数理方程期末试题参考答案

2020 年毕业班期末试题

一、(共 18 分) 求解下列 *Cauchy* 问题。

(1)

$$\begin{cases} u_{tt} = 4u_{xx}, \quad (t > 0, -\infty < x < +\infty) \\ u|_{t=0} = x^2, \quad u_t|_{t=0} = \cos 2x \end{cases}$$

(2)

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 20 \\ u(0,y) = y^2, \quad u(x,0) = \sin x \end{cases}$$

解：

(1)

$$\begin{aligned} u(t,x) &= \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\mathrm{d}\xi \\ &= \frac{(x-2t)^2 + (x+2t)^2}{2} + \frac{\sin 2(x+2t) - \sin 2(x-2t)}{8} \end{aligned}$$

(2)

$$u(x,y) = y^2 + \sin x + 20xy$$

二、(共 18 分) 求以下固有值问题的固有值和固有函数。

(1)

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, (0 < x < \pi) \\ Y'(0) = 0, Y'(\pi) = 0 \end{cases}$$

(2)

$$\begin{cases} x^2 Y''(x) + xY'(x) + \lambda Y(x) = 0, (1 < x < b) \\ Y(1) = 0, Y'(b) = 0 \end{cases}$$

解：

(1) 固有值为 $\lambda = n^2$, 对应的固有函数为 $Y_n(x) = \cos nx, n = 0, 1, 2, 3, \dots$.

(2) 固有值为

$$\lambda_n = \left(\frac{2n+1}{2\ln b}\pi\right)^2$$

固有函数为

$$Y_n(x) = \sin(\sqrt{\lambda_n}\ln x)$$

其中 $n = 0, 1, 2, \dots$.

三、(共 18 分)

(1) 求周期边界条件下

$$\begin{cases} u_{tt} = u_{xx}, (t > 0, 0 < x < 1) \\ u(t,0) = u(t,1), u_x(t,0) = u_x(t,1) \end{cases}$$

的分离变量解 $u = T(t)X(x)$

(2) 求解

$$\begin{cases} u_{tt} = u_{xx}, (t > 0, 0 < x < 1) \\ u(t,0) = u(t,1), u_x(t,0) = u_x(t,1) \\ u(0,x) = \sin 2\pi x, u_t(0,x) = 2\pi \cos 2\pi x \end{cases}$$

解：

(1)

$$u(t,x) = \sum_{n=1}^{\infty} (C_n \cos 2n\pi t + D_n \sin 2nxt) \cdot \cos 2n\pi x + (C_n' \cos 2n\pi t + D_n' \sin 2n\pi t) \cdot \sin 2n\pi x$$

(2)

$$u(t,x) = \cos 2\pi t \sin 2\pi x + \sin 2\pi t \cos 2\pi x = \sin 2\pi(t+x)$$

四、(共 14 分) 求解

$$\begin{cases} u_t = u_{xx} + u(t > 0, -\infty < x < +\infty) \\ u|_{t=0} = \delta(x+1) \end{cases}$$

解：

$$\begin{aligned} u(t,x) &= e^t \cdot \left[\frac{1}{2\sqrt{\pi t}} \cdot e^{-\frac{x^2}{4t}} * \delta(x+1) \right] \\ &= \frac{e^t}{2\sqrt{\pi t}} \cdot e^{-\frac{(x+1)^2}{4t}} \end{aligned}$$

五、(共 18 分)

(1) P_n 为 n 阶勒让德函数，写出 $P_0(x), P_1(x), P_2(x)$, 并计算积分 $\int_{-1}^{1}(20+x)P_2(x)\mathrm{d}x$.

(2) 求解以下定解问题，其中（r, θ, φ）为球坐标.

$$\begin{cases} \Delta_3 u = 0, (r < 2) \\ u|_{r=2} = 3\cos 2\theta \end{cases}$$

解：

(1) 积分值为 0.

(2)

$$u(r,\theta)=r^2\cdot P_2(\cos\theta)-1$$

六、(共 14 分) 已知平面区域 $D=\{(x,y)\mid -\infty<x<+\infty, y<1\}$

(1) 写出 D 内泊松方程第一边值问题的格林函数所满足的定解问题，并求出格林函数。

(2) 求解定解问题: 其中常数 $a>0$

$$\left\{\begin{array}{l} u_{xx}+a^2u_{yy}=0,(-\infty<x<+\infty, y<1) \\ \left.u\right|_{y=1}=\varphi(x) \end{array}\right.$$

解：

(1)

$$\begin{aligned} G\left(M;M_0\right) &= \frac{1}{2\pi}\left[\ln\frac{1}{r\left(M,M_0\right)}-\ln\frac{1}{r\left(M,M_1\right)}\right] \\ &= \frac{1}{2\pi}\ln\frac{r\left(M,M_1\right)}{r\left(M,M_0\right)} \\ &= \frac{1}{4\pi}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y-2+\eta)^2} \end{aligned}$$

(2)

$$u(x,y)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\varphi(\xi)\left(\frac{y}{a}-\frac{1}{a}\right)}{(x-\xi)^2+\left(\frac{y}{a}-\frac{1}{a}\right)^2}d\xi=-\frac{a}{\pi}\int_{-\infty}^{+\infty}\frac{\varphi(\xi)(y-1)}{a^2(x-\xi)^2+(y-1)^2}d\xi$$

参　考　公　式

(1) 直角坐标系: $\Delta_3 u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$

柱坐标系: $\Delta_3 u=\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}+\dfrac{\partial^2 u}{\partial z^2}$

球坐标系: $\Delta_3 u=\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial u}{\partial \theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u}{\partial \varphi^2}$

(2) 若ω是$J_\nu(\omega a)=0$ 的一个正根，则有模平方$N_{\nu 1}^2=\|J_\nu(\omega x)\|_1^2=\dfrac{a^2}{2}J_{\nu+1}^2(\omega a)$

若ω是$J_\nu'(\omega a)=0$ 的一个正根， 则有模平方$N_{\nu 2}^2=\|J_\nu(\omega x)\|_2^2=\dfrac{1}{2}\left[a^2-\dfrac{\nu^2}{\omega^2}\right]J_\nu^2(\omega a)$

(3) 勒让德多项式: $P_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n, n=0,1,2,3,\ldots$

母函数: $\left(1-2xt+t^2\right)^{-\frac{1}{2}}=\sum\limits_{n=0}^{+\infty}P_n(x)t^n$, 递推公式: $P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$

(4)$\dfrac{1}{\pi}\displaystyle\int_0^{+\infty}e^{-a^2\lambda^2 t}e^{i\lambda x}d\lambda=\dfrac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda=\dfrac{1}{2a\sqrt{\pi t}}\exp\left(-\dfrac{x^2}{4a^2 t}\right)$

(5) 2 维泊松方程基本解为$u=\dfrac{1}{2\pi}\ln r$, 这里(r,θ) 为极坐标

(6) 由平面区域D内$Poisson$方程第一边值问题的格林函数$G\left(M;M_0\right)$, 求得$Poisson$方程第一边值问题解$u(M)$ 的公式是 (其中S为D的边界).

$$u(M)=-\int_S\varphi\left(M_0\right)\frac{\partial G}{\partial n}\left(M;M_0\right)dS+\iint_D f\left(M_0\right)G\left(M;M_0\right)dM_0$$