

关于拉普拉斯算子在三大坐标系下的表达的应用，大家应该有所体会。而关于这个表达式考试时是否会给出，这个目前没有明确的消息，按照往年的试卷来看是会给出的。但是还是建议大家具备自己能够推导出的能力。这里把之前在群里分享的我的推导思路整理一下，供大家参考。

$$\begin{aligned} \Delta_2 u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \Delta_2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \end{aligned} \quad \begin{aligned} x &= r \cos \theta & y &= r \sin \theta \\ r &= \sqrt{x^2 + y^2} & \theta &= \arctan \frac{y}{x} \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{1}{r} \frac{\partial u}{\partial r} \cos \theta & \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \sin \theta \\ \frac{\partial \theta}{\partial x} &= \frac{1}{r^2} \left(-\frac{y}{x} \right) = -\frac{\sin \theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{\cos \theta}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \cos \theta \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \cos \theta \right) \cdot \frac{\partial \theta}{\partial x} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \sin \theta \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \sin \theta \right) \cdot \frac{\partial \theta}{\partial y} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial^2 u}{\partial r^2} \cos \theta + \frac{\partial^2 u}{\partial r \partial \theta} \cdot \left(-\frac{\sin \theta}{r} \right) + \frac{\partial^2 u}{\partial \theta^2} \cdot \left(-\frac{\sin \theta}{r} \right) \right) \cdot \frac{\partial \theta}{\partial x} \\ &+ \left(\frac{\partial^2 u}{\partial r \partial \theta} \cos \theta + \frac{\partial^2 u}{\partial r^2} \cdot \left(-\frac{\sin \theta}{r} \right) + \frac{\partial^2 u}{\partial \theta^2} \cdot \left(-\frac{\cos \theta}{r} \right) \right) \cdot \left(-\frac{\sin \theta}{r} \right) \\ &= \frac{\partial^2 u}{\partial r^2} \cos^2 \theta + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{\sin^2 \theta}{r^2} - \frac{\partial^2 u}{\partial r \partial \theta} \cdot \frac{2 \sin \theta \cos \theta}{r} + \frac{\partial u}{\partial r} \cdot \frac{\sin^2 \theta}{r} + \frac{\partial u}{\partial \theta} \cdot \frac{2 \sin \theta \cos \theta}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \sin \theta \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \sin \theta \right) \cdot \frac{\partial \theta}{\partial y} \\ &= \left(\frac{\partial^2 u}{\partial r^2} \sin \theta + \frac{\partial^2 u}{\partial r \partial \theta} \cdot \frac{\cos \theta}{r} + \frac{\partial^2 u}{\partial \theta^2} \cdot \left(\frac{\cos \theta}{r} \right) \right) \cdot \sin \theta \\ &+ \left(\frac{\partial^2 u}{\partial r \partial \theta} \sin \theta + \frac{\partial^2 u}{\partial r^2} \cdot \frac{\cos \theta}{r} + \frac{\partial^2 u}{\partial \theta^2} \cdot \left(-\frac{\sin \theta}{r} \right) \right) \cdot \frac{\cos \theta}{r} \\ &= \frac{\partial^2 u}{\partial r^2} \sin^2 \theta + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{\cos^2 \theta}{r^2} + \frac{\partial^2 u}{\partial r \partial \theta} \cdot \frac{2 \sin \theta \cos \theta}{r} + \frac{\partial u}{\partial r} \cdot \frac{\cos^2 \theta}{r} - \frac{\partial u}{\partial \theta} \cdot \frac{2 \sin \theta \cos \theta}{r} \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{1}{r^2} + \frac{\partial^2 u}{\partial \varphi^2} \cdot \frac{1}{r^2 \sin^2 \theta} + \frac{\partial u}{\partial r} \cdot \frac{2}{r} + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta}{r \sin \theta}$$

$$x = r \sin \theta \cos \varphi \quad y = r \sin \theta \sin \varphi \quad z = r \cos \theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad \theta = \arccos \frac{z}{r} \quad \varphi = \arctan \frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin \theta \cos \varphi \quad \frac{\partial r}{\partial y} = \sin \theta \sin \varphi \quad \frac{\partial r}{\partial z} = \cos \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{r} \cdot \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial x}{\partial x} = \frac{\cos \theta \cos \varphi}{r} \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta \sin \varphi}{r} \quad \frac{\partial \theta}{\partial z} = \frac{-\sin \theta}{r}$$

$$\frac{\partial \varphi}{\partial x} = \frac{-\sin \varphi}{r \sin \theta} \quad \frac{\partial \varphi}{\partial y} = \frac{\cos \varphi}{r \sin \theta} \quad \frac{\partial \varphi}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \varphi} \frac{\partial \varphi}{\partial x} = \frac{\partial u}{\partial r} \cdot \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \cdot \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \cdot \left(\frac{-\sin \theta \sin \varphi}{r} \right)$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial r} \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right) \cdot \frac{\partial r}{\partial x} \\ &+ \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right) \cdot \frac{\partial \theta}{\partial x} \\ &+ \frac{\partial}{\partial \varphi} \left(\frac{\partial u}{\partial r} \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right) \cdot \frac{\partial \varphi}{\partial x} \end{aligned}$$

$$+ \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial r} \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right) \cdot \frac{\partial \theta}{\partial x}$$

$$+ \frac{\partial}{\partial \varphi} \left(\frac{\partial u}{\partial r} \sin \theta \cos \varphi + \frac{\partial u}{\partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial u}{\partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right) \cdot \frac{\partial \varphi}{\partial x}$$

$$= \left(\frac{\partial^2 u}{\partial r^2} \sin \theta \cos \varphi + \frac{\partial^2 u}{\partial r \partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial r \partial \varphi} \frac{-\sin \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \theta^2} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{-\sin \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \varphi^2} \frac{-\sin \theta \sin \varphi}{r} \right)$$

$$+ \sin \theta \cos \varphi + \left(\frac{\partial^2 u}{\partial r \partial \theta} \cdot \sin \theta \cos \varphi + \frac{\partial^2 u}{\partial r} \cos \theta \cos \varphi + \frac{\partial^2 u}{\partial \theta} \cdot \frac{\cos \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial \theta} \frac{-\sin \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{-\sin \theta \sin \varphi}{r} \right)$$

$$+ \frac{\partial^2 u}{\partial \varphi} \frac{\sin \theta \cos \varphi}{r \sin \theta} \cdot \frac{\cos \theta \cos \varphi}{r} + \left(\frac{\partial^2 u}{\partial r \partial \varphi} \sin \theta \cos \varphi + \frac{\partial^2 u}{\partial r} \sin \theta (-\sin \varphi) + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{\cos \theta \cos \varphi}{r} \right)$$

$$+ \frac{\partial^2 u}{\partial \theta} \cdot \frac{-\cos \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \varphi^2} \cdot \frac{-\sin \theta \sin \varphi}{r \sin \theta} + \frac{\partial^2 u}{\partial \varphi} \frac{-\cos \varphi}{r \sin \theta} \cdot \frac{-\sin \theta \sin \varphi}{r}$$

$$= \frac{\partial^2 u}{\partial r^2} \cdot \sin^2 \theta \cos^2 \varphi + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{\cos^2 \theta \cos^2 \varphi}{r^2} + \frac{\partial^2 u}{\partial \varphi^2} \cdot \frac{\sin^2 \varphi}{r^2 \sin^2 \theta} + \frac{\partial^2 u}{\partial r \partial \theta} \cdot \frac{2 \sin \theta \cos \theta \cos^2 \varphi}{r}$$

$$+ \frac{\partial^2 u}{\partial r \partial \varphi} \cdot \frac{-2 \sin \theta \cos \theta \sin^2 \varphi}{r} + \frac{\partial^2 u}{\partial \theta \partial \varphi} \cdot \frac{-2 \cos \theta \sin \theta \cos \varphi \sin \varphi}{r^2 \sin \theta} + \frac{\partial^2 u}{\partial \theta} \cdot \frac{\cos^2 \theta \cos^2 \varphi}{r} \sin^2 \theta$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} \frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial^2 u}{\partial \varphi} \frac{\partial \varphi}{\partial y} = \frac{\partial^2 u}{\partial r^2} \sin \theta \sin \varphi + \frac{\partial^2 u}{\partial \theta} \frac{\cos \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \varphi} \frac{\cos \varphi}{r \sin \theta}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 u}{\partial r^2} \sin \theta \cos \varphi + \frac{\partial^2 u}{\partial r \partial \theta} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial r \partial \varphi} \frac{-\sin \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \theta^2} \frac{\cos \theta \cos \varphi}{r} + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{-\sin \theta \sin \varphi}{r} + \frac{\partial^2 u}{\partial \varphi^2} \frac{-\sin \theta \sin \varphi}{r} \right)$$

$$\begin{aligned}
 & \sin\theta \sin\varphi + \left(\frac{\partial^2 u}{\partial r \partial \theta} \sin\theta \sin\varphi + \frac{\partial u}{\partial r} \cos\theta \sin\varphi + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{\cos\theta \sin\varphi}{1-r} + \frac{\partial u}{\partial \theta} \cdot \frac{-\sin\theta \sin\varphi}{1-r} \right. \\
 & + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{\cos\varphi}{1-\sin\theta} + \frac{\partial u}{\partial \varphi} \frac{-\cos\theta \cos\varphi}{1-\sin\theta} \left. \right) \frac{\cos\theta \sin\varphi}{1-r} + \left(\frac{\partial^2 u}{\partial r \partial \varphi} \cdot \sin\theta \sin\varphi + \frac{\partial u}{\partial r} \sin\theta \cos\varphi + \frac{\partial^2 u}{\partial \theta \partial \varphi} \cdot \frac{\cos\theta \sin\varphi}{1-r} + \frac{\partial u}{\partial \theta} \cdot \frac{\cos\theta \cos\varphi}{1-r} + \frac{\partial^2 u}{\partial \varphi^2} \frac{\cos\varphi}{1-\sin\theta} \right. \\
 & + \frac{\partial u}{\partial \varphi} \cdot \frac{-\sin\varphi}{1-\sin\theta} \left. \right) \frac{\cos\varphi}{1-\sin\theta} \\
 & = \frac{\partial^2 u}{\partial r^2} \sin^2\theta \sin^2\varphi + \frac{\partial^2 u}{\partial \theta^2} \frac{\cos^2\theta \sin^2\varphi}{1-r^2} + \frac{\partial^2 u}{\partial \varphi^2} \frac{\cos^2\varphi}{1-\sin^2\theta} + \frac{\partial u}{\partial r} \frac{2\sin\theta \cos\theta \sin\varphi}{1-r} \\
 & + \frac{\partial^2 u}{\partial r \partial \theta} \frac{2\sin\theta \cos\varphi}{1-r} + \frac{\partial^2 u}{\partial \theta \partial \varphi} \frac{2\cos\theta \sin\varphi \cos\varphi}{1-\sin^2\theta} + \frac{\partial u}{\partial r} \frac{\cos^2\theta \sin^2\varphi + \cos^2\varphi}{1-r} \\
 & + \frac{\partial u}{\partial \theta} \left(\frac{-2\sin\theta \cos\varphi \sin\varphi}{1-r^2} + \frac{\cos\theta \cos\varphi}{1-\sin\theta} \right) + \frac{\partial u}{\partial \varphi} \frac{-2\sin\varphi \cos\varphi}{1-\sin^2\theta} \\
 \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \varphi} \frac{\partial \varphi}{\partial z} = \frac{\partial u}{\partial r} \cos\theta + \frac{\partial u}{\partial \theta} \frac{-\sin\theta}{1-r} \\
 \frac{\partial u}{\partial z} &= \left(\frac{\partial^2 u}{\partial r^2} \cos\theta + \frac{\partial^2 u}{\partial r \partial \theta} \frac{-\sin\theta}{1-r} + \frac{\partial u}{\partial \theta} \frac{\sin\theta}{1-r^2} \right) \cos\theta
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{\partial^2 u}{\partial r \partial \theta} \cos\theta + \frac{\partial u}{\partial r} (-\sin\theta) + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{-\sin\theta}{1-r} + \frac{\partial u}{\partial \theta} \frac{\partial u}{\partial \theta} \cdot \frac{-\cos\theta}{1-r} \right) \frac{\sin\theta}{1-r} \\
 & = \frac{\partial^2 u}{\partial r^2} \frac{\partial^2 u}{\partial r} \cdot \cos^2\theta + \frac{\partial^2 u}{\partial \theta^2} \cdot \frac{\sin^2\theta}{1-r^2} + \frac{\partial^2 u}{\partial r \partial \theta} \cdot \frac{-2\sin\theta \cos\theta}{1-r} + \frac{\partial u}{\partial r} \frac{\sin^2\theta}{1-r} \\
 & + \frac{\partial u}{\partial \theta} \cdot \frac{2\sin\theta \cos\theta}{1-r} \\
 \frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial r} \frac{1}{1-r} + \frac{\partial^2 u}{\partial \theta^2} \frac{1}{1-\sin^2\theta} \\
 & + \frac{\partial u}{\partial r} \cdot \frac{2}{1-r} + \frac{\partial u}{\partial \theta} \frac{\cos\theta}{1-\sin\theta}
 \end{aligned}$$