2019-2020春 数理方程B 毕业班考试答案

一.(1) $\begin{cases} u_{tt}=4u_{xx}, & t>0,\ -\infty<x<+\infty \\ u|_{t=0}=x^2, & u_t|_{t=0}=\cos 2x \end{cases}$

解：通解 $u(t,x)=f(x-2t)+g(x+2t)$

代入初值条件：

$$\begin{cases} u(0,x)=f(x)+g(x)=x^2 & ① \\ u_t(0,x)=-2f'(x)+2g'(x)=\cos 2x & ② \end{cases}$$

②积分得

$$-f(x)+g(x)=\frac{1}{4}\sin 2x \qquad ③$$

联立①③解得

$$\begin{cases} f(x)=\frac{1}{2}\left(x^2-\frac{1}{4}\sin 2x\right) \\ g(x)=\frac{1}{2}\left(x^2+\frac{1}{4}\sin 2x\right) \end{cases}$$

$$\Rightarrow u(t,x)=\frac{1}{2}\left[(x-2t)^2-\frac{1}{4}\sin 2(x-2t)\right]$$
$$+\frac{1}{2}\left[(x+2t)^2+\frac{1}{4}\sin 2(x+2t)\right]$$

(2) $\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 20 \\ u(0,y) = y^2, \quad u(x,0) = \sin x \end{cases}$

解: $\dfrac{\partial^2 u}{\partial x \partial y} = 20 \xrightarrow{\text{积分}} u(x,y) = 20xy + f(x) + g(y)$

代入初值条件

$$\begin{cases} u(0,y) = g(y) = y^2 \\ u(x,0) = f(x) = \sin x \end{cases}$$

$$\Rightarrow u(t,x) = 20xy + y^2 + \sin x$$

二. (1) $\begin{cases} Y''(x)+\lambda Y(x)=0, & 0<x<\pi \\ Y'(0)=0, & Y'(\pi)=0 \end{cases}$

解：由 S-L 定理 $\Rightarrow \lambda \geq 0$

不含 I.III 类边界条件 $\Rightarrow \lambda=0$ 对应 $Y_0(x)=1$

$\lambda=k^2>0$ 时，通解为

$Y(x)=A\cos kx+B\sin kx$

代入边界条件：

$$\begin{cases} Y'(0)=Bk=0 \Rightarrow B=0 \\ Y'(\pi)=-Ak\sin k\pi=0 \Rightarrow k=n, \quad n=1,2,3,\cdots \end{cases}$$

$\Rightarrow$ 固有值 $\lambda_n=n^2$，相应的固有函数 $Y_n(x)=\cos nx$

(2) $\begin{cases} x^2Y''(x)+xY'(x)+\lambda Y(x)=0, & 1<x<b \\ Y(1)=0, & Y'(b)=0 \end{cases}$

解：由 S-L 定理，含 I 类边界条件 $\Rightarrow \lambda=k^2>0$

令 $x=e^t$，$t=\ln x$，则

$$\frac{dY}{dx}=\frac{dY}{dt}\cdot\frac{dt}{dx}=\frac{1}{x}\cdot\frac{dY}{dt}$$

$$\frac{d^2Y}{dx^2}=\frac{1}{x}\cdot\frac{d^2Y}{dt^2}\cdot\frac{dt}{dx}-\frac{1}{x^2}\cdot\frac{dY}{dt}=\frac{1}{x^2}\left(\frac{d^2Y}{dt^2}-\frac{dY}{dt}\right)$$

$$\Rightarrow x^2Y''(x)+xY'(x)=\frac{d^2Y}{dt^2}$$

$\Rightarrow$ 原方程化为 $\frac{d^2Y}{dt^2}+\lambda Y=0$

通解 $Y(t)=A\cos kt+B\sin kt$

$$Y(x)=A\cos k\ln x+B\sin k\ln x$$

代入边界条件：

$$\begin{cases} Y(1)=A=0 \\ Y'(b)=B\cos k\ln b\cdot\frac{k}{b}=0 \Rightarrow k\ln b=\frac{\pi}{2}+n\pi, \end{cases}$$

$n=0,1,2,\cdots$

$\Rightarrow$ 固有值 $\lambda_n=\left(\frac{2n+1}{2\ln b}\pi\right)^2$,

相应的固有函数 $Y_n(x)=\sin\sqrt{\lambda_n}\ln x$

三. (1) $\begin{cases} u_{tt} = u_{xx}, t>0, 0<x<1 \\ u(t,0) = u(t,1), u_x(t,0) = u_x(t,1) \end{cases}$

解：令 $u(t,x) = T(t)X(x)$，有

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

解固有值问题

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(1), X'(0) = X'(1) \end{cases}$$

由S-L定理，含I类周期边界条件，$\lambda = k^2 > 0$

通解 $X(x) = A\cos kx + B\sin kx$

代入边界条件：

$$\begin{cases} X(0) = A = A\cos k + B\sin k = X(1) \\ X'(0) = kB = -kA\sin k + kB\cos k = X'(1) \end{cases}$$

整理得 $\begin{pmatrix} \cos k - 1 & \sin k \\ -\sin k & \cos k - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

要使上述方程组有非零解，系数行列式应为0：

$$\begin{vmatrix} \cos k-1 & \sin k \\ -\sin k & \cos k-1 \end{vmatrix} = (\cos k-1)^2+\sin^2 k = 2-2\cos k = 0$$

$\Rightarrow \cos k = 1$，$k = 2n\pi$，$n=1,2,3,\cdots$

$\Rightarrow$ 固有值 $\lambda_n = 4n^2\pi^2$，

相应的固有函数 $X_n(x) = \begin{Bmatrix} \cos 2n\pi x \\ \sin 2n\pi x \end{Bmatrix}$

解关于 t 的方程：$T''(t)+\lambda T(t)=0$

$\Rightarrow T_n(t) = C_n\cos 2n\pi t + D_n \sin 2n\pi t$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty}(C_n\cos 2n\pi t+D_n\sin 2n\pi t)\cdot\cos 2n\pi x + (C_n'\cos 2n\pi t+D_n'\sin 2n\pi t)\cdot\sin 2n\pi x$$

(2) $\begin{cases} u_{tt}=u_{xx}, \quad t>0,\ 0<x<1 \\ u(t,0)=u(t,1),\ u_x(t,0)=u_x(t,1) \\ u(0,x)=\sin 2\pi x,\ u_t(0,x)=2\pi\cos 2\pi x \end{cases}$

解：由(1)已知分离变量解的形式

代入初始条件：

$$\begin{cases} u(0,x) = \sum_{n=1}^{\infty} C_n \cos 2n\pi x + C_n' \sin 2n\pi x = \sin 2\pi x \\ \qquad \text{由固有函数的正交性 } C_1' = 1\text{，其余系数为}0 \\ u_t(0,x) = \sum_{n=1}^{\infty} 2n\pi \left(D_n \cos 2n\pi x + D_n' \sin 2n\pi x\right) = 2\pi \cos 2\pi x \\ \qquad \text{由固有函数的正交性 } D_1 = 1\text{，其余系数为}0 \end{cases}$$

$$\Rightarrow u(t,x) = \cos 2\pi t \sin 2\pi x + \sin 2\pi t \cos 2\pi x = \sin 2\pi (t+x)$$

四. $\begin{cases} u_t = u_{xx} + u, & t>0, -\infty<x<+\infty \\ u|_{t=0} = \delta(x+1) \end{cases}$

解：作傅立叶变换：

$$\begin{cases} \hat{u}_t = (-\lambda^2+1)\hat{u} \\ \hat{u}|_{t=0} = e^{i\lambda} \end{cases}$$

$\Rightarrow$ 通解 $\hat{u}(t,\lambda) = C e^{(1-\lambda^2)t}$

代入初值条件 $\hat{u}(0,\lambda) = C = e^{i\lambda}$

$\Rightarrow \hat{u}(t,\lambda) = e^{(1-\lambda^2)t} \cdot e^{i\lambda} = e^t \cdot e^{-t\lambda^2} \cdot e^{i\lambda}$

计算反变换：

$$F^{-1}[e^{-t\lambda^2}] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t\lambda^2} \cdot e^{i\lambda x} d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t(\lambda - \frac{ix}{2t})^2} d\lambda \cdot e^{-\frac{x^2}{4t}}$$

$$= \frac{1}{2\sqrt{\pi t}} \cdot e^{-\frac{x^2}{4t}}$$

$$\Rightarrow u(t,x) = e^t \cdot \left[\frac{1}{2\sqrt{\pi t}} \cdot e^{-\frac{x^2}{4t}} * \delta(x+1)\right]$$

$$= \frac{e^t}{2\sqrt{\pi t}} \cdot e^{-\frac{(x+1)^2}{4t}}$$

五. (1) $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

$$\int_{-1}^{1}(20+x)P_2(x)\,dx$$

$$=20\int_{-1}^{1}P_0(x)P_2(x)\,dx+\int_{-1}^{1}P_1(x)P_2(x)\,dx=0$$

(2) $\begin{cases}\Delta_3 u=0, & r<2\\ u|_{r=2}=3\cos 2\theta\end{cases}$

解：球内轴对称问题通解：

$$u(r,\theta)=\sum_{n=0}^{\infty}C_n\left(\frac{r}{2}\right)^n P_n(\cos\theta)$$

代入边界条件：

$$u(2,\theta)=\sum_{n=0}^{\infty}C_nP_n(\cos\theta)=3\cos 2\theta$$

其中 $3\cos 2\theta=3\cdot(2\cos^2\theta-1)=6\cos^2\theta-3$

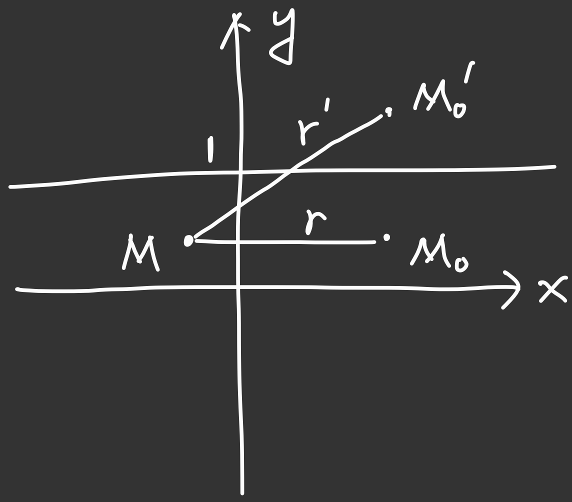
$$=4\cdot\frac{1}{2}(3\cos^2\theta-1)-1=4P_2(\cos\theta)-P_0(\cos\theta)$$

$$\Rightarrow u(2,\theta)=\sum_{n=0}^{\infty}C_nP_n(\cos\theta)=4P_2(\cos\theta)-P_0(\cos\theta)$$

$\Rightarrow C_0=-1$, $C_2=4$, 其余为 0

$$\Rightarrow u(r,\theta)=r^2\cdot P_2(\cos\theta)-1$$

六. (1) $\begin{cases} \Delta_2 G = -\delta(M-M_0) \\ G|_{\partial D} = G|_{y=1} = 0 \end{cases}$



$$G(M;M_0) = \frac{1}{2\pi}\ln r - \frac{1}{2\pi}\ln r'$$

$$= \frac{1}{2\pi}\ln\frac{r}{r'} = \frac{1}{2\pi}\ln\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y-2+\eta)^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{4\pi}\ln\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y-2+\eta)^2}$$

(2) $\begin{cases} u_{xx} + a^2 u_{yy} = 0, \quad -\infty < x < +\infty, \ y < 1 \\ u|_{y=1} = \varphi(x) \end{cases}$

令 $s = x$, $t = \frac{y}{a}$, 则问题化为

$$\begin{cases} u_{ss} + u_{tt} = 0, \quad -\infty < s < +\infty, \ t < \frac{1}{a} \\ u|_{t=\frac{1}{a}} = \varphi(s) \end{cases}$$

由(1)知基本解为 $G(M;M_0) = \frac{1}{4\pi}\ln\frac{(s-\xi)^2+(t-\eta)^2}{(s-\xi)^2+(t-\frac{2}{a}+\eta)^2}$

$$\left.\frac{\partial G}{\partial n}\right|_{\eta=\frac{1}{a}} = \left.\frac{\partial G}{\partial \eta}\right|_{\eta=\frac{1}{a}} = \frac{1}{\pi}\frac{t-\frac{1}{a}}{(s-\xi)^2+(t-\frac{1}{a})^2}$$

$$\Rightarrow u(s,t) = -\int_{-\infty}^{+\infty} \varphi(\xi)\cdot \frac{\partial G}{\partial n}\Big|_{\eta=\frac{1}{a}} \cdot d\xi$$

$$= -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(\xi)(t-\frac{1}{a})}{(s-\xi)^2+(t-\frac{1}{a})^2} d\xi$$

$$\Rightarrow u(x,y) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(\xi)(\frac{y}{a}-\frac{1}{a})}{(x-\xi)^2+(\frac{y}{a}-\frac{1}{a})^2} d\xi$$

$$= -\frac{a}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(\xi)(y-1)}{a^2(x-\xi)^2+(y-1)^2} d\xi$$