computational method ⇒

significant digits ⇒ are digits starting with leftmost non zero ending with rightmost correct digit. ~~including~~ including final zeros that are exact

errors ⇒ abs error = |exact value - approximate value|

relative error = $\dfrac{|\text{exact value} - \text{approx. value}|}{|\text{exact value}|}$

(relative error) (exact value) = $\dfrac{\text{exact value}}{\dfrac{-\text{approxi}}{\dfrac{-\text{mate}}{\text{value}}}}$

Accuracy an precision ⇒

Decimal places ⇒ 12.345

significant digits ⇒ 0.076 → 2 sig digits

precision ⇒ how precise the instrument is like 0.1 m or 0.0001 m etc.

eg 1.7 + 5.65 = 4.65 → two sig digits

it could be either 4.6 or 4.7

multiplicant and division ⇒

$(1.23)(4.5) = 5.535$

→ two significant digits

Rounding and chopping ⇒

0.217 to 0.22 (two decimal)

0.475 to 0.48 (using the round to even rule)

chopping $0.217 \to 0.21$

$0.475 \to 0.47$

round to even rule ⇒ rounds no ending in 5 to nearest even no.

rounding to nearest even

7+1

$0.475 \Rightarrow 0.48$

$0.365 \Rightarrow 0.36$

even ⇒ remains same

Taylor Series

~~$e^x = 1+$~~

applications $\Rightarrow$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$|x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} ; |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k , (-1 < x \leq 1)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

Thm 1 $\Rightarrow$ $f(x) \sim f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2$

$+ \frac{f'''(c)}{3!}(x-c)^3 + \cdots$

$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$

Taylor series at point c

if $c = 0$ its called the lauren seris

$f(x) \sim f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$+ \cdots$

Mean value theorem $\Rightarrow$ if f is continous in $[a, b]$ and diff in $(a\ b)$ then

$f(b) = f(a) + (b-a)f'(\xi_e)$

for som ξ_e in (a, b)

hence $f'(\xi_e) = \frac{f(b)-f(a)}{b-a}$

Rolle's theorem $\Rightarrow$ f is cont in $[a, b]$ and diff on (a, b) if $f(a) = f(b) = 0$

then $f'(c) = 0$ for sum c in $[a, b]$

Loss of Significance ⇒ eliminating the loss of ~~significan~~ significant values

Significant digits

$$x = 0.3721489 \times 10^{-5}$$

least significant digit.

most significant digit

Cannot report more precision than was originally present in your measured quantity.

occurs when substracting two nearly equal numbers → lead to reduction in no of signific. digits.

if for a fn ~~f(x) and a~~ given $f(x, y, z)$ and errors in x, y, z are then ⇒

$$\delta f_{max} = \left|\frac{\partial f}{\partial x}\delta(x)\right| + \left|\frac{\partial f}{\partial y}\delta(y)\right| + \left|\frac{\partial f}{\partial z}(\delta(z))\right|$$

relative max error

Bisection method ⇒

f → real or complex

r for which $f(r) = 0$ → root

$f(x) = 6x^2 - 7x + 2$

$f(x) = (2x-1)(3x-2)$ → simple

But complex and complicated fn ?

like $f(x) = 4.7x^2 + 2.74x + 1.2$

let f be a fn that has values of opposite ~~side~~ sign at the two ends of the interval

⇒ f is cont. on one interval

let $a < b$ and $f(a) f(b) < 0$

the f has a root in a and b

$u = f(a)$, $v = f(b)$, $uv < 0$

let $c = \frac{1}{2}(a+b)$

we ~~compt~~ compute $f(c) = w$

it can happen that $f(c) = 0$, if not then wu or $wv < 0$

if wv is less than 0

Calculate in that interval i.e (a, c) or (c, b) repeat the process until you get ~~___~~ 0 in n decimal places (defined in question).

Newton Rapson Method $\Rightarrow$ let x_0 be an approximate root of the equation $f(x) = 0$, if $x_0 = x_1 + h$ be the exact root then $f(x_1) = 0$

Expanding $f(x_0 + h)$ by taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is small so neglecting h^2 and higher powers of h

$$f(x_0) + hf'(x_0) = 0$$

$$h = \frac{f(x_0)}{-f'(x_0)}$$

then $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with x_1, a still better approximation x_2 is given

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, 2, 3 \ldots$$

Convergence of newton raphson method

$\Rightarrow$ suppose x_n from the root by a small quantity ε_n then

$$x_0 = \alpha + \varepsilon_0$$

and $x_{n+1} = \alpha + \varepsilon_{n+1}$

then the general equation

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\cancel{\varepsilon_{n+1} = \varepsilon_n - f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}}$$

$$\varepsilon_{n+1} = \varepsilon_n - \frac{\left(f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + \ldots\right)}{\left(f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\right)}$$

$$= \frac{\varepsilon_n - \varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \cdots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \cdots}$$

as $f(\alpha) = 0$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]}$$ [neglecting higher powers of x_n]

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2f'(\alpha)}$$

convergence of second order

convergence for bisection method $\Rightarrow$

$$|r - c_0| \leq \frac{b_0 - a_0}{2}$$ error or distance from the actual root

error bound after n steps

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

the no of iterations needed to achieve a specified ~~toller~~ tolerance ε (i.e, $|r - c_n| < \varepsilon$) rearranging the inequality

$$b - a < 2\varepsilon \cdot 2^n$$

$$n > \frac{\log(b-a) - \log(2\varepsilon)}{\log(2)}$$

Types of errors

⇒) ~~Gnhe~~ Inherent → already present in the statement of a ~~problem~~ problem

Rounding errors → error from rounding off the numbers during the computations.

~~Truncation~~

Truncation error → infinite to finite

like

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad \text{---} \; \infty x$$

truncated to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + x'$$

truncation error =

$$X - X'$$

Absolute error

$$e_a = |X - X'| \quad \begin{matrix} \text{real} \\ = \text{error} \\ \text{approx} \end{matrix}$$

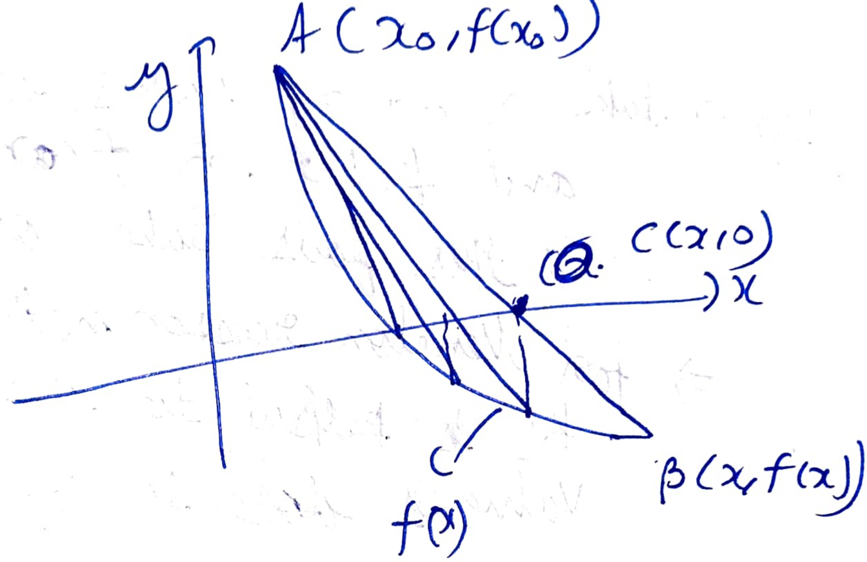
Relative error

$$= \left| \frac{X - X'}{X} \right|$$

Percentage error

$$e_p = 100\left|\frac{X - X'}{X}\right|$$

Secant Method $\Rightarrow$ does not need the condition $f(x_0)f(x_1) < 0$



slope of AB = slope of AC

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0 - f(x_0)}{x - x_0}$$

$$x - x_0 = \frac{-f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\cancel{x = x_0 + \frac{f(x_0)(x_0 - x_1)}{f(x_1) - f(x_0)}}$$

$$x = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

or $x = x_2$

so we get

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$$

if $f(x_{n+1}) = f(x_n)$ then this method fail

I/p → take x_0 and x_1 values ~~and~~ and put it in $f(x_0) f(x_1) < 0$
if this fails take other values

⇒ for Newton raphson method
this is helpful to choose which value is ~~close to 0~~. close to zero.

Interpolation

The technique or method of estimating unknown values from ~~guess~~ Given ~~[illegible]~~ set of observation.

x	f(x)
1971	1000
→ 1974?	
1981	1025
1991	1680
⇒ 1995?	
2001	1121
2011	1200

- equal interval
  - Newton forward
  - Newton's backward
- unequal interval
  - Newton's divided
  - Lagrange's interpolation.

Q

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

forward

backward

1925

Newton's forward $\Rightarrow a + hu = 1895$

$a = 1891, \quad h = 10$

so $u = 0.4$

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}$$

$$\Delta^4 f(a) + \text{———}$$

$$f(1895) = 46 + 0.4 \times 20 + 0.4(0.4-1)(-5)$$

$$+ \frac{0.4(0.4-1)(0.4-2)}{3!}(2)$$

$$+ \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 54.8528$$

Newton Backward

for 1925 $= a + hu$

$a = 1931,\ h = 10$

~~f(1925) =~~ $u = -0.6$

$$\cancel{f(1925) = f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a)}$$

$$= 101 - 0.6 \times 8 - \frac{0.6(1-0.6)(-4)}{2} - \frac{0.6(1-0.6)(2-0.6)(-1)}{6} - \frac{0.6(1-0.6)(2-0.6)(3-0.6)}{4!}$$

$$f(a+hu) = f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \cdots$$

$= 96.8368$

Lagrange's interpolation for unequal interval

Q

x	5	6	9	11
$f(x)$ y	12	13	14	16

find y when $x = 10$

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12)$$

$$+ \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14)$$

$$+ \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}16$$

$$f(10) = \frac{4 \times 1 \times (-1)}{-1 \times -4 (-6)} 12$$

$$+ \frac{5(1)(-1)}{1(-3)(-5)}(13) + \frac{5(4)(-1)}{4(3)(-2)} 14$$

$$+ \frac{5(4)(1)}{6(5)(2)}(16)$$

$$= 4 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= 4 - \frac{38}{3} = \frac{50}{3} = 14.66$$

Newton's Divided difference ⇒

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
5	12	$\frac{13-12}{6-5} = 1$	$\frac{\frac{1}{3} - 1}{9-5} = -\frac{1}{6}$	$\frac{\frac{2}{15} + \frac{1}{6}}{11-5} = \frac{1}{20}$
6	13	$\frac{14-13}{9-6} = \frac{1}{3}$	$\frac{1 - \frac{1}{3}}{11-6} = \frac{2}{15}$	
9	14	$\frac{16-14}{11-9} = 1$		
11	16			

$$f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x)$$
$$+ \text{---} \quad \text{---}$$

$$f(10) = 12 + (10-5)(1) + (10-5)(10-6)\left(\frac{-1}{6}\right)$$
$$+ (10-5)(10-6)(10-9)\left(\frac{1}{20}\right)$$

$$= 14.66$$

Numerical Integration :-

The area bounded by the curve $f(x)$ and x-axis between a and b is defined by ⇒

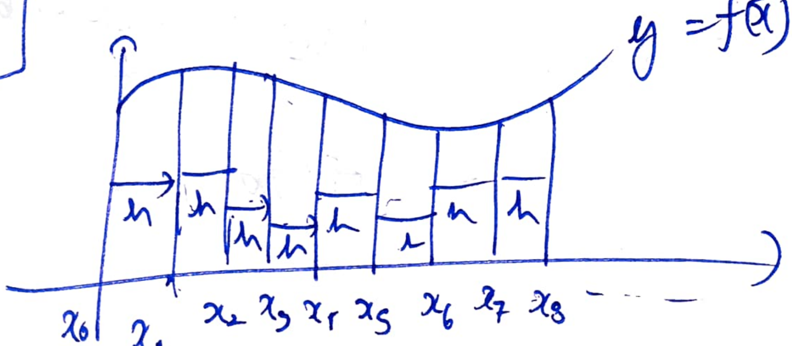
$$I = \int_a^b f(x)\,dx \quad - (1)$$

divide the interval (a, b) into n equal intervals with length h (step size)

i.e $(a, b) = (a = x_0, x_1, x_2 ---- x_{n-1}; x_n = b)$

$$\cancel{a = x_0}$$
$$\cancel{b = x_0 + h}$$
$$\cancel{c = x_1 + h}$$

$$a = x_0$$
$$x_1 = x_0 + h$$
$$x_2 = x_1 + h$$
$$\vdots$$
$$x_n = x_{n-1} + h$$



Eq. 1 can be ~~evaluated~~ ~~to~~ evaluated by

$\Rightarrow$

① Trapezoid rule

$$\int_a^b f(x)\,dx = h\left(\frac{y_0+y_n}{2} + y_1 + \cdots y_{n-1}\right)$$

~~applicable~~ any no of interval

applicable

② Simpson $\frac{1}{3}$ rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots) + 2(y_2+y_4+\cdots)\right]$$

only applicable if total no of interval is even.

③ Simpson $\frac{3}{8}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\cdots) + 2(y_3+y_6+\cdots)\right]$$

only applicable if total no interval is multiple of 3.

Numerical operators $\Rightarrow$

① forward difference operator.

$$\Delta f(a) = f(a+h) - f(a)$$

or $\Delta f(a+h) = f(a+2h) - f(a+h)$

in general

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = f(x+2h) - f(x+h) - (f(x+h) - f(x))$$

if $f(x) = a_0 x^n + a_1 x^{n-1} + a_n;$

$$\Delta^n f(x) = a_0 n! h^n$$

eg $f(x) = 3x^6 + 2x^5 + 7x^3 + 2x^2 + 5$

$$\Delta^6 f(x) = 3 \cdot 6! \, h^6$$

② $\nabla f(x) = f(x) - f(x-h)$ (backward difference operator)

$$\nabla^2 f(x) = \nabla f(x) - \nabla f(x-h)$$

$$= f(x) - 2f(x-h) + f(x-2h)$$

Identity operator $\Rightarrow I$ defined by $If(x) = f(x)$

The shifting operator

$$Ef(x) = f(x+h)$$

$$E^{-1} f(x) = f(x-h)$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$= Ef(x) - If(x)$$

$$\Delta f(x) = (E - I) f(x)$$

$$\Delta = E - I$$

$$\nabla f(x) = f(x) - f(x-h)$$

$$= If(x) - E^{-1} f(x)$$

$$\nabla = (I - E^{-1})$$

Q. Evaluate $E(x^2)$

$$(x+h)^2 = x^2 + h^2 + 2hx$$

taking $h = 1$, $\Rightarrow x^2 + 1 + 2x$

$\Rightarrow$ central difference operator (δ)

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

averaging operator

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{1}{2}h\right) + f\left(x-\frac{1}{2}h\right)\right]$$

Hence

$$\mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\Rightarrow \mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$$

$$\mu^2 f(x) = \frac{1}{2}\mu\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)^2\right] f(x)$$

Solution of linear algebric method

① Gauss elimination method

② Gauss jordan method

$$Ax = B \Rightarrow (A|\vec{b}) = X$$

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

reduced row + / or upper triangular matrix

Gauss elimination method → echleon form

$$\begin{array}{ccc} k_{11} & k_{12} & k_{13} \\ \end{array}$$

$$\left(\begin{array}{ccc|c} \cancel{a_{11}} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}\,k_{22} & a_{23}\,k_{23} & b_2 \\ 0 & 0 & a_{33}\,k_{33} & b_3 \end{array}\right) = (A:B) \text{ or } (A|\vec{b})$$

→ only row operation

Gauss jordan → we get a ~~dia~~ diagonal matrix

Some things to remember

make $a_{11} = 1$ by anything and then make a_{21}, a_{31} by the help of a_{11} ~~before doing~~ that after doing that

LU decomposition $\Rightarrow$

$Ax = B$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{33} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} g & h & i \\ 0 & j & k \\ 0 & 0 & l \end{pmatrix}$$

$$(LU)X = B \Rightarrow L\overbrace{(UX)}^{Y} = B$$

$LY = B$ and $UX = Y$

$\downarrow$

we get X

① ~~Dolittle's method~~ Doolittle's method $\Rightarrow$

'$A = LU$' $\Rightarrow l_{ii} = 1$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} d & e & f \\ 0 & g & h \\ 0 & 0 & i \end{bmatrix}$$

② ~~Gro Gro~~ Crout's factorization $\Rightarrow$

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \quad U = \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

③ Cholesky's method

$A = LL^T \rightarrow U = L^T \rightarrow$ transpose

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \quad L^T = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

$$LL^T X = B$$

Splines

~~Linear~~

linear, quadratic, cubic

linear ↓

$$s_i(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

$$\underbrace{1}_{\text{1st term}} \leq i \leq \underbrace{n}_{\text{last term}}$$

and it should be aceding order.

$$\cancel{S_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

$$\cancel{= 0 + \left(1 - 0\right.}$$

example =)

$i = 1 \xrightarrow{S_1} 2 \xrightarrow{S_2} 3$

x	0	1	2
y	0	1	8
	y_1	y_2	y_3

$$S_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\cancel{= 0 + \left(\frac{1-8}{1-0}\right)(x-0) - 7(x-0) = -7x}$$

$$= 0 + \left(\frac{1-0}{1-0}\right)(x-0) = x$$

$$x \in [0, 1]$$

$$S_2(x) = y_2 + \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

$$= 1 + \frac{8-1}{2-1}(x-1)$$

$$= 1 + 7(x-1)$$

Quadratic spline ⇒

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(x_{i+1} - x_i)}(x - x_i)^2 + z_i(x - x_i) + y_i$$

$$z_{i+1} = -z_i + 2\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)$$

$$0 \leq i \leq n-1$$

we take $z_0 = 0$ mainly

example we need to find the value of $y(1.5)$ if we have ~~for~~ $(-2, 0), (0, 2), (2, 7)$

$$Q_1(x) = \frac{z_2 - z_1}{2(x_2 - x_1)}(x - x_1)^2 + z_1(x - x_1) + y_1$$

~~$z_2(x) = -z_1$~~

$z_1 = 0$ (assume)

$$z_2 = -z_1 + 2\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$z_2 = 2\left(\frac{2-0}{0+2}\right) = 2$$

$$Q_1(x) = \frac{2-\cancel{0}}{2(0+2)}(x+2)^2 + 0(x+2) + 0$$

$$= \frac{\cancel{2}}{\cancel{4}\,2}(x+2)^2 = \frac{(x+2)^2}{2}$$

~~$Q_2(x$~~ ~~$z_2 - z_1 +$~~

$$z_3 = -z_2 + 2\left(\frac{y_3-y_2}{x_3-x_2}\right)$$

$$= -2 + 2\left(\frac{7-2}{2-0}\right)$$

$$= -2 + \cancel{2}\left(\frac{5}{\cancel{2}}\right) = 3$$

$$Q_2(x) = \frac{z_3-z_2}{2(x_3-x_2)}(x-x_2)^2 + z_2(x-x_2) + y_2$$

$$= \frac{3-2}{2(2-0)}(x-0)^2 + 2(x-0) + 2$$

$$= \frac{x^2}{4} + 2x + 2 \quad \text{for } 0 \le x \le 2$$

$$y(1.5) = \frac{(1.5)^2}{4} + 2(1.5) + 2 = 5.5625$$

Cubic Spline ⇒

$$f(x) = \frac{(x_{i+1} - x)^3}{6h_i} M_i + \frac{(x - x_i)^3}{6h_i} M_{i+1}$$

$$+ \frac{x_{i+1} - x}{h_i}\left(y_i - \frac{h^2}{6} M_i\right)$$

$$+ \frac{x - x_i}{h_i}\left(y_{i+1} - \frac{h^2}{6} M_{i+1}\right)$$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h_i^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)$$

$i = 1$ to $n-1$

Since graph is linear for $x < x_0$ and $x > x_n$

$M_0 = 0$, $M_n = 0$

for example

x	1	2	3	4	5
y	1	0	1	0	1
	y_0	y_1	y_2	y_3	y_4

$h = 1, \quad n = 5, \quad M_0 = 0, \quad M_5 = 0$

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)$$

$i = 1$

$$M_0 + 4M_1 + M_2 = \frac{6}{}\left(y_0 - 2y_1 + y_2\right)$$

$M_0 = 0,$

$\cancel{M_0 + 4M_1 + M_2}$

$$4M_1 + M_2 = 6\left(y_0 - 2y_1 + y_2\right)$$

$$= 6\left(1 - 2(0) + 1\right)$$

$$4M_1 + M_2 = 12 \quad - A$$

$i = 2$

$$M_1 + 4M_2 + M_3 = 6\left(0 - 2(1) + 0\right)$$

$$M_1 + 4M_2 + M_3 = -12 \quad - B$$

$i = 3$

$$M_2 + 4M_3 + \underbrace{M_5}_{\to 0} = 6\left(1 - 2(0) + 1\right)$$

$$M_2 + 4M_3 = 12 \quad - C$$

$\cancel{(1 - 4M_3 + 4M_3 + M_5 = 6}$

$\cancel{4M_1 + M_2 = 12}$

and $M_3 = \dfrac{12 - M_2}{4}$ in B

$\cancel{\dfrac{M_1}{4} + \dfrac{4M_2 \times 4}{4} + \dfrac{12 - M_2}{4} = -12}$

$\cancel{M_1 + 16M_2 + 12 - M_2 = -48}$

$\cancel{M_1 - 15M_2 = -60}$

$\cancel{\text{and } (4M_1 + M_2 = 12) \times 15}$

$\cancel{60M_1 + 15M_2 = 180}$

$\cancel{\text{we get } 61M_1 = 120}$

15
12
180

$+$

$M_1 = \dfrac{12 - M_2}{4}$ in B

$\dfrac{12 - M_2}{4} + \dfrac{4 \times 4 M_2}{4} + \dfrac{12 - M_2}{4} = -12$

$12 - M_2 + 16M_2 + 12 - M_2 = -48$

$\cancel{14M_2 = -60}$

$\cancel{M_2 = \dfrac{30}{7}}$

$\cancel{M_1 = \dfrac{12 - \frac{30}{7}}{4}}$

$M_1 = \dfrac{54}{28} \dfrac{6}{4} = \dfrac{3}{2}$

$M_2 = \frac{-36}{7}$

x	y
1	1
2	0
3	1
4	0
5	1

$M_1 = \frac{30}{7}, \quad M_3 = \frac{30}{7}$

T cubic equation

~~for~~ $x \in [1,2]$

$$f(x) = \frac{(x_{i+1} - x)^3}{6h} M_i + \frac{(x - x_i)^3}{6h} M_{i+1}$$

$$+ \frac{x_{i+1} - x}{h} \left(y_i - \frac{h^2}{6} M_i \right)$$

$$+ \frac{x - x_i}{h} \left(y_{i+1} - \frac{h^2}{6} M_{i+1} \right)$$

for $i = 0 \quad x \in [1, 2]$

$$f_1(x) = \frac{(x_1 - x)^3}{6h_i} M_0 + \frac{(x - x_0)^3}{6h_i} M_1$$

$$+ \frac{x_1 - x}{h_i} \left(y_0 - \frac{h^2}{6} M_0 \right)$$

$$+ \frac{x - x_0}{h_i} \left(y_1 - \frac{h^2}{6} M_1 \right)$$

we know ~~$x_1 - x_0 = h$~~ $x_{i+1} - x_i = h_i$

$h_0 = 1 = h_1 = h_2 = h_3$

$$f_0(x) = \frac{(2-x)^3}{6}(0) + \frac{(x-1)^3}{6}\left(\frac{30}{7}\right)$$

$$+ \frac{2-x}{(1)}\left(1 - \frac{1}{6}(0)\right)$$

$$+ \frac{(x-1)}{1}\left(y_1 - \frac{h^2}{6}\left(\frac{30}{7}\right)\right)$$

$$f_0(x) = \frac{\cancel{30}^{5}}{7\cancel{42}}(x-1)^3 + 2 - x$$

$$+ (x-1)\left(-\frac{\cancel{30}^{5}}{7\cancel{42}}\right)$$

$$= \frac{5}{7}(x-1)^3 + 2 - x - \frac{5}{7}(x-1)$$

$i = 1 \Rightarrow x[2, 3]$

$$\cancel{f_1(x) = \frac{(x_2-x)^3}{6h} M_1 + \frac{(x-x_1)^3}{6h} M_2}$$

$$\cancel{+ \frac{x_2 - x}{h}\left(y_1 - \frac{h^2}{6} M_1\right)}$$

$$+ \frac{x - x_1}{h}\left(y_2 - \frac{h^2}{6} M_2\right)$$

$i=1,\ M_1=\frac{30}{7}$ $i=0,1$; $i=1,2$; $i=2,3$; $i=3,4$ 0 0 1 0 $, M_2=-\frac{36}{7}$

$$f_1(x)=\frac{1}{6(1)}(x_2-x)^3 M_1+\frac{1}{6(1)}(x-x_1)^3 M_2+\frac{(x_2-x)}{h}\left(y_1-\frac{h^2}{6}M_1\right)+\frac{(x-x_1)}{(1)}\left(y_2-\frac{h^2}{6}M_2\right)$$

$$=\frac{1}{\cancel{6}}(3-x)^3\left(\frac{\cancel{30}^{5}}{7}\right)+\frac{1}{\cancel{6}}(x-1)^3\left(\frac{-\cancel{36}^{6}}{7}\right)+\frac{(3-x)}{h}\left(0-\frac{1}{\cancel{6}}\left(\frac{\cancel{30}^{5}}{7}\right)\right)+(x-1)\left(1-\frac{1}{\cancel{6}}\left(-\frac{\cancel{36}^{6}}{7}\right)\right)$$

$$=(3-x)^3\left(\frac{5}{7}\right)+(x-2)^3\left(-\frac{6}{7}\right)+\left(\frac{3-x}{1}\right)\left(-\frac{5}{7}\right)+(x-2)\left(\frac{7+6}{7}\right)$$

$$=\frac{5}{7}(3-x)^3-\frac{6}{7}(x-2)^3-\frac{5}{7}(3-x)+\frac{13}{7}(x-2)$$

$\cancel{i=2 \Rightarrow f(x) = \frac{1}{6}}$

$i=2, \quad M_2 = \frac{-36}{7}, \quad M_3 = \frac{30}{7}$

$$f_2(x) = \frac{1}{6(1)}(x_3 - x)^3 M_2 + \frac{1}{6}(x - x_2)^3 M_3$$

$$+ (x_3 - x)\left(y_2 - \frac{1}{6}M_2\right)$$

$$+ (x - x_2)\left(y_3 - \frac{1}{6}M_3\right)$$

$$= \frac{1}{6}(3-x)^3\left(\frac{-36}{7}\right) + \frac{1}{6}(x-3)^3\left(\frac{30}{7}\right)$$

$$+ (3-x)\left(1 - \frac{1}{6}\left(\frac{-36}{7}\right) + (x-3)\right.$$

$$\left.\left(0 - \frac{1}{6}\left(\frac{30}{7}\right)\right)\right)$$

$$= -\frac{6}{7}(3-x)^3 + \frac{1}{6}(x-3)^3\left(\frac{30}{7}\right)$$

$$+ (3-x)\left(1 + \frac{6}{7}\right) + (x-3)\left(\frac{-5}{7}\right)$$

Similarly $f_3(x)$ and get the value of $f(2.5)$

for $x \in [2, 3]$

ODEs

⇒ Picard's method

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

ex ⇒ $\frac{dy}{dx} = x + y^2$, $y(0) = 0$

also find $y(0.1)$

Sol ⇒ $f(x, y) = x + y^2$ | $x_0 = 0$, $y_0 = 0$

put $n = 0$, $y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$

$$y_1 = y_0 + \int_0^x (x+0)\,dx$$

$$y_1 = y_0 + \left[\frac{x^2}{2}\right]_0^x$$

$$y_1 = \frac{x^2}{2}$$

$n=1$ $\quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$

$$y_2 = \int_0^x \left(x + \frac{x^4}{4}\right)dx$$

$$= \cancel{\frac{x^2}{2} + x}$$

$$y_2 = \frac{x^2}{2} + \frac{x^5}{20}$$

$$\cancel{y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx}$$

$$\cancel{= \int_0^x \left(\frac{x^2}{2} + \frac{x^5}{20} + x\right)}$$

$$y_3 = \frac{x^3}{6} + \frac{x^6}{120} + \frac{x^2}{2}$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$y_3 = \int_0^x (x + y_2^2)\, dx$$

$$y_3 = \int_0^x \left(x + \left(\frac{x^2}{2} + \frac{x^5}{20}\right)^2\right) dx$$

$$y_3 = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{x^7}{20}\right) dx$$

$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}$$

$y_3(0.1) = $

Euler and Euler modified Method ⇒

Euler's method ⇒

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Q Solve $\frac{dy}{dx} = x + y$ with $y = 1$ at $x = 0$

find y for $x = 0.1$ $\quad \boxed{x_0 = 0}$

$h = \frac{0.1}{5} = 0.02$

$x_0 = 0, y_0 = 1$

$x_0 = 0$

$x_1 = 0.02$ $\quad f(x, y)$

$x_2 = 0.04$

$x_3 = 0.06$

$x_4 = 0.08$

$x_5 = 0.10$

$$y_1 = y_0 + 0.02 f(x_0, y_0)$$

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1 = y_0 + 0.02 = \cancel{1.0.2}$$

1.02

$$y_2 = y_1 + 0.02(0.02 + 1.02)$$

$$\cancel{= 1.02 + \cdot 0} \qquad = 1.0408$$

$$y_3 = y_2 + 0.02(0.04 + 1.0408)$$

$$= 1.0408 + 0.02(1.0808)$$

$$= 1.062416$$

$$y_4 = \overset{1.062416}{y_3} + 0.02(0.06 + \cancel{1.04} \; 1.062416)$$

$$y_4 = 1.0848632$$

$$y_5 = 1.0848632 + 0.02(0.08 + 1.062416)$$

$$y_5 = 1.10771265$$

$$y_6 = y_5 + 0.02(0.1 + 1.10771264)$$

$$= 1.131866893$$

Euler's modified method.

$$y_{n+1}^* = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)\right]$$

Q $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$

find $y(0.02)$ and $y(0.04)$

by euler's modified method

$h = 0.02$

~~$y(0) =$~~ $x_0 = 0$, $y_0 = 1$

$x_1 = 0.02$, $y_1 =$

$x_2 = 0.04$, $y_2 =$

$$y_1^* = y_0 + 0.02(x_0^2 + y_0)$$

$$y_1^* = 1 + 0.02(0 + 1)$$

$$y_1^* = 1.02$$

$$y_1 = 1 + \frac{0.02}{2}\left[0 + 1 + 0.01 + 1.02\right]$$

$= 1.0203$

$$y_2^* = y_1 + h f(x_1, y_1)$$
$$= y_1 + h((0.02)^2 + 1.0203)$$
$$= 1.0203 + 0.02(0.0004 + 1.0203)$$
$$= 1.040714$$

$$y_2 = \overset{1.0203}{y_1} + \frac{0.02}{2}\left((0.02)^2 + 1.0203 + (0.04)^2 + 1.040714\right)$$

$$= 1.04093014$$

Runga kutta method of 4 th order

Consider initial value problem $\frac{dy}{dx} = f(x,y)$

when $y(x_0) = y_0$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h/2, y_n + k_1/2)$$

$$k_3 = h f(x_n + h/2, y_n + k_2/2)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + k$$

Q Given $\frac{dy}{dx} = x + y^2$, $y(0) = 1$

find $y(0.2)$ ~~when~~ where $h = 0.1$

$x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $y_1 =$

$x_2 = 0.2$, $y_2 =$

$y(0) = 1$ (x_0, y_0)

$\cancel{K_0 = hf(x_0, y_0)}$

$\cancel{K_0 = 0.1(0 + 1^2)}$

$\cancel{K_0 = 0.1}$

$\cancel{K_1 = 0.1(}$

$k_1 = hf(x_0, y_0)$

$= 0.1(0 + 1^2)$

$k_1 = 0.1$

$k_2 = 0.1\left(\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right)$

$= 0.1\left(\left(\frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2\right)$

~~$= 0.11025$~~

$= 0.1\left(0.05 + (1.1025)\right)$

$= 0.11525$

$x+y^2$

$K_3 = \cancel{h(f(x_0 + \frac{h}{4}))}$

$= h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.1 \left[(0 + \frac{0.1}{2}) + (1 + \frac{0.011525}{2})^2 \right]$

$= 0.116857064$

$\cancel{K = \frac{1}{6}[(K_1}$

$K_4 = hf(x_0 + h, y_0 + k_3)$

$= h(x_0 + h + (y_0 + k_3)^2)$

$= 0.1 ((0 + 0.1) + (1 + 0.116857064)^2)$

$\cancel{= 0.1 (0.1 +}$

$= 0.13473697$

$\cancel{K = \frac{1}{6}[0.1 + 0.11525 + 0.116857064 + 0.13473697]}$

$\cancel{K = 0.0778}$

$\Rightarrow K_a = \frac{1}{6}[K_1 + 2k_2 + 2k_3 + k_4)$

then $y_1 = y_0 + K_a$

and $y_2 = y_1 + K_b$

Milne's predictor and corrector method ⇒

Consider IVP $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

$y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$

where x_0, x_1, x_2 and x_3 are equidistant value of x with unit step size h.

Milne's predictor formula ⇒

$$y_4^P = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

Milne's corrector formula ⇒

$$y_4^C = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^P)$$

Q Using Milne's P and C method find y when $x = 0.8$ given $\frac{dy}{dx} = x - y^2$

$y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$

x	y	$f(x,y)$
0	0	~~$0 - (0.2)^2$~~ 0
0.2	0.02	$0.2 - (0.02)^2$
0.4	0.0795	$0.4 - (0.0795)^2$
0.6	0.1762	$0.6 - (0.1762)^2$ $= 0.5689$

0.8 $y_4^p = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$y_4^c = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^p)$

$\cancel{y_4^p = 0 + \frac{4(0.2)}{3}(2 + 4}$

$\cancel{y_4^p = 0 + \frac{4(0.2)}{3}(0.3992 - 0.3936}$

$+ 2$

$0.3049 = y_4^p$

$y_4^c =$

Adam bashforth predictor and corrector formula.

$y_4^p = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$

$y_4^c = y_3 + \frac{h}{24}[9f_4^p + 19f_3 - 5f_2 + f_1]$

~~Example~~ $\Rightarrow$

Gauss Quadrature formula

Ex $I = \int_0^1 x\,dx$ by GQF with $n = 4$

upto 5 decimal places

sol $\rightarrow$ we have to transform

intervals $[0, 1]$ to $[-1, 1]$ by using

substitution $\rightarrow$ limits of the integral

$$x = \frac{b+a}{2} + \frac{b-a}{2}u = \frac{1+0}{2} + \frac{1-0}{2}u$$

$$\cancel{x = \frac{0+1}{2} + \frac{0-1}{2} =}$$

$$x = \frac{u+1}{2} \Rightarrow dx = \frac{1}{2}du$$

$$du = 2dx$$

$$\int_0^1 x\,dx = \int_{-1}^{1}\left(\frac{u+1}{2}\right)\frac{du}{2} = \frac{1}{4}\int_{-1}^{1}(u+1)\,du$$

$$= \frac{1}{4}\sum_{i=1}^{4} w_i\,\phi(u_i) \rightarrow w_i \text{ are weights}$$

$$I = \sum_{i=1}^{n} w_i\,\phi(u_i)$$

Romberg's algorithm/ method

ex $I = \int_0^1 \frac{1}{1+x}\,dx$ $\quad h = \frac{0+1}{2}$

subsequent h are 0.25, 0.125 etc.

take $h = 0.5, 0.25, 0.125$

for $h = 0.5$

$f(x) = \frac{1}{1+x}$, $x_0 = 0$, $x_n = 1.0$

x	0	0.5	1
$f(x)$	1	0.666	0.5

~~trampe~~ from trapezoidal rule,

~~$x_0 = 0$~~

$$I(h) = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \text{---} y_{n-1})\right]$$

$$= \frac{0.5}{2}(1 + 0.5 + 2(0.667))$$

$$= \cancel{0.7085}\ 0.709$$

$h = 0.25$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.80	0.667	0.571	0.5

$$I\left(\frac{h}{2}\right) = \frac{0.25}{2}\left(1 + 0.5 + 2(0.80 + 0.667 + 0.571)\right)$$

$$= 0.697$$

$I\left(\frac{h}{4}\right) =$

$$
\begin{array}{r}
0.625 \\
125 \\
\hline
0.750 \\
125 \\
\hline
875
\end{array}
\qquad
\begin{array}{r}
0.375 \\
0.125 \\
\hline
00
\end{array}
$$

$h = 0.125$

$\cancel{I\left(\frac{h}{4}\right) = \frac{0.125}{2}\Big(}$

x	0	0.125	0.250	0.375	0.5
$f(x)$	1	0.889	0.8	0.727	0.667

0.625	0.750	0.875	1
0.615	0.570	0.534	0.500

$$I\left(\frac{h}{4}\right) = \frac{0.125}{2}\Big(1 + 0.5 + 2(0.889 + 0.800 + 0.727 + 0.667 + 0.615 + 0.570 + 0.534)\Big)$$

$= 0.694$

$I(h) = 0.708$, $I\left(\frac{h}{2}\right) = 0.697$

$I\left(\frac{h}{4}\right) = 0.694$

$$\begin{tikzcd}
I(h) \arrow[dr] & & \\
& I\left(h, \frac{h}{2}\right) \arrow[dr] & \\
I\left(\frac{h}{2}\right) \arrow[ur] \arrow[dr] & & I\left(h, \frac{h}{2}, \frac{h}{4}\right) \arrow[d] \\
& I\left(\frac{h}{2}, \frac{h}{4}\right) \arrow[ur] & \text{final ans} \\
I\left(\frac{h}{4}\right) \arrow[ur] & &
\end{tikzcd}$$

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$

Newton Cotes formula $\Rightarrow$

$$I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

let

$$u = \frac{x - x_0}{h}$$

$x_0 \to 0$, $x_0 + nh \to n$

$x = x_0 + uh$

$dx = h\,du$

$$I = \int_0^n f(x_0 + uh)\,h\,du$$

we know by newton's forward interpolation method.

$$I = f(x_0) + \frac{u}{1!}\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0)$$

$$+ \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \text{---} \text{---}$$

so

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n \left(f(x_0) + \frac{u}{1!}\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \text{---} \right) du$$

$$= h\left[n f(x_0) + \frac{n^2}{2}\Delta f(x_0) + \frac{\Delta^2 f(x_0)}{2!}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \text{---} \text{---} \right]$$

if $n=1 \rightarrow$ trapezoidal rule.

$n=2 \rightarrow$ Simpson $1/3^{rd}$ rule

$n=3 \rightarrow$ Simpson $3/8^{th}$ rule

Q Golden Search rule ⇒

$[a, b] \rightarrow f(x)$

find x_1 and x_2 in a, b

$$x_1 = b - \frac{b-a}{\phi}, \quad x_2 = a + \frac{b-a}{\phi}$$

$\phi = 1.618 \rightarrow$ Golden search ratio eg

$$f(x) = x^4 - 14x^3 + 60x^2 - 70x$$

In the range $[0, 2]$

$$x_1 = b - \frac{b-a}{\phi} = 2 - \frac{2}{1.618} =$$

$$x_2 = a + \frac{b-a}{\phi} = 0 + \frac{2}{1.618}$$

$$\cancel{f(x_1) = 0.7639}$$

$$\cancel{f(x_2) = 1.2360}$$

$$x_1 = 0.7639$$

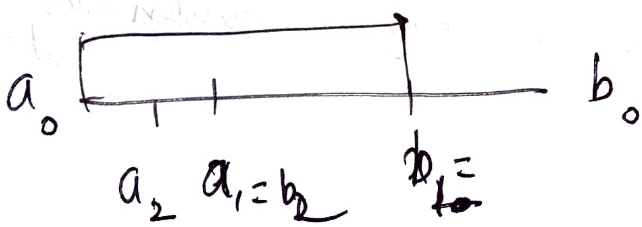
$$x_2 = 1.2360 = b_1$$

$$f(x_1) = -24.3606 = -24.36 = a_1$$

$$f(x_2) = -18.96 = -18.96 = b_1$$

$$f(a_1) < f(x_2)$$

$$\cancel{[x_2, b]}$$

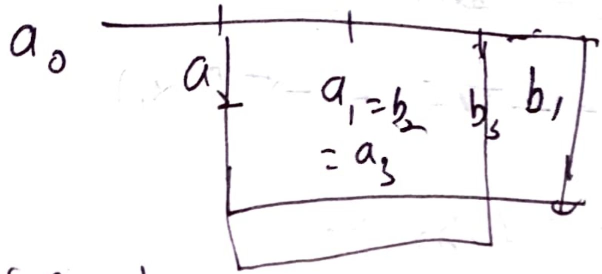


$$a_2 = b_1 - \frac{b_1 - a_0}{\phi} = 0.4720$$

$f(a_2) = -21.09$

$f(b_2) = f(a_1) = -24.36$

$f(a_2) > f(b_2)$



$\cancel{f(a_3) =}$

$f(a_3) = f(b_2) = -24.36$

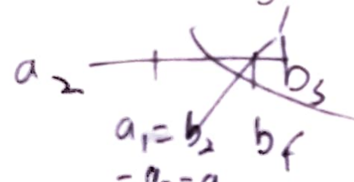
$$b_3 = a_2 + \frac{b_1 - a_2}{\phi}$$

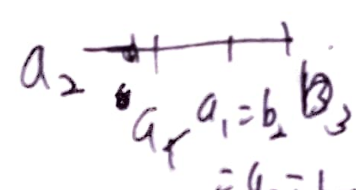
$= 0.944$

$f(b_3) = -23.59$

$f(b_3) > f(a_3)$

$\boxed{[a_2, b_3]}$





$$a_4 = b_3 - \frac{b_3 - a_2}{\phi}$$

$$\cancel{b_4 = a_2 + \frac{b_3 - a_2}{\phi}}$$

Steepest descent method Iteration 1 =)

St-1 Start with arbitrary initial point X_1. set iteration

Step-2 find the search direction S

$$S_i = -\nabla f_i = -\nabla f(x_1)$$

$$\nabla f = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix}_{X_1}$$

Step 3 Determine the optimal step length λ_1^* we minimize the

$$f(X_1 + \lambda_1^* S_1) \frac{\partial f}{\partial \lambda_1^*} = 0$$

$$\Downarrow$$

$$\lambda_1^* = ?$$

Step(4) $X_2 = X_1 + \lambda_1^* S_1$

$$\Delta f_2 = \begin{Bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{Bmatrix} \text{ or }$$

$$\begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix}_{X_2} \neq \{\text{initial point}\}$$

Step 5 Iteration 2 =)

Q $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$
starting from the point $X_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

As Iteration 1

$$\nabla f_1 = \begin{cases} \frac{\partial f}{\partial x_1} = 1 - 0 + 4x_1 + 2x_2 \\ \frac{\partial f}{\partial x_2} = 0 - 1 + 0 + 2x_1 + 2x_2 \end{cases}$$

$$\nabla f_1 = \begin{Bmatrix} 1 + 4x_1 + 2x_2 = \frac{\partial f}{\partial x_1} \\ -1 + 2x_1 + 2x_2 = \frac{\partial f}{\partial x_2} \end{Bmatrix} \quad X_1 = \begin{bmatrix} 0 \to x_1 \\ 0 \to x_2 \end{bmatrix}$$

$$S_1 = -\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ +1 \end{pmatrix}$$

$$f(X_1 + \lambda_1^* S_1) \frac{\partial f}{\partial \lambda_1^*} = 0$$

$$f\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda^* \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] = f\begin{bmatrix} -\lambda_1^* \to x_1 \\ \lambda_1^* \to x_2 \end{bmatrix}$$

$$+\left[-\alpha_1^* - \alpha_1^* + (-\alpha_1^*)^2 + 2(-\alpha_1^*)(\alpha_1^*) + (\alpha_1^*)^2\right]$$

$$= f\left[-2\alpha_1^* + \cancel{2\alpha_1^{*2}} - \cancel{2\alpha_1^{*2}} + \alpha_1^{*2}\right]$$

$$= f(-2\alpha_1^* + \alpha_1^{*2})$$

$$\frac{\partial f}{\partial \alpha_1^*} = -2 + 2\alpha_1^* = 0$$

$$\alpha_1^* = 1$$

$$X_2 = X_1 + \alpha_1^* S_1$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}(1)$$

$$X_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \neq X_1$$

$$\nabla f_2 = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix}_{X_2}$$ and repeat the steps

Runge kutta method of second order

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0, \; y(x_1)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

Numerical solution of PDE
parabolic, hyperbolic, elliptical

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} = 0$$

$D = b^2 - 4ac$

$D = 0 \rightarrow$ parabola, 5
$D > 0 \rightarrow$ hyperbola, 2 charge sol
$D < 0 \rightarrow$ ellipse $\rightarrow$ No sol

Newton's method for minization

Q ~~$f(x) = x$~~

$f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$ by taking the starting point as $X_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

sol $\Rightarrow$ $X_2 = ?$

$$[J_1] = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$[J_1]^{-1} = \frac{1}{|J_1|} \operatorname{adj} [J_1]$$

$$\operatorname{adj} J_1 = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

$$g_1 = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix}_{X_1} = \begin{pmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{pmatrix}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$X_2 = X_1 - [J_1]^{-1} g_1$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{Bmatrix} -1 \\ 3/2 \end{Bmatrix}$$

$$g_2 = \begin{Bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{Bmatrix}_{X_2} = \begin{Bmatrix} 1 + 4x_1 + 2x_2 \\ -1 + 2x_1 + 2x_2 \end{Bmatrix}_{\begin{Bmatrix} -1 \\ 3/2 \end{Bmatrix}}$$

$$g_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$X_3 = X_2$$