Analog communication

① Fourier transform

The fourier transform converts a time domain signal $g(t)$ into its frequency domain representation $G(f)$. This frequency-domain representation gives the complex amplitudes of the sinusoidal components (sine and cosine waves) that make up the original signal

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t} \rightarrow \text{analysis equation}$$

Inverse fourier transform ⇒ converts back the frequency domain representation back to time domain representation.

$$g(t) = \int_{-\infty}^{\infty} ~~g~~ G(f)\, e^{j2\pi f t} \rightarrow \text{synthesis equation}$$

The ~~basic~~ advantage of transforming the time domain behaviour ~~of a system~~, in to the frequency domain is that resolution into eternal sinusoids presents the behaviour as superposition of steady state effects.

decomposing the signal into ~~sinosoids~~ sinusoids signal → superposition (or sum) of multiple ~~that~~ steady state sinusoids, each oscillating forever or eternally.

· for fourier transform of signal to exist, the following conditions must be met →

① The function $g(t)$ is ~~sig~~ single valued with finite no of maxima and minimas.

② The function $g(t)$ has a finite no. of discontinuities.

③ The function $g(t)$ is absolutely integrable

i.e $$\int_{-\infty}^{\infty} |g(t)|\,dt < \infty$$

we may ignore the question of existence of fourier transform of a time function $g(t)$ when it is a real signal (eg. voice or video signal) for physical realisibility the energy of signal defined by $\int_{-\infty}^{\infty} |g(t)|^2 dt$ must satisfy the condition

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$$

Notations

we can represent $2\pi f t$ as ωt

f → is preffered because it results in mathematical symmetry of $G(f)$ and $g(t)$

→ most voice and video signal are in hertz.

we can write the notations for $G(f)$ and $g(t)$ as $G(f) = F[g(t)]$ and $g(t) = F^{-1}[G(f)]$

continuous spectrum $\Rightarrow$ we know that $g(t)$ of finite energy is expressed as the ~~sum of~~ continuous sum of exponential function (sinusoidal wave ~~after~~ ~~since~~ since ~~$e^{i\theta} = \sin\theta$~~ $e^{j\theta} = \cos\theta + j\sin\theta$)

with frequencies in the interval $-\infty$ to ∞

the amplitude of a component of frequency f is proportional to $G(f)$, at any ~~pref~~ frequency f, the fn $e^{j2\pi ft}$ is weighted by the factor $G(f)\,df$ which contributes to the whole function $g(t)$ (the contribution is very less)

In general the FT ~~is~~ $G(f)$ is a complex function of frequency f so we may express it as $G(f) = |G(f)|\, e^{j\theta(f)}$

where $|G(f)|$ is continuous amplitude spectrum and $\theta(f)$ is continuous phase spectrum for real valued functions $g(t)$

$$G(-f) = G^*(f)$$

if $g(t)$ is real valued function

$$|G(-f)| = |G(f)| \rightarrow \text{even fn}$$

$$\theta(-f) = -\theta(f) \rightarrow \text{odd fn}$$

⇒ Rectangular pulse.

box fun ⇒ of time duration of T, ~~ampl~~ amp

$$\text{rect}(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0, & t < -\frac{1}{2} \text{ or } t > \frac{1}{2} \end{cases}$$

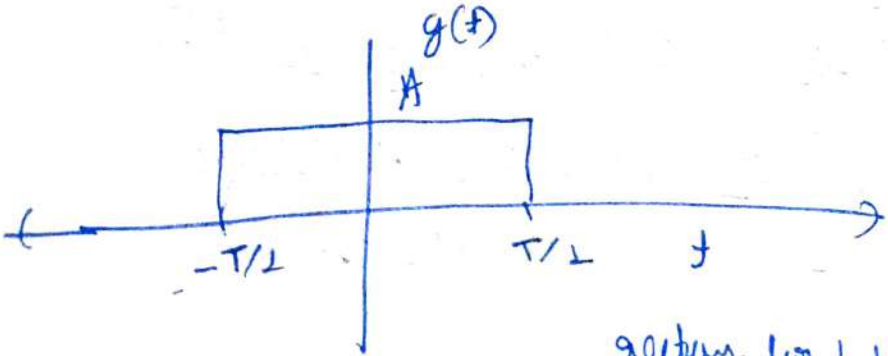
we can write $g(t)$ as

$$g(t) = A\,\text{rect}\left(\frac{t}{T}\right)$$

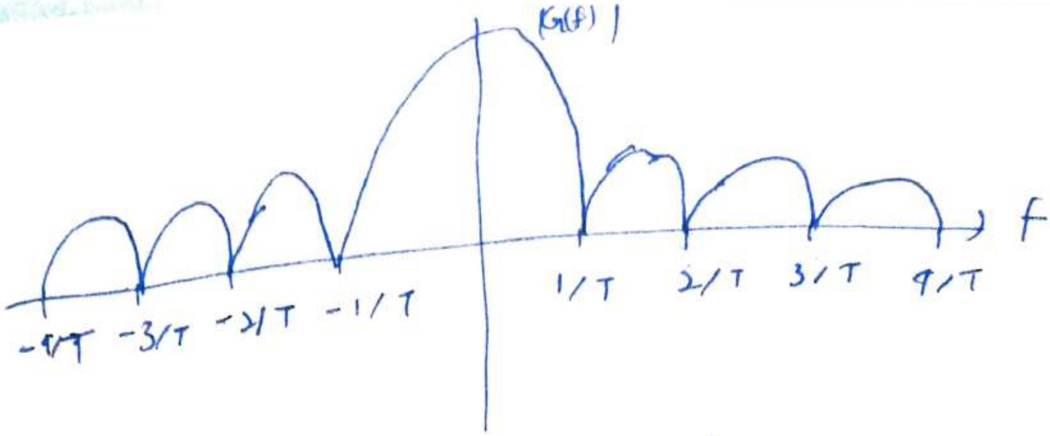
$$G(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi f t}\, dt$$

$$= AT\left(\frac{\sin(\pi f T)}{\pi f T}\right)$$

$$= \cancel{AT\,\text{sinc}(\pi f T)} \quad AT\,\text{sinc}(fT)$$



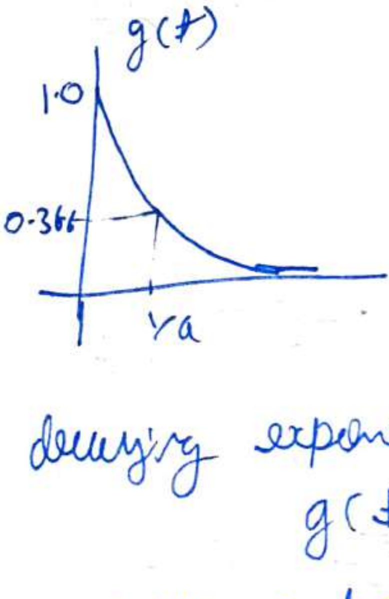
rectangular pulse

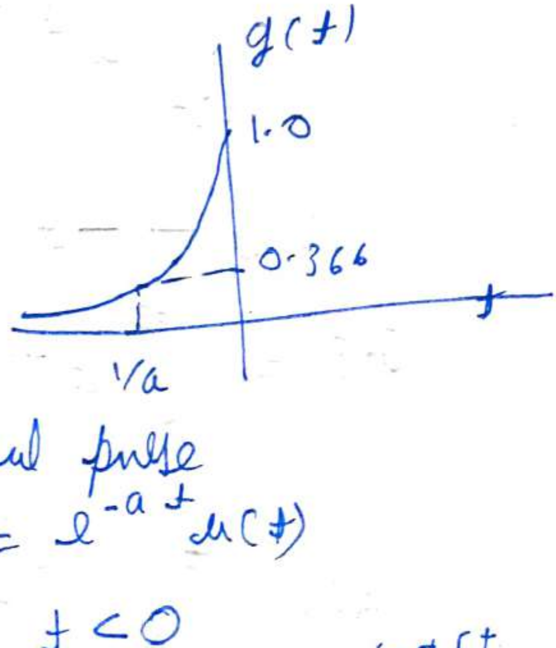


Amplitude Spectrum

⇒ Exponential pulse → we define the pulse mathematically using unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$





decaying exponential pulse

$$g(t) = e^{-at} u(t)$$

$g(t) = 0$ for $t < 0$

$$G(f) = \int_0^\infty e^{-at} e^{-j2\pi ft} \, dt$$

$$= \int_0^{\infty} e^{-t(a+j2\pi f)} u(t)\, dt$$

$$= \frac{1}{a+j2\pi f}$$

for decaying exponential pulse ~~of Fig (2.4)~~

~~let $e^{-at} u(t) = g(t)$~~

$$g(t) = e^{+at} u(-t)$$

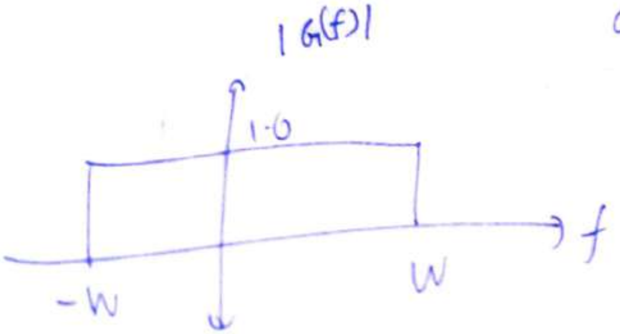
$$G(f) = \int_{-\infty}^{0} e^{at} e^{-j2\pi f t}\, dt \; u(-t)$$

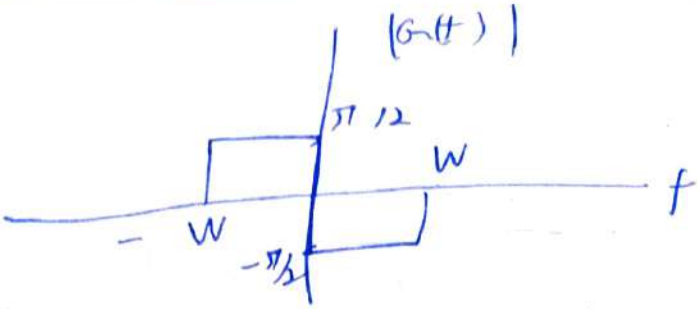
$$= \int_{-\infty}^{0} e^{t(a-j2\pi f)}\, dt$$

$$t \to -t$$

$$G(f) = \int_0^{\infty} e^{-t(a-j2\pi f)}\, dt$$

$$= \frac{1}{a-j2\pi f}$$





properties of fourier transform.

① linearity

$$g_1(t) \rightleftharpoons G_1(f)$$
$$g_2(t) \rightleftharpoons G_2(f)$$
$$c_1 g_1(t) + c_2 g_2(t) \underset{FC}{\rightleftharpoons} \cancel{c_1 g_1(t)}$$

$$c_1 G_1(f) + c_2 G_2(f)$$

consider $\cancel{g(t) = \begin{cases} e^{-at}, & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases}}$

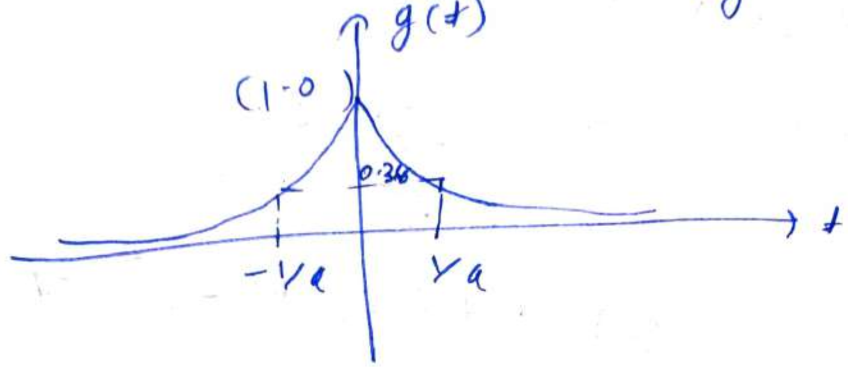
$$g(t) = \begin{cases} e^{-at}, & t > 0 \\ 1, & t = 0 \\ e^{at}, & t < 0 \end{cases}$$
$$= e^{(-a|t|)}$$

sum of truncated decaying and rising exponential pulse therefore by linearity property

$$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2}$$

thus $e^{+-a(t}$

$$e^{-a|t|} \rightleftharpoons \frac{2a}{a^2+(2\pi f)^2}$$ real and symmetric ①



for the difference between them i.e

$$g(t) = \begin{cases} \exp(-at), & t>0 \\ 0, & t=0 \\ -\exp(at), & t<0 \end{cases}$$

if we use $sgn(t)$ it might be possible

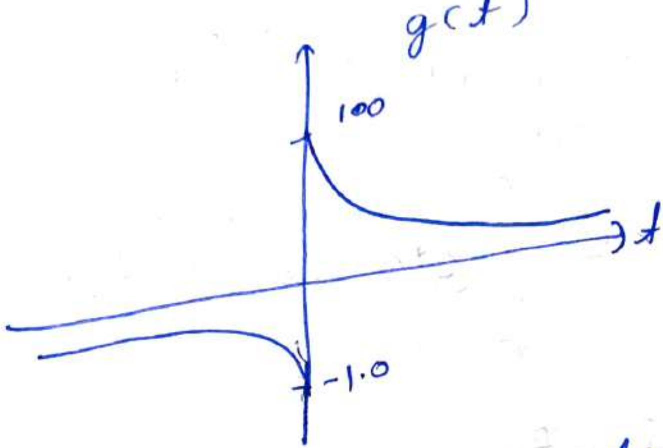
$$sgn(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

$$g(t) = e^{-a|t|}\, sgn(t)$$

$$F(g(t)) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j2\pi ft} sgn(t)\, dt$$

$$= \frac{1}{a+j2\pi f} - \frac{1}{a-j2\pi f}$$

$$= \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$ — ① ~~complex~~ imaginary and
• odd symmetric



eq 1 and 2 are fourier transform pairs

property - 2 Dilation

$$g(t) \rightleftharpoons G(f)$$

$$g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

T.P

$$[F(g(at))] = \int_{-\infty}^{\infty} g(at)\, e^{(-j2\pi f t)}\, dt$$

let $\tau = at$

$$[F(g(at))] = \frac{1}{a} \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi\left(\frac{f}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} G\left(\frac{f}{a}\right)$$

if $a<0$ it will be $-\frac{1}{a}$

property 3 Conjugation $g(t) \rightleftharpoons G(f)$
→ complex valued fn

$$g^*(t) \rightleftharpoons G^*(-f)$$

TP

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df$$

$$g^*(t) = \int_{-\infty}^{\infty} G^*(f)\, e^{-j2\pi f t}\, df$$

$$f \to -f$$

$$g^*(t) = -\int_{\infty}^{-\infty} G^*(-f)\, e^{j2\pi f t}\, df$$

$$g^*(t) = \int_{-\infty}^{\infty} G^*(-f)\, e^{j2\pi f t}\, df$$

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

$$G^*(f) = \int_{-\infty}^{\infty} g^*(t)\, e^{j2\pi f t}\, dt$$

$$\Rightarrow t - t$$

$$G^*(f) = \int_{\infty}^{-\infty} g^*(-t)\, e^{-j2\pi f t}\, dt$$

$$\cancel{G^*(f) = \int_{\infty}^{-\infty}}$$

$$\cancel{G^*(f) = -\int_{-\infty}^{\infty} g^*(-t)\, e^{-j2\pi f t}\, dt}$$

$$\cancel{G^*(f) \rightleftharpoons -g^*(-t)}$$

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

$$G^*(f) = \int_{-\infty}^{\infty} g^*(t)\, e^{j2\pi f t}\, dt$$

$t \rightarrow -t$

$$G^*(f) = -\int_{\infty}^{-\infty} g^*(-t)\, e^{-j2\pi f t}\, dt$$

$$G^*(f) = \int_{-\infty}^{\infty} g^*(-t)\, e^{-j2\pi f t}\, dt$$

$$\therefore\ G^*(f) \rightleftharpoons g^*(-t)$$

property of Duality

If $g(t) \rightleftharpoons G(f)$

$G(t) \rightleftharpoons g(-f)$

T.P

$$g(-t) = \int_{-\infty}^{\infty} G(f)\, e^{-j2\pi f t}\, df$$

$-t \to -f$

$$g(-f) = \int_{-\infty}^{\infty} G(t)\, e^{-j2\pi f t}\, dt$$

Example Sinc pulse

$g(t) = A\,\mathrm{Sinc}(2Wt)$

$\Rightarrow \quad A\,\mathrm{Sinc}(2Wt) \rightleftharpoons \frac{A}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$

Time shifting $\Rightarrow$ property

$g(t) \rightleftharpoons G(f)$ then

$g(t-t_0) \rightleftharpoons G(f)\, e^{-j2\pi f t_0}$

Proof $\Rightarrow$

$$\cancel{F[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)\, e^{-j2\pi f(t-t_0)}\, dt}$$

let $\cancel{t - t_0 = \tau}$

$$\cancel{F[g(\tau)] = \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi f\tau}\, d\tau}$$

$=$

let $t - t_0 = \tau \Rightarrow t = \tau + t_0 \Rightarrow dt = d\tau$

$$F[(g(t - t_0))] = \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi f t}\, d\tau$$

$$[F(g(t - t_0))] = \int_{-\infty}^{\infty} g(\tau)\, e^{-j2\pi f(t_0 + \tau)}\, d\tau$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t - t_0)\, e^{-j2\pi f \tau}\, d\tau$$

$$F(g(t - t_0)) = e^{-j2\pi f t_0}\, G(f)$$

frequency shifting $\Rightarrow g(t) \rightleftharpoons G(f)$

$$e^{j2\pi f_c t}\, g(t) \rightleftharpoons G(f - f_c)$$

Radio frequency RF pulse

Sinusoidal wave from $T/2$ to $-T/2$

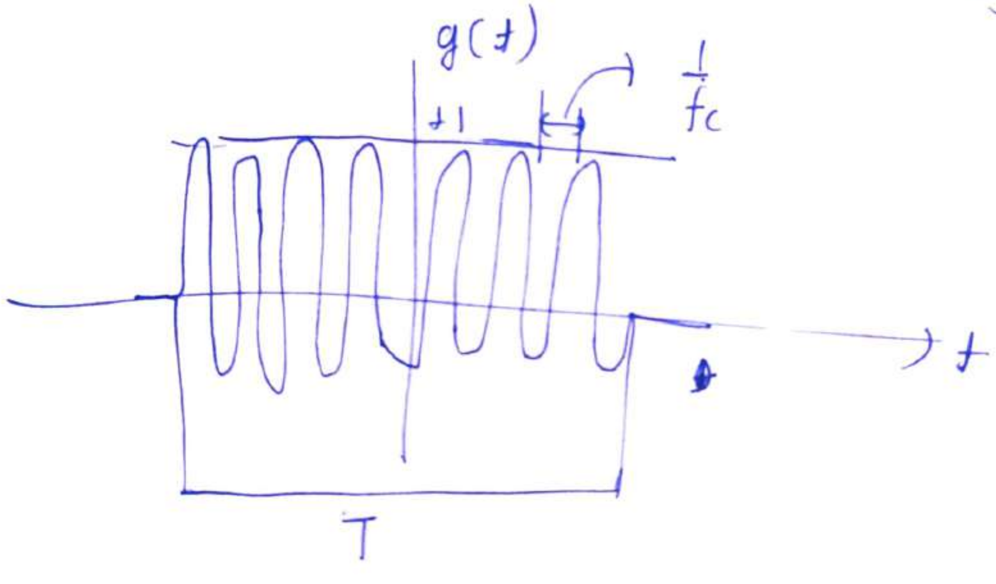
$$g(t) = \text{rect}\left(\frac{t}{T}\right) \cos(2\pi f_c t)$$

$$\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$$

we get

$$\text{rect}\left(\frac{t}{T}\right) \cos 2\pi f_c t \rightleftharpoons \frac{T}{2}\left[\text{Sinc}(T(f-f_c))\right.$$

$$\left. + \text{Sinc}(T(f+f_c))\right]$$



$$G(f) = \begin{cases} \frac{T}{2}\,\text{Sinc}[T(f-f_c)], & f>0 \\ 0, & f=0 \\ \frac{T}{2}\,\text{Sinc}[T(f+f_c)], & f<0 \end{cases}$$

Property ⇒ differentiation in time domain

$$g(t) \rightleftharpoons G(f)$$

$$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f\, G(f)$$

for higher derivatives

$$\frac{d^n}{dt^n} g(t) \rightleftharpoons (j2\pi f)^n G(f) \quad — ②$$

unit guassian pulse ⇒

g(t) and G(f) have diff mathematical forms but this is a diff case (only for a particular pulse)

$$-j2\pi t\, g(t) \rightleftharpoons \frac{d}{df} G(f)$$

$$2\pi t\, g(t) \rightleftharpoons j\frac{d}{df} G(f) \quad — ③$$

from 2 and 3 LHS

$$\frac{d}{dt} g(t) = -2\pi t\, g(t) \quad — ④$$

and 2 and 3 RHS

$$\frac{d}{dt} G(f) = -2\pi f\, G(f)$$

hence g(t) and G(f) have exactly the same mathematical form

we get ~~G(f) = g(t)~~ $G(f) = g(f)$

solving the eq 4 we get

$g(t) = \exp(-\pi t^2)$ → unit gaussian ~~eqn~~ pulse wm ✓ wv

property ⇒ Integration in time domain

$g(t) \rightleftharpoons G(f)$ and $G(0) = 0$

$$\int_{-\infty}^{t} g(\tau)\,d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f)$$

verification

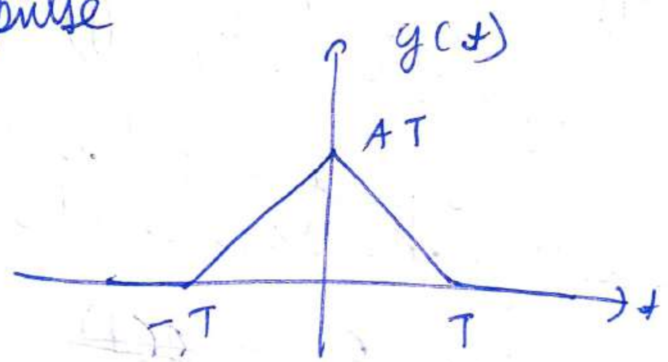
$$g(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} g(\tau)\,d\tau\right]$$

applying the diff. property

$$G(f) = j2\pi f \left\{ F\left[\int_{-\infty}^{t} g(\tau)\,d\tau\right]\right\}$$

$$F\left[\int_{-\infty}^{t} g(\tau)\,d\tau\right] = \frac{G(f)}{j2\pi f}$$

Triangular pulse



$$G_2(f) = \frac{1}{j2\pi f} G_1(f)$$

~~$G_1(f) \longrightarrow g_1(t) = AT\,\mathrm{sinc}(fT)\, e^{j2\pi fT}$~~

$G_1(f) = AT\,\mathrm{sinc}(fT)\, e^{j2\pi fT} - AT\,\mathrm{sinc}(fT)\, e^{-j\pi fT}$

$= 2j\, AT\,\mathrm{sinc}(fT)\, \sin(\pi fT)$

Real and imaginary part of a time function

$g(t) = \mathrm{Re}[g(t)] + j\,\mathrm{Img}[g(t)]$

$g^*(t) = \mathrm{Re}[g(t)] - j\,\mathrm{Img}[g(t)]$

$\mathrm{Re}[g(t)] = \frac{1}{2}[g(t) + g^*(t)]$

$\mathrm{Im}[g(t)] = \frac{1}{2}[g(t) - g^*(t)]$

$\mathrm{Re}[g(t)] \rightleftharpoons \frac{1}{2}[G(f) + G^*(-f)]$

$\mathrm{Im}[g(t)] \rightleftharpoons \frac{1}{2}[G(f) - G^*(-f)]$

Property 11 Modulation theorem $\Rightarrow$

let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$

$g_1(t)\, g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)\, d\lambda$

~~$g_1(t)\, g_2(t) \rightleftharpoons \int$~~

$$g_1(t)g_2(t) \rightleftharpoons G_{12}(f)$$

$$G_{12}(f) = \int_{-\infty}^{\infty} g_1(t)\, g_2(t)\, e^{-j2\pi f t}\, dt$$

for $g_2(t)$ we next substitute the inverse F-T

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f')\, e^{j2\pi f' t}\, df'$$

Here we get $G_{12}(f) = \cancel{\int_{-\infty}^{\infty}}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(t)\, G_2(f')\, e^{-j2\pi(f-f')t}\, df'\, dt$$

let $\lambda = f - f'$ by eliminating f' and interchanging the order of integration we get

$$G_{12}(f) = \int_{-\infty}^{\infty} G_2(f-\lambda)\left[\int_{-\infty}^{\infty} g_1(t)\, e^{-j2\pi\lambda t}\, dt\right] d\lambda$$

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)\, d\lambda$$

$$G_{12}(f) = \int_{-\infty}^{\infty} G_1(\lambda)\, G_2(f-\lambda)\, d\lambda$$

$$G_{12}(f) = G_1(f) * G_2(f)$$

property 12 convolution theorem =)

$$\int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\, d\tau \rightleftharpoons G_1(f)\, G_2(f)$$

$\cancel{g_1(\tau) *}$

$\cancel{g_1(\tau)}$ $g_1(t) * g_2(t) \rightleftharpoons G_1(f)\, G_2(f)$

property 13 correlation theorem

$$\int_{-\infty}^{\infty} g_1(t)\, g_2^*(t-\tau)\, dt \rightleftharpoons G_1(f)\, G_2^*(f)$$

property 14 Rayleigh's energy theorem

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

Dirac Delta function

~~under~~ for energy signals

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$$

~~o~~ not applicable for power signals $\Rightarrow$ expand the dirichlet condition i.e for the signals

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |g(t)|^2 dt < \infty$$

It turns out the objective are met by dirac delta fn.

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

---

property $\int_{-\infty}^{\infty} g(t)\, \delta(t - t_0) dt = g(t_0)$

Since $\delta(t)$ fn is even fn we may write it as

$$\int_{-\infty}^{\infty} g(\tau)\, \delta(t-\tau) d\tau = g(t)$$

or $g(t) * \delta(t) = g(t)$

fourier transform of delta fn is given by

~~$F\delta g$ $F[g(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt$~~

~~at $t = 0$~~

~~$F[g(t)]$~~

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt$$

at $t = 0$

$$F[\delta(t)] = 1$$

$$\delta(t) \rightleftharpoons 1$$

application of delta function.

① dc signal

duality property and noting that delta fn is an even fn

$$1 \rightleftharpoons \delta(f)$$

$$\int_{-\infty}^{\infty} e^{-j2\pi ft}\, dt = \delta(f)$$

Since ~~is~~ it is real valued

$$\int_{-\infty}^{\infty} \cos(2\pi ft)\, dt = \delta(f)$$

(2) complex exponential function

$e^{+j2\pi f_c t} \rightleftharpoons$

applying frequency shifting property

$$e^{j2\pi f_c t} \rightleftharpoons \delta(f-f_c)$$

(3) Sinusoidal function.

$$\cos(2\pi f_c t) = \frac{1}{2}[e^{j2\pi f_c t} + e^{-j2\pi f_c t}]$$

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f-f_c) + \delta(f+f_c)]$$

$$\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[\delta(f-f_c) - \delta(f+f_c)]$$

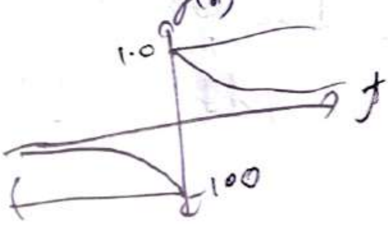
(4) Signum function

$$sgn(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

does not satisfy dirichlet condition.
no FT

However add symmetric exp fn is

$$g(t) = \begin{cases} e^{-at}, & t>0 \\ 0, & t=0 \\ -e^{at}, & t<0 \end{cases}$$



seeing it as a limiting form of exp pulse

$$G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$

The $|G(f)|$ is shown as dashed curve. In the limit as a tends to 0

$$F[sgn(t)] = \lim_{a \to 0} \frac{-4j\pi f}{a^2 + (2\pi f)^2}$$

$$= \frac{1}{j\pi f}$$

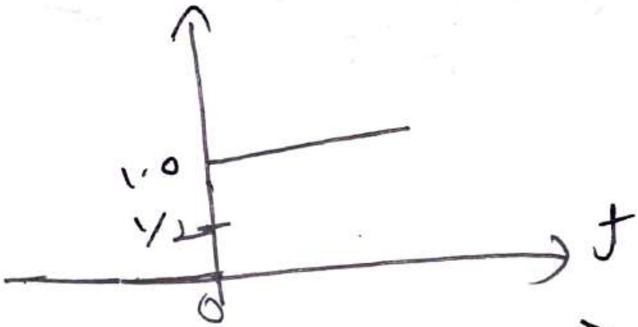
$$sgn(t) \rightleftharpoons \frac{1}{j\pi f}$$

(5) unit step function

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$

u(t) can be represented as

$$u(t) = \frac{1}{2}[sgn(t) + 1]$$



$$u(t) \rightleftharpoons \frac{1}{2}\left[\frac{1}{j\pi f} + \delta(f)\right]$$

⑥ Integration in the time domain

in general case where $G(0) \neq 0$

$$y(t) = \int_{-\infty}^{t} g(\tau)\,d\tau$$

convolution of $g(t)$ and unit step fn $u(t)$, as shown by

$$y(t) = \int_{-\infty}^{\infty} g(\tau)\,u(t-\tau)\,d\tau$$

$$= \cancel{G(f)\,\delta(f)}$$

$$Y(f) = G(f)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

according the shifting property of a delta fn formulated in frequency domain we have $G(f)\,\delta(f) = G(0)\,\delta(f)$

$$Y(f) = \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\,\delta(f)$$

Transmission of signals through linear system

• Time response

In time domain, the response of a linear time invariant LTI system to an arbitrary input $x(t) \longrightarrow h(t)$
impulse response

$h(t) \rightarrow$ system's output when unit impulse or $\delta(t)$ is applied at $t=0$

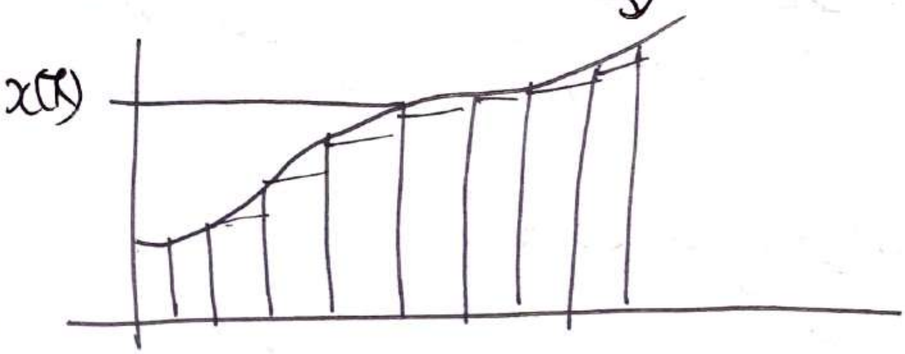
→ ~~due~~ if impulse is applied at any other time τ the response will be $h(t-\tau)$ (simply shifted in time)

to find response to $x(t) \rightarrow x(t)$ is approximated as series of small pulses each of duration ~~$\Delta\tau$~~ $\Delta\tau$

Input $x(t)$ $\longrightarrow$ $\boxed{\text{impulse response } h(t)}$ $\longrightarrow$ output $y(t)$

~~each pulse~~

$x(t)$



Each pulse can be thought of as an approximation of delta fn $\delta(t-\tau)$ weighted by $x(\tau)\Delta\tau$

The response to each pulse by system, at τ time is $x(\tau)h(t-\tau)\Delta\tau$

$y(t)$ is superposition of all these ~~responses~~ responses as $\Delta\tau$ approaches 0

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Causality ⇒ for a system to be physically realizable in real-time applications, it must be causal. This means the output at any given time depends only on present and past inputs, not future inputs. However in situations where signals are processed from stored data (not in real time), non-causal systems can be utilized.

Stability ⇒ A system is stable if it produces a bounded output for every bounded input.

BIBO stability criteria

$M \to$ finite

$$|x(t)| \leq M \;\; \forall\, t \in x$$

we know

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

applying BIBO stability

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau \right|$$

$$\downarrow$$

$$|y(t)| \le \int_{-\infty}^{\infty} |h(t-\tau)|\, |x(\tau)|\, d\tau$$

$$|y(t)| \le \int_{-\infty}^{\infty} M\, |h(t-\tau)|\, d\tau$$

stability condition ⇒ the condition of stability of system is

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

frequency response

let ⇒ $x(t) = e^{j2\pi f t}$

$$y(t) = \int_{-\infty}^{\infty} h(t)\, e^{[j2\pi f(t-\tau)]}\, d\tau$$

$$= e^{j2\pi f t} \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f \tau}\, d\tau$$

$$y(t) = e^{j2\pi f t}\, H(f)$$

or $H(f) = \left.\dfrac{y(t)}{x(t)}\right|_{x(t) = e^{j2\pi f t}}$

Next consider a signal

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

then ~~y(t) = H(f) x(t)~~

$$y(t) = \int_{-\infty}^{\infty} \underbrace{H(f)\, X(f)}_{Y(f)} \exp(j2\pi f t)\, df$$

$$Y(f) = H(f)\, X(f) \longrightarrow \text{frequency response}$$

$H(f)$ can be written as

$$H(f) = |H(f)|\, e^{j\beta(f)} \rightarrow \text{phase response, odd fn}$$

$|H(f)|$ ↳ magnitude response, even fn

Gain of the system ⇒
representing $H(f)$ in log. form

$$\ln H(f) = \alpha(f) + j\beta(f) \rightarrow \text{phase}$$

$\alpha(f)$ ↳ ⇒ $\ln|H(f)|$
gain in nepers

gain in decibel

$$\alpha_{dB}(f) = 20 \log_{10} |H(f)|$$

$$\alpha_{dB}(f) = 8.69\, \alpha(f) \text{(nepers)}$$

Bandwidth of a system where the system's amplitude response remains relatively constant for diff types of systems it is defined as ⇒

- for low-pass system, the bandwidth is the frequency at which ~~the~~ $|H(f)|$ drops to half (or $\frac{1}{\sqrt{2}}$) of its value at zero frequency. In terms of gain this corresponds to a 3 dB drop from maximum gain
- for band pass system → the bandwidth is the range of frequencies where the amplitude response stays within $\frac{1}{\sqrt{2}}$ times its mid-band value.

Ideal low pass filters ⇒

filters ⇒ are systems used in signal processing to modify a signal's frequency spectrum by either passing certain frequencies and rejecting others region

passband
↓
frequencies are passed with little or no distortion

stopband
↓
frequencies in this range are rejected by filter.

Ideal low-pass filter

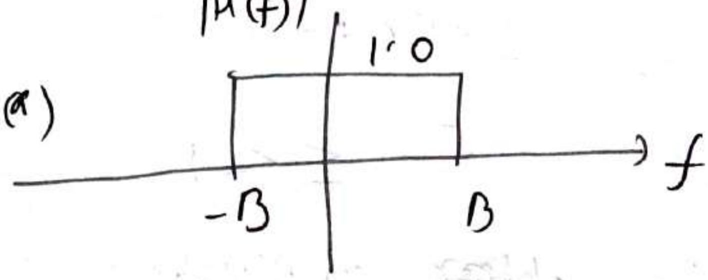
① amplitude ~~response~~ response → Inside the passband (from $-B$ to B, where B is cutoff from) → the amplitude response is constant (1)

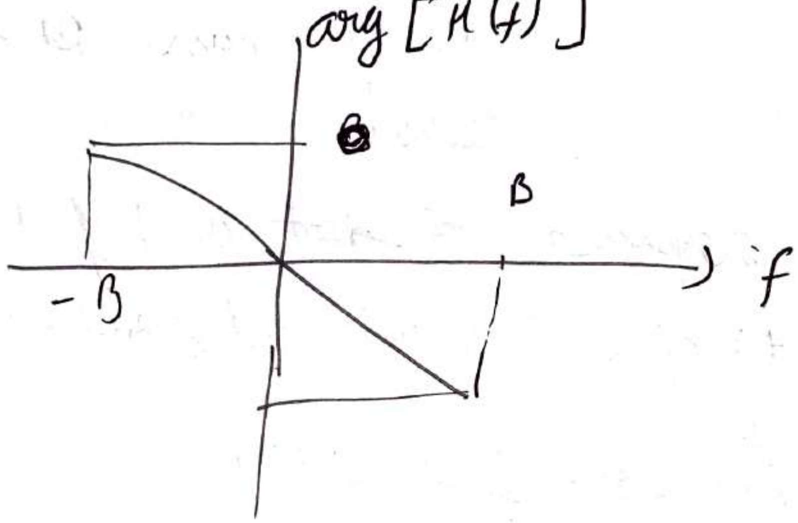
② phase response ⇒ The phase response varies linearly ~~wit~~ with frequency within the passband. outside the passband (for frequencies f such that $|f| > B$) outside the phase response can take arbitrary values.

$H(f)$ for low pass filter

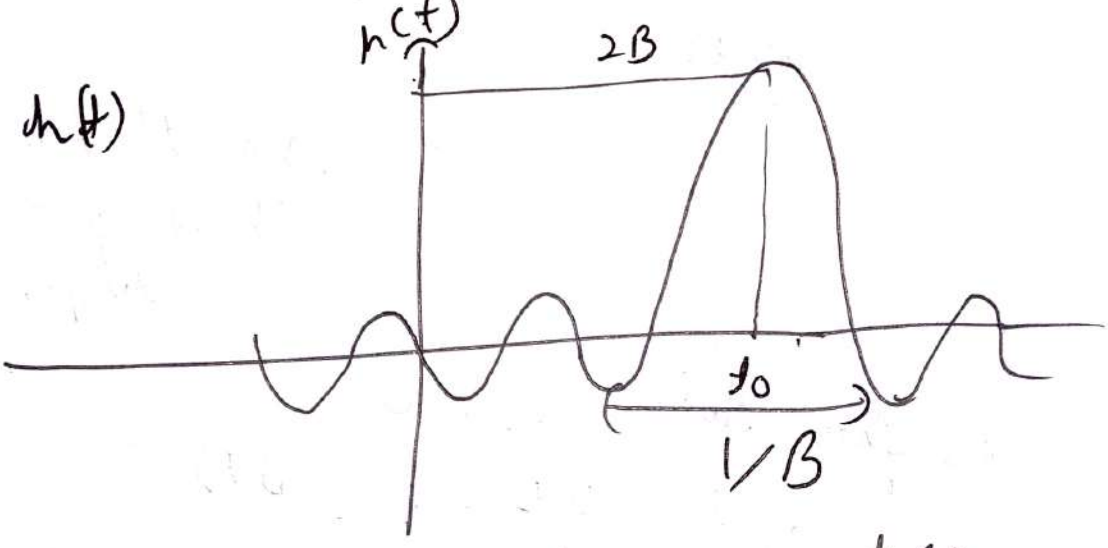
$$H(f) = \begin{cases} e^{-j2\pi f t_0}, & -B \le f \le B \\ 0, & \text{if } |f| > B \end{cases}$$

(a)





h(t)



The parameter B defines the bandwidth of the filter. The ideal low pass filter is of course non causal. By observing the impulse response h(t),

$$h(t) = \int_{-B}^{B} e^{j2\pi f[(t-t_0)]} \, df$$

$$= \frac{\sin[2\pi B(t-t_0)]}{2\pi(t-t_0)}$$

$$= 2B \, \text{sinc}[2B(t - t_0)]$$

$$|\text{sinc}[2B(t - t_0)]| \leq < 1 \quad \text{for } t < 0$$

for ~~appox~~ approximate value of t_0 we can make it causal.

Pulse response of ~~ideal~~ ideal filters ⇒

$x(t)$ of unit amplitude and duration T

↓

ideal low pass filter

↓

$y(t)$

$h(t) \rightarrow t_0$ has no effect on the shape of filter response $y(t)$

therefore taking $t_0 = 0$

$$h(t) = 2B \, \text{sinc}(2Bt)$$

input $x(t) = 1$ for $-\frac{T}{2} \leq t \leq \frac{T}{2}$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \, h(t - \tau) \, d\tau$$

$$= 2B \int_{-\infty}^{\infty} \operatorname{Sin} c[2B(t-\tau)]\, d\tau$$

$$= 2B \int_{-T/2}^{T/2} \left(\frac{\operatorname{Sin}[2\pi B(t-\tau)]}{2\pi B(t-\tau)} \right) d\tau$$

take $\lambda = 2\pi B(t-\tau)$ $\rightarrow 2\pi B(t-\frac{T}{2})$, $\rightarrow 2\pi B(t+T/2)$

$$d\lambda = -2\pi B\, d\tau$$

$$d\tau = -\frac{d\lambda}{2\pi B}$$

$$= \frac{-\cancel{2}\cancel{B}}{\cancel{2}\pi \cancel{B}} \int_{2\pi B(t+\frac{T}{2})}^{2\pi B(t-T/2)} \left(\frac{\operatorname{Sin} \lambda}{\lambda} \right) d\lambda$$

$$= \frac{1}{\pi} \int_{2\pi B(t-\frac{T}{2})}^{2\pi B(t+T/2)} \left(\frac{\operatorname{Sin} \lambda}{\lambda} \right) d\lambda$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi B(t+T/2)} \left(\frac{\operatorname{Sin} \lambda}{\lambda} \right) d\lambda - \int_0^{2\pi B(t-T/2)} \left(\frac{\operatorname{Sin} \lambda}{\lambda} \right) d\lambda \right]$$

$$= \frac{1}{\pi} \left[\operatorname{Si}[2\pi B(t+T/2)] - \operatorname{Si}[2\pi B(t+T/2)] \right]$$

$$\left(\operatorname{Si}(u) = \int_0^{u} \frac{\operatorname{Sin} z}{z}\, dz \right)$$

we have few observation

(1) The sine integral $Si(u)$ is an oscillatory function of u, having odd symmetry about $u=0$

(2) It has its maxima at multiples of π.

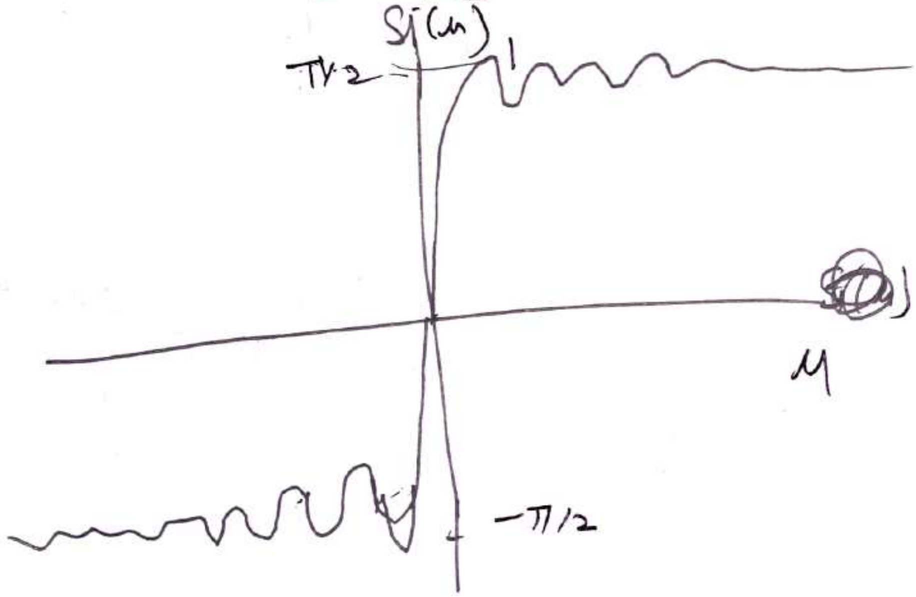
(3) It approaches the limiting value $(\frac{\pi}{2})$ for $u \to \infty$

the max value of $Si(u)$ ~~occurs~~ occurs at π

$$1.8519 = (1.179) \times \frac{\pi}{2}$$

the maxima minima of filter response $y(t)$ is

$$t_{max} = \pm\frac{T}{2} \pm \frac{1}{2B}$$



Si(u)

$$y(t_{max}) = \frac{1}{\pi}\left[Si(\pi) - Si(\pi - 2\pi BT)\right]$$

(by putting the value of t_{max})

$$= \frac{1}{\pi}\left[Si(\pi) + Si(2\pi BT - \pi)\right]$$

let
$$Si(2\pi BT - \pi) = \frac{\pi}{2}(1 \pm \Delta)$$

where Δ is abs value of deviation in the value of $Si(2\pi BT - \pi)$ expressed as a fraction of the final value $+\pi/2$.

Thus $$Si(\pi) = (1.179)\left(\frac{\pi}{2}\right)$$

we get ~~value~~ to redefine it as

$$y(t_{max}) = \frac{1}{2}(1.179 + 1 \pm \Delta)$$
$$\approx 1.09 \pm \frac{1}{2}\Delta$$

for a time- ~~simplifi~~ bandwidth product $BT >> 1$, the fractional deviation Δ has very small value → the percentage of overshoot in filter response is approx 9%

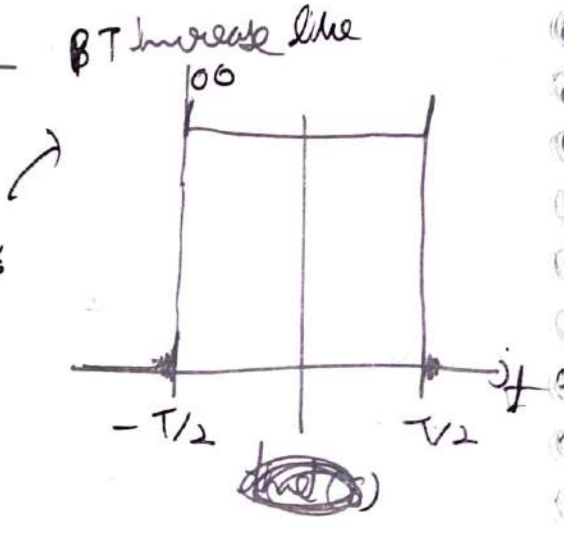
→ the overshoot is practically ~~independent~~ independent of bandwidth B

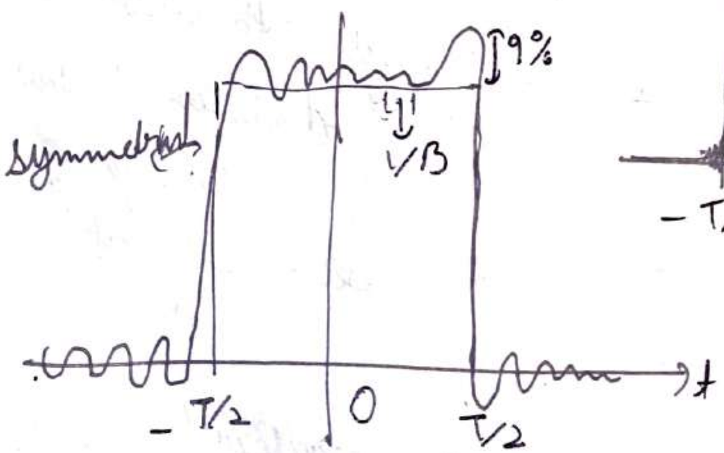
Gibbs phenomenon → sharp edges of input signal ~~causes~~ causes ringing near the edges.

The percentage overshoot (eg. 9% here) refers to how much the peak of this oscillation exceeds the final value around the edges

so it smoothes out the sharp edges (eg falling edges of square wave)

Note → when using an ideal low pass filter we must use ~~a time~~. $BT \geq 1$ to ensure that the waveform of filter input is recognizable from output.





modulation

- Amplitude modulation
  - DSB(SC)
  - SSB
  - Vestigial sideband
- angle modulation

modulation is defined as the process of by which some characteristics of a carrier wave is varied in accordance with ~~to~~ the information bearing signal

⇒ consider a sinusoidal carrier wave c(t)

$$c(t) = A_c \cos(2\pi f_c t)$$

amplitude sensit.

~~where~~

~~Amplitude~~ AM wave

$$s(t) = A_c[1 + K_a m(t)]\cos(2\pi f_c t)$$

The envelope of AM wave is part of signal that contains the actual information from the message signal m(t). given by

$$A_c|1 + k_a m(t)|$$

This envelope resembles the message signal but only if certain condition are met.

conditions for proper amplitude modulation.

- ① condition 1 :

  The. ~~amplitude of~~ $|K_a m(t)| < 1$

  for all t

  this ensures that carrier signal remains +ve and modulated signal does not experience overmodulation

② Condition 2 : carrier frequency

$f_c >> W \rightarrow$ frequency component of message signal $(m(t))$

Overmodulation ⇒ when amplitude sensitivity K_a is too large and makes $|K_a m(t)| > 1$ for some values of t, overmodulation occurs

Amp of modulation signal to go -ve which results in which results in carrier phase reversal (carrier signal flips in phase)

Percentage modulation : →

$$|k_a m(t)|_{max} \times 100$$

~~demod~~ after the conditions the demodulation of AM is achieved by using envelope detector which is defined as a device whose output traces the envelope of the AM wave acting as the input ~~signal~~ signal.

frequency domain description

$$m(t) \rightleftharpoons M(f)$$

$$S(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)] + \frac{k_a A_c}{2}[M(f-f_c) + M(f+f_c)]$$

where

$$\cos(2\pi f_c t) = \frac{1}{2}[e^{j2\pi f_c t} + e^{-j2\pi f_c t}]$$

$$e^{j2\pi f_c t} \rightleftharpoons \delta(f-f_c)$$

and $m(t) e^{j2\pi f_c t} \rightleftharpoons M(f-f_c)$

Observations →

① In the modulation process, the -ve frequency part of the message signal's spectrum ($-W$ to 0) is shifted to +ve - frequency range after modulation.

bcz the -ve ~~freq~~ freq cannot be ~~direct~~ directly measured

② For the modulated AM, the part of spectrum above the carrier frequency f_c is called upper sideband → Spectrum shifted to frequencies $f_c + W$

and part below f_c is lower sideband shifted to frequencies $f_c - W$

③ Total bandwidth B_T of AM is determined by highest and lowest freq components

Highest freq ⇒ $f_c + W$

Lowest freq ⇒ $f_c - W$

$$B_T = (f_c + W) - (f_c - W)$$

$$= 2W$$

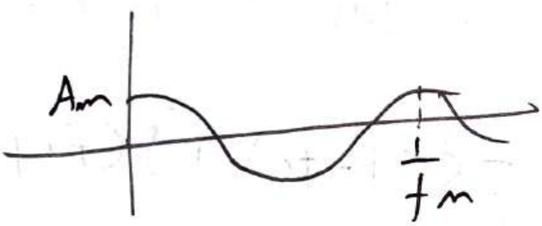
example single tone modulation ⇒

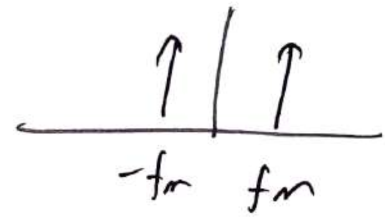
$$m(t) = A_m \cos(2\pi f_m t)$$

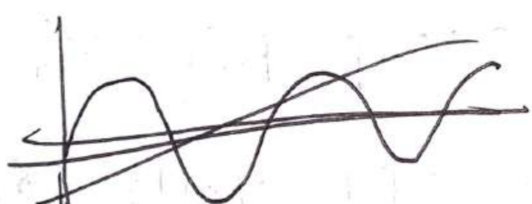
$$s(t) = A_c[1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

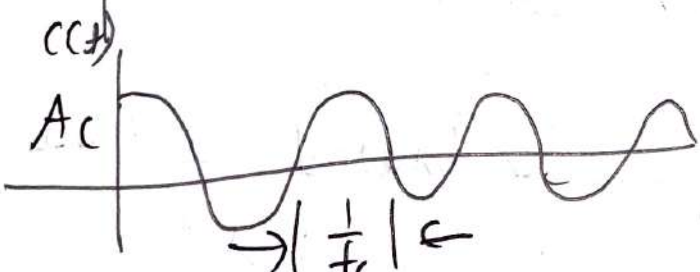
where $\mu = k_a A_m$ → modulation factor

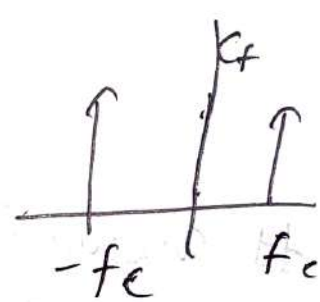
$|k_a A_m| < 1$

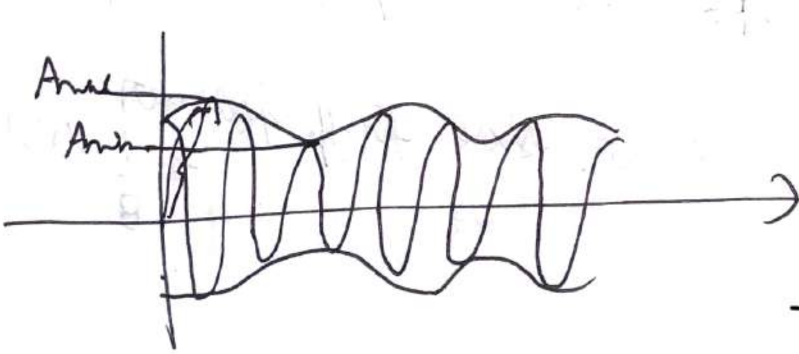


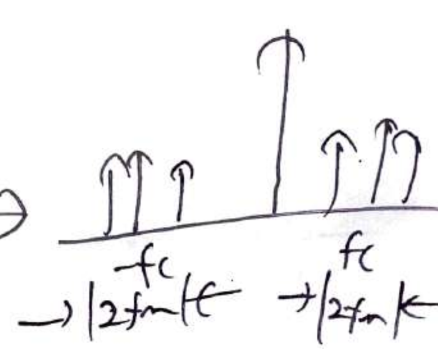












$$\frac{A_{max}}{A_{min}} = \frac{A_c(1+\mu)}{A_c(1-\mu)}$$

$$\mu = \frac{A_{max} - A_{min}}{A_{max} + A_{min}}$$

Hence $s(t) = A_c \cos(2\pi f_c t)$

$$+ \frac{1}{2}\mu A_c \cos[2\pi(f_c+f_m)t]$$

$$+ \frac{1}{2}\mu A_c \cos[2\pi(f_c-f_m)t]$$

$$S(f) = \frac{A_c}{2}[\delta(f-f_c) + \delta(f+f_c)]$$

$$+ \frac{1}{4}\mu A_c[\delta(f-f_c-f_m) + \delta(f-f_c-f_m)]$$

$$+ \frac{1}{4}\mu A_c[\delta(f+f_c-f_m) + \delta(f+f_c+f_m)]$$

average power delivered to 1 Ω

carrier power $= \frac{1}{2} A_c^2$

upper side freq power $= \frac{1}{8}\mu^2 A_c^2$

lower side freq power $= \frac{1}{8}\mu^2 A_c^2$

$H(f) \rightarrow$ transfer fn

ratio of total sideband power to total power in modulated wave is equal to

$$\frac{\mu^2}{\mu^2+2}$$

if $\mu = 1 \rightarrow 100\%$ modulation

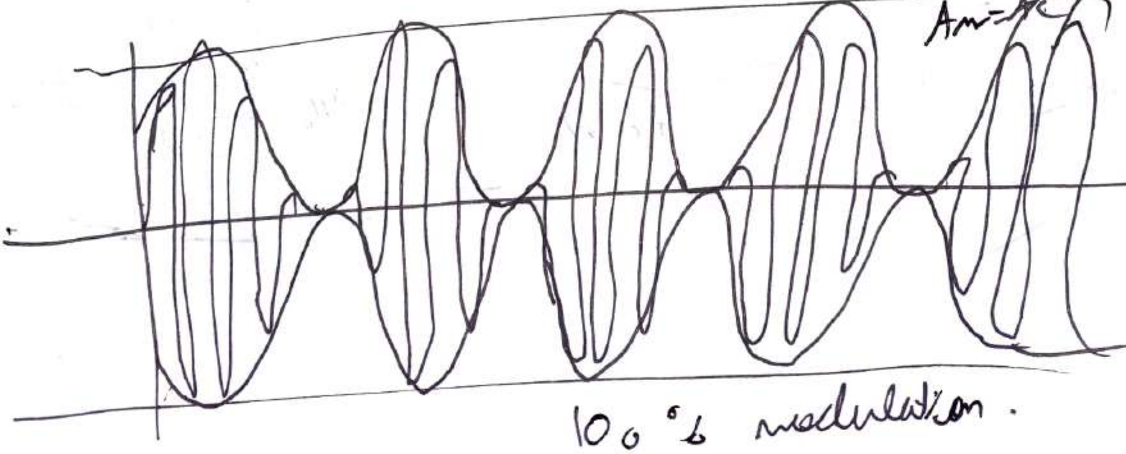
$$\boxed{\frac{A_m}{A_c} = \mu}$$

$A_c > A_m$

Envelope detector $\Rightarrow$

Amplitude is the peak displacement of a signal while the envelope of a signal is a curve tracing the signal's peak.

50% modulation



100% modulation.

An envelope detector is simple circuit used to demodulate AM signal. It capture the envelope of modulated signal, which represent the original message signal

The effectiveness ⇒

① $f_c >> W$
② percentage modulation $< 100\%$

Operation of the envelope detector ⇒

① Charging phase ⇒ +ve half cycle diode is forward biased and conducts current

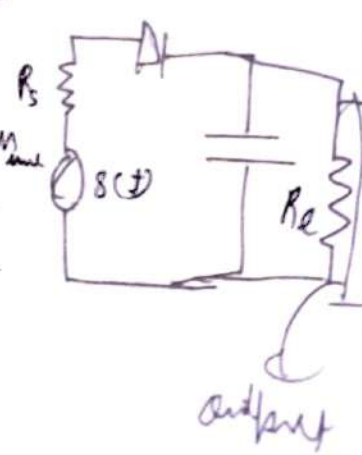
- The capacitor C charges rapidly to the peak value

DISCHARGING
② When input signal is below capacitor voltage the ~~capacitor~~ diode becomes reversed biased.

- The capacitor discharges slowly through R_L until next positive half cycle

$$R_L C >> \frac{1}{f_c W}$$

$$(r_f + R_s) C << \frac{1}{f_c}$$



DSS-CM
Double sideband -- supressed carrier modulation

$$S(t) = c(t)\, m(t)$$

$$S(f) = \frac{1}{2} A [M(f - f_c) + M(f + f_c)]$$

$m(t) \Rightarrow -W \leq f \leq W$

transmission bandwidth $= 2W$

power in either lower or upper side -frequency is 50% of total energy power

~~Band~~ transmission bandwidth
$= 2W$

message spectrum is if modulation is done with wave of A_m

$$M(f) = \frac{1}{2} A_m \delta(f - f_m) + \frac{1}{2} A_m \delta(f + f_m)$$

operating on carrier of A_c and f_c

two side frequency for positive freq

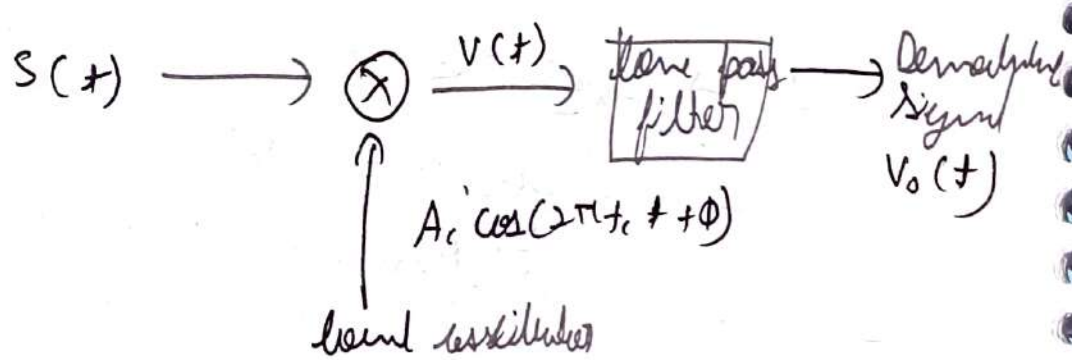
$\frac{1}{4} A_c A_m \delta(f - (f_c + f_m))$; $\frac{1}{4} A_c A_m \delta(f - (f_c - f_m))$

other shifted spectrum

$\frac{1}{4} A_c A_m \delta(f + (f_c - f_m))$; $\frac{1}{4} A_c A_m \delta(f - (f_c + f_m))$

Coherent detection =)

~~modulated~~ (s(t)) → product ~~modulator~~



let $s(t) = A_c \cos(2\pi f_c t)\, m(t)$

$$v(t) = A_c A_c' \cos(2\pi f_c t) \cos(2\pi f_c t + \phi)\, m(t)$$

$$= \frac{1}{2} A_c A_c' \cos(4\pi f_c t + \phi)\, m(t)$$

$$+ \frac{1}{2} A_c A_c' \cos\phi\, m(t)$$

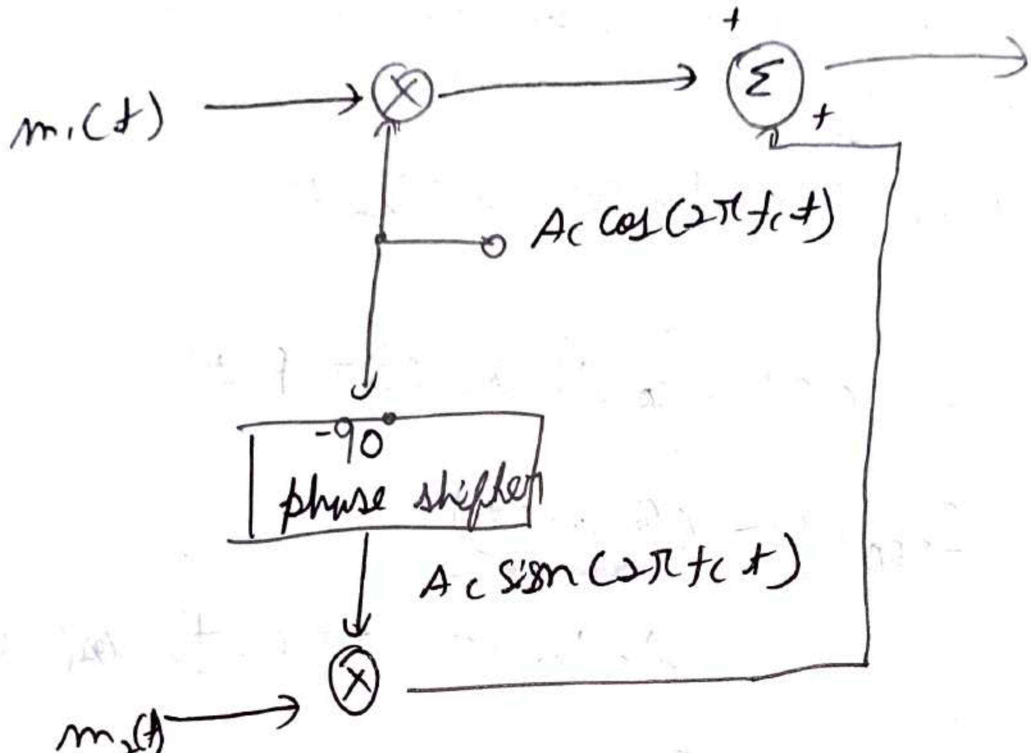
filtered out

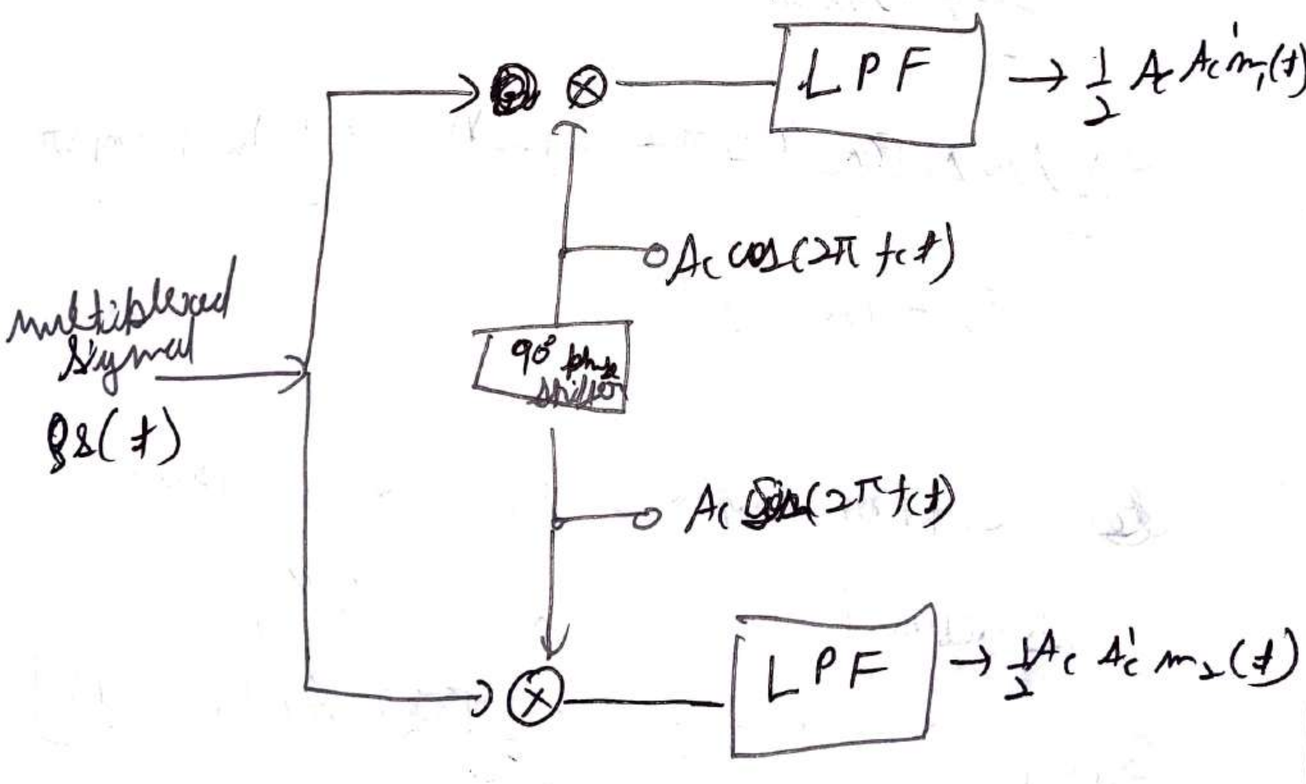
$$v_0(t) = \frac{1}{2} A_c A_c' \cos\phi\, m(t)$$

considering $\cos\phi = 1$

$$v_0(t) = \frac{1}{2} A_c A_c' m(t)$$

Quadrature - Carrier Multiplexing =>





SSB modulation → only one sideband

→ transmission bandwidth = W

let $m(t) = A_m \cos(2\pi f_m t)$

and $c(t) = A_c \cos(2\pi f_c t)$

$S_{DSB}(t) = c(t) m(t)$

$= A_c A_m \cos(2\pi f_m t) \cos(2\pi f_c t)$

$= \cancel{\frac{1}{2} A_c A_m \cos}$

$= \frac{1}{2} A_m A_c [\cos(2\pi(f_c + f_m)t)] + \frac{1}{2} A_m A_c \cos 2\pi (f_c - f_m) t]$

~~Suppress~~ suppress $f_c - f_m$ (lower sideband) or $f_c + f_m$ (USB)

$S_{USSB}(t) = \frac{1}{2} A_c A_m \cos[2\pi (f_c + f_m) t)]$

$S_{USSB}(t) = \frac{1}{2} A_c A_m [(\cos 2\pi f_c t \cos 2\pi f_m t)$

$- \sin 2\pi f_c t \sin 2\pi f_m t]$

On the other hand if $f_c - f_m$ is taken

$$S_{oSSB}(t) = \frac{1}{2} A_c A_m \cos 2\pi f_c t \cos 2\pi f_m t$$

$$+ \frac{1}{2} A_c A_m \sin 2\pi f_c t \sin 2\pi f_m t$$

$$S_{SSB}(t) = \frac{1}{2} A_c A_m \cos 2\pi f_c t \cos 2\pi f_m t \mp$$

$$\frac{1}{2} A_c A_m \sin 2\pi f_c t \sin 2\pi f_m t$$

~~dividing~~ dividing it into two stage

stage 1 → message signal is periodic

stage 2 → it is non periodic

~~m(t)~~ consider a periodic message signal defined by

$$m(t) = \sum_n a_n \cos 2\pi f_m t$$

~~c(t)~~ c(t) is common to all sinusoidal components of m(t) we can infer

$$S_{SSB}(t) = \frac{1}{2} A_c \cos(2\pi f_c t) \sum_n a_n \cos 2\pi f_m t$$

$$\mp \frac{1}{2} A_c \sin(2\pi f_c t) \sum_n a_n \sin 2\pi f_m t$$

another periodic sig.

$$\hat{m}(t) = \sum_n \sin(2\pi f_m t)$$

we get

$$S_{SSB}(t) = \frac{1}{2} A_c m(t) \cos 2\pi f_c t \mp \frac{1}{2} A_c \hat{m}(t) \sin 2\pi f_c t$$

where $\hat{m}(t)$ can be derived from $m(t)$ by phase shifting of -90°

① we know from fourier analysis under certain conditions the fourier series representation of periodic signal converges to fourier transform of a non periodic signal.

② $\hat{m}(t)$ is ~~the~~ Hilbert transform of $m(t)$ $\quad H(f) = -j\,\mathrm{sgn}(f)$

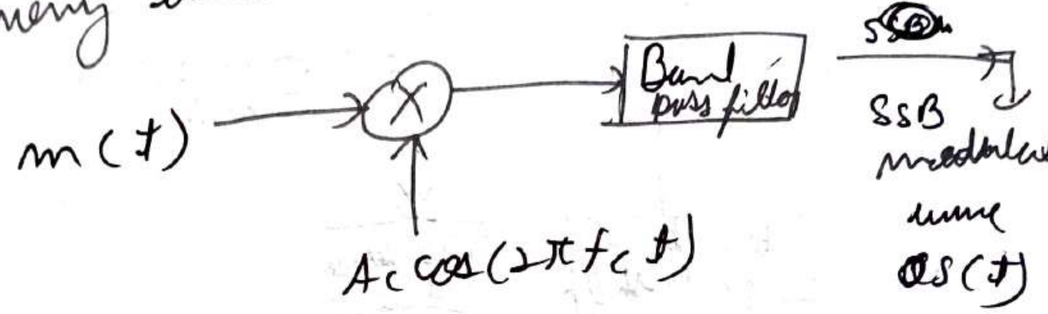
where $\mathrm{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$

In other words hilbert transform is a wide band phase shifter

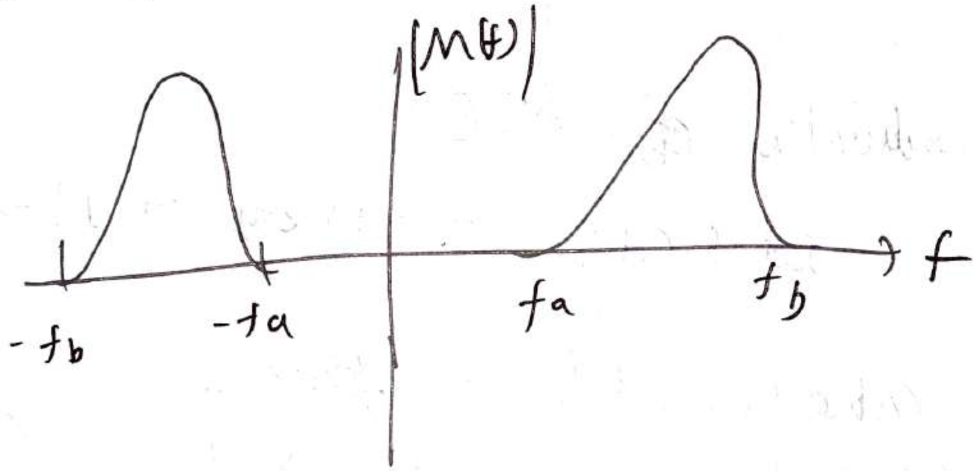
whose frequency response is characterised in two parts.

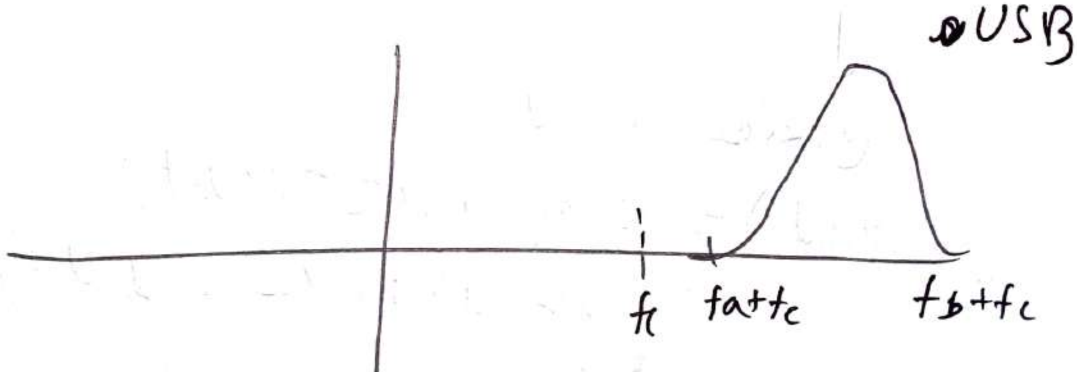
~~modulators for SSB~~

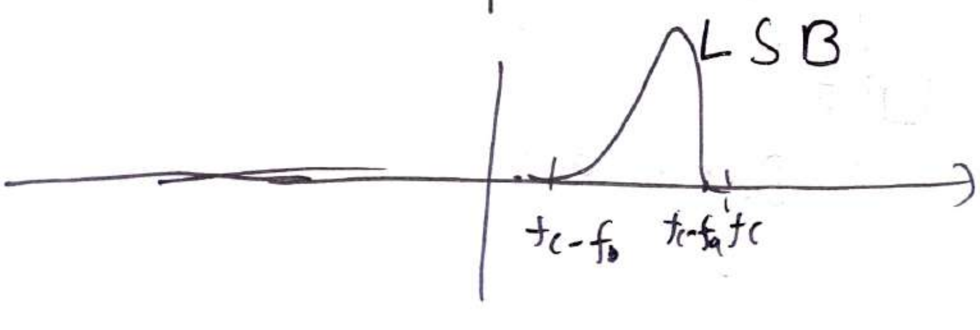
modulators for SSB
frequency discrimination



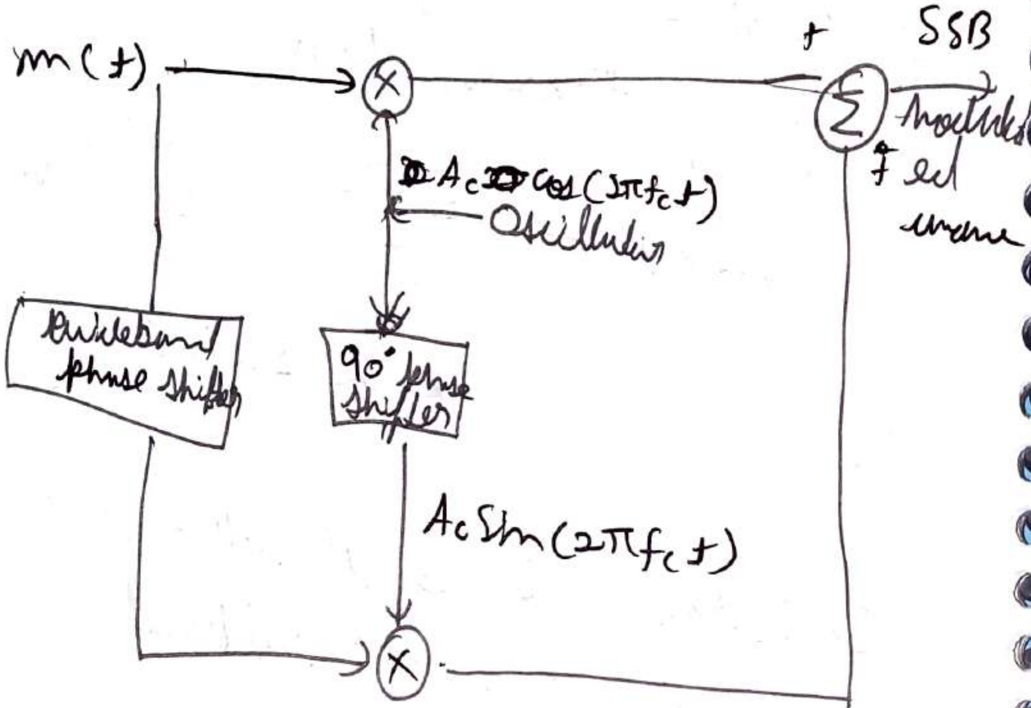
Band pass filter will either filter out the upper or lower sideband







phase discrimination method.



Demodulation of SSB $\Rightarrow$

$$\text{let } s(t) = m(t)\cos(\omega_c t) - \hat{m}(t)\sin(\omega_c t)$$

coherent detection process

let $s(t) = m(t)\cos(\omega_c t)$ (Simplified Version)

$\downarrow$

$\otimes\, 2\cos(\omega_c t)$

$$r(t) = 2m(t)\cos^2(\omega_c t)$$
$$= m(t)(1 + \cos 2\omega_c t)$$

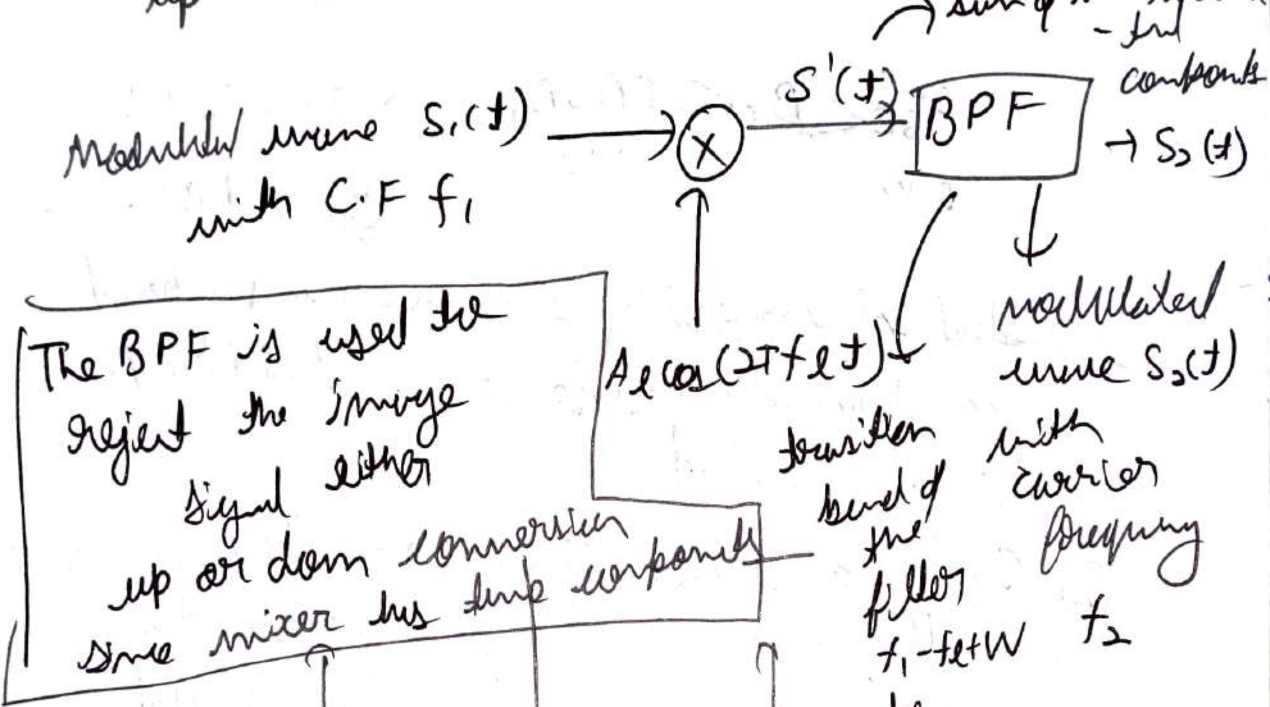
$\downarrow$

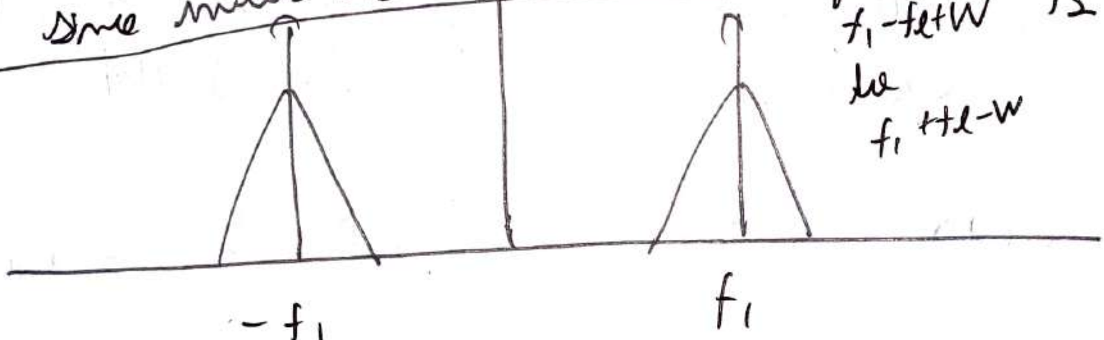
LPF

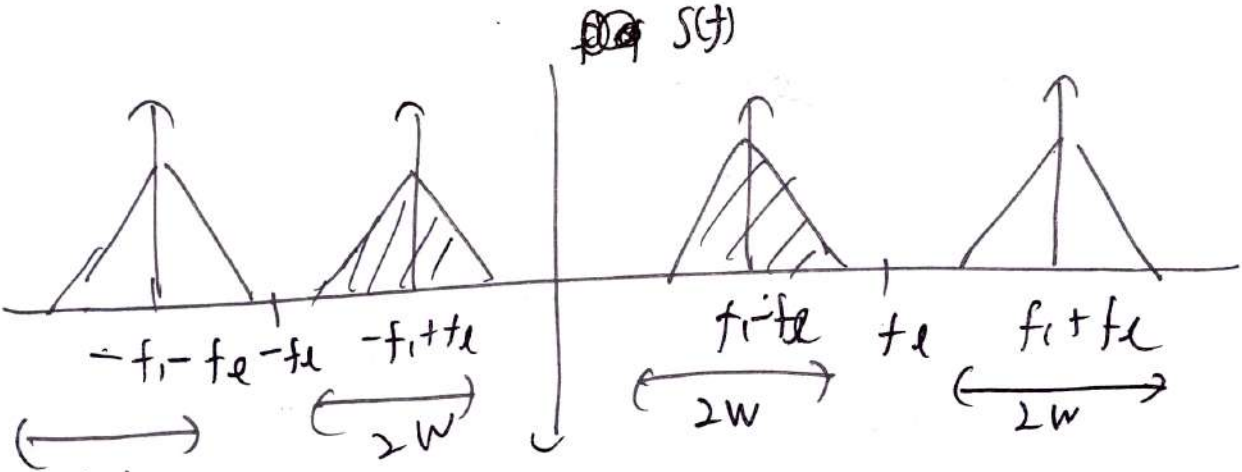
$= m(t)$

Frequency translation =)

suppose we have a modulated wave $s_1(t)$ whose spectrum is centred around carrier freq f_1 and requirement is to translate upwards or ~~downwt~~ downwards.







up conversion $f_2 = f_1 + f_l$

down conversion $f_2 = f_1 - f_l$

Vestigial sideband modulation

~~① Instead of tont~~

for some frequencies (like tv and and computer data)

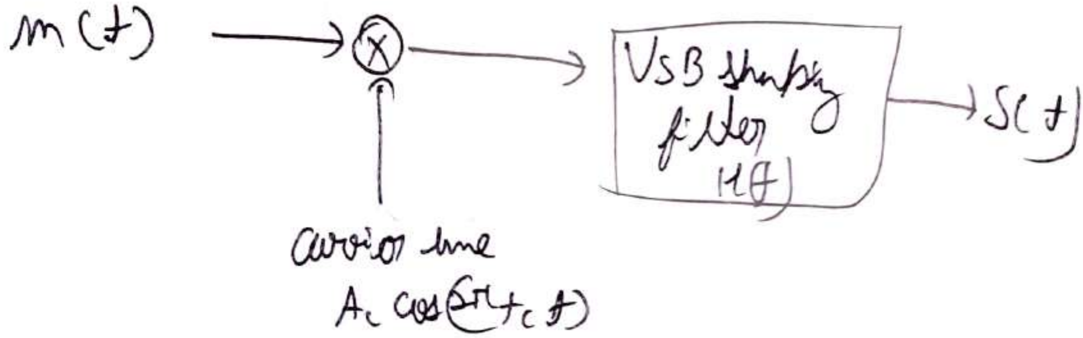
$SSB < B_T < DSB$ → $B_T = f_v + W$

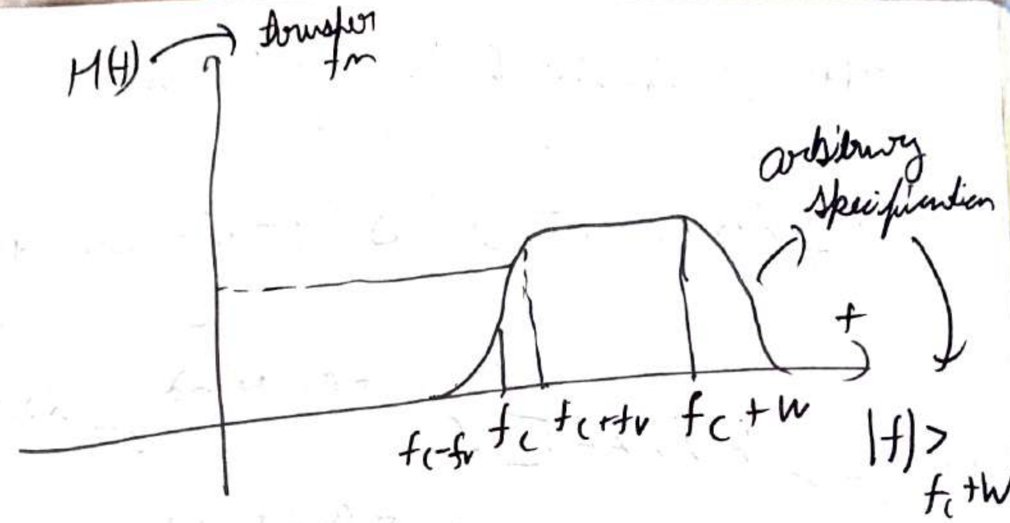
W 2W

vestige bandwidth

① Instead of completely removing a sideband, a trace or vestige of that sideband is transmitted;

② Instead of transmitting the other sideband in full, almost the whole of this second band is also transmitted.





$$H(f+f_c) + H(f-f_c) = 1$$
$$-W \le f \le W$$

where $H(f) = u(f-f_c) - H_v(f-f_c)$

u ↓ unit step

H_v ↓ vestige transfer fn

Sinusoidal VSB

$m(t) = A_m \cos(2\pi f_m t)$

CW $\quad c(t) = A_c \cos(2\pi f_c t)$

let ~~$f_c - f_m$ and $f_c + f_m$~~ $(f_c + f_m)$ and $-(f_c + f_m)$ be attenuated by a factor k and $(f_c - f_m)$ $-(f_c - f_m)$ by $1-k$

$$S(f) = \frac{1}{4} A_m A_c k \left(\delta(f - (f_c + f_m)) + \delta(f + (f_c + f_m)) \right)$$

$$+ \frac{1}{4} A_m A_c (1-k) \left(\delta(f - (f_c - f_m)) + \delta(f - (f_c - f_m)) \right)$$

$$s(t) = \frac{1}{4} k A_c A_m \left[e^{j2\pi(f_c + f_m)t} + e^{-j2\pi(f_c - f_m)t} \right]$$

$$+ \frac{1}{4} A_m A_c k \left[e^{j2\pi(f_c - f_m)t} + e^{-j2\pi(f_c - f_m)t} \right]$$

$$= \frac{1}{2} k A_c A_m \cos(2\pi(f_c + f_m)t)$$

$$+ \frac{1}{2}(1-k) A_c A_m \cos(2\pi(f_c - f_m)t)$$

$$= \frac{1}{2} k A_c A_m \left(\cos(2\pi f_c t)\cos(2\pi f_m t) - \sin(2\pi f_c t)\sin(2\pi f_m t) \right)$$

$$+ \frac{1}{2} A_c A_m \left(\cos(2\pi f_c t)\cos(2\pi f_m t) + \sin(2\pi f_c t) \sin(2\pi f_m t) \right)$$

$$- \frac{1}{2} k A_c A_m \left(\cos(2\pi f_c t)\cos(2\pi f_m t) + \sin(2\pi f_c t)\sin(2\pi f_m t) \right)$$

$$= -\frac{2}{2} k A_c A_m \sin(2\pi f_c t) \sin(2\pi f_m t)$$

$$+ \frac{1}{2} A_c A_m \left(\cos(2\pi f_c t) \cos 2\pi f_m t \right)$$

$$+ \frac{1}{2} A_c A_m \left(\sin 2\pi f_c t \sin(2\pi f_m t) \right)$$

$= \cancel{(1-2k)\sin 2}$

$= (1-2k)\frac{A_c A_m}{2} (\sin(2\pi f_c t) \sin(2\pi f_m t))$

$+ \frac{1}{2}\frac{A_c A_m}{2} \cos(2\pi f_c t) \cos(2\pi f_m t)$

if $k = 1/2 \rightarrow$ DSB-SC

$k = 0 \rightarrow$ lower SSB

$k = 1 \rightarrow$ upper SSB

$k \in (0, \frac{1}{2}) \rightarrow$ ~~attend~~ attenuated version of upper side frequency

↓

LSB

vestige of $s(t)$

$k \in (\frac{1}{2}, 1) \rightarrow$ attenuated version of LSB

↓

USB

defines the vestige of $s(t)$

Demodulation of VSB
(Coherent detection)

$$v(t) = \cancel{A_c \cos} A_c' s(t) \cos(2\pi f_c t)$$

$$V(f) = \frac{1}{2} A_c' [S(f-f_c) + S(f+f_c)]$$

$$S(f) = \frac{1}{2} A_c [M(f-f_c) + M(f+f_c)] H(f)$$

$$S(f-f_c) = \frac{1}{2} A_c [M(f-f_c-f_c) + M(f)] H(f)$$

$$= \frac{1}{2} A_c [M(f-2f_c) + M(f)]$$

$$\cancel{\oplus} S(f+f_c) = \frac{1}{2} A_c [M(f) + M(f+2f_c)] H(f+f_c)$$

$$V(f) = \frac{1}{4} A_c A_c' M(f) [H(f-f_c) + H(f+f_c)]$$

$$+ \underbrace{\frac{1}{4} A_c A_c' [M(f-2f_c) H(f-f_c) + M(f+2f_c) H(f+f_c)]}_{Z(f)}$$

$$V(f) = \frac{1}{4} A_c A_c' M(f) + Z(f)$$

Since $H(f-f_c) + H(f+f_c) = 1$

~~Example → 3.4~~ Demodulation of Sinusoidal VSB

from Sinusoidal VSB

$$s(t) = (1-2k)\frac{A_c A_m}{2}\sin(2\pi f_c t)\sin(2\pi f_m t) + \frac{A_c A_m}{2}\cos(2\pi f_c t)\cos(2\pi f_m t)$$

&

$$v(t) = A_c' s(t)\cos(2\pi f_c t)$$

$$= \frac{(1-2k)}{2} A_c' A_c A_m (\sin(2\pi f_c t))\sin(2\pi f_m t)\cos(2\pi f_c t)$$

$$+ \frac{A_c A_m A_c'}{2}\cos^2(2\pi f_c t)\cos(2\pi f_m t)$$

$$\cancel{= \frac{(1-2k)}{2} A_c' A_c A_m (\sin(2\pi f_c t))\sin(2\pi f_m t)}$$

$$\cancel{+ \frac{A_c A_c' A_m}{2}\left[\cos(2\pi f_m t)\cos(4\pi f_c t)\right.}$$

$$v(t) = \frac{1}{4} A_c A_c' A_m \cos(2\pi f_m t)$$

$$+ \frac{1}{4} A_c A_c' A_m\left[\cos(2\pi f_m t)\cos(4\pi f_c t) + (1-2k)\sin(2\pi f_m t)\sin(4\pi f_c t)\right]$$

$$V_0(t) = \frac{1}{4} A_c A_c' A_m \cos(2\pi f_m t)$$

Envelope detection of VSB + Carrier

$$S_{VSB+C}(t) = A_c \cos(2\pi f_c t) + k_a S(t)$$

Angle Modulation

Phase modulation

~~freq of CS changes w.r.t m(t)~~

phase of CS changes w.r.t m(t)

$$y(t) = A_c \cos(\omega(t)\, t + \phi)$$

$$\omega(t) = F(m(t))$$

Frequency modulation

~~freq of~~

freq of CS changes w.r.t m(t)

~~y(t) = A~~

$$y(t) = A_c \cos(\omega_c t + \phi(t))$$

$$\phi(t) = F(m(t))$$

Advantages ⇒

- Noise reduction
- Improved system fidelity
- Efficient use of power
  - ↳ Amp of CW remains constant
- → Better BW usage

→ Radio broadcasting

→ TV sound transmission

→ cellular radio

Frequency modulation $\Rightarrow$

$\Rightarrow$ let C·S $c(t) = E_c \cos(\omega_c t + \phi)$ $\rightarrow E_c \cos(2\pi f_c t + \phi)$

$E_c \cos(\theta(t))$

$$\theta = 2\pi f_c t + 2\pi k_f \int_0^t x(t)\,dt$$

or

$$\theta = 2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt$$

sensitivity

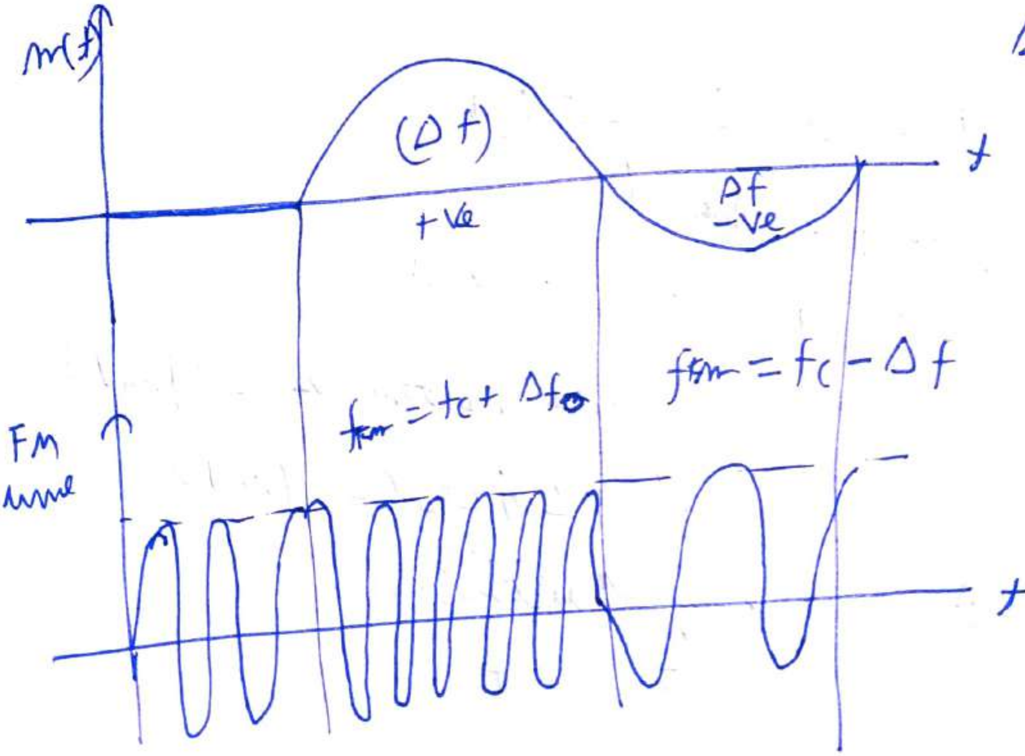
freq ↑

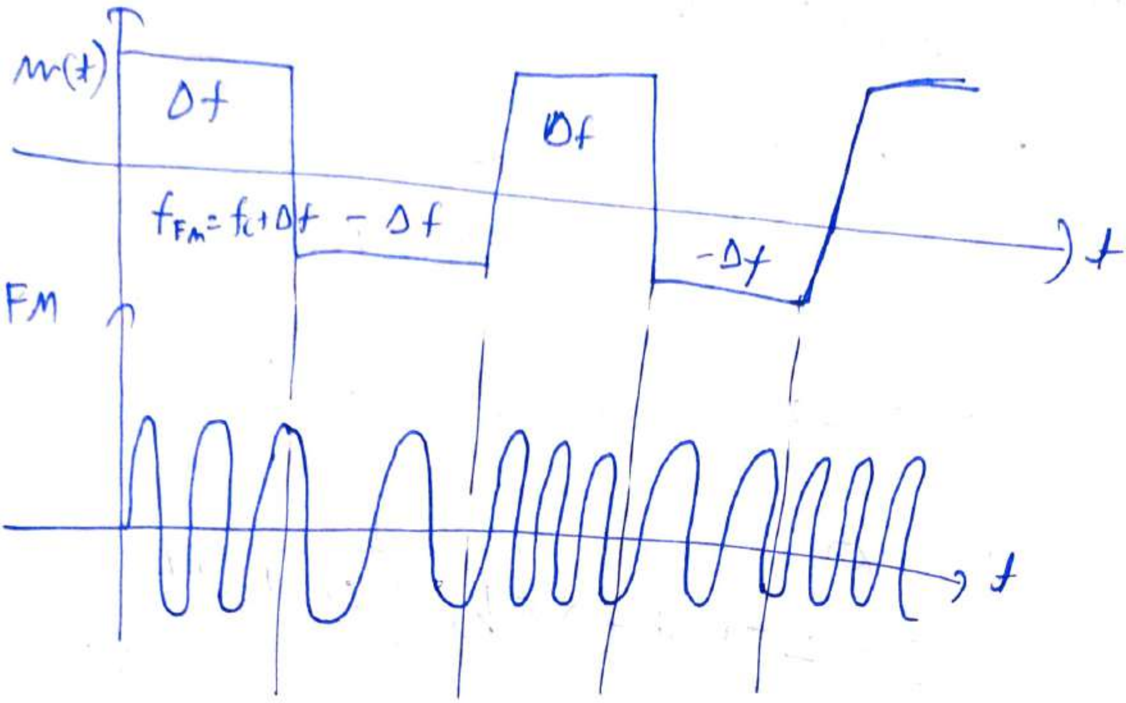
we get

$\cancel{c(t)\ or}$ (S) $s(t) = A_c \cos\left[2\pi f_c t + \underline{2\pi k_f \int_0^t m(t)\,dt}\right]$

deviation of freq

Δf





~~phase modulation~~

⇒ ~~m~~

frequency deviation and modulating Index of FM

$$m(t) = A_m \cos(2\pi f_m t)$$

→

$$c(t) = A_c \sin(2\pi f_c t)$$

freq of FM signal

$$F_i(t) = f_c + K_f x(t)$$

$$= f_c + K_f A_m \cos(2\pi f_m t)$$

$$= f_c + \Delta f \cos(2\pi f_m t)$$

max freq deviation

$$= f_c \pm \Delta f$$

→ FM signal

$$s(t) = A_c \sin\left(\omega_c t + K_f 2\pi \int_0^t m(t)\,dt\right)$$

$$= A_c \sin\left(\omega_c t + K_f \frac{2\pi A_m}{2\pi f_m} \sin 2\pi f_m t\right)$$

$$= A_c \sin\left(\omega_c t + K_f \frac{A_m}{f_m} \sin 2\pi f_m t\right.$$

$$= A_c \sin\left(\omega_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t)\right)$$

→ modulation index of FM $= \dfrac{\Delta f}{f_m}$

$$y_{FM}(t) = E_c \sin\left(\omega_c t + m_f \sin(2\pi f_m t)\right)$$

⇒ Deviation ratio $= \dfrac{\text{Max deviation}}{\text{Max modulating freq}}$

⇒ % Modulation of FM $= \dfrac{\text{Actual freq deviation}}{\text{Max allowed deviation}}$

avg Power

$$P_{AM} = \frac{A^2}{2R}\left(1+\frac{m^2}{2}\right)$$

$$DSB\text{-}SC = \frac{A^2 m^2_{rms}}{2R}$$

$$SSB = \frac{A^2 m^2_{rms}}{TR}$$

$$VSB = \frac{A^2 m^2_{rms}}{3R}$$

The antenna current in AM system

$$I_m = I_c\sqrt{1+m^2}$$

$$\text{Modulating wave} \longrightarrow \boxed{\text{Integrator}} \longrightarrow \boxed{\text{Phase modulator}} \longrightarrow FM$$

$$\int_0^t m(t)\,dt$$

$$A_c\cos(2\pi f_c t)$$

Transmission Bandwidth in angle modulation

$$\cancel{m = \frac{\Delta f}{f_m}}$$

~~Three categories of modulation index~~

Carson's rule

$$B = 2(\Delta f + f_m)$$

Carson's rule $\Rightarrow$

① In theory there are infinite no of ~~sidebands~~ side frequencies, however ~~they are finite in~~ practice the significant side frequencies ~~are~~ that contribute to the signal are finite and can be calculated for a specific level of distortion

② when modulating single tone wave of frequency f_m the significant side frequency ~~dev~~ that deviate from f_c by more than Δf rapidly decrease thus limiting bandwidth

~~for a~~ Large MI β

→ BW approaches $2\Delta f$ as additional SF become negligible

~~→ BW~~

small MI B

→ BW approaches $2f_m$ since only the carrier and a pair of SF are present

Carson's rule $\Rightarrow$ for a single tone modulating wave

$$B_T \approx 2\Delta f + 2f_m$$

Carson's rule → gnolhrule for WB FM system

→ the more precise method is considering max no of sig. SF that exceed a specified amp. (1% of carrier Amp)

- $B_T = 2 n_{max} f_m$

n_{max} is largest integer satisfying $|J_n(\beta)| > 0.01$ where $J_n(\beta)$ ↓ bessel's fn of first kind

Generalised Carson's rule

$$B_T = 2(\Delta f + W)$$

Narrow Band frequency Modulation

Since FM → non linear hence spectral analysis of FM → more difficult.

consider single tone modulation that produces a narrow band FM wave

consider

$m(t) = A_m \cos(2\pi f_m t)$

Instantaneous frequency ⇒

$f_i(t) = f_c + k_f A_m \cos(2\pi f_m t)$

$= f_c + \underbrace{\Delta f}_{\text{prop to } f_m} \cos(2\pi f_m t)$

$\theta_i(t) = 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t)$ — eq 1

$\beta = \frac{\Delta f}{f_m}$

$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$

for FM wave of $s(t)$ of eq 1 MI

$\beta < 1$ and we may use

~~cos(A+B~~

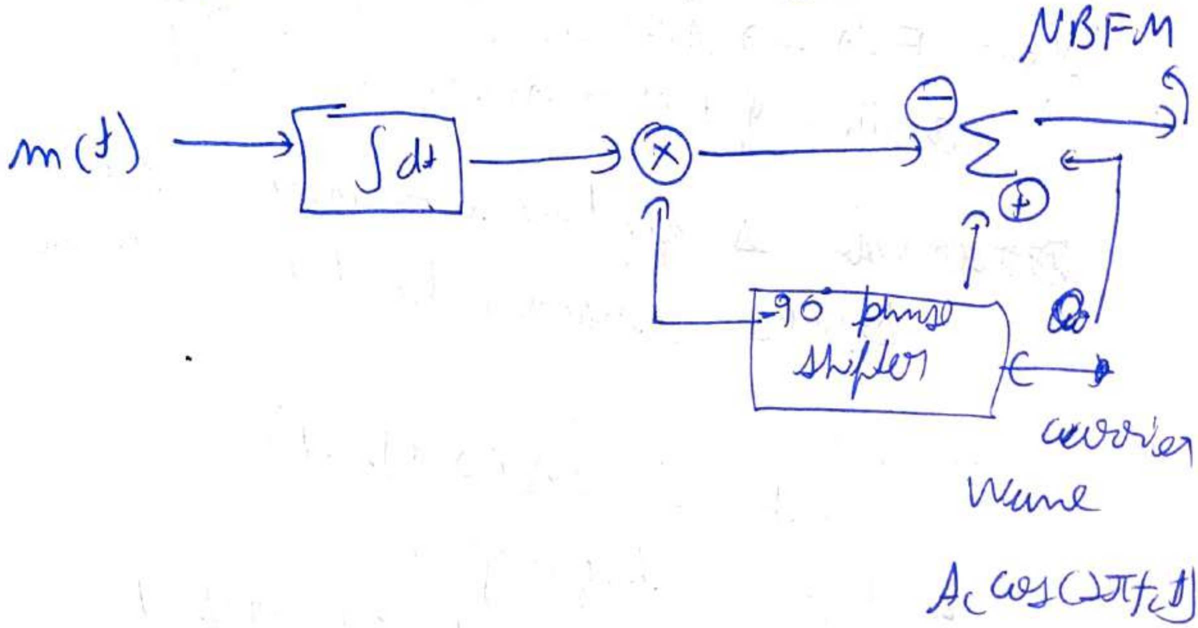
$s(t) = A_c \cos 2\pi f_c t \cos[\beta \sin(2\pi f_m t)]$

$- A_c \sin 2\pi f_c t \sin[\beta \sin(2\pi f_m t)]$

$\cos[\beta \sin(2\pi f_m t)] = 1$

$\sin[\beta \sin(2\pi f_m t)] \simeq \beta \sin(2\pi f_m t)$

$$s(t) \approx A_c \cos 2\pi f_c t - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$



proceeding further with

$$\beta \leq 0.3 \text{ rad}$$

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2} \beta A_c \cos(f_c + f_m) t - \frac{1}{2} \beta A_c \cos(f_c - f_m) t]$$

which is similar to AM wave

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2} \mu A_c \{ \cos((f_c + f_m) t\, 2\pi) + \cos(2\pi (f_c - f_m) t) \}$$

NBFM $\rightarrow B_T = 2 f_m \rightarrow$ same for AM

Wideband frequency Modulation

How can we simplify the spec analysis of ~~WBS~~ W B FM $\Rightarrow$ complex baseband representation of a modulated ~~wave~~ signal.

assuming f_c is large enough compared to the bandwidth of FM wave

$$s(t) = Re\left[A_c e^{j2\pi f_c t + j\beta \sin(2\pi f_m t)}\right]$$

from eq 1

$$s(t) = Re\left[\tilde{s}(t) e^{j2\pi f_c t}\right] \quad -①$$

$$\cancel{\tilde{s}(t) = j\beta \sin 2\pi f_m t}$$

$$\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_m t)}$$

$K \in Z$

let $t = t + \frac{K}{f_m} \rightarrow$

$$\tilde{s}(t) = A_c e^{j\beta \sin\left(2\pi\left(t + \frac{K}{f_m}\right) f_m\right)}$$

$$= A_c e^{j\beta \sin(2\pi f_m t + 2\pi K)}$$

$$= A_c e^{j\beta \sin(2\pi f_m t)}$$

hence $f_m \rightarrow$ fundamental freq of $\tilde{s}(t)$

we therefore expand $\tilde{s}(t)$

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi f_m t)n}$$

where

$$c_n = f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t)\, e^{-j2\pi n f_m t}\, dt$$

$$= f_m A_c \int_{-1/2f_m}^{1/2f_m} e^{[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]}\, dt$$

$$= f_m A_c \int_{-1/2f_m}^{1/2f_m} e^{[j\beta \sin(2\pi f_m t) - j2\pi n f_m t]}\, dt$$

let $2\pi f_m t = x \Rightarrow dx = dt/2\pi$

$1/2f_m \to \pi$

$-1/2f_m \to -\pi$

$$= \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j\beta \sin x - jnx}\, dx$$

$$= \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j[\beta \sin x - nx]}\, dx$$

we know

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin - nx)} dx$$

~~we~~ where $J_n(\beta) \to n^{th}$ order bessel's function

$C_n = A_c J_n(\beta)$

hence $\tilde{s}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{(j 2\pi n f_m t)}$

and we can rewrite the eq as

~~$s(t) = Re\left[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n(f_m t + f_c)t} \right.$~~

$$s(t) = Re\left[A_c \sum_{n=\infty} J_n(\beta) e^{2\pi(n f_m + f_c)t} \right]$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(f_c + n f_m) t \, 2\pi$$

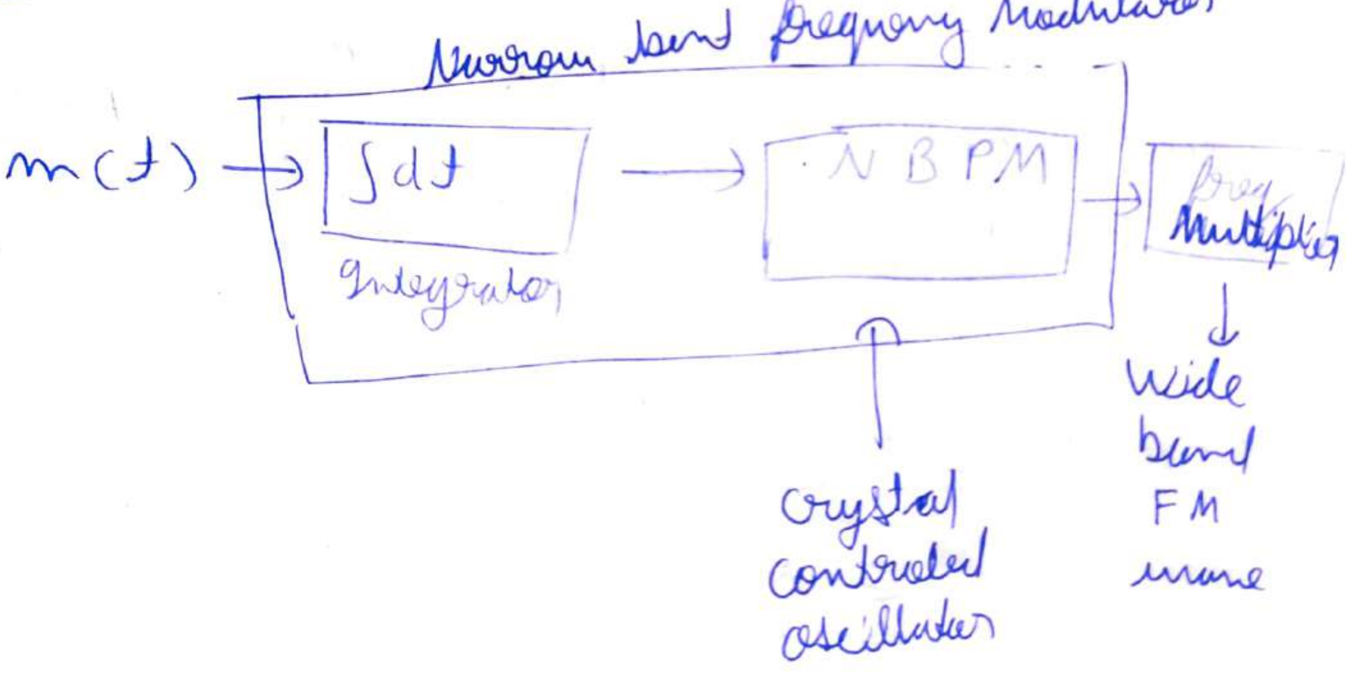
$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} \frac{J_n(\beta)}{2} \left[\delta(f - (f_c + n f_m)) + \delta(f + (f_c + n f_m)) \right)$$

$s(t) \Longleftrightarrow S(f)$

~~Demodulation:~~

Generation of FM signals ⇒

⇒ Direct method → Sinusoidal oscillator, with one of the reactive elements (eg capacitive element) in the tank circuit of the oscillator is being directly controlled by message signal.



for indirect method

Indirect method $\Rightarrow$

Armstrong method

The message signal is first integrated ~~and~~ in integrator then ~~passed itso into Narrow band phase modulator which is crystal controlled~~ used to phase modulate a crystal controll-ed oscillator so that carrier frequency do not deviate much. In order to minimize the distortion β is kept small thereby resulting in NB FM wave. The Narrow band FM wave is then passed through frequency multiplier to get desired wide band FM wave

~~$v(t) = a_1 s(t) + a_2 s(t)^2 + a_3 c$~~

frequency multiplier

$s(t) \longrightarrow$ memoryless non linear device $\xrightarrow{v(t)}$ BPF with mid band freq nf_c $\longrightarrow s'(t)$

$f_c' = nf_c$ and $\beta' = n\beta$

$$v(t) = a_1 s(t) + a_2 s(t)^2 + a_3 s^3(t) + \text{—} \; a_n s^n(t)$$

$$s'(t) = A_c \cos\left[2\pi f_c' t + 2\pi k_f' \int_0^t m(t)\,dt\right]$$

$$f_c' = n f_c$$

$$k_f' = n k_f$$

Demodulation of F M Signals ⇒

⇒ frequency discriminator

given $s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt\right)$

$$\frac{ds(t)}{dt} = -2\pi A_c\left[f_c + k_f\, m(t)\right] \sin\left(2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt\right)$$

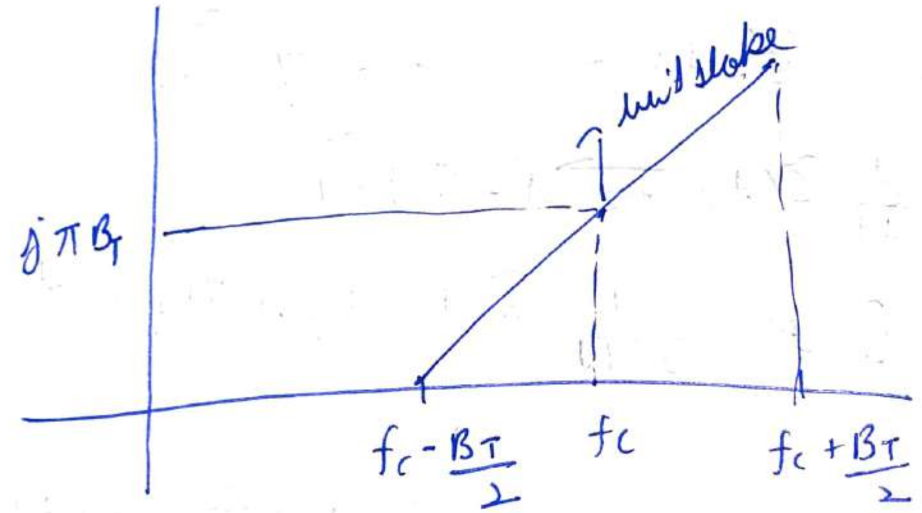
we know $\frac{d}{dt} \rightleftharpoons j2\pi f$

in practical terms it is difficult to built a circuit with TF ⇒ $j2\pi f$.

Instead a circuit that approximates a TF over BP signal bandwidth particularly

$$f_c - (B_T/2) \le |f| \le f_c + (B_T/2)$$

$$H_1(f) = \begin{cases} j2\pi\,[f - (f_c - B_T/2)], & f_c - \frac{B_T}{2} \le |f| \le f_c + \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$



proceeding with complex baseband representation of signal processing

complex envelope of FM signal

$$\tilde{S}(t) = A_c\, e^{\left(j2\pi k_f \int_0^t m(t)\,dt\right)} \quad \text{— (A)}$$

slope circuit corresponding to eq A

$$H_1(f) = \begin{cases} j\,[2\pi(f + B_T/2)], & -\frac{B_T}{2} \le |f| \le \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases}$$

$\tilde{S}_1(t) \rightarrow$ complex envelope of response of slope circuit

$$S_1(f) = \frac{1}{2}\tilde{H}_1(f)\,\tilde{S}(f) =$$

$$= \begin{cases} j\pi\left(f + \frac{1}{2}B_T\right)\tilde{S}(f), & -\frac{B_T}{2} \le f \le \frac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases}$$

~~Argue $\tilde{s}(t)$~~

we know $\tilde{s}(t)$ is inverse of $\tilde{S}_1(f)$

$$\frac{d}{dt}\tilde{s}(t) \rightleftharpoons j2\pi f\tilde{S}(f)$$

$$\tilde{s}_1(t) = \frac{1}{2}\frac{d}{dt}\tilde{s}(t) + \frac{1}{2}j\pi B_T\tilde{s}(t)$$

$$\tilde{s}_1(t) = \frac{1}{2}j2\pi A_c B_T\left[1 + \left(\frac{2k_f}{B_T}\right)m(t)\right]\left(e^{j2\pi k_f\int_0^t m(t)dt}\right)$$

finally the actual response of slope circuit due to FM wave $s(t)$ is given by

$$s_1(t) = \text{Re}\left[\tilde{s}_1(t)e^{j2\pi f_c t}\right]$$

$$= \frac{1}{2}\pi A_c B_T\left[1 + \left(\frac{2k_f}{B_T}\right)m(t)\right]\cos\left(2\pi f_c t + 2\pi k_f\int_0^t m(t)dt + \frac{\pi}{2}\right)$$

for envelope detector

$$v_1(t) = \frac{1}{2}\pi A_c B_T \left[1 + 2\left(\frac{k_f}{B_T}\right) m(t)\right]$$

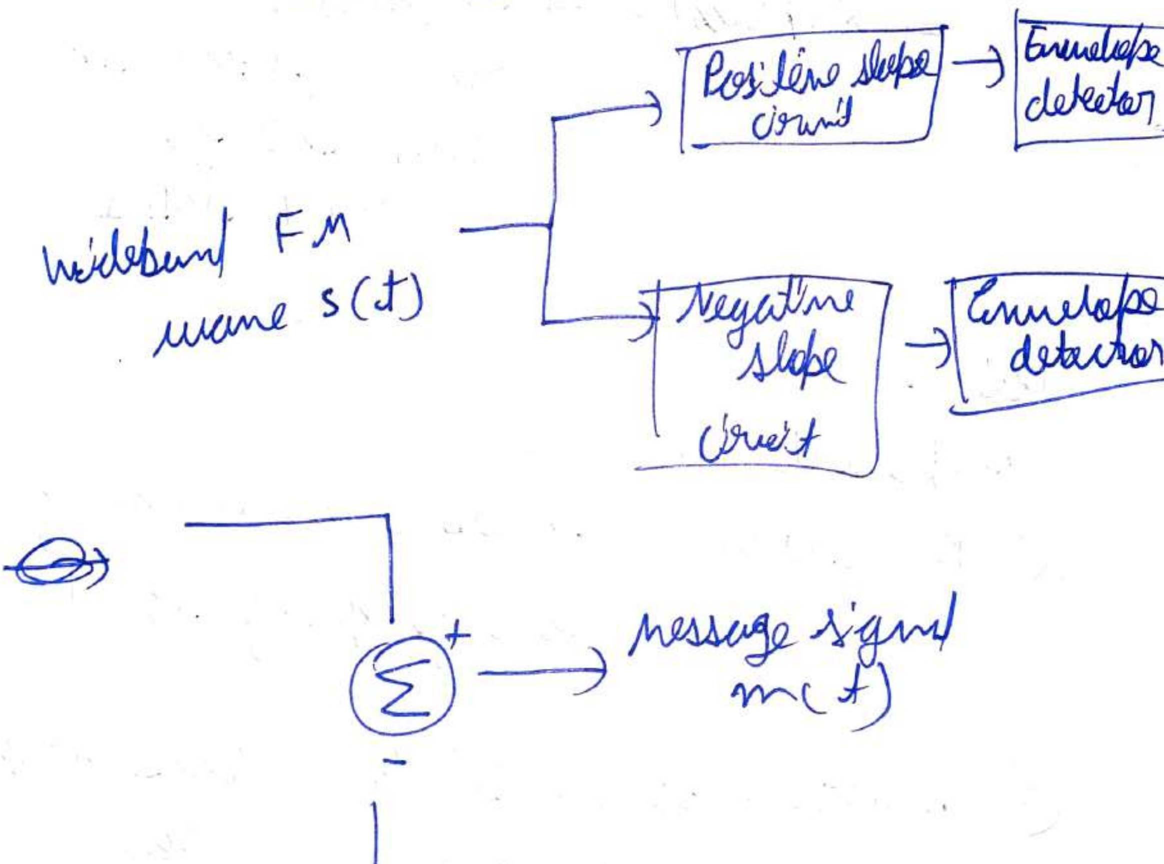
$\downarrow$

remove the bias $\frac{\pi A_c B_T}{2}$

$$v_2(t) = \frac{1}{2}\pi A_c B_T \left[1 - \left(\frac{2k_f}{B_T}\right) m(t)\right]$$

$$v(t) = v_1(t) - v_2(t) = c\, m(t)$$

→ negative slope circuit ~~Envelo~~ Envelope detector



~~Phase locked loop~~

* Super heterodyne reciever
fulfills

⇒ carrier frequency tuning → the purpose is which to select desired signal

⇒ ~~Filtering~~ Filtering → seperate the desired signal from other modulated signal.

⇒ Amplification → Amplifies the weak recieved signal to compensate for power losses.

~~Basic~~.

Components ⇒

1. RF section ⇒ Amplifies the incoming radio frequency signal

2. Mixer and local oscillator ⇒ combination provides heterodyning function
converts the incoming signal to lower fixed (IF) → this freq is easy to process and filter than the original
$f_{IF} = f_{LO} - f_{RF}$

③ IF section ⇒ $f \rightarrow f_{IF} \rightarrow$ filtered and amplified

④ Demodulator ⇒ Extracts the audio or video Info from modulated wave for AM, envelope detector is used

⑤ power amplifier → boosts the demodulated signal to a sufficient level

Phase locked loop ⇒

FM wave →
$s(t)$

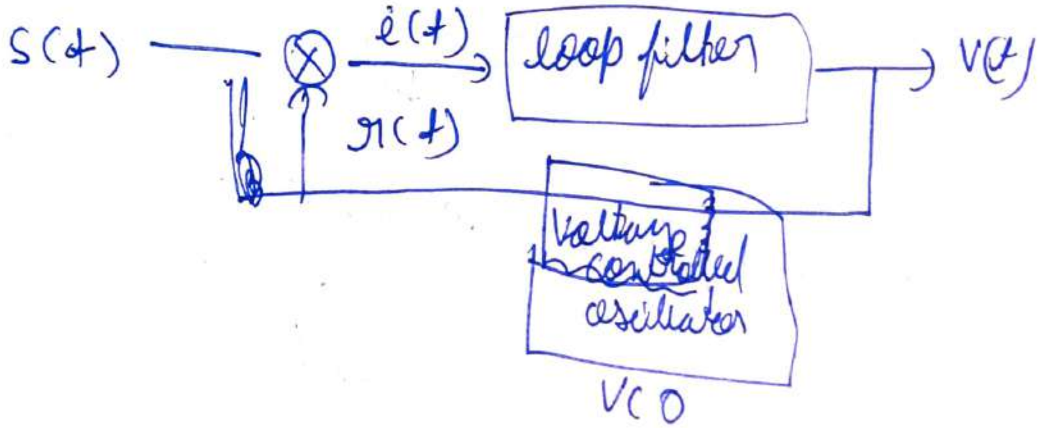
$$s(t) = A_c \sin[2\pi f_c t + \phi_1(t)]$$

to demonstrate the operation of a low pass filter VCO has been adjusted to the factors

① VCO is set precisely at the unmodulated carrier frequency f_c of FM wave $s(t)$

② The VCO output ~~[illegible]~~ has a 90° phase shift w.r.t unmodulated carrier wave.

$$\phi_1(t) = 2\pi k_f \int_0^t m(t)\,dt$$



$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)]$$

$$\phi_2(t) = 2\pi k_v \int_0^t v(\tau)\,d\tau$$

frequency ~~sent~~ sensitivity factor of VCO

multiplication of incoming s(t) by r(t) gives

① HF component ⇒ → discarded by loop filter

$$A_v A_c k_m \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)]$$

→ k_m is multiplier gain

② Low F component

$$k_m A_v A_c \sin[\phi_1(t) - \phi_2(t)]$$

$$e(t) = k_m A_c A_v \sin[\phi_e(t)]$$

$$\phi_e(t) = \phi_1(t) - \phi_2(t)$$

$$= \phi_1(t) - 2\pi k_v \int_0^t v(\tau)\, d\tau$$

when $[\phi_e(t)] = 0 \Rightarrow$ phase locked loop is said to be in phase lock.

near phase locked $\phi_e(t) < 1$

for this

$$\sin[\phi_e(t)] \approx \phi_e(t)$$

$$e(t) \approx k_m A_c A_v \phi_e(t)$$

$$\approx \frac{K_0}{k_v} \phi_e(t)$$

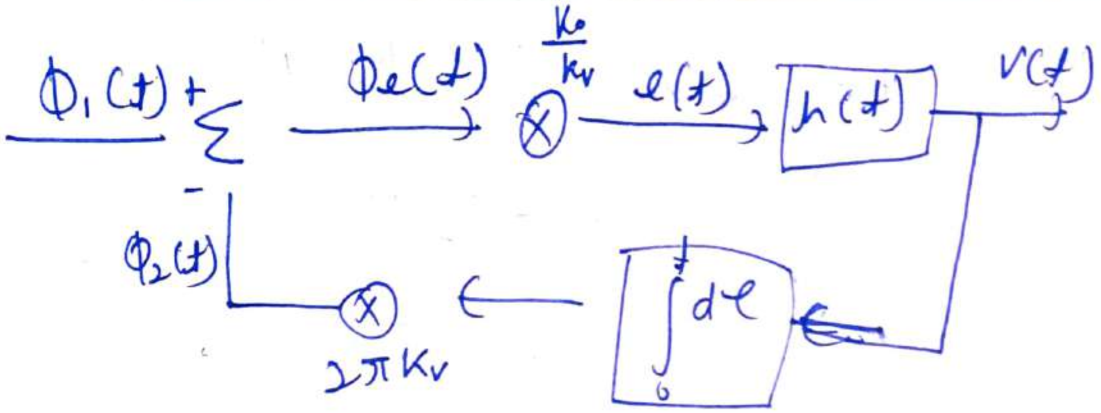
$$K_0 = k_m k_v A_c A_v$$

$\downarrow$

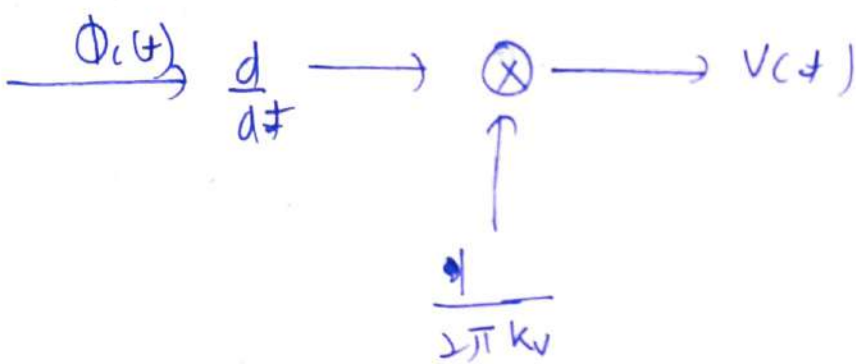
loop gain parameter

now $$v(t) = \int_{-\infty}^{\infty} e(\tau)\, h(t-\tau)\, d\tau$$

$$= e(t) * h(t)$$



linearised model of phase locked loop.



approximate form of model

assuming that the loop gain K_0 is large compared to unity

$$v(t) \approx \frac{1}{2\pi K_v}\left(\frac{d}{dt}\phi_1(t)\right)$$

$$\approx \frac{1}{2\pi K_v}\frac{d}{dt}\left(2\pi K_f \int_0^t m(\tau)\, d\tau\right)$$

$$= \frac{K_f}{K_v} m(t)$$

Random Variables and Processes

probability thm.

$P \rightarrow$ function $\rightarrow$ assign a +ve no to event A in sample space S.

① $0 \leq P[A] \leq 1$

② $P[S] = 1$

if A and B are mutually exclusive events

$P[A \cup B] = P[A] + P[B]$

and $P[\bar{A}] = 1 - P[A]$

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

and if A_1, A_2, A_3 --- A_m are mutually exclusive events.

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$$\sum_{m=1}^{m} P[A_i] = 1$$

Conditional probability

$$P[B|A] = \frac{P[A \cap B]}{P[A]}$$

probability of B when A has already occured

$$P[A \cap B] = P[B|A]P[A]$$

joint probability of A and B

and

$$P[A \cap B] = P[A|B]\ P[B]$$

$$P[B|A] = \frac{P[A|B]\ P[B]}{P[A]}$$

<u>Binary symmetric channel</u>

discrete memoryless channel used to transmit data

① $P[A_0] = p_0$ and $P[A_1] = p_1$

probability of sending 0 and recieving 0

probability of sending 1 and recieving 1 let $P[B_1 | A_0] = P[B_0 | A_1] = p$

sending 0 recieving 1 — sending 1 recieving 0

requirement → to determine the ~~pre~~ posteriori probabilities $P[A_0 | B_0]$ and $P[A_1 | B_1]$

we know

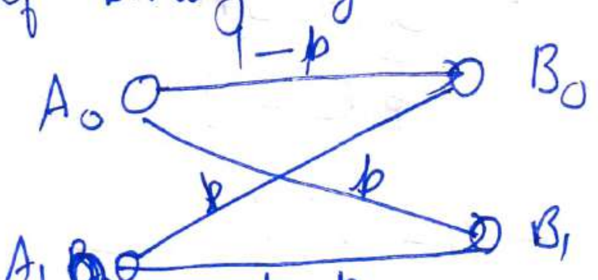
$$P[B_0 | A_0] + P[B_1 | A_0] = 1$$

that is to say

$$P[B_0 | A_0] = 1 - p$$

$$\therefore P[B_1 | A_1] = 1 - p$$

Transition probability diagram of binary symmetric channel.



binary symmetric channel.

① probability of recieving 0 is given by

$$P[B_0] = P[B_0 | A_0] P[A_0] + P[B_0 | A_1] P[A_1]$$

$$= (1-p)p_0 + p\, p_{01}$$

for recieving 1 is given by

$$P[B_1] = P[B_1 | A_0] P[A_0] + P[B_1 | A_1] P[A_1]$$

therefore

$$P[A_0 | B_0] = \frac{P[B_0 | A_0] P[A_0]}{P[B_0]}$$

$$= \frac{(1-p)p_0}{(1-p)p_0 + p\, p_{01}}$$

$$P[A_1 | B_1] = \frac{P[B_1 | A_1] P[A_1]}{P[B_1]}$$

$$= \frac{(1-p)p_1}{p\, p_0 + (1-p)p_1}$$

Random variable ⇒ we assign the events in sample ~~sample~~ space a variable $X(S)$ or just X

- alternatively a function whose domain is a sample space and whose range is a set of real no. is called random variable

continuous | discrete

discrete ↓ probability mass function describes the probability of each possible value of random variable

(eg) coin tossing exp

$$P[X=x] = \begin{cases} 1/2, & x=0 \\ 1/2, & x=1 \\ 0, & \text{otherwise} \end{cases}$$

Distribution fn

Probability that X (random variable) takes values less than or equal x.

$F_X(x)$, so

$$F_X(x) = P[X \le x]$$

cumulative distribution function (cdf) for random variable X

properties

① $F_X(x)$ is bounded between 0 and 1

② $F_X(x)$ is a monotone - non decreasing fn of x i.e

$$F_X(x_1) \le F_X(x_2) \quad \text{if } x_1 < x_2$$

if it is continuously differentiable then alternative description of the probability of random variable X is often useful.

$f_X(x) = \frac{d}{dx} F_X(x)$

) probability density fn.

it arises from the fact that

$x_1 < X \leq x_2$ equals

$P[x_1 < X \leq x_2] = P[X \leq x_2] - P[X \leq x_1]$

$= F_X(x_2) - F_X(x_1)$

$= \int_{x_1}^{x_2} f_X(x)\,dx$

the probability of an interval is area under $f_X(x)$

replacing $x_1 = -\infty$ and some things

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

Since $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

~~Sev~~
Several random variables

$F_{X,Y}(x, y)$ Same sample space

X, Y may be component of a vector or not.

joint distribution function $F_{X,Y}(x, y)$

→ monotone non decreasing

i.e $F_{X,Y}(x, y) = P[X \leq x, Y \leq y]$

if continous everywhere

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

joint probability density function.

non negative

also the total value

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

PDF of single random variable say (X) can be obtained from its joint probability density fn

$$F_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy$$

diff both sides

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$f_X(x), f_Y(y) \longrightarrow$ marginal densities

conditional probability density of Y given that $\boxed{X = x}$ $\longrightarrow$ why?

$$f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

variable

arbitrary but fixed.

it satisfies the property of non negativity $f_Y(y|x) \geq 0$

and $\int_{-\infty}^{\infty} f_Y(y|x)\, dx = 1$

if X and Y are statistically independent then

$$f_Y(y|x) = f_Y(y)$$

hence

$$\cancel{f_{x,y}}\ f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

~~equivo~~ ~~equivo~~ equivalently

$$P[X \in A, Y \in B] = P[X \in A]\ P[Y \in B]$$

Statistical Averages

mean of random variable X is defined by

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

E → statistical expectation operator.

i.e μ_X ~~loc~~ locates the centre of gravity of area under probability density curve of random variable X.

function of random variable

$$Y = g(X) \curvearrowright R.V$$

& R.V

$$E[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, dx$$

~~if the ka~~

Moments

If we take $g(X) = X^n \longrightarrow n^{th}$ moment of DF

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) \, dx$$

if $n=1$ mean ~~ct~~ of random variable

$n=2$ mean square of X

Central moments

$$E[(X-\mu_X)^n] = \int_{-\infty}^{\infty} (x-\mu_X)^n f_X(x) \, dx$$

if $n=1$ then it will be 0

$n=2$ then second central moment

$$Var[X] = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x) \, dx$$

sq root $\rightarrow$ standard deviation

Variance σ_x^2 of RV $\to$ X is a measure of Randomness.

by specifying the value σ_x^2 we constrain ~~it the~~. about its mean μ_x

- Chebyshev Inequality

$$\boxed{P(|X - \mu_x| \geq \varepsilon) \leq \frac{\sigma_x^2}{\varepsilon^2}}$$

$\varepsilon \to$ +ve no

Now we know

$$\sigma_x^2 = E[X^2 - 2\mu_x X + \mu_x^2]$$
$$= E[X^2] - 2\mu_x E[X] + \mu_x^2$$
$$= E[X^2] - \mu_x^2 \quad (E[X] = \mu_x)$$

Characteristic fn

$$\Phi_x(\nu) = E[e^{j\nu X}]$$
$$= \int_{-\infty}^{\infty} e^{j\nu x} f_X(x)\,dx$$

$\downarrow$

FT of PDF $f_X(x)$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(x)\, e^{-jvx}\, dx$$

Random process

signals like ~~that~~ computer, voice ~~so~~ signals are ~~random~~

time dependent

random

Sample space ~~or~~ ensamble comprised of function of time is called random process

$$X(t, s), \quad -T \leq t \leq T$$

for a fixed sample point s_j the graph of function $X(t, s_j)$ vs time is called a realization or sample function.

* for a random variable, the outcome of a random expt is mapped into a number
* for a random process, the outcome of a random experiment is mapped into a waveform i.e a fn of time.

Types of Random process

1. continuous Random process → T and S are continuous
2. Discrete random process → T is continuous and S is discrete
3. Discrete random sequence → T and S are discrete
4. continuous Random sequence → T is discrete but S is conti-nous.

Stationary ~~radom~~ random processes

If a radom process is divided in a different time intervals then each then varios sections of the process exhibit essentialy the same statistical properties.

↓

then it is stationary or else non stationary.

to be more precise let $X(t)$ be a random process that is observed at time ~~period~~ t_1

let $F_{X(t_1)}(x)$ be DF of the different sample function of the random process at the t_1. Suppose random process is obs at the $t_1 + e$ ⟹

$$F_{X(t_1+e)}(x)$$

If $F_{X(t_1+e)}(x) = F_{X(t_1)}(x)$

$\forall\, t_1, e$

stationary to the first order

as a consequence mean and Var is also time independent for such a process

Next consider sampling the RP $X(t)$ at two points t_1 and t_2

$$F_{X(t_1)X(t_2)}(x_1, x_2)$$

$$F_{X(t_1+\tau)X(t_2+\tau)}(x_1, x_2) = F_{X(t_1)X(t_2)}(x_1, x_2)$$

if $\forall\ t_1, t_2, \tau$

Stationary to second order

covariance and correlation do not depend upon time

Mean, Correlation and Covariance functions

$f_{X(t_1)}(x) = f_{X(t_2)}(x)$

$\mu_X(t) = \mu_X \quad \forall t$

~~Var [X(t)]~~

$Var[X(t_1)] = Var[X(t_2)]$

autocorrelation fn of X(t) as product of $X(t_1), X(t_2)$ at time t_1, t_2

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \underbrace{f_{X(t_1)X(t_2)}(x_1, x_2)}_{\text{joint PDF}} dx_1 \, dx_2$$

$X(t) \rightarrow$ stationary to second order if joint DF ~~$f_{X_1X_2}(x_1, x_2)$~~

$f_{X(t_1)X(t_2)}(x_1, x_2)$ depends upon diff between observation times t_1 and t_2

$$R_X(t_1, t_2) = R_X(t_2 - t_1)$$

$$\forall \; t_1, t_2$$

$C_x(t_1, t_2) = E[(X(t_1) - \mu_x)(X(t_2) - \mu_x)]$

$= R_x(t_2 - t_1) - \mu_x^2$

auto covariance fn

practical consideration ve liye enough hai

Properties

autocorrelation

$R_x(\tau) = E[X(t+\tau)X(t)] \ \forall t$

for ~~certain~~ convenience.

① $R_x(0) = E[X^2(t)] \rightarrow$ MS value

② $R_x(\tau) = R_x(-\tau)$

~~$R_x(\tau) = E[X(t)$~~

③ $|R_x(\tau)| \leq R_x(0)$ max value

$X(t) \xrightarrow[\text{rapidly change}]{}$ $R_x(\tau)$ decrease from $R_x(0)$ max value

decorrelation time τ_0

such that $\tau > \tau_0$

mag of autocorrelation fn

$R_x(\tau)$ remains ~~value~~ below some prescribed value.

~~Cross~~ Cross correlation fn

$\Rightarrow X(t)$ and $Y(t)$

$R_X(t,u)$, $R_Y(t,u)$

$$R_{XY}(t,u) = E[X(t)Y(u)]$$

If wide sense stationary
then C.C

$$R_{XY}(t,u) = R_{XY}(\tau)$$

$$\tau = t - u$$

Transmission of a random process through a linear filter

$X(t) \longrightarrow$ $\boxed{\text{impulse response } h(t)}$ $\longrightarrow Y(t)$

$Y(t) \longrightarrow$ probability distribution tough hai nikalna

so I/O in time domain form ~~to~~ freq
defining the A.C fn and
mean of $Y(t)$ in
terms of $X(t)$

$$\mu_y(t) = E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, E[x(t-\tau_1)]\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, \mu_x(t-\tau_1)\, d\tau_1$$

$x(t) \to$ WSS , $\mu_x(t)$ is const

$$\mu_y = \mu_x \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1$$

$$= \mu_x\, H(0) \rightarrow \text{zero freq (dc) response}$$

$$R_y(t,u) = E[y(t)\, y(u)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\, x(u-\tau_2)\, d\tau_2\right]$$

$$R_y(t,u) = \left[\int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 \int_{-\infty}^{\infty} d\tau_2\, h(\tau_2)\; E[x(t-\tau_1)\, x(u-\tau_2)]\right]$$

$\to$ WSS

$\tau = t - u$

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_x(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

Power Spectral density

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f \tau}\, df$$

properties of power spectral density

① $S_x(0) = \int_{-\infty}^{\infty} R_x(\tau)\, d\tau$

② $E[x^2(t)] = \underbrace{\int_{-\infty}^{\infty} S_x(f)\, df}_{\text{area under PSD}}$

③ PSD of a WSS random process is always non negative

$$S_x(f) \geq 0 \quad \forall f$$

④ $S_x(-f) = S_x(f)$

Relation among the power spectral densities of input and output random processes.

$$S_y(f) = \int_{-\infty}^{\infty} R_y(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

~~$y(t) \longrightarrow$~~

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\cancel{S_y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)}$$

$$S_y(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_x(\tau - \tau_1 + \tau_2)\, e^{-j2\pi f\tau}$$

$$\cdot\, d\tau_1\, d\tau_2\, d\tau$$

let $\tau - \tau_1 + \tau_2 = \tau_0$, or

$$\tau = \tau_0 + \tau_1 - \tau_2$$

$$S_y(f) = H(f)\, H^*(f)\, S_x(f)$$

$$S_y(f) = |H(f)|^2\, S_x(f)$$

$$\boxed{H(f) = \frac{1}{1 + j2\pi RCf}}$$

Gaussian processes

$$Y = \int_0^T g(t)\, x(t)\, dt$$

linear functional of a random process $x(t)$ weighted ~~over~~ with $g(t)$

- Collection of random ~~pr~~ variables where every linear combination of the process is normally distributed.
- Large no of independent, identically distributed Random variables tends to follow a gaussian distribution.
- in other words the process $x(t)$ is a gaussian process if every linear functional of $x(t)$ is a gaussian Random V.

$$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i$$

where $Y_i = \underbrace{\frac{1}{\sigma_X}(X_i - \mu_X)}_{\text{normalised } X}$

V_N is the average of normalised variable as N increases.

If $N \to \infty$ $V_N \to$ tends to follow Gaussian distribution with mean 0 variance 1.

<u>properties</u>

$$Y(t) = \int_0^T h(t-\tau)\, X(\tau)\, d\tau$$

$X(\tau) \to$ gaussian process

$$Z = \int_0^{\infty} g_Y(t) \int_0^T h(t-\tau)\, X(\tau)\, d\tau\, dt$$

Z must be a random variable

$$z = \int_0^T g(\tau)\, x(T-\tau)$$

where $g(\tau)$

$$g(\tau) = \int_0^\infty g_Y(t)\, h(t-\tau)\, dt$$

property $\rightarrow$ 2 $X(t_1), X(t_2), X(t_3) - - X(t_n)$ ~~observed~~ is observed through random process $X(t)$ at $t_1, t_2, t_3, - t_n$ if $X(t)$ is guassian then this set of random variables is jointly guassian

of any $n \Rightarrow$ set of means $\rightarrow$ by specifying their n-fold joint probability density function being completely determined by

$\mu_{X(t_i)} = E[X(t_i)]$

and the set of autocovariance fn

$C_X(t_k, t_i) = E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})]$

$\rightarrow$ extended into two or more RP consider the composite set of RV $X(t_1), X(t_2), - - X(t_n), Y(u_1), Y(u_2) - -$ $Y(u_n)$ $\rightarrow$ they are jointly guassian if their composite set is jointly guassian for any n and m

Cross covariance fn

$$E\left[\left[(X(t_i) - \mu_{X(t_i)})\right]\left[Y(u_i) - \mu_{Y}(u_K)\right]\right]$$

$$= R_{XY}(t_i, u_K) - \mu_{X(t_i)} Y(u_K)$$

→ cross correlation fn.

Property 3 → If a gaussian process is ~~wide~~ WSS then it also stationary in strict sense

Property 4 If ~~RV $X(t_1), \ldots X(t_n)$~~

R.V $\overset{n}{\underset{i=1}{X}}(t_i)$ are uncorrelated then

→ no relationship b/w

$\left(\overset{n}{\underset{i=1}{X}}(t_i) = X(t_1), X(t_2) \ldots \ldots X(t_n) \right)$

i.e

$$E\left[(X(t_i) - \mu_{X(t_i)})(X(t_K) - \mu_X(t_K))\right]$$

$= 0$

then these are statistically independent

More into ~~Gaussi~~ Gaussian R.V

PDF $\Rightarrow$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{\left(\frac{-(y-\mu_Y)^2}{2\sigma_Y^2}\right)}$$

for $\mu_Y = 0, \sigma_y = 1$

$$f_Y(y) = \cancel{\frac{1}{\sqrt{2\pi}} e^{\left(\frac{-y^2}{2}\right)}}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

denoted as $N(0,1) \leftarrow$ normalised Gaussian distribution

Central limit theorem $\Rightarrow$ sampling distribution of the sum (or average) of a iid random variable follows gaussian distribution.

Standard form

Asymptotic form of CLT $\Rightarrow$ The CLT states that the probability distribution of a normalized sum approaches a standard normal distribution $N(0,1) \rightarrow$ as $N \rightarrow \infty$

Distribution type $\rightarrow$ iid R.V

$$V_n = \frac{\sum_{i=1}^{n} (x_i - \mu_x)}{\sigma \sqrt{n}} \rightarrow N(0,1)$$

mean

SD

as $N \rightarrow \infty$

Some examples $\Rightarrow$

① A random variable X is said to be uniformly distributed over the interval (a, b) if its PDF is

$$f_x(x) = \begin{cases} 0, & x \le a \\ \frac{1}{b-a}, & a < x \le b \\ 0, & x > b \end{cases}$$

$$F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$$

$$= \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x \le b \\ 0, & x > b \end{cases}$$

② Binomial Random variable

let probility of heads in a coin tossing exp $= p$

and X_n be the RV representing outcome of n^{th} toss

Since the outcome of one coin toss is independent of other hence its a set of independent Bernoulli trials.

$$\cancel{Y = \sum_{i=n}}$$

$$Y = \sum_{i=1}^{N} X_i$$

no of heads on N tosses

first ~~conclude~~ ~~we~~ consider the probabity of getting y heads in a row and $N-y$ tails afterwards if the tosses are independent hence

$P[y$ heads then $N-y$ tails$]$

$= p\,p\,p\,p \ldots (1-p)(1-p) \ldots$

$= p^y (1-p)^{N-y}$

the probability of getting anywhere in N trials . then possible arrangements

i.e the probability that $Y = y$ is

$$\cancel{P[y} \quad P[Y = y] = \binom{N}{y} p^{y} (1-p)^{N-y}$$

where $\binom{N}{y} = {}^{N}C_y = \frac{N!}{(N-y)!\, y!}$

binomial distribution.

Expl

3 let $Y = g(X) = \cos(X)$

where X is a random variable uniformly distributed in $(-\pi, \pi)$

$$f_X(x) = \begin{cases} \frac{1}{2\pi}, & x \in (-\pi, \pi) \\ 0, & x \in (-\infty, -\pi) \cup (+\pi, \infty) \end{cases}$$

$$E[Y] = \int_{-\pi}^{\pi} \cos x \left(\frac{1}{2\pi}\right) dx$$

$$= \frac{-1}{2\pi} \sin \Big|_{-\pi}^{\pi} = 0$$

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx}$$

~~Gaussian~~

(7) Gaussian random variable

let X be a gaussian random variable of mean μ_X and variance σ_X^2, the PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

$$x \in (-\infty, \infty)$$

diff both sides n times

$$\frac{d}{d\nu^n} \phi_X(\nu)\Big|_{\nu=0} = (j)^n \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

$$\left(\phi_X(\nu) = \int_{-\infty}^{\infty} f_X(x) \exp(j\nu x)\,dx\right)$$

$$\frac{d^n}{d\nu^n} \phi_X(\nu)\Big|_{\nu=0} = (j)^n \boxed{E[X^n]}$$

$$(-j)^n \frac{d^n}{d\nu^n} \phi(\nu)\Big|_{\nu=0} = E[X^n] \quad -(1)$$

we will be writing $\phi(\nu)$ as $\phi(v)$

i.e $\nu \longrightarrow v$ (to be more relaxed) well to get characteristic fun of Gaussian random variable X of mean μ_X and

$\sigma_X^2 \rightarrow$ var

$\cancel{\phi(v)} =$

$$\phi_X(v) = \cancel{E[e^{jvx}]}$$

$$= E[\exp(jvx)]$$

$$\phi_X(v) = \int_{-\infty}^{\infty} e^{jvx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{jvx} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right)$$

→ e and exp are ~~sence~~ same

$$\phi_X(v) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} e^{jvx} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right) dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} \exp\left(jvx - \frac{(x-\mu_X)^2}{2\sigma_X^2}\right) dx$$

simplify $(x-\mu_X)^2$

$$= x^2 - 2\mu_X x + \mu_X^2$$

in $\exp\left(jvx - \left(\frac{x^2}{2\sigma_X^2} + \frac{\mu_X^2}{2\sigma_X^2} - \frac{2\mu_X x}{2\sigma_X^2}\right)\right)$

$\Rightarrow \exp\left(-\left(\frac{x^2}{2\sigma_x^2} + \left(\frac{\mu_x}{\sigma_x^2} - jv\right)x + \frac{\mu_x^2}{2\sigma_x^2}\right)\right)$

$\Rightarrow$ we get

$$\Phi_x(v) = \exp\left(jv\mu_x - \frac{1}{2}v^2\sigma_x^2\right) \quad -②$$

~~eq 1 and 2 uniquely determines by~~

eq 1 and shows that higher order moments of GRV are uniquely determined by μ_x and variance σ_x^2.

Central moments $\Rightarrow$ shows how RV X deviates from its mean μ.

k^{th} central moment is $\Rightarrow$

$$\mu_k' = E[(X-\mu_x)^k]$$

$$= \begin{cases} 1 \times 3 \times 5 \times \cdots (n-1)\sigma_x^n & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

~~$n \in (2i+1)$~~

⑤ Ex $\Rightarrow$ Moments of a Bernoulli random variable

$$P(X=x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$$

the expected value of X is

$$E[X] = \sum_{k=0} k\,P(X=k)$$

$$= 0(1-p) + 1\cdot p$$

with $\mu_x = E[x]$ the var of X is given by

$$\sigma_x^2 = \sum_{k=0}^{1} (k-\mu_x)^2 \cdot P[X=k]$$

$$= (0-p)^2 (1-p) + (1-p)^2 p$$

$$= p(1-p)$$

(6) Consider a sinusoidal signal with random phase ⇒

$$X(t) = A\cos(2\pi f_c t + \phi)$$

where A and f_c are constants
ϕ is a random variable
i.e uniformly distributed over the interval $(-\pi, \pi)$ i.e

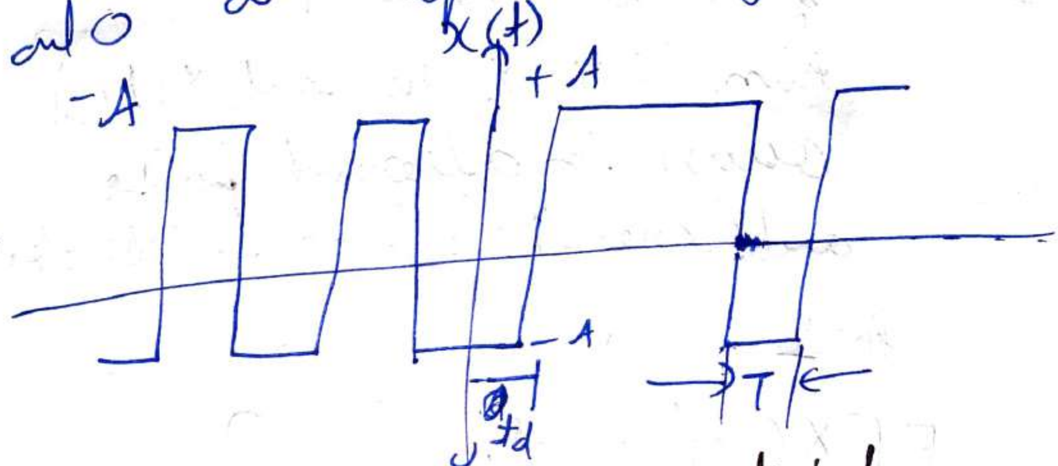
$$f_\phi(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \le \theta \le \pi \\ 0, & \text{elsewhere} \end{cases}$$

$$R_x(\tau) = E[x(t+\tau)x(t)]$$

$$= E[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \phi) \cos(2\pi f_c t + \phi)]$$

$= \frac{A^2}{2} \cos(2\pi f_c t)$

Random binary signal $\Rightarrow$
pulses are not synchronised
1 and 0 are represented by amp +A and -A



td is starting ~~pt value~~ point
and a sample point of a uniformly
distributed random variable Td
with pdf $f_{T_d}(t_d) = \begin{cases} \frac{1}{T}, & 0 \leq t_d \leq T \\ 0, & \text{elsewhere} \end{cases}$

during the time interval
$(n-1)T < t - t_d < nT$, $n \in \mathbb{Z}$
the presence of 1 and 0 is independent
in the interval i.e they are
equally likely to occurs.

Since the amplitude levels -A and
+A occurs with equal probability
~~that~~ $\Rightarrow$ $E[X(t)] = \frac{1}{2}(A) + \frac{1}{2}(-A)$
$= 0$ $\forall t$

autocorrelation of $R_X(t_k, t_i)$, we have first consider t_k & t_i

two time points of R.P. X

Case I If $|t_k - t_i| \geq T$

then RV $X(t_k)$ and $X(t_i)$ occur in different pulse and are therefore independent

thus

$$E[X(t_i)X(t_k)] = 0$$

Case II when $|t_k - t_i| < T$ with

$t_k = 0$ and $t_i > t_k$

hence they occur in same pulse interval if and only if

t_d satisfies $t_d < T - |t_k - t_i|$

we thus obtain the conditional expectation

$$E[X(t_k)X(t_i)|t_d] = \begin{cases} A^2, & t_d < T - |t_k - t_i| \\ 0, & \text{elsewhere} \end{cases}$$

averaging the result

$$E[X(t_k)X(t_i)|t_d] = \int_0^{T-|t_k-t_i|} A^2 f_{T_d}(t_d)\,dt_d$$

$$= \int_0^{T-|t_k-t_i|} \frac{A^2}{T}\,dt_d$$

$$= A^2\left[\frac{T-|t_k-t_i|}{T}\right]dt$$

$$; T-|t_k-t_i| > 0$$

hence from
Case I and II we get

$$R_X(\tau) = \begin{cases} A^2\left(1-\frac{|\tau|}{T}\right), & |\tau|<T \\ 0, & |\tau| \geq T \end{cases}$$

6 (7) for a pair quadrature modulated process $X_1(t)$ and $X_2(t)$

$$X_1(t) = X(t)\cos(2\pi f_c t + \phi)$$

$$X_2(t) = X(t)\sin(2\pi f_c t + \phi)$$

~~$R_{X_1X_2}(t) = E[X(t)\cos(2\pi f_c t + \phi)$~~

$$R_{X_1X_2}(t) = E[X_1(t)\,X_2(t-\tau)]$$

ergodic processes ⇒

~~example~~

ensamble average ⇒ ~~the~~

let $X(t)$ be a random process
lets take a fixed time and observe
the process at $t = t_K$ then
all the sample functions of the
random process $X(t)$ will be
observed at the $t = t_K$. then
mean of all those points is
ensamble ~~ensample~~ average.

$$\mu_X(t_K) = E[X(t_K)] = \cancel{\int_{-\infty}^{\infty} x f_X(x)} dx$$

$$= \int_{-\infty}^{\infty} x f_{X(t_K)}(x) dx$$

Time average ⇒ for the average
take $X(t)$ random process ⇒
grab one of ~~rand~~ sample functions
and take its average over a
period T.
($\mu_X(T)$ is a random variable).

$$\mu_x(T) = \frac{1}{T}\int_0^T x(t)\,dt$$

take E op both sides

$$E[\mu_x(T)] = \frac{1}{T}\int_0^T E[x(t)]\,dt$$

$$= \frac{1}{T}\int_0^T \mu_x\,dt$$

$$= \mu_x$$

* An unbiased estimator is a statistical term that means an estimator on averages ~~prod~~ parameter it is value of the estimate. In ~~this~~ this case $\mu_x(T)$ is being used to estimate μ_x

* If the process is <u>ERGODIC</u> in mean then the conditions are satisfied

① the time average approaches ensemble average as $T \to \infty$ i.e

$$\lim_{T\to\infty} \mu_x(T) = \mu_x$$

② The variance of $\mu_x(T)$ is zero as $N \to \infty$

$$\lim_{T\to\infty} Var[\mu_x(T)] = 0$$

time averaged autocorrelation

$$R_x(\tau, T) = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)\,dt$$

$\rightarrow$ $x(t)$ sample fun

$x(t)$ is ergodic in autocorrelation fn if the following time conditions are satisfied

$$\lim_{T\to\infty} R_x(\tau, T) = R_x(\tau)$$

$$\lim_{T\to\infty} \text{var}[R_x(\tau, T)] = 0$$

for a RP to be ergodic it has to stationary.

<u>Spectrum of random signal</u>

consider the random ~~function~~ process $X(t)$
let $x(t)$ be a sample function
let $x_T(t)$ be a sample function from $-T < t < T$ hence
fourier transform of $x_T(t)$ will be

$$\mathcal{E}_T(f) = \int_{-\infty}^{\infty} x_T(t)\,e^{-j2\pi ft}\,dt$$

$\mathcal{E}_T(f)$ is the ~~sample function~~ fourier transform of $x_T(t)$

~~effectively the FT of family of sample function will be $G_T(f)$ indexed by parameter~~

effectively the FT of family of RVs $X(t)$ indexed by time t (basically random process) is $G_T(f)$ indexed by frequency f

power spectral density of sample function $x(t)$ in interval $t \in (-t, t)$

is $\dfrac{|\mathcal{E}_T(f)|^2}{2T}$

we want to estimate PSD of random process $X(t)$. PSD tells us how the power of a signal is distributed across different frequencies. Since the signal is random one FT is not sufficient. hence PSD is needed.

Ensemble average method. ⇒ In this method we take the average of all PSD of sample functions of the R.P. $X(t)$

$$S_X(f) = \lim_{T\to\infty} \frac{E[|G_T(f)|^2]}{2T}$$

(2) Discrete fourier transform method →

In this method we break one long ~~long~~ signal in smaller chunks.

DFT is used to break down this signals in frequency components to get the information regarding its power.

We use this method if the process is ergodic

if $\{x_n : n = 0, 1 \ldots N-1\}$ are uniformly spaced samples of fn $x(t)$ at $t = nT_s$ then

$$\varepsilon_k = \sum_{n=0}^{N-1} x_n W^{kn}$$

where $W = e^{-j2\pi/N}$ and

ε_k are sample of frequency domain response at $f = k/NT_s$

The PSD will be estimated by.

① Partition the sample function $x(t)$ into M section each of length NT_s and sample interval T_s.

② Perform DFT on each section of each length NT_s. Let ε_{k+mN}, where $m = 0, 1, 2 \ldots M-1$ represent ~~no~~ M DFT outputs. ~~one set~~.

③ Average the magnitude squared of ~~Each~~ Each DFT. then PSD estimate is given by.

$$\hat{S}_x\left(\frac{k}{NT_s}\right) = \frac{1}{M} \sum_{m=0}^{M-1} \left| \varepsilon_{k+mN} \right|^2$$

$$\hat{S}_x\left(\frac{k}{NT_s}\right) = \frac{1}{M} \sum_{m=0}^{M-1} \left| \sum_{n=0}^{N-1} x_{n+mN} W^{kn} \right|^2$$

$$, \quad k = 0, \ldots \rightarrow M-1$$

NOISE

In communication System

- External Noise
  - Extraterrestial noise
  - Industrial Noise
  - ~~Intern~~ Atmosphere Noise
- Internal Noise
  - Shot Noise
  - Thermal Noise
  - White Noise

Extraterrestrial Noise ⟶ beyond the earth's atmosphere if the source is a star then it is cosmic noise
under normal conditions there is constant noise radiation from Sun because of its very high temp
It constantly emits radiations of various frequencies.

Industrial Noise ⇒ Automobiles, industries, aircraft ignitions, leakages from high voltage lines.

Atmospheric Noise ⇒ lightning discharges and thunderstorm includes radio frequencies

Internal Noise ⇒

① Shot Noise ⇒ It ~~arises~~ arises from electronic devices due to discrete nature of current flow in these devices. like in (diode, transistors).

⇒ Consider a use of ~~photodiode circuit~~ photodetector circuit in which ~~detects~~ current is generated everytime an electron hits it from a source of const intensity. total current flowing will be

$$X(t) = \sum_{k=-\infty}^{\infty} h(t-\tau_k)$$

$h(t-\tau_k)$ → current pulse generated at time $t = t_k$

$X(t)$: random process (Shot noise)

The no of electrons $N(t)$ emitted in time $(0, t)$ → contributes a discrete stochastic process.

the mean value of no of electrons ν emitted between $(t, t+t_0)$ is

$$E[\nu] = \lambda t_0 \quad \text{rate of process}$$

total no of e^- in $(t, t+t_0)$

$$N(t+t_0) - N(t) = \nu$$

follows a poisson distribution

hence Probability ~~that~~ that k no of e^- are ~~emitt~~ emitted between $(t, t+t_0)$

$$\text{so } P[\nu = k] = \frac{(\lambda t_0)^k}{k!} e^{-\lambda k}$$

Here the ~~sto~~ stastical char of $X(t)$ is

$\Rightarrow$ mean of $X(t)$

$$\mu_x = \int_{-\infty}^{\infty} \lambda\, h(t)\, dt$$

waveform of current pulse

$\lambda \rightarrow$ rate of process

$\Rightarrow$ autocovariance of $X(t)$

$$\underbrace{C_x(\tau) = \lambda \int_{-\infty}^{\infty} h(t)\, h(t+\tau)\, dt}_{\text{Campbell's thm}}$$

Campbell's thm.

if $h(t) \Rightarrow$ rectangular pulse of amp A and duration T

mean is $\lambda A T$ covariance is

$$C_x(\tau) = \begin{cases} \lambda A^2(T-|\tau|) & , |\tau| < T \\ 0 & , |\tau| \geq T \end{cases}$$

$\downarrow$

triangular form

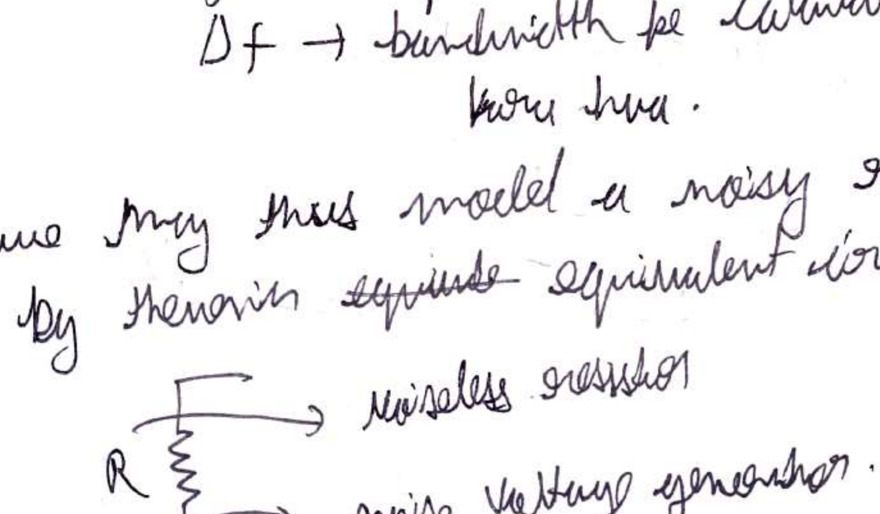
Thermal Noise $\Rightarrow$ electrical noise due the ~~thermal~~ heat generated from random motion of electrons in conductor.

$$E[V_{Th}]^2 = 4kTR\Delta f \text{ volts}^2$$

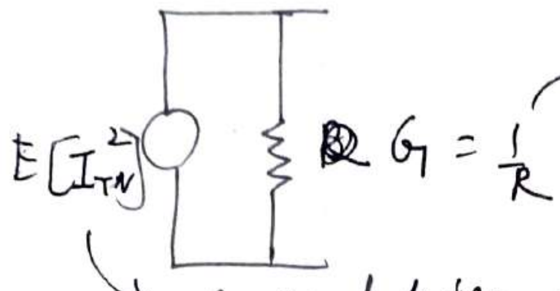
$\hookrightarrow$ mean sq. value of thermal noise

$\Delta f \rightarrow$ bandwidth to calculate from there.

we may thus model a noisy resistor by thevenin ~~equivale~~ equivalent circuit



alternatively we can also design a ~~Norton~~ Norton equivalent circuit.



$$E[I_{TN}^2] = \frac{1}{R^2} 4KTR\,\Delta f$$
$$= 4KTG\,\Delta f \; (A^2) \rightarrow \text{dimension}$$

White Noise $\Rightarrow$ PSD of white noise is independent of operating frequency

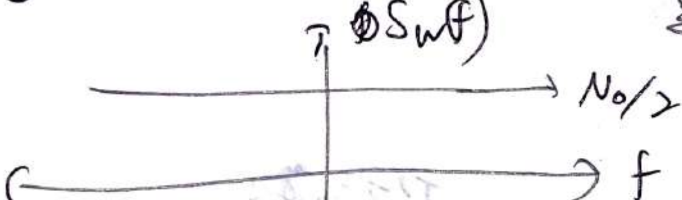
White Noise

$\Uparrow$ based on.

Noise in Comm. Systems

$$S_W(f) = \frac{N_0}{2}$$

~~$S_W(f) = N_0$~~



$N_0 = K\,T_{se} \rightarrow$ equivalent Noise temp of receiver

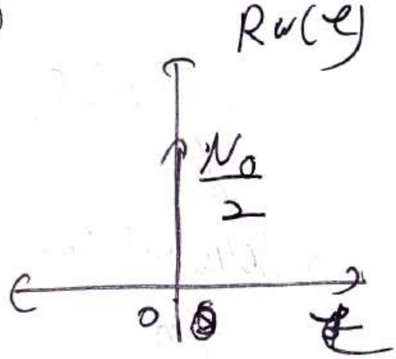
Te → Temp at which the Noiseless resistor has to be maintained so that by ~~connect~~ connecting it to the input of noiseless ~~resistor~~ system it produces the same noise power at the output of the system as that produced by all the sources of the system.

~~$R_W(\tau)$~~ =

Now autocorrelation of w(t)

$$R_W(\tau) = \cancel{\frac{N_0}{2}\delta(t)}$$

$$= \frac{N_0}{2}\delta(\tau)$$



if w(t) is gaussian then time samples are statistically independent.

Ex ⇒ Ideal low pass filtered white Noise

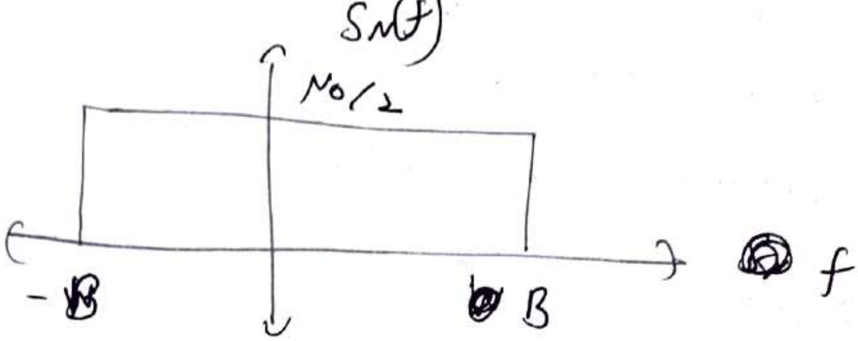
let w(t) $\mu_W = 0$, $S_W(f) = \frac{N_0}{2}$

↓ LPF with Bandwidth B

n(t) and passband amp of 1

$n(t) \Rightarrow$

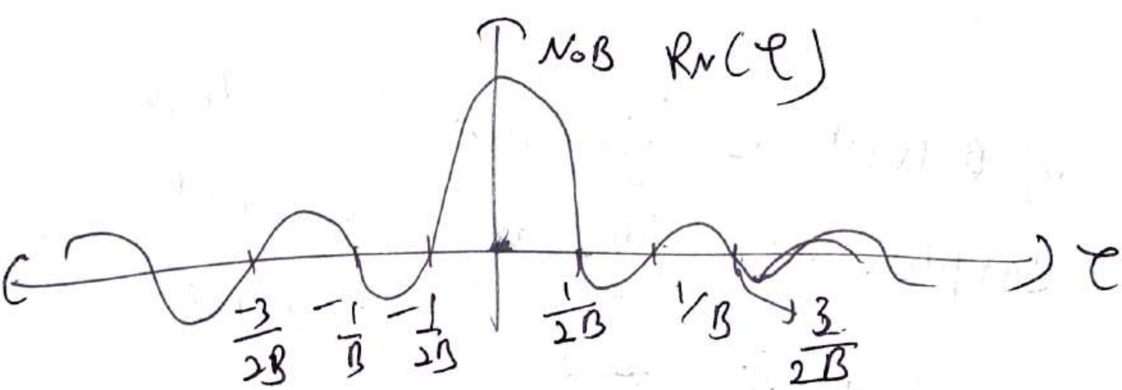
$$S_N(f) = \begin{cases} \frac{N_0}{2}, & -B < f < B \\ 0, & |f| > B \end{cases}$$



$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f\tau} df$$

autocorrelation fn

$$= N_0 B \, \text{Sinc}(2B\tau)$$



RC low pass filtered white noise

we know freq. response of the filter $H(f) = \dfrac{1}{1 + j2\pi f(RC)}$

$$R_N(\tau) = \int_{-\infty}^{\infty} \frac{N_0}{2\left(1+\left(\frac{2\pi f}{\alpha}\right)^2\right)} e^{j2\pi f \tau} \, df$$

$$\text{let } u = \frac{2\pi f}{\alpha}$$

$$\frac{\alpha}{2\pi} du = df$$

$$= \frac{N_0}{2} \cdot \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+u^2} e^{j\alpha u \tau} \, d\cancel{u}$$

$$= \frac{N_0 \alpha}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\cancel{1}+u^2} e^{j\alpha u \tau} \, du$$

$$\pi e^{-|x|} = \int_{-\infty}^{\infty} \frac{e^{jux}}{1+u^2}$$

$$= \frac{N_0}{4RC} \exp\left(-\frac{|\tau|}{RC}\right)$$

NEB $\Rightarrow$

Noise equivalent bandwidth of a system refers to how much noise power a filter passes compared to an ideal rectangular filter.

$$B_{NEB} = \frac{\int_0^{\infty} |H(f)|^2 \, df}{|H(f)|_{max}}$$

NEB for ideal low pass RC filter

$$H(f) = \frac{1}{1 + j2\pi f RC}$$

$$\text{so} \quad B_{NEB} = \frac{\displaystyle\int_0^{\infty} \frac{1}{1+(2\pi f RC)^2}\,df}{H(0)}$$

$$\left(H(f)_{max} = H(0) \right]$$

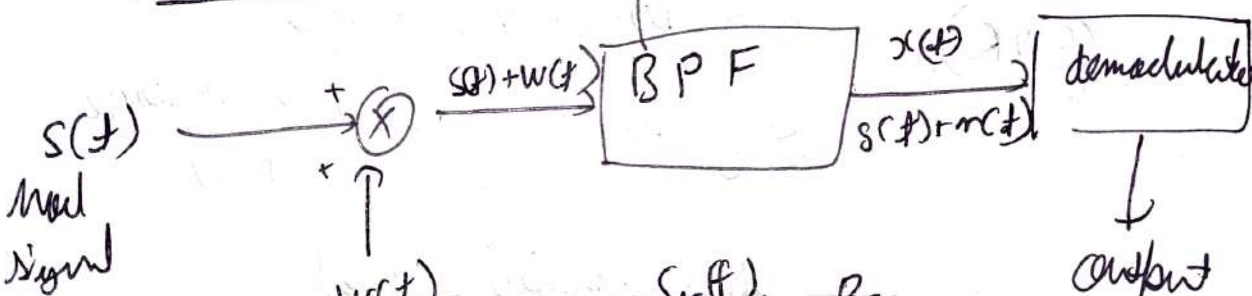
$$B_{NEB} = \int_0^{\infty} \frac{df}{1+(2\pi f RC)^2}$$

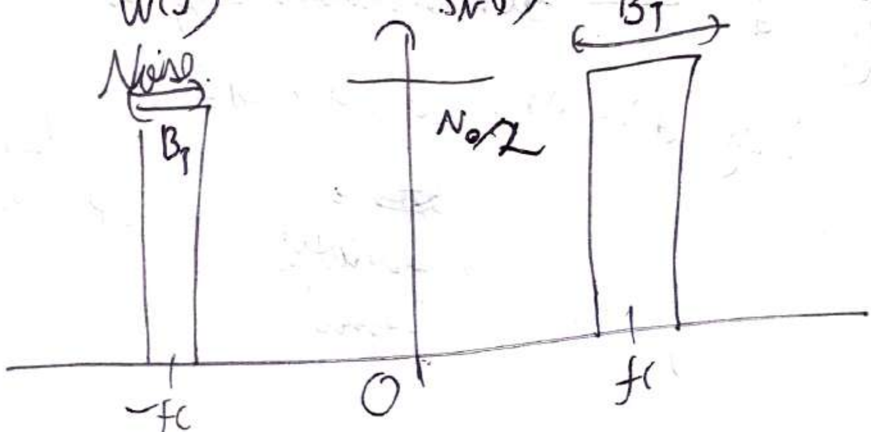
$$= \frac{1}{4RC}$$

midband ~~passband~~ frequency $= f_c$

transmission BW $= B_T$

Reciever model $\Rightarrow$





total area under the curve of PSD

$$S_N(f) = \frac{N_0}{\cancel{2}} \times \cancel{2} B_T = N_0 B_T$$

average noise power.

$n(t) \longrightarrow$ Narrow band noise — quadrature noise comp

$$\boxed{n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin 2\pi f_c t}$$

Inphase noise component

$x(t) \longrightarrow$ filtered signal

$$x(t) = s(t) + n(t)$$

(i) pre signal to noise ratio
↓
channel (signal to noise ratio)

$$\frac{\text{Avg power of mod signal}}{\text{Avg power of noise}}$$

$$(SNR)_c = \frac{\text{Avg power of Mod signal}}{\text{Avg power of noise in message BW}}$$

$y(t) \longrightarrow$ demodulated signal

$$\therefore y(t) = m_d(t) + n_d(t)$$

$m_d(t)$ → ~~...~~ demodulated signal

$n_d(t)$ → demodulated O/P noise

(ii) Post SNR (O/P SNR)

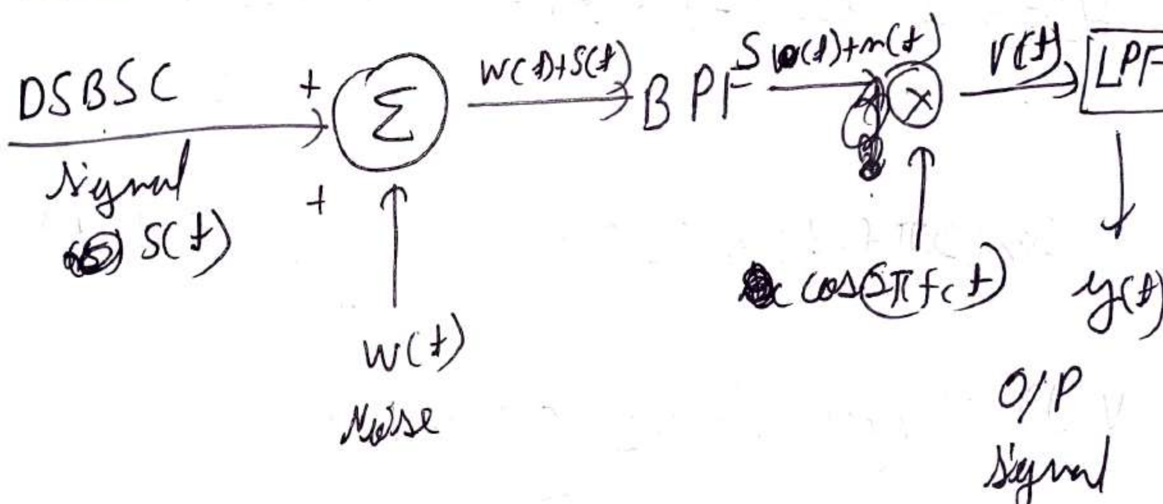
$$(SNR)_o = \frac{\text{Avg power of demodulated O/P}}{\text{Avg power of output noise}}$$

$$FOM = \frac{(SNR)_o}{(SNR)_c}$$

↓

figure of merit.

Noise in DSBSC Receiver :-



let $m(t) \rightarrow$ P be avg power in $m(t)$

$c(t) \longrightarrow$ carrier Signal

$$S(t) = m(t)\, c(t)$$

$$\therefore S(t) = m(t)\, A_c \cos(2\pi f_c t)$$

here we ~~know~~ know that $x(t) = S(t) + n(t)$

and $n(t) = n_I \cos 2\pi f_c t - n_Q \sin 2\pi f_c t$

$$x(t) = m(t)\, A_c \cos(2\pi f_c t) + n_I \cos(2\pi f_c t)$$
$$- n_Q \sin(2\pi f_c t)$$

$$v(t) = \left[m(t)\, A_c \cos 2\pi f_c t + n_I \cos(2\pi f_c t) - n_Q \sin(2\pi f_c t) \right] \cos(2\pi f_c t)$$

$$\left(v(t) = x(t) \cos 2\pi f_c t \right)$$
$\searrow$ output of product modulator $(\otimes)$

$$\cancel{v(t)} = \cos^2(2\pi f_c t) = \frac{1 + \cos(4\pi f_c t)}{2}$$

$$\sin(2\pi f_c t)\cos(2\pi f_c t) = \frac{1}{2} \sin(4\pi f_c t)$$

$$\cancel{v(t) = \left[\frac{m(t)}{2} A_c \cos(4\pi f_c t) + \frac{n_I}{2} \cos 4\pi \right.}$$

$$v(t) = \left[\frac{m(t)}{2} A_c (1 + \cos 4\pi f_c t) + \frac{n_I}{2} (1 + \cos(4\pi f_c t)) - \frac{n_Q}{2} \sin(4\pi f_c t) \right]$$

when it passed through LPF

we get

$$\cancel{v(t)} \quad y(t) = \frac{m(t)A_c}{2} + \frac{n_I}{2}$$

to find ~~SDN~~ $(SNR)_c$

Avg power of $s(t) \Rightarrow$

$$\cancel{E[m^2(t)A_c\cos^2(2\pi f_c t)]}$$

$$\cancel{= P.\frac{A_c^2}{2}}$$

$$E = [A_c^2 m^2(t) \cos^2(2\pi f_c t)]$$

$$= \frac{A^2 P}{2}$$

average power of noise in message BW $= \frac{N_0}{\cancel{2}} \times \cancel{2}W = N_0 W$

$$(SNR)_c = \frac{A^2 P}{2 N_0 W}$$

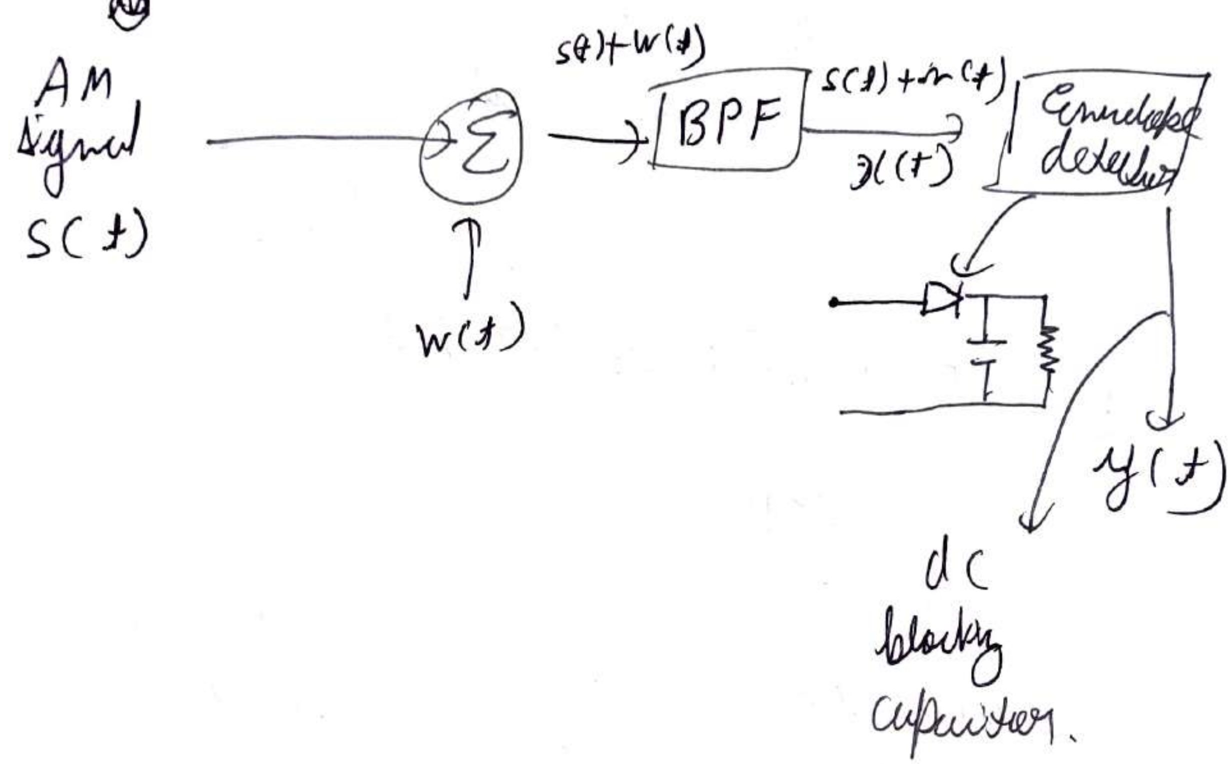
$$(SNR)_o = \frac{E\left[m^2(t)\frac{A_c^2}{4}\right]}{E\left[\frac{n_I^2}{4}\right]}$$ → half of I/P Noise power

$$= \frac{\frac{P A_c^2}{\cancel{4}}}{\frac{n_I^2}{\cancel{4}}} = \frac{P A_c^2}{\cancel{n_I^2}} = \frac{P A_c^2}{\frac{N_0 W}{2}}$$

$$FOM = \frac{(SNR)_o}{(SNR)_c} = \frac{A_c^2 P \, \cancel{N_0 W}}{\cancel{2N_0W \cdot 2P/A_c^2}}$$

$$= 1$$

Noise in AM receivers



$$s(t) = A_c[1 + k_a m(t)] \cos 2\pi f_c t$$

$$x(t) = s(t) + n(t)$$

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$

$$x(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t) + n_I \cos(2\pi f_c t) - n_Q \sin(2\pi f_c t)$$

$$= [(A_c + A_c k_a m(t) \cancel{\cos(2\pi f_c t)} + n_I) \cos(2\pi f_c t) - n_Q (\sin(2\pi f_c t))]$$

(Note if I write n_I, or n_Q instead of $n_I(t)$ or $n_Q(t)$ then please replace it)

O/P of ED is

$$y(t) = \sqrt{\{(A_c + A_c k_a m(t) + n_I)^2 + n_Q^2(t)\}}$$

$$A_c + A_c k_a m(t) + n_I >> n_Q(t)$$

$$y(t) \simeq A_c + A_c k_a m(t) + n_I(t)$$

dC comp is filtered using Capacitor

$$y(t) \simeq A_c k_a m(t) + n_I(t)$$

~~$(SNR)_c = E$~~

let the Avg Power of $m(t)$ be 'P'

the avg power of $s(t)$

$$= E[(A_c^2[1 + m(t) k_a]^2 \cos(2\pi f_c t)^2]$$

$$= \cancel{E((A_c \cos(2\pi f_c t) + m(t) k_a \cos 2\pi f_c t)^2]}$$

$$= E\left[A_c^2 \cos^2(2\pi f_c t)\left[1 + m(t)\right]^2\right]$$

$$= \frac{A_c^2}{2} \cdot E\left[(1 + m(t) k_a)^2\right]$$

$$\cancel{1 + k_a^2 m(t)} = \frac{(1 + k_a^2 P) A_c^2}{2}$$

Avg power of Noise $= N_0 W$

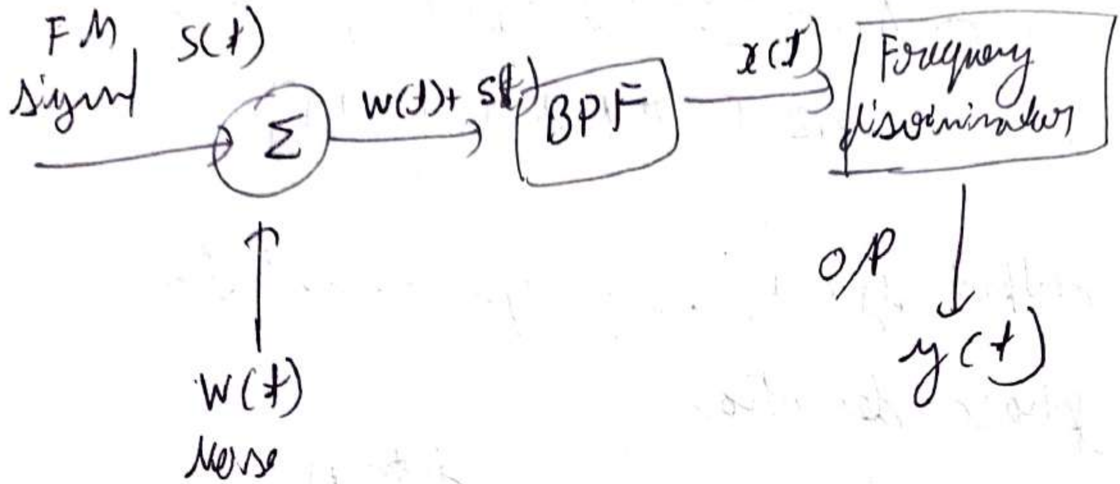
$$(SNR)_c = \frac{(1 + k_a^2 P) A_c^2}{2 N_0 W}$$

$$(SNR)_o = \frac{E\left[A_c^2 k_a^2 m^2(t)\right]}{E\left[n_I(t)\right]} \rightarrow 2 N_0 W$$

$$(SNR)_o = \frac{A_c^2 k_a^2 P}{2 N_0 W}$$

$$FOM = \frac{(SNR)_o}{(SNR)_c} = \frac{A_c^2 k_a^2 P \cancel{2 N_0 W}}{\cancel{2 N_0 W} (1 + k_a^2 P) A_c^2}$$

Noise in FM — Receiver Reciever



$x(t) = s(t) + n(t)$

~~where n(t)~~

$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right]$

message signal

$\phi(t) = 2\pi k_f \int_0^t m(\tau)\, d\tau$

$s(t) = A_c \cos\left[2\pi f_c t + \phi(t)\right]$

Now $n(t) \rightarrow$ in terms of mag and phase
$r(t)$ $\psi(t)$

$n(t) = r(t) \cos(2\pi f_c t + \psi(t))$

$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$

$$x(t) = \cancel{A_c(t)}\, s(t) + n(t)$$

$$= A_c \cos[2\pi f_c t + \phi(t)] + r(t)$$

$$\cos[2\pi f_c t + \psi(t)]$$

the output $y(t)$ is proportional to phase deviation.

$$\therefore y(t) = \cancel{\frac{1}{2\pi}} \frac{d\phi(t)}{dt}$$

$$= \frac{1}{2\pi} \frac{d\,\theta(t)}{dt}$$

$$\theta(t) = \phi(t) + \frac{n_Q(t)}{A_c}$$

$$y(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} + \frac{d\, n_Q(t)}{A_c\, dt}$$

$$= \left[\frac{\cancel{2\pi}\, k_f m(t)}{\cancel{2\pi}} + \frac{1}{2\pi A_c} \frac{d}{dt} n_Q(t)\right]$$

$$= \underbrace{k_f m(t)}_{\text{demodulated signal}} + \underbrace{\frac{1}{2\pi A_c} \frac{d}{dt} n_Q(t)}_{\text{demodulated noise}}$$

Avg Power of dem. signal $= E[K_f^2 m^2(t)]$

$= K_f^2 P$

Avg power of dem noise $= \frac{2}{3} \frac{N_0 W^2}{A_c^2}$

$$(SNR)_o = \frac{K_f^2 P}{\frac{2}{3} \frac{N_0 W^3}{A_c^2}} = \frac{3}{2} \frac{K_f^2 P}{N_0 W^2}$$

~~FOM~~ $\cancel{= \frac{(SNR)_o}{(SNR)_c} =}$

$$(SNR)_c = \frac{A_c^2/2}{N_0 W}$$

$\rightarrow$ s(t) avg power

$\rightarrow$ power of noise

$$FOM = \frac{\frac{3}{2} \frac{K_f^2 P}{N_0 W^2}}{\frac{A_c^2}{2 N_0 W}} = \frac{3 K_f^2 P}{W A_c^2}$$

Capture effect $\Rightarrow$ The signal can be affected by another signal whose freq is close to the carrier freq of desired signal.

→ Then the reciever may lock on the interference signal and suppress the desired signal.

→ ~~Then~~ If the strength of interference signal and desired signal are equal then reciever might block interference signal for sometime and ~~the~~ desired signal for sometime.

FM threshholds effect ⇒

⇒ $(SNR)_0$ is valid only if $CNR >> 1$

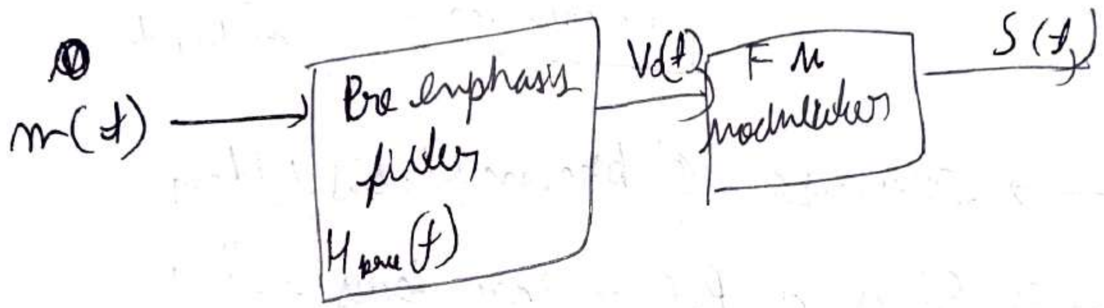
⇒ if $CNR < 1$ then FM signal is corrupted by noise and FM reciever will break down ⇒ FM threshhold effect

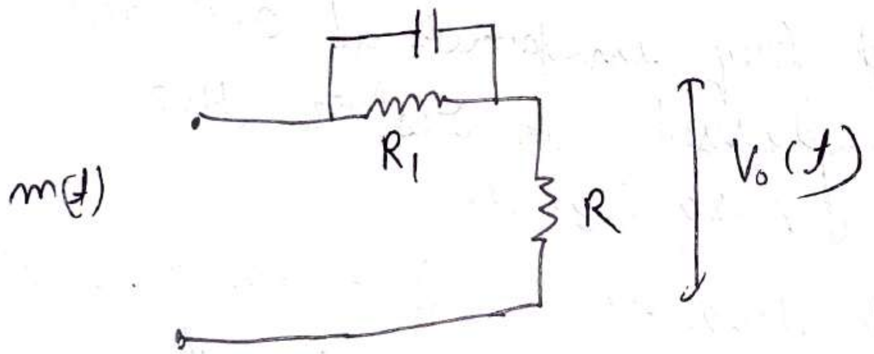
⇒ can be avoided if $CNR >> 1$

Pre emphasis and De-emphasis in FM.

→ commonly used in FM transmitter and FM receiver to improve threshold (avoid threshold effect)

Pre emphasis :



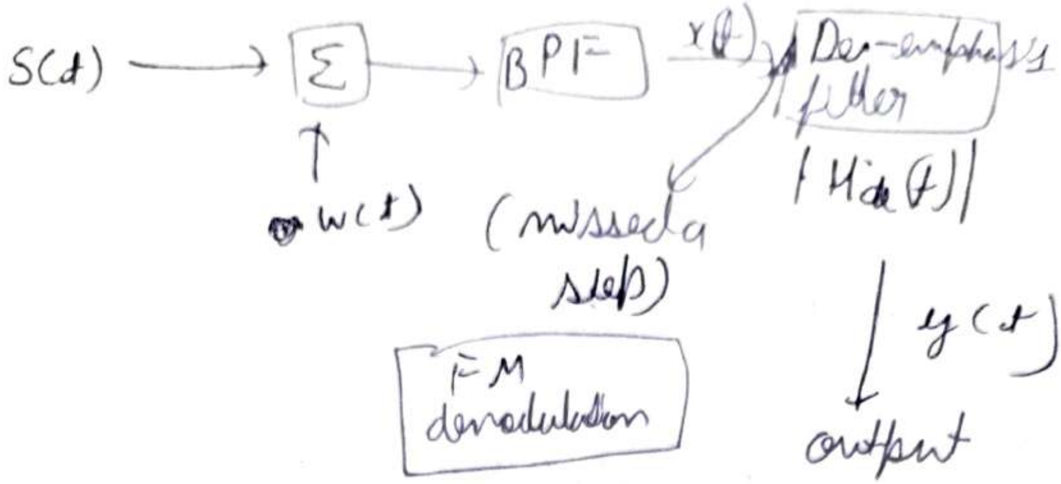


pre emphasis circuit

→ It's a high pass filter with transfer fn H(f).

→ to improve $(SNR)_o$, at FM modulator the high freq. comp of m(t) are emphasized.

De emphasis:



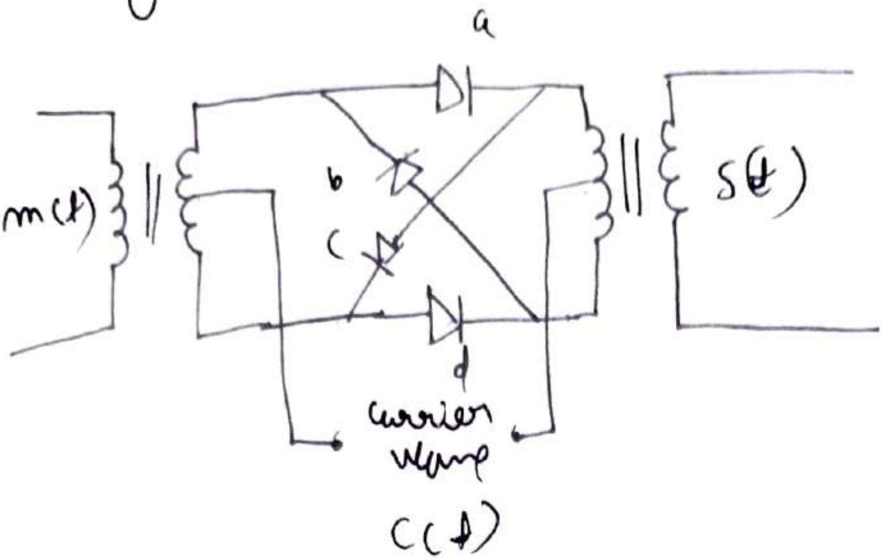
→ reverse of pre emphasis filter

→ It is a RC – low pass filter

→ High freq components at O/P are de-emphasised to restore the original msg signal.

→ High freq components of noise is also reduced, which increases the $(SNR)_0$

Ring Modulator for DSBSC

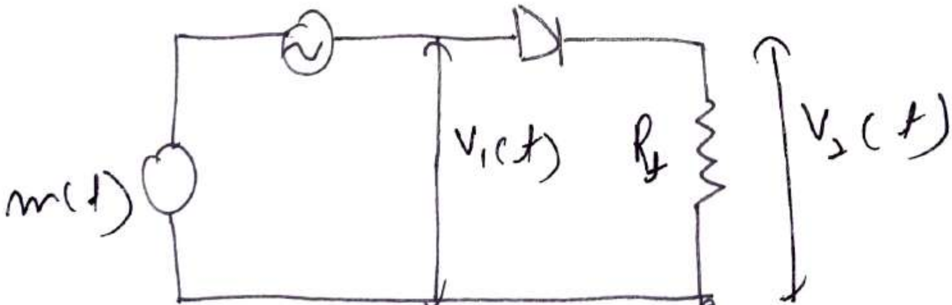


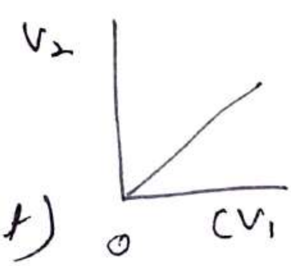
a and d will r_f in +ve half cycle
b and c will be r_b

a and d will be r_b in −ve half cycle b and c will be r_f

Switching modulator of AM generation

$c(t) = A_c \cos 2\pi f_c t$





$$v_1(t) = A_c \cos(2\pi f_c t) + m(t)$$

$$v_2(t) = \frac{A_2}{2}\left[1 + \frac{4}{\pi A_c} m(t)\right] \cos 2\pi f_c t$$