专题 常用曲线坐标系下的流体力学基本方程

一. 正交曲线坐标系

直角坐标系 $x_1 \quad x_2 \quad x_3$

正交曲线坐标系 $q_1 \quad q_2 \quad q_3$

对于一个矢量 $\vec{R} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

定义其在正交曲线坐标系下的基矢为 $\vec{e_i} = \frac{\partial \vec{R}}{\partial q_i} = \frac{\partial x_1}{\partial q_i}\hat{i} + \frac{\partial x_2}{\partial q_i}\hat{j} + \frac{\partial x_3}{\partial q_i}\hat{k}$

注意，正交曲线坐标系的基矢量在空间各处大小，方向均不同，且一般不为单位矢量。

那么，一个矢量 $\vec{a}$ 在正交曲线系下可以表示为

$$\vec{a} = a_1'\vec{e_1} + a_2'\vec{e_2} + a_3'\vec{e_3}$$

为了方便，我们一般用沿基矢方向的单位矢量来分解一个矢量。

为此，我们定义 $\hat{n}_i$ 为单位基矢量，h_i 为尺度因子

$$\hat{n}_i = \frac{\vec{e_i}}{h_i}, \quad h_i = |\vec{e_i}| = \sqrt{\left(\frac{\partial x_1}{\partial q_i}\right)^2 + \left(\frac{\partial x_2}{\partial q_i}\right)^2 + \left(\frac{\partial x_3}{\partial q_i}\right)^2}$$

现在，可以用单位基矢量表示矢量 $\vec{a}$：

$$\vec{a} = a_1\hat{n}_1 + a_2\hat{n}_2 + a_3\hat{n}_3$$

1. 单位基矢的偏导数

我们利用二阶连续偏导数可以交换求导次序的性质，有

$$\frac{\partial \vec{e_1}}{\partial q_2} = \frac{\partial \vec{e_2}}{\partial q_1} = \frac{\partial^2 \vec{R}}{\partial q_1 \partial q_2}$$

而 $\begin{cases} \frac{\partial \vec{e_1}}{\partial q_2} = \frac{\partial}{\partial q_2}(h_1\hat{n}_1) = h_1\frac{\partial \hat{n}_1}{\partial q_2} + \frac{\partial h_1}{\partial q_2}\hat{n}_1 \\ \frac{\partial \vec{e_2}}{\partial q_1} = \frac{\partial}{\partial q_1}(h_2\hat{n}_2) = h_2\frac{\partial \hat{n}_2}{\partial q_1} + \frac{\partial h_2}{\partial q_1}\hat{n}_2 \end{cases}$ 注意到正交曲线系的性质指出 $\frac{\partial \hat{n}_i}{\partial q_j} \parallel \hat{n}_j$，因此两部分分别对应相等

$$\Rightarrow \begin{cases} h_1\frac{\partial \hat{n}_1}{\partial q_2} = \frac{\partial h_2}{\partial q_1}\hat{n}_2 \\ h_2\frac{\partial \hat{n}_2}{\partial q_1} = \frac{\partial h_1}{\partial q_2}\hat{n}_1 \end{cases} \Rightarrow \frac{\partial \hat{n}_i}{\partial q_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial q_i}\hat{n}_j \quad (i \neq j)$$

利用 $\hat{n}_i=\hat{n}_j\times\hat{n}_k$ （i,j,k 为 1,2,3 的一个顺序排列）有

$$\frac{\partial \hat{n}_i}{\partial q_i}=\frac{\partial}{\partial q_i}(\hat{n}_j\times\hat{n}_k)=\frac{\partial \hat{n}_j}{\partial q_i}\times\hat{n}_k+\hat{n}_j\times\frac{\partial \hat{n}_k}{\partial q_i}$$

$$=\frac{1}{h_j}\frac{\partial h_i}{\partial q_j}\hat{n}_i\times\hat{n}_k+\hat{n}_j\times\frac{1}{h_k}\frac{\partial h_i}{\partial q_k}\hat{n}_i$$

$$=-\frac{1}{h_j}\frac{\partial h_i}{\partial q_j}\hat{n}_j-\frac{1}{h_k}\frac{\partial h_i}{\partial q_k}\hat{n}_k$$

至此，我们得到了正交曲线系下单位基矢的偏导数：

$$\begin{cases}\dfrac{\partial \hat{n}_i}{\partial q_j}=\dfrac{1}{h_i}\dfrac{\partial h_j}{\partial q_i}\hat{n}_j & (i\neq j)\\[2ex] \dfrac{\partial \hat{n}_i}{\partial q_i}=-\dfrac{1}{h_j}\dfrac{\partial h_i}{\partial q_j}\hat{n}_j-\dfrac{1}{h_k}\dfrac{\partial h_i}{\partial q_k}\hat{n}_k\end{cases}$$

2. 正交曲线系下的 Nabla 算符 ∇

利用梯度与方向导数的关系 $\frac{\partial}{\partial n}=\hat{n}\cdot\nabla$，可以得到正交曲线系下的方向导数

$$\hat{n}_i\cdot\nabla=\frac{\partial}{\partial e_i}=\frac{1}{h_i}\frac{\partial}{\partial q_i}$$

由此得到正交曲线系下的 Nabla 算符

$$\nabla=\frac{\hat{n}_1}{h_1}\frac{\partial}{\partial q_1}+\frac{\hat{n}_2}{h_2}\frac{\partial}{\partial q_2}+\frac{\hat{n}_3}{h_3}\frac{\partial}{\partial q_3}$$

(1) 梯度

$$\nabla a=\frac{\hat{n}_1}{h_1}\frac{\partial a}{\partial q_1}+\frac{\hat{n}_2}{h_2}\frac{\partial a}{\partial q_2}+\frac{\hat{n}_3}{h_3}\frac{\partial a}{\partial q_3}$$

(2) 散度

$$\nabla\cdot\vec{a}=\left(\frac{\hat{n}_1}{h_1}\frac{\partial}{\partial q_1}+\frac{\hat{n}_2}{h_2}\frac{\partial}{\partial q_2}+\frac{\hat{n}_3}{h_3}\frac{\partial}{\partial q_3}\right)\cdot(a_1\hat{n}_1+a_2\hat{n}_2+a_3\hat{n}_3)$$

$$=\frac{\hat{n}_1}{h_1}\left(\frac{\partial a_1}{\partial q_1}\hat{n}_1+\frac{\partial a_2}{\partial q_1}\hat{n}_2+\frac{\partial a_3}{\partial q_1}\hat{n}_3+a_1\frac{\partial \hat{n}_1}{\partial q_1}+a_2\frac{\partial \hat{n}_2}{\partial q_1}+a_3\frac{\partial \hat{n}_3}{\partial q_1}\right)+\cdots$$

$$=\frac{\hat{n}_1}{h_1}\left[\frac{\partial a_1}{\partial q_1}\hat{n}_1+a_1\left(-\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\hat{n}_2-\frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\hat{n}_3\right)+a_2\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\hat{n}_1+a_3\frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\hat{n}_1\right]+\cdots$$

$$=\frac{\hat{n}_1}{h_1}\left(\frac{\partial a_1}{\partial q_1}+\frac{a_2}{h_2}\frac{\partial h_1}{\partial q_2}+\frac{a_3}{h_3}\frac{\partial h_1}{\partial q_3}\right)\hat{n}_1+\cdots$$

$$= \frac{1}{h_1}\left(\frac{\partial a_1}{\partial q_1} + \frac{a_2}{h_2}\frac{\partial h_1}{\partial q_2} + \frac{a_3}{h_3}\frac{\partial h_1}{\partial q_3}\right)$$

$$+ \frac{1}{h_2}\left(\frac{\partial a_2}{\partial q_2} + \frac{a_3}{h_3}\frac{\partial h_2}{\partial q_3} + \frac{a_1}{h_1}\frac{\partial h_2}{\partial q_1}\right)$$

$$+ \frac{1}{h_3}\left(\frac{\partial a_3}{\partial q_3} + \frac{a_1}{h_1}\frac{\partial h_3}{\partial q_1} + \frac{a_2}{h_2}\frac{\partial h_3}{\partial q_2}\right)$$

$$= \frac{1}{h_1h_2h_3}\left[\left(h_2h_3\frac{\partial a_1}{\partial q_1} + a_1h_3\frac{\partial h_2}{\partial q_1} + a_1h_2\frac{\partial h_3}{\partial q_1}\right) + \cdots\right]$$

$$= \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{h_1h_2h_3}{h_i}a_i\right)$$

(3) 旋度

$$\nabla\times\vec{a} = \left(\frac{\hat{n}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{n}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{n}_3}{h_3}\frac{\partial}{\partial q_3}\right)\times\left(a_1\hat{n}_1 + a_2\hat{n}_2 + a_3\hat{n}_3\right)$$

$$= \frac{\hat{n}_1}{h_1}\frac{\partial}{\partial q_1}\times\left(a_1\hat{n}_1 + a_2\hat{n}_2 + a_3\hat{n}_3\right) + \cdots$$

$$= \frac{\hat{n}_1}{h_1}\times\left(\frac{\partial a_1}{\partial q_1}\hat{n}_1 + \frac{\partial a_2}{\partial q_1}\hat{n}_2 + \frac{\partial a_3}{\partial q_1}\hat{n}_3 + a_1\frac{\partial \hat{n}_1}{\partial q_1} + a_2\frac{\partial \hat{n}_2}{\partial q_1} + a_3\frac{\partial \hat{n}_3}{\partial q_1}\right) + \cdots$$

$$= \frac{\hat{n}_1}{h_1}\times\left[\frac{\partial a_2}{\partial q_1}\hat{n}_2 + \frac{\partial a_3}{\partial q_1}\hat{n}_3 + a_1\left(-\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\hat{n}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial q_3}\hat{n}_3\right)\right] + \cdots$$

$$= \frac{1}{h_1}\left[\left(\frac{a_1}{h_3}\frac{\partial h_1}{\partial q_3} - \frac{\partial a_3}{\partial q_1}\right)\hat{n}_2 + \left(\frac{\partial a_2}{\partial q_1} - \frac{a_1}{h_2}\frac{\partial h_1}{\partial q_2}\right)\hat{n}_3\right]$$

$$+ \frac{1}{h_2}\left[\left(\frac{a_2}{h_1}\frac{\partial h_2}{\partial q_1} - \frac{\partial a_1}{\partial q_2}\right)\hat{n}_3 + \left(\frac{\partial a_3}{\partial q_2} - \frac{a_2}{h_3}\frac{\partial h_2}{\partial q_3}\right)\hat{n}_1\right]$$

$$+ \frac{1}{h_3}\left[\left(\frac{a_3}{h_2}\frac{\partial h_3}{\partial q_2} - \frac{\partial a_2}{\partial q_3}\right)\hat{n}_1 + \left(\frac{\partial a_1}{\partial q_3} - \frac{a_3}{h_1}\frac{\partial h_3}{\partial q_1}\right)\hat{n}_2\right]$$

$$= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{n}_1 & h_2\hat{n}_2 & h_3\hat{n}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix}$$

至此，我们得到了正交曲线系下的基本运算关系.

下面讨论的流体作如下假设：

① 均匀 $\rho = const$　② 不受体积力

二、连续性方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

由于 $\rho = const$，上述方程化为 $\nabla \cdot \vec{v} = 0$

(1) 柱坐标

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} h_r = 1 \\ h_\theta = r \\ h_z = 1 \end{cases}$$

$$\nabla \cdot \vec{v} = \frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta} v_\theta + \frac{1}{r}\frac{\partial}{\partial z}(r v_z)$$

$$= \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

(2) 球坐标、

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow \begin{cases} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\sin\theta \end{cases}$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2\sin\theta} \frac{\partial}{\partial r}(r^2\sin\theta v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(r\sin\theta v_\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \varphi}(r v_\varphi)$$

$$= \frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi} = 0$$

三、动量方程

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v}\vec{v}) = \rho \vec{f} + \nabla \cdot \overleftrightarrow{\sigma}$$

由之前的假设 $\rho = const$，$\vec{f} = \vec{0}$，上述方程化为

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v}\vec{v}) = \frac{1}{\rho} \nabla \cdot \overleftrightarrow{\sigma}$$

由本构关系

$$\overleftrightarrow{\sigma} = -p\overleftrightarrow{\delta} + 2\mu \overleftrightarrow{e}$$

$$\nabla\cdot\overleftrightarrow{\sigma} = -\nabla\cdot(p\overleftrightarrow{\delta}) + 2\mu\,\nabla\cdot\overleftrightarrow{e}$$

其中第二项可以化简

$$(\nabla\cdot\overleftrightarrow{e})_i = \partial_j e_{ij} = \partial_j \frac{1}{2}(\partial_i v_j + \partial_j v_i) = \frac{1}{2}(\partial_i\partial_j v_j + \partial_j\partial_j v_i)$$

写成实体形式，并利用连续性方程 $\nabla\cdot\vec{v}=0$，得

$$\nabla\cdot\overleftrightarrow{e} = \frac{1}{2}\left[\nabla(\nabla\cdot\vec{v}) + \nabla^2\vec{v}\right] = \frac{1}{2}\nabla^2\vec{v}$$

关于方程左边第二项：

$$[\nabla\cdot(\vec{v}\vec{v})]_i = \partial_j(v_i v_j) = v_j\partial_j v_i + v_i\partial_j v_j$$

写成实体形式，并利用连续性方程 $\nabla\cdot\vec{v}=0$ 得

$$\nabla\cdot(\vec{v}\vec{v}) = (\vec{v}\cdot\nabla)\vec{v} + (\nabla\cdot\vec{v})\vec{v} = (\vec{v}\cdot\nabla)\vec{v}$$

至此，我们将满足假设的动量方程化简如下：

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla\cdot(p\overleftrightarrow{\delta}) + \nu\,\nabla^2\vec{v} \qquad \left(\nu = \frac{\mu}{\rho}\right)$$

下面我们讨论具体的曲线坐标系下方程的形式：

(1) 柱坐标

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} h_r = 1 \\ h_\theta = r \\ h_z = 1 \end{cases} \qquad \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_z \end{pmatrix} \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \hat{\theta} & -\hat{r} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
(\vec{v}\cdot\nabla)\vec{v} &= (v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z})\cdot\left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z\right)(v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}) \\
&= \left(v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + v_z\partial_z\right)(v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}) \\
&= v_r\frac{\partial v_r}{\partial r}\hat{r} + v_r\frac{\partial v_\theta}{\partial r}\hat{\theta} + v_r\frac{\partial v_z}{\partial r}\hat{z} \\
&+ \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}\hat{r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}\hat{\theta} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}\hat{z} + \frac{v_\theta v_r}{r}\hat{\theta} - \frac{v_\theta^2}{r}\hat{r} \\
&+ v_z\frac{\partial v_r}{\partial z}\hat{r} + v_z\frac{\partial v_\theta}{\partial z}\hat{\theta} + v_z\frac{\partial v_z}{\partial z}\hat{z}
\end{aligned}$$

$$= \left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right)\hat{r}$$

$$+ \left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right)\hat{\theta}$$

$$+ \left(v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right)\hat{z}$$

$$\nabla^2 = \left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z\right)\left(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \hat{z}\partial_z\right)$$

$$= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$$

$$\nabla^2 \vec{v} = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2\right)\left(v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}\right)$$

$$= \frac{\partial^2 v_r}{\partial r^2}\hat{r} + \frac{\partial^2 v_\theta}{\partial r^2}\hat{\theta} + \frac{\partial^2 v_z}{\partial r^2}\hat{z} + \frac{1}{r}\frac{\partial v_r}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial r}\hat{\theta} + \frac{1}{r}\frac{\partial v_z}{\partial r}\hat{z}$$

$$+ \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - v_r\right)\hat{r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\hat{\theta} + \frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2} - v_\theta\right)\hat{\theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\hat{r}$$

$$+ \frac{\partial^2 v_r}{\partial z^2}\hat{r} + \frac{\partial^2 v_\theta}{\partial z^2}\hat{\theta} + \frac{\partial^2 v_z}{\partial z^2}\hat{z}$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)\hat{r}$$

$$+ \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right)\hat{\theta}$$

$$+ \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\hat{z}$$

$$-\frac{1}{\rho}\nabla\cdot(p\overleftrightarrow{\delta}) = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{z}\right)$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial v_r}{\partial t}\hat{r} + \frac{\partial v_\theta}{\partial t}\hat{\theta} + \frac{\partial v_z}{\partial t}\hat{z}$$

(2) 球坐标

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\sin\theta \end{cases} \qquad \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\varphi \end{pmatrix}\begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\varphi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \hat{\theta} & -\hat{r} & 0 \\ \sin\theta\,\hat{\varphi} & \cos\theta\,\hat{\varphi} & -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta} \end{pmatrix}$$

$$(\vec{v}\cdot\nabla)\vec{v} = (v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi})\cdot(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\varphi}}{r\sin\theta}\partial_\varphi)(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi})$$

$$= (v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\varphi}{r\sin\theta}\partial_\varphi)(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi})$$

$$= v_r\frac{\partial v_r}{\partial r}\hat{r} + v_r\frac{\partial v_\theta}{\partial r}\hat{\theta} + v_r\frac{\partial v_\varphi}{\partial r}\hat{\varphi}$$

$$+ \frac{v_\theta}{r}(\frac{\partial v_r}{\partial\theta} - v_\theta)\hat{r} + \frac{v_\theta}{r}(\frac{\partial v_\theta}{\partial\theta} + v_r)\hat{\theta} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta}\hat{\varphi}$$

$$+ \frac{v_\varphi}{r\sin\theta}(\frac{\partial v_r}{\partial\varphi} - \sin\theta\, v_\varphi)\hat{r} + \frac{v_\varphi}{r\sin\theta}(\frac{\partial v_\theta}{\partial\varphi} - \cos\theta\, v_\varphi)\hat{\theta}$$

$$+ \frac{v_\varphi}{r\sin\theta}(\frac{\partial v_\varphi}{\partial\varphi} - \sin\theta\, v_r - \cos\theta\, v_\theta)\hat{\varphi}$$

$$= (v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{v_\theta^2 + v_\varphi^2}{r})\hat{r}$$

$$+ (v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi} + \frac{v_r v_\theta}{r} - \frac{v_\varphi^2\cot\theta}{r})\hat{\theta}$$

$$+ (v_r\frac{\partial v_\varphi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_r v_\varphi}{r} - \frac{v_\theta v_\varphi\cot\theta}{r})\hat{\varphi}$$

$$\nabla^2 = (\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\varphi}}{r\sin\theta}\partial_\varphi)\cdot(\hat{r}\partial_r + \frac{\hat{\theta}}{r}\partial_\theta + \frac{\hat{\varphi}}{r\sin\theta}\partial_\varphi)$$

$$= \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 + \frac{1}{r}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta$$

$$\nabla^2\vec{v} = (\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 + \frac{2}{r}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta)(v_r\hat{r} + v_\theta\hat{\theta} + v_\varphi\hat{\varphi})$$

$$= \frac{\partial^2 v_r}{\partial r^2}\hat{r} + \frac{\partial^2 v_\theta}{\partial r^2}\hat{\theta} + \frac{\partial^2 v_\varphi}{\partial r^2}\hat{\varphi} + \frac{2}{r}\frac{\partial v_r}{\partial r}\hat{r} + \frac{2}{r}\frac{\partial v_\theta}{\partial r}\hat{\theta} + \frac{2}{r}\frac{\partial v_\varphi}{\partial r}\hat{\varphi}$$

$$+ \frac{1}{r^2}(\frac{\partial^2 v_r}{\partial\theta^2} - v_r)\hat{r} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\hat{\theta} + \frac{1}{r^2}(\frac{\partial^2 v_\theta}{\partial\theta^2} - v_\theta)\hat{\theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\hat{r} + \frac{\partial^2 v_\varphi}{\partial\theta^2}\hat{\varphi}$$

$$+ \frac{\cot\theta}{r^2}(\frac{\partial v_r}{\partial\theta} - v_\theta)\hat{r} + \frac{\cot\theta}{r^2}(\frac{\partial v_\theta}{\partial\theta} + v_r)\hat{\theta} + \frac{\cot\theta}{r^2}\frac{\partial v_\varphi}{\partial\theta}\hat{\varphi}$$

$$+ \frac{1}{r^2\sin^2\theta}(\frac{\partial^2 v_r}{\partial\varphi^2} - v_r\sin^2\theta - v_\theta\sin\theta\cos\theta - 2\frac{\partial v_\varphi}{\partial\varphi}\sin\theta)\hat{r}$$

$$+ \frac{1}{r^2\sin^2\theta}(\frac{\partial^2 v_\theta}{\partial\varphi^2} - v_\theta\cos^2\theta - v_r\sin\theta\cos\theta - 2\frac{\partial v_\varphi}{\partial\varphi}\cos\theta)\hat{\theta}$$

$$+ \frac{1}{r^2\sin^2\theta}(\frac{\partial^2 v_\varphi}{\partial\varphi^2} - v_\varphi\sin^2\theta - v_\varphi\cos^2\theta + 2\frac{\partial v_r}{\partial\varphi}\sin\theta + 2\frac{\partial v_\theta}{\partial\varphi}\cos\theta)\hat{\varphi}$$

$$= \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2 v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2 v_\theta \cot\theta}{r^2} \right.$$

$$\left. - \frac{2}{r^2 \sin\theta}\frac{\partial v_\varphi}{\partial \varphi} \right) \hat{r}$$

$$+ \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \varphi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta} \right.$$

$$\left. - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\varphi}{\partial \varphi} \right) \hat{\theta}$$

$$+ \left(\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial v_\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\varphi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\cot\theta}{r^2}\frac{\partial v_\varphi}{\partial \theta} - \frac{v_\varphi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial \varphi} \right.$$

$$\left. + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\theta}{\partial \varphi} \right) \hat{\varphi}$$

$$-\frac{1}{\rho} \nabla \cdot (p \breve{\delta}) = -\frac{1}{\rho} \left(\frac{\partial p}{\partial r} \hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta}\frac{\partial p}{\partial \varphi} \hat{\varphi} \right)$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial v_r}{\partial t} \hat{r} + \frac{\partial v_\theta}{\partial t} \hat{\theta} + \frac{\partial v_\varphi}{\partial t} \hat{\varphi}$$