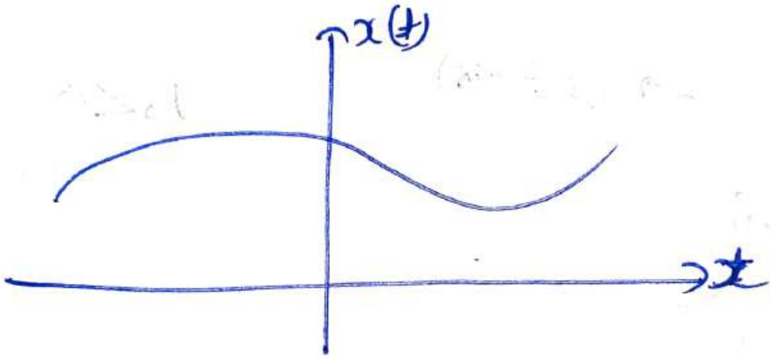
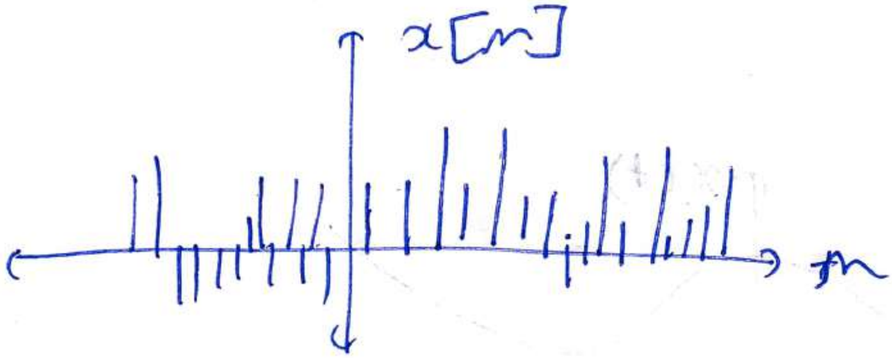
Signal and System

Signal → continous → $x(t)$

Signal → discrete → $x[n]$, $n \in \mathbb{Z}$

~~conti~~ continous





$$p(t) = v(t)\, i(t) = \frac{1}{R} v^2(t)$$

across a resistor

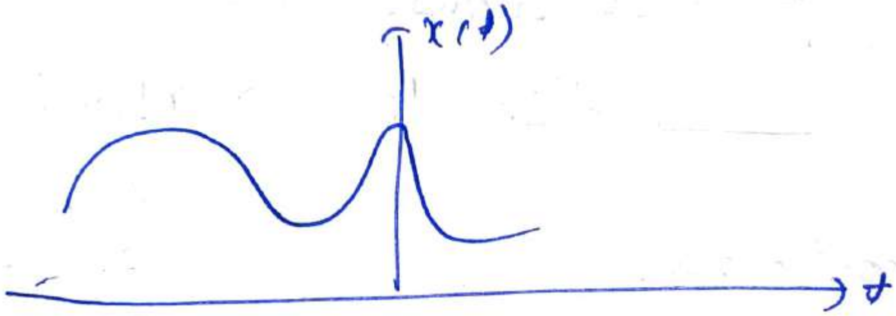
$$\int_{t_1}^{t_2} p(t) = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\, dt \rightarrow \text{energy over T.I}$$

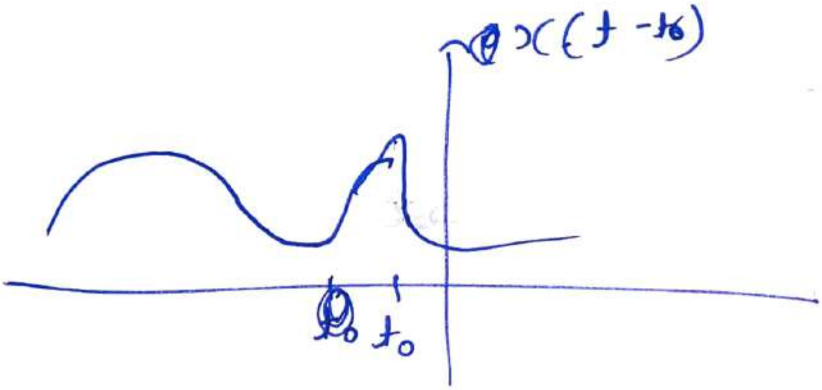
$$t_1 \leq t \leq t_2$$

average power

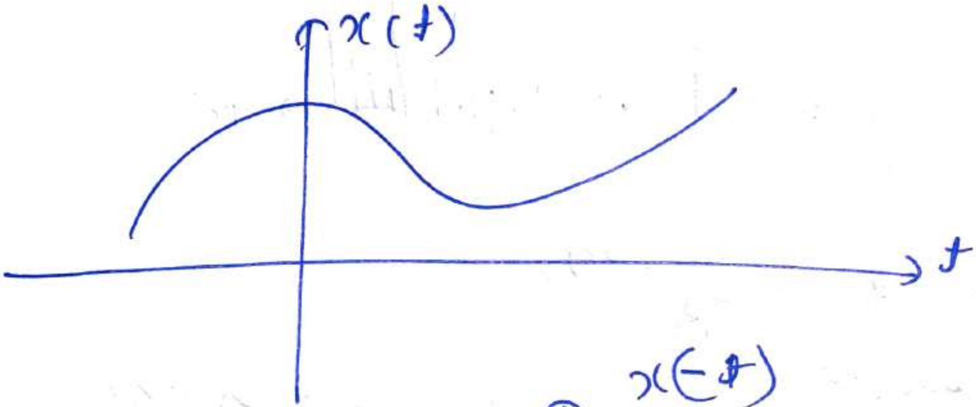
$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} p(t) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\, dt$$

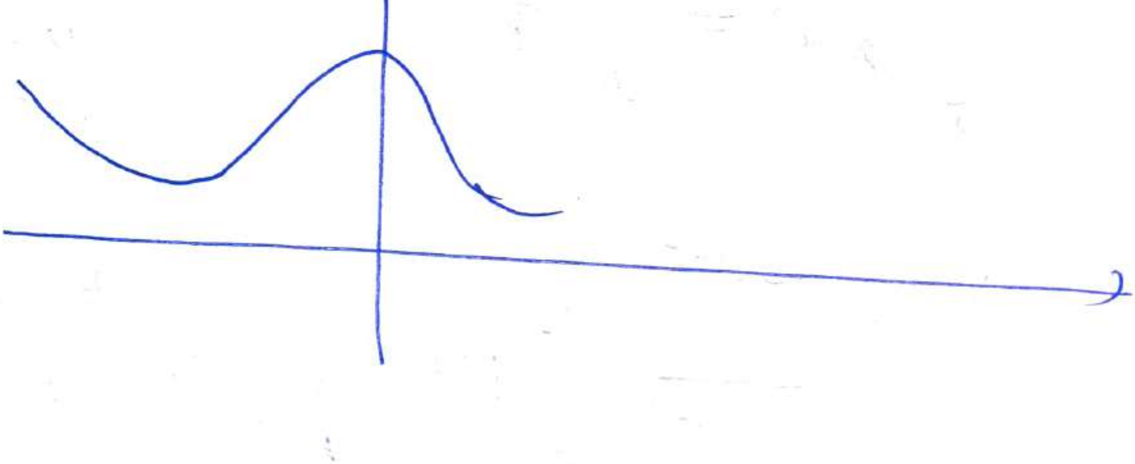
transformation in Signals

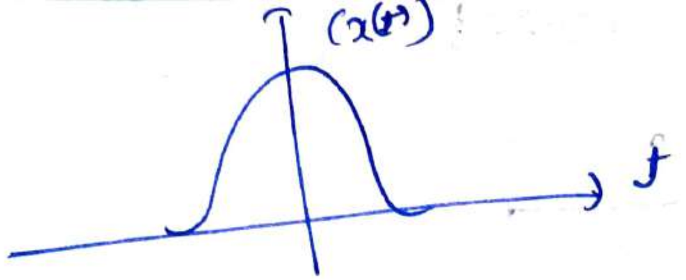


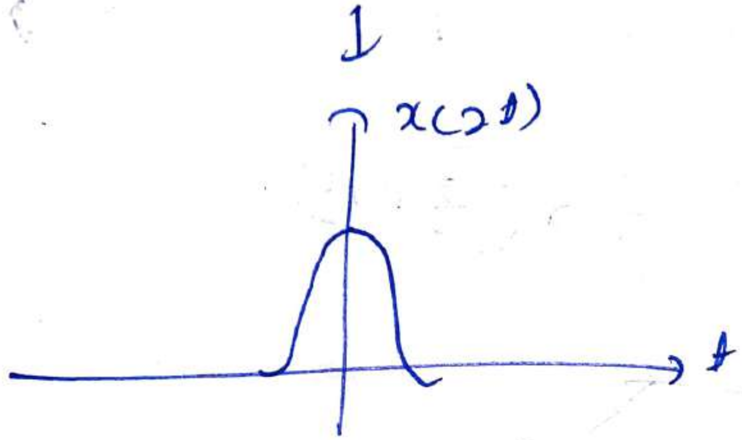


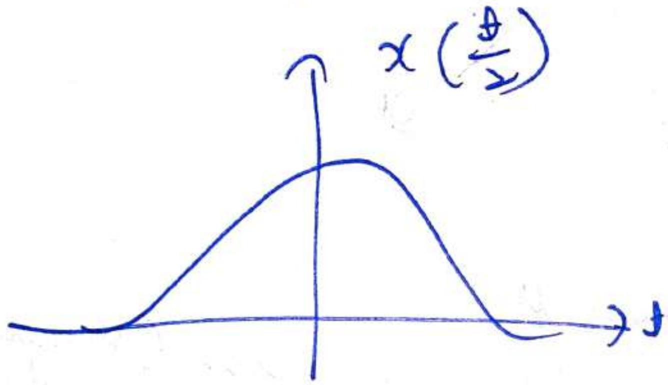
$t_0 < 0$

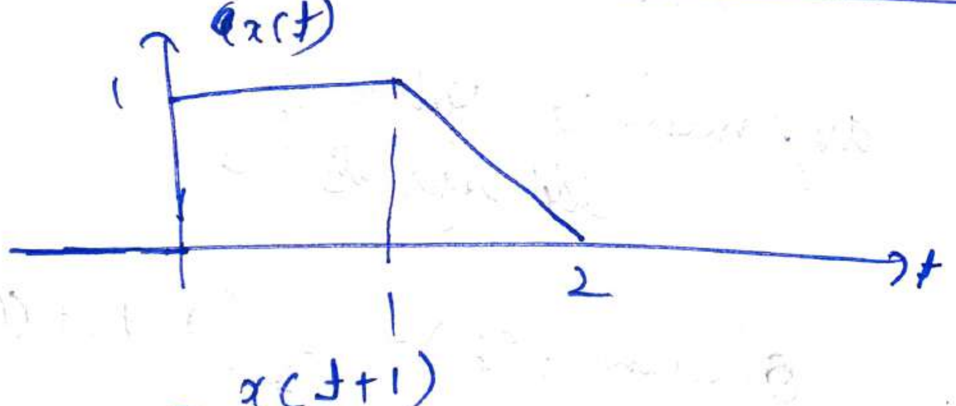


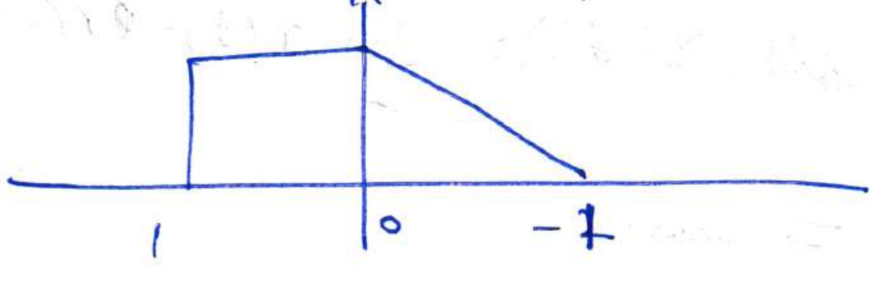


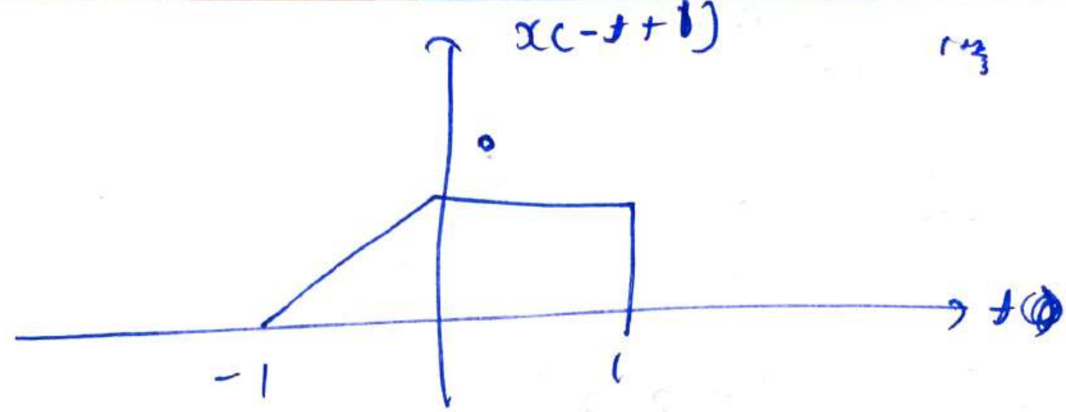

(x(t))
t

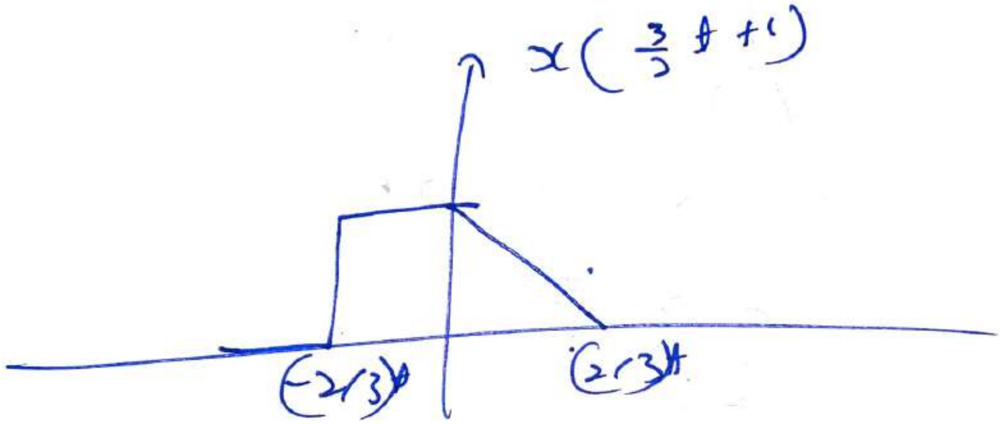

x(2t)
t


x(t/2)
t


x(t)
1
1
2
t


x(t+1)
1
0
-1





Even and odd signals

$x(-t) = x(t) \rightarrow$ even (symmetric about y axis)

$x(t) = -x(-t) \rightarrow$ odd (symmetric about origin)

av/mean/dc values of odd signals $= 0$

$\text{even}(x(t)) = \frac{1}{2}\left(x(t) + x(-t)\right)$

$\text{odd}(x(t)) = \frac{1}{2}\left(x(t) - x(-t)\right)$

dc + even = even

eg $10 + t^2 \Rightarrow$ even

③ dc + ~~odd~~ ⇒ NENO

eg $10 + t^3$

④ $E \times E$ = even

⑤ $E \times O$ = odd

⑥ $O \times O$ = even

⑦ $\frac{d}{dt}(E)$ = odd

⑧ $\frac{d}{dt}(O)$ = even

⑨ $\int E\,dt$ = odd

⑩ $\int O\,dt$ = even

periodic and non periodic signals ⇒

$\cancel{x(n+}$ $x(t + nT_0) = x(t)$

fundamental period

$x[n + Nm] = x[n]$

dc signal is periodic for all t

T_0 undefined

time shifting on periodic signal will remain the signal unharmed

$x(t) \rightarrow$ periodic

$x(t \pm 2) \rightarrow$ same signal

fundamental ~~T~~ TP $\Rightarrow$

$$x(t) = A_0 e^{j\omega_0 t}$$

$$x(t+T_0) = A_0 e^{j\omega_0 t} e^{j\omega_0 T_0}$$

$$\cancel{A e^{j\omega_0 t}} = \cancel{A e^{j\omega_0 t}} e^{j\omega_0 T_0}$$

$$e^{j\omega_0 T_0} = 1 = e^{j2\pi K}$$

$$\omega_0 T_0 = 2\pi K$$

$$T_0 = \frac{2\pi K}{\omega_0}$$

or $T_0 = \frac{2\pi}{\omega_0}$

$x_3(t) = x_1(t) + x_2(t)$

$x_1(t) \rightarrow$ periodic, $x_2(t) \rightarrow$ periodic

$\rightarrow$ periodic

eg $x(t) = \sin 6\pi t + \cos 5\pi t$

$$T_1 = \frac{2\pi}{6\pi} = \frac{1}{3} \qquad T_2 = \frac{2}{5}$$

① $\frac{T_1}{T_2} \Rightarrow$ rational periodic ✓

② $$T_0 = \frac{T_1}{T_2} = \frac{LCM(1,2)}{HCF(3,5)} = \frac{2}{1}$$

T.P $= 2$

①) time scaling on P.S

$$x_3(t) = x_1(2t)$$

$$x_1(t) = \cos 4\pi t \rightarrow T_0 = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$x_1(2t) = \cos 2\pi (2t)$$

$$x_3(t) = \cos 2\pi t \rightarrow T_0 = \frac{2\pi}{2\pi} = 1$$

$$x(2t) = \frac{T_0}{|2|}$$

$$x(at) = \frac{T_0}{|a|} \quad a \neq 0$$

Energy and power continuous time signals

total energy ⇒

$$E = \int_{-\infty}^{\infty} p(t)\,dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

Avg power ~~$E = \frac{1}{}$~~

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt \rightarrow NP$$

$$P = \frac{1}{T_0} \oint_{T_0} |x(t)|^2\,dt$$

⟩ Periodic

if total Energy of signal is finite then its power will be ~~infinite~~ 0 ⇒ energy signal

if T. Energy is ∞ then then avg Power ⇒ finite
⇓
power signal

① $\boxed{E \cdot S}$ are absolutely integratable.

F.T

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \text{finite}$$

② P.S over a TP is Integratable

③ $P = (RMS)^2 \Rightarrow RMS = \sqrt{P}$

④ periodic signal have finite power (over a TP)

Unit Impulse Signal

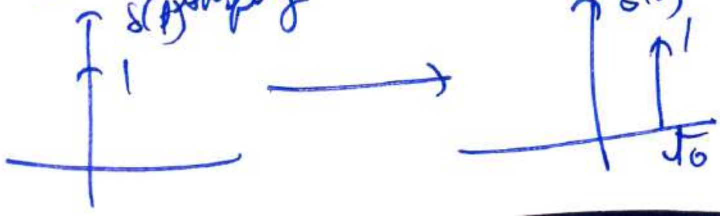
$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

weight or strength of an impulse

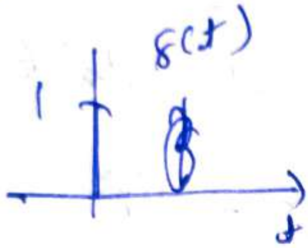
$$\int_{-\infty}^{\infty} y(t)\, dt = \int_{-\infty}^{\infty} A_0\, \delta(t)\, dt = A_0 \leftarrow \text{weight or step}$$

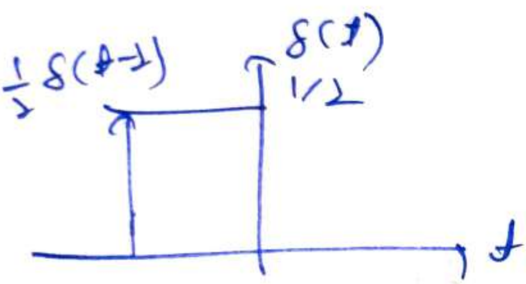
⑥ $\delta(t) \xrightarrow[\delta(t)\ \text{shifting}]{\text{time}} \delta(t - t_0)$

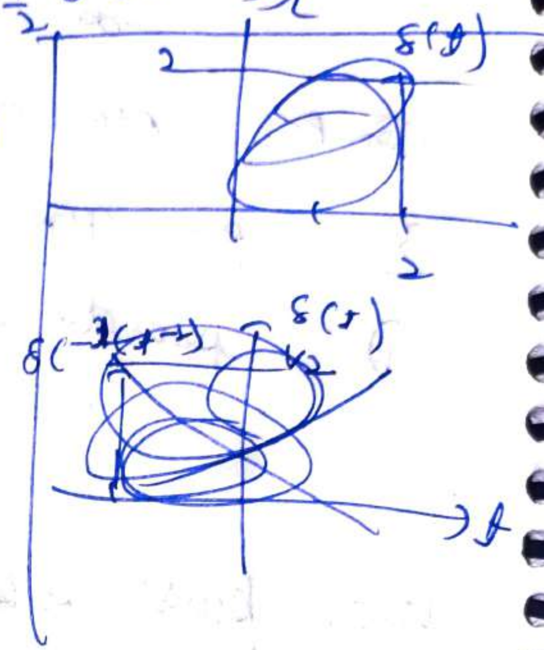


$\delta(at) \longrightarrow \frac{1}{|a|}\delta(t)$

eg $\delta(-2t+4) = \delta(-2(t-2))$
$= \frac{1}{2}\delta(t-2)$







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(8) multiplication

$x(t)\,\delta(t-t_0) = x(t-t_0)\,\delta(t-t_0)$

(9) $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

↳ const.

(10) $\int_{-\infty}^{\infty} x(t) \frac{d^n}{dt^n} \delta(t-t_0)\, dt$

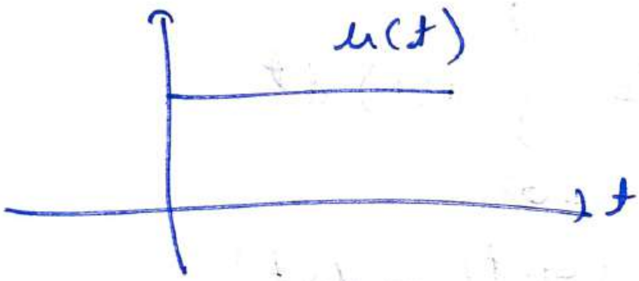
$$= (-1)^n \frac{d^n}{dt^n} x(t) \Big|_{t=t_1}$$

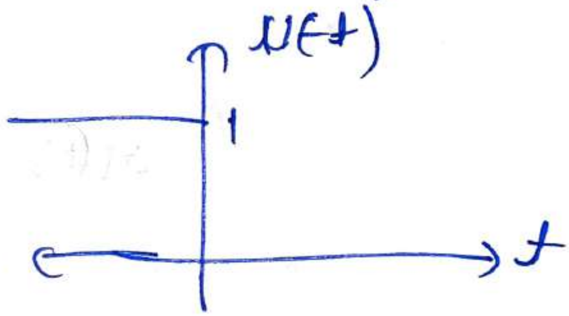
~~condition~~ condition $x(t) \big|_{t=\infty} = 0$ or finite

Unit step signal

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Mone $u(t) + u(-t) = ?$





$u(t) + u(-t)$ $\Rightarrow$

Gibbs phenomenon

point of discont.

(ii) $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

$= \lim_{T \to \infty} \frac{1}{2T} \left[\int_{-T}^{0} 0 \, dt + \int_{0}^{T} 1 \, dt \right]$

$= \frac{1}{2} \rightarrow P \rightarrow$ finite

Power Signal

$E \rightarrow \infty$

unit ramp signal

$r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$

$r(t) = \int_{-\infty}^{t} u(t) \, dt$

or $\frac{d r(t)}{dt} = u(t)$

$r(t) = t u(t)$

$r(t) \rightarrow$ power signal after calculating

$$\delta(t) \underset{d}{\overset{I}{\rightleftarrows}} u(t) \underset{d}{\overset{I}{\rightleftarrows}} r(t)$$

Exponential signals $\Rightarrow$

$$x(t) = A_0 e^{st}$$

$$x(t) = e^{st}$$

$$\text{let } s = \underbrace{\sigma}_{Re} + \underbrace{j\omega}_{Im}$$

$$x(t) = e^{(\sigma + j\omega)t}$$

$$x(t) = e^{\sigma t} * e^{j\omega t}$$

$$x(t) = e^{\sigma t}(\cos\omega t + j\sin\omega t$$

Discrete time convolution

$x_1[n] = \{1, 2, -5\}$

$x_2[n] = \{2, 0, 1\}$

$x_2[n]$ \ $x(n)$	1	2	-5
2	2	4	-10
0	0	0	0
1	1	2	-5

$y[n] = \{2, 4, -9, 2, -5\}$

continuous time convolution $\Rightarrow$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, dt$$

$h(t) \downarrow h(\tau)$ $\quad$ $x(t) \downarrow x(\tau)$

$\downarrow$

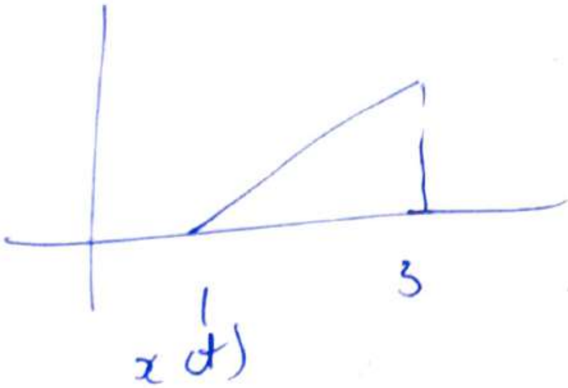
$h(-\tau)$

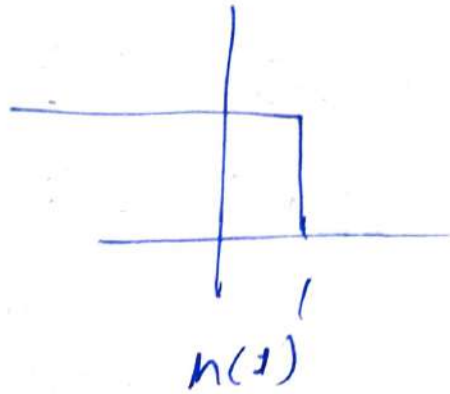
$\downarrow$

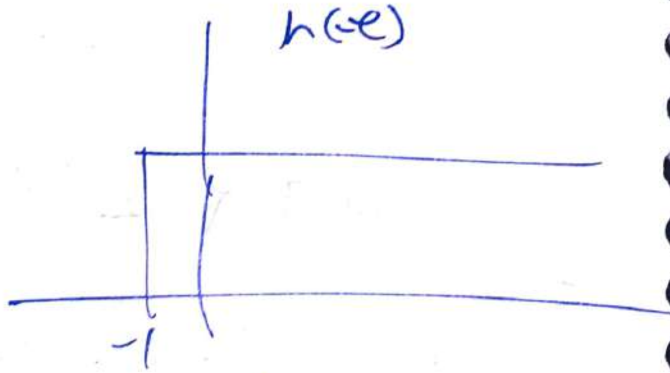
$h(-(\tau - t))$

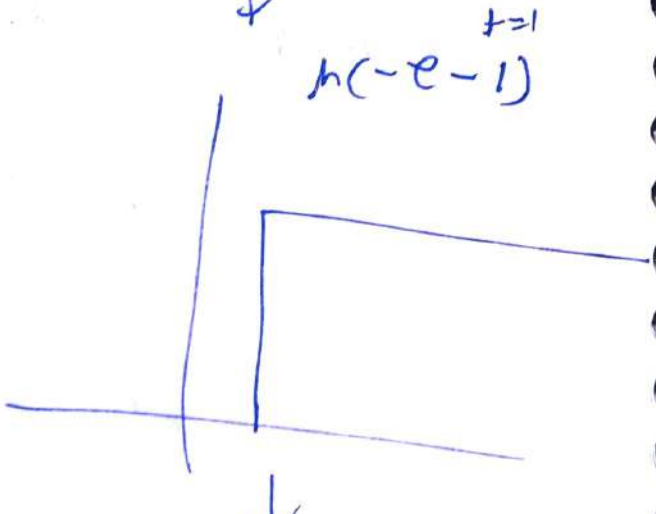
$\downarrow$

$h(t-\tau)$


1
3
x(t)


1
h(t)


h(-τ)
-1


t=1
h(-τ-1)

sampling theorem $\rightarrow$ CTS $\rightarrow$ DTS

$\omega_s > 2\omega_m$
over sampled

$\omega_s < 2\omega_m$
$\downarrow$
undersampled

Nyquist state

Minimum $\rightarrow$ sampling rate

$$\left.\begin{array}{l}\omega_s = 2\omega_m \\ f_s = 2f_m\end{array}\right\}$$

$T_s \rightarrow$ Nyquist interval

$T_s = \dfrac{1}{f_s}$

$f \rightarrow$ Nyquist rate

Z - transform
(discrete the signals)

$\cancel{x[n] \xleftrightarrow{ZT} X[Z]}$

~~x~~

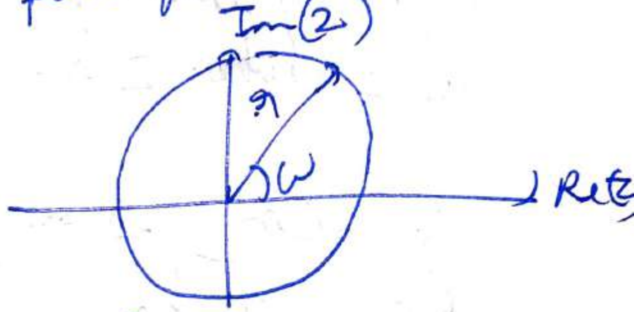
→ bidirectional

$$x[n] \xrightarrow{ZT} X(Z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$$

$Z = re^{j\omega}$ ← polar form

$|Z| = r$,

$\angle Z = \omega$



$$x[n] \xleftrightarrow{ZT} X(Z) = \sum_{n=0}^{\infty} x[n] Z^{-n}$$

↑
unilateral
Z-Transform

~~x[z~~

$$x[n] \xleftrightarrow{ZT} X[Z] = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$$

~~put Z = re^{jω}~~

put $Z = re^{j\omega}$

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n] (re^{j\omega})^{-n}$$ → convergence factor

$$= \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n}$$

If $X(z)$ exist then $x[n].r^{-n}$ should be bounded

If $r=1$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \text{DTFT of } x[n]$$

ROC

region of convergence is the range of z values where z transform converges.

(1) If $x[n]$ is right handed signal then ROC will be $|z| > |P_{max}|$

(2) If $x[n]$ is left handed signal then ROC will be $|z| < |P_{min}|$

(3) If $x[n]$ is two sided then ROC will be $|P_1| < |z| < |P_2|$

(4) ROC cannot contain any poles

(5) ROC must be connected region.

(6) If $x[n]$ is finite length sequence then ROC is the entire z-plane except possibly $z=0$ or $z=\infty$

~~Casual~~ Casuality condition :-

A discrete time LTI system is causal if and only if

(1) ROC is the exterior of a circle outside the ~~outer~~ outermost pole.

$$|Z| > |P_{max}|$$

if $|Z| < |P_{min}|$ → anticausal

~~$|\ \ < |Z| <$~~

$|P_{min}| < |Z| < |P_{max}|$
→ non causal

(2) If $H(z)$ is expressed as a ratio of polynomial in z, then order of numerator can not be greater than the order of denominator.

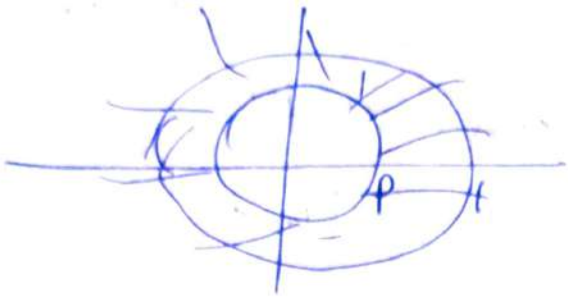
Stability condition ⇒

An LTI system is stable iff ROC of its system fn $H(z)$ included the unit circle

If ROC include unit circle it means $x[n]$ is bounded and its DTFT exists.

Causal and Stable

$|z| > \underset{<1}{\underbrace{P_{max}}}$, $|z| = 1$



Example

① $x[n] = a^n u[n]$

② $x[n] = -a^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(1) $$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= (a z^{-1})^0 + (a z^{-1})^1 + (a z^{-1})^2 + \text{———} \infty$$

$S_\infty = \dfrac{a}{1-r}$, $|r| < 1$

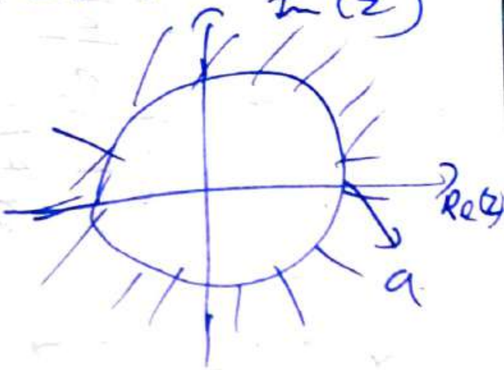
$a = (a z^{-1})^0 = 1$ $r = (a z^{-1})^1$

$S_\infty = \dfrac{1}{1 - a z^{-1}}$

$X(z) = \dfrac{1}{1-az^{-1}}$ $\quad |az^{-1}| < 1$

$|a| < |z|$

$|z| > |a|$



$a^n u[n] \xleftrightarrow{ZT} \dfrac{z}{z-a}$

② $x[n] = -a^n u(-n-1)$

$$u[-n-1] = u[-(n+1)] = \begin{cases} 1, & n \le -1 \\ 0, & n > -1 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{-1} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-a^n)\, z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n$$

let $n = -m$

$-\infty \to \infty$, $-1 \to 1$

$$= -\sum_{m=1}^{\infty} (az^{-1})^{-m}$$

$= (az^{-1})^{-1} + (az^{-1})^{-2} + (az^{-1})^{-3} + \dots \infty$

$$= -\left(\frac{z}{a^{+1}} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \text{———} \infty\right)$$

$$= -\left[\frac{\frac{z}{a}}{1-\frac{z}{a}}\right] = -\left[\frac{\frac{z}{a}}{\frac{a-z}{a}}\right] = \left[\frac{z}{a-z}\right]$$

$$X(z) = -\left[\frac{z}{a-z}\right] \qquad |z| < |a|$$

Q find z transform of

(1) $x[n] = \delta[n]$

(2) $x[n] = \{-4, -2, \underset{\uparrow}{-1}, 5, 4\}$

~~(1) $X[n] =$~~

(1) $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$= z^{-n} \big|_{n=0}$$

$$X(z) = 1$$

$$\delta[n] \xleftrightarrow{z} 1 \Rightarrow$$ ROC will be entire z-plane

(2) $x[n] = \{-4, -2, -1, 5, 4\}$

$\quad\quad\quad\quad -2 \;\; -1 \;\; \underset{n=0}{\uparrow} \;\; 1 \;\; 2$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=-2}^{2} x[n] z^{-n}$$

$$= x[-2] z^{+2} + x[-1] z^{+1} + x[0] z^{0} + x[1] z^{1} + x[2] z^{-2}$$

$$X(z) = -4z^2 - 2z - 1 + 5z^{-1} + 4z^{-2}$$

at $z = 0$, $z = \infty$ poles found

$|z| \neq 0, \infty$ ROC will be entire z plane except at $z = 0, \infty$

properties of Z transform

linearity $x_1[n] \xleftrightarrow{ZT} X_1(z)$, ROC = R_1
$x_2[n] \xleftrightarrow{ZT} X_2(z)$, ROC = R_2

$$\alpha x_1[n] + \beta x_2[n] \xleftrightarrow{ZT} \alpha X_1(z) + \beta X_2(z)$$

$\alpha, \beta \rightarrow$ constant , ROC = $R_1 \cap R_2$

$$ROC = R_1 \cap R_2$$

ROC : $R_1 \cap R_2$ is valid till there is no pole zero cancellation

eg $$\frac{\cancel{(z-1)}(z-2)}{\cancel{(z-1)}(z-3)(z-5)}$$

ROC will be depend on these

Q $x[n] = \left(\frac{4}{5}\right)^{|n|}$

$$\left(\frac{4}{5}\right)^{|n|} = \begin{cases} \left(\frac{4}{5}\right)^{n}, & n \geq 0 \\ \left(\frac{4}{5}\right)^{-n}, & n < 0 \end{cases}$$

$$x[n] = \left(\frac{4}{5}\right)^n u[n] + \left(\frac{4}{5}\right)^{-n} u[-n-1]$$

let $x_1[n] = \left(\frac{4}{5}\right)^n u[n]$

$$x_2[n] = \left(\frac{4}{5}\right)^{-n} u[-n-1]$$

$$a^n u[n] \xleftrightarrow{ZT} \frac{z}{z-a}, \quad |z| > |a|$$

$$\left(\frac{4}{5}\right)^n u[n] \longleftrightarrow \frac{z}{z-\left(\frac{4}{5}\right)}, \quad |z| > \frac{4}{5}$$

$$-a^n u[-n-1] \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$|z| < |a|$$

$$\left(\frac{4}{5}\right)^{-n} u[-n-1]$$

$$= \left(\frac{5}{4}\right) u[-n-1]$$

$$-\left[-\left(\frac{5}{4}\right)^{-n} u[-n-1]\right] \xleftrightarrow{ZT} \frac{-z}{z-\frac{5}{4}}, \quad |z| < \left|\frac{5}{4}\right|$$

by linearity property

$$x_1[n] + x_2[n] = \frac{z}{\left(z-\frac{4}{5}\right)} - \frac{z}{z-\left(\frac{5}{4}\right)}$$

$$= \frac{-\frac{9}{20} z}{\left(z-\frac{4}{5}\right)\left(z-\frac{5}{4}\right)}$$

$\frac{1}{5} < |z| < \frac{5}{4}$

Time shifting

$x[n] \xleftrightarrow{ZT} X(z) \quad R$

$$\left.\begin{array}{l} x[n-n_0] \xleftrightarrow{ZT} z^{-n_0} \cdot X(z) \\ x[n+n_0] \xleftrightarrow{ZT} z^{n_0} X(z) \end{array}\right\} \text{ROC} : R$$

It is valid

except for addition or deletion of $z = 0$ or

$z = \infty$

<u>eg</u> if $x[n] = \delta[n]$ then ZT $x[n-2]$

and $x[n+1] = ?$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

(i) ~~$\delta[n]$~~ $\delta[n-2]$

$$\cancel{X(z) = \sum_{n=-\infty}^{\infty} \delta[n-2] z^{-}}$$

$\delta[n] \xleftrightarrow{ZT} 1$

$\delta[n-2] \xleftrightarrow{ZT} z^{-2} \rightarrow$ entire z plane except at $z=0$

$\delta[n+1] \xleftrightarrow{ZT} z^{1} \rightarrow$ entire z plane except $z = \infty$

$\delta(n+1) \xleftrightarrow{ZT} z$

$\delta[(n-1)+1] \xleftrightarrow{ZT} z^{-1}z, =1$, ROC → entire z plane

multiplication by a^n

$x[n] \xleftrightarrow{ZT} X(n)$, ROC = R

$a^n x[n] \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$, ROC = $|a|R$

ZT of $\cos(\omega_0 n)\, u(n)$

$$x[n] = \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) u[n]$$

$$= \underbrace{\frac{1}{2} e^{j\omega_0 n} u[n]} + \underbrace{\frac{1}{2} e^{-j\omega_0 n} u[n]}$$

$$u[n] \xleftrightarrow{ZT} \frac{z}{z-1}$$

$$(e^{j\omega_0})^n u[n] \xleftrightarrow{ZT} \frac{\frac{z}{e^{j\omega_0}}}{\frac{z}{e^{j\omega_0}} - 1}$$

$$\left|\frac{z}{e^{j\omega_0}}\right| > 1$$

$\frac{1}{2} \cdot e^{j\omega_0 n} u[n] \xleftrightarrow{ZT} \frac{1}{2} \cdot \frac{z}{z - e^{j\omega_0}}$

~~$e^{j\omega_0 n} u[n] \xleftrightarrow{Z} \frac{1}{2}$, $|z| > |e^{j\omega_0}|$~~

ROC $|z| > |e^{j\omega_0}|$

$|z| > 1$

$\frac{1}{2} e^{-j\omega_0 n} u[n] \Leftrightarrow \frac{1}{2} \frac{z}{z - e^{-j\omega_0}}, |z| > 1$

$\frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$

$\Big\updownarrow$ ZT

$\frac{1}{2} \frac{z}{z - e^{+j\omega_0}} + \frac{1}{2} \cdot \frac{z}{z - e^{-j\omega_0}}, |z| > 1$

Time reversal property

$x[n] \xleftrightarrow{ZT} X(z)$, ROC=R

$x[-n] \xleftrightarrow{ZT} Z(z^{-1})$, ROC=$\frac{1}{R}$

find ZT $x[n] = u[-n]$

$u[n] \xleftrightarrow{ZT} \frac{z}{z-1}, |z| > 1$

$u[-n] \xleftrightarrow{ZT} \frac{z^{-1}}{z^{-1}-1}, |z^{-1}| > 1$

$$u[-n] \xleftrightarrow{ZT} \frac{1}{1-z}, \quad \frac{1}{|z|} > 1$$

$$1 > |z|$$

~~Properties of~~

Multiplication by n or diff in Z domain

$$x[n] \xleftrightarrow{ZT} X(Z), \quad ROC = R$$

$$n\,x[n] \xleftrightarrow{ZT} -Z \frac{d}{dz}[X(Z)]$$

$$ROC = R$$

$$n^m x[n] \xleftrightarrow{ZT} (-z)^m \frac{d^m}{dz^m}[X(Z)]$$

find ZT of $n a^n u[n]$

$$a^n u[n] \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$n a^n u[n] \xleftrightarrow{ZT} -z \cdot \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$n a^n u[n] \xleftrightarrow{ZT} -z\left(\frac{(z-a)(1) - z}{(z-a)^2}\right)$$

$$n a^n u[n] \xleftrightarrow{Z} \frac{+za}{(z-a)^2} \quad |z| > |a|$$

Convulution

$$x_1(n) \xleftrightarrow{ZT} X_1(Z), \quad ROC = R_1$$

$$x_2(n) \xleftrightarrow{ZT} X_2(Z), \quad ROC = R_2$$

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(Z) \cdot X_2(Z)$$

$$ROC = R_1 \cap R_2$$

eg ⇒ $x_1[n] = \{2, 3, 4\}$ if nothing mentioned then take origin as 1st

$x_2[n] = \{-3, \underset{\uparrow}{4}, 5\}$

find convolution of $x_1(n) * x_2(n)$

$= ?$

$x_1[n] = 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)$

$x_2[n] = -3\delta(n+1) + 4\delta(n) + 5\delta(n-1)$

$X_1(Z) = 2Z^0 + 3Z^{-1} + 4Z^{-2}$, R_1 = entire Z plane except $Z = 0$

$X_2(Z) = -3Z^1 + 3Z^0 + 5Z^{-1}$

$R_2 \Rightarrow (Z = 0, Z = \infty)$ — entire Z plane

$x_1[n] * x_2[n]$

$= X_1(Z) X_2(Z)$

$ROC = R_1 \cap R_2$

~~$= -6z$~~

$= -6z - 1 + 10z^{-1} + 31z^{-2} + 20z^{-3}$

~~ROC = $\mathbb{C} / \{0, \infty$~~

ROC $= \mathbb{C} / \{0\}$

Initial value theorem

$\lim_{n \to 0} x[n] = x(0) = \lim_{z \to \infty} X(z)$

Note $x[n]$ must be causal

$$x[n] = \begin{cases} 0, & n < 0 \\ \neq 0, & n \geq 0 \end{cases}$$

eg If $X(z)$ is z transform of $x[n]$ then Initial value of $x[n]$ is

~~$x[n] =$~~

$$X(z) = \frac{12z^2}{(3z^2 + 2z - 1)}$$

$$x(0) = \lim_{z \to \infty} X(z)$$

$$x(0) = \lim_{z \to \infty} \frac{12z^2}{3z^2 + 2z - 1}$$

$$x(0) = \lim_{z\to\infty} \frac{12}{3 + \cancel{\frac{2}{z}}^{\to 0} - \cancel{\frac{1}{z^2}}^{\to 0}}$$

$$x(0) = 4$$

$\to$ N.C Initial value theorem invalid

$$x[n] = u[-n] + u[n]$$

$$x[0] = ?$$

Final value theorem: If $x[n] \to$ causal and bounded then

$$\lim_{n\to\infty} x[n] = x[\infty] = \lim_{z\to 1} (1-z^{-1})X(z)$$

eg Z transform of a signal is given its final value is

$$C(z) = \frac{z^{-1}(1-z^{-4})}{4(1-z^{-1})^2}$$

$$x[\infty] = \lim_{z\to 1} \left(1-\frac{1}{z}\right) \frac{1}{z} \frac{\left(1-\frac{1}{z^4}\right)}{4\left(1-\frac{1}{z}\right)^2}$$

$$= \lim_{z\to 1} \frac{\cancel{(z-1)}}{4z^2} \frac{(z^4-1)\cancel{z^2}}{z^4(z-1)^{\cancel{2}}}$$

$$= \lim_{z\to 1} \frac{(z^4-1)}{(4)z^4(z-1)} = \frac{1}{4}\left(\frac{z^3}{z^3(z-1)+z^4}\right)$$

$$= \frac{1}{4}$$

general equation

$$X(z) = \frac{k(z-z_1)(z-z_2)\ldots}{(z-p_1)^{m}(z-p_2)^{m}\ldots}$$

$m \to$ multiplicity

for causal signal

$h(n) = 0,$ ~~$n > 0$~~ $n < 0$

anticausal $h(n) = 0, n > 0$

non causal $h(n) \neq 0, n \in (-\infty, \infty)$

for stability $\to$

causal check $n = \infty$ $\to$ if finite then stable

anticausal $n = -\infty$

Non causal check $n = -\infty$ and ∞

Z - transform → Inverse

eg find all possible sequences $x(n)$ associated with z transform

$$X(z) = \frac{5z}{(z-2)(z-3)}$$

→ always divide by z

• $$\frac{X(z)}{z} = \frac{5}{(z-2)(z-3)}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$A = \frac{1}{2-3} = -1$$

$$B = \frac{1}{3-2} = 1$$

$$= \frac{-1}{z-2} + \frac{1}{z-3}$$

$$\frac{X(z)}{z} = 5\left(\frac{1}{z-3} - \frac{1}{z-2}\right)$$

$$\frac{X(z)}{z} = \frac{5}{z-3} - \frac{5}{z-2}$$

$$X(z) = \frac{5z}{z-3} - \frac{5z}{z-2}$$

poles $z = 2, 3$

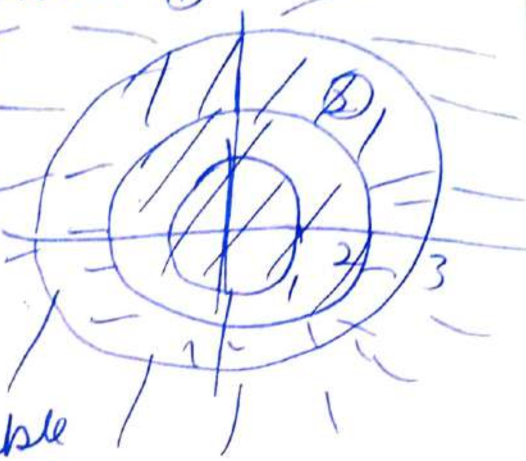
ROC

① $|z| > 3$ → causal, stable

non stable ①

② $2 > |z| > 3$, unstable, non causal

③ $|z| < 2$ → anti causal, stable



$$a^n u[n] \xleftrightarrow{ZT} \frac{z}{z-a}$$

$$\cancel{-5(2^n u[n]) \xleftrightarrow{ZT} \frac{-5z}{z-2}}$$

$$x(n) = -5\,2^n u(n) + 5(3)^n u(n)$$

(eg) $X(z) = \ln(1+az^{-1})$ then $x[n] = ?$

$$x[n] \xleftrightarrow{ZT} X(z)$$

$$x[n] \xleftrightarrow{ZT} \ln(1+az^{-1})$$

$$n x(n) \xleftrightarrow{ZT} -z\frac{d}{dz}X(z)$$

$$n x[n] \xleftrightarrow{ZT} -z \cdot \frac{1}{1+az^{-1}} \cancel{(0+a}$$

$$\times (0 + a(-1)z^{-2})$$

$$n x[n] \xleftrightarrow{ZT} \frac{\cancel{-}z(-a)z^{-2}}{1+az^{-1}} = \frac{az^{-1}}{1+az^{-1}}$$

$$n x[n] \xleftrightarrow{ZT} \frac{a}{z+a}$$

$a^{n-1} u(n-1)$

$$-n x[n] \xleftrightarrow{ZT} \frac{-a}{z-(-a)}$$

$$a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}, \quad |z| > a$$

$$a^{n-1} u(n-1) \xleftrightarrow{ZT} z^{-1} \frac{z}{z-a}, \quad |z| > a$$

$$a^{n-1} u(n-1) \xleftrightarrow{ZT} \frac{1}{z-a}$$

replace a by (−a).

$$a^n u(n-1) \xleftrightarrow{ZT} \frac{a}{z-a}$$

$$(-a)^n u(n-1) \longleftrightarrow \frac{-a}{z+a}$$

$$-1(-a)^n u(n-1) \longleftrightarrow \frac{a}{z+a}$$

$$n x(n) = -1(-a)^n u(n-1)$$

$$x(n) = \frac{-(-a)^n u(n-1)}{n}$$

$$x(n) = \frac{(-1)^{n+1} a^n u(n-1)}{n}$$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Note $\delta(n-k) \xleftrightarrow{ZT} Z^{-k}$

Power series method

find Inverse Z - transform

① $X(Z) = \frac{Z}{2Z^2 - 3Z + 1}, \quad |Z| > \frac{1}{2}$

② $X(Z) = \frac{Z}{2Z^2 - 3Z + 1}, \quad |Z| < \frac{1}{2}$

Causal $\div$ denominator → highest power to lowest power

Anti causal $\div$ Denominator → lowest power to highest power

$X(Z)^2$

$$\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3}$$

$$2z^2 - 3z + 1 \,\big)\, z$$

$$z - \frac{3}{2} + \frac{1}{2}z^{-1}$$

$$- \quad + \quad -$$

$$\cancel{\frac{3}{2}} - \frac{1}{2}z^{-1}$$

$$\cancel{\frac{3}{2}} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}$$

$$- \quad + \quad -$$

$$\cancel{\frac{7}{4}z^{-1}} - \frac{3}{4}z^{-2}$$

$$\cancel{\frac{7}{4}z^{-1}} - \frac{21}{8}z^{-2} + \frac{7}{8}z^{-3}$$

$$- \quad + \quad -$$

$$\frac{15}{8}z^{-2} - \frac{7}{8}z^{-3}$$

$$X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \text{——}$$

$$\delta(n-k) \longleftrightarrow z^{-k}$$

$$x(n) = \frac{1}{2}\delta(n-1) + \frac{3}{4}\delta(n-2) + \frac{7}{8}\delta(n-3) + \text{——}$$

(ii) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$, $|z| < \frac{1}{2}$

Re arrange denominator

$X(z) = \dfrac{z}{\cancel{1+3z^2}}$

$= \dfrac{z}{1 - 3z + 2z^2}$

$$\begin{array}{r|l} & z + 3z^2 + 7z^3 \\ 1 - 3z + 2z^2 & z \\ & z - 3z^2 + 2z^3 \\ & - \quad + \quad - \\ \hline & 3z^2 - 2z^3 \\ & 3z^2 - 9z^3 + 6z^4 \\ & - \quad + \quad - \\ \hline & 7z^3 - 6z^4 \\ & 7z^3 - 21z^4 + 14z^5 \\ & - \quad + \quad - \\ \hline & 15z^4 - 14z^5 \end{array}$$

$X(z) = z + 3z^2 + 7z^3 + \ldots$

$x(n) = \delta(n+1) + 3\,\delta(n+2) + 7\,\delta(n+3) + \ldots$

LTI System =)

Impulse response

input $x(n) \longrightarrow \boxed{\text{System}} \longrightarrow$ output $y(n)$

$h(n)$

step response

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z)\,H(z)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

Transfer function System function

(eg)

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$H(z) = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$= \frac{8z^2}{8z^2 - 6z + 1}$$

$$= \frac{8z^2}{2z(4z-1) \cancel{\,} -1(4z-1)} = \frac{8z^2}{(4z-1)(2z-1)}$$

$$= \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{z}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}}$$

$$z = \frac{1}{4}$$

$$\frac{\cancel{2}\,1/4}{\cancel{\left(-\frac{1}{4}\right)}} = -1 = A, \quad B = \frac{1/\cancel{2}}{\frac{1}{\cancel{4}\,2}} = 2$$

$$\frac{-1}{z-\frac{1}{4}} + \frac{\cancel{3}\,2}{z-\frac{1}{2}} = \frac{H(z)}{z}$$

$$H(z) = \frac{2z}{z-\frac{1}{2}} \cancel{\,} - \frac{z}{z-\frac{1}{4}}$$

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

~~$a^n u(n)$~~ $\xleftarrow{ZT}$

$x(n) = u(n)$

$y(n) = x(n) * h(n)$

$Y(z) = H(z)\, X(z)$

$$Y(z) = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)} \cdot \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)}$$

$$\frac{z^2}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)(z-1)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-1}$$

$z = \frac{1}{4}$

$$\frac{\frac{1}{16}}{\left(-\frac{1}{4}\right)\left(-\frac{3}{4}\right)} = A = \frac{1}{3}$$

$$\frac{\frac{1}{4}}{-\left(\frac{1}{4}\right)\frac{1}{2}} = B = -2$$

$$\frac{1}{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} = \frac{8}{3} = C$$

$$= \frac{1}{3}\left(\frac{1}{z-\frac{1}{4}}\right) + 2\left(\frac{1}{z-\frac{1}{3}}\right) + \frac{8}{3}\left(\frac{1}{z-1}\right)$$

⇒ <u>Discrete the fourier transform</u>

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow \text{Analysis eqn}$$

$X(e^{j\omega})$ ↓ continuous + periodic

$x(n)$ ↓ aperiodic + discrete

C — A

P — D.

D ⟷ P

P ⟶ D

$X(e^{j\omega})$ is a periodic fn in ω with a period 2π

proof → Replace ω by $\omega + 2\pi$

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} (1)$$

convergence condition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

It will converge $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

$\downarrow$

absolutely summable

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

* $$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \omega_0)$$

$x(n) \longrightarrow$ periodic + discrete

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-jk\omega_0 n}$$

$$\omega_0 = \frac{2\pi k}{N}$$

$N \longrightarrow$ period of $x(n)$

Inverse DTFT

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

any interval of 2π

Synthesis eqn

find DTFT of following signals

(1) $x(n) = \delta(n)$

(2) $x(n) = a^n u(n)$, $|a| < 1$

① $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

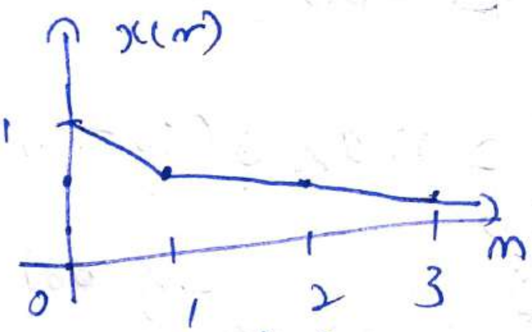
$= \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$

$= e^{-j\omega(0)} = 1$

DTFT $= 1$

$\delta(n) \xleftrightarrow{DTFT} 1$

② $x(n) = a^n u(n)$, $|a| < 1$



$a = \frac{1}{2}$

$\left(\frac{1}{2}\right)^n u(n)$

$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$

$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$

$$x(e^{j\omega}) = (ae^{j\omega})^0 + (ae^{-j\omega})^1 + (ae^{-j\omega})^2$$

<u>GP series</u>

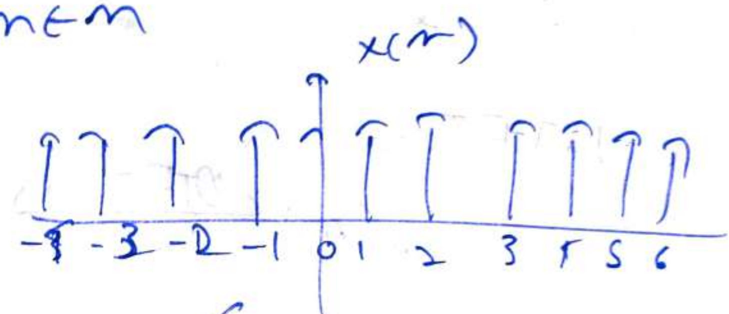
$$= \frac{a}{1-r}$$

$$x(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}, \quad |a|<1$$

Q find DTFT

$x(n) = 1 \ \forall \ n \in m$



~~aperiodic~~ periodic with $N=1$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-\omega_0)$$

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\omega_0 kn}$$

$$a_k = \frac{1}{1}\sum_{n=0}^{0} x(n) e^{-j\omega_0 kn}$$

$$= x(0) e^{-j\omega_0 k(0)}$$

$$a_k = 1$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - \omega_0)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0)$$

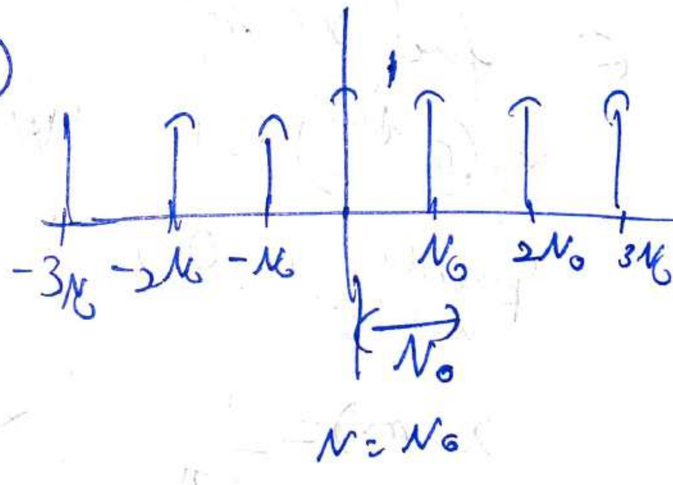
$$\omega_0 = \frac{2\pi k}{N} = 2\pi k$$

$$= \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - 2\pi k)$$

$$1 \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\, \delta(\omega - 2\pi k)$$

Q: DTFT of impulse train

$$x(n) = \sum_{k=-\infty}^{\infty} \delta(n - kN_0)$$



$N = N_0$

$$\omega_0 = \frac{2\pi k}{N_0}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta\left(\omega - \frac{2\pi k}{N_0}\right)$$

$$a_k = \frac{1}{N_0} \sum_{n=0}^{N-1} x(n)\, e^{-j\omega_0 n k}$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \delta(n)\, e^{-j\omega_0 n k}$$

$$= \frac{1}{N_0}(1)$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{1}{N_0} 2\pi\, \delta\left(\omega - \frac{2\pi k}{N_0}\right)$$

$$\sum_{k=-\infty}^{\infty} \delta(n-kN_0) \quad \text{OR} \quad 1 \xleftrightarrow{DTFT}$$

$N = N_0$

$$\sum_{k=-\infty}^{\infty} \frac{2\pi\, \delta\left(\omega - \frac{2\pi k}{N_0}\right)}{N_0}$$

$$\sum_{k=-\infty}^{\infty} \frac{2\pi}{N_0}\, \delta\left(\omega - \frac{2\pi k}{N_0}\right)$$

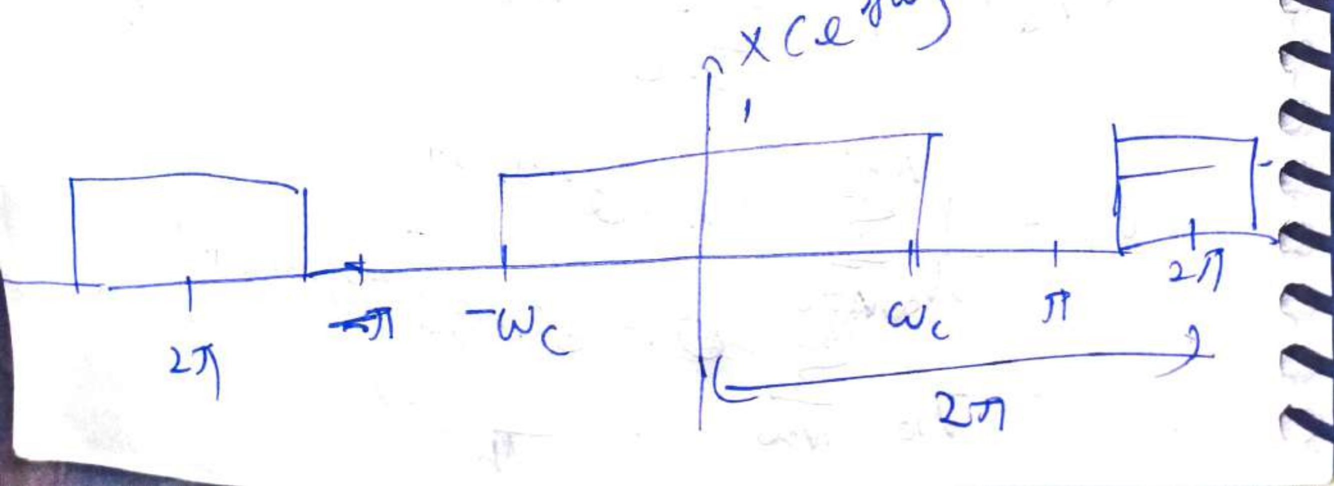
general formula

Q find $x(n)$

$X(e^{j\omega}) = 1, \ |\omega| \le \omega_c$

periodic with period 2π $\quad 0, \ \omega_c < |\omega| \le \pi$

$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$



$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi}\left\{ \int_{-\pi}^{-\omega_c} 0\, \cancel{e^{j\omega n}}^{\,0}\, d\omega + \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega \right.$$

$$\left. + \int_{\omega_c}^{0} 0\, \cancel{e^{j\omega n}}^{\,0}\, d\omega \right]$$

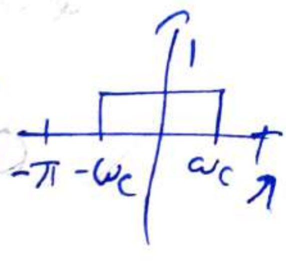
$$x(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]\frac{1}{n}$$

$$= \frac{1}{\pi n}\sin \omega_c n$$

~~Sin(ωc~~ $\dfrac{\sin(\omega_c n)}{\pi n} \xleftrightarrow{DTFT}$



$$\approx 1\, \mathrm{rect}\left(\frac{\omega}{2\omega_c}\right)$$

property

(1) linearity : If $x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

$$a x_1(n) + b x_2(n) \xleftrightarrow{DTFT} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

Example $\Rightarrow$ $x_1(n) = 4 a^n u(n)$, $|a|<1$

and $x_2(n) = 5\delta(n)$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

+

$$\delta(n) \xleftrightarrow{DTFT} 1$$

$$X_1(e^{j\omega}) = \frac{4}{1-ae^{-j\omega}}$$

$$X_2(e^{j\omega}) = 5$$

$$4a^n u(n) + 5\delta(n) \xleftrightarrow{DTFT} \frac{4}{1-ae^{-j\omega}} + 5$$

(2) Time shifting $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} X(e^{j\omega})$

$x(n+n_0) \xleftrightarrow{DTFT} e^{j\omega n_0} X(e^{j\omega})$

Ex $\Rightarrow$ $x(n) = \delta(n-2)$

$\delta(n) \xleftrightarrow{DTFT} 1$

$\delta(n-2) \xleftrightarrow{DTFT} e^{-2j\omega}.(1)$

(3) frequency shifting $\Rightarrow$

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$e^{j\omega_0 n} x(n) \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$

$e^{-j\omega_0 n} x(n) \xleftrightarrow{DTFT} (X e^{+j(\omega+\omega_0)})$

eg $x(n) = \sin(\omega_0 n)$

$$\Rightarrow x(n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

$$x(n) = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]$$

$$1 \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi k)$$

$$e^{j\omega_0 n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi k)$$

$-$

$$e^{-j\omega_0 n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega + \omega_0 - 2\pi k)$$

$2j$

3) Time ~~scaling~~ scaling

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega}) \rightarrow 2\pi$$

$$x\left(\frac{n}{k}\right) \xleftrightarrow{DTFT} X(e^{jk\omega})$$

Time period $= \frac{2\pi}{k}$

eg $x(n) = \{1, 2, 3\}$ then DTFT of
$x(n/2) = ?$

$x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-3)$

$X(e^{j\omega}) = 1 + 2e^{-j\omega(1)} + 3e^{-j\omega(3)}$

$x\left(\frac{n}{2}\right) \xleftrightarrow{DTFT} X(e^{j(2\omega)})$

$(Xe^{j(2\omega)}) = 1 + 2e^{-j(2\omega)} + 3e^{-3j(2\omega)}$

Time reversal

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$x(-n) \xleftrightarrow{DTFT} X(e^{j(-\omega)})$

- if $x(n) = \{1, 2, 3\}$
$\uparrow_{n=0} \; \uparrow_{n=1} \; \uparrow_{n=2}$

OR $x(n) = 1\delta(n) + 2\delta(n-1) + 3\delta(n-2)$

$X(e^{j\omega}) = 1 + 2e^{-j\omega(1)} + 3e^{-j\omega(2)}$

$X(e^{-j\omega}) = 1 + 2e^{-j(-1\omega)} + 3e^{-j\omega(-2)}$

$= 1 + 2e^{j\omega} + 3e^{2j\omega}$

(6) Accumulation

$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

$\cancel{\sum_{n=-\infty}^{n} x(m)}$

$$\sum_{m=-\infty}^{n} x(m) \xleftrightarrow{DTFT} \frac{X(e^{j\omega})}{1-e^{-j\omega}} + \pi X(e^{j0}) \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

eg DTFT of $x(n) = u(n)$

$$u(n) = \sum_{m=-\infty}^{n} \delta(m)$$

$x(m) = \delta(m)$

$X(e^{j\omega}) = 1$

$X(e^{j0}) = 1$

$$\sum_{m=-\infty}^{n} \delta(m) \xleftrightarrow{DTFT} \frac{1}{1-e^{-j\omega}} + \pi(1) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

⑦ Differentiation in frequency domain

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

$$n\,x(n) \xleftrightarrow{DTFT} j\frac{d}{d\omega}\left(X(e^{j\omega})\right)$$

Ex ⇒ $x(n) = a^n u(n)$, $|a|<1$

then DTFT of $n\,x(n)$

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\omega}}$$

$$n\,a^n u(n) \xleftrightarrow{DTFT} j\frac{d}{d\omega}\left(\frac{1}{1-ae^{-j\omega}}\right)$$

$$= j\left[\frac{0-(0+je^{-j\omega})}{(1-ae^{-j\omega})^2}\right]$$

$$n\,a^n u(n) \longrightarrow j^2\frac{(-ae^{-j\omega})}{(1-ae^{-j\omega})^2}$$

$$n\,a^n u(n) \xrightarrow{DTFT} \frac{ae^{-j\omega}}{(1-ae^{-j\omega})^2}$$

(3) modulation

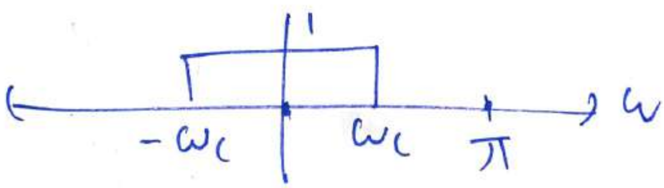
$$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

$$x_1(n)\,x_2(n) \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[X_1(e^{j\omega}) * X_2(e^{j\omega})\right]$$

for $\quad x(n) = \underbrace{\left(\frac{\sin(\omega_c n)}{\pi n}\right)}_{x_1(n)} \underbrace{\left(\frac{\sin(\omega_c n)}{\pi n}\right)}_{x_2(n)}$

$$\Big\updownarrow \text{DTFT}$$



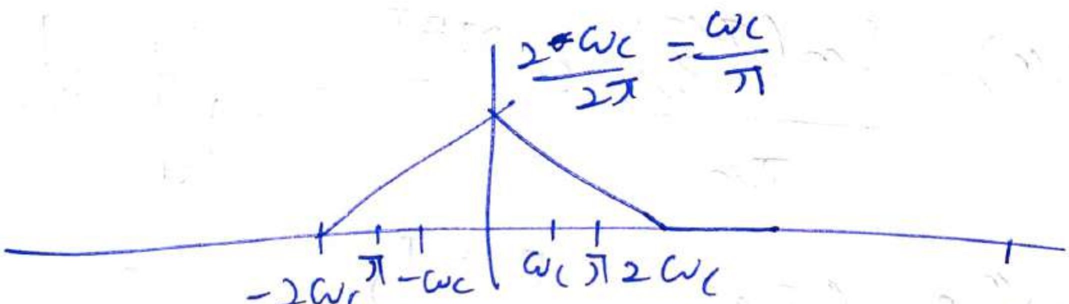
$$X_1(e^{j\omega}) = u(\omega+\omega_c) - u(\omega-\omega_c)$$

$$\frac{\sin(\omega_c n)}{\pi n} \cdot \frac{\sin(\omega_c n)}{\pi n} \longleftrightarrow \left[u(\omega+\omega_c) - u(\omega-\omega_c)\right] * \left[u(\omega+\omega_c) - u(\omega-\omega_c)\right]$$

$u(t-a) * u(t-b) = r(t-a-b)$

$$\left(\frac{\sin(\omega_c n)}{\pi n}\right)^2 \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[r(\omega+2\omega_c) - r(\omega) - r(\omega) + r(\omega-2\omega_c)\right.$$

$$\left[\frac{\sin(\omega_c n)}{\pi n}\right]^2 \xleftrightarrow{DTFT} \frac{1}{2\pi}\left[r(\omega+2\omega_c) - 2r(\omega) + r(\omega-2\omega_c)\right]$$



Convolution

$$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$$

$$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

$$x_1(n) * x_2(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})\, X_2(e^{j\omega})$$

eg $x(n) = (-1)^n$ and $h(n) = \frac{\sin\left(\frac{\pi n}{3}\right)}{\pi n}$

then $y(n) = x(n) * h(n)$

$$\boxed{x(n) = (-1)^n} = e^{-j\pi n}$$

$$\cos(\pi n) - j\,\cancel{\sin(\pi n)}^{0}$$

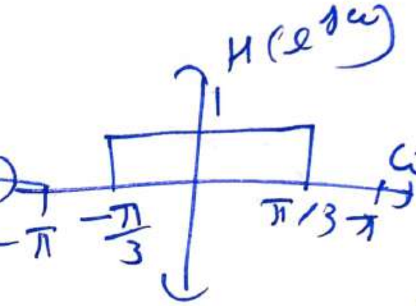
$$e^{-j\omega_0 n} x(n) \longleftrightarrow X(e^{j(\omega + \omega_0)})$$

$$1 \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi \, \delta(\omega - 2\pi k)$$

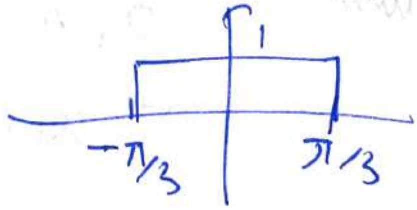
$$e^{-j\pi n} \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi \, \delta(\omega + \pi - 2\pi k)$$

$$(-1)^n \xleftrightarrow{DTFT} \sum_{k=-\infty}^{\infty} 2\pi \, \delta(\omega + \pi - 2\pi k)$$

$$h(n) = \frac{\sin\left(\frac{\pi n}{3}\right)}{\pi n} \xleftrightarrow{DTFT}$$



$$(-1)^n * \frac{\sin\left(\frac{\pi}{3} n\right)}{\pi n} \xleftrightarrow{DTFT}$$



$$(-1)^n * \frac{\sin\left(\frac{\pi}{3} n\right)}{\pi n} \xleftrightarrow{DTFT} 0$$

Parseval and other properties later

Parseval's thm →

$x_1(n) \xleftrightarrow{FT} X(e^{j\omega})$

$x_2(n) \xleftrightarrow{FT} X(e^{j\omega})$

$$\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$$

If $x_1(n) = x_2(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(e^{j\omega})|^2\, d\omega$$

↓

energy of the signal

Convolution thm ⇒

~~$x_1(n) = x_2(n) = x(n)$~~

~~then $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi}$~~

$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$

$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$

$r_{x_1x_2}(n) \xleftrightarrow{DTFT} S_{x_1x_2}(\omega) = X_1(\omega)\, X_2(-\omega)$

$$r_{x_1x_2}(m) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(k-m)$$

A D
↓ ↓
C P

$$S_{x_1x_2} = \sum_{m=-\infty}^{\infty} r_{x_1x_2}(m) e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1(k)x_2(k-m) \right) \cdot e^{-j\omega m}$$

let $k-m = m$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{m=-\infty}^{\infty} x_2(m) e^{-j\omega(k-m)}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) e^{-j\omega k} \sum_{m=-\infty}^{\infty} x_2(m) e^{j\omega m}$$

$$= X_1(\omega) X_2(-\omega)$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \qquad 0 \le k \le N-1$$

$X(k)$ ↓ P + D, $e^{-j\frac{2\pi}{N}kn}$ ↓ P + D

let $e^{-j\frac{2\pi}{N}} = W_N$ → twiddle factor

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}$$

$$X(K) = x(0)W_N^{K\cdot 0} + x(1)W_N^{K\cdot 1} + x(2)W_N^{K\cdot 2} + \cdots x(N-1)W_N^{K(N-1)}$$

$$X(0) = x(0)W_N^{0\cdot 0} + x(1)W_N^{0\cdot 1} + x(2)W_N^{0\cdot 2} + \cdots + x(N-1)W_N^{0(N-1)}$$

$$X(1) = x(0)W_N^{1\cdot 0} + x(1)W_N^{1\cdot 1} + x(2)W_N^{1\cdot 2} + \cdots + x(N-1)W_N^{1\cdot(N-1)}$$

$$X(2) = x(0)W_N^{2\cdot 0} + x(1)W_N^{2\cdot 1} + x(2)W_N^{2\cdot 2} + \cdots x(N-1)W_N^{2\cdot(N-1)}$$

$$\vdots$$

$$X(N-1) = x(0)W_N^{(N-1)(0)} + x(1)W_N^{(N-1)\cdot 1} + x(2)W_N^{(N-1)2} + \cdots + x(N-1)W_N^{(N-1)(N-1)}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & W_N^{0\cdot 2} & \cdots & W_N^{0(N-1)} \\ W_N^{1\cdot 0} & W_N^{1\times 1} & W_N^{1\cdot 2} & \cdots & W_N^{1(N-1)} \\ W_N^{2\cdot 0} & W_N^{2\times 1} & W_N^{2\cdot 2} & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & W_N^{(N-1)2} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$X = W_N \cdot x$$

Tewiddle factor

$$W_N = e^{-j\frac{2\pi}{N}} \quad W_2 = e^{-j\pi}$$

$$W_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & e^{-j\pi} \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & \cos\pi - j\sin\pi \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$w_3 = e^{-j\frac{2\pi}{3}} = \cos\left(\frac{-2\pi}{3}\right) - j\sin\left(\frac{-2\pi}{3}\right)$$
$$-\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$W_3 = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{4\pi}{3}} \\ 1 & e^{-j\frac{4\pi}{3}} & e^{-j\frac{8\pi}{3}} \end{bmatrix}$$

$2\pi + \frac{2\pi}{3}$

$\frac{6\pi + 2\pi}{3}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}$$

$2\pi + \frac{\pi}{2}$

$4\pi + \frac{\pi}{2}$

Twiddle factor

$W_N = e^{-j\frac{2\pi}{N}}$ $\quad N = 4$

$W_N = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$ $\quad W_4 = 0 - j = -j$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

$W_4^2 = -1 - 0$
$= -1$

$W_4^3 = 0 + j = j$

$W_4^4 = 1 + 0$

$W_4^5 = 0 - j$

$W_4^6 = -1.$

$W_4^9 = 0 - j$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

$$x(n) \xleftrightarrow[DFT]{4\ point} X(k)$$

(eg) find the 4 point DFT

$x(n) = \{0, 1, 2, 3\}$

$X = W_N x$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$-1+2-3$

$-j+2$

$$= \begin{bmatrix} 6 \\ -j-2+3j \\ -2 \\ j-2-3j \end{bmatrix}$$

eg 3 point DFT

$$x(n) = \begin{cases} 1/2 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \{\underset{-1}{\tfrac{1}{2}}, \underset{\underset{0}{\uparrow}}{\tfrac{1}{2}}, \underset{1}{\tfrac{1}{2}}\}$$

$$X(k) = \sum_{n=-1}^{1} x(n)\, e^{-j\frac{2\pi}{N} k n} \quad , -1 \le k \le +1$$

$$= x(-1)\, e^{-j\frac{2\pi}{N} k(-1)} + x(0)\, e^{-j\frac{2\pi}{N} k(0)} + x(1)\, e^{-j\frac{2\pi}{N} k(1)}$$

$$= \frac{1}{2} e^{+j\frac{2\pi}{N} k} + \frac{1}{2} + \frac{1}{2} e^{-j\frac{2\pi}{N} k}$$

$$X(k) = \cos\left(\frac{2\pi}{N} k\right) + \frac{1}{2}$$

$$X(-1) = \cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

$$X(0) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$X(1) = -\frac{1}{2} + \frac{1}{2} = 0$$

$$X(k) = \left\{ 0, \frac{3}{2}, 0 \right\}$$

Properties If $x(n)$ is periodic with ~~N → total~~ periodicity of N samples then N point DFT, $X(k)$ is also periodic with periodicity of N samples

let $x(n) = x(n+K)$, $\forall n$

$X(k) = X(k+N)$, $\forall k$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$k \longrightarrow k + N$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(k+N)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}\, \cancel{e^{-j2\pi n}}^{\,1}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$X(k+N) = X(k)$$

linearity

~~$x(n) = \{1, 0, 2, 3\}$ (4)~~

$x_1(n)$ and $x_2(n)$ are periodic with period "N"

$$x_1(n) \longleftrightarrow X_1(k)$$

$$x_2(n) \longleftrightarrow X_2(k)$$

$$a\,x_1(n) + b\,x_2(n) \longleftrightarrow a\,X_1(k) + b\,X_2(k)$$

Time reversal

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$(x(-n))_{N} \xleftrightarrow{DFT} (X(-k))_{N}$$

eg $x(n) = \{1, 2, 3, 4\}$ $N=4$

4 DFT

$X(k) = \{10, -2+2j, -2, -2-2j\}$

$N/2$

$x(-n) = \{1, 4, 3, 2\}$

$X(-k) = \{10, -2-2j, -2, -2+2j\}$

Circular time shift

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$(x(n-n_0))_{MOD\,N} \xleftrightarrow{DFT} \left(e^{-j\frac{2\pi}{N}n_0 k} X(k)\right)_{MOD\,N}$$

$$(x(n+n_0)) \xrightarrow{DFT} \left(e^{j\frac{2\pi}{N}n_0 k} X(k)\right)_{MOD\,N}$$

eg $x(n) = \{1, 2, 3, 4\}$ $\xleftrightarrow{4\ point\ DFT}$

$x(n-2) = ?$ $\quad X(k) = \{10, -2+2j, -2, -2-2j\}$

$(x(n-2))_{MOD\,N} = \{3, 4, 1, 2\}$

$\longleftrightarrow e^{-j\frac{2\pi}{4}2k} \{10, -2+2j, -2, -2-2j\}$

$e^{-j\pi k} = (-1)^{k}$

Circular frequency shift

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$e^{j\frac{2\pi}{N}k_0 n} x(n) \longleftrightarrow (X(k-k_0))_{MOD\ N}$$

$$e^{-j\frac{2\pi}{N}k_0 n} x(n) \longleftrightarrow (X(k+k_0))_{MOD\ N}$$

Complex conjugate

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$x^*(n) \xleftrightarrow{DFT} X^*(N-k)$$

Note If $x(n)$ is real

$$x(n) = x^*(n)$$

$$X(k) = X^*(N-k)$$

$\frac{N}{2}$

eg $x(n) = \{1, 2, 3, 4\}$, $X(k) = \{10, -2+2j, -2, -2-2j\}$

real

Real + comp conj

$$X(0) = X^*(4-0)$$

$$X^*(0) = X^*(4)$$

$$X(4) = X^*(0)$$

$$X(1) = X^*(3)$$

$$X(3) = X^*(1)$$

Circular convolution : column method

$x(n) = \{1, 2, 3, 4\}$ $\quad h(n) = \{0, 1, 2, 1\}$

$x(n) \; N \; h(n) = ?$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 12 \\ 8 \\ 8 \end{bmatrix}$$

$x(n) \; N \; h(n) = \{12, 12, 8, 8\}$

Shortcut

	1	2	3	4	
0	0	0	0	0	
1	4	1	2	3	4
2	6	8	2	4	6 8
1	2	3	4	1	2 3 4
	12	12	8	8	

Circular conv : Row method

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 12 \\ 8 \\ 8 \end{bmatrix}$$

Circular conv using formula $\Rightarrow$

$x(n) = \{1, 2, 3, 4\}$

$h(n) = \{0, 1, 2, 1\}$ $\quad \frac{N}{2} = 2$

$x(n) \; N \; h(N) = ?$

$$z(n) = x(n) \; N \circledast h(N)$$

$$= \sum_{m=0}^{N-1} x(m)\, h((n-m))_{\text{mod } N}$$

$$z(0) = \sum_{m=0}^{3} x(m)\, h((-m))_{\text{mod } 4}$$

$x(m) = \{1, 2, 3, 4\}$

$h(-m) = \{0, 1, 2, 1\}$

$z(0) = 0 + 2 + 6 + 4$

$= 12$

$$z(1) = \sum_{m=0}^{3} x(m)\, h((1-m))_{\text{MOD } 4}$$

$$h(1-m) = h(-m+1)$$
$$= h(-(m-1)) \rightarrow \text{right shift}$$

$$h(-m) = \{0, 1, 2, 1\}$$

$$h(1-m) = \{1, 0, 1, 2\}$$

$$x(m) = \cancel{\{1, 0, 1, 2\}}\ \{1, 2, 3, 4\}$$

$$z(1) = 1 + 0 + 3 + 8 = 12$$

$$z(2) = \sum_{m=0}^{3} x(m)\, h(2-m)$$

$$x(m) = \{1, 2, 3, 4\}$$

$$h(2-m) = h(-(m-2))$$
$$= \{2, 1, 0, 1\}$$

$$z(2) = 2 + 2 + 0 + 4$$
$$= 8$$

$$z(3) = \sum_{m=0}^{3} x(m)\, h((3-m))_{\text{MOD } 4}$$

$$h(3-m) = \{2, 1, 0, 1\}$$
$$x(m) = \{1, 2, 3, 4\}$$
$$z(3) = 8$$

$z(n) = \{12, 12, 8, 8\}$

Note
- $x(n) \longrightarrow N_1$ elements
  $h(n) \longrightarrow N_2$ elements

linear conv:	Circular conv
$N_1 + N_2 - 1$	$\max(N_1, N_2)$

properties of DFT ⇒

$$x(n) \xleftrightarrow{DFT} X(k)$$
$$h(n) \xleftrightarrow{DFT} H(k)$$

$$x(n) \, (N) \, h(n) \xleftrightarrow{DFT} X(k) * H(k)$$

$x(n) = \{1, 2, 3, 4\}$ | $X(K) = \{10, -2+2j, -2, -2-2j\}$

$y(n) = \{0, 1, 2, 1\}$ | $Y(K) = \{4, -2, 0, -2\}$

$$x(n) * h(n) \xleftrightarrow{DFT} X(k) H(k)$$

DFT of $x(n) * y(n) = ?$

	1	2	3	4
0	0	0	0	0
1	1	2	3	4
2	2	4	6	8
1	1	2	3	4

	1	2	3	4
0	0	0	0	0
1	4	1	2	3
2	3	4	1	2
1	2	3	4	1

	1	2	3	4	
0	0	0	0	0	
1	4	1	2	3	4
2	6	8	2	4	
1	2	3	1	1	2 3 4
	12	12	8	8	

$$X(k)\,Y(k) = \{40, 4-4j, 0, 4+4j\}$$

Modulation

$$x(n) \xleftrightarrow{N\,DFT} X(k)$$

$$h(n) \xleftrightarrow{N\,DFT} Y(k)$$

$$x(n)\,h(n) \xleftrightarrow{DFT} \frac{1}{N}\{X(k) * Y(k)\}$$

Parseval's theorem

$$x(n) \xleftrightarrow{DFT} X(k)$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N} |X(k)|^2$$

Q $N=4$ point DFT of a sequence $x(n)$ is

$X[k] = \{3, 2+j, 1, 2-j\}$

the value of $\sum_{n=0}^{3} |x(n)|^2 = ?$

$$= \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$= \frac{1}{4} \left[9 + \cancel{3+4j+1} + 3-2j \right]$$

$$= \cancel{\frac{1}{4} \left[16 + 2j \right]}$$

$$=$$

$$= \frac{1}{4} \left[9+5+1+5 \right]$$

$$= 5$$

~~Sungram~~

Laplace transform

→ It is used to solve differential equation

Definition If $x(t)$ is cont time signal →

$$LT[x(t)] = X(S) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

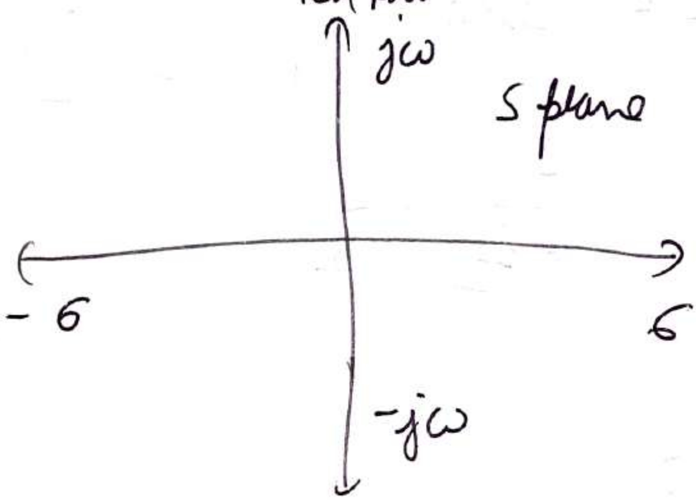
~~Laplace tr~~

ROC: The set of values of 's' for which laplace of $x(t)$ exist is called ROC

$S \rightarrow$ a complex no

$S = \sigma + j\omega \rightarrow$ natural freq

Real part



LT of ~~δc~~ $\delta(t)$ ~~and~~ $u(t)$ and $r(t)$

(i) LT of $\delta(t)$

$$X(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt \qquad e^{-st} \text{ at } t=0$$

$$X(s) = 1$$

$$\delta(t) \xleftrightarrow{LT} 1$$

(ii) $x(t) = u(t)$

$$LT \text{ of } [x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$\Rightarrow \int_{0}^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

ROC $\Rightarrow$ $s>0$ $= \left[\frac{e^{-\infty} - e^{0}}{-s}\right]$

$$= \left[\frac{\cancel{-}1}{\cancel{-}s}\right] = \frac{1}{s}$$

$\mathrm{Re}(s) > 0$

(ii) $x(t) = r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$r(t) = t\, u(t)$$

So $\Rightarrow$ $\int_{-\infty}^{\infty} t\, u(t)\, e^{-st}\, dt$

$$= \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \left(e^{-st}\,dt\right)$$

$$= \left[\frac{t \cdot e^{-st}}{-s}\right]_0^\infty + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty \quad \cancel{R\ s > 0}$$

$$= \left[\frac{\infty e^{-\infty} - 0}{-s}\right] + \frac{1}{s}\left[\frac{e^{-\infty} - e^{0}}{-s}\right]$$

$$= 0 + \frac{1}{s^2} = \frac{+1}{s^2}$$

ROC $Re(s) > 0$

$$x(t) \longleftrightarrow \frac{1}{s^2}\ ; \quad \text{ROC } Re(s) > 0$$

~~LT of $e^{-at}u(t)$ and $-e^{-at}u(-t)$~~

LT of $e^{-at}u(t)$ and $-e^{-at}u(-t)$

① LT of $e^{-at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(a+s)t}\, dt$$

$$\Rightarrow \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty}$$

$$= \left[\frac{e^{-(a+s)\infty} - e^{-(a+s)(0)}}{-(a+s)}\right]$$

$$X(s) = \frac{1}{a+s}$$

$$\text{ROC} \Rightarrow a+s > 0$$
$$s > -a$$
$$\text{Re}(s) > -a$$

(ii) $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-at} e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$= -\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0}$$

$$L[e^{-at}u(t)] = \frac{1}{s+a}$$

$$L[e^{+at}u(t)] = \frac{1}{s-a}$$

$$L[-e^{-at}u(-t)] = \frac{1}{s+a}$$

$$L[-e^{at}u(-t)] = \frac{1}{s-a}$$

$$= -\left[\frac{-e^{(s+a)\infty} + e^{0}}{-(s+a)}\right]$$

$$X(s) = +\frac{1}{s+a} \qquad ROC: \Rightarrow \begin{array}{c} s+a<0 \\ s<-a \\ Re(s)<-a \end{array}$$

LT of $A\sin\omega t\, u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$= \cancel{\int_{-\infty}^{\infty} A\sin\omega t}$$

$$= \int_{0}^{\infty} A\sin\omega t\, e^{-st}\,dt$$

$$= \cancel{\int_{0}^{\infty} A\left(e^{-j\omega t} + e^{+j\omega t}\right.}$$

$$= \int_{0}^{\infty} A\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) e^{-st}\,\cancel{\sin\omega t}\,dt$$

$$= \frac{A}{2j}\int_{0}^{\infty} e^{j\omega t}\, e^{-st}\,dt$$

$$- \frac{A}{2j} \int_0^{\infty} e^{-j\omega t} e^{-st} dt$$

$$X(s) = \frac{A}{2j} \left[\int_0^{\infty} e^{j\omega t} e^{-st} dt - \int_0^{\infty} e^{-j\omega t} e^{-st} dt \right]$$

$$= \frac{A}{2j} \left[\int_0^{\infty} e^{(j\omega t - st)} dt - \int_0^{\infty} e^{-(j\omega t + st)} dt \right]$$

$$= \frac{A}{2j} \left[\left[\frac{e^{(j\omega - s)t}}{j\omega - s} \right]_0^{\infty} - \left[\frac{e^{-(j\omega + s)t}}{-(j\omega + s)} \right]_0^{\infty} \right]$$

$$= \frac{A}{2j} \left[\left(\frac{0 - 1}{j\omega - s} \right) - \left[\frac{0 - 1}{-(j\omega + s)} \right] \right]$$

$$= \frac{-A}{2j} \left[\frac{1}{j\omega - s} + \frac{1}{j\omega + s} \right]$$

$$= \frac{-A}{2j} \left[\frac{j\omega + \cancel{s} + j\omega - \cancel{s}}{(j\omega)^2 - (s)^2} \right]$$

$$= \frac{-A}{\cancel{2}\cancel{j}} \left[\frac{\cancel{2}\cancel{j}\omega}{-\omega^2 - s^2} \right]$$

$$= \frac{A\omega}{\omega^2 + s^2}$$

ROC $\Rightarrow$

$j\omega - s < 0$ | $j\omega + s > 0$

$s > j\omega$ | $s > -j\omega$

LT of $A\cos(\omega t)\, u(t)$

$L(A\cos\omega t\, u(t))$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= A\int_{0}^{\infty} \cos\omega t\, e^{-st}\, dt$$

$$= A\int_{0}^{\infty} \left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) e^{-st}\, dt$$

$$= \frac{A}{2}\int_{0}^{\infty} e^{(j\omega t - st)}\, dt + \frac{A}{2}\int_{0}^{\infty} e^{-(j\omega + s)t}\, dt$$

$$= \frac{A}{2}\, \frac{e^{j\omega t - st}}{(j\omega - s)}\Bigg|_{0}^{\infty} + \frac{A}{2}\, \frac{e^{-(j\omega + s)t}}{-(j\omega + s)}\Bigg|_{0}^{\infty}$$

$$= \frac{A}{2}\, \frac{1}{j\omega - s}\left[0 - 1\right] - \frac{A}{(j\omega + s)2}\left[0 - 1\right]$$

$$= -\frac{A}{2}\left[\frac{1}{j\omega - s} - \frac{1}{j\omega + s}\right] =$$

$$\cancel{= -\frac{A}{2}\left[\frac{2j\omega}{(j\omega)^2 - s^2}\right] = \frac{-Aj\omega}{-\omega^2 + s^2}}$$

$$= \frac{A}{2}\left[\frac{1}{j\omega+s} - \frac{1}{j\omega-s}\right]$$

$$= \frac{A}{2}\left(\frac{j\omega - s - j\omega - s}{(j\omega)^2 - s^2}\right)$$

$$= \frac{\cancel{-2}\,A}{\cancel{2}}\,\frac{s}{-(\omega^2+s^2)}$$

$$X(s) = \frac{As}{\omega^2+s^2}$$

LT of $t^n u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$x(t) = t^n$$

$$X(s) = \int_{-\infty}^{\infty} t^n e^{-st}\, dt$$

let $st = x \Rightarrow t = \frac{x}{s}$

$$dt = \frac{dx}{s}$$

$$= \int_{-\infty}^{\infty} \left(\frac{x}{s}\right)^n e^{-x} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_{-\infty}^{\infty} x^n e^{-x} ds$$

$$= \frac{1}{s^{n+1}} \int_{-\infty}^{\infty} x^{(n+1)-1} e^{-x} ds \qquad \left\{ \underbrace{\Gamma(n+1) = \int_{-\infty}^{\infty} x^{(n+1)-1} e^{-s} ds}_{\text{gamma fn.}} \right.$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} = \begin{cases} \frac{\Gamma(n+1)}{s^{n+1}}, & n \notin Z \\ \frac{n!}{s^{n+1}}, & n \in Z \end{cases}$$

Properties of LT

① Linearity property

$x_1(t) \xleftrightarrow{LT} X_1(s)$ ROC : R_1

$x_2(t) \xleftrightarrow{LT} X_2(s)$ ROC : R_2

$$a x_1(t) + b x_2(t) \xleftrightarrow{LT} a X_1(s) + b X_2(s)$$

$$ROC \Rightarrow R_1 \cap R_2$$

Q $x(t) = e^{-2t} u(t) + \left(-e^{2t} u(-t)\right)$

$$= \frac{1}{s+2} - \frac{1}{s-2} \qquad -2 < \sigma < 2$$

$X(s) = X_1(s) + X_2(s)$

② Time scaling

If $x(t) \xleftrightarrow{LT} X(s)$ ROC: R

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$ ROC: $\frac{R}{a}$

Q If $x(t) \xleftrightarrow{LT} \frac{2s}{s^2+2}$

then $L(x(3t))$?

$$L[x(3t)] = \frac{1}{3} \frac{2\frac{s}{3}}{\left(\frac{s}{3}\right)^2+2}$$

$$= \frac{2}{9} \frac{9s}{\frac{s^2+18}{9}}$$

$$= \frac{2s}{s^2+18}$$

③ Time shifting

If $x(t) \xleftrightarrow{LT} X(s)$

$x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$

$x(t+t_0) \xleftrightarrow{LT} e^{st_0} X(s)$

④ ~~Fr~~

Q $x(t) = \cos\left(t - \frac{2\pi}{7}\right)$

$$\cos t \xleftrightarrow{LT} \frac{s}{s^2+1}$$

$$\cos\left(t - \frac{2\pi}{7}\right) \xleftrightarrow{LT} \frac{e^{-s\frac{2\pi}{7}} s}{s^2+1}$$

④ frequency shifting

$$x(t) \xleftrightarrow{LT} X(s) \quad ROC: R$$

$$e^{s_0 t} x(t) \xrightarrow{LT} X(s - s_0) \quad ROC: R + Re(s_0)$$

$$e^{-s_0 t} x(t) \xleftrightarrow{LT} X(s + s_0)$$

Q $x(t) = e^{-2t} u(t+1)$

$$u(t) \xleftrightarrow{LT} \frac{1}{s}$$

$$u(t+1) \longleftrightarrow \frac{e^{s}}{s}$$

$$e^{-2t} u(t+1) \longleftrightarrow \frac{e^{(s+2)}}{(s+2)}$$

Diff in time domain

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\frac{d}{dt} x(t) \xleftrightarrow{LT} s \cdot X(s)$$

$$\frac{d^n}{dt^n} x(t) \xleftrightarrow{LT} s^n X(s)$$

Q If $x(t) = \delta(t) - \delta(t-2)$ find LT

$\frac{d^2x(t)}{dt^2}$

$$X(s) = [1 - e^{-2s} \cdot 1]$$

~~$X(s) = [1-$~~

$$\frac{d^2x(t)}{dt^2} = s^2[1-e^{-2s}]$$

⑥ Diff in S – domain

$$x(t) \xrightarrow{LT} X(s) \quad ROC = R$$

$$t\,x(t) \xleftrightarrow{LT} \frac{d\,X(s)}{ds} \quad ROC: R$$

$$t^n x(t) \xleftrightarrow{LT} \frac{d^n X(s)}{ds^n} (-1)^n$$

Q $x(t) = \sin at$

$$L(t\,x(t)) = L[t \sin at]$$

$$L[\sin at]$$

$$= \frac{a}{s^2+a^2}$$

$$= -\frac{d}{ds}\frac{a}{s^2+a^2} = -a\left(\frac{-2s}{(s^2+a^2)^2}\right)$$

Integration in time domain

Q If $x(t) = 5\cos 6t$ then find

$$\int_{-\infty}^{t} x(\tau)\,d\tau = ?$$

$$x(t) \xleftrightarrow{LT} X(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$$

Q If $x(t) = 5\cos 6t$ then find

$$\int_{-\infty}^{t} x(\tau)\,d\tau = ?$$

$$X(s) = \frac{5s}{s^2 + 36}$$

Integration in s domain (Division by t)

$$x(t) \xleftrightarrow{LT} X(s) \quad ROC: R$$

$$\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s)\,ds$$

Q LT of $\frac{\sin \pi t}{\pi t} u(t) = ?$

$\underline{\sin \pi t}$ $LT[\sin \pi t] = \frac{\pi}{\pi^2 + s^2}$

so $LT\left[\frac{\sin \pi t}{t}\right] = \int_s^{\infty} \frac{\pi}{\pi^2 + s^2} ds$

$= \left[\frac{\cancel{\pi}}{\cancel{\pi}} \tan^{-1}\left(\frac{s}{\pi}\right)\right]_s^{\infty}$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{\pi}\right)$

so $X(s) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{s}{\pi}$

<u>Convolution</u> $\Rightarrow$

$x_1(t) \xleftrightarrow{LT} X_1(s)$

$x_2(t) \xleftrightarrow{LT} X_2(s)$

$x_1(t) * x_2(t) \longleftrightarrow X_1(s) X_2(s)$

Q $x(t) = t\,u(t) * e^{-3t} u(t)$

$X_1(s) = \frac{1}{s^2}$, $X_2(s) = \frac{1}{s+3}$

$X(s) = \frac{1}{s^2(s+3)}$

Q $x(t) = 4\cos 2t\, u(t)$

then $x(t) * \frac{d}{dt}(x(t)) = ?$

$\frac{d}{dt}$ $L[4\cos 2t\, u(t)]$

$\Rightarrow \frac{4s}{s^2+4}$

$\frac{d}{dt}(x(t)) \xleftrightarrow{LT} \frac{s\,4s}{s^2+4}$

$L\left(\frac{d}{dt}x(t)\right) = \frac{4s^2}{s^2+4}$

~~L x t~~ so

$X(s) = \left(\frac{4s^2}{s^2+4}\right)\left(\frac{4s}{s^2+4}\right)$

Q <u>Application of LT</u>

Q A continuous time LTI signal is initially relaxed and represented by the eqn

$y''(t) + 3y'(t) + 2y(t) = 2x(t)$

$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \underline{2x(t)}$

(i) Determine the transfer fn of the system
(ii) Determine impulse response of the system
(iii) find response of the system to an input $x(t) = 4e^{-3t}u(t)$

$y(t) = ?$

$x(t) \longrightarrow$ LTI system $\longrightarrow y(t)$

Transfer fn $= \dfrac{LT[y(t)]}{LT[x(t)]} = H(S)$

(i) $y''(t) + 3y'(t) + 2y(t) = 2x(t)$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = 2x(t)$$

using property $\frac{d}{dt}(x(t)) \xleftrightarrow{LT} sX(s)$

$$s^2Y(s) + 3sY(s) + 2Y(s) = 2X(s)$$

$$(s^2 + 3s + 2) = 2\frac{X(s)}{Y(s)}$$

$$H(S) = \frac{2}{s^2+3s+2}$$

s^2+3s+2

$s^2+2s+s+2$

$s(s+2)+1$

$$= \frac{2}{(s+1)(s+2)}$$

$$H(s) = \frac{2}{(s+1)(s+2)}$$

$$\cancel{\frac{2}{(s+1)(s+2)} = 2 \Big[}$$

$$\frac{1}{(s+1)(s+2)} = \left[\frac{A}{s+1} + \frac{B}{s+2}\right]$$

$s = -1$

$$\frac{1}{(2-1)} = A = 1$$

$s = -2$

$$\frac{1}{-1} = B = -1$$

So it will be $H(s) = 2\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$

(ii) $h(t) = 2e^{-t} - 2e^{-2t} = 2e^{-t}u(t) - 2e^{-2t}u(t)$

(iii) $y(t) = ?$, $x(t) = 4e^{-3t}u(t)$

$$X(s) = \frac{4}{s+3}$$

$$Y(s) = H(s)\,X(s)$$

$$= \frac{2}{(s+1)(s+2)} \cdot \frac{4}{s+3}$$

$$= \frac{8}{(s+1)(s+2)(s+3)}$$

$y(t) =$

$$\frac{8}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{s+2} + \frac{C}{(s+3)}$$

$s = -1$

$$A = \frac{8}{(1)(2)} = 4$$

$s = -2$

$$B = \frac{-8}{1} = -8$$

$s = -3$

$$C = \frac{8}{(-2)(-1)} = 4$$

$$y(t) = 4e^{-t}u(t) - 8e^{-2t}u(t) + 4e^{-3t}u(t)$$

Q Solve using LT

$$y''(t) + 4y'(t) + 4y(t) = 6e^{-t}$$

$$y(0) = -2, \quad y'(0) = 8$$

$$\Rightarrow L[y''(t)] = \cancel{s^2 L[y(t)]}$$

$$= s^2 Y(s) - s\,Y(0) - Y'(0)$$

$$L[y'(t)] = s\,Y(s) - y(0)$$

$$s^2 Y(s) - s\,Y(0) - Y'(0) + sY(s) - Y(0)$$

$$+ 4Y(S) = \frac{6}{s+1}$$

$$s^2 Y(s) + 2s - \cancel{8} + 4s\,Y(s) + \cancel{8} + 4Y(s) = \frac{6}{s+1}$$

$$Y(S)\,[s^2 + 4s + 4] = \frac{6}{s+1} - 2s$$

$$Y(s) = \frac{6}{(s+2)^2(s+1)} - \frac{2s}{(s+2)^2}$$

$$= \frac{6}{(s+2)^2(s+1)} - \frac{2[(s+2)-2]}{(s+2)^2}$$

$$= \frac{6}{(s+2)^2(s+1)} - \frac{2}{s+2} + \frac{4}{(s+2)^2}$$

$$\frac{6}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$$

~~$s=-1$~~

$$6 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

$s=-1$

$A=6$

$s=-2$

$C=-6$

$s=0$

~~$6 = (6)\,4 + 2B + -6$~~

~~$B=3$~~

$6 = 24 + 2B - 6$

$2B = -12$

$B=-6$

so

$$\frac{6}{(s+1)(s+2)^2} = \frac{6}{s+1} - \frac{6}{s+2} - \frac{6}{(s+2)^2}$$

$$Y(s) = \frac{6}{s+1} - \frac{6}{s+2} - \frac{6}{(s+2)^2} - \frac{2}{(s+2)} + \frac{4}{(s+2)^2}$$

$$Y(s) = \frac{6}{s+1} - \frac{2}{(s+2)^2} - \frac{8}{(s+2)}$$

$y(t) = 6e^{-t}u(t) - 2e^{-2t}u(t) - 8e^{-2t}u(t)$

Q $y'''(t) - 3y''(t) + 3y(t) - y(t) = t^2 e^t$

hint $\Rightarrow$

$L[y'''(t)] = s^3 Y(s) - s^2 Y(0) - s\,y'(0) - y''(0)$

$$\frac{1}{(s-1)^6} = 2e^t \frac{t^5}{5!} u(t)$$

$$\frac{1}{s^{n+1}} \xleftrightarrow{LT} \frac{t^n}{n!}$$

fourier series

$x(t) = x(t+T)$

$$X(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 k t}$$

$k = 0 \longrightarrow$ constant or DC term

$k = 1, -1 \longrightarrow$ fundamental component

$k = 2, -2 \longrightarrow$ 2nd harmonic "

$k = 3, -3 \longrightarrow$ 3rd ~~harmonic~~ harmonic "

$k = -N, N \longrightarrow$ N^{th} harmonic

$C_k \rightarrow$ complex of F~~.~~S ~~ser~~ coefficient

$$C_k = \frac{1}{T} \int_{\langle T \rangle} x(t) \, e^{-j\omega_0 k t} \, dt \qquad \omega_0 = \frac{2\pi}{T}$$

No $\Rightarrow$ $x(t)$ is cont and periodic $\xrightarrow{FS}$ $X(t)$ is discrete and NP

If $x(t)$ is real

$$C_k = C_{-k}^*$$

or $(C_k)^* = (C_{-k}^*)^*$

$$C_k^* = C_{-k}$$

Ex ~~x(t) cos² t~~ $x(t) = \cos^2 t$

$$x(t) = \frac{1}{2} + \frac{1}{2} \, \frac{e^{j2t} + e^{-j2t}}{2}$$

$$= \underbrace{\frac{1}{2}}_{C_0} + \underbrace{\frac{1}{4}}_{C_1} e^{j2t(1)} + \underbrace{\frac{1}{4}}_{C_{-1}} e^{+j2t(-1)}$$

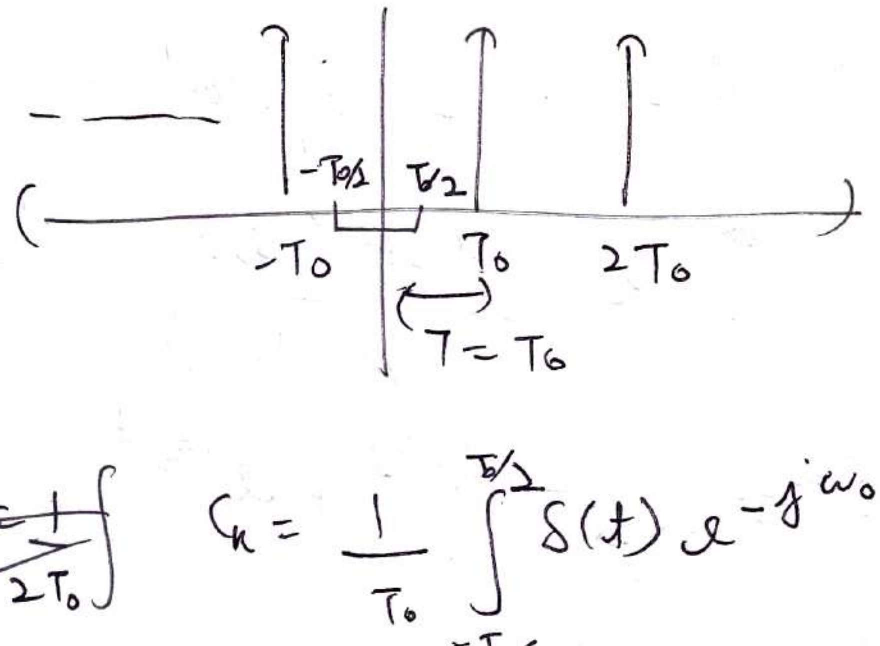
$=$

Q ~~Ex~~ The FS representation of an Impulse train ~~is given by~~ $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

is ?

$$f(t) \rightleftharpoons f(t+T)$$

$$s(t) = \dots + \delta(t+T_0) + \delta(t) + \delta(t-T_0) + \delta(t-2T_0) \dots$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\cancel{C_k = \frac{1}{2T_0}\int} \qquad C_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-j\omega_0 k t}\, dt$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$C_k = \frac{1}{T_0}$$

$$\cancel{s(t)} \quad s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{+j\omega_0 k T}$$

Q(2) If $x(t) = \sin(3t) + \cancel{e^{j\frac{2}{7}t}} + e^{j\frac{2}{7}t}$

then $C_2 = ?$

$$x(t) = \cancel{e^{j3t}}\ \frac{e^{j3t} - e^{-j3t}}{2j} + e^{j\frac{2}{7}t}$$

$$= \frac{e^{j3t}}{2j} - \frac{e^{-j3t}}{2j} + e^{j\frac{2}{7}t}$$

$\omega_1 = 3$ rad/s , $\omega_2 = \frac{2}{7}$ rad/sec

$$\omega_0 = \frac{HCF(3,2)}{LCM(1,7)} = \frac{1}{7} \text{ rad/sec}$$

$$= \frac{e^{j7\times\frac{3}{7}t}}{2j} - \frac{e^{-j\frac{3}{7}\times7t}}{2j} + e^{j\frac{2}{7}t}$$

$C_2 = 1 \rightarrow$ amp

$C_{-21} = \frac{-1}{2j}$, $C_{21} = \frac{1}{2j}$

Q 3 find exponential FS of PF

$f(t) = e^{t}, \ 0 < t < 2\pi$ with $f(t+2\pi) = f(t)$

$f(t+2\pi) = f(t)$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 kt}$$

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} e^{t} e^{-j\omega_0 kt}\, dt$$

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} e^{(1-j\omega_0 k)t}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(1-jk)t}\, dt \qquad \omega_0 = \cancel{\frac{2\pi}{2\pi}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-jk)t}}{(1-jk)}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{(1-jk)2\pi} - 1}{(1-jk)}\right]$$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi(1-jk)} \left(e^{(1-jk)2\pi} - 1\right) e^{jkt}$$

properties of CTFT

① linearity

$$x(t) \longrightarrow C_k$$

$\downarrow$

T_0

$$g(t) \longrightarrow g_k$$

$\downarrow$

T_0

$$x(t) + g(t) \longrightarrow C_k + g_k$$

② Time shifting

$$x(t) \longrightarrow C_k$$

$$x(t-t_0) \longrightarrow e^{-j\omega_0 t_0 k} \cdot C_k$$

$$x(t+t_0) \longrightarrow e^{j\omega_0 t_0 k} C_k$$

③ frequency shifting

$$x(t) \longrightarrow C_k$$

$$\cancel{e}^{-j\frac{2\pi}{m}m k} \cancel{x(t)} \longrightarrow C_{k+m}$$

e

$$\cancel{e^{j\omega_0 m t} x(t) \longrightarrow}$$

$$\cancel{e^{j\omega_0 M t} x(t) \longrightarrow C_{k+M} \quad C_{k+m}}$$

$$\cancel{e^{j\omega_0 M t} x(t) \longrightarrow C_{k+M}}$$

$$e^{j\omega_0 M t} x(t) \longrightarrow C_{k-M}$$

$$e^{-j\omega_0 M t} x(t) \longrightarrow C_{k+M}$$

4. conjugation

$$x(t) \longrightarrow C_k$$

$$x^*(t) \longrightarrow C_{-k}$$

Imp point if $x(t)$ is real $x(t) = x^*(t)$

$$C_k = C_{-k}^*$$

5. time reversal

$$x(t) \longrightarrow C_k$$

$$x(-t) \longrightarrow C_{-k}$$

6. time scaling

$$x(t) \longrightarrow C_k$$

$$x(\alpha t) \longrightarrow C_k$$

where $\alpha > 0$, $x(\alpha t)$ is periodic with $\frac{T_0}{\alpha}$

⑦ Periodic convolution

$$x(t) \longrightarrow C_k$$
$$g(t) \longrightarrow g_k$$

$$x(t) * g(t) \longrightarrow T_0 C_k g_k$$

⑧ multiplication

$$x(t)\, g(t) \longrightarrow C_k * g_k$$
$$= \sum_{l=-\infty}^{\infty} C_{(k-l)} g_l$$

⑨ differentiation

$$x(t) \longrightarrow C_k$$
$$\frac{d}{dt} x(t) \longrightarrow j\omega_0 k\, C_k$$

⑩ $x(t) \longrightarrow$ Real and even

$C_k \longrightarrow$ Real and even

$x(t) \longrightarrow$ Real and odd

$C_k \longrightarrow$ Purely Imaginary and odd

⑫ Parseval's theorem.

$$\frac{1}{T_0} \int_{\langle T_0 \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

when $R=1$, otherwise you have to $\times$ or $/$ by R depending upon the type of signal.

Trigonometric form of CTFS

$f(t) \to T_0$, $\omega_0 = \frac{2\pi}{T_0}$

~~$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b$~~

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{(1)}$$

Average value of $f(t)$ $= a_0 = \frac{1}{T_0}\int_{<T_0>} f(t)\,dt$

$$a_n = \frac{2}{T_0}\int_{<T_0>} f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{<T_0>} f(t)\sin n\omega_0 t\,dt$$

Harmonic form OR amplitude phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \quad \text{(2)}$$

expand it from there compare it to the eq 1 and get a_n and b_n $\longrightarrow$ A_n (Mag), $\phi_n \to$ phase

Discrete time fourier Series (DTFS)

$$x(n) = \sum_{k=0}^{N-1} C_k e^{+j\omega_0 k n}, \quad \omega_0 = \frac{2\pi}{N_0}$$

Discrete + Periodic

$$C_k = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x(n) e^{-j\omega_0 k n}$$

$C_k = C_{k+N_0}$ → in general $C_k = C_{k+mN_0}$, $m \in Z$

periodic

Q $x(n) = \{0, 1, 2, 3\}$ $N_0 = 4$

find $\overset{3}{\underset{k=0}{C_k}}$

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$C_0 = \frac{1}{4} \sum_{n=0}^{3} x(n) e^{-j\omega_0 k n}$$

$$= \frac{1}{4}\left[x(0) + x(1)e^{-j\omega_0 (0)} + x(2) + x(3)\right]$$

$$= \frac{1}{4}[0 + 1 + 2 + 3] = \frac{3}{2}$$

$$C_1 = \frac{1}{4}\left[x(0) + x(1)e^{-j\omega_0} + x(2)e^{-j2\omega_0} + x(3)e^{-j3\omega_0}\right]$$

$$= -\frac{1}{2} + \frac{1}{2}j$$

Similarly get it for $k = 2, 3$

~~sample no~~

$$C_k = C_{k+mN_0}$$

$$C_{302} = ? \Rightarrow k + mN_0 = 302$$

$$m = \frac{302 - k}{N_0} \in Z \quad k = 0, 1, 2, 3$$

$$C_{302} = C_2$$

Q Find the Discrete time F.S

(1) $x(n) = \cos \frac{\pi}{4} n$

(2) $x(n) = \cos \frac{\pi}{3} n + \sin \frac{\pi}{4} n$

(3) $x(n) = \cos^2 \frac{\pi}{8} n$

(1) $N_0 = \frac{2\pi}{\pi/4} = 8 \qquad \omega_0 = \frac{\pi}{4}$

$$x(n) = \frac{1}{2} e^{+j\frac{\pi}{4}n(1)} + \frac{1}{2} e^{+j\pi/4 n(1)}$$

$$C_1 = \frac{1}{2}, \quad C_{-1} = \frac{1}{2} = C_{-1+8} = C_7$$

$C_{400} = ?$

$C_{400} = C_0$? why

$C_k = C_{k+mN_0}$

$k + mN_0 = 400$

$$m = \frac{400-k}{N_0} \to 0$$

(b) $x(n) = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$

$N_1 = \frac{2\pi}{\frac{\pi}{3}} = 6$, $N_2 = \frac{2\pi}{\frac{\pi}{4}} = 8$

$LCM(6, 8) = 24$

$$\omega_0 = \frac{2\pi}{24} = \frac{\pi}{12}$$

~~$x(n) = \cos\frac{\pi}{12} \times 4n + \sin\frac{3\pi}{12}n$~~

$$x(n) = \cos\frac{4\pi}{12}n + \sin\frac{3\pi}{12}n$$

$$x(n) = \frac{e^{-j\frac{4\pi}{12}n}}{2} + \frac{e^{j\frac{4\pi}{12}n}}{2} - \frac{1}{2j} e^{-j\frac{3\pi}{12}n} + \frac{1}{2j} e^{+j\frac{3\pi}{12}n}$$

$$C_4 = \frac{1}{2}, \quad C_{-4} = \frac{1}{2}$$

$$C_3 = \frac{1}{2j}, \quad C_{-3} = -\frac{1}{2j}$$

$$N = 24$$

w ⇒ FFT

w ⇒ Synthe

w ⇒ Complete

~~KHW~~ ⇒ CTFT

Revise everything

Properties of DTFS

① linearity

$$x(n) \xrightarrow{FSC} C_K$$

$$g(n) \xrightarrow{FSC} g_K$$

$$x(n) + g(n) \longrightarrow C_K + g_K$$

② time ~~shifting~~ shifting

$$x(n) \xrightarrow{FSC} C_K$$

$$x(n-n_0) \xrightarrow{FSC} e^{-j\omega_0 n_0 K} C_K$$

$$x(n+n_0) \longrightarrow e^{j\omega_0 n_0 K} C_K$$

~~③ time scaling x(~~

(3) freq shifting

$$x(n) \longrightarrow C_k$$

$$e^{+j\frac{2\pi}{N}mn}\, x(n) \longrightarrow C_{k-m}$$

(4) Conjugation

$$x(n) \longrightarrow C_k$$

$$x^*(n) \longrightarrow C^*_{-k}$$

for $x(n)$ is real

$$C_k = C^*_{-k}$$

(5) time reversal

$$x(n) \longrightarrow C_k$$

$$x(-n) \longrightarrow C_{-k}$$

(6) Periodic convolution

$$x(n) \longrightarrow C_k$$

$$g(n) \longrightarrow g_k$$

$$x(n) * g(n) \longrightarrow N\, C_k g_k$$

(7) multiplication

$$x(n)\, g(n) \longrightarrow C_k * g_k$$

(8)

(8) differentiation

$$x(n) \longrightarrow C_k$$

$$x(n) - x(n-1) \longrightarrow \left(1 - e^{-j\omega k(1)}\right) C_k$$

(9) $x(n) \longrightarrow$ Real and even

$C_k \longrightarrow$ Real and even

$x(n) \longrightarrow$ Odd and Real

$C_k \longrightarrow$ Imaginary and odd

(10) $$\frac{1}{N} \sum_{\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |C_k|^2$$

<u>Continous time fourier transform</u>

$$FT(C(x(t)) \longrightarrow X(\omega) / X(j\omega) / X(f)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

<u>~~Dirlecha~~ dirlecht's condisions</u>

(1) $x(t)$ should be single valued in any finite time interval T

(2) $x(t)$ should have finite no of discontinuities in time interval T

(3) $x(t)$ should have finite no of maxima and minima in any finite time interval T

(4) $x(t)$ should be absolutely integrable

$$\int_{\infty}^{\infty} |x(t)|\, dt < \infty$$

Q $e^{-at}u(t) \xrightarrow{FT} ?$

$$X(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{a+j\omega}$$

Similarly

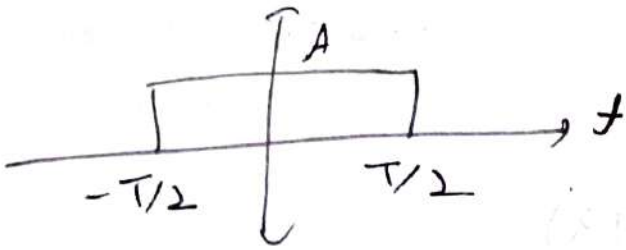
$$e^{at}u(-t) = \frac{1}{a-j\omega}$$

hence

$$\left(e^{|a|t|}\right) \xleftrightarrow{FT} \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$

Q find the F T of



$x(t) = A \, \mathrm{rect}\left(\frac{t}{T}\right)$

so $X(\omega) = \int_{-T/2}^{T/2} A e^{-j\omega t} \, dt$

$= A \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T/2}^{T/2}$

$= A \left[\frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}\right]$

$= \frac{2A}{+\omega} \left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right]$

$= \frac{2A}{\omega} \sin\left(\frac{\omega T}{2}\right)$

$= \frac{AT \sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} = AT \, \mathrm{Sinc}\left(\frac{\omega T}{2}\right)$

Q

(1) $f(t) = \delta(t)$

$F(\omega) = 1$ (proof is left for exercise for the reader)

(2) $f(t) = \delta(t+3)$

$$\delta(t+3) \longrightarrow e^{+j3\omega}(1)$$

$F(\omega) = 1$

~~$F(\omega$~~

$FT(\delta(t+3)) = e^{+j3\omega}$

(3) $f(t) = \delta(3t) - \delta'(2t)$

$$\delta(3t) \xleftrightarrow{FT} \frac{1}{3} \times \delta\left(\frac{\omega}{3}\right)$$

$$\delta(t) \xrightarrow{FT} 1$$

$$\delta(3t) \xleftrightarrow{FT} \frac{1}{3}$$

~~$\delta'(2t) \xrightarrow{FT}$~~

$$\delta'(t) \xrightarrow{F} j\omega\, X(\omega)$$

$$\delta'(2t) \xrightarrow{T} \frac{j\omega}{2} X\left(\frac{\omega}{3}\right)$$

$$= \frac{j\omega}{2}$$

$X(\omega) = \frac{1}{3} - \frac{j\omega}{2}$

properties of FT

(1) linearity

$$\alpha x_1(t) + \beta x_2(t) \xleftrightarrow{FT} \alpha X_1(\omega) + \beta X_2(\omega)$$

eg $e^{-a(t)}$

(2) time shifting property

$$x(t-t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

$$x(t+t_0) \xleftrightarrow{FT} e^{j\omega t_0} X(\omega)$$

eg $\delta(t+3) - \delta(t-3)$

$\Rightarrow e^{+j3\omega} - e^{-j3\omega}$

(3) frequency shifting

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$e^{-j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega + \omega_0)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)$$

(4) time scaling

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

(5) time reversal property

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(-t) \xleftrightarrow{FT} X(-\omega)$$

(6) differentiation in time domain

$$\frac{d}{dt}(x(t)) \xleftrightarrow{FT} j\omega X(\omega)$$

$$\frac{d^n}{dt^n}(x(t)) \xleftrightarrow{FT} (j\omega)^n X(\omega)$$

(7) differentiation in freq domain

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$t\,x(t) \xleftrightarrow{FT} j\frac{d}{d\omega}(X(\omega))$$

$$t^n x(t) \xleftrightarrow{FT} j^n \frac{d^n}{d\omega^n}(X(\omega))$$

(8) Integration

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{X(\omega)}{j\omega} + \pi X(0)\,\delta(\omega)$$

eg $u(t)$

$$u(t) = \int_{-\infty}^{t} \delta(t)\,dt$$

$$\delta(t) \xleftrightarrow{FT} 1$$

$$\int_{-\infty}^{t} \delta(t)\,dt \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$u(t) \xleftrightarrow{FT} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$x(-t) \xleftrightarrow{FT} X(-\omega)$$

$$u(-t) \xleftrightarrow{FT} 1$$

Signum fn

$$x(t) = u(t) - u(-t)$$

$$= \frac{1}{j\omega} \cancel{+ \pi\delta(\omega)} + \frac{1}{j\omega} \cancel{- \pi\delta(-\omega)}$$

$$\cancel{= \pi(\delta(\omega) - \delta(-\omega))}$$

$$= \frac{2}{j\omega}$$

(9) Duality

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$$

$$\boxed{\begin{aligned} \delta(t) &\xrightarrow{FT} 1 \\ 1 &\xrightarrow{FT} 2\pi\,\delta(\omega) \end{aligned}} \longrightarrow \text{Imp}$$

(10) modulation

$$x_1(t) \xleftrightarrow{FT} X(\omega)$$

$$x_2(t) \xleftrightarrow{FT} X_2(\omega)$$

$$x_1(t)\,x_2(t) \xleftrightarrow{FT} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

$$\boxed{\delta(t-a) * \delta(t-b) = \delta(t-a-b)}$$

(11) Convolution

$$x_1(t) \xleftrightarrow{FT} X_1(\omega)$$

$$x_2(t) \xleftrightarrow{FT} X_2(\omega)$$

$$x_1(t) * x_2(t) \xleftrightarrow{FT} X_1(\omega)\,X_2(\omega)$$

(12) parseval's Thm

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

Introduction to Systems.

System ⇒ A system is defined as an entity ~~as~~ that manipulates one or more signals to accomplish a function, thereby producing a new ~~signal~~ signal.

Input Signal → System → output Signal

System

- Continuous time system
- Discrete time system

- linear or non linear
- Causal or non causal
- Time variant or time invariant system
- Stable or unstable system
- Static or dynamic system

linear and non linear system

A system is said to be linear if it follows two properties.

① ~~Addivity~~ additivity → more than enough to check the linearity

If $x_1(t) \longrightarrow y_1(t)$

$\cancel{x_1(t)} + x_2(t) \longrightarrow y_2(t)$

$x_1(t) + x_2(t) \xrightarrow{\text{system}} y_1(t) + y_2(t)$

② Homogeneity

$x(t) \xrightarrow{\text{system}} y(t)$

$\alpha x(t) \xrightarrow{\text{system}} \alpha y(t)$

$\alpha \rightarrow$ real

Q Check whether the system is linear or not?

(i) $y(t) = e^{x(t)}$

$y_1(t) = e^{x_1(t)}$

$y_2(t) = e^{x_2(t)}$

$y_1(t) + y_2(t) = e^{x_1(t)} + e^{x_2(t)}$

$[x_1(t) + x_2(t)] \xrightarrow{\text{system}} e^{(x_1(t) + x_2(t))}$

$y_1(t) + y_2(t) \neq y[x_1(t) + x_2(t)]$

Non linear.

Causal and Non causal System

Causal system ⇒ If present output of system depends on present or past or present and past both the inputs then the system is Causal system.

Non causal system : If present output of system depends on future inputs also then system is non causal.

eg $y(t) = x(t-2) + x(t+T)$

$y(2) = x(0) + x(6)$ → future input

NC

Time Invarient system → If the time shift in the input signal results in corresponding time shift in the output otherwise system is called time variant system.

Q eg $y(t) = t\,x(t)$

$y(t) = t\,x(t-t_0)$) shifting the input by t_0

$y(t-t_0) = (t-t_0)\,x(t-t_0)$

$y(t) \neq y(t-t_0)$ → Time variant system.

Stable and unstable system

Stable system → If a system gives a bounded output for bounded input, then the system is stable.

Unstable system → If a system gives unbounded output for bounded input, then system is unstable

Q eg $y(t) = \underbrace{\sin[\overset{\to \infty, \infty}{x(t)}]}_{-1 \text{ to } 1}$ → stable

Static and dynamic system ⇒

Static (memory less) → If present output of the system depends on ~~prep~~ present input, then system is static

Dynamic (memoryless) → If present output of system depends on past or future input, then system is dynamic.

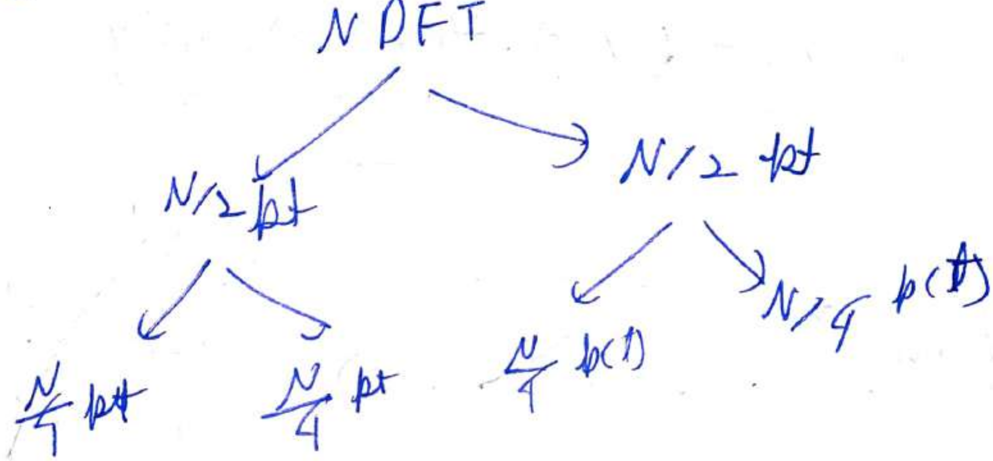
i) $y(t) = 3x(t)+5$

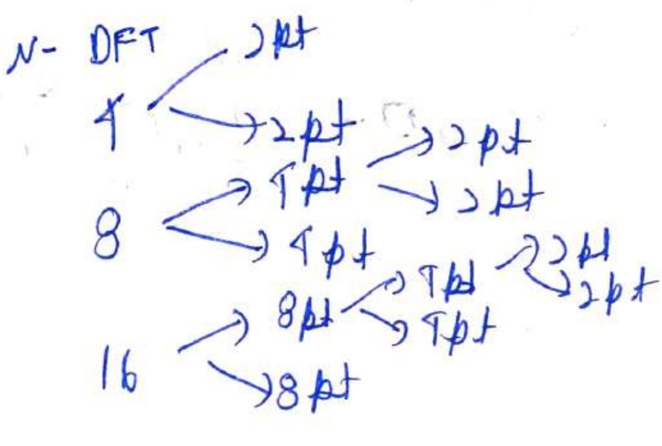
$y(2) = 3x(2)+5$

→ Static

FFT → fast fourier transform

we have seen in DFT that ~~we needs~~ we use the twiddle matrix to calculate ~~[illegible]~~ the DFT of signal ~~[illegible]~~ which in computer is ~~reat~~ a really inefficient process. hence we use FFT to perform that operation





$$N = 2^x$$

↓

base radix

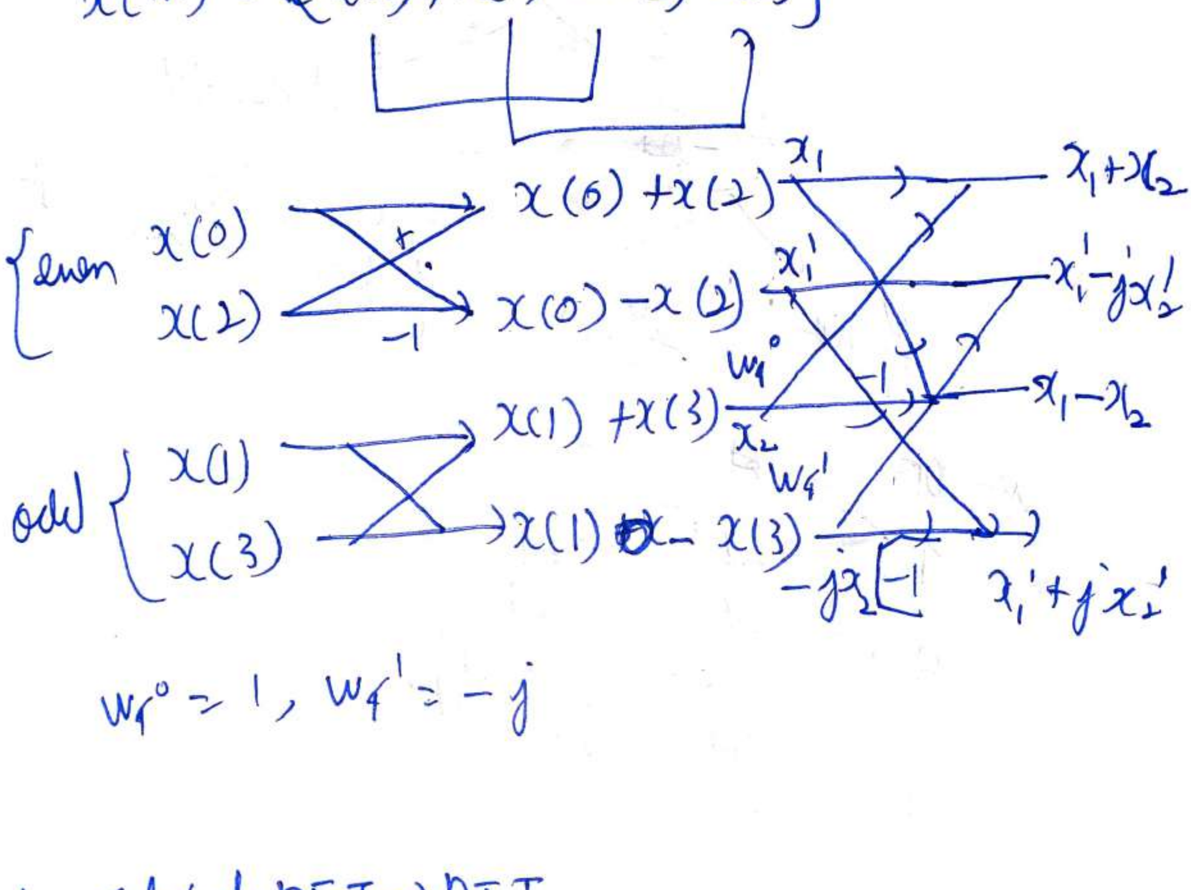
Radix 2

DIT → Decimation in time $x(n)$

DFT → Decimation in frequency $X(k)$

4 point DIT – FFT graph

$x(n) = \{x(0), x(1), x(2), x(3)\}$



$w_4^0 = 1,\ w_4^1 = -j$

So 4 point DFT → DIT

$\{X(0), X(1), X(2), X(3)\}$

$= \{x_1+x_2,\ x_1'-jx_2',\ x_1-x_2,\ x_1'+jx_2'\}$

8 point DIT → FFT graph

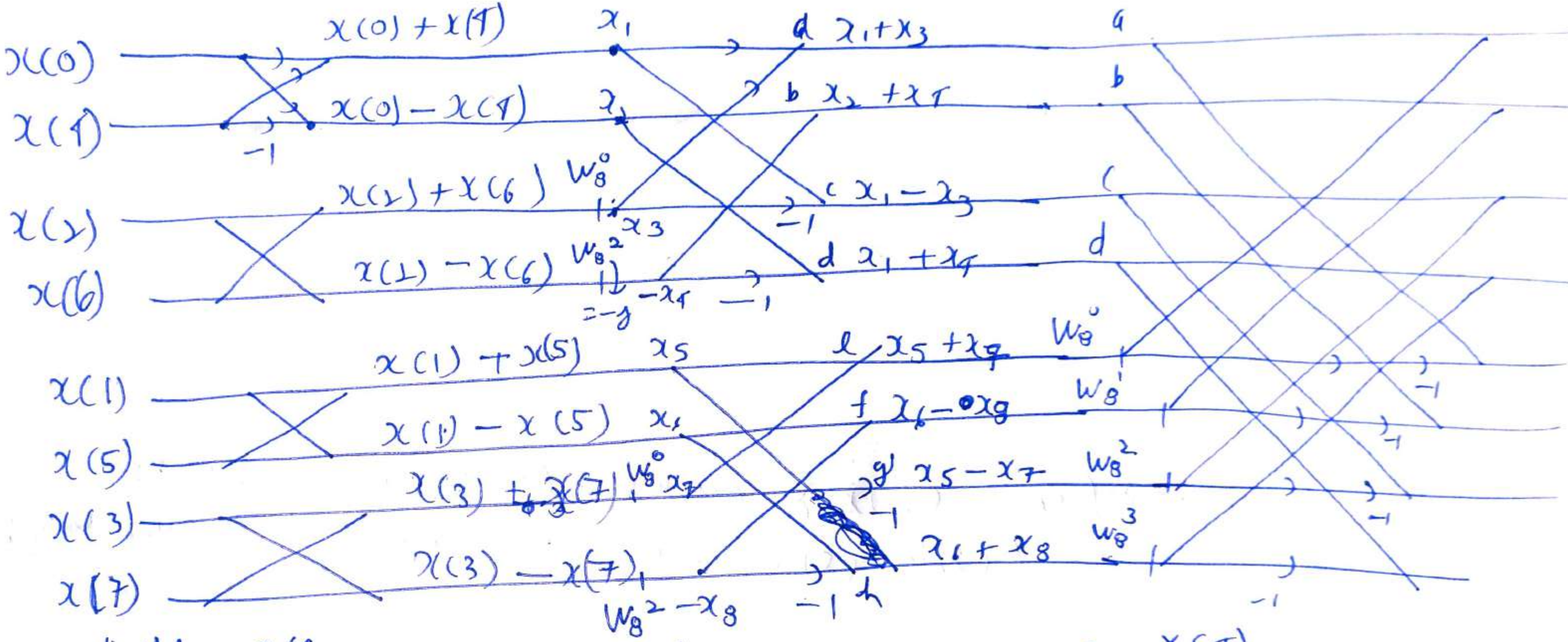
$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$ $N=8$

even $x_e(n) = \{x(0), x(2), x(4), x(6)\}$ $N=4$

odd $x_o(n) = \{x(1), x(3), x(5), x(7)\}$ $N=4$

2pt $x_1(n) = \{x(0), x(4)\}$, $x_2(n) = \{x(2), x(6)\}$

$x_3(n) = \{x(1), x(5)\}$, $x_4(n) = \{x(3), x(7)\}$.



first four O/P

$a+e$	$X(0)$
$b+W_8^1 f$	$X(1)$
$c-g$	$X(2)$
$d+W_8^3 h$	$X(3)$

$a-e$	$X(4)$
$b-W_8^1 f$	$X(5)$
$c+g$	$X(6)$
$d-W_8^3 h$	$X(7)$