关于拉普拉斯算子在三大坐标系下的表达的应用，大家应该有所体会。而关于这个表达式考试时是否会给出，这个目前没有明确的消息，按照往年的试卷来看是会给出的。但是还是建议大家具备自己能够推导出的能力。这里把之前在群里分享的我的推导思路整理一下，供大家参考。

$$\Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\Delta_2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$r = \sqrt{x^2+y^2} \quad \theta = \arctan\frac{y}{x}$$

$$\cancel{\partial u} \ \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\cdot\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right)\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right)\cdot\frac{\partial \theta}{\partial x}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\cos\theta + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\left(-\frac{\sin\theta}{r}\right) + \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r^2}\right)\cdot\cos\theta$$

$$+ \left(\frac{\partial^2 u}{\partial r\partial\theta}\cdot\cos\theta + \frac{\partial u}{\partial r}(-\sin\theta) + \frac{\partial^2 u}{\partial \theta^2}\left(-\frac{\sin\theta}{r}\right) + \frac{\partial u}{\partial \theta}\cdot\left(-\frac{\cos\theta}{r}\right)\right)\cdot\left(-\frac{\sin\theta}{r}\right).$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\sin^2\theta}{r^2} - \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{2\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\cdot\cancel{r^2}\,\frac{\sin^2\theta}{r} + \frac{\partial u}{\partial \theta}\cdot\frac{2\sin\theta\cos\theta}{r^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2} = \cancel{\frac{\partial u}{\partial r}}\ \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\cdot\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right)\cdot\frac{\partial \theta}{\partial y}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{\cos\theta}{r} + \frac{\partial u}{\partial \theta}\left(-\frac{\cos\theta}{r^2}\right)\right)\cdot\sin\theta +$$

$$\left(\frac{\partial^2 u}{\partial r\partial\theta}\cdot\sin\theta + \frac{\partial u}{\partial r}\cdot\cos\theta + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\cos\theta}{r} + \frac{\partial u}{\partial \theta}\frac{-\sin\theta}{r}\right)\cdot\frac{\cos\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\cos^2\theta}{r^2} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\cdot\frac{\cos^2\theta}{r} - \frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{1}{r^2} + \frac{\partial^2 u}{\partial \varphi^2}\cdot\frac{1}{r^2\sin\theta} + \frac{\partial u}{\partial r}\cdot\frac{2}{r} + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r^2\sin\theta}$$

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \theta = \arctan\frac{\sqrt{x^2+y^2}}{z} \qquad \varphi = \arctan\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\varphi \qquad \frac{\partial r}{\partial y} = \sin\theta\sin\varphi \qquad \frac{\partial r}{\partial z} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{x^2+y^2}{z^2}}\cdot\frac{1}{z}\cdot\frac{2x}{2\sqrt{x^2+y^2}} = \frac{\cos\theta\cos\varphi}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta\sin\varphi}{r} \qquad \frac{\partial \theta}{\partial z} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \varphi}{\partial x} = \frac{-\sin\varphi}{r\sin\theta} \qquad \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\theta} \qquad \frac{\partial \varphi}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial x} = \frac{\partial u}{\partial r}\cdot\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\cdot\left(-\frac{\sin\varphi}{r\sin\theta}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\,\frac{-\sin\varphi}{r\sin\theta}\right)\cdot\frac{\partial r}{\partial x}$$
$$+ \frac{\partial}{\partial \theta}\cdot\left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\,\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\cdot\frac{-\sin\varphi}{r\sin\theta}\right)\frac{\partial \theta}{\partial x}$$
$$+ \frac{\partial}{\partial \varphi}\cdot\left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\,\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\,\frac{-\sin\varphi}{r\sin\theta}\right)\frac{\partial \varphi}{\partial x}$$

$$+ \frac{\partial}{\partial \theta}\cdot\left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\,\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\cdot\frac{-\sin\varphi}{r\sin\theta}\right)\frac{\partial \theta}{\partial x}$$
$$+ \frac{\partial}{\partial \varphi}\cdot\left(\frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial u}{\partial \theta}\,\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \varphi}\,\frac{-\sin\varphi}{r\sin\theta}\right)\frac{\partial \varphi}{\partial x}$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta\cos\varphi + \frac{\partial^2 u}{\partial r\partial\theta}\,\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \theta}\cdot\frac{-\cos\theta\cos\varphi}{r^2} + \frac{\partial^2 u}{\partial r\partial\varphi}\,\frac{-\sin\varphi}{r\sin\theta} + \frac{\partial u}{\partial \varphi}\,\frac{\sin\varphi}{r^2\sin\theta}\right)$$
$$\cdot\sin\theta\cos\varphi + \left(\frac{\partial^2 u}{\partial r\partial\theta}\cdot\sin\theta\cos\varphi + \frac{\partial u}{\partial r}\cos\theta\cos\varphi + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\cos\theta\cos\varphi}{r} + \frac{\partial u}{\partial \theta}\,\frac{-\sin\theta\cos\varphi}{r} + \frac{\partial^2 u}{\partial\theta\partial\varphi}\,\frac{-\sin\varphi}{r\sin\theta}\right.$$
$$\left.+ \frac{\partial u}{\partial \varphi}\,\frac{\sin\varphi\cos\theta}{r\sin^2\theta}\right)\frac{\cos\theta\cos\varphi}{r} + \left(\frac{\partial^2 u}{\partial r\partial\varphi}\sin\theta\cos\varphi + \frac{\partial u}{\partial r}\sin\theta(-\sin\varphi) + \frac{\partial^2 u}{\partial\theta\partial\varphi}\,\frac{\cos\theta\cos\varphi}{r}\right.$$
$$\left.+ \frac{\partial u}{\partial \theta}\cdot\frac{-\cos\theta\sin\varphi}{r} + \frac{\partial^2 u}{\partial \varphi^2}\cdot\frac{-\sin\varphi}{r\sin\theta} + \frac{\partial u}{\partial \varphi}\,\frac{-\cos\varphi}{r\sin\theta}\right)\cdot\frac{-\sin\varphi}{r\sin\theta}.$$

$$= \frac{\partial^2 u}{\partial r^2}\cdot\sin^2\theta\cos^2\varphi + \frac{\partial^2 u}{\partial \theta^2}\cdot\frac{\cos^2\theta\cos^2\varphi}{r^2} + \frac{\partial^2 u}{\partial \varphi^2}\,\frac{\sin^2\varphi}{r^2\sin^2\theta} + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{2\sin\theta\cos\theta\cos^2\varphi}{r}$$
$$+ \frac{\partial^2 u}{\partial r\partial\varphi}\,\frac{-2\sin\varphi\cos\varphi}{r} + \frac{\partial^2 u}{\partial\theta\partial\varphi}\cdot\frac{-2\cos\theta}{r^2\sin\theta}\quad\frac{-2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta} + \frac{\partial u}{\partial r}\,\frac{\cos^2\theta\cos^2\varphi}{r}\sin^2\varphi$$
$$+ \frac{\partial u}{\partial \theta}\cdot\left(-\frac{2\sin\theta\cos\theta\cos^2\varphi}{r^2} + \frac{\cos\theta\sin^2\varphi}{r^2\sin\theta}\right) + \frac{\partial u}{\partial \varphi}\cdot\frac{2\sin\varphi\cos\varphi}{r^2\sin^2\theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \varphi}\cdot\frac{\partial \varphi}{\partial y} = \frac{\partial u}{\partial r}\sin\theta\sin\varphi + \frac{\partial u}{\partial \theta}\,\frac{\cos\theta\sin\varphi}{r} + \frac{\partial u}{\partial \varphi}\cdot\frac{\cos\varphi}{r\sin\theta}.$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial r^2}\sin\theta\sin\varphi + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{\cos\theta\sin\varphi}{r} + \frac{\partial u}{\partial \theta}\cdot\frac{-\cos\theta\sin\varphi}{r^2} + \frac{\partial^2 u}{\partial r\partial\varphi}\,\frac{\cos\varphi}{r\sin\theta} + \frac{\partial u}{\partial \varphi}\,\frac{-\cos\varphi}{r^2\sin\theta}\right)$$

$$\cdot \sin\theta\sin\varphi + \left(\frac{\partial^2 u}{\partial r\partial\theta}\sin\theta\sin\varphi + \frac{\partial u}{\partial r}\cos\theta\sin\varphi + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{\cos\theta\sin\varphi}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{-\sin\theta\sin\varphi}{r}\right.$$

$$\left. + \frac{\partial^2 u}{\partial\theta\partial\varphi}\frac{\cos\varphi}{r\sin\theta} + \frac{\partial u}{\partial\varphi}\cdot\frac{-\cos\theta\cos\varphi}{r\sin\theta}\right)\frac{\cos\theta\sin\varphi}{r} + \left(\frac{\partial^2 u}{\partial r\partial\varphi}\cdot\sin\theta\sin\varphi\right.$$

$$+ \frac{\partial u}{\partial r}\sin\theta\cos\varphi + \frac{\partial^2 u}{\partial\theta\partial\varphi}\cdot\frac{\cos\theta\sin\varphi}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta\cos\varphi}{r} + \frac{\partial^2 u}{\partial\varphi^2}\frac{\cos\varphi}{r\sin\theta}$$

$$\left. + \frac{\partial u}{\partial\varphi}\cdot\frac{-\sin\varphi}{r\sin\theta}\right)\frac{\cos\varphi}{r\sin\theta}$$

$$= \frac{\partial^2 u}{\partial r^2}\sin^2\theta\sin^2\varphi + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{\cos^2\theta\sin^2\varphi}{r^2} + \frac{\partial^2 u}{\partial\varphi^2}\cdot\frac{\cos^2\varphi}{r^2\sin^2\theta} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{2\sin\theta\cos\theta\sin^2\varphi}{r}$$

$$+ \frac{\partial^2 u}{\partial r\partial\varphi}\frac{2\sin\varphi\cos\varphi}{r} + \frac{\partial^2 u}{\partial\theta\partial\varphi}\frac{2\cos\theta\sin\varphi\cos\varphi}{r^2\sin\theta} + \frac{\partial u}{\partial r}\frac{\cos^2\theta\sin^2\varphi + \cos^2\varphi}{r}$$

$$+ \frac{\partial u}{\partial\theta}\left(\frac{-2\sin\theta\cos\theta\sin^2\varphi}{r^2} + \frac{\cos\theta\cos^2\varphi}{r^2\sin\theta}\right) + \frac{\partial u}{\partial\varphi}\frac{-2\sin\varphi\cos\varphi}{r^2\sin^2\theta}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial z} + \frac{\partial u}{\partial\varphi}\cdot\frac{\partial\varphi}{\partial z} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial\theta}\frac{-\sin\theta}{r}$$

$$\frac{\partial^2 u}{\partial z^2} = \left(\frac{\partial^2 u}{\partial r^2}\cos\theta + \frac{\partial^2 u}{\partial r\partial\theta}\frac{-\sin\theta}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{\sin\theta}{r^2}\right)\cos\theta$$

$$+ \left(\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta + \frac{\partial u}{\partial r}(-\sin\theta) + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{-\sin\theta}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{-\cos\theta}{r}\right)\cdot\frac{-\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cdot\cos^2\theta + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{\sin^2\theta}{r^2} + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{-2\sin\theta\cos\theta}{r} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$+ \frac{\partial u}{\partial\theta}\cdot\frac{2\sin\theta\cos\theta}{r^2}$$

$$\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{1}{r^2} + \frac{\partial^2 u}{\partial\varphi^2}\frac{1}{r^2\sin^2\theta}$$

$$+ \frac{\partial u}{\partial r}\cdot\frac{2}{r} + \frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta}{r^2\sin\theta}$$