Polynomials

$$p(x) = \underbrace{a_n}_{\text{leading coefficient}} x^{\overset{\text{degree}}{n}} + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

deg P = n

<u>ex</u> = $x^2 - 6x + 5$

$z^3 - iz^2 - z + i$

$p(x) = 2$ degree 0

$p(x) = 0$ zero polynomial deg ⇒ $-\infty$

$$p(x) = \sum_{i=0}^{n} a_i x^i = a_0 x^0 + a_1 x^1 + \cdots + a_n x^n$$

<u>ex</u> $(z^2+1) + (z^3 - iz^2 - z + i)$

$= z^3 + (1-i)z^2 - z + (1+i)$

② $(z^2+1)(z^3 - iz^2 - z + i)$

$= z^5 - iz^4 - z^3 + iz^2 + z^3 - iz^2 - z + i$

$= z^5 - iz^4 - z + i$

<u>remarks</u> :→ $\deg(P+Q) \leq \max\{\deg P, \deg Q\}$

$\deg(PQ) = \deg P + \deg Q$

polynomials satisfy properties like comm,
(with $+$, $\cdot$)
assoc. but they are not a field.

Roots of polynomials

Def → A number x_0 is called a root of $p(z)$ if $p(x_0) = 0$

ex $z_0 = 1$ is a root of

$Q(z) = z^3 - iz^2 - z + i$

since $Q(1) = 0$

ex $(x - x_0)$ is called a linear factor
in fact this is a polynomial of degree 1
a product of n linear factors

$(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_n)$

is a polynomial of degree n, for which $x_1, x_2, \ldots, x_n$ are roots.

ex $(z+1)(z-1)(z-i) = (z^2-1)(z-i) =$
$= z^3 - iz^2 - z + i = Q(z)$

the roots of $Q(z)$ are $1, -1, i$

ex $\Rightarrow$ $P(x) = x^2 - 6x + 5$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36-20}}{20} = \frac{6 \pm 4}{2} \begin{cases} 5 \\ 1 \end{cases}$$

$\Rightarrow P(x) = (x-5)(x-1)$

ex $p(x) = x^2 - 2x + 1 = (x-1)(x-1)$

$x_1 = 1$ is ~~an~~ a root with multiplicity 2.

The fundamental theorem ~~o~~ of algebra

Every polynomial of degree n over $\mathbb{C}$ has n roots (counting multiplicity)

Question $\rightarrow$ how do we find roots?

for $n=1$ $\quad p(x) = ax + b \quad x_1 = -\frac{b}{a}$

for $n=2$; $p(x) = ax^2 + bx + c$ · quadratic formula

for $n = 3, 4$ ~~$p(x) = ax^2 + bx + c$~~

$\exists$ formulas

for $n \geq 5$ no formula by radicals

There are "tricks" for finding roots in specific cases. In general there are numerical methods.

Thm (Vieta's formulas)

the sum of the roots $x_1 + x_2 + \cdots + x_n$

$$= \frac{-a_{n-1}}{a_n}$$

the product of the roots $x_1 x_2 \cdots x_n =$

$$(-1)^n \frac{a_0}{a_n}$$

(for $n=2$; $x_1 + x_2 = \frac{-b}{a}$, $x_1 x_2 = \frac{c}{a}$)

Thm If all the coefficients are ~~integ~~. integers and $\frac{p}{q}$ is a rational root then p divides a_0 and q divides a_n

Thm If $p(z)$ has real coefficients and $z_0 \in \mathbb{C}$ is a root, then $\bar{z}_0$ is also a root

~~(This is~~

Proof $\Rightarrow$ write $P(z) = \sum_{j=0}^{n} a_j z^j$

(this is the same as : $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$)

z_0 is a root $\Rightarrow P(z_0) = 0 \Rightarrow \overline{P(z_0)} = 0$

$$0 = \overline{P(z_0)} = \overline{\sum_{j=0}^{n} a_j (z_0)^j}$$

$$= \sum_{j=0}^{n} \overline{a_j (z_0)^j} = \sum_{j=0}^{n} a_j (\overline{z_0})^j$$

($a_j \in \mathbb{R}$ for every j)

$$= \sum_{j=0}^{n} a_j (\overline{z_0})^j = P(\overline{z_0})$$

$\rightarrow \overline{z_0}$ is a root

ex. $p(x) = x^5 - x^4 - 10x^3 + 10x^2 + 9x - 9$

~~write $y = x^2$~~

$(x-1)$ is a linear factor of $x^5 - x^4 - 10x^3 + 10x^2 + 9x - 9$

$x-1 \,)\, x^5 - x^4 - 10x^3 + 10x^2 + 9x - 9 \,(x^4 - 10x^2 + 9$

$+ \cancel{x^5} - \cancel{x^4}$

$-10\cancel{x^3} + 10\cancel{x^2}$

$-10x^3 + 10x^2$

$+ \quad -$

$9x - 9$

$9x - 9$

0

so it will be

$p(x) = (x-1)(x^4 - 10x^2 + 9)$

write $y = x^2$: $y^2 - 10y + 9$

$y^2 - 9y - 1y + 9$

$y(y-9) - 1(y-9)$

$= (y-1)(y-9)$

roots are $y = 1, 9$

$x^2 = 1$, $x^2 = 9$

$x = \pm 1$; $x = \pm 3$

hence

$p(x) = (x-1)(x-1)(x+1)(x+3)(x-3)$

Matrix → A matrix is a chart of elements arranged in rows and ~~cont~~ columns

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \pi \\ & i & -3 \end{pmatrix}$$

A is a matrix over $\mathbb{C}$

In general

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} \\ a_{21} & a_{22} & a_{22} & - & - & a_{2n} \\ a_{31} & a_{32} & a_{33} & - & - & a_{3n} \\ | & | & | & \ddots & & | \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} \end{pmatrix}$$

more notation ⇒

a_{ij} is the ~~one~~ entry in row i, column j

more notation $A = A_{m \times n} = (a_{ij})$

$A \in M_{m \times n}(\mathbb{C})$

if $n = m$ then $A \in M_n(\mathbb{C})$

remarks ① if there are many rows / columns we write $a_{25,16}$

② In this course all entries will be $\mathbb{R}/\mathbb{C}$

Terminology ⇒

- Size of a matrix : $m \times n$
- Square matrix : $A = A_{n \times n}$
- main diagonal : all elements
- 0 matrix : a matrix with
  $a_{ij} = 0 \; \forall \, i, j$
- identity matrix : square where all diagonal entries are $a_{ii} = 1$ and all off diagonal entries are $a_{ij} = 0 \; (i \neq j)$

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- ~~diagonal~~ diagonal matrix : ex $\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

  every $a_{ij} = 0 \; (i \neq j)$
- Scalar ~~matrix~~ matrix : ex $\Rightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

  $a_{ij} = 0 \; (i \neq j)$

  $a_{ii} = c, \; \forall \, i$
- transpose matrix $\Rightarrow$ ex : $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix} \; A^t = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$

  $(A^t)_{ij} = (A)_{ji}$ note that $A = A_{m \times n}$

  $\Rightarrow A^t = A^t_{n \times m}$
- symmetric matrix $A = A^t \; (a_{ij} = a_{ji})$

  ex $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 8 & 9 \end{pmatrix}$

~~A : Do~~

- $-A$: ex $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} \Rightarrow -A = \begin{pmatrix} -1 & -2 \\ 3 & -4 \end{pmatrix}$

- Skew symmetric : $[A = -A^t]$

$$A = \begin{pmatrix} 0 & 7 \\ -7 & 6 \end{pmatrix} \qquad A^t = \begin{pmatrix} 0 & -7 \\ 7 & 0 \end{pmatrix}$$

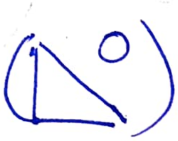
$a_{ij} = -a_{ji}$

(implies $a_{ii} = 0 \ \forall i$)

- upper triangular : ex $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

$a_{ij} = 0$ for $i > j$

- lower - triangular $a_{ij} = 0$ for $j > i$



- row vector : a matrix with 1 row

$(2 \ 5 \ 3)$

- column vector : a matrix with 1 column

$$\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$$

Operations on matrices

equality $\Rightarrow$ $A = B$ if they have exactly the same size and exactly the same entries

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

addition / substraction : element-wise

ex $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 2 \\ -3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 7 & 1 & 0 \end{pmatrix}$

- **Scalar times a matrix** $\Rightarrow$ entry wise

ex $3 \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}$

properties

1. $A + B = B + A$
2. $(A+B) + C = A + (B+C)$
3. $A + 0 = A$
4. $A + (-A) = 0$
5. $\alpha(A+B) = \alpha A + \alpha B$
6. $(\alpha+\beta)(A) = \alpha A + \beta A$
7. $(\alpha\beta) A = \alpha(\beta A)$
8. $(A+B)^t = A^t + B^t$
9. $(A-B)^t = A^t - B^t$
10. $(\alpha(A))^t = \alpha A^t$
11. $(A^t)^t = A$

proof of 8

Sizes

matrix	Size
A	$m\times n$
B	$m\times n$
$A+B$	$m\times n$
$(A+B)^t$	$n\times m$
A^t	$n\times m$
B^t	$n\times m$
A^t+B^t	$n\times m$

entries ij

~~$A+B$~~

$\Rightarrow ((A+B)^t)_{ij} = (A+B)_{ji}$

$(A)_{ji} + (B)_{ji} = (A^t)_{ij} + (B^t)_{ij}$

$= (A^t+B^t)_{ij}$

$((A+B)^t)_{ij}$

$\downarrow$

ijth entry of $(A+B)^t$

$\Downarrow$

$(A+B)_{ji}$

$\downarrow$

(ji)th entry of $(A+B)$

Matrix and Multiplication

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 2 & -7 \end{pmatrix}$$

$A_{3\times 2}$ $B_{2\times 3}$

$$= \begin{pmatrix} 1 & 4 & -3 \\ 1 & 8 & -5 \\ 1 & 12 & -7 \end{pmatrix}$$

Definition of matrix mult :→

- Defined for $A_{m\times n}$ and $B_{n\times l}$.
- The product is $C_{m\times l}$
- The entries of C are

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

columns

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

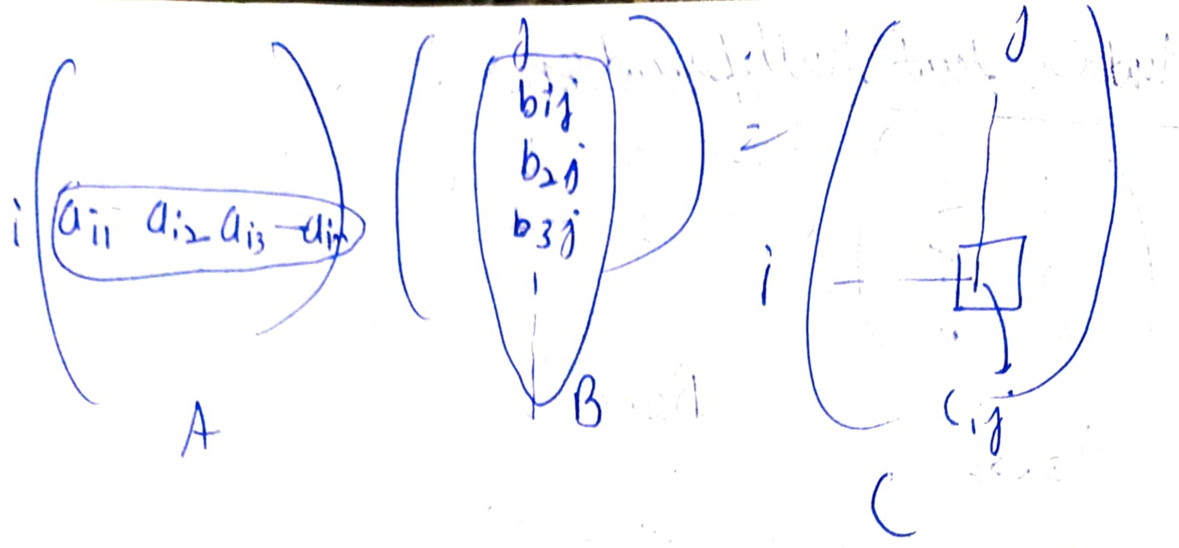
rows

$= a_{11} b_{11} + a_{12} b_{21}$ $(i=1, j=1)$

if it was

3×3

$a_{11} b_{11} + a_{12} b_{21} + a_{13} b_{31}$



__remarks__ $AB \neq BA$ __ex.__ $\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\-1&-1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

formulas like $(A+B)^2 \neq A^2+2AB+B^2$

there could be zero divisors

$A\neq 0,\ B\neq 0$ but $A\cdot B = 0$!

no cancellations $AB = AC \Rightarrow B = C$

$\left(AB = AC \Rightarrow AB - AC = 0 \Rightarrow A(B-C) = 0 \right)$

__Properties of matrix multiplication__ ⇒

Thm ⇒ let $A_{m\times n}$ $B_{n\times l}$ $D_{n\times l}$ $C_{l\times q}$ be matrices Then: (1) $(AB)C = A(BC)$

(2) $A(B+D) = AB+AD$

(3) $(D+B)C = DC+BC$

(4) $\alpha(AB) = A(\alpha B)$

(5) $AI_n = I_m A = A$ (7) $(AB)^t = B^t A^t$

(6) $AO = OA = O$

proof of 7

elements:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

matrix	size
A	$m \times n$
B	$n \times l$
AB	$m \times l$
$(AB)^t$	$l \times m$
A^t	$n \times m$
B^t	$l \times n$
$B^t A^t$	$l \times m$

$$((AB)^t)_{ij} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$(B^t A^t)_{ij} = \sum (B^t)_{ik} (A^t)_{kj}$$

$$= \sum_{k=1}^{n} b_{ki} \; a_{jk}$$

hence

$$((AB)^t)_{ij} = (B^t A^t)_{ij}$$

proof of 5

$$A I_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \\ a_{31} & \vdots & & \ddots & \\ \vdots & \vdots & & & a_{nn} \\ a_{n1} & a_{n2} & & & \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & \ddots \; 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} = A$$

Remark: multiplication of a matrix and a ~~col~~ column vector is just a special case:

$$A_{m \times n} B_{n \times 1} = C_{m \times 1}$$

power of matrices: If $A_{n \times n}$ then we define $A^0 = I$

$$A^1 = A$$

$$A^2 = AA$$

$$A^3 = AAA$$

$$\vdots$$

substituting a matrix to a polynomial

ex $P(x) = 3x^3 - 5x + 2$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

$$P(A) = 3A^3 - 5A + 2I$$

$$= 3\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I$$

$$A^3 = A^2A = IA = A$$

$p(A)$

$$P(A) = 3A^3 - 5.A + 2I$$

$$= 3\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} - 5\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 6 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 0 & -5 \end{pmatrix} + \cancel{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 \\ 0 & 4 \end{pmatrix}$$