System of linear equations

A linear equation:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

coefficient — unknown (variables) — constant

$$\sum_{i=1}^{n} a_i x_i = b$$

ex $3x + 4y = 7 \longrightarrow$ the eqn of line in the plane

$2x + 3y + 5z = 10$ (the eqn of ~~the~~ a plane ~~in plane~~ in R^3)

$x^2 + y = 5$ $\longrightarrow$ Not linear.

$x - xy = 4$

a solution vector: $(x_1, x_2, \ldots x_n)$ such that if we substitute the x_i's into equation, they satisfy it.

eg ex $(1, -1, 0, 2)$ is a solution of

$$2x_1 - x_2 + 3x_3 - 4x_4 = -5$$

so is $(0, 5, 0, 0)$

ex $\begin{cases} x - y = 3 \\ 2x + y = 9 \end{cases}$ this is a system of 2 equations with 2 unknowns

ex $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 13 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - 2x_2 + x_3 + 2x_4 = 13 \end{cases}$

3 eqns with 4 unknowns

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ 13 \end{pmatrix}$$

~~A $\vec{x}$~~

A $\vec{x}$ $\vec{b}$

System of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \text{---} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{---} + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \text{---} a_{mn}x_n = b_m \end{cases}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

the coefficient matrix $A\vec{x} = \vec{b}$

Def :→ If $\vec{b} = \vec{0}$ (all b_i's are 0) then system is called homogenous.

If $\vec{b} \neq \vec{0}$ then its called non-homogenous

Q why write in the form $A\vec{x} = \vec{b}$?

ex

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 13 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - 2x_2 + x_3 + 2x_4 = 13 \end{cases}$$

3 eqn with 4 unknowns

$$x_3 = 8 - 2x_1 + x_2$$

$$\Rightarrow \begin{cases} x_1 + 2x_2 + 3(8\ x_2 - 2x_1) + 4x_4 = 13 \\ 3x_1 - 2x_2 + (8 + x_2 - 2x_1) + 2x_4 = 13 \end{cases}$$

$$x_2 = x_1 + 2x_4 - 5$$

$$\Rightarrow -5x_1 + 5(x_1 + 2x_4 - 5) + 4x_4 = -11$$

$$14x_4 = 14$$

$$x_4 = 1$$

$x_2 = x_1 - 3$ and $x_3 = 5 - x_1$

x_1 is called a "degree of freedom" a <u>general solution</u> is of the form

$$(x,\ x-3,\ 5-x,\ 1)$$

for example, taking $x = 2$ we get

$$(2, -1, 3, 1)$$

this is a ~~Specificf~~ <u>Specific solution</u>

$$\begin{cases} x - y = 3 \\ 2x + y = 9 \end{cases}$$

$y = x - 3$

$\Rightarrow 2x + x - 3 = 9$

$\Rightarrow 3x = 12$

$x = 4$

$y = 1$

there could be no solutions:

$$\begin{cases} x - y = 3 \\ x - y = 4 \end{cases}$$

The method of row - reductions

(Gauss)

Def: Two systems are equivalent if they have the same solution set.

The idea: Find an equivalent system which is simple (many are 0 or 1).

ex
$$\begin{cases} x_1 + x_3 = 5 \\ x_2 + x_3 = 2 \\ x_4 = 1 \end{cases}$$

Thm The solutions of a system remain the same if we:

1. Swap two equations
2. ~~Multiply Mul~~ Multiply an equation by any scalar $\neq 0$.

③ add a scalar multiple of one equation to another equation.

These 3 operations are called ⇒

"elementary operations"

when we write $A\vec{x} = \vec{b}$ these operations

① swap two rows of A, and the corresponding entries in $\vec{b}$

$$R_i \longleftrightarrow R_j$$

② multiply a row and the corresponding entry of $\vec{b}$ by $\alpha \neq 0$

$$R_i \longleftrightarrow \alpha R_i \ (\alpha \neq 0)$$

③ add a multiple of one row to another, and do the same for $\vec{b}$

$$R_i \longrightarrow R_i + \alpha R_j$$

Def ⇒ A, B ~~were~~ are called row-equivalent if one can be derived from the other by a finite number of elementary ~~ro~~ row operations

Notation $(A \mid \vec{b}) \equiv A^*$

this is called the ~~augmentation~~ ~~agt~~ augmented matrix of system $A\vec{x} = \vec{b}$

eg
$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 = 13 \\ 2x_1 - x_2 + x_3 = 8 \\ 3x_1 - 2x_2 + x_3 + 2x_4 = 13 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 13 \\ 8 \\ 13 \end{pmatrix}}_{\vec{b}}$$

$$\underbrace{\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 2 & -1 & 1 & 0 & 8 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right)}_{A^*} \xrightarrow{R_2 \to \frac{1}{2} R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 4 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & -4 & -9 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{3} R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & -4 & -9 \\ 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{13}{3} \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & -4 & -9 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5/2 & -5/2 & -4 & -9 \\ 1 & -2/3 & 1/3 & \frac{2}{3} & \frac{13}{3} \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & \cancel{0}\,4 & 13 \\ 0 & -\frac{5}{2} & -\frac{5}{2} & -4 & -9 \\ 0 & -\frac{8}{3} & -\frac{8}{3} & -\frac{10}{3} & -\frac{26}{3} \end{array}\right) \xrightarrow[R_3 \to -3R_3]{R_2 \to -2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & 5 & 5 & 8 & 18 \\ 0 & 8 & 8 & 10 & 26 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & 1 & 1 & 8/5 & 18/5 \\ 0 & 8 & 8 & 10 & 26 \end{array}\right) \xrightarrow{R_3 \to R_3 - 8R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & \cancel{4}\,8/5 & 13 \\ 0 & 1 & 1 & & -18/5 \\ 0 & 0 & 0 & -14/5 & -17/5 \end{array}\right)$$

leading coeff / pivot elements

$$\xrightarrow{R_3 \to -\frac{5}{14} R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & 1 & 1 & 8/5 & 18/5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

row echelon form

~~No~~ of 0's to the of each leading ~~coef~~ coeff increases from row to row.

$$\xrightarrow{R_2 \to R_2 - \frac{8}{5}R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 9 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 4R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

reduced row echelon form

(canonical)

all leading coeff's are 1, all elements in their columns are 0.

summary ⇒ (i) Goal : canonical augmented matrix.

(remark : it is unique)

(ii) Method: elementary row operations.

(remark : we can do several operations in a single step)

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 2 & -1 & 1 & 0 & 8 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 0 & -5 & -5 & -8 & -18 \\ 0 & -8 & -8 & -10 & -26 \end{array}\right)$$

(iii) At the end we get an ~~equivalent~~ system whose ~~equivent~~ equivalent system whose solution is easy.

<u>Determining the no of solutions</u> →

<u>Def</u> ⇒ Let A be a matrix. The number of non zero rows in a row echelon form obtained from A using elementary row operations is called the rank of A, denoted rank (A) or $r(A)$.

remarks $\Rightarrow$ ① The rank does not depend on the way we do row reduction, or on the ~~the~~ row ~~ech~~ echelon form.

② matrices that ~~are~~ corresponds to equivalent systems have the same rank.

③ one can define ~~to~~ column operations, column — echelon form, rank in terms of columns....
the rank by rows is the same as the rank by columns.

Thm $\Rightarrow$ Let $A\vec{x} = \vec{b}$ be a system of m equations with n ~~unkn~~ unknowns
Then:

① The system has a unique solution if and ~~if~~ iff $r(A) = r(A^*) = n$

② if $r(A) \neq r(A^*)$ then there is no solution.

③ If $r(A) = r(A^*) < n$ then the system has ∞ - many solutions, with $n - r(A)$ degrees of freedom.

Ex
$$\begin{cases} \lambda x + y + z = 1 \\ x + \lambda y + z = \lambda \\ x + y + \lambda z = \lambda^2 \end{cases}$$

$m = 3$, $n = 3$, λ is a parameter

for which values of λ are there ?

1. a unique solution.
2. ∞ many solution.
3. no solution.

$$\left(\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - \lambda R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda-\lambda^2 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda^3 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & -(\lambda-1) & -\lambda(\lambda-1) \\ 0 & -(\lambda-1) & -(\lambda-1)(\lambda+1) & -(\lambda-1)(\lambda^2+\lambda+1) \end{array}\right)$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & 1 & 1 & -\lambda \\ 0 & 0 & -\lambda-2 & -\lambda^2-2\lambda-1 \end{array}\right) \xrightarrow[R_3 \to \frac{1}{-\lambda-2} R_3]{\lambda \neq -2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & 1 & -1 & -\lambda \\ 0 & 0 & 1 & \dfrac{-\lambda^2-2\lambda-1}{-\lambda-2} \end{array}\right)$$

if $\lambda \neq 1, -2$ then $r(A) = r(A^*) = 3$

there is going to be a unique solution and we know how to find it!

if $\lambda = 1$ then $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$$x + y + z = 1 \Rightarrow z = 1 - x - y$$

$$r(A^*) = r(A) = 1 < 3 \qquad (x, y, 1-x-y)$$

there $3 - 1 = 2$ degrees of freedom

if $\lambda = -2$ then $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right)$

$$r(A) = 2 \neq 3 = r(A^*)$$

$0x + 0y + 0z = -1 \Rightarrow$ no solution!

$\therefore \vec{x} = \vec{0}$

ex $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Homogenous Vs non-homogenous System

Reminder ⇒ $A\vec{x} = \vec{0}$ is called homogenous

Observation ⇒ A homogenous System always has a Solution, ~~called the~~ called the trivial solution: $\vec{x} = \vec{0}$.

ex $\begin{pmatrix} 1 & 2 & 3 & 8 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Thm If a homogenous system has a non-zero solution, then it has ∞ many solution.

proof $A\vec{x} = \vec{b}$

let $\vec{x_0} \neq \vec{0}$ be a solution, then $\alpha\vec{x_0}$ is also a solution:

$$A(\alpha\vec{x_0}) = \alpha(A\vec{x_0}) = \alpha(\vec{0}) = \vec{0}$$

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

~~for~~ Summary ⟹ for a ~~general~~ general system: ∃ no /unique sol/ ∞ solutions.

for a homogeneous System: trivial / ∞ Soln.

$(1, 1, -1, 0)$ is a soln and in fact the soln set is. $\underline{(x, x, -x, 0)}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ 13 \end{pmatrix}$$

<u>Soln</u> $(x, x-3, 5-x, 1)$

Thm ⟹ ① if $A\vec{x} = \vec{b}$ has ∞ solutions then so ~~does~~ does $A\vec{x} = \vec{0}$

② if $A\vec{x} = \vec{b}$ has a unique soln, so does $A\vec{x} = \vec{0}$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 13 \\ 2 & -1 & 1 & 0 & 8 \\ 3 & -2 & 1 & 2 & 13 \end{array}\right) \xrightarrow[\text{many steps}]{} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 2 & 0 \end{array}\right) \xrightarrow[\text{same no of many steps}]{} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Remark: the converse to the previous thm is not true:

counter example: $\begin{cases} x-y=3 \\ x-y=4 \end{cases}$ non-homogenous no soln.

$\begin{cases} x-y=0 \\ x-y=0 \end{cases}$ corresponding homogenous ∞ soln's

Thm If $\vec{x_0}$ is a specific soln of $A\vec{x}=\vec{b}$

then:

the soln set of $A\vec{x}=\vec{b}$ = $\vec{x_0}$ + the solution set of $A\vec{x}=\vec{0}$

in the example:

$$\begin{pmatrix} x \\ x-3 \\ 5-x \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 5 \\ 1 \end{pmatrix} + \begin{pmatrix} x \\ +x \\ -x \\ 0 \end{pmatrix}$$

proof: Show ① take $\vec{y_0}$ a soln of $A\vec{x}=\vec{0}$ and show that $\vec{x_0}+\vec{y_0}$ is a soln of $A\vec{x}=\vec{b}$:

② if $\vec{x_1}$ is any soln of $A\vec{x}=\vec{b}$ then there is a soln of $A\vec{x}=\vec{0}$ st $\vec{x_1}=\vec{x_0}+\vec{y_1}$ take $\vec{y_1}=\vec{x_1}-\vec{x_0}$

$$A(\vec{y}_1) = A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = 0$$

~~Corollary~~

Corollary: If we know the solⁿ to $A\vec{x} = b$ then we can find the solⁿ to $A\vec{x} = \vec{0}$

ex

$$\begin{pmatrix} x \\ x-3 \\ 5-x \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} x-2 \\ x-2 \\ 2-x \\ 0 \end{pmatrix}$$