complex No →

$\mathbb{N}$ $x+5=3$

$\mathbb{Z}$ $x \cdot 5 = 12$

$\mathbb{Q}$ $x^2 = 3$

$\mathbb{R}$ $x^2+1=0$

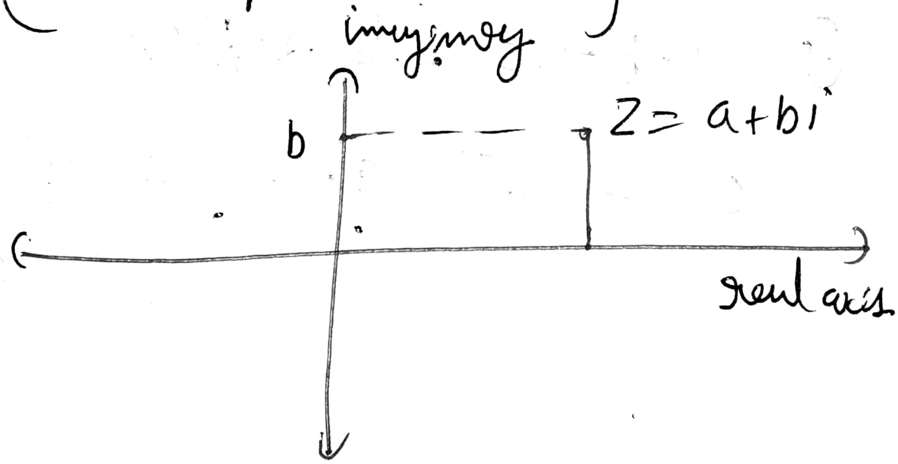
$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$

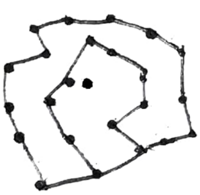
in $\mathbb{C}$ any equation has a solution

$i^2 = \sqrt{-1}\sqrt{-1} = -1$

doesnt satisfy $\sqrt{ab} = \sqrt{a}\sqrt{b} = 1$

$c = \{a+bi \mid a, b \in \mathbb{R}\}$





addition $\Rightarrow$ $(a+bi)+(c+di) = (a+c)+(b+d)i$

ex $\Rightarrow$ $(1+2i)+(3-5i) = 4-3i$

<u>Substraction</u> $\Rightarrow$ $(a+bi)-(c+di) = (a-c)+(b-d)i$

<u>multiplication</u>: $(a+bi)(c+di) = ac - bd + (ad+bc)i$

remarks $i^2 = (0+1i)(0+1i) = -1$

$(a+0i)(b+0i) = ab$

Thm $\Rightarrow$ $\mathbb{C}$ is a field

proof HW! for example $\Rightarrow$

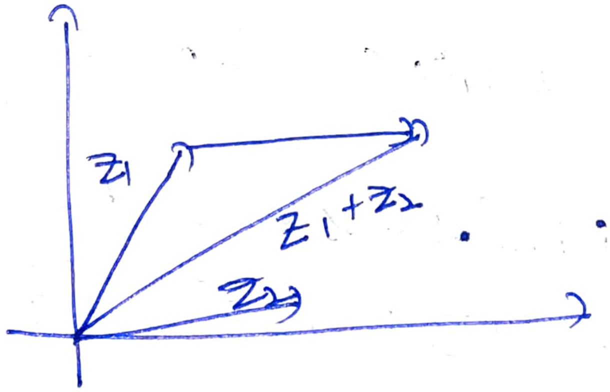
$z_1 z_2 = (a+bi)\cdot(c+di) = (ac-bd)+(ad+bc)i$

$= (ca-db)+(da+cb)i$

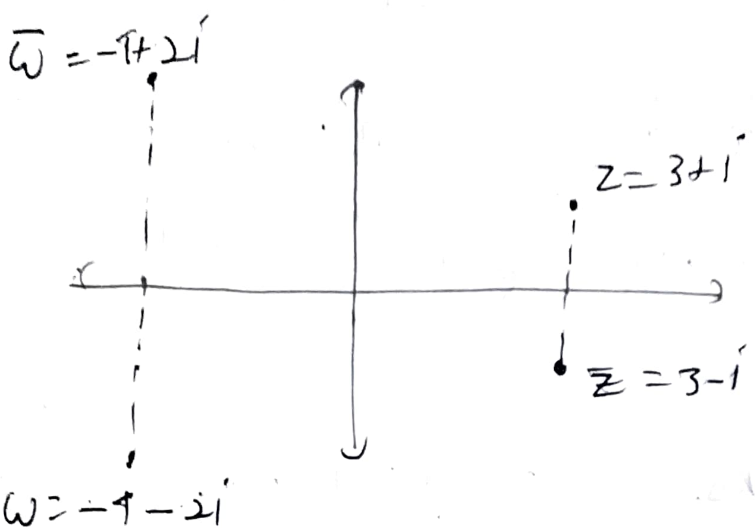
$= (c+di)\cdot(a+bi) = z_2 z_1$

in particular $\Rightarrow$

$z+(0+0i) = z$, $z\cdot(1+0i) = z$



Def For $z = a+bi$ we define the conjugate to be $\bar{z} = a - bi \Rightarrow$



<u>properties</u> :⇒

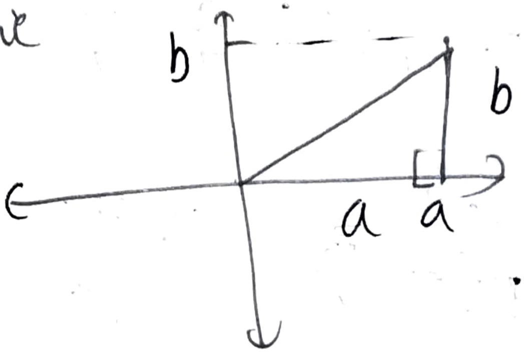
1. $\overline{z \pm w} = \bar{z} \pm \bar{\omega}$
2. $\overline{z\omega} = \bar{z} \cdot \bar{\omega}$
3. $z + \bar{z} = 2\,\mathrm{Re}(z)$
4. $z - \bar{z} = 2\,\mathrm{Im}(z)\,i$
5. $z = \bar{z} \Rightarrow \mathrm{Im}(z) = 0$
6. $z\bar{z} \geq 0$, $z\bar{z} = 0 \Leftrightarrow z = 0$

<u>proof of (6):</u>

$$z\bar{z} = (a+bi)(a-bi)$$
$$\Rightarrow a^2+b^2 \geq 0$$
$$a^2+b^2 = 0 , \; a, b = 0$$

Def: For $z = a+bi$ we define the modulus to be $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$

geometric



$z = a + bi$

Pythagorean

$a^2 + b^2 = |z|^2$

$|z|$ is the distance of z from 0 (and the length of the vector corresponding to z).

properties: (1) ~~$z\ |z|$~~ $z\bar{z} = |z|^2$

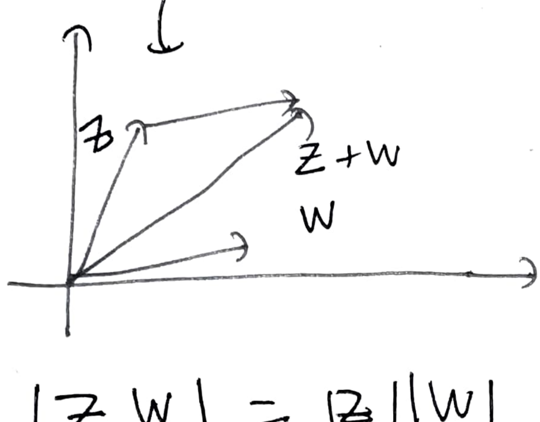
(2) $|\bar{z}| = |z|$

(3) $|z| \geq 0$, $|z| = 0 \iff z = 0$

(4) $|Re(z)| \leq |z|$, $|Im(z)| \leq |z|$

(5) $|z + w| \leq |z| + |w|$ triangle inequality

(6)



(6) $|zw| = |z||w|$

ex $\Rightarrow$ $\dfrac{2+3i}{4-i} \cdot \dfrac{4+i}{4+i} = \dfrac{5+14i}{16+1} = \dfrac{5}{17} + \dfrac{14}{17}i$

Def $\Rightarrow$ $\cancel{\dfrac{z}{w} = \dfrac{z\bar{w}}{|zw|^2}}$ $\dfrac{z}{w} = \dfrac{z\bar{w}}{|w|^2}$

Remark $\dfrac{1}{w} = \dfrac{\bar{w}}{|w|^2}$ is called the (mult.)

inverse of w, and denoted w^{-1}

properties ① $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\bar{z}}{\bar{w}}$

② $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$

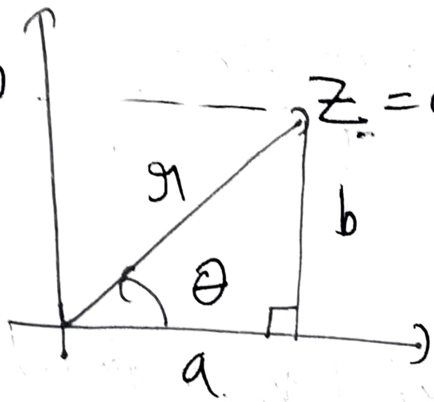
proof of (2):

$$\left|\frac{z}{w}\right| = \left|\frac{z\bar{w}}{|w|^2}\right| = \left|\frac{1}{|w|^2} z\bar{w}\right| =$$

$$= \left|\frac{1}{|w|^2}\right| |z|\,|\bar{w}| = \frac{1}{|w|^2} |z|\,|w|$$

~~$z = a+bi = r\cos$~~

Polar form



determine z
polar form of z;

$z = r \operatorname{cis} \theta$

given a, b:

$$\begin{cases} r = \sqrt{a^2 + b^2} \\ \theta = \arctan\left(\frac{b}{a}\right) \end{cases}$$

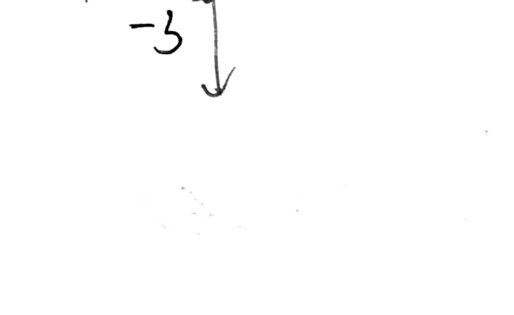
$\cos\theta = \frac{a}{r}$, $\sin\theta = \frac{b}{r}$ $\Rightarrow \begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$

$z = a + bi = r\cos\theta + i\sin\theta$
$= r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta$
$r = |z|$ and θ

ex: $z = 3 \operatorname{cis} 270°$

~~$= 3(\cos 270° + i\sin 270°)$~~ $(3\cos 270° + i 3\sin 270°)$

$a = 3\cos 270° = 0$, $b = 3\sin 270°$
$= -3$



$z = 1 + \sqrt{3}i$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$, $\theta = \arctan = \frac{\sqrt{3}}{1}$

$= \frac{\pi}{3} = 60°$

<u>Remarks:</u> (i) 0 does not have argument we just write $z = 0$

(2) $\theta = 45°$ and $\theta = 405°$ give the same complex number. We define the argument as a set : $\{\theta + 360°k \mid k \in \mathbb{Z}\}$

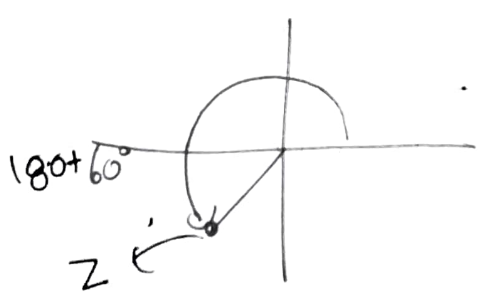
(iii) Be careful while θ :

ex $z = -1 - \sqrt{3}i$

$r = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

$\arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{-\sqrt{3}}{-1}\right) = \arctan\sqrt{3}$

a calculator will give $60°$ but



it will be $180 + 60° = 240°$

Multiplication, division, powers and roots

Thm $\Rightarrow$ Suppose $z_1 = r_1 \operatorname{cis}\theta_1$

$z_2 = r_2 \operatorname{cis}\theta_2$

Then: ① $z_1 z_2 = r_1 \operatorname{cis}(\theta_1 + \theta_2)$

② $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

ex $\Rightarrow$ $\dfrac{1+\sqrt{3}i}{-3i} = \dfrac{2 \operatorname{cis} 60^\circ}{3 \operatorname{cis} 270^\circ} = \dfrac{2}{3} \operatorname{cis}(60^\circ - 270^\circ)$

$= \dfrac{2}{3} \operatorname{cis}(-210^\circ) = \dfrac{2}{3} \operatorname{cis}(150^\circ)$

$= \dfrac{2}{3}(\cos 150^\circ + i \sin 150^\circ)$

$= \dfrac{2}{3}\left(-\dfrac{\sqrt{3}}{2}\right) + i \dfrac{2}{3} \dfrac{1}{2}$

$= -\dfrac{1}{\sqrt{3}} + \dfrac{1}{3} i$

proof of 1:) $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)$
$r_2(\cos\theta_2 + i\sin\theta_2)$

$= \cancel{r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))}$

$= r_1 r_2 ((\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) +$
$(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2) i)$

$= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$

$= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Thm $\Rightarrow$ (De - Moivre)

$$z^n = r^n \operatorname{cis}(n\theta)$$

ex $(\sqrt{3} + i)^{15}$

~~$\Rightarrow r = \sqrt{3}$~~

$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\theta = \arctan \frac{1}{\sqrt{3}} = 30^\circ$

$(2 \operatorname{cis} 30^\circ)^{15} = 2^{15} \operatorname{cis}(15 \cdot 30^\circ)$

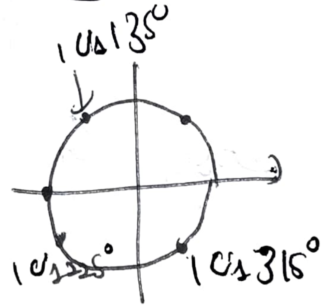
$= 2^{15} \operatorname{cis}(450^\circ)$

~~Def an nth root of z ∈ ℂ is a~~

Def $\rightarrow$ an n^{th} root of $z \in \mathbb{C}$ is a $w \in \mathbb{C}$ s.that $w^n = z$

ex $w = 1 \operatorname{cis} 45^\circ$

$z = w^4 = 1 \operatorname{cis} 180^\circ$



$w_1^4 = w_2^4 = w_3^4 = z$

in general for every $z = r \operatorname{cis} \theta \in \mathbb{C}$ there are n roots of order n they all have modulus $\sqrt[n]{r}$ and arguments:

$$\frac{\theta + 360^\circ k}{n}, \quad k = 0, 1, 2, 3, \dots n-1$$

$$\sqrt[n]{z} = \left\{ \sqrt[n]{r} \operatorname{cis}\left(\frac{\theta + 360^\circ k}{n}\right) \middle| k = 0, 1, 2, 3, \dots, n-1 \right\}$$

ex ⇒ find all roots of unity of order 6

$$\sqrt[6]{1} = \operatorname{cis} \frac{360^\circ k}{6}, \quad \cancel{k = 0, 1,} \ k = 0, 1, 2, 3, 4, 5, \cancel{6}$$

$\operatorname{cis} 0 = 1$

$\operatorname{cis} 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\operatorname{cis} 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\operatorname{cis} 180^\circ = -1$

$\operatorname{cis} 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$\operatorname{cis} 300^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

remark: their sum is 0

Thm → The sum of the n n^{th} ~~roots~~ roots of unity is 0, for $n \geq 2$