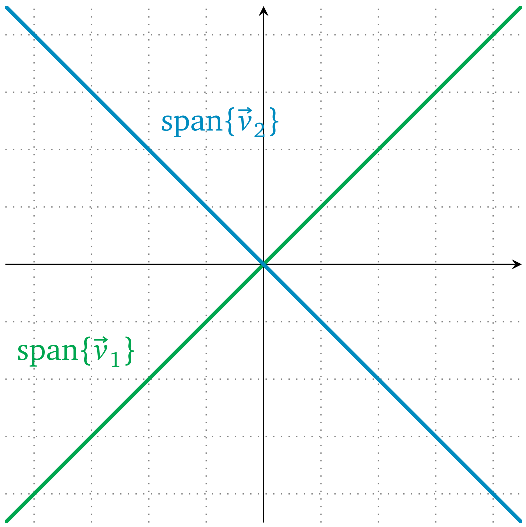
The goal of this problem is to

- Practice applying a new definition in a familiar context ($\mathbb{R}^2$).
- Recognize spans as lines and planes through the origin.

15 Let $\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix}2\\2\end{bmatrix}$.

15.1 Draw $\operatorname{span}\{\vec{v}_1\}$.

15.2 Draw $\operatorname{span}\{\vec{v}_2\}$.



15.3 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$.

$\operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$

We can see this since for any $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$,

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{x}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right) + \frac{y}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{x+y}{2}\vec{v}_1 + \frac{x-y}{2}\vec{v}_2$$

15.4 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_3\}$. $\operatorname{span}\{\vec{v}_1, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1\}$, a line through the origin with direction vector $\vec{v}_1$.

15.5 Describe $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \operatorname{span}\{\vec{v}_1, \vec{v}_2\} = \mathbb{R}^2$

16 Let $\ell_1 \subseteq \mathbb{R}^2$ be the line with equation $x - y = 0$ and $\ell_2 \subseteq \mathbb{R}^2$ the line with equation $x - y = 4$.

16.1 If possible, describe ℓ_1 as a span. Otherwise explain why it's not possible.

$\ell_1 = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$, since $\begin{bmatrix}x\\y\end{bmatrix} \in \ell_1$ if and only if $x = y$, which in turn is true if and only if it is a scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$.

16.2 If possible, describe ℓ_2 as a span. Otherwise explain why it's not possible.

This is not possible. $\begin{bmatrix}0\\0\end{bmatrix}$ is an element of the span of *any* set of vectors, since we can use all zeroes as the scalars in a linear combination, but $\begin{bmatrix}0\\0\end{bmatrix} \notin \ell_2$.

16.3 Does the expression $\operatorname{span}(\ell_1)$ make sense? If so, what is it? How about $\operatorname{span}(\ell_2)$?

Both of these expressions do make sense. One can compute the span of any set of vectors, and these lines are just special set of points in $\mathbb{R}^2$ which we are already used to thinking of as vectors.

$\operatorname{span}(\ell_1) = \ell_1$, since all of the vectors on the line ℓ_1 are already multiples of $\begin{bmatrix}1\\1\end{bmatrix}$, as we discovered earlier.

$\operatorname{span}(\ell_2)$ equals all of $\mathbb{R}^2$. It's easy to see that the vectors $v = \begin{bmatrix}4\\0\end{bmatrix}$ and $w = \begin{bmatrix}0\\-4\end{bmatrix}$ are both on ℓ_2, and the span of these two vectors alone is all of $\mathbb{R}^2$.

Connect geometric figures to spans.

The goal of this problem is to

- Identify a relationship between lines and spans.
- Describe a line through the origin as a span.
- Identify when a line cannot be described as a span.
- Apply the definition of $\operatorname{span} X$ even when X is infinite.

Set Addition

DEF

If A and B are sets of vectors, then the ***set sum*** of A and B, denoted $A+B$, is

$$A+B=\{\vec{x} \,:\, \vec{x}=\vec{a}+\vec{b} \text{ for some } \vec{a}\in A \text{ and } \vec{b}\in B\}.$$

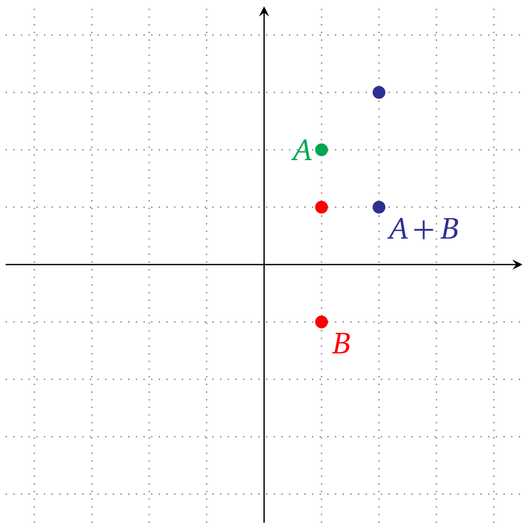
Describing geometry using sets.

The goal of this problem is to

- Practice applying a new definition in a familiar context ($\mathbb{R}^2$).
- Gain an intuitive understanding of set addition.
- Describe lines that don't pass through $\vec{0}$ using a combination of set addition and spans.

17 Let $A=\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$, $B=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}$, and $\ell=\operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

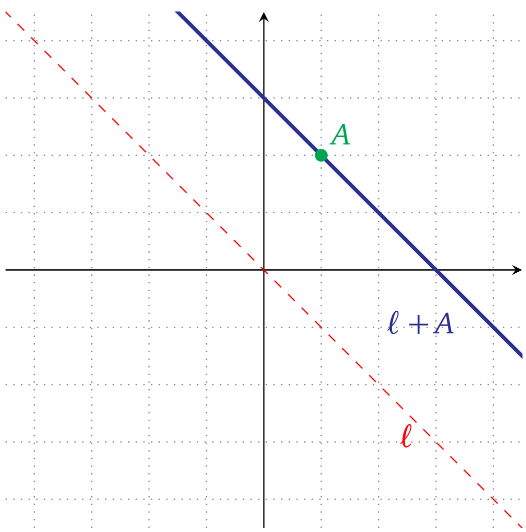
17.1 Draw A, B, and $A+B$ in the same picture.



17.2 Is $A+B$ the same as $B+A$?

Yes. Since A and B are such small sets we could just compute all the vectors in $A+B$ and $B+A$ and see that they're equal. However, we know that real numbers can be added up in any order, and the coordinates of an element of $A+B$ or $B+A$ are simply sums of the corresponding coordinates of elements of A and B.

17.3 Draw $\ell+A$.



17.4 Consider the line ℓ_2 given in vector form by $\vec{x}=t\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\2\end{bmatrix}$. Can ℓ_2 be described using only a span? What about using a span and set addition?

ℓ_2 cannot be described using only a span, for the same reason as the line ℓ_2 in Problem 16.2 couldn't be. We know that the origin $\begin{bmatrix}0\\0\end{bmatrix}$ must be an element of any span, but it is not a point on ℓ_2.

ℓ_2 can be described as a span plus a set addition though. Specifically, $\ell_2=\ell+A$.

18 Suppose you are now in a three-dimensional world for the carpet ride problem, and you have three modes of transportation:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

You are only allowed to use each mode of transportation **once** (in the forward or backward direction) for a fixed amount of time (c_1 on $\vec{v}_1$, c_2 on $\vec{v}_2$, c_3 on $\vec{v}_3$).

1. Find the amounts of time on each mode of transportation (c_1, c_2, and c_3, respectively) needed to go on a journey that starts and ends at home *or* explain why it is not possible to do so.
2. Is there more than one way to make a journey that meets the requirements described above? (In other words, are there different combinations of times you can spend on the modes of transportation so that you can get back home?) If so, how?
3. Is there anywhere in this 3D world that Gauss could hide from you? If so, where? If not, why not?
4. What is $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}\right\}$?

Span in higher dimensions.

The goal of this problem is to

- Examine subtleties that exist in three dimensions that are missing in two dimensions.
- Apply linear algebra tools to answer open-ended questions.

Linearly Dependent & Independent (Geometric)

DEFINITION

We say the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if for at least one i,

$$\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n\}.$$

Otherwise, they are called ***linearly independent***.

Geometric definition of linear independence/dependence.

19 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

Apply the (geometric) definition of linear independence/dependence.

The goal of this problem is to
- Develop a mental picture linking linear dependence and "redundant" vectors.
- Practice applying a new definition.
- Find multiple linearly independent subsets of a linearly dependent set.

19.1 Describe $\operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

The xy-plane in $\mathbb{R}^3$. That is, the set of all vectors in $\mathbb{R}^3$ with z-coordinate equal to zero.

19.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ linearly independent? Why or why not?

No. $\vec{w} = \vec{u} + \vec{v}$, and so $\vec{w} \in \operatorname{span}\{\vec{u}, \vec{v}\}$.

Let $X = \{\vec{u}, \vec{v}, \vec{w}\}$.

19.3 Give a subset $Y \subseteq X$ so that $\operatorname{span} Y = \operatorname{span} X$ and Y is linearly independent.

$Y = \{\vec{u}, \vec{v}\}$ is one example that works.

19.4 Give a subset $Z \subseteq X$ so that $\operatorname{span} Z = \operatorname{span} X$ and Z is linearly independent and $Z \neq Y$.

$Z = \{\vec{u}, \vec{w}\}$ and $Z = \{\vec{v}, \vec{w}\}$ both have the same span as Y above.

Trivial Linear Combination

DEF

The linear combination $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is called ***trivial*** if $\alpha_1 = \cdots = \alpha_n = 0$. If at least one $\alpha_i \neq 0$, the linear combination is called ***non-trivial***.

Link trivial/non-trivial linear combinations to linear independence/dependence.

20 Recall $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$.

20.1 Consider the linearly dependent set $\{\vec{u}, \vec{v}, \vec{w}\}$ (where $\vec{u}, \vec{v}, \vec{w}$ are defined as above). Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set? $\vec{0} = \vec{u} + \vec{v} - \vec{w}$.

20.2 Consider the linearly independent set $\{\vec{u}, \vec{v}\}$. Can you write $\vec{0}$ as a non-trivial linear combination of vectors in this set?

No. Suppose

$$a_1\vec{u} + a_2\vec{v} = \vec{0}$$

was a non-trivial linear combination. Then at least one of a_1 or a_2 is non-zero. If a_1 is non-zero, then

$$\vec{u} = -\frac{a_2}{a_1}\vec{v}$$

and so $\vec{u} \in \operatorname{span}\{\vec{v}\}$. If a_2 is non-zero, then

$$\vec{v} = -\frac{a_1}{a_2}\vec{u}.$$

and so $\vec{v} \in \operatorname{span}\{\vec{u}\}$. In either case, $\{\vec{u}, \vec{v}\}$ would be linearly dependent.

We now have an equivalent definition of linear dependence.

Link algebraic and geometric definitions of linear independence/dependence.

The goal of this problem is to
- Understand how the algebraic and geometric definitions of linear independence/dependence relate.
- Practice writing mathematical arguments.

Linearly Dependent & Independent (Algebraic)

DEF

The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are ***linearly dependent*** if there is a non-trivial linear combination of $\vec{v}_1, \dots, \vec{v}_n$ that equals the zero vector. Otherwise they are linearly independent.

21

21.1 Explain how the geometric definition of linear dependence (original) implies this algebraic one (new).

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent according to the geometric definition. Fix i so that $\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}$.

By the definition of span, we know that

$$\vec{v}_i = \beta_1\vec{v}_1 + \cdots + \beta_{i-1}\vec{v}_{i-1} + \beta_{i+1}\vec{v}_{i+1} + \cdots + \beta_n\vec{v}_n.$$

Thus

$$\vec{0} = -\vec{v}_i + \beta_1\vec{v}_1 + \cdots + \beta_{i-1}\vec{v}_{i-1} + \beta_{i+1}\vec{v}_{i+1} + \cdots + \beta_n\vec{v}_n,$$

and this is a non-trivial linear combination since the coefficient of $\vec{v}_i$ is $-1 \neq 0$.

21.2 Explain how this algebraic definition of linear dependence (new) implies the geometric one (original).

Suppose the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is linearly dependent in this new sense. That means there are scalars $a_1, a_2, \ldots, a_n$, at least one of which is non-zero, such that

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = 0.$$

Suppose $a_i \neq 0$. Then

$$\vec{v}_i = \tfrac{-a_1}{a_i}\vec{v}_1 + \cdots + \tfrac{-a_{i-1}}{a_i}\vec{v}_{i-1} + \tfrac{-a_{i+1}}{a_i}\vec{v}_{i+1} + \cdots + \tfrac{-a_n}{a_i}\vec{v}_n.$$

This means $\vec{v}_i \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n\}$, which is precisely the geometric definition of linear dependence.

Since we have geometric def $\implies$ algebraic def, and algebraic def $\implies$ geometric def ($\implies$ should be read aloud as 'implies'), the two definitions are *equivalent* (which we write as algebraic def $\iff$ geometric def).

22

Linear dependence and infinite solutions.

The goal of this problem is to

- Connect linear dependence with infinite solutions.
- Connect linear independence with unique solutions.

Suppose for some unknown $\vec{u}, \vec{v}, \vec{w}$, and $\vec{a}$,

$$\vec{a} = 3\vec{u} + 2\vec{v} + \vec{w} \qquad \text{and} \qquad \vec{a} = 2\vec{u} + \vec{v} - \vec{w}.$$

22.1 Could the set $\{\vec{u}, \vec{v}, \vec{w}\}$ be linearly independent?

No. If both equations are true, they would combine to show

$$3\vec{u} + 2\vec{v} + \vec{w} = 2\vec{u} + \vec{v} - \vec{w}.$$

Collecting all the terms on the left side, we get:

$$\vec{u} + \vec{v} + 2\vec{w} = \vec{0},$$

which is a non-trivial linear combination of vectors in the given set equalling the zero vector.

Suppose that

$$\vec{a} = \vec{u} + 6\vec{r} - \vec{s}$$

is the *only* way to write $\vec{a}$ using $\vec{u}, \vec{r}, \vec{s}$.

22.2 Is $\{\vec{u}, \vec{r}, \vec{s}\}$ linearly independent?

Yes. If it were not, there would exist scalars a_1, a_2, a_3, not all of which are zero, such that:

$$a_1\vec{u} + a_2\vec{r} + a_3\vec{s} = \vec{0}.$$

But then

$$\vec{u} + 6\vec{r} - \vec{s} + (a_1\vec{u} + a_2\vec{r} + a_3\vec{s})$$

would be another way to write $\vec{a}$ using only the same three vectors.

22.3 Is $\{\vec{u}, \vec{r}\}$ linearly independent?

Yes. If it were not, we would necessarily have $\vec{u} = \beta\vec{r}$ for some scalar β. But then

$$(\beta+6)\vec{r} - \vec{s}$$

would be another way to write $\vec{a}$ using only the same three vectors.

22.4 Is $\{\vec{u}, \vec{v}, \vec{w}, \vec{r}\}$ linearly independent?

No. We know from earlier that $\vec{u} + \vec{v} + 2\vec{w} = \vec{0}$, and so $\vec{u} + \vec{v} + 2\vec{w} + 0\vec{r} = \vec{0}$ is a non-trivial linear combination of the vectors in this set that equals the zero vector.

Linear Independence and Dependence, Creating Examples

23

1. Fill in the following chart keeping track of the strategies you used to generate examples.

	Linearly independent	Linearly dependent
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

2. Write at least two generalizations that can be made from these examples and the strategies you used to create them.

Dot Product

Norm

DEFINITION

The ***norm*** of a vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the length/magnitude of $\vec{v}$. It is written $\|\vec{v}\|$ and can be computed from the Pythagorean formula

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Dot Product

DEFINITION

If $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ are two vectors in n-dimensional space, then the ***dot product*** of $\vec{a}$ an $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Equivalently, the dot product is defined by the geometric formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

Practicing dot products.

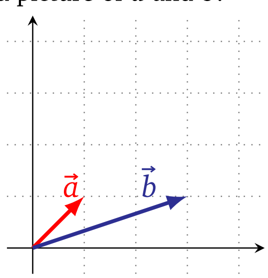
The goal of this problem is to

- Use both the algebraic and geometric definitions of the dot product as appropriate to compute dot products.
- Gain an intuition that positive dot product means "pointing in similar directions", negative dot product means "pointing in opposite directions", and zero dot product means "pointing in orthogonal directions".

24

Let $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

24.1 (a) Draw a picture of $\vec{a}$ and $\vec{b}$.



(b) Compute $\vec{a} \cdot \vec{b}$. $\vec{a} \cdot \vec{b} = (1)(3) + (1)(1) = 4.$

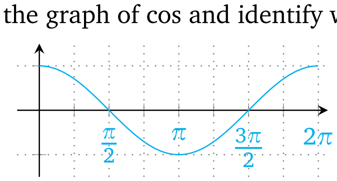
(c) Find $\|\vec{a}\|$ and $\|\vec{b}\|$ and use your knowledge of the multiple ways to compute the dot product to find θ, the angle between $\vec{a}$ and $\vec{b}$. Label θ on your picture.

$\|\vec{a}\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$ and $\|\vec{b}\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$.

Using the two definitions of the dot product we have:

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\| \|\vec{b}\| \cos\theta \\ \implies 4 &= (\sqrt{2})(\sqrt{10}) \cos\theta \\ \implies \theta &= \arccos\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

24.2 Draw the graph of cos and identify which angles make cos negative, zero, or positive.



Cosine is positive for angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

cos is positive for angles in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, as well as all shifts of this interval by a multiple of 2π in either direction.

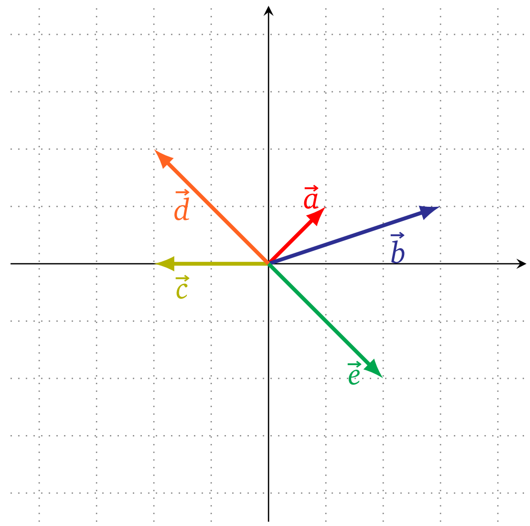
24.3 Draw a new picture of $\vec{a}$ and $\vec{b}$ and on that picture draw

(a) a vector $\vec{c}$ where $\vec{c} \cdot \vec{a}$ is negative.

(b) a vector $\vec{d}$ where $\vec{d} \cdot \vec{a} = 0$ and $\vec{d} \cdot \vec{b} < 0$.

(c) a vector $\vec{e}$ where $\vec{e} \cdot \vec{a} = 0$ and $\vec{e} \cdot \vec{b} > 0$.

(d) Could you find a vector $\vec{f}$ where $\vec{f} \cdot \vec{a} = 0$ and $\vec{f} \cdot \vec{b} = 0$? Explain why or why not.



(d) $\vec{f} = \vec{0}$ is the only possibility. For any vector $\vec{f} = \begin{bmatrix} x \\ y \end{bmatrix}$, we can compute:

$$\vec{f} \cdot \vec{a} = x + y \qquad \text{and} \qquad \vec{f} \cdot \vec{b} = 3x + y.$$

If these both equal zero, the first equation says that $y = -x$, and in turn the second one says $x = 0$ (and so $y = 0$ as well).

24.4 Recall the vector $\vec{u}$ whose coordinates are given at the beginning of this problem.

(a) Write down a vector $\vec{v}$ so that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$. (Hint, how does this relate to the dot product?)

$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ is one such vector.

Since $\cos(\pi/2) = 0$, from the second definition of the dot product above we know we are looking for a $\vec{v}$ such that $\vec{u} \cdot \vec{v} = 0$. Using the first definition of the dot product, we can see that the $\vec{v}$ given above is one possibility.

(b) Write down another vector $\vec{w}$ (in a different direction from $\vec{v}$) so that the angle between $\vec{w}$ and $\vec{u}$ is $\pi/2$.

$\vec{w} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ is a possible answer.

$\vec{u} \cdot \vec{w} = 0$, and $\vec{w}$ is clearly not parallel to $\vec{v}$ from above.

(c) Can you write down other vectors different than both $\vec{v}$ and $\vec{w}$ that still form an angle of $\pi/2$ with $\vec{u}$? How many such vectors are there?

Yes. $\begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$ is one possibility.

There are actually infinitely many such vectors; any linear combination of $\vec{w}$ and $\vec{v}$ will work.

To see this, note that any such vector $\vec{x}$ is of the form

$$\vec{x} = t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t - s \\ t + s \\ -3t - s \end{bmatrix},$$

for scalars t and s. We can then compute

$$\vec{u}\cdot\vec{x} = (1)(t-s)+(2)(t+s)+(1)(-3t-s) = 0,$$

and so any such vector $\vec{x}$ forms an angle of $\pi/2$ with $\vec{u}$.

THEOREM

For a vector $\vec{v}\in\mathbb{R}^n$, the formula

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

always holds.

Distance

DEF The ***distance*** between two vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u}-\vec{v}\|$.

Unit Vector

DEF A vector $\vec{v}$ is called a ***unit vector*** if $\|\vec{v}\| = 1$.

---

25 Let $\vec{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\1\\3\end{bmatrix}$.

25.1 Find the distance between $\vec{u}$ and $\vec{v}$.

$\vec{u}-\vec{v} = \begin{bmatrix}0\\1\\-2\end{bmatrix}$, and so $\|\vec{u}-\vec{v}\| = \sqrt{5}$.

25.2 Find a unit vector in the direction of $\vec{u}$.

$\frac{1}{\sqrt{6}}\vec{u} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix}$.

$\|\vec{u}\| = \sqrt{6}$, and so if we multiply $\vec{u}$ by $\frac{1}{\sqrt{6}}$, the length of the resulting vector will be 1.

25.3 Does there exist a *unit vector* $\vec{x}$ that is distance 1 from $\vec{u}$?

No. $\|\vec{u}\| = \sqrt{6}$, and so the shortest length that a vector whose distance from $\vec{u}$ is 1 can have is $\sqrt{6}-1$, which is greater than 1.

25.4 Suppose $\vec{y}$ is a unit vector and the distance between $\vec{y}$ and $\vec{u}$ is 2. What is the angle between $\vec{y}$ and $\vec{u}$?

The angle between $\vec{u}$ and $\vec{y}$ is $\arccos\left(\frac{3}{2\sqrt{6}}\right)$.

By assumption, $2 = \|\vec{u}-\vec{y}\|$, and so

$$\begin{aligned}4 &= \|\vec{u}-\vec{y}\|^2\\ &= (\vec{u}-\vec{y})\cdot(\vec{u}-\vec{y})\\ &= \vec{u}\cdot\vec{u} - 2(\vec{u}\cdot\vec{y}) + \vec{y}\cdot\vec{y}\\ &= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{y} + \|\vec{y}\|^2\\ &= 6 - 2\vec{u}\cdot\vec{y} + 1.\end{aligned}$$

Then we rearrange to find that $\vec{u}\cdot\vec{y} = \frac{3}{2}$.

Using this in the second definition of the dot product, we see:

$$\frac{3}{2} = \left(\sqrt{6}\right)(1)\cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{y}$.

Practice using norms.

The goal of this problem is to
- Practice finding the distance between two vectors.
- Produce a unit vector pointing in the same direction as another vector.
- Intuitively apply the triangle inequality: $\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$.

Orthogonal

DEF Two vectors $\vec{u}$ and $\vec{v}$ are ***orthogonal*** to each other if $\vec{u}\cdot\vec{v} = 0$. The word orthogonal is synonymous with the word perpendicular.

Apply the definition of *orthogonal*.

The goal of this problem is to
- Gain an intuitive understanding of *orthogonal vectors*.
- Produce orthogonal vectors via guess-and-check.
- Apply the Pythagorean theorem to orthogonal vectors to find lengths.

26

26.1 Find two vectors orthogonal to $\vec{a}=\begin{bmatrix}1\\-3\end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}-6\\-2\end{bmatrix}$.

It is impossible for two non-parallel vectors to both be orthogonal to $\vec{a}$. If $\vec{b}=\begin{bmatrix}x\\y\end{bmatrix}$ is orthogonal to $\vec{a}$, then we must have that $x-3y=0$, or in other words that $x=3y$. Any $\vec{b}$ satisfying this is a multiple of $\begin{bmatrix}3\\1\end{bmatrix}$.

26.2 Find two vectors orthogonal to $\vec{b}=\begin{bmatrix}1\\-3\\4\end{bmatrix}$. Can you find two such vectors that are not parallel?

Two such vectors are $\begin{bmatrix}7\\1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\2\\1\end{bmatrix}$.

These two vectors are not parallel.

26.3 Suppose $\vec{x}$ and $\vec{y}$ are orthogonal to each other and $\|\vec{x}\|=5$ and $\|\vec{y}\|=3$. What is the distance between $\vec{x}$ and $\vec{y}$?

The distance between them must be $\sqrt{34}$.

One way to see this is with Pythagoras' theorem. Two perpendicular line segments of lengths 3 and 5 form the two shorter sides of a right angle triangle, and so the length of the third side is $\sqrt{5^2+3^2}=\sqrt{34}$.

An equivalent way to see this is to use what we know about dot products to calculate $\|\vec{x}-\vec{y}\|$ as follows:

$$\|\vec{x}-\vec{y}\|=\sqrt{(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})}=\sqrt{\|\vec{x}\|^2-2(\vec{x}\cdot\vec{y})+\|\vec{y}\|^2}=\sqrt{5^2+2(0)+3^2},$$

where in the last step we've used the fact that $\vec{x}$ and $\vec{y}$ are orthogonal, so $\vec{x}\cdot\vec{y}=0$.

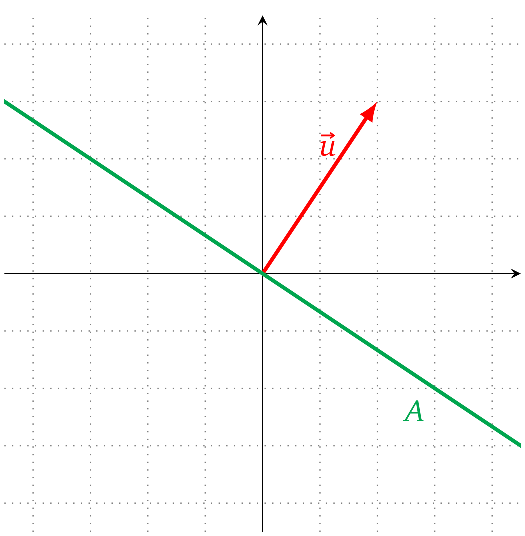
27

Generate lines using orthogonality.

The goal of this problem is to

- Visually see how the set of all vectors orthogonal to a given vector forms a line.
- Given a line defined as the set of all vectors orthogonal to a given vector, express the line using an equation or span.

27.1 Draw $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$ and *all* vectors orthogonal to it. Call this set A.



27.2 If $\vec{x}=\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{x}$ is orthogonal to $\vec{u}$, what is $\vec{x}\cdot\vec{u}$? $\vec{x}\cdot\vec{u}=0$, by the definition of orthogonality.

27.3 Expand the dot product $\vec{u}\cdot\vec{x}$ to get an equation for A.

A is the line with vector equation $\vec{x} = t\begin{bmatrix}3\\-2\end{bmatrix}$.

If $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix} \in A$, then $\vec{x}\cdot\vec{u} = 2x + 3y = 0$.

27.4 If possible, express A as a span. $A = \operatorname{span}\left\{\begin{bmatrix}3\\-2\end{bmatrix}\right\}$.

Normal Vector

DEF A ***normal vector*** to a line (or plane or hyperplane) is a non-zero vector that is orthogonal to all direction vectors for the line (or plane or hyperplane).

---

28 Let $\vec{d} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{p} = \begin{bmatrix}1\\-1\end{bmatrix}$, and define the lines

$$\ell_1 = \operatorname{span}\{\vec{d}\} \qquad \text{and} \qquad \ell_2 = \operatorname{span}\{\vec{d}\} + \{\vec{p}\}.$$

Express lines in normal form.

The goal of this problem is to

- Express lines, including lines that don't pass through $\vec{0}$, in normal form.
- See the $\vec{q}$ in $\vec{n}\cdot(\vec{x}-\vec{q}) = 0$ is similar to the $\vec{p}$ in the vector form $\vec{x} = t\vec{d}+\vec{p}$ in that it accommodates lines that don't pass through $\vec{0}$.
- Use the dot product to represent a line compactly.

28.1 Find a vector $\vec{n}$ that is a normal vector for both ℓ_1 and ℓ_2.

$\vec{n} = \begin{bmatrix}2\\-1\end{bmatrix}$ is one possibility.

This vector is orthogonal to $\vec{d}$, which is a direction vector for both lines.

28.2 Let $\vec{v} \in \ell_1$ and $\vec{u} \in \ell_2$. What is $\vec{n}\cdot\vec{v}$? What about $\vec{n}\cdot(\vec{u}-\vec{p})$? Explain using a picture.

$\vec{n}\cdot\vec{v} = \vec{n}\cdot(\vec{u}-\vec{p}) = 0$.

This is because any $\vec{v} \in \ell_1$ is a multiple of $\vec{d}$, which is orthogonal to $\vec{n}$. Similarly, for any $\vec{u} \in \ell_2$, the vector $\vec{u}-\vec{p}$ is a direction vector for ℓ_2, and so it is orthogonal to $\vec{n}$.

$\vec{n}\cdot\vec{u} = 3$, since any such $\vec{u}$ is of the form $\vec{u} = \vec{p} + t\vec{d}$ for some scalar t, and so

$$\vec{n}\cdot\vec{u} = \vec{n}\cdot(\vec{p}+t\vec{d}) = \vec{n}\cdot\vec{p} + t(\vec{n}\cdot\vec{d}) = 3 + t(0) = 3.$$

28.3 A line is expressed in *normal form* if it is represented by an equation of the form $\vec{n}\cdot(\vec{x}-\vec{q}) = 0$ for some $\vec{n}$ and $\vec{q}$. Express ℓ_1 and ℓ_2 in normal form.

A normal form of ℓ_1 is $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\vec{x} = 0$.

A normal form of ℓ_2 is $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\left(\vec{x} - \begin{bmatrix}1\\-1\end{bmatrix}\right) = 0$. In the previous part we saw that $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$ for all $\vec{x} \in \ell_2$, or in other words $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$.

28.4 Some textbooks would claim that ℓ_2 could be expressed in normal form as $\begin{bmatrix}2\\-1\end{bmatrix}\cdot\vec{x} = 3$. How does this relate to the $\vec{n}\cdot(\vec{x}-\vec{p}) = 0$ normal form? Where does the 3 come from?

Let $\vec{n} = \begin{bmatrix}2\\-1\end{bmatrix}$ and let $\vec{x} \in \ell_2$. From the previous part, we know

$$0 = \vec{n}\cdot(\vec{x}-\vec{p}) = \vec{n}\cdot\vec{x} - \vec{n}\cdot\vec{p} = \vec{n}\cdot\vec{x} - 3.$$

Therefore

$$\vec{n}\cdot\vec{x} = 3.$$

---

29 Let $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Planes in normal form.

The goal of this problem is to

- Observe that the set of all vectors orthogonal to another in $\mathbb{R}^3$ is a plane.
- Translate descriptions of sets into precise mathematical statements using set-builder notation.
- Express a plane in multiple ways.

29.1 Use set-builder notation to write down the set, X, of all vectors orthogonal to $\vec{n}$. Describe this set geometrically.

$X = \left\{\vec{x} \in \mathbb{R}^3 : \vec{x}\cdot\vec{n} = 0\right\}$.

Geometrically, this is a plane through the origin and orthogonal to $\vec{n}$.

29.2 Describe X using an equation. $x+y+z=0$.

29.3 Describe X as a span. $X=\operatorname{span}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ is one way to do this.

Projections

Projection

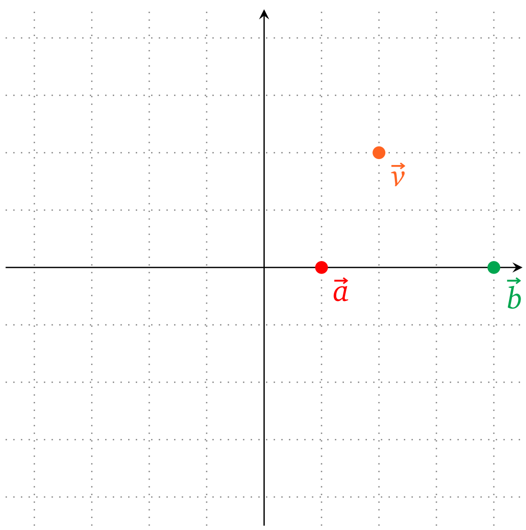
DEF Let X be a set. The ***projection*** of the vector $\vec{v}$ onto X, written $\operatorname{proj}_X \vec{v}$, is the closest point in X to $\vec{v}$.

Apply the definition of projection.

The goal of this problem is to
- Use the definition of projection to compute projections onto finite sets and lines.
- Pick an appropriate representation of a line to solve a projection problem.

30 Let $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}4\\0\end{bmatrix}$, $\vec{v}=\begin{bmatrix}2\\2\end{bmatrix}$ and $\ell=\operatorname{span}\{\vec{a}\}$.

30.1 Draw $\vec{a}$, $\vec{b}$, and $\vec{v}$ in the same picture.



30.2 Find $\operatorname{proj}_{\{\vec{b}\}}\vec{v}$, $\operatorname{proj}_{\{\vec{a},\vec{b}\}}\vec{v}$.

$\operatorname{proj}_{\{\vec{b}\}}\vec{v}=\vec{b}$. Since there is only one point in $\{\vec{b}\}$, it must be the closest point to $\vec{v}$.

$\operatorname{proj}_{\{\vec{a},\vec{b}\}}\vec{v}=\vec{a}$. We can simply compute $\|\vec{v}-\vec{a}\|=\sqrt{5}$ and $\|\vec{v}-\vec{b}\|=\sqrt{8}$, so $\vec{a}$ is closer to $\vec{v}$.

30.3 Find $\operatorname{proj}_\ell \vec{v}$. (Recall that a quadratic at^2+bt+c has a minimum at $t=-\frac{b}{2a}$).

$\operatorname{proj}_\ell \vec{v}=2\vec{a}=\begin{bmatrix}2\\0\end{bmatrix}$.

Any point in ℓ is of the form $t\vec{a}$ for some scalar t. The distance between such a point and $\vec{v}$ is

$$\|\vec{v}-t\vec{a}\|=\sqrt{\|v\|^2-2t(\vec{v}\cdot\vec{a})+t^2\|a\|^2}=\sqrt{8-4t+t^2}$$

The quadratic inside the square root has a minimum at $t=2$, so $2\vec{a}$ is the closest point in the line to $\vec{v}$.

Project onto lines.

The goal of this problem is to
- Use orthogonality to compute the projection onto a line.
- Project onto lines that don't pass through $\vec{0}$.

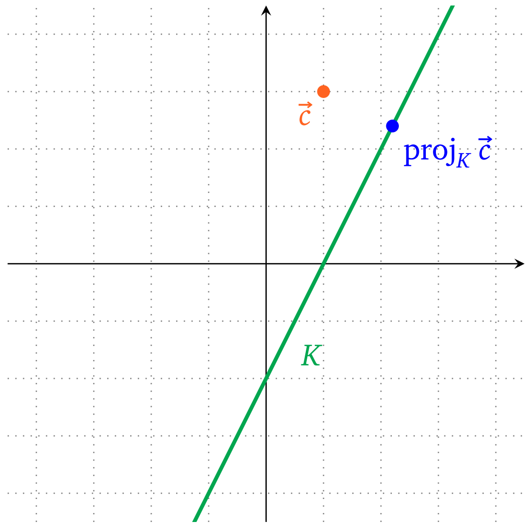
30.4 Is $\vec{v}-\operatorname{proj}_\ell\vec{v}$ a normal vector for ℓ? Why or why not?

Yes.

By the previous part, $\vec{v}-\operatorname{proj}_\ell\vec{v}=\vec{v}-2\vec{a}=\begin{bmatrix}0\\2\end{bmatrix}$. This vector is orthogonal to $\vec{a}$, and therefore to ℓ.

31 Let K be the line given in vector form by $\vec{x} = t\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}$ and let $\vec{c} = \begin{bmatrix}1\\3\end{bmatrix}$.

31.1 Make a sketch with $\vec{c}$, K, and $\text{proj}_K \vec{c}$ (you don't need to compute $\text{proj}_K \vec{c}$ exactly).



31.2 What should $(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix}$ be? Explain.

$(\vec{c} - \text{proj}_K \vec{c}) \cdot \begin{bmatrix}1\\2\end{bmatrix} = 0.$

From our picture we can see that $c - \text{proj}_K \vec{c}$ is perpendicular to the line K, and so the dot product of this vector with any direction vector for K should be zero.

31.3 Use your formula from the previous part to find $\text{proj}_K \vec{c}$ *without* computing any distances.

$\text{proj}_K \vec{c} = \frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$

If $\text{proj}_K \vec{c} = \begin{bmatrix}x\\y\end{bmatrix}$, the formula from the previous part tells us

$$\left(\begin{bmatrix}1\\3\end{bmatrix} - \begin{bmatrix}x\\y\end{bmatrix}\right) \cdot \begin{bmatrix}1\\2\end{bmatrix} = 1 - x + 6 - 2y = 0 \qquad \iff \qquad x + 2y = 7$$

So we need a point on K that satisfies this equation. In other words, we need

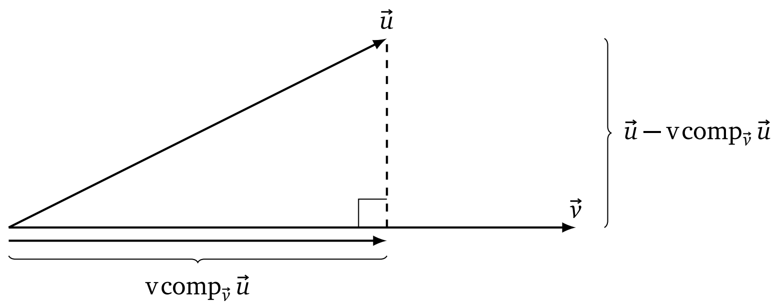
$$(t+1) + 2(2t) = 7 \qquad \implies \qquad t = \frac{6}{5}.$$

The point on K for this value of t is $\frac{1}{5}\begin{bmatrix}11\\12\end{bmatrix}$.

Vector Components

DEFINITION

Let $\vec{u}$ and $\vec{v} \neq \vec{0}$ be vectors. The ***vector component of $\vec{u}$ in the $\vec{v}$ direction***, written $\text{vcomp}_{\vec{v}}\, \vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u} - \text{vcomp}_{\vec{v}}\, \vec{u}$ is orthogonal to $\vec{v}$.



Component of a vector in the direction of another.

The goal of this problem is to
- Read and apply a new definition.
- Use orthogonality to obtain a formula for components in terms of dot products.

32 Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unknown vectors.

32.1 List two conditions that $\operatorname{vcomp}_{\vec{b}} \vec{a}$ must satisfy.

$\operatorname{vcomp}_{\vec{b}} \vec{a}$ must be a scalar multiple of $\vec{b}$.

$\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}$ must be orthogonal to $\vec{b}$, or in other words $(\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}) \cdot \vec{b} = 0$.

32.2 Find a formula for $\operatorname{vcomp}_{\vec{b}} \vec{a}$.

$$\operatorname{vcomp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}.$$

From the previous part, we should have $\operatorname{vcomp}_{\vec{b}} \vec{a} = t\vec{b}$ for some scalar t, and $(\vec{a} - \operatorname{vcomp}_{\vec{b}} \vec{a}) \cdot \vec{b} = 0$.

Combining these, we get:

$$0 = (\vec{a} - t\vec{b}) \cdot \vec{b} = \vec{a} \cdot \vec{b} - t\vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{b} - t(\vec{b} \cdot \vec{b}).$$

Solving for t, we get $t = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}$.

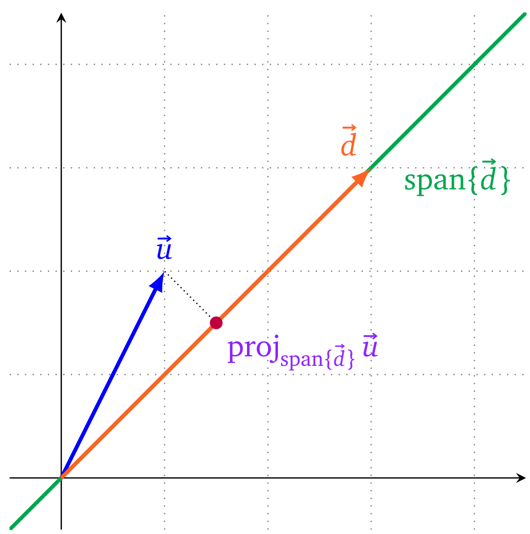
Relate components and projections.

The goal of this problem is to

- Find a connection between components and projections onto spans.
- Recognize that $\operatorname{vcomp}_{\vec{u}} \vec{v} = \operatorname{vcomp}_{-\vec{u}} \vec{v}$.

33 Let $\vec{d} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

33.1 Draw $\vec{d}$, $\vec{u}$, $\operatorname{span}\{\vec{d}\}$, and $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ in the same picture.



33.2 How do $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ and $\operatorname{vcomp}_{\vec{d}} \vec{u}$ relate? They are equal.

33.3 Compute $\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u}$ and $\operatorname{vcomp}_{\vec{d}} \vec{u}$.

Using our formula from the previous problem

$$\operatorname{proj}_{\operatorname{span}\{\vec{d}\}} \vec{u} = \operatorname{vcomp}_{\vec{d}} \vec{u} = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \tfrac{9}{18} \vec{d} = \tfrac{1}{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

33.4 Compute $\operatorname{vcomp}_{-\vec{d}} \vec{u}$. Is this the same as or different from $\operatorname{vcomp}_{\vec{d}} \vec{u}$? Explain.

$$\operatorname{vcomp}_{-\vec{d}} \vec{u} = \frac{\vec{u} \cdot (-\vec{d})}{\|-\vec{d}\|^2} (-\vec{d}) = \tfrac{-9}{18} (-\vec{d}) = \tfrac{1}{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \operatorname{vcomp}_{\vec{d}} \vec{u}.$$

We expect them to be equal since $\vec{d}$ and $-\vec{d}$ are in the same direction as one another.

Subspaces and Bases

Subspace

DEFINITION

A non-empty subset $V \subseteq \mathbb{R}^n$ is called a ***subspace*** if for all $\vec{u}, \vec{v} \in V$ and all scalars k we have

(i) $\vec{u} + \vec{v} \in V$; and

(ii) $k\vec{u} \in V$.

Subspaces give a mathematically precise definition of a "flat space through the origin."

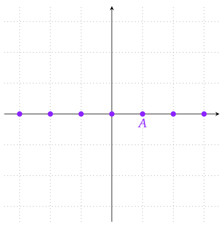
34 For each set, draw it and explain whether or not it is a subspace of $\mathbb{R}^2$.

Visualizing subspaces.

The goal of this problem is to

- Read and apply the definition of subspace.
- Identify from a picture whether or not a set is a subspace.
- Write formal arguments showing whether or not certain sets are subspaces.

34.1 $A = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} a \\ 0 \end{bmatrix} \text{ for some } a \in \mathbb{Z} \right\}$.

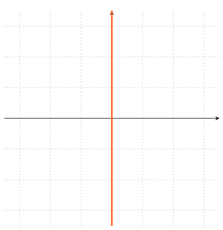


A is not a subspace, since for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in A$ but $\frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin A$.

34.2 $B = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

B is not a subspace, since for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ are both in B, but their sum is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ which is not in B.

34.3 $C = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.



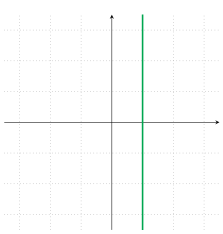
C is a subspace.

(i) Let $\vec{u}, \vec{v} \in C$. Then $\vec{u} = \begin{bmatrix} 0 \\ t \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ s \end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = \begin{bmatrix} 0 \\ s+t \end{bmatrix} \in C$.

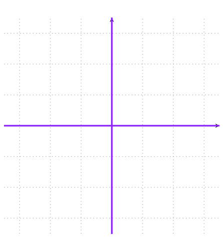
(ii) Let $\vec{u} = \begin{bmatrix} 0 \\ t \end{bmatrix} \in C$. For any scalar α we have $\alpha\vec{u} = \begin{bmatrix} 0 \\ \alpha t \end{bmatrix} \in C$.

34.4 $D = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.



D is not a subspace, since for example $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in D$, but $0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin D$.

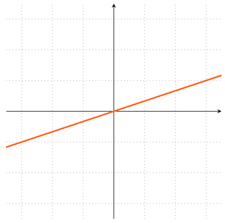
34.5 $E = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.



E is not a subspace, since for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are both in E, but their sum is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in E.

34.6 $F = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}3\\1\end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

F is a subspace.



(i) Let $\vec{u}, \vec{v} \in F$. Then $\vec{u} = t\begin{bmatrix}3\\1\end{bmatrix}$ and $\vec{v} = s\begin{bmatrix}3\\1\end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix}3\\1\end{bmatrix} \in F$.

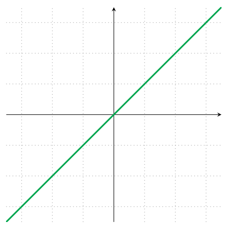
(ii) Let $\vec{u} = t\begin{bmatrix}3\\1\end{bmatrix} \in F$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix}3\\1\end{bmatrix} \in F$.

34.7 $G = \operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

G is a subspace.

By definition of a span, $G = \left\{ \vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix}1\\1\end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$.

The proof that G is a subspace now proceeds similarly to the proof for F above.



(i) Let $\vec{u}, \vec{v} \in G$. Then $\vec{u} = t\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{v} = s\begin{bmatrix}1\\1\end{bmatrix}$ for some $s, t \in \mathbb{R}$.

But then $\vec{u} + \vec{v} = (s+t)\begin{bmatrix}1\\1\end{bmatrix} \in G$.

(ii) Let $\vec{u} = t\begin{bmatrix}1\\1\end{bmatrix} \in G$. For any scalar α we have $\alpha\vec{u} = (\alpha t)\begin{bmatrix}1\\1\end{bmatrix} \in G$.

34.8 $H = \operatorname{span}\{\vec{u}, \vec{v}\}$ for some unknown vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$.

H is a subspace.

(i) Let $\vec{x}, \vec{y} \in H$. Then $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v}$ and $\vec{y} = \beta_1\vec{u} + \beta_2\vec{v}$ for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2$. But then

$$\vec{x} + \vec{y} = \alpha_1\vec{u} + \alpha_2\vec{v} + \beta_1\vec{u} + \beta_2\vec{v} = (\alpha_1 + \beta_1)\vec{u} + (\alpha_2 + \beta_2)\vec{v} \in H.$$

(ii) Let $\vec{x} = \alpha_1\vec{u} + \alpha_2\vec{v} \in H$. For any scalar β we have $\beta\vec{u} = (\beta\alpha_1)\vec{u} + (\beta\alpha_2)\vec{v} \in H$.

Basis (DEF)

A ***basis*** for a subspace $\mathcal{V}$ is a linearly independent set of vectors, $\mathcal{B}$, so that $\operatorname{span}\mathcal{B} = \mathcal{V}$.

Dimension (DEF)

The ***dimension*** of a subspace V is the number of elements in a basis for V.

---

35 Let $\vec{u} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $V = \operatorname{span}\{\vec{u}, \vec{v}, \vec{w}\}$.

Apply the definitions of basis and dimension to a simple example.

The goal of this problem is to learn

- To apply the definition of basis and dimension.
- Intuition that a plane is two dimensional.
- A basis is not unique, but always has the same size (this is not proved).
- Spans are never bases—you must not confuse a subspace with its basis!

35.1 Describe V. V is the xy-plane in $\mathbb{R}^3$.

35.2 Is $\{\vec{u}, \vec{v}, \vec{w}\}$ a basis for V? Why or why not?

No. The set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent since $\vec{w} = \vec{u} + \vec{v}$.

35.3 Give a basis for V. $\{\vec{u}, \vec{v}\}$.

35.4 Give another basis for V. $\{\vec{u}, \vec{w}\}$ or $\{\vec{v}, \vec{w}\}$.

35.5 Is $\operatorname{span}\{\vec{u}, \vec{v}\}$ a basis for V? Why or why not?

No. $\operatorname{span}\{\vec{u}, \vec{v}\}$ is an infinite set of vectors which includes $\vec{0}$, so it cannot be linearly independent and therefore isn't a basis.

35.6 What is the dimension of V?

A basis for V has two vectors so it is two-dimensional. We also know this because V is the xy-plane in $\mathbb{R}^3$ and all planes are two-dimensional.

36

Let $\vec{a} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{c} = \begin{bmatrix}7\\8\\8\end{bmatrix}$ (notice these vectors are linearly independent) and let $P = \operatorname{span}\{\vec{a}, \vec{b}\}$ and $Q = \operatorname{span}\{\vec{b}, \vec{c}\}$.

36.1 Give a basis for and the dimension of P.

$\{\vec{a}, \vec{b}\}$ is a basis for P, and so its dimension is 2.

36.2 Give a basis for and the dimension of Q.

$\{\vec{b}, \vec{c}\}$ is a basis for Q, and so its dimension is 2.

36.3 Is $P \cap Q$ a subspace? If so, give a basis for it and its dimension.

Yes. $\{\vec{b}\}$ is a basis for $P \cap Q$, and so its dimension is 1.

P and Q are both planes and are not parallel (since $\vec{a}, \vec{b}, \vec{c}$ are linearly independent). The intersection of any two non-parallel planes in $\mathbb{R}^3$ is a line. We know that $\vec{0}$ and $\vec{b}$ are on this line, and therefore the line is $\operatorname{span}\{\vec{b}\}$

36.4 Is $P \cup Q$ a subspace? If so, give a basis for it and its dimension.

No. For example $\vec{a}$ and $\vec{c}$ are both in $P \cup Q$, but $\vec{a} + \vec{c} \notin P \cup Q$.

Proof: A vector is in $P \cup Q$ if it is in P or Q, so we must show that $\vec{a} + \vec{c} \notin P$ and $\vec{a} + \vec{c} \notin Q$

$\vec{a} + \vec{c} \notin P$ since if it were, we would also have $(\vec{a} + \vec{c}) - \vec{a} = \vec{c} \in P$. We know this is impossible since the vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly independent, and so $\vec{c}$ does not equal a linear combination of $\vec{a}$ and $\vec{b}$.

An analogous argument shows that $\vec{a} + \vec{c} \notin Q$.

The relationship between subspaces, bases, unions, and intersections.

The goal of this problem is to learn

- Recognize intersections of subspaces as subspaces.
- Recognize the union of subspaces need not be a subspace.
- Visualize planes in $\mathbb{R}^3$ to solve problems without computations.

Matrices

37

Let $A = \begin{bmatrix}1 & 2\\3 & 3\end{bmatrix}$, $\vec{x} = \begin{bmatrix}x\\y\end{bmatrix}$, and $\vec{b} = \begin{bmatrix}-2\\-1\end{bmatrix}$.

37.1 Compute the product $A\vec{x}$.

$A\vec{x} = \begin{bmatrix}x + 2y\\3x + 3y\end{bmatrix}$.

37.2 Write down a system of equations that corresponds to the matrix equation $A\vec{x} = \vec{b}$.

$$\begin{aligned} x + 2y &= -2\\ 3x + 3y &= -1 \end{aligned}$$

37.3 Let $\begin{bmatrix}x_0\\y_0\end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *intersecting lines* (hint: think about systems of equations).

The lines represented by the equations $x + 2y = -2$ and $3x + 3y = -1$ from the system of equations above intersect at the point $\begin{bmatrix}x_0\\y_0\end{bmatrix}$.

37.4 Let $\begin{bmatrix}x_0\\y_0\end{bmatrix}$ be a solution to $A\vec{x} = \vec{b}$. Explain what x_0 and y_0 mean in terms of *linear combinations* (hint: think about the columns of A).

x_0 and y_0, when used as scalars in a linear combination of the columns of A, make the vector $\vec{b}$. In other words:

$$x_0\begin{bmatrix}1\\3\end{bmatrix} + y_0\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}.$$

Relate matrix equations and systems of linear equations.

The goal of this problem is to

- Use matrix-vector multiplication to represent a system of equations with compact notation.
- View a matrix equation as a statement about (i) linear combinations of column vectors and (ii) a system of equations coming from the rows.

Rephrase previous questions using matrix equations.

The goal of this problem is to

- Rephrase the question of linear independence as the special matrix equation $A\vec{x} = \vec{0}$.

38

Let $\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, $\vec{w} = \begin{bmatrix}7\\8\\9\end{bmatrix}$.

38.1 How could you determine if $\{\vec{u}, \vec{v}, \vec{w}\}$ was a linearly independent set?

The set is linearly independent if and only if no non-trivial linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$ equals $\vec{0}$. That is, if x, y, z are scalars such that $x\vec{u}+y\vec{v}+z\vec{w} = \vec{0}$, then $x = y = z = 0$.

In other words, the only solution of the following system of equations is $x = y = z = 0$.

$$\begin{aligned} x + 4y + 7z &= 0 \\ 2x + 5y + 8z &= 0 \\ 3x + 6y + 9z &= 0 \end{aligned}$$

38.2 Can your method be rephrased in terms of a matrix equation? Explain.

The system of linear equations above can be represented by the matrix equation $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$ and $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix}$.

So another way to say the above is that the set is linearly independent if and only if the only solution to the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

39

Interpret matrix equations.

The goal of this problem is to

- Use knowledge about systems of linear equations to answer questions about matrix equations.

Consider the system represented by

$$\begin{bmatrix}1&-3&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{b}.$$

39.1 If $\vec{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

This system has no solutions, since, if we expand the matrix equation into a system of equations, the third equation would be $0 = 3$, which is impossible.

39.2 If $\vec{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, is the set of solutions to this system a point, line, plane, or other?

A line. The system would be

$$\begin{aligned} x - 3y &= 1 \\ z &= 1 \\ 0 &= 0 \end{aligned}$$

A vector $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix}$ that satisfies this system must have $z = 1$, and by the first equation in the system any value of x determines the value of y, and vice versa. In other words the system has one free variable, and so its set of solutions is a line.

40

Apply matrix equations to planes.

The goal of this problem is to

- Rephrase properties of a plane in terms of matrix equations.
- Be able to describe one application of the transpose.

Let $\vec{d}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$. Let $\mathcal{P}$ be the plane given in vector form by $\vec{x} = t\vec{d}_1 + s\vec{d}_2$. Further, suppose M is a matrix so that $M\vec{r} \in \mathcal{P}$ for any $\vec{r} \in \mathbb{R}^2$.

40.1 How many rows does M have?

Three. It must have three rows in order for $M\vec{r}$ to be an element of $\mathbb{R}^3$.

40.2 Find such an M.

$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$ is one possible answer, since if $\vec{r} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $M\vec{r} = a\vec{d}_1 + b\vec{d}_2$.

Another less interesting answer is the 3×2 zero matrix.

40.3 Find necessary and sufficient conditions (phrased as equations) for $\vec{n}$ to be a normal vector for $\mathcal{P}$.

$\vec{n}$ is normal to $\mathcal{P}$ if and only if $\vec{n} \neq \vec{0}$, $\vec{n} \cdot \vec{d}_1 = 0$, *and* $\vec{n} \cdot \vec{d}_2 = 0$

40.4 Find a matrix K so that non-zero solutions to $K\vec{x} = \vec{0}$ are normal vectors for $\mathcal{P}$. How do K and M relate?

$K = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$. K and M are transposes of one another.

The conditions $\vec{n} \cdot \vec{d}_1 = 0$ and $\vec{n} \cdot \vec{d}_2 = 0$ from the previous part translate to the following system of equations:

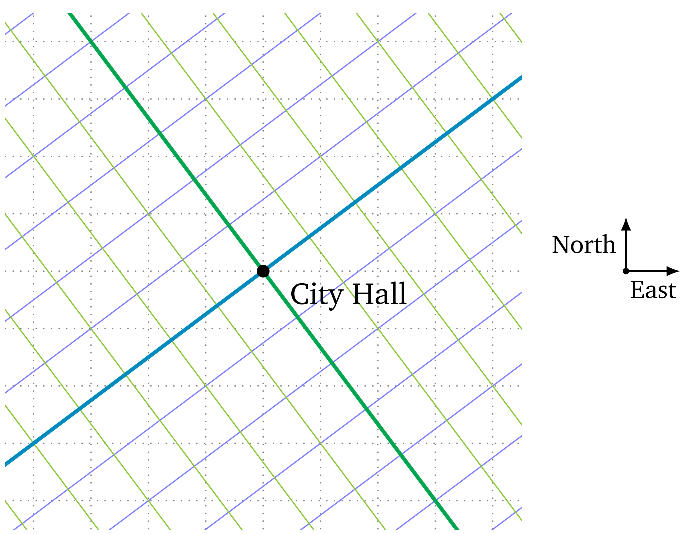
$$\begin{aligned} x + y + 2z &= 0 \\ -x + y &= 0. \end{aligned}$$

This system of equations can be represented by the matrix equation

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Change of Basis & Coordinates

41 The mythical town of Oronto is not aligned with the usual compass directions. The streets are laid out as follows:



Instead, every street is parallel to the vector $\vec{d}_1 = \frac{1}{5}\begin{bmatrix} 4 \text{ east} \\ 3 \text{ north} \end{bmatrix}$ or $\vec{d}_2 = \frac{1}{5}\begin{bmatrix} -3 \text{ east} \\ 4 \text{ north} \end{bmatrix}$. The center of town is City Hall at $\vec{0} = \begin{bmatrix} 0 \text{ east} \\ 0 \text{ north} \end{bmatrix}$.

Locations in Oronto are typically specified in *street coordinates*. That is, as a pair (a, b) where a is how far you walk along streets in the $\vec{d}_1$ direction and b is how far you walk in the $\vec{d}_2$ direction, provided you start at city hall.

Motivate change of basis.

The goal of this problem is to
- Describe points in multiple bases when given a visual description of the basis or when given the basis vectors numerically.
- Recognize ambiguity when faced with the question, "Which basis is better?"

41.1 The points $A=(2,1)$ and $B=(3,-1)$ are given in street coordinates. Find their east-north coordinates.

$A=(1,2)$ and $B=(3,1)$ in east-north coordinates.

We obtain A for example by finding the vector $2\vec{d}_1+\vec{d}_2$.

41.2 The points $X=(4,3)$ and $Y=(1,7)$ are given in east-north coordinates. Find their street coordinates. $X=(5,0)$ and $Y=(5,5)$ in street coordinates.

41.3 Define $\vec{e}_1=\begin{bmatrix}1 \text{ east}\\ 0 \text{ north}\end{bmatrix}$ and $\vec{e}_2=\begin{bmatrix}0 \text{ east}\\ 1 \text{ north}\end{bmatrix}$. Does $\operatorname{span}\{\vec{e}_1,\vec{e}_2\}=\operatorname{span}\{\vec{d}_1,\vec{d}_2\}$?

Yes. Both of these sets spans all of $\mathbb{R}^2$.

41.4 Notice that $Y=5\vec{d}_1+5\vec{d}_2=\vec{e}_1+7\vec{e}_2$. Is the point Y better represented by the pair $(5,5)$ or by the pair $(1,7)$? Explain.

It is equally well represented by either pair. For example, the street coordinates might be more useful for a resident of Oronto, while the east-north coordinates might be more useful for someone looking at Oronto on a world map.

Representation in a Basis

DEFINITION

Let $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ be a basis for a subspace V and let $\vec{v}\in V$. The ***representation of $\vec{v}$ in the $\mathcal{B}$ basis***, notated $[\vec{v}]_{\mathcal{B}}$, is the column matrix

$$[\vec{v}]_{\mathcal{B}}=\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}$$

where $\alpha_1,\dots,\alpha_n$ uniquely satisfy $\vec{v}=\alpha_1\vec{b}_1+\cdots+\alpha_n\vec{b}_n$.

Conversely,

$$\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix}_{\mathcal{B}}=\alpha_1\vec{b}_1+\cdots+\alpha_n\vec{b}_n$$

is notation for the linear combination of $\vec{b}_1,\dots,\vec{b}_n$ with coefficients $\alpha_1,\dots,\alpha_n$.

---

42 Let $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\mathcal{C}=\{\vec{c}_1,\vec{c}_2\}$ where $\vec{c}_1=\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}$ and $\vec{c}_2=\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}}$ be another basis for $\mathbb{R}^2$.

Change of basis notation.

The goal of this problem is to

- Practice using change-of-basis notation.
- Compute representations of vectors in different bases.
- Find a matrix that computes a change of basis.

42.1 Express $\vec{c}_1$ and $\vec{c}_2$ as a linear combination of $\vec{e}_1$ and $\vec{e}_2$. $\vec{c}_1=2\vec{e}_1+\vec{e}_2$ and $\vec{c}_2=5\vec{e}_1+3\vec{e}_2$.

42.2 Express $\vec{e}_1$ and $\vec{e}_2$ as a linear combination of $\vec{c}_1$ and $\vec{c}_2$. $\vec{e}_1=3\vec{c}_1-\vec{c}_2$ and $\vec{e}_2=-5\vec{c}_1+2\vec{c}_2$.

42.3 Let $\vec{v}=2\vec{e}_1+2\vec{e}_2$. Find $[\vec{v}]_{\mathcal{E}}$ and $[\vec{v}]_{\mathcal{C}}$.

$[\vec{v}]_{\mathcal{E}}=\begin{bmatrix}2\\2\end{bmatrix}$ and $[\vec{v}]_{\mathcal{C}}=\begin{bmatrix}-4\\2\end{bmatrix}$.

The second one is since

$$\vec{v}=2\vec{e}_1+2\vec{e}_2=2(3\vec{c}_1-\vec{c}_2)+2(-5\vec{c}_1+2\vec{c}_2)=-4\vec{c}_1+2\vec{c}_2.$$

42.4 Can you find a matrix X so that $X[\vec{w}]_{\mathcal{C}}=[\vec{w}]_{\mathcal{E}}$ for any $\vec{w}$?

$X=\begin{bmatrix}2&5\\1&3\end{bmatrix}$ is such a matrix.

We know X must be a 2×2 matrix, so suppose $X=\begin{bmatrix}a&b\\c&b\end{bmatrix}$ for some $a,b,c,d\in\mathbb{R}$.

From the first part above, we know

$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{C}}=\begin{bmatrix}2\\1\end{bmatrix}_{\mathcal{E}}\quad\text{and}\quad\begin{bmatrix}0\\1\end{bmatrix}_{\mathcal{C}}=\begin{bmatrix}5\\3\end{bmatrix}_{\mathcal{E}},$$

and so we need X to satisfy

$$X\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \quad\text{and}\quad X\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}5\\3\end{bmatrix}.$$

But $X\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\c\end{bmatrix}$ and $X\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}$, so we can now immediately solve for a,b,c,d to find that X must be the matrix $\begin{bmatrix}2&5\\1&3\end{bmatrix}$.

42.5 Can you find a matrix Y so that $Y[\vec{w}]_{\mathcal{E}}=[\vec{w}]_{\mathcal{C}}$ for any $\vec{w}$?

$Y=\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$ is such a matrix.

Using similar reasoning to the previous part, we know Y must be a 2×2 matrix, so suppose $Y=\begin{bmatrix}a&b\\c&b\end{bmatrix}$ for some $a,b,c,d\in\mathbb{R}$.

From the second part above, we know

$$\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{E}}=\begin{bmatrix}3\\-1\end{bmatrix}_{\mathcal{C}} \quad\text{and}\quad \begin{bmatrix}0\\1\end{bmatrix}_{\mathcal{E}}=\begin{bmatrix}-5\\2\end{bmatrix}_{\mathcal{C}},$$

and so we need Y to satisfy

$$Y\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix} \quad\text{and}\quad Y\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-5\\2\end{bmatrix},$$

But $Y\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a\\c\end{bmatrix}$ and $Y\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}$, so we can now immediately solve for a,b,c,d to find that Y must be the matrix $\begin{bmatrix}3&-5\\-1&2\end{bmatrix}$.

42.6 What is YX?

$YX=\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

DEFINITION

Orientation of a Basis

The ordered basis $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ is ***right-handed*** or ***positively oriented*** if it can be continuously transformed to the standard basis (with $\vec{b}_i\mapsto\vec{e}_i$) while remaining linearly independent throughout the transformation. Otherwise, $\mathcal{B}$ is called ***left-handed*** or ***negatively oriented***.

---

43 Let $\{\vec{e}_1,\vec{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and let $\vec{u}_\theta$ be a unit vector. Let θ be the angle between $\vec{u}_\theta$ and $\vec{e}_1$ measured counter-clockwise starting at $\vec{e}_1$.

Visually understand orientation.

The goal of this problem is to
- Determine the orientation of a basis from a picture.
- Recognize the order of vectors in a basis relates to the orientation of that basis.

43.1 For which θ is $\{\vec{e}_1,\vec{u}_\theta\}$ a linearly independent set? Every θ that is not a multiple of π.

43.2 For which θ can $\{\vec{e}_1,\vec{u}_\theta\}$ be continuously transformed into $\{\vec{e}_1,\vec{e}_2\}$ and remain linearly independent the whole time?

Every $\theta\in(0,\pi)$.

For $\theta\in(\pi,2\pi)$, a continuous transformation of $\vec{u}_\theta$ to $\vec{e}_2$ would have to cross the x-axis, at which point $\{\vec{e}_1,\vec{u}_\theta\}$ would cease to be linearly independent.

43.3 For which θ is $\{\vec{e}_1,\vec{u}_\theta\}$ right-handed? Left-handed?

It is right-handed for $\theta\in(0,\pi)$, and left handed for $\theta\in(\pi,2\pi)$.

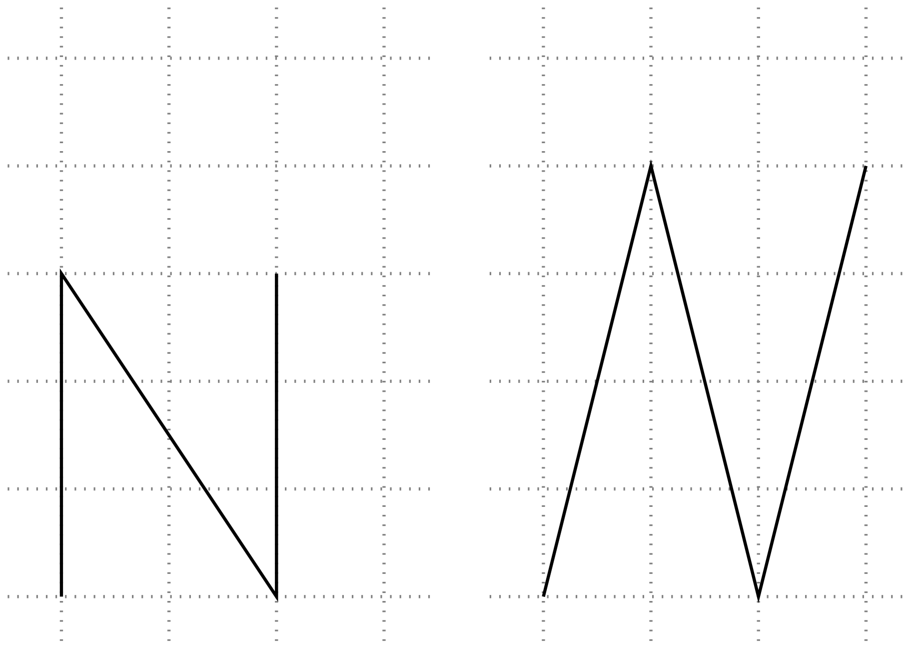
43.4 For which θ is $\{\vec{u}_\theta,\vec{e}_1\}$ (in that order) right-handed? Left-handed?

It is right-handed for $\theta\in(\pi,2\pi)$, and left handed for $\theta\in(0,\pi)$.

43.5 Is $\{2\vec{e}_1,3\vec{e}_2\}$ right-handed or left-handed? What about $\{2\vec{e}_1,-3\vec{e}_2\}$?

$\{2\vec{e}_1,3\vec{e}_2\}$ is right-handed and $\{2\vec{e}_1,-3\vec{e}_2\}$ is left-handed.

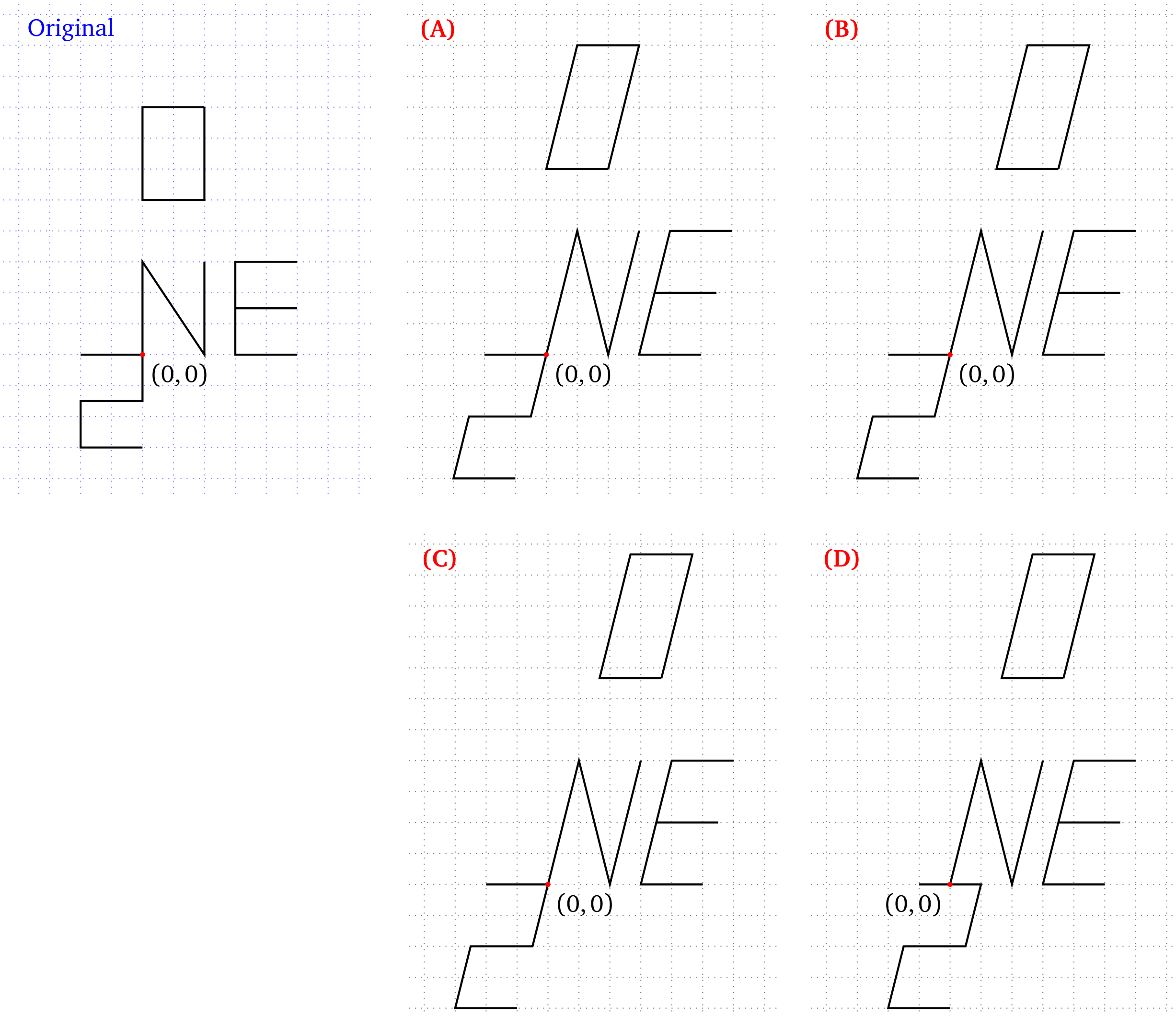
44 The citizens of Oronto want to erect a sign welcoming visitors to the city. They've commissioned letters to be built, but at the last council meeting, they decided they wanted italicised letters instead of regular ones. Can you help them?



Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Work with your group to write out your solution and approach. Make a list of any assumptions you notice your group making or any questions for further pursuit.

45 Some council members were wondering how letters placed in other locations in the plane would be transformed under $A = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$. If other letters are placed around the "N," the council members argued over four different possible results for the transformed letters. Which choice below, if any, is correct, and why? If none of the four options are correct, what would the correct option be, and why?



46 $\mathcal{R}:\mathbb{R}^2\to\mathbb{R}^2$ is the transformation that rotates vectors counter-clockwise by 90°.

Apply geometric transformations to vectors.

The goal of this problem is to
- Given a transformation described in words, compute the result of the transformation applied to particular vectors.
- Use linear combinations to compute the result of rotations applied to unknown vectors.
- Distinguish between a general transformation and a matrix transformation.

46.1 Compute $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$. $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$.

46.2 Compute $\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}$. How does this relate to $\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}$?

$$\mathcal{R}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}=\mathcal{R}\begin{bmatrix}1\\0\end{bmatrix}+\mathcal{R}\begin{bmatrix}0\\1\end{bmatrix}.$$

46.3 What is $\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)$?

$$\mathcal{R}\left(a\begin{bmatrix}1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\end{bmatrix}\right)=a\begin{bmatrix}0\\1\end{bmatrix}+b\begin{bmatrix}-1\\0\end{bmatrix}.$$

Rotating a vector and then multiplying by a scalar gives the same result as multiplying first then rotating. Similarly, adding two vectors and then rotating their sum gives the same result as rotating them and then adding.

46.4 Write down a matrix R so that $R\vec{v}$ is $\vec{v}$ rotated counter-clockwise by 90°.

$R=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is such a matrix.

DEFINITION

Linear Transformation

Let V and W be subspaces. A function $T:V\to W$ is called a ***linear transformation*** if

$$T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})\qquad\text{and}\qquad T(\alpha\vec{v})=\alpha T(\vec{v})$$

for all vectors $\vec{u},\vec{v}\in V$ and all scalars α.

47 47.1 Classify the following as linear transformations or not.

Apply the definition of a linear transformation to examples.

The goal of this problem is to
- Distinguish between a linear transformation and a non-linear transformation.
- Provide a proof of whether a transformation is linear or not.

(a) $\mathcal{R}$ from before (rotation counter-clockwise by 90°).

A linear transformation. We proved this in the previous problem.

(b) $W:\mathbb{R}^2\to\mathbb{R}^2$ where $W\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x^2\\y\end{bmatrix}$.

Not a linear transformation, since for example $W\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq\begin{bmatrix}2\\0\end{bmatrix}=2W\begin{bmatrix}1\\0\end{bmatrix}$.

(c) $T:\mathbb{R}^2\to\mathbb{R}^2$ where $T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x+2\\y\end{bmatrix}$.

Not a linear transformation, since for example $T\left(2\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}4\\0\end{bmatrix}\neq 2T\begin{bmatrix}1\\0\end{bmatrix}$.

(d) $\mathcal{P}:\mathbb{R}^2\to\mathbb{R}^2$ where $\mathcal{P}\begin{bmatrix}x\\y\end{bmatrix}=\operatorname{vcomp}_{\vec{u}}\begin{bmatrix}x\\y\end{bmatrix}$ and $\vec{u}=\begin{bmatrix}2\\3\end{bmatrix}$.

A linear transformation.
We found a general formula for $\operatorname{vcomp}_{\vec{u}}$ in a previous exercise:

$$\operatorname{vcomp}_{\vec{u}}\vec{x}=\frac{\vec{u}\cdot\vec{x}}{\vec{u}\cdot\vec{u}}\vec{u}=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}.$$

For any two vectors $\vec{x}$ and $\vec{y}$, we have

$$\begin{aligned}\operatorname{vcomp}_{\vec{u}}(\vec{x}+\vec{y})&=\frac{\vec{u}\cdot(\vec{x}+\vec{y})}{13}\vec{u}\\&=\frac{\vec{u}\cdot\vec{x}}{13}\vec{u}+\frac{\vec{u}\cdot\vec{y}}{13}\vec{u}\\&=\operatorname{vcomp}_{\vec{u}}\vec{x}+\operatorname{vcomp}_{\vec{u}}\vec{y}.\end{aligned}$$

For any $\vec{x}$ and scalar α, we have

$$\operatorname{vcomp}_{\vec{u}}(\alpha\vec{x}) = \frac{\vec{u}\cdot(\alpha\vec{x})}{13}\vec{u} = \frac{\alpha(\vec{u}\cdot\vec{x})}{13}\vec{u} = \alpha \operatorname{vcomp}_{\vec{u}}\vec{x}.$$

Image of a Set

DEFINITION

Let $L:\mathbb{R}^n\to\mathbb{R}^m$ be a transformation and let $X\subseteq\mathbb{R}^n$ be a set. The ***image of the set** X **under** L*, denoted $L(X)$, is the set

$$L(X) = \{\vec{y}\in\mathbb{R}^m : \vec{y} = L(\vec{x}) \text{ for some } \vec{x}\in X\}.$$

Work with *Images*.

The goal of this problem is to

- Compute images of sets under transformations.
- Develop geometric intuition for transformations of $\mathbb{R}^n$ in terms of inputs and outputs.
- Relate *images* to graphical problems like italicising N.

48 Let $S = \left\{\begin{bmatrix}x\\y\end{bmatrix} : 0\le x\le 1 \text{ and } 0\le y\le 1\right\}\subseteq\mathbb{R}^2$ be the filled-in unit square and let $C = \{\vec{0},\vec{e}_1,\vec{e}_2,\vec{e}_1+\vec{e}_2\}\subseteq\mathbb{R}^2$ be the corners of the unit square.

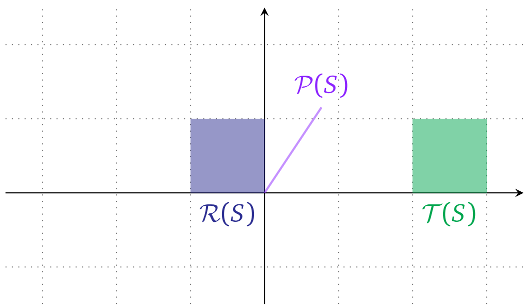
48.1 Find $\mathcal{R}(C)$, $W(C)$, and $T(C)$ (where $\mathcal{R}$, W, and T are from the previous question).

$\mathcal{R}(C) = \{\vec{0},\vec{e}_2,-\vec{e}_1,-\vec{e}_1+\vec{e}_2\}$.

$W(C) = C$.

$T(C) = \left\{\begin{bmatrix}2\\0\end{bmatrix},\begin{bmatrix}3\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\}$.

48.2 Draw $\mathcal{R}(S)$, $T(S)$, and $\mathcal{P}(S)$ (where $\mathcal{R}$, T, and $\mathcal{P}$ are from the previous question).



48.3 Let $\ell = \{\text{all convex combinations of } \vec{a} \text{ and } \vec{b}\}$ be a line segment with endpoints $\vec{a}$ and $\vec{b}$ and let A be a linear transformation. Must $A(\ell)$ be a line segment? What are its endpoints?

$A(\ell)$ must be a line segment, with endpoints $A(\vec{a})$ and $A(\vec{b})$.

For any scalars α_1 and α_2, by the linearity of A we have: $A(\alpha_1\vec{a}+\alpha_2\vec{b}) = \alpha_1A(\vec{a})+\alpha_2A(\vec{b})$.

If $\alpha_1+\alpha_2 = 1$, then the linear combination on the right is also convex, and so $A(\ell)$ is the set of convex combinations of $A(\vec{a})$ and $A(\vec{b})$. This is precisely the straight line segment joining $A(\vec{a})$ and $A(\vec{b})$.

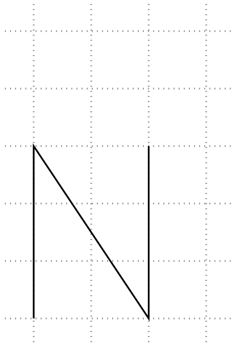
Note that if $A(\vec{a}) = A(\vec{b})$ (for example, if A is the zero transformation), then $A(\ell)$ will consist of the single point, which we think of as a "degenerate" line segment in this situation.

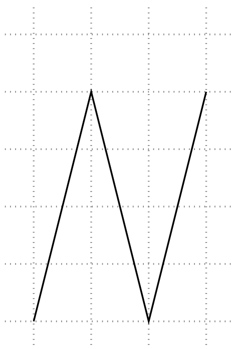
48.4 Explain how images of sets relate to the *Italicizing N* task.

The task asked us to find a linear transformation such that the image of the regular "N" is the italicized "N".

By the previous exercise, we now know it suffices to find a linear transformation that sends the four endpoints of line segments on the regular "N" to the corresponding four endpoints on the italicized "N".

49





Suppose that the “N” on the left is written in regular 12-point font. Find a matrix A that will transform the “N” into the letter on the right which is written in an *italic* 16-point font.

Two students—Pat and Jamie—explained their approach to italicizing the N as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the “N” taller, find a matrix that italicizes the taller “N,” and a combination of those two matrices will give the desired matrix A.*

1. Do you think Pat and Jamie’s approach allowed them to find A? If so, do you think they found the same matrix that you did during Italicising N?
2. Try Pat and Jamie’s approach. Either (a) come up with a matrix A using their approach, or (b) explain why their approach does not work.

Decompose a transformation into a composition of simpler transformations.

The goal of this problem is to

- Decompose a transformation into simpler ones.
- Produce examples showing matrix multiplication is not commutative.

Connect function composition and matrix multiplication.

The goal of this problem is to
- Distinguish between matrices and linear transformations.
- Explain the relationship between matrix multiplication and composition of linear transformations.

50 Define $\mathcal{P}$ to be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

50.1 Find a matrix P so that $P\vec{x} = \mathcal{P}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

$P = \frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ is such a matrix.

The matrix P corresponding to $\mathcal{P}$ is a 2×2 matrix, so suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some $a, b, c, d \in \mathbb{R}$. Then we know that if $\{\vec{e}_1, \vec{e}_2\}$ is the standard basis for $\mathbb{R}^2$,

$$P(\vec{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad P(\vec{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}.$$

We know from an earlier exercise that $\mathcal{P}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$. Therefore, the first column of P is

$$\begin{bmatrix} a \\ c \end{bmatrix} = \mathcal{P}(\vec{e}_1) = \frac{2}{13}\vec{u} = \frac{1}{13}\begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

and the second column of P is

$$\begin{bmatrix} b \\ d \end{bmatrix} = \mathcal{P}(\vec{e}_2) = \frac{3}{13}\vec{u} = \frac{1}{13}\begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

50.2 Find a matrix R so that $R\vec{x} = \mathcal{R}(\vec{x})$ for all $\vec{x} \in \mathbb{R}^2$.

$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is such a matrix.

Using the same reasoning as the previous part, we can compute

$$\mathcal{R}(\vec{e}_1) = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathcal{R}(\vec{e}_2) = -\vec{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Therefore, the matrix R for $\mathcal{R}$ is the matrix with the two vectors above as its respective columns.

50.3 Write down matrices A and B for $\mathcal{P} \circ \mathcal{R}$ and $\mathcal{R} \circ \mathcal{P}$.

$A = \frac{1}{13}\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix}$ and $B = \frac{1}{13}\begin{bmatrix} -6 & -9 \\ 4 & 6 \end{bmatrix}$ are two such matrices.

Using the same reasoning as above, we can compute

$$(\mathcal{P}\circ\mathcal{R})(\vec{e}_1) = \mathcal{P}(\mathcal{R}(\vec{e}_1)) = \mathcal{P}(\vec{e}_2) = \frac{1}{13}\begin{bmatrix} 6 \\ 9 \end{bmatrix} \quad \text{and} \quad (\mathcal{P}\circ\mathcal{R})(\vec{e}_2) = \mathcal{P}(\mathcal{R}(\vec{e}_2)) = \mathcal{P}(-\vec{e}_1) = \frac{1}{13}\begin{bmatrix} -4 \\ -6 \end{bmatrix}.$$

Therefore, the matrix A for $\mathcal{P} \circ \mathcal{R}$ is the matrix with the two vectors above as its respective columns.

Similarly, for $\mathcal{R} \circ \mathcal{P}$, we can compute:

$$(\mathcal{R} \circ \mathcal{P})(\vec{e}_1) = \mathcal{R}(\mathcal{P}(\vec{e}_1)) = \mathcal{R}\left(\frac{1}{13}\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \frac{1}{13}\begin{bmatrix} -6 \\ 4 \end{bmatrix}$$
$$(\mathcal{R} \circ \mathcal{P})(\vec{e}_2) = \mathcal{R}(\mathcal{P}(\vec{e}_2)) = \mathcal{R}\left(\frac{1}{13}\begin{bmatrix} 6 \\ 9 \end{bmatrix}\right) = \frac{1}{13}\begin{bmatrix} -9 \\ 6 \end{bmatrix}.$$

Therefore, the matrix B for $\mathcal{R} \circ \mathcal{P}$ is the matrix with these two vectors as its respective columns.

50.4 How do the matrices A and B relate to the matrices P and R?

$A = PR$ and $B = RP$.

We can compute these matrix products to see this, but from the previous parts, we know that for any vector $\vec{x}$

$$A\vec{x} = (\mathcal{P} \circ \mathcal{R})(\vec{x}) = \mathcal{P}(\mathcal{R}(\vec{x})) = \mathcal{P}(R\vec{x}) = PR\vec{x}$$

and

$$B\vec{x} = (\mathcal{R} \circ \mathcal{P})(\vec{x}) = \mathcal{R}(\mathcal{P}(\vec{x}) = \mathcal{R}(P\vec{x}) = RP\vec{x}.$$

Using $\vec{x} = \vec{e}_1$ shows that first column of A must equal the first column of PR, and using $\vec{x} = \vec{e}_2$ shows that the second column of A must equal the second column of PR, and therefore $A = PR$. For the same reason, we must also have $B = RP$.

Range

DEFINITION

The ***range*** (or ***image***) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$\text{range}(T) = \{\vec{y} \in W \ : \ \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\}.$$

Null Space

DEFINITION

The ***null space*** (or ***kernel***) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to the zero vector under T. That is,

$$\text{null}(T) = \{\vec{x} \in V \ : \ T\vec{x} = \vec{0}\}.$$

Understanding ranges and null spaces.

The goal of this problem is to
- Read and apply the definition of range and null space.
- Geometrically visualize the range and null space of a projection.

51 Let $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (like before).

51.1 What is the range of $\mathcal{P}$?

range$(\mathcal{P}) = $ span$\{\vec{u}\}$.

$\mathcal{P}(\vec{x})$ is by definition the vector in span$\{\vec{u}\}$ that is closest to $\vec{x}$, so in particular $\mathcal{P}(\vec{x}) \in$ span$\{\vec{u}\}$ for all $\vec{x} \in \mathbb{R}^2$. Therefore range$(\mathcal{P}) \subseteq$ span$\{\vec{u}\}$.

On the other hand, $\mathcal{P}(\alpha\vec{u}) = \alpha\mathcal{P}(\vec{u}) = \alpha\vec{u}$ for any scalar α, and so range$(\mathcal{P}) =$ span$\{\vec{u}\}$.

51.2 What is the null space of $\mathcal{P}$?

$$\text{null}(\mathcal{P}) = \text{span}\left\{\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right\}.$$

A vector $\vec{x}$ projects to $\vec{0}$ if and only if $\vec{x}$ is on the line orthogonal to span$\{\vec{u}\}$ passing through the origin.

Practicing proofs.

The goal of this problem is to
- Practice proving an abstract set (the range or the null space) is a subspace.

52 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary linear transformation.

52.1 Show that the null space of T is a subspace.

(i) Let $\vec{u}, \vec{v} \in$ null(T). Applying the linearity of T we see $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}$, and so $\vec{u} + \vec{v} \in$ null(T).

(ii) Let $\vec{u} \in$ null(T) and let α be any scalar. Again using the linearity of T we see $T(\alpha\vec{u}) = \alpha T(\vec{u}) = \alpha\vec{0} = \vec{0}$, and so $\alpha\vec{u} \in$ null(T).

52.2 Show that the range of T is a subspace.

Since range$(T) = T(\mathbb{R}^n)$ and $\mathbb{R}^n$ is non-empty, we know that range(T) is non-empty. Therefore, to show that range(T) is a subspace, we need to show (i) that it's closed under vector addition, and (ii) that it is closed under scalar multiplication.

(i) Let $\vec{x}, \vec{y} \in$ range(T). By definition, there exist $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $T(\vec{u}) = \vec{x}$ and $T(\vec{v}) = \vec{y}$. Since T is linear, $\vec{x} + \vec{y} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v})$, and so $\vec{x} + \vec{y} \in$ range(T).

(ii) Let $\vec{x} \in$ range(T) and let α be a scalar. By definition, there exists $\vec{u} \in \mathbb{R}^n$ such that $T(\vec{u}) = \vec{x}$, and so by the linearity of T, $\alpha\vec{x} = \alpha T(\vec{u}) = T(\alpha\vec{u})$. Therefore $\alpha\vec{x} \in$ range(T).

Formalizing the connection between matrices and linear transformations.

The goal of this problem is to
- Distinguish between linear transformations and matrices.
- Explain how to relate matrices and linear transformations.
- Practice using formal language and notation, avoiding category errors.

Induced Transformation

DEFINITION

Let M be an $n \times m$ matrix. We say M ***induces*** a linear transformation $\mathcal{T}_M : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$[\mathcal{T}_M\vec{v}]_{\mathcal{E}'} = M[\vec{v}]_{\mathcal{E}},$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^m$ and $\mathcal{E}'$ is the standard basis for $\mathbb{R}^n$.

53 Let $M=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, let $\vec v=\vec e_1+\vec e_2\in\mathbb{R}^2$, and let $\mathcal{T}_M$ be the transformation induced by M.

53.1 What is the difference between "$M\vec v$" and "$M[\vec v]_{\mathcal{E}}$"?

"$M\vec v$" is ambiguous notation, as it is only defined if $\vec v$ is a specific list of numbers. There are infinitely many different bases of $\mathbb{R}^2$ and so $\vec v$ has infinitely many different representations as a list of numbers (a different basis produces a different representation). For example, let $\mathcal{B}=\{\vec v,\vec e_1\}$. We now have $[\vec v]_{\mathcal{E}}=\begin{bmatrix}1\\1\end{bmatrix}$ and $[\vec v]_{\mathcal{B}}=\begin{bmatrix}1\\0\end{bmatrix}$. There's no way to decide whether $M\vec v$ should mean $M\begin{bmatrix}1\\1\end{bmatrix}$ or $M\begin{bmatrix}1\\0\end{bmatrix}$ or any of the infinitely many other options.

"$M[\vec v]_{\mathcal{E}}$" is unambiguous, as $[\vec v]_{\mathcal{E}}$ is an explicit representation of $\vec v$ in a particular basis.

53.2 What is $[\mathcal{T}_M\vec e_1]_{\mathcal{E}}$?

It is the first column of M.

By definition, $[\mathcal{T}_M\vec e_1]_{\mathcal{E}}=M[\vec e_1]_{\mathcal{E}}=M\begin{bmatrix}1\\0\end{bmatrix}$, which equals the first column of M.

53.3 Can you relate the columns of M to the range of $\mathcal{T}_M$?

The range of $\mathcal{T}_M$ equals the span of the columns of M.

By the previous part, the first column of M is in the range of $\mathcal{T}_M$. By a similar argument, the second column of M is also in the range of $\mathcal{T}_M$, since it equals $[\mathcal{T}_M\vec e_2]_{\mathcal{E}}$. Therefore the span of the columns of M is a subset of the range of $\mathcal{T}_M$.

On the other hand, if $\vec v_1=\begin{bmatrix}a\\c\end{bmatrix}$ and $\vec v_2=\begin{bmatrix}b\\d\end{bmatrix}$ are the columns of M and $\vec x=\alpha_1\vec v_1+\alpha_2\vec v_2$ is an element of $\operatorname{span}\{\vec v_1,\vec v_2\}$, then

$$[\vec x]_{\mathcal{E}}=\alpha_1\begin{bmatrix}a\\c\end{bmatrix}+\alpha_2\begin{bmatrix}b\\d\end{bmatrix}=\alpha_1[\mathcal{T}_M\vec e_1]_{\mathcal{E}}+\alpha_2[\mathcal{T}_M\vec e_2]_{\mathcal{E}}=[\mathcal{T}_M(\alpha_1\vec e_1+\alpha_2\vec e_2)]_{\mathcal{E}}.$$

Therefore $\vec x$ is in the range of $\mathcal{T}_M$.

Fundamental Subspaces

DEFINITION

Associated with any matrix M are three fundamental subspaces: the ***row space*** of M, denoted $\operatorname{row}(M)$, is the span of the rows of M; the ***column space*** of M, denoted $\operatorname{col}(M)$, is the span of the columns of M; and the ***null space*** of M, denoted $\operatorname{null}(M)$, is the set of solutions to $M\vec x=\vec 0$.

54 Consider $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$.

Fundamental subspaces of a matrix.

The goal of this problem is to
- Compute row and column spaces of a matrix.
- Recognize that row and column spaces may be unrelated.
- Geometrically relate the row space to the null space.
- Connect the fundamental subspaces of a matrix to the range and null space of a transformation.

54.1 Describe the row space of A.

$$\operatorname{row}(A)=\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\},\text{ which is the } xy\text{-plane in } \mathbb{R}^3.$$

54.2 Describe the column space of A.

$$\operatorname{col}(A)=\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}=\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}=\mathbb{R}^2.$$

54.3 Is the row space of A the same as the column space of A?

No.

Although they are both two dimensional spaces, $\operatorname{row}(A)$ is a subspace of $\mathbb{R}^3$ and all vectors in it have three coordinates (with the third always being zero), while $\operatorname{col}(A)$ is a subspace of $\mathbb{R}^2$ and all vectors in it have two coordinates. Therefore, these two spaces are different.

54.4 Describe the set of all vectors orthogonal to the rows of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to the rows of A if and only if its dot product with both rows is zero. That is

$$\vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = y = 0.$$

$\vec{x}$ satisfies these equations if and only if $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$ for some real number t, or in other words if $\vec{x}$ is on the z-axis.

54.5 Describe the null space of A.

The z-axis in $\mathbb{R}^3$.

A vector $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in null(A) if and only if

$$A\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}.$$

These are the same conditions as in the previous part, so the set of vectors satisfying this is the z-axis.

54.6 Describe the range and null space of T_A, the transformation induced by A.

$\text{range}(T_A) = \text{col}(A) = \mathbb{R}^2$ and $\text{null}(T_A) = \text{null}(A)$, which is the z-axis in $\mathbb{R}^3$.

By Problem 53.3, the range of an induced transformation equals the span of the columns of the matrix. In other words, $\text{range}(T_A) = \text{col}(A)$.

Next, by definition $\vec{v} \in \text{null}(T_A)$ when $[T_A\vec{v}]_{\mathcal{E}} = A[\vec{v}]_{\mathcal{E}} = \vec{0}$. In other words, $\vec{v} \in \text{null}(T_A)$ if and only if $[\vec{v}]_{\mathcal{E}} \in \text{null}(A)$. We know from the previous part that $\text{null}(A)$ is the z-axis in $\mathbb{R}^3$.

55

Fundamental subspaces and row reduction.

The goal of this problem is to
- Explain why row reduction doesn't change the row space or the null space.

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \qquad C = \text{rref}(B) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

55.1 How does the row space of B relate to the row space of C?

They are equal.

Row operations replace rows with linear combinations of rows. Therefore, since C is the matrix B after the application of some row operations, $\text{row}(C) \subseteq \text{row}(B)$.

Since row operations are all reversible, we also know that B can be obtained from C by applying row operations, so $\text{row}(B) \subseteq \text{row}(C)$.

Therefore, $\text{row}(B) = \text{row}(C)$.

55.2 How does the null space of B relate to the null space of C?

They are equal.

A vector is in $\text{null}(B)$ or $\text{null}(C)$ if and only if it is orthogonal to all vectors in $\text{row}(B)$ or all vectors in $\text{row}(C)$, respectively. But $\text{row}(B) = \text{row}(C)$ by the previous part, so their null spaces must also be equal.

55.3 Compute the null space of B.

$$\text{null}(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

We compute $\text{null}(C)$, since it equals $\text{null}(B)$ by the previous part.

$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in null(C) if and only if $C\vec{x} = \begin{bmatrix} x - z \\ y + 2z \end{bmatrix} = \vec{0}$. The complete solution to this matrix equation is

$$\text{null}(C) = \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \in \mathbb{R}^3 \,:\, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

56

Fundamental subspaces and row reduction.

The goal of this problem is to
- Recognize that row reduction may change the column space of a matrix.

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \qquad Q = \text{rref}(P) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

56.1 How does the column space of P relate to the column space of Q?

They are not equal, but have the same dimension.

56.2 Describe the column space of P and the column space of Q.

$\text{col}(P) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, which is the y-axis in $\mathbb{R}^2$.

$\text{col}(Q) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, which is the x-axis in $\mathbb{R}^2$.

Rank

DEFINITION

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the ***rank*** of T, denoted rank(T), is the dimension of the range of T.

For an $m \times n$ matrix M, the ***rank*** of M, denoted rank(M), is the dimension of the column space of M.

57

Rank of linear transformations.

The goal of this problem is to
- Apply the definition of *rank* to compute the rank of a linear transformation.
- Use geometric intuition to compute the rank of a linear transformation.
- Relate the rank of a linear transformation to the rank of its matrix.

Let $\mathcal{P}$ be projection onto span$\{\vec{u}\}$ where $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and let $\mathcal{R}$ be rotation counter-clockwise by 90°.

57.1 Describe range$(\mathcal{P})$ and range$(\mathcal{R})$.

range$(\mathcal{P})$ = span$\{\vec{u}\}$, and range$(\mathcal{R}) = \mathbb{R}^2$.

For $\mathcal{P}$, by the definition of projection $\mathcal{P}(\vec{x})$ is the vector in span$\{\vec{u}\}$ that is closest to $\vec{x}$, so in particular $\mathcal{P}(\vec{x}) \in \text{span}\{\vec{u}\}$ for all $\vec{x} \in \mathbb{R}^2$. Therefore range$(\mathcal{P}) \subseteq \text{span}\{\vec{u}\}$.

On the other hand, $\mathcal{P}(\alpha\vec{u}) = \alpha\mathcal{P}(\vec{u}) = \alpha\vec{u}$ for any scalar α, and so range$(\mathcal{P})$ = span$\{\vec{u}\}$.

For $\mathcal{Q}$, we have that any vector $\vec{x} \in \mathbb{R}^2$, $\vec{x} = \mathcal{Q}(\vec{y})$, where $\vec{y}$ is the rotation of $\vec{x}$ clockwise by 90°. Therefore range$(\mathcal{Q}) = \mathbb{R}^2$.

57.2 What is the rank of $\mathcal{P}$ and the rank of $\mathcal{R}$?

rank$(\mathcal{P}) = 1$ and rank$(\mathcal{R}) = 2$.

By the previous part, we know range$(\mathcal{P})$ is 1-dimensional and range$(\mathcal{Q})$ is 2-dimensional.

57.3 Let P and R be the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. What is the rank of P and the rank of R?

rank$(P) = 1$ and rank$(R) = 2$.

By Problem 50, $P = \frac{1}{13}\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ and $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are the matrices corresponding to $\mathcal{P}$ and $\mathcal{R}$. Then we compute:

$$\text{rref}(P) = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{rref}(R) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

These matrices have 1 and 2 pivots, respectively.

57.4 Make a conjecture about how the rank of a transformation and the rank of its corresponding matrix relate. Can you justify your claim?

They are equal.

By Problem 53.3, the range of a transformation is equal to the column space of its corresponding matrix, and therefore the dimensions of these two spaces are equal. In other words, the rank of a transformation is equal to the dimension of the column space of its corresponding matrix.

We already know that the dimension of the column space of a matrix is equal to the number of pivots in its reduced row echelon form, and that is by definition the rank of the matrix.

58

Rank of matrices.

The goal of this problem is to

- Use the definition of rank to compute the rank of matrices.

58.1 Determine the rank of (a) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$ (b) $\begin{bmatrix}1&2\\3&4\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}$ (d) $\begin{bmatrix}3\\3\\2\end{bmatrix}$ (e) $\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$.

For each part, we compute the reduced row echelon form of the matrix and count the number of pivots.

(a) $\mathrm{rank}\left(\begin{bmatrix}1&1\\2&2\end{bmatrix}\right)=1$, since $\mathrm{rref}\left(\begin{bmatrix}1&1\\2&2\end{bmatrix}\right)=\begin{bmatrix}1&1\\0&0\end{bmatrix}$ has one pivot.

(b) $\mathrm{rank}\left(\begin{bmatrix}1&2\\3&4\end{bmatrix}\right)=2$, since $\mathrm{rref}\left(\begin{bmatrix}1&2\\3&4\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ has two pivots

(c) $\mathrm{rank}\left(\begin{bmatrix}1&1&0\\0&0&1\end{bmatrix}\right)=2$. This matrix is already in reduced row echelon form, and has two pivots.

(d) $\mathrm{rank}\left(\begin{bmatrix}3\\3\\2\end{bmatrix}\right)=1$, since $\mathrm{rref}\left(\begin{bmatrix}3\\3\\2\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}$ has one pivot.

(e) $\mathrm{rank}\left(\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right)=3$, since $\mathrm{rref}\left(\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ has three pivots.

Connect rank to existing concepts.

The goal of this problem is to

- Connect $\mathrm{rank}(A)$ to the number of solutions to $A\vec{x}=\vec{0}$.
- Connect $\mathrm{rank}(A)$ to linear independence or dependence of the columns of A.

59 Consider the homogeneous system

$$\begin{array}{llll} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =0 \end{array} \tag{1}$$

and the non-augmented matrix of coefficients $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 1 \end{bmatrix}$.

59.1 What is $\text{rank}(A)$?

$\text{rank}(A) = 2$, since $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ as two pivots.

59.2 Give the general solution to system (1).

$\vec{x}$ is a solution to the system if $\vec{x} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ for some real number t.

If $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a solution to the system, then we must have $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, from which it follows that $z = 0$ and $x = -2y$. In other words, any scalar multiple of $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is a solution.

59.3 Are the column vectors of A linearly independent?

No. The second column is two times the first column.

59.4 Give a non-homogeneous system with the same coefficients as (1) that has

(a) infinitely many solutions

(b) no solutions.

(a)

$$\begin{array}{llll} x & +2y & +z & =1 \\ x & +2y & +3z & =1 \\ -x & -2y & +z & =-1 \end{array}$$

(b)

$$\begin{array}{llll} x & +2y & +z & =0 \\ x & +2y & +3z & =0 \\ -x & -2y & +z & =1 \end{array}$$

Connect the rank of a matrix to the linear independence/dependence of its columns.

The goal of this problem is to

- Determine the linear independence/dependence of the columns of a matrix based on its size and rank.

60

60.1 The rank of a 3×4 matrix A is 3. Are the column vectors of A linearly independent?

No. A 3×4 matrix has four columns, each of which are vectors in $\mathbb{R}^3$. It is not possible for four different vectors in $\mathbb{R}^3$ to be linearly independent.

60.2 The rank of a 4×3 matrix B is 3. Are the column vectors of B linearly independent?

Yes. Since $\operatorname{rank}(B) = 3$, there are three pivots in $\operatorname{rref}(B)$. Pivot positions in $\operatorname{rref}(B)$ indicate a maximal linearly independent subset of the columns of B. Since there are three columns in B and three pivots, the three columns of B must be linearly independent.

Rank-nullity Theorem

THEOREM

The ***nullity*** of a matrix is the dimension of the null space.

The rank-nullity theorem for a matrix A states

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = \#\text{ of columns in } A.$$

61

Relate linear transformation concepts with matrix concepts.

The goal of this problem is to

- Rephrase the rank-nullity theorem as stated for matrices as the rank-nullity theorem for linear transformations.

61.1 Is there a version of the rank-nullity theorem that applies to linear transformations instead of matrices? If so, state it.

Yes. If $T : V \to W$ is a linear transformation, then $\operatorname{rank}(T) + \dim(\operatorname{null}(T)) = \dim(V)$.

If A is the matrix corresponding to T, then $\operatorname{rank}(T) = \operatorname{rank}(A)$ by Problem 57.4.

$\operatorname{null}(T) = \operatorname{null}(A)$ by Problem 54.6, since $T = T_A$, and so $\dim(\operatorname{null}(T)) = \operatorname{nullity}(A)$.

Finally, the number of columns of A is equal to the dimension of the domain of T.

Apply the rank-nullity theorem.

The goal of this problem is to

- Apply the rank-nullity theorem to compute the rank or nullity of unknown matrices.

62

The vectors $\vec{u}, \vec{v} \in \mathbb{R}^9$ are linearly independent and $\vec{w} = 2\vec{u} - \vec{v}$. Define $A = [\vec{u}|\vec{v}|\vec{w}]$.

62.1 What is the rank and nullity of A^T?

$\operatorname{rank}(A^T) = 2$ and $\operatorname{nullity}(A^T) = 7$.

A^T is the matrix with rows $\vec{u}$, $\vec{v}$, and $\vec{w}$. Since $\vec{w} = 2\vec{u} - \vec{v}$, the third row of A^T can be reduced to a row of zeros by the row operation $R_3 \mapsto R_3 - 2R_1 + R_2$. Neither of the first two rows can be reduced to rows of zeros since they are linearly independent. Therefore $\operatorname{rref}(A^T)$ has two pivots, meaning $\operatorname{rank}(A^T) = 2$.

The rank-nullity theorem then says that $2 + \operatorname{nullity}(A^T) = 9$, and so $\operatorname{nullity}(A^T) = 7$

62.2 What is the rank and nullity of A?

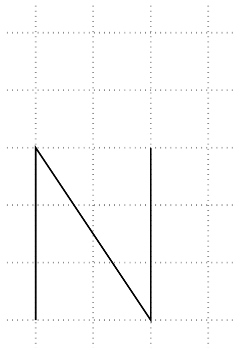
$\operatorname{rank}(A) = 2$ and $\operatorname{nullity}(A) = 1$.

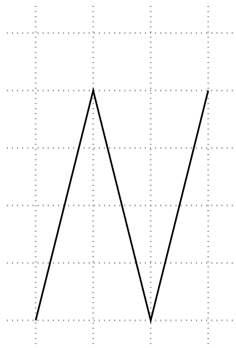
We know that $\operatorname{rank}(A)$ equals the number of pivots in $\operatorname{rref}(A)$, which in turn equals the dimension of $\operatorname{col}(A)$. Since A has two linearly independent columns, $\dim(\operatorname{col}(A)) = 2$.

Again, the rank-nullity theorem then says that $2 + \operatorname{nullity}(A) = 3$, and so $\operatorname{nullity}(A) = 1$

63 "We've made a terrible mistake," a council member says. "Can we go back to the regular N?"

Recall the original Italicising N task.





Suppose that the "N" on the left is written in regular 12-point font. Find a matrix A that will transform the "N" into the letter on the right which is written in an *italic* 16-point font.

Pat and Jamie explained their approach to the Italicizing N task as follows:

> *In order to find the matrix A, we are going to find a matrix that makes the "N" taller, find a matrix that italicizes the taller "N," and a combination of those two matrices will give the desired matrix A.*

The Oronto city council has asked you to *unitalicise* the N. Your new task is to find a matrix C that transforms the "N" on the right to the "N" on the left.

1. Use any method you like to find C.
2. Use a method similar to Pat and Jamie's method, only use it to find C instead of A.

64

64.1 Apply the row operation $R_3 \mapsto R_3 + 2R_1$ to the 3×3 identity matrix and call the result E_1.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3+2R_1} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_1.$$

64.2 Apply the row operation $R_3 \mapsto R_3 - 2R_1$ to the 3×3 identity matrix and call the result E_2.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3 \mapsto R_3-2R_1} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = E_2.$$

Elementary Matrix

DEF A matrix is called an ***elementary matrix*** if it is an identity matrix with a single elementary row operation applied.

Elementary matrices.

The goal of this problem is to

- Define *elementary matrices*.
- Relate elementary matrices to row reduction.
- Use the "reversibility" of elementary row operations to create inverses to elementary matrices.

$$A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$$

64.3 Compute E_1A and E_2A. How do the resulting matrices relate to row operations?

$$E_1A = \begin{bmatrix}1&2&3\\4&5&6\\9&12&15\end{bmatrix} \text{ and } E_2A = \begin{bmatrix}1&2&3\\4&5&6\\5&4&3\end{bmatrix}.$$

E_1A is the result applying the row operation $R_3 \mapsto R_3 + 2R_1$ to A, and similarly E_2A is the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to A.

64.4 Without computing, what should the result of applying the row operation $R_3 \mapsto R_3 - 2R_1$ to E_1 be? Compute and verify.

It should be the identity matrix, since the row operation $R_3 \mapsto R_3 - 2R_1$ should undo the operation $R_3 \mapsto R_3 + 2R_1$.

64.5 Without computing, what should E_2E_1 be? What about E_1E_2? Now compute and verify.

They should both be the identity matrix.

The solution to part 3 above lead us to believe that applying E_1 to a matrix has the effect of applying the row operation $R_3 \mapsto R_3 + 2R_1$ to it. Applying that row operation to E_2 would produce the identity matrix, so we expect that E_1E_2 should equal the identity matrix.

Similar reasoning leads us to believe that E_2E_1 should also equal the identity matrix.

Indeed, we can compute

$$E_1E_2 = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_2E_1.$$

Apply the definition of *inverse matrix*.

The goal of this problem is to

- Use the definition of *inverse matrix* to identify whether two matrices are inverses of each other.

Matrix Inverse

DEF The ***inverse*** of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A and is notated A^{-1}.

65 Consider the matrices

$$A=\begin{bmatrix}1&2&0\\0&1&0\\-3&-6&1\end{bmatrix}\qquad B=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\qquad C=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

$$D=\begin{bmatrix}1&-2&0\\0&1&0\\3&0&1\end{bmatrix}\qquad E=\begin{bmatrix}1&0&0\\0&2&0\\0&1&1\end{bmatrix}\qquad F=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

65.1 Which pairs of matrices above are inverses of each other?

A and D are inverses of each other, and F is its own inverse.

66

Compute inverses.

The goal of this problem is to
- Use elementary matrices to compute matrix inverses.
- Decompose an invertible matrix into the product of elementary matrices.

$$B=\begin{bmatrix}1&4\\0&2\end{bmatrix}$$

66.1 Use two row operations to reduce B to $I_{2\times 2}$ and write an elementary matrix E_1 corresponding to the first operation and E_2 corresponding to the second.

$$\begin{bmatrix}1&4\\0&2\end{bmatrix}\xrightarrow{R_2\mapsto\frac{1}{2}R_2}\begin{bmatrix}1&4\\0&1\end{bmatrix}\xrightarrow{R_1\mapsto R_1-4R_2}\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

The two elementary matrices are $E_1=\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}$ and $E_2=\begin{bmatrix}1&-4\\0&1\end{bmatrix}$.

66.2 What is E_2E_1B?

$$E_2E_1B=\begin{bmatrix}1&-4\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&4\\0&2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

66.3 Find B^{-1}.

$$B^{-1}=E_2E_1=\begin{bmatrix}1&-2\\0&\frac{1}{2}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2&-4\\0&1\end{bmatrix}.$$

By the previous part we already know that $(E_2E_1)B=I$. We can also check that $B(E_2E_1)=I$, meaning E_2E_1 is the inverse of B.

66.4 Can you outline a procedure for finding the inverse of a matrix using elementary matrices?

Suppose A is a matrix that can be row reduced to the identity. Let $E_1,E_2,\ldots,E_n$ be the elementary matrices corresponding to the sequence of row operations that reduces A to I. Then as we have seen, we have $E_nE_{n-1}\cdots E_2E_1A=I$.

Thus $E_nE_{n-1}\cdots E_2E_1$ is the inverse of A.

67

Solve systems with inverses.

The goal of this problem is to
- Relate inverse matrices to the previous methods for solving equations, row reduction.
- Symbolically write the solution to a matrix equation using inverses.

$$A=\begin{bmatrix}1&2&-1\\2&2&4\\1&3&-3\end{bmatrix}\qquad \vec b=\begin{bmatrix}1\\2\\3\end{bmatrix}\qquad C=[A|\vec b]\qquad A^{-1}=\begin{bmatrix}9&-3/2&-5\\-5&1&3\\-2&1/2&1\end{bmatrix}$$

67.1 What is $A^{-1}A$?

$A^{-1}A=I$. This is true by the definition of an inverse, but we can also verify it by hand.

67.2 What is $\mathrm{rref}(A)$?

$$\mathrm{rref}(A)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I.$$

67.3 What is $\mathrm{rref}(C)$? (Hint, there is no need to actually do row reduction!)

$$\operatorname{rref}(C)=\left[I\middle|A^{-1}\vec b\right]=\left[\begin{array}{ccc|c}1&0&0&-9\\0&1&0&6\\0&0&1&2\end{array}\right].$$

We know that the reduced row echelon form of C must be of the form $[I|\vec c]$ for some $\vec c$, and we know that multiplying on the left by A^{-1} is equivalent to applying the sequence of row operations that reduces A to $\operatorname{rref}(A)=I$. So the same sequence of row operations applied to $\vec b$, the last column of C, will produce the vector $\vec c=A^{-1}\vec b=\begin{bmatrix}-9\\6\\2\end{bmatrix}$.

67.4 Solve the system $A\vec x=\vec b$.

The system has one solution: $\vec x=A^{-1}\vec b=\begin{bmatrix}-9\\6\\2\end{bmatrix}$.

We can read this solution from the reduced row echelon form of the augmented matrix C representing this system. We can also multiply both sides of the equation on the left by A^{-1}:

$$A\vec x=\vec x \implies A^{-1}A\vec x=A^{-1}\vec b \implies \vec x=A^{-1}\vec b.$$

68

Inverses and composition.

The goal of this problem is to
- Create a correct formula for $(XY)^{-1}$ and explain it algebraically or in terms of function composition.
- Relate invertibility of a matrix and its induced transformation.

68.1 For two square matrices X,Y, should $(XY)^{-1}=X^{-1}Y^{-1}$?

No.

By the definition of an inverse we need $(XY)^{-1}(XY)=I$, so that multiplying by $(XY)^{-1}$ undoes multiplication by XY. To do this, we must first undo multiplication by X, then undo multiplication by Y. That is, we must first multiply by X^{-1} then multiply by Y^{-1}.

In other words, we expect that $(XY)^{-1}=Y^{-1}X^{-1}$. We can then verify this by computing

$$(XY)(Y^{-1}X^{-1})=XYY^{-1}X^{-1}=XIX^{-1}=XX^{-1}=I$$

and

$$(Y^{-1}X^{-1})(XY)=Y^{-1}X^{-1}XY=Y^{-1}IY=Y^{-1}Y=I.$$

68.2 If M is a matrix corresponding to a non-invertible linear transformation T, could M be invertible?

No.

Suppose M^{-1} exists. Then $M^{-1}M=MM^{-1}=I$. Let S be the linear transformation induced by M^{-1}. Since M is the matrix for T we must have $S\circ T=T\circ S=\mathrm{id}$. But then S would be the inverse of T, which is impossible.

More Change of Basis

69

Inverses and change of basis.

The goal of this problem is to
- Use inverses to answer change-of-basis questions.
- Explain why the inverse of a change-of-basis matrix is another change of basis matrix.

Let $\mathcal B=\{\vec b_1,\vec b_2\}$ where $\vec b_1=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec b_2=\begin{bmatrix}1\\-1\end{bmatrix}$ and let $X=[\vec b_1|\vec b_2]$ be the matrix whose columns are $\vec b_1$ and $\vec b_2$.

69.1 Compute $[\vec e_1]_{\mathcal B}$ and $[\vec e_2]_{\mathcal B}$.

$[\vec e_1]_{\mathcal B}=\frac12\begin{bmatrix}1\\1\end{bmatrix}$ and $[\vec e_2]_{\mathcal B}=\frac12\begin{bmatrix}1\\-1\end{bmatrix}$.

This is because $\vec e_1=\frac12(\vec b_1+\vec b_2)$ and $\vec e_2=\frac12(\vec b_1-\vec b_2)$

69.2 Compute $X[\vec e_1]_{\mathcal B}$ and $X[\vec e_2]_{\mathcal B}$. What do you notice?

$X[\vec{e}_1]_\mathcal{B} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 + \frac{1}{2}\vec{b}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ and

$X[\vec{e}_2]_\mathcal{B} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\vec{b}_1 - \frac{1}{2}\vec{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

We notice that multiplying by X turns the representations of these two vectors in the basis $\mathcal{B}$ into representations in the standard basis.

69.3 Find the matrix X^{-1}. How does X^{-1} relate to change of basis?

$X^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

X^{-1} should undo what X does. In the previous part we saw that X takes vectors represented in $\mathcal{B}$ and represents them in the standard basis. So X^{-1} should do the reverse, and take vectors represented in the standard basis and represent them in the basis $\mathcal{B}$.

Inverses and change of basis in arbitrary dimensions.

The goal of this problem is to
- Recognize $[\vec{b}_1|\cdots|\vec{b}_n]$ as a change-of-basis matrix.
- Explain why changing basis is an invertible operation.
- Explain how the representation of the standard basis vectors as columns of 0's and one 1 is a result of representing a vector in its own basis and not something special about the standard basis.

70 Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Given a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$, the matrix $X = [\vec{b}_1|\vec{b}_2|\cdots|\vec{b}_n]$ converts vectors from the $\mathcal{B}$ basis into the standard basis. In other words,

$$X[\vec{v}]_\mathcal{B} = [\vec{v}]_\mathcal{E}.$$

70.1 Should X^{-1} exist? Explain.

Yes. X converts vectors from the $\mathcal{B}$ basis to the standard basis, and this process can be undone. X^{-1} is the matrix that does this.

70.2 Consider the equation

$$X^{-1}[\vec{v}]_? = [\vec{v}]_?.$$

Can you fill in the "?" symbols so that the equation makes sense?

$X^{-1}[\vec{v}]_\mathcal{E} = [\vec{v}]_\mathcal{B}$.

As we said in the previous part X^{-1} should undo what X does, meaning it should convert vectors from the standard basis into the $\mathcal{B}$ basis.

70.3 What is $[\vec{b}_1]_\mathcal{B}$? How about $[\vec{b}_2]_\mathcal{B}$? Can you generalize to $[\vec{b}_i]_\mathcal{B}$?

$[\vec{b}_1]_\mathcal{B} = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$, and $[\vec{b}_2]_\mathcal{B} = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, where each of these vectors have n coordinates.

In general, $[\vec{b}_i]_\mathcal{B}$ should be the column vector with zeroes in all coordinates except for a 1 in the i^{th} coordinate.

Representations of transformations.

The goal of this problem is to
- Represent a transformation as a matrix in different bases.
- Recognize that some bases give *nicer* matrix representations than others.
- Connect the definition of similar matrices to change-of-basis.

71 Let $\vec{c}_1 = \begin{bmatrix}2\\1\end{bmatrix}_\mathcal{E}$, $\vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_\mathcal{E}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$, and $A = \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$. Note that $A^{-1} = \begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$ and that A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis.

71.1 Compute $[\vec{c}_1]_\mathcal{C}$ and $[\vec{c}_2]_\mathcal{C}$. $[\vec{c}_1]_\mathcal{C} = \begin{bmatrix}1\\0\end{bmatrix}$ and $[\vec{c}_2]_\mathcal{C} = \begin{bmatrix}0\\1\end{bmatrix}$.

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $\vec{c}_1$ direction by a factor of 2 and doesn't stretch in the $\vec{c}_2$ direction at all.

71.2 Compute $T\begin{bmatrix}2\\1\end{bmatrix}_\mathcal{E}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_\mathcal{E}$. $T\begin{bmatrix}2\\1\end{bmatrix}_\mathcal{E} = T\vec{c}_1 = 2\vec{c}_1 = \begin{bmatrix}4\\2\end{bmatrix}_\mathcal{E}$ and $T\begin{bmatrix}5\\3\end{bmatrix}_\mathcal{E} = T\vec{c}_2 = \vec{c}_2 = \begin{bmatrix}5\\3\end{bmatrix}_\mathcal{E}$.

71.3 Compute $[T\vec{c}_1]_\mathcal{C}$ and $[T\vec{c}_2]_\mathcal{C}$. $[T\vec{c}_1]_\mathcal{C} = [2\vec{c}_1]_\mathcal{C} = \begin{bmatrix}2\\0\end{bmatrix}$ and $[T\vec{c}_2]_\mathcal{C} = [\vec{c}_2]_\mathcal{C} = \begin{bmatrix}0\\1\end{bmatrix}$.

71.4 Compute the result of $T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$ and express the result in the $\mathcal{C}$ basis (i.e., as a vector of the form $\begin{bmatrix}?\\?\end{bmatrix}_{\mathcal{C}}$).

$T\begin{bmatrix}\alpha\\\beta\end{bmatrix}_{\mathcal{C}}=\begin{bmatrix}2\alpha\\\beta\end{bmatrix}_{\mathcal{C}}$.

If $\vec{v}$ is a vector such that $[\vec{v}]_{\mathcal{C}}=\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, then $\vec{v}=\alpha\vec{c}_1+\beta\vec{c}_2$. Since T is linear, we can then compute

$$T\vec{v}=T(\alpha\vec{c}_1+\beta\vec{c}_2)=\alpha T(\vec{c}_1)+\beta T(\vec{c}_2)=2\alpha\vec{c}_1+\beta\vec{c}_2=\begin{bmatrix}2\alpha\\\beta\end{bmatrix}_{\mathcal{C}}.$$

71.5 Find $[T]_{\mathcal{C}}$, the matrix for T in the $\mathcal{C}$ basis.

$[T]_{\mathcal{C}}=\begin{bmatrix}2&0\\0&1\end{bmatrix}$.

From the results of the previous parts, we know that we must have $[T]_{\mathcal{C}}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$ and $[T]_{\mathcal{C}}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$, so these must be the first and second columns of $[T]_{\mathcal{C}}$, respectively.

71.6 Find $[T]_{\mathcal{E}}$, the matrix for T in the standard basis.

$[T]_{\mathcal{E}}=\begin{bmatrix}7&-10\\3&-4\end{bmatrix}$

There are two methods to determine this.

Method 1: Since $\vec{e}_1=3\vec{c}_1-\vec{c}_2$ and $\vec{e}_2=-5\vec{c}_1+2\vec{c}_2$, we compute

$$[T\vec{e}_1]_{\mathcal{E}}=[T(3\vec{c}_1-\vec{c}_2)]_{\mathcal{E}}=3[T(\vec{c}_1)]_{\mathcal{E}}-[T(\vec{c}_2)]_{\mathcal{E}}=3\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}$$

and

$$[T\vec{e}_2]_{\mathcal{E}}=[T(-5\vec{c}_1+2\vec{c}_2)]_{\mathcal{E}}=-5[T(\vec{c}_1)]_{\mathcal{E}}+2[T(\vec{c}_2)]_{\mathcal{E}}=-5\begin{bmatrix}4\\2\end{bmatrix}+2\begin{bmatrix}5\\3\end{bmatrix}=\begin{bmatrix}-10\\-4\end{bmatrix}.$$

These two vectors are the respective columns of $[T]_{\mathcal{E}}$, as usual.

Method 2: Since A changes vectors from the $\mathcal{C}$ basis to the standard basis and A^{-1} changes vectors from the standard basis to the $\mathcal{C}$ basis, we know $[T]_{\mathcal{E}}=A[T]_{\mathcal{C}}A^{-1}$. Using $[T]_{\mathcal{C}}$ from the previous part, we compute

$$[T]_{\mathcal{E}}=\begin{bmatrix}2&5\\1&3\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}3&-5\\-1&2\end{bmatrix}=\begin{bmatrix}7&-10\\3&-4\end{bmatrix}.$$

DEFINITION

Similar Matrices

The matrices A and B are called ***similar matrices***, denoted $A\sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A\sim B$ if there is an invertible matrix X so that

$$A=XBX^{-1}.$$

Determinants

DEFINITION

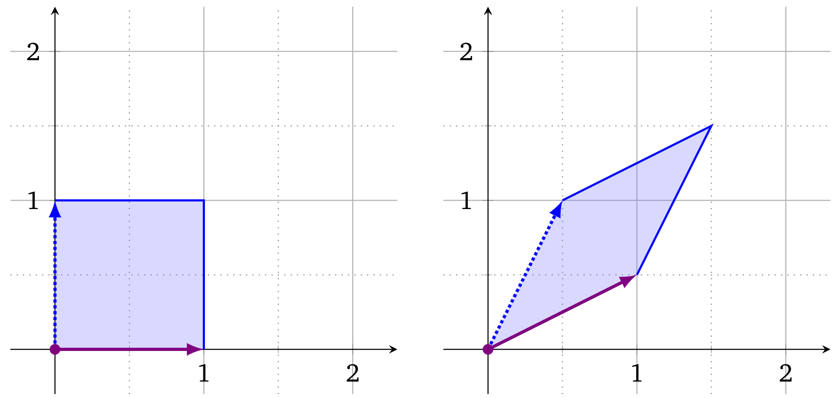
Unit n-cube

The ***unit n-cube*** is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin. That is

$$C_n=\left\{\vec{x}\in\mathbb{R}^n:\vec{x}=\sum_{i=1}^n\alpha_i\vec{e}_i\text{ for some }\alpha_1,\dots,\alpha_n\in[0,1]\right\}=[0,1]^n.$$

The sides of the unit n-cube are always length 1 and its volume is always 1.

72 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).



72.1 What is $T\begin{bmatrix}1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}$, $T\begin{bmatrix}1\\1\end{bmatrix}$?

$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$, $T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$, and $T\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}$?

We can see first two directly in the picture.

Using the linearity of T, we can compute

$$T\begin{bmatrix}1\\1\end{bmatrix}=T\left(\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\right)=T\begin{bmatrix}1\\0\end{bmatrix}+T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}=\begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}.$$

72.2 Write down a matrix for T.

The matrix for T in the standard basis is $\begin{bmatrix}1 & \frac{1}{2}\\\frac{1}{2} & 1\end{bmatrix}$.

72.3 What is the volume of the image of the unit square (i.e., the volume of $T(C_2)$)? You may use trigonometry.

The volume is $\frac{3}{4}$.

Volumes of images.

The goal of this problem is to
- Apply the definitions of *unit n-cube* and *image of a set*.
- Use tools from outside of linear algebra class to compute the area of a polygon.
- Be comfortable using the word "volume" in $\mathbb{R}^2$.

Determinant

DEF The ***determinant*** of a linear transformation $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$, denoted $\det(\mathcal{T})$ or $|\mathcal{T}|$, is the oriented volume of the image of the unit n-cube. The determinant of a square matrix is the determinant of its induced transformation.

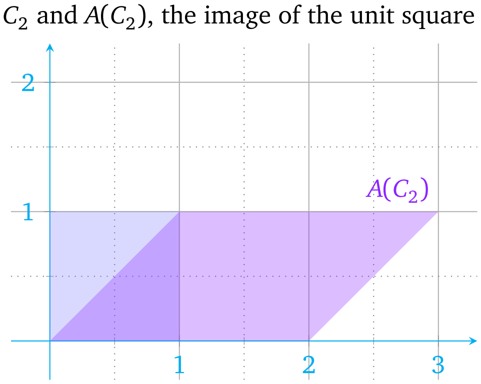
73 We know the following about the transformation A:

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}\quad\text{and}\quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}.$$

Apply the definition of determinant.

The goal of this problem is to
- Compute a determinant from the definition.
- Practice finding the area of a parallelogram.

73.1 Draw C_2 and $A(C_2)$, the image of the unit square under A.



73.2 Compute the area of $A(C_2)$. The area of this parallelogram is 2.

73.3 Compute $\det(A)$.

$\det(A) = 2$.

The parallelogram with sides $\begin{bmatrix}2\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ is positively oriented, so $\det(A) = +2$.

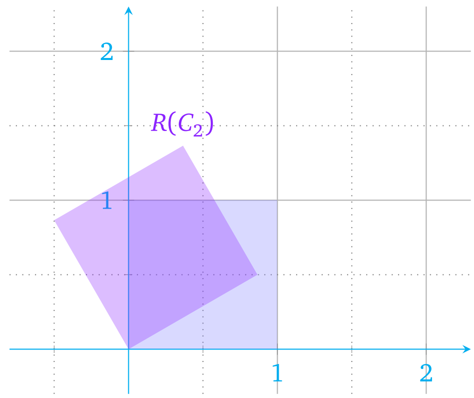
Apply the definition of determinant.

The goal of this problem is to
- Compute a determinant from the definition by applying geometric reasoning.

---

74 Suppose R is a rotation counter-clockwise by 30°.

74.1 Draw C_2 and $R(C_2)$.



74.2 Compute the area of $R(C_2)$.

The area is 1.

R rotates the entire unit square, which does not change its area.

74.3 Compute $\det(R)$.

Since R preserves orientation, $\det(R)$ must be positive. Since R does not change the area of the unit square, $\det(R) = +1$.

Apply the definition of determinant.

The goal of this problem is to
- Compute a determinant from the definition when orientation is reversed.

---

75 We know the following about the transformation F:

$$F\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad F\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}.$$

75.1 What is $\det(F)$?

$\det(F) = -1$.

F does not change the area of the unit square, but reverses its orientation, so $\det(F) = -1$.

Relate determinants of transformations and matrices.

The goal of this problem is to
- Relate the image of the unit cube under a transformation T to the columns of T's matrix representation.
- Relate the determinant of a matrix and its transpose.

Volume Theorem I (THM)

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the column vectors of M.

Volume Theorem II (THM)

For a square matrix M, $\det(M)$ is the oriented volume of the parallelepiped (n-dimensional parallelogram) given by the row vectors of M.

76

76.1 Explain volume theorem I using the definition of determinant.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a transformation with standard matrix $M = [\vec{c}_1|\cdots|\vec{c}_n]$, then $T(\vec{e}_i)$ (represented in the standard basis) will be c_i, the ith column of M. The image of the unit cube will be a parallelepiped with sides $T(\vec{e}_1) = \vec{c}_1, \ldots, T(\vec{e}_n) = \vec{c}_n$, and so $\det(T)$ will be the oriented volume of the parallelepiped with sides given by $\vec{c}_1, \ldots, \vec{c}_n$.

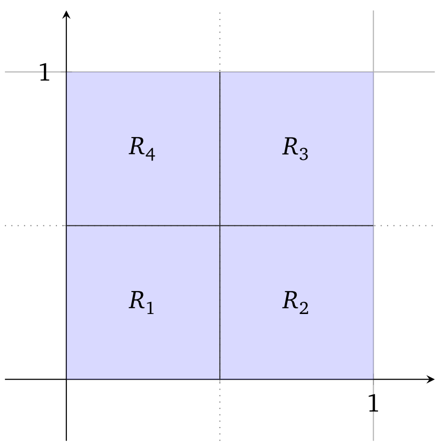
76.2 Based on volume theorems I and II, how should $\det(M)$ and $\det(M^T)$ relate for a square matrix M?

$\det(M) = \det(M^T)$. Since the transpose switches columns for rows, this is an immediate consequence of volume theorems I and II.

Determinants and areas.

The goal of this problem is to
- Use determinants to compute areas/volumes of images of arbitrary sets.
- See determinants as a "change of area/volume" *factor*.
- Explain the multiplicative property of determinants in terms of area/volume changes.

77



Let $R = R_1 \cup R_2 \cup R_3 \cup R_4$. You know the following about the linear transformations M, T, and S.

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$T : \mathbb{R}^2 \to \mathbb{R}^2$ has determinant 2

$S : \mathbb{R}^2 \to \mathbb{R}^2$ has determinant 3

77.1 Find the volumes (areas) of R_1, R_2, R_3, R_4, and R.

The volumes of R_1, R_2, R_3, and R_4 are $\frac{1}{4}$. The volume of R is 1.

77.2 Compute the oriented volume of $M(R_1)$, $M(R_2)$, and $M(R)$.

The oriented volumes of $M(R_1)$ and $M(R_2)$ are $\frac{1}{2}$ and the oriented volume of $M(R)$ is 2.

77.3 Do you have enough information to compute the oriented volume of $T(R_2)$? What about the oriented volume of $T(R + \{\vec{e}_2\})$?

Yes. The oriented volume of $T(R_2) = \frac{1}{2}$ and the oriented volume of $T(R+\{\vec{e}_2\}) = 2$.

We don't have enough information to determine what $T(R_2)$ looks like, but we do know (i) $T(R_2)$ will be a parallelogram, (ii) $T(R)$ has oriented volume 2, and (iii) $T(R)$ is made of four translated copies of $T(R_2)$. From this we deduce that the oriented volume of $T(R_2) = \frac{1}{4}\big(\text{oriented volume of } T(R)\big) = (\frac{1}{4})(2)$.

To find the oriented volume of $T(R + \{\vec{e}_2\})$, we use linearity to observe

$$T(R + \{\vec{e}_2\}) = T(R) + T(\{\vec{e}_2\}) = T(R) + \{T\vec{e}_2\}.$$

This shows that $T(R + \{\vec{e}_2\})$ is just a translation of $T(R)$ and therefore has the same oriented volume.

77.4 What is the oriented volume of $S \circ T(R)$? What is $\det(S \circ T)$?

They are both equal to 6.

$S \circ T(R) = S(T(R))$. We already know $T(R)$ has a volume of 2, and so $S(T(R))$ has a volume of 6, since S scales the volumes of all regions by 3. The oriented volume of $S \circ T(R)$ is the determinant of $S \circ T$ by definition.

Determinants of elementary matrices.

The goal of this problem is to
- Memorize the determinant of each type of elementary matrix.
- Justify why the determinant of an elementary matrix of type "add a multiple of one row to another" is always 1.
- Outline a method to compute determinants of arbitrary matrices.

78

- E_f is $I_{3\times3}$ with the first two rows swapped.
- E_m is $I_{3\times3}$ with the third row multiplied by 6.
- E_a is $I_{3\times3}$ with $R_1 \mapsto R_1 + 2R_2$ applied.

78.1 What is $\det(E_f)$?

$\det(E_f) = -1$.

$\det(I_{3\times3}) = 1$, and swapping one pair of rows of a matrix changes the sign of its determinant.

78.2 What is $\det(E_m)$?

$\det(E_m) = 6$.

Multiplying one row of a matrix by a constant multiplies its determinant by the same constant.

78.3 What is $\det(E_a)$?

$\det(E_a) = 1$.

Adding a multiple of one row of a matrix to another row has no effect on its determinant.

78.4 What is $\det(E_f E_m)$? $\det(E_f E_m) = \det(E_f)\det(E_m) = (-1)(6) = -6$.

78.5 What is $\det(4I_{3\times3})$? $\det(4I_{3\times3}) = 4^3 = 64$.

78.6 What is $\det(W)$ where $W = E_f E_a E_f E_m E_m$?

$\det(W) = \det(E_f)\det(E_a)\det(E_f)\det(E_m)\det(E_m) = (-1)(1)(-1)(6)(6) = 36$.

79

$$U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

79.1 What is $\det(U)$?

$\det(U) = -12$.

79.2 V is a square matrix and $\operatorname{rref}(V)$ has a row of zeros. What is $\det(V)$?

$\det(V) = 0$.

Reasoning about determinants via elementary matrices.

The goal of this problem is to
- Develop a shortcut for computing determinants of triangular matrices.
- Reason about the determinant of a matrix when given its reduced row echelon form.

80

80.1 V is a square matrix whose columns are linearly dependent. What is $\det(V)$?

$\det(V) = 0$.

80.2 P is projection onto $\operatorname{span}\left\{\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$. What is $\det(P)$?

$\det(P) = 0$.

The image of the unit square under P is a line segment, which has zero volume.

Determinants of singular matrices.

The goal of this problem is to
- Reason geometrically about why a transformation that isn't one-to-one has a zero determinant.

81

Suppose you know $\det(X) = 4$.

81.1 What is $\det(X^{-1})$?

$\det(X^{-1}) = \frac{1}{4}$.

We know that $XX^{-1} = I$. Therefore we must have that $\det(XX^{-1}) = \det(X)\det(X^{-1}) = \det(I) = 1$, and so $\det(X^{-1}) = \frac{1}{4}$.

81.2 Derive a relationship between $\det(Y)$ and $\det(Y^{-1})$ for an arbitrary matrix Y.

$\det(Y^{-1}) = \frac{1}{\det(Y)}$.

Using the same reasoning as the previous part, we know that $YY^{-1} = I$. Therefore we must have $\det(Y)\det(Y^{-1}) = \det(YY^{-1}) = \det(I) = 1$, and so $\det(Y^{-1}) = \frac{1}{\det(Y)}$.

Determinants and invertibility.

The goal of this problem is to
- Produce the determinant of X^{-1} give the determinant of X.
- Explain why a transformation T is invertible if and only if $\det(T) \neq 0$.

81.3 Suppose Y is not invertible. What is $\det(Y)$?

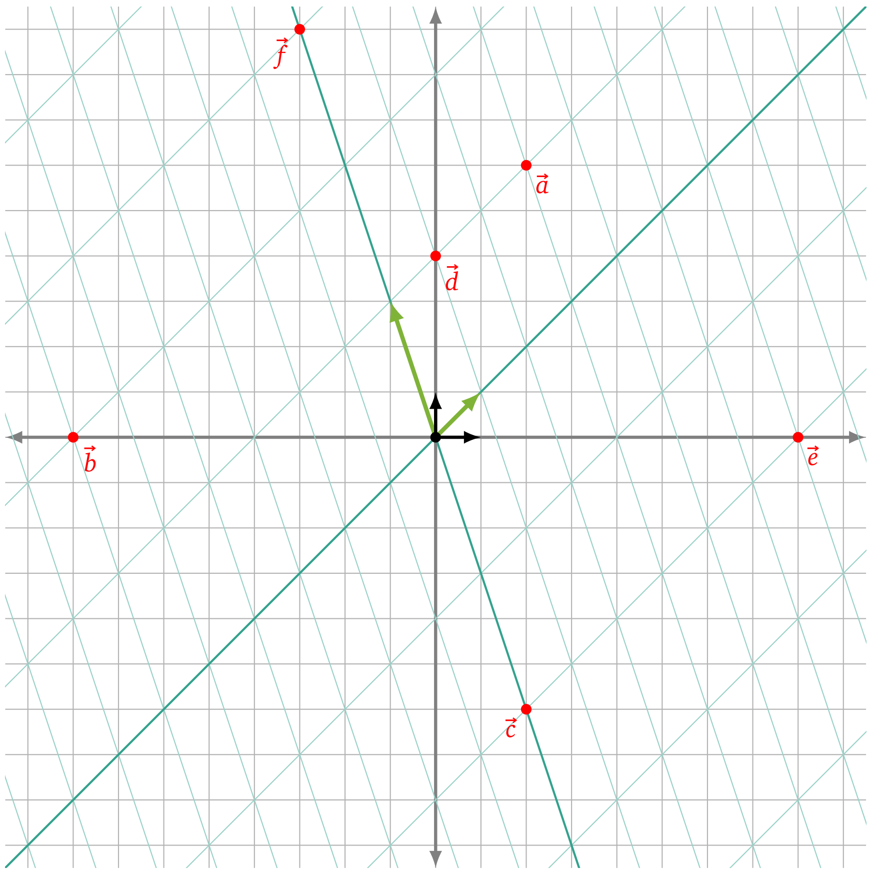
$\det(Y) = 0$.

If Y is not invertible, it has linearly dependent columns. Therefore the parallelepiped formed by the columns of Y will be "flattened" and have zero volume.

This is consistent with our previous findings. For a square matrix Y, $\det(Y^{-1}) = \frac{1}{\det(Y)}$. This formula always works, except when $\det(Y) = 0$.

82 The subway system of Oronto is laid out in a skewed grid. All tracks run parallel to one of the green lines shown. Compass directions are given by the black lines.

While studying the subway map, you decide to pick two bases to help: the green basis $\mathcal{G} = \{\vec{g}_1, \vec{g}_2\}$, and the black basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$.



1. Write each point above in both the green and the black bases.

2. Find a change-of-basis matrix X that converts vectors from a green basis representation to a black basis representation. Find another matrix Y that converts vectors from a black basis representation to a green basis representation.

3. The city commission is considering renumbering all the stops along the $y = -3x$ direction. You deduce that the commission's proposal can be modeled by a linear transformation.

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that stretches in the $y = -3x$ direction by a factor of 2 and leaves vectors in the $y = x$ direction fixed.

   Describe what happens to the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ when T is applied given that

$$[\vec{u}]_{\mathcal{G}} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad [\vec{v}]_{\mathcal{G}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix} \qquad [\vec{w}]_{\mathcal{B}} = \begin{bmatrix} -8 \\ -7 \end{bmatrix}.$$

4. When working with the transformation T, which basis do you prefer vectors be represented in? What coordinate system would you propose the city commission use to describe their plans?

Eigenvectors

Eigenvector

DEFINITION

Let X be a linear transformation or a matrix. An ***eigenvector*** for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the ***eigenvalue*** of X corresponding to the eigenvector $\vec{v}$.

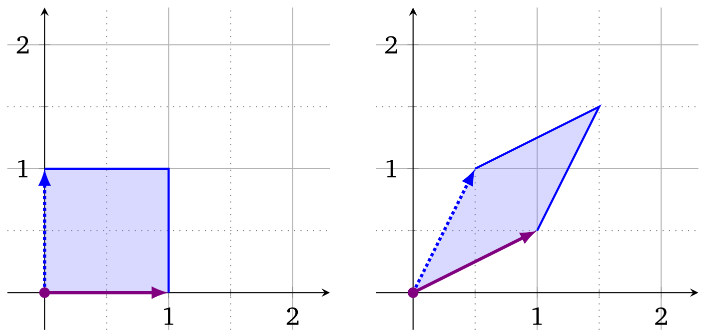
---

83 The picture shows what the linear transformation T does to the unit square (i.e., the unit 2-cube).

Apply the definition of eigenvector/value geometrically.

The goal of this problem is to

- Find eigenvectors/values from transformations defined geometrically.
- Produce new eigenvectors from existing ones by scaling.



83.1 Give an eigenvector for T. What is the eigenvalue?

$\vec{v} = \begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for T, with corresponding eigenvalue $\frac{3}{2}$.

We can see from the image that $T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}$.

83.2 Can you give another?

Any scalar multiple of $\begin{bmatrix}1\\1\end{bmatrix}$ is also an eigenvector for T.

For any scalar α, we have $T\left(\alpha\begin{bmatrix}1\\1\end{bmatrix}\right) = \alpha T\begin{bmatrix}1\\1\end{bmatrix} = \alpha\frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}$, meaning $\alpha\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for T with eigenvalue $\frac{3}{2}\alpha$.

More interestingly, since $T\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\1\end{bmatrix}$ is an eigenvector with corresponding eigenvalue $\frac{1}{2}$.

---

84 For some matrix A,

Apply the definition of eigenvector/value algebraically.

The goal of this problem is to

- Identify numerically whether a vector is an eigenvector.
- Use numerical evidence to compute an eigenvalue.
- Reason about the matrix $A-\lambda I$ given λ is an eigenvalue.

$$A\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2/3\end{bmatrix} \qquad \text{and} \qquad B = A - \tfrac{2}{3}I.$$

84.1 Give an eigenvector and a corresponding eigenvalue for A.

$\begin{bmatrix}3\\3\\1\end{bmatrix}$ is an eigenvector for A, with corresponding eigenvalue $\frac{2}{3}$.

84.2 What is $B\begin{bmatrix}3\\3\\1\end{bmatrix}$?

$B\begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$.

We compute

$$B\begin{bmatrix}3\\3\\1\end{bmatrix}=(A-\frac{2}{3}I)\begin{bmatrix}3\\3\\1\end{bmatrix}=A\begin{bmatrix}3\\3\\1\end{bmatrix}-\frac{2}{3}I\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2/3\end{bmatrix}-\begin{bmatrix}2\\2\\2/3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

84.3 What is the dimension of $\mathrm{null}(B)$?

The most we can say is that $\mathrm{nullity}(B)\geq 1$.

We know $\begin{bmatrix}3\\3\\1\end{bmatrix}\in\mathrm{null}(B)$ by the previous part, and so the dimension of $\mathrm{null}(B)$ is at least 1. It could be larger, but we do not have enough information to say for sure.

84.4 What is $\det(B)$? $\det(B)=0$.

Explore the matrix $C-\lambda I$.

The goal of this problem is to

- Relate the matrix $C-\lambda I$ to the problem of finding eigenvectors/values.
- Relate the equation $C\vec{x}=\lambda\vec{x}$ to the null space of the matrix $E_\lambda=C-\lambda I$.
- Use the determinant to determine when a parameterized family of matrices is invertible or not.
- Numerically compute eigenvalues/vectors without extra geometric information.

85 Let $C=\begin{bmatrix}-1&2\\1&0\end{bmatrix}$ and $E_\lambda=C-\lambda I$.

85.1 For what values of λ does E_λ have a non-trivial null space?

$\lambda=-2$ and $\lambda=1$.

E_λ has a non-trivial null space exactly when its determinant is zero. We compute:

$$\det(E_\lambda)=\det\left(\begin{bmatrix}-1-\lambda&2\\1&-\lambda\end{bmatrix}\right)=(-1-\lambda)(-\lambda)-(1)(2)=\lambda^2+\lambda-2=(\lambda-1)(\lambda+2).$$

This equals zero exactly when $\lambda=-2$ or $\lambda=1$.

85.2 What are the eigenvalues of C?

-2 and 1.

The scalar λ is an eigenvalue of C if and only if $C\vec{v}=\lambda\vec{v}$ for some $\vec{v}\neq\vec{0}$. Thus

$$\vec{0}=C\vec{v}-\lambda\vec{v}=(C-\lambda I)\vec{v}=E_\lambda\vec{v},$$

and so E_λ has a non-trivial null space if and only if λ is an eigenvalue of C.

85.3 Find the eigenvectors of C.

$\begin{bmatrix}-2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$ (along with all non-zero scalar multiples of these).

We know from the previous part that finding an eigenvector with corresponding eigenvalue -2 amounts to finding the non-zero vectors in the null space of $E_{-2}=\begin{bmatrix}1&2\\1&2\end{bmatrix}$. Computing,

$$\mathrm{null}(E_{-2})=\mathrm{span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}.$$

Similarly, the eigenvectors with corresponding eigenvalue 1 are the non-zero vectors in the null space of $E_1=\begin{bmatrix}-2&2\\1&-1\end{bmatrix}$, and we compute that $\mathrm{null}(E_1)=\mathrm{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

Apply the definition of the characteristic polynomial.

The goal of this problem is to

- Compute a characteristic polynomial by applying the definition.
- Relate the characteristic polynomial to eigenvalues.

DEFINITION

Characteristic Polynomial

For a matrix A, the ***characteristic polynomial*** of A is

$$\mathrm{char}(A)=\det(A-\lambda I).$$

86 Let $D = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$.

86.1 Compute $\operatorname{char}(D)$.

$\operatorname{char}(D) = (-2-\lambda)(3-\lambda)$.

We compute:

$$\det(D-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix}\right) = (1-\lambda)(-\lambda)-(2)(3) = \lambda^2-\lambda-6 = (-2-\lambda)(3-\lambda).$$

86.2 Find the eigenvalues of D.

The eigenvalues of D are -2 and 3. The eigenvalues of D are the roots of $\operatorname{char}(D)$.

87 Suppose $\operatorname{char}(E) = -\lambda(2-\lambda)(-3-\lambda)$ for some unknown 3×3 matrix E.

Getting information from the characteristic polynomial.

The goal of this problem is to
- Use the characteristic polynomial to determine eigenvalues and invertibility of matrices.
- Relate the characteristic polynomial to determinants.

87.1 What are the eigenvalues of E?

0, 2, and -3.

The eigenvalues of E are the roots of $\operatorname{char}(E)$.

87.2 Is E invertible?

No.

Since 0 is an eigenvalue of E, there must be a non-zero vector $\vec v$ such that $E\vec v = 0\vec v = \vec 0$. This means $\operatorname{nullity}(E) > 0$, which implies E is not invertible.

87.3 What can you say about $\operatorname{nullity}(E)$, $\operatorname{nullity}(E-3I)$, $\operatorname{nullity}(E+3I)$?

$\operatorname{nullity}(E) = 1$, $\operatorname{nullity}(E-3I) = 0$, $\operatorname{nullity}(E+3I) = 1$

Notice that evaluating the characteristic polynomial at λ give the determinant of $E-\lambda I$. From this, we can determine the invertibility of any matrix of the form $E-\lambda I$.

Since E is not invertible, $\operatorname{nullity}(E)\geq 1$. Since $E-3I$ is invertible, $\operatorname{nullity}(E-3I) = 0$, and since $E+3I$ is not invertible, $\operatorname{nullity}(E+3I)\geq 1$.

To pin down the nullities of E and $E+3I$ exactly takes more work. E has three distinct eigenvalues and so E must have three linearly independent eigenvectors $\vec v_0, \vec v_2, \vec v_{-3}$. Further, $\operatorname{span}\{\vec v_i\}\subseteq \operatorname{null}(E-\lambda_i I)$. Since $\vec v_0, \vec v_1, \vec v_{-3}\in\mathbb{R}^3$, it must be the case that $\operatorname{null}(E-\lambda_i I)$ is one dimensional for $i\in\{0,2,-3\}$.

88 Consider

Eigen bases.

The goal of this problem is to
- Compute eigenvalues when given eigenvectors.
- Compute a characteristic polynomial without using a determinant when given eigenvalues.
- Compute the result of a transformation when vectors are written in an eigen basis.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and notice that $\vec v_1, \vec v_2, \vec v_3$ are eigenvectors for A. Let T_A be the transformation induced by A.

88.1 Find the eigenvalues of T_A.

The eigenvalues of T_A are 2, -1, and 1.

We compute that $T_A\vec v_1 = 2\vec v_1$, $T_A\vec v_2 = -\vec v_2$, and $T_A\vec v_3 = \vec v_3$, so 2, -1, and 1 are eigenvalues of T_A. By the last part of the previous problem, there are no other eigenvalues.

88.2 Find the characteristic polynomial of T_A.

$\operatorname{char}(A) = (2-\lambda)(1-\lambda)(-1-\lambda) = -(\lambda-2)(\lambda+1)(\lambda-1)$.

Sine we know the roots of the characteristic polynomial are the eigenvalues and we know $\operatorname{char}(T_A)$ is a cubic, we can immediately write down $\operatorname{char}(T_A) = (2-\lambda)(1-\lambda)(-1-\lambda)$ without computing a determinant.

88.3 Compute $T_A\vec w$ where $w = 2\vec v_1 - \vec v_2$.

$T_A\vec{w} = 4\vec{v}_1 + \vec{v}_2$.

Using the computations we did in the first part above, we find

$$T_A\vec{w} = T_A(2\vec{v}_1 - \vec{v}_2) = 2T_A\vec{v}_1 - A\vec{v}_2 = 2(2\vec{v}_1) - (-\vec{v}_2) = 4\vec{v}_1 + \vec{v}_2.$$

88.4 Compute $T_A\vec{u}$ where $\vec{u} = a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ for unknown scalar coefficients a, b, c.

$T_A\vec{u} = 2a\vec{v}_1 - b\vec{v}_2 + c\vec{v}_3$.

Using the same reasoning as the previous part, we compute

$$T_A\vec{u} = aA\vec{v}_1 + bA\vec{v}_2 + cA\vec{v}_3 = 2a\vec{v}_1 - b\vec{v}_2 + c\vec{v}_3.$$

Notice that $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

88.5 If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ is $\vec{x}$ written in the $\mathcal{V}$ basis, compute $T_A\vec{x}$ in the $\mathcal{V}$ basis.

$[T_A\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

If $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, then $\vec{x} = \vec{v}_1 + 3\vec{v}_2 + 4\vec{v}_3$. Using the previous part, we then have that $T_A\vec{x} = 2\vec{v}_1 - 3\vec{v}_2 + 4\vec{v}_3$, so $[T_A\vec{x}]_{\mathcal{V}} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

---

89 Recall from Problem 88 that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and $\mathcal{V} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Let T_A be the transformation induced by A. The matrix P^{-1} takes vectors in the standard basis and outputs vectors in their $\mathcal{V}$-basis representation.

Diagonalizing matrices.

The goal of this problem is to

- Diagonalize a matrix.
- Explain diagonalization in terms of change of basis.
- Use diagonalization to compute large matrix powers.

89.1 Describe in words what P does.

P undoes what P^{-1} does, which is to say that it takes vectors in the $\mathcal{V}$ basis and outputs vectors in their representation in the standard basis.

89.2 Describe how you can use P and P^{-1} to compute $T_A\vec{y}$ for any $\vec{y} \in \mathbb{R}^3$.

Computing $[T_A]_{\mathcal{V}}[\vec{y}]_{\mathcal{V}}$ is easy, since $[T_A]_{\mathcal{V}}$ just multiplies each coordinate of $[\vec{y}]_{\mathcal{V}}$ by a scalar. We know that $P^{-1}[\vec{y}]_{\mathcal{E}} = [\vec{y}]_{\mathcal{V}}$ and that $P[\vec{x}]_{\mathcal{V}} = [\vec{x}]_{\mathcal{E}}$ and so given any vector $\vec{v}$ represented by $[\vec{v}]_{\mathcal{E}}$ in the standard basis, we have

$$A[\vec{v}]_{\mathcal{E}} = P[T_A]_{\mathcal{V}}P^{-1}[\vec{v}]_{\mathcal{E}}$$

since

$$P[T_A]_{\mathcal{V}}P^{-1}[\vec{v}]_{\mathcal{E}} = P[T_A]_{\mathcal{V}}[\vec{v}]_{\mathcal{V}} = P[T_A\vec{v}]_{\mathcal{V}} = [T_A\vec{v}]_{\mathcal{E}} = A[\vec{v}]_{\mathcal{E}}.$$

89.3 Can you find a matrix D so that

$$PDP^{-1} = A?$$

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$D = [A]_{\mathcal{V}}$, so by the previous part we have that that for any vector $\vec{v}$

$$A[\vec{v}]_{\mathcal{E}} = PDP^{-1}[\vec{v}]_{\mathcal{E}}.$$

89.4 $[\vec{x}]_{\mathcal{V}} = \begin{bmatrix}1\\3\\4\end{bmatrix}$. Compute $T_A^{100}\vec{x}$. Express your answer in both the $\mathcal{V}$ basis and the standard basis.

$$T_A^{100}\vec{x} = \begin{bmatrix}2^{100}\\3\\4\end{bmatrix}_{\mathcal{V}}.$$

By the previous problem, we know how T_A acts on vectors represented in the $\mathcal{V}$ basis: it multiplies the first coordinate by 2, the second by -1, and leaves the third coordinate unchanged. So we compute

$$T_A^{100}\begin{bmatrix}1\\3\\4\end{bmatrix} = T_A^{99}\begin{bmatrix}2\\-3\\4\end{bmatrix} = TA^{98}\begin{bmatrix}2^2\\3\\4\end{bmatrix} = T_A^{97}\begin{bmatrix}2^3\\-3\\4\end{bmatrix} = \ldots$$

To express $T_A^{100}\vec{x}$ in the standard basis, we use P.

$$[T_A^{100}\vec{x}]_{\mathcal{E}} = P[T_A^{100}\vec{x}]_{\mathcal{V}} = \begin{bmatrix}2^{100}-1\\2^{100}+7\\2^{100}-6\end{bmatrix}.$$

Diagonalizable

DEF A matrix is ***diagonalizable*** if it is similar to a diagonal matrix.

---

90 Let B be an $n\times n$ matrix and let T_B be the induced transformation. Suppose T_B has eigenvectors $\vec{v}_1,\ldots,\vec{v}_n$ which form a basis for $\mathbb{R}^n$, and let $\lambda_1,\ldots,\lambda_n$ be the corresponding eigenvalues.

Eigenvectors and diagonalization.

The goal of this problem is to

- Explain how the existence of a basis of eigenvectors implies diagonalizability.
- Explain how if there isn't a basis of eigenvectors a matrix is not diagonalizable.
- Not confuse diagonalizability and invertibility.

90.1 How do the eigenvalues and eigenvectors of B and T_B relate?

The eigenvalues of B and T_B are the same. The eigen vectors are also the same, except eigenvectors for B must be written in the standard basis. E.g., $[\vec{v}_1]_{\mathcal{E}},\ldots,[\vec{v}_n]_{\mathcal{E}}$ are eigenvectors for B.

90.2 Is B diagonalizable (i.e., similar to a diagonal matrix)? If so, explain how to obtain its diagonalized form.

Yes.

$\mathcal{V}=\{\vec{v}_1,\ldots\vec{v}_n\}$ is a basis consisting of eigenvectors for T_B. By definition, $B[\vec{v}_i]_{\mathcal{E}} = \lambda_i[\vec{v}_i]_{\mathcal{E}}$ for each i.

Let P be the matrix that takes vectors represented in the $\mathcal{V}$ basis and outputs their representations in the standard basis $\mathcal{E}$. Then, for example, we should have that

$$P^{-1}BP\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = P^{-1}BP[\vec{v}_1]_{\mathcal{V}} = P^{-1}B[\vec{v}_1]_{\mathcal{E}} = \lambda_1P^{-1}[\vec{v}_1]_{\mathcal{E}} = \lambda_1[\vec{v}_1]_{\mathcal{V}} = \begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix}.$$

Therefore, the first column of $P^{-1}BP$ is $\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix}$. By similar reasoning, the i^{th} column of $P^{-1}BP$ consists of all zeroes except for λ_i in the i^{th} position. In other words, B is similar to the diagonal matrix D with $\lambda_1,\lambda_2,\ldots,\lambda_n$ along the diagonal, in that order.

90.3 What if one of the eigenvalues of T_B is zero? Would B be diagonalizable?

Yes.

The argument in the previous part does not depend on any of the eigenvalues being non-zero.

90.4 What if the eigenvectors of T_B did not form a basis for $\mathbb{R}^n$. Would B be diagonalizable?

No.

The argument we used in the first part definitely would not work.

Consider the converse, and assume B is similar to diagonal matrix D. That is, suppose there is an invertible matrix P such that $B = PDP^{-1}$. Then, if $\vec{v}_1$ is the first column of P and λ_1 is the first entry on the diagonal of D, we would have

$$B[\vec{v}_1]_{\mathcal{E}} = PDP^{-1}[\vec{v}_1]_{\mathcal{E}} = PD[\vec{v}_1]_{\mathcal{V}} = PD\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \lambda_1 P\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = \lambda_1[\vec{v}_1]_{\mathcal{E}},$$

meaning that $[\vec{v}_1]_{\mathcal{E}}$ is an eigenvector of B. Similarly, all of the columns of P would be eigenvectors of B, with eigenvalues equal to the corresponding entry on the diagonal of D. Since P is invertible, its columns must be linearly independent, and therefore the n columns of P would form a basis of $\mathbb{R}^n$ consisting of eigenvectors of B.

Eigenspace

DEFINITION

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$. The ***eigenspace*** of A corresponding to the eigenvalue λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

The ***geometric multiplicity*** of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The ***algebraic multiplicity*** of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times $x - \lambda_i$ occurs as a factor).

Non-diagonalizable matrices.

The goal of this problem is to
- Explain why not all matrices are diagonalizable.
- Memorize an example of a non-diagonalizable matrix.

91 Let $F = \begin{bmatrix}3 & 1\\0 & 3\end{bmatrix}$ and $G = \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$.

91.1 Is F diagonalizable? Why or why not?

No.

F is diagonalizable if and only if there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of F, so we begin by computing all eigenvectors of F. $\mathrm{char}(F) = (3-\lambda)^2$, so the only eigenvalue of F is 3, meaning that the eigenvectors of F are precisely the non-zero vectors in $\mathrm{null}(F - 3I)$. We check that

$$\mathrm{null}(F-3I) = \mathrm{null}\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\},$$

which is one-dimensional, and so there cannot be a basis of $\mathbb{R}^2$ consisting of eigenvectors of F.

91.2 Is G diagonalizable? Why or why not?

Yes. G is already diagonal (and is necessarily similar to itself).

91.3 What are the geometric and algebraic multiplicities of each eigenvalue of F? What about the multiplicities for each eigenvalue of G?

The only eigenvalue for F is 3. Its geometric multiplicity is 1, and its algebraic multiplicity is 2.

The only eigenvalue for G is 3. Its geometric and algebraic multiplicity is 2.

91.4 Suppose A is a matrix where the geometric multiplicity of one of its eigenvalues is smaller than the algebraic multiplicity of the same eigenvalue. Is A diagonalizable? What if all the geometric and algebraic multiplicities match?

If one of the geometric multiplicities is smaller than the corresponding algebraic multiplicity, A cannot be diagonalizable.

Since the characteristic polynomial of an $n \times n$ matrix has degree n, it has at most n real roots. Since each root is an eigenvalue, we have

$$\sum \text{algebraic multiplicities} \leq n.$$

If one of the geometric multiplicities is smaller than the algebraic multiplicities, we have

$$\sum \text{geometric multiplicities } < n,$$

and so there cannot be a basis for $\mathbb{R}^n$ consisting of eigenvectors.

For the converse statement, we need the fundamental theorem of algebra: *a degree n polynomial has exactly n complex roots, counting multiplicity.*

If we allow eigenvalues to be complex numbers, then

$$\sum \text{algebraic multiplicities } = n,$$

and so if all geometric and algebraic multiplicities are equal, we have

$$\sum \text{geometric multiplicities } = n.$$

Thus, there would be a basis of eigenvectors.