数理方程　历年真题汇总

说明

1. 这里收录了若干套中国科学技术大学数理方程(A/B)考试试题，对扫描质量较差的黑心书店版本试卷内容进行LaTeX科技排版，方便读者阅读使用.
2. 按照考试时间先后排序，其次为A、B卷. 修读数理方程B的同学可以完成大部分数理方程A的试题.
3. 本试题集的主要作用是供同学们考试之前模拟使用，越靠近现在的考卷越能接近现在的出题风格.
4. 参考答案仅给出结果，不保证正确性，希望读者自行思考，同时熟悉题目类型. 建议助教在考前习题课针对一些易错题集中讲解.
5. 不同试卷的参考公式不一，教学组没有明确考试会给哪些公式，读者备考时尽量多记诵一些以防万一.
6. 不同读者的复习备考方法不尽相同，敬请读者根据自己的需求使用本试题集.
7. 感谢鄢雯哲助教核对试卷！感谢吴天助教的指导！预祝读者在期末考试取得满意的成绩！

2019-2020春季学期 数理方程B助教

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欢迎访问课程主页: 2020春数理方程B 001549.02

2001-2002学年第一学期数理方程期末试题

注：考试时间两小时, 前七题中选做六题, 第八题必做. 试卷中$a>0$是常数.

一. (15分)解定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+2x,\ (t>0,-\infty<x<\infty),\\ u(t,x)|_{t=0}=0,\ \frac{\partial u}{\partial t}|_{t=0}=3x^2.\end{cases}$$

二. (15分)线性偏微分算子$L=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x\partial y}-2\frac{\partial^2}{\partial y^2}$,

1. 求方程$L[u]=0$的通解;

2. 解定解问题

$$\begin{cases}L[u]=0,\ (y>0,-\infty<x<+\infty),\\ u(x,y)|_{y=0}=\sin x,\ \frac{\partial}{\partial y}|_{y=0}=0.\end{cases}$$

三. 解定解问题(15分)

1.

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ (t>0,0<x<l),\\ u(t,x)|_{x=0}=\frac{\partial u}{\partial x}|_{x=l}=0,\\ u(t,x)|_{t=0}=\phi(x),\ (\phi(0)=0).\end{cases}$$

2.

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ (t>0,0<x<l),\\ u(t,x)|_{x=0}=u_0,\ \frac{\partial u}{\partial x}|_{x=l}=\frac{q_0}{k},\\ u(t,x)|_{t=0}=u_0.\end{cases}$$

其中u_0,q_0,k为常数.

四. (15分)

1. 求解Laplace方程的边值问题

$$\begin{cases}\Delta_2 u=0,\ (r=\sqrt{x^2+y^2}<1),\\ \frac{\partial u}{\partial r}|_{r=1}=\cos^2\theta-\sin^2\theta.\end{cases}$$

2. 如果把边界条件改为$\frac{\partial}{\partial r}|_{r=1}=f(\theta)$，$f(\theta)=f(\theta+2\pi)$且有一阶连续导数及分段二阶连续导数，上述边值问题是否一定有解？为什么？

五. (15分)解定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},\ (t>0,x>0),\\ \left(u-\frac{\partial u}{\partial x}\right)|_{x=0}=0,\\ u(t,x)|_{t=0}=1,\ \frac{\partial u}{\partial t}|_{t=0}=0.\end{cases}$$

六. (15分)

1. 解定解问题

$$\begin{cases}\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}=-\delta(x-\xi,y-\eta),\ (x>0,\xi<+\infty;\ y>0,\eta<+\infty),\\ G(x,y;\xi,\eta)|_{x=0}=G(x,y;\xi,\eta)|_{y=0}=0.\end{cases}$$

2. 利用1)中的$G(x,y;\xi,\eta)$写出定解问题

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ (x>0;\ y>0),\\ u(x,y)|_{x=0}=\phi(y),\ u(x,y)|_{y=0}=\psi(x).\ (\phi(0)=\psi(0))\end{cases}$$

解的积分公式.

七. (15分)求初值问题

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\Delta_2 u+b_1\frac{\partial u}{\partial x}+b_2\frac{\partial u}{\partial y}+cu+f(t,x,y),\ (t>0,-\infty<x,y<+\infty),\\ u(t,x,y)|_{t=0}=\phi(x,y).\end{cases}$$

的基本解，并利用基本解写出此定解问题解的积分公式 (b_1,b_2,c是常数).

八. (10分)用分离变量法求解边值问题

$$\begin{cases}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+x\frac{\partial}{\partial x}(x\frac{\partial}{\partial x})=0,\ (1<x<e,0<y<1,0<z<+\infty),\\ u(x,y,z)|_{x=1}=u(x,y,z)|_{x=e}=0,\\ \frac{\partial u}{\partial y}|_{y=0}=\frac{\partial u}{\partial y}|_{y=1}=0,\\ (u-\frac{\partial}{\partial z})|_{z=0}=\psi(x,y),\ \text{且}z\to\infty\text{时},\ u(x,y,z)\text{有界}.\end{cases}$$

参考公式

$\int_0^{+\infty}e^{-a^2x^2}\cos bx dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}$；$L[\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}]=\frac{e^{-a\sqrt{p}}}{\sqrt{p}}$；$L[l^n]=\frac{n!}{p^{n+1}}$，$n=0,1,2,3,\cdots$；

$L[e^{\lambda t}f(t)]=\bar{f}(p-\lambda)$；$L[f(t-\tau)]=e^{-p\tau}\bar{f}(p)$, 其中$\bar{f}(p)=L[f(t)]$.

2001-2002学年第二学期数理方程期末试题

一. (20分)

1. 利用镜像法写出上半圆$(x^2+y^2<a^2, y>0)$内场位方程第一边值问题的Green函数.

2. 利用达朗贝尔公式求出一维波动方程初值问题的基本解.

二. (45分)解下列定解问题

1.

$$\begin{cases} \Delta_2 u = 0,\ (r<1, 0<\phi<\pi/4), \\ u|_{\phi=0} = \frac{\partial u}{\partial \phi}|_{\phi=\pi/4} = 0, \\ u|_{r=1} = \sin 2\phi + \sin 6\phi. \end{cases}$$

2.

$$\begin{cases} \Delta_3 u = 0,\ (r \neq 1), \\ u|r=1 = f(\theta), \\ \lim\limits_{r\to\infty} u = 0. \end{cases}$$

3.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},\ (t>0, -\infty<x<\infty), \\ \frac{\partial u}{\partial x}|_{x=0} = q(t),\ u|_{t=0} = 0, \\ u_x(t,\infty) = u(t,\infty) = 0. \end{cases}$$

三. (20分)

1. 解定解问题$(G = G(t, x; \xi))$

$$\begin{cases} G_{tt} = a^2 G_{xx} + \delta(x-\xi),\ (0<t, 0<x<l, 0<\xi<l), \\ G|_{x=0} = G|_{x=l} = 0, \\ G|_{t=0} = 0,\ G_t|_{t=0} = 0. \end{cases}$$

2. 利用1)得到的$G(t, x; \xi)$, 写出定解问题

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x),\ (t>0, 0<x<l), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = 0,\ u_t|_{t=0} = 0 \end{cases}$$

的解.

四. (15分)(任选一题)

1. 设$G(x,y,z;\xi,\eta,\zeta)$为场位方程第三边值问题的Green函数，即定解问题

$$\begin{cases} \Delta_3 G = -\delta(x-\xi, y-\eta, z-\zeta),\ ((x,y,z)\in V, (\xi,\eta,\zeta)\in V), \\ (\alpha G + \beta\frac{\partial G}{\partial n})|_S = 0,\ \alpha,\beta\text{是任意常数},\ S\text{是}V\text{的边界} \end{cases}$$

的解, 试利用第二Green公式, 推出定解问题

$$\begin{cases} \Delta_3 u = 0,\ ((x,y,z)\in V), \\ (\alpha u + \beta\frac{\partial u}{\partial n})|_S = \phi(x,y,z),\ \alpha,\beta\text{是任意常数},\ S\text{是}V\text{的边界} \end{cases}$$

的解的积分表达式.

2. 利用积分变换求出三维波动方程初值问题的基本解.

参考公式

1. 设$u(x,y,z)$和$v(x,y,z)$在区域V及边界曲面S上有一阶连续偏导数, 在V内有二阶连续偏导数, 则有

$$\iiint_V (u\Delta v - v\Delta u)dV = \iint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial v}\right)dS$$

2.
$$L[f(t-\tau)] = e^{-p\tau}L[f(t)],\ L\left[\frac{1}{\sqrt{\pi t}}e^{-\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}$$

3.
$$\int_{-\infty}^{+\infty} e^{a\lambda-\beta^2\lambda^2}d\lambda = \frac{\sqrt{\pi}}{\beta}e^{\frac{\alpha^2}{4\beta^2}},\ \beta\neq 0$$

4.
$$\int_0^{+\infty} e^{-a^2x^2}\cos bx dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}$$

2002-2003学年第二学期数理方程期末试题

一. (20分)解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},\ (0 < x < l, t > 0), \\ u|_{t=0} = 0,\ \frac{\partial u}{\partial t}|_{t=0} = \sin\frac{\pi}{l}x + \sin\frac{2\pi}{l}x, \\ u|_{x=0} = 0,\ u|_{x=l} = 0. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - u,\ (r = \sqrt{x^2+y^2} < 1, t > 0), \\ u|_{t=0} = x^2 + y^2, \\ u|_{r=1} = e^{-t}. \end{cases}$$

三. (15分)用Laplace变换求解

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} + c^2 u = 0,\ (x > 0, y > 0),\ c > 0\text{为常数}, \\ u|_{x=0} = y, \\ u|_{y=0} = 0. \end{cases}$$

四. (10分)求边值问题

$$\begin{cases} \frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-\xi, y-\eta),\ (0 < x,\xi < +\infty, 0 < y,\eta < +\infty), \\ G|_{x=0} = 0, G|_{y=0} = 0 \end{cases}$$

的解$G(x,y;\xi,\eta)$.

五. (20分)现有初值问题

$$\begin{cases} \frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u + f(t,x,y),\ ((x,y) \in R^2, t > 0), \\ u|_{t=0} = \phi(x,y), \end{cases}$$

1. 求此初值问题的基本解$U(t,x,y)$;

2. 利用此基本解写出上述初始问题解的积分表达式.

六. (15分)设$L[u]=x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}$, $xy\neq 0$, 试

1. 求出方程$L[u]=0$的特征曲线族$\phi(x,y)=c_1$, $\psi(x,y)=c_2$;

2. 在区域$x>0,y>0$内求方程$L[u]=0$的通解;

3. 求定解问题

$$\begin{cases} L[u]=0,\ (x>0,xy>1,y>x), \\ u|_{xy=1}=\frac{1}{x^2}, \\ u|_{y=x^2}=x^2. \end{cases}$$

参考公式

1. 在柱坐标(r,θ,z)下,

$$\Delta_3 u=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$$

2. 在球坐标(r,θ,ϕ)下,

$$\Delta_3 u=\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]$$

3. ν阶Bessel方程$x^2y''+xy'+(x^2-\nu^2)y=0$, 在$0<x<+\infty$上得基础解组为$J_\nu(x),N_\nu(x)$, 其中

$$J_\nu(x)=\sum_{l=0}^{+\infty}(-1)^k\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

2003-2004学年数理方程A期末试题

一. (20分)解定解问题:

$$\begin{cases} \Delta_2 u = 0, \\ u(r,\theta)|_{r=1} = 1 + \cos\theta + \cos 2\theta. \end{cases}$$

二. (20分)解定解问题:

$$\begin{cases} u_{tt} = u_{xx} + 2xt, \\ u|_{x=0} = 0,\ u|_{x=1} = -\frac{1}{3}t^3, \\ u|_{t=0} = u_t|_{t=0} = 0. \end{cases}$$

三. (20分)将$y(x) = x^2 - 1, (|x| \le 1)$按零阶贝塞尔函数展开.

四. (20分)解初值问题:

$$\begin{cases} u_t = u_{xx} - 2u_x + u + f(t,x), \\ u|_{t=0} = \varphi(x). \end{cases}$$

五. (10分)用V表示区域: $\{x^2+y^2+z^2<1, z>0\}$, S表示V的边界, 求 $\begin{cases} \Delta_3 u = 0, \\ u|_S = 0 \end{cases}$ 的基本解.

六. (10分) 验证:

$$u(t,x) = \int_0^l \phi(\xi) G(t,x;0,\xi) d\xi + \int_0^t d\tau \int_0^l f(\tau,\xi) G(t,x;\tau,\xi) d\xi$$

是定解问题

$$\begin{cases} u_t = Lu + f(t,x), \\ u|_{x=0} = u|_{x=l} = 0, \\ u|_{t=0} = \phi(x) \end{cases}$$

的解. 其中$G(t,x;\tau,\xi) = G(t-\tau,x;\xi)$, $G(0,x;\xi) = \delta(x;\xi)$是该定解问题的基本解.

2003-2004学年第一学期数理方程B期末试题

一. (20分)解定解问题

$$\begin{cases} u_{tt} - u_{xx} = \sin 2x, \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = 0, \ u_t|_{t=0} = 6x^2. \end{cases}$$

二. (20分)解定解问题

$$\begin{cases} \Delta_3 u = 0, \ (1 < r < 2, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi), \\ u|_{r=1} = 1 + \cos^2\theta, \\ u_r|_{r=2} = 0. \end{cases}$$

三. (20分)解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + u, \ (t > 0, 0 < x < 1), \\ u|_{x=0}\text{有界}, \ u|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

四. (20分)解定解问题

$$\begin{cases} u_t = a^2 u_{xx} + b u_x + cu + f(t,x), \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x). \end{cases}$$

其中a, b, c为常数.

五. (20分)求平面区域$D : x > 0, y > 0$的格林函数$G(x, y; \xi, \eta)$, 并求下列定解问题的解:

$$\begin{cases} \Delta_2 u = -f(M), \ M(x,y) \in D : x > 0, y > 0, \\ u|_l = \varphi(M), \ M(x,y) \in l : l\text{为}D\text{的边界}. \end{cases}$$

注: $\Delta_3 u = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial u}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 u}{\partial\varphi^2}$.

2004-2005学年第二学期数理方程A期末试题

一. (30分)填空题

1. 设$0 < x_0 < l, \delta(x-x_0)$在$[0,l]$上按照正弦函数系$\{\sin\frac{n\pi x}{l}\}$的展开式为

$$\delta(x-x_0) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad},$$

$\delta'(x-x_0)$在$[0,l]$上按照余弦函数系$\{\cos\frac{n\pi x}{l}\}$的展开式为

$$\delta'(x-x_0) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

2. $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2\delta(x,y)$的Fourier变换是____________________.

3. 已知$f(x)$的Fourier变换为$\mathcal{F}[f(x)] = \frac{A}{2}(\delta(\lambda+\lambda_0)+\delta(\lambda-\lambda_0))$, 则

$$f(x) = \underline{\qquad\qquad\qquad\qquad\qquad\qquad}.$$

4. $\Delta_2 u = f(x,y)$在平面区域$D: 0 < \arg z < 1/3\pi$内第一边值问题的Green函数是________________.

5. 固有值问题

$$\begin{cases} y'' + \lambda y = 0,\ (0 < x < 1), \\ y(0) = 0,\ y(1) = 0 \end{cases}$$

的固有值为________________, 固有函数为________________, 固有函数的模平方为________________.

二. 解下列初值问题:

1. (10分)

$$\begin{cases} \frac{\partial u}{\partial t} - e^{-x}\frac{\partial u}{\partial x} = 0,\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x. \end{cases}$$

2. (10分)

$$\begin{cases} u_{xx} - u_{yy} + \cos x = 0,\ (-\infty < x, y < +\infty), \\ u(x,0) = 0,\ u_y(x,0) = 4x. \end{cases}$$

3. (10分)

$$\begin{cases} 3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x\partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0,\ (-\infty < x < +\infty, y > 0), \\ u|_{y=0} = 0,\ \frac{\partial u}{\partial y}|_{y=0} = \varphi(x). \end{cases}$$

三. 解下列定解问题:

1. (10分)

$$\begin{cases} u_t - u_{xx} + hu = f(t,x),\ (t>0, -\infty < x < +\infty), \\ u(0,x) = 0. \end{cases}$$

2. (15分)

$$\begin{cases} \Delta_2 u = x^2 - y^2,\ (r^2 = x^2 + y^2 < a^2), \\ \left(\frac{\partial u}{\partial r} + u\right)|_{x^2+y^2=a^2} = 0. \end{cases}$$

3. (15分)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),\ (t>0, 0<r=\sqrt{x^2+y^2}<b), \\ u|_{r=0}\text{有界},\ \frac{\partial u}{\partial r}|_{r=b} = 0, \\ u|_{t=0} = \varphi(r),\ \frac{\partial u}{\partial t}|_{t=0} = 0. \end{cases}$$

注: $\displaystyle\int_0^{+\infty} e^{-a^2x^2}\cos bx dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}},\ (a>0).$

2005-2006学年第一学期数理方程B期末试题

一. (30分)填空

1. 方程$u_{xy}+u_y=1$的通解是________________________.

2. 固有值问题: $y''+\lambda y=0, y(0), y'(\pi)=0$的固有值$\lambda_n=$ ________________________, 对应的固有函数$y_n(x)=$________________________.

3. 设$P_{2006}(x)$是2006阶勒让德多项式, 计算$\int_{-1}^{1} x^{2005}P_{2006}(x)dx=$________________________.

4. 计算$\delta(x-a)$的傅里叶变换$F(\delta(x-a))=$________________________.

5. 试将函数$f(x)=x^3(-1<x<1)$按勒让德多项式展开: $f(x)=$________________________.

二. (15分)求解定解问题

$$\begin{cases} u_{tt}=u_{xx}+2x,\ (t>0,-\infty<x<+\infty), \\ u(0,x)=0,\ u_t(0,x)=0. \end{cases}$$

三. (15分)求解定解问题

$$\begin{cases} u_t=u_{xx},\ (t>0,0<x<\pi), \\ u(t,0)=0,\ u(t,\pi)=100, \\ u(0,x)=\frac{100}{\pi}x+\delta(x-\frac{\pi}{x}). \end{cases}$$

四. (15分)求解定解问题

$$\begin{cases} \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial t^2},\ (0<x<l,t>0), \\ u(t,0)\text{有限},\ u(t,l)=0, \\ u|_{t=0}=f(x),\ u_t|_{t=0}=0. \end{cases}$$

五. (10分)

1. 求出区域$D=\{(x,y):x^2+y^2<1,y>0\}$上的格林函数$G(x,y;\xi,\eta),(\xi,\eta)\in D$, 即求解定解问题

$$\begin{cases} \Delta_2 G=-\delta(x-\xi,y-\eta),\ (x,y)\in D, \\ G(x,y)|_{x^2+y^2=1}=0,\ G(x,0)=0. \end{cases}$$

2. 写出定解问题

$$\begin{cases} \Delta_2 u=-f(x,y),\ (x,y)\in D, \\ u(x,y)|_{x^2+y^2=1}=0,\ u(x,0)=\phi(x) \end{cases}$$

的解的积分表达式.

六. (15分)

1. 求出方程$u_t = a^2 u_{xx} + bu$的柯西问题的基本解$U(t,x)$, 其中a和b是常数, 即求定解问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, \ (t > 0, -\infty < x < +\infty), \\ u(0,x) = \delta(x). \end{cases}$$

2. 求解柯西问题

$$\begin{cases} u_t = a^2 u_{xx} + bu, \ (t > 0, -\infty < x < +\infty), \\ u(0,x) = 1 + x^2. \end{cases}$$

参考公式

1. 勒让德方程式$(1-x^2)y'' - 2xy' + n(n+1)y = 0$, $(n = 0, 1, 2, \cdots, -1 < x < 1)$; 勒让德多项式: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, 特别地, $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2-1), P_3(x) = \frac{1}{2}(5x^2-3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

2. 贝塞尔方程是$x^2y'' + xy' + (x^2 - \nu^2)y = 0$, $(\nu \geq 0, 0 < x < a)$, 贝塞尔函数具有微分关系式:
$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$$
和
$$\frac{d}{dx}\left[\frac{J_\nu(x)}{x^\nu}\right] = -\frac{J_{\nu+1}(x)}{x^\nu}.$$
贝塞尔函数在第一、二类边界条件下的模平方$N_\nu^2 = \int_0^n xJ_\nu^2(\omega x)dx$分别是
$$N_{\nu 1}^2 = \frac{a^2}{2}J_{\nu+1}^2(\omega a), \ N_{\nu 2}^2 = \frac{1}{2}\left[a^2 - \left(\frac{\nu}{\omega}\right)^2\right]J_\nu^2(\omega a).$$

3. 积分$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$. $f(x)$的傅里叶变换定义为$\mathcal{F}(\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}dx$. $F(\lambda) = e^{-a|\lambda|}$的傅里叶反变换是 $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x}d\lambda = \frac{a}{\pi(x^2+a^2)}$, $F(\lambda) = e^{-\lambda^2 t}$的傅里叶反变换是$f(x) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}$.

2005-2006学年第二学期数理方程A期末试题

一. (10分)求解定解问题

$$\begin{cases} x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \\ u|_{y=0} = x^2. \end{cases}$$

二. (12分)求解定解问题

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 1, \\ u(x,0) = 3x^2,\ u_y(x,0) = \frac{x}{2}. \end{cases}$$

三. (12分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1. $$\begin{cases} \frac{1}{x}(xY')' + \lambda Y = 0,\ (0 < x < 1), \\ |Y(0)| < +\infty,\ Y(1) = 0. \end{cases}$$

2. $$\begin{cases} Y'' + \lambda Y = 0,\ (0 < x < 2), \\ Y(0) = 0,\ Y'(2) = 0. \end{cases}$$

四. (14分, 超纲)写出泛函

$$J[u(x,y)] = \iint_{x^2+y^2\le 1} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2xyu\right] dxdy$$

的Euler方程并求出满足边界条件$u|_{x^2+y^2=1} = 1$的极小元.

五.(8分) 将函数$f(x) = \delta(x)$在$[-1,1]$上按Legendre多项式$P_n(x)$展开.

六. (14分)求定解问题

$$\begin{cases} u_{tt} = u_{xx} + \cos 3\pi x,\ (x \in [0,1], t > 0), \\ u_x(t,0) = u_x(t,1) = 0, \\ u_t(0,x) = 0,\ u(0,x) = 2\cos \pi x + 4\cos 2\pi x. \end{cases}$$

七. (14分)求函数$f_1(x) = \delta(x-1), f_2(x) = e^{ix}, f_3(x) = \cos x$的Fourier变换$\mathcal{F}[f_1(x)], \mathcal{F}[f_2(x)], \mathcal{F}[f_3(x)]$并利用Fourier变换求初值问题

$$\begin{cases} u_t = 2u_{xx} + u + f(t,x),\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \phi(x). \end{cases}$$

的基本解, 再利用相应公式解出此初值问题.

八. (10分)已知半空间的场位方程的第一边值问题为:

$$\begin{cases}\Delta_3 u = -f(x,y,z),\ (x>0),\\ u|_{x=0} = \phi(y,z).\end{cases} \quad (1)$$

1. 写出此边值问题的Green函数G满足的定解问题, 并求出Green函数G.

2. 当在半空间的场位方程的第一边值问题(1)中取$f(x,y,z)=0$时, 倒出解$u(x,y,z)$的积分公式.

九.(6分) 用球函数将以下函数展开:

$$f(\theta,\varphi) = \sin^2\theta\left(\cos^2\varphi + 15\cos\theta\cos 2\varphi\right)$$

参考公式

1. $$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n,\ (|t|<1, |x|\le 1)$$

2. $$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x), (m\le n);\ P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, (n=0,1,2,\cdots)$$

2006-2007学年第一学期数理方程B期末试题

一. (20分)求解定解问题

$$\begin{cases} u_{tt}-u_{xx}=x+t,\ (t>0,-\infty<x<+\infty), \\ u|_{t=0}=\sin x,\ u_t|_{t=0}=4x. \end{cases}$$

二. (20分)求解定解问题

$$\begin{cases} \Delta_3 u=0,\ (1<r<2), \\ u|_{r=1}=0,\ u|_{r=2}=1+\cos\theta, \end{cases}$$

其中(r,θ,φ)为球坐标.

三. (24分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1.

$$\begin{cases} Y''(x)+\lambda Y(x)=0,\ (0<x<1), \\ Y'(0)=Y'(1)=0. \end{cases}$$

2.

$$\begin{cases} x^2Y''+xY'+(\lambda x^2-1)Y=0,\ (0<x<b), \\ |Y(0)|<+\infty,\ Y(b)=0. \end{cases}$$

3.

$$\begin{cases} \Delta_2 u+\lambda u=0,\ (0<x<2,0<y<3), \\ u|_{x=0}=u|_{x=2}=u|_{y=0}=u|_{y=3}=0. \end{cases}$$

四. 设初值问题

$$(*)\begin{cases} u_t=2u_x+f(t,x),\ (t>0,-\infty<x<+\infty), \\ u|_{t=0}=\varphi(x). \end{cases}$$

1. (10分)求上述初值问题的基本解$U(t,x)$.

2. (10分)求初值问题$(*)$的解.

五. 设平面区域$D=\{(x,y)|y>x\}$,

1. (10分)求D内格林函数G:

$$\begin{cases}\Delta_2 G=-\delta(x-\xi,y-\eta),\ ((x,y)\in D,(\xi,\eta)\in D),\\ G|_{y=x}=0.\end{cases}$$

2. (6分)求边值问题

$$\begin{cases}\Delta_2 u=-f(x,y),((x,y)\in D),\\ u|_{y=x}=\varphi(x)\end{cases}$$

的解.

参考公式

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

2006-2007学年第二学期数理方程A期末试题

一. (8分)设$u = u(x, y)$, 求偏微分方程

$$\frac{\partial^2 u}{\partial x \partial y} = x$$

的通解.

二. (15分)求解以下固有值问题 (计算结果种要明确指出固有值和固有函数)

1.

$$\begin{cases} Y'' + \lambda Y = 0, \ (0 < x < 2), \\ Y(0) = Y(2) = 0. \end{cases}$$

2.

$$\begin{cases} Y'' + \lambda Y = 0, \ (-\infty < x < +\infty), \\ Y(x) = Y(x+2). \end{cases}$$

3.

$$\begin{cases} x^2 Y'' + xY' + (\lambda x^2 - 1)Y = 0, \ (0 < x < 1), \\ |Y(0)| < +\infty, Y(1) = 0. \end{cases}$$

三. (12分)求解定解问题

$$\begin{cases} u_{tt} = 4u_{xx}, \ (t > 0, -\infty < x < +\infty), \\ u(0, x) = x^2, \ u_t(0, x) = x + 1. \end{cases}$$

四. (10分)求解定解问题

$$\begin{cases} (x^2 + 1)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \\ u|_{x=0} = y^2. \end{cases}$$

五. (15分)求解定解问题

$$\begin{cases} u_{tt} = u_{xx} + e^t \sin 2x, \ (t > 0, 0 < x < \pi), \\ u(t, 0) = u(t, \pi) = 0, \\ u(0, x) = \sin x + 4\sin 5x, \ u_t(0, x) = 0. \end{cases}$$

六. (14分)

1. (9分)求解初值问题:

$$\begin{cases} U_t = U_{xx} + 2U_x,\ (t>0, -\infty < x < +\infty), \\ U|_{t=0} = \delta(x). \end{cases}$$

2. (5分)求初值问题:

$$\begin{cases} u_t = u_{xx} + 2u_x + f(t,x),\ (t>0, -\infty < x < +\infty), \\ u|_{t=0} = \varphi(x) \end{cases}$$

的解的积分表达式.

七. (10分, 超纲) 写出泛函

$$J[u(x,y,z)] = \iiint\limits_{x^2+y^2+z^2\le 1} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - 2u \right] dxdydz$$

的Euler方程并求出满足边界条件$u|_{x^2+y^2+z^2=1} = x^2 + y^2$的极值元.

八. (8分) 设平面区域$D = \{(x,y)|y>0, x^2+y^2<1\}$, 试求定解问题

$$\begin{cases} \Delta_2 G = -\delta(x-\xi, y-\eta),\ ((x,y)\in D, (\xi,\eta)\in D), \\ G|_L = 0,\ (L\text{为}D\text{的边界}) \end{cases}$$

的解$G(x,y;\xi,\eta)$.

九. (8分) 设$V = \{(x,y,z)|x^2+y^2+z^2>1\}$的方程:

$$U_{xx} + 4U_{yy} + 9U_{zz} = \delta(x-1, y-2, z-3),\ (\text{其中}(x,y,z)\in V)$$

的满足$\lim\limits_{x^2+y^2+z^2\to+\infty} U = 0$的所有解.

参考公式

1.

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,\ n = 0,1,2,\cdots$$

2.

$$\frac{1}{\pi}\int_0^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda x d\lambda = \frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right)$$

2007-2008学年第二学期数理方程A期末试题

一. (14分)设$u = u(t, x)$, 求解以下定解问题:

1.
$$\begin{cases} u_{tx} = x,\ (t > 0, x > 0), \\ u(0, x) = 1 + \sin x,\ u(t, 0) = 1. \end{cases}$$

2.
$$\begin{cases} u_{tt} = 9u_{xx},\ (t > 0, -\infty < x < +\infty), \\ u(0, x) = \cos x,\ u_t(0, x) = x^2. \end{cases}$$

二. (10分)求解定解问题

$$\begin{cases} 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0, \\ u|_{x=0} = y^2 - z. \end{cases}$$

三. (14分)求解定解问题

$$\begin{cases} u_t = 4u_{xx},\ (t > 0, 0 < x < 2), \\ u(t, 0) = u(t, 2) = 0, \\ u(0, x) = \delta(x - 1). \end{cases}$$

四. (10分)求解以下固有值问题(计算结果中要明确指出固有值和固有函数)

1.
$$\begin{cases} [(1 - x^2)y']' + \lambda y = 0,\ (0 < x < 1), \\ y(0) = 0,\ |y(1)| < +\infty. \end{cases}$$

2.
$$\begin{cases} \Delta_2 u + \lambda u = 0,\ (0 < x < 1, 0 < y < 2), \\ \frac{\partial u}{\partial x}|_{x=0} = u|_{x=1} = u|_{y=0} = \frac{\partial u}{\partial y}|_{y=2} = 0. \end{cases}$$

五. (8分,超纲)写出泛函

$$J[y(x)] = \int_1^2 (y'^2 - 2xy)dx$$

的Euler方程并求出满足边界条件$y(1) = 0, y(2) = -1$的极值元.

六. (12分)求解定解问题

$$\begin{cases} \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0,\ (0<r<1,0<z<1), \\ |u(0,z)|<+\infty,\ u(1,z)=0, \\ u(r,0)=0,\ u(r,1)=1-r. \end{cases}$$

七. (14分)求初值问题

$$\begin{cases} u_t=u_{xx}+2u_y+u+f(t,x,y),\ (t>0,-\infty<x,y<+\infty), \\ u|_{t=0}=\phi(x,y) \end{cases}$$

的解的积分表达式.

八. (8分)设空间区域$V=\{(x,y,z)|x>0,y>0\}$, 试求定解问题

$$\begin{cases} \Delta_3 G=-\delta(x-\xi,y-\eta,z-\zeta),\ ((x,y,z)\in V,(\xi,\eta,\zeta)\in V), \\ G|_S=0,\ (\text{其中}S\text{是}V\text{的边界}) \end{cases}$$

的解$G(x,y,z;\xi,\eta,\zeta)$.

九. (10分)求解定解问题

$$\begin{cases} u_{tt}=u_{xx}+\sin\frac{3}{2}x,\ (t>0,0<x<\pi), \\ u(t,0)=0,\ u_x(t,\pi)=1, \\ u(0,x)=x+\sin\frac{x}{2}+5\sin\frac{5x}{2},\ u_t(0,x)=\sin\frac{3x}{2}. \end{cases}$$

参考公式

1.
$$\left(x^\gamma J_\gamma\right)'=x^\gamma J_{\gamma-1}, N_{\gamma 1n}^2=\frac{a^2}{2}J_{\gamma+1}^2\left(\omega_{1n}a\right)$$

2.
$$\frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda=\frac{1}{2a\sqrt{\pi t}}exp\left(-\frac{x^2}{4a^2t}\right)$$

2008-2009学年第二学期数理方程A期末试题

一. (12分)求下面方程的通解:

$$u_{xx} - u_{yy} = x^2 - y^2.$$

二. (13分)求解定解问题:

$$\begin{cases} (x^2+1)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, \\ u|_{x=0} = y^2. \end{cases}$$

三. (15分)求解定解问题:

$$\begin{cases} u_{tt} = u_{xx},\ (0 < x, \xi < 1), \\ u_x|_{x=0} = u_x|_{x=1} = 0, \\ u|_{t=0} = 0,\ u_t|_{t=0} = \delta(x-\xi). \end{cases}$$

四. (10分)求矩形域$[0,a] \times [0,b]$上问题

$$\begin{cases} u_{xx} + u_{yy} + u_x + \lambda u = 0, \\ u|_{x=0} = u|_{x=a} = u|_{y=0} = u|_{y=b} = 0 \end{cases}$$

的固有值和固有函数.

五. (15分)求解以下定解问题, 其中(r,θ,φ)为球坐标:

$$\begin{cases} \Delta_3 u = 1,\ (r < 1), \\ u|_{r=1} = \cos 2\theta. \end{cases}$$

六. (15分)先求下面Cauchy问题的基本解, 再求该定解问题解的积分公式:

$$\begin{cases} u_t = u_{xx} + 2u_x + f(t,x),\ (t > 0, -\infty < x < +\infty), \\ u(0,x) = \phi(x). \end{cases}$$

七. (20分)设D为圆心在原点, 半径r_0的圆盘, 考虑定解问题

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\Delta_2 u, \ (t>0, M(x,y)\in D), \\ \frac{\partial u}{\partial n}|_{M(x,y)\in\partial D} = 0, \\ u(0,M) = \phi(x,y). \end{cases}$$

1. 求$u(t,M)$.

2. 证明$\int_D u(t,M)dM = \int_D \phi(M)dM$.

3. 对任意$M\in D$, 求极限$\lim\limits_{t\to\infty} u(t,M)$.

4. 试从物理上说明2,3小题的意义.

参考公式

1. $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n, \ n=0,1,2,\cdots.$

2. $\dfrac{1}{\pi}\displaystyle\int_0^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda x d\lambda = \dfrac{1}{2a\sqrt{\pi t}}\exp\left(-\dfrac{x^2}{4a^2 t}\right).$

3. 柱坐标下:

$$\Delta_3 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$

4. 球坐标下:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

2013-2014学年第二学期数理方程B期末试题

一. (16分)求下列偏微分方程的通解$u = u(x, y)$:

1. $\dfrac{\partial^2 u}{\partial x \partial y} = x^2 y.$

2. $y\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial u}{\partial x} = xy.$

二. (10分)求下列固有值问题的解, 要求明确指出固有值及其所对应的固有函数:

$$\begin{cases} x^2 y'' + xy' + \lambda x^2 y = 0, \ (0 < x < 2), \\ |y(0)| < +\infty, \ y'(2) = 0. \end{cases}$$

三. (12分)求第一象限$D = \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$的第一边值问题的Green函数.

四. (12分)用积分变换法求解下列方程:

$$\begin{cases} u_t = a^2 u_{xx} + u, \ (-\infty < x < +\infty, t > 0), \\ u(0, x) = \varphi(x). \end{cases}$$

五. (15分)用分离变量法求解下列方程:

$$\begin{cases} \Delta_2 u = 0, \ (r < 2), \\ u|_{r=2} = \sin\theta + 2\sin 5\theta - 7\cos 4\theta. \end{cases}$$

六. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{tt} = 4u_{xx}, \ (0 < x < 1, t > 0), \\ u(t, 0) = 0, \ u(t, 1) = 1, \\ u(0, x) = \varphi(x) + x, \ u_t(0, x) = \delta(x - \frac{1}{2}). \end{cases}$$

七. (15分)用分离变量法求解下列方程:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = z, \ (x^2 + y^2 + z^2 < 1), \\ u|_{x^2+y^2+z^2=1} = 0 \end{cases}$$

八. (5分)求解下列定解问题:

$$\begin{cases} 4u_{xx} = u_{tt} + 2u_t + u, \ (-\infty < x < +\infty, t > 0), \\ u(0, x) = 2\cos x, \ u_t(0, x) = 2x. \end{cases}$$

提示: 先对泛定方程进行变换成为一个较为简单的泛定方程, 再根据初始条件进行求解.

参考公式: 包括极坐标和球坐标下的Laplace算子表达式, Fourier级数及其系数的公式, Laplace和Fourier所有性质和变换公式及求解过程中用到的反变换公式, 勒让德方程的固有值和固有函数以及勒让德函数$n = 1$ 5时的表达式.

注: 本卷为考后回忆版本, 未给具体公式内容, 请同学自行参考其它卷子的相关公式.

2014-2015学年第二学期数理方程A期末试题

一. (15分)设$a \neq b$为实常数, 考察二阶线性齐次方程:

$$u_{xx} - (a+b)u_{xy} + abu_{yy} = 0,\ (-\infty < x, y < +\infty).$$

1. 是判断方程的类型(椭圆/双曲线/抛物线).

2. 试将该方程化成标准型.

3. 求出该方程的解.

4. 求出该方程满足的条件: $u(x,-ax) = \varphi(x),\ u(x,-bx) = \psi(x)$的特解, 其中$\varphi(0) = \psi(0)$.

二. (10分)考察一阶线性非齐次方程:

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} = y,\ (-\infty < x, y < +\infty).$$

1. 求出此方程的特征线.

2. 求出此方程满足条件$u(0,y) = 1 + y^2$的解.

三. (20分)考察定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + f(t,x),\ (0 < x < \pi, t > 0), \\ u|_{x=0} = 0,\ u|_{x=\pi} = 0, \\ u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x). \end{cases}$$

1. 当$f(t,x) = 0$时, 求此定解问题的解u_1.

2. 当$f(t,x) = \sin 2x \sin \omega t$(其中$\omega \neq 4$), $\varphi(x) = 0,\ \psi(x) = 0$时, 求此定解问题的解$u_2$以及$\lim\limits_{\omega \to 4} u_2(x,t,\omega)$的值.

四. (20分)考察定解问题:

$$\begin{cases} \Delta_3 u = 0,\ (r < a, 0 < \theta < 2\pi, 0 < z < h), \\ u|_{r=a} = 0, \\ u|_{z=0} = g_1(r,\theta),\ u|_{z=h} = g_2(r,\theta). \end{cases}$$

1. 当$g_1(r,\theta) = 0,\ g_2(r,\theta) = f(r)$时,求此定解问题的解.

2. 当$g_1(r,\theta) = \varphi(r,\theta),\ g_2(r,\theta) = \psi(r,\theta)$时, 可作分离变量: $u = R(r)\Theta(\theta)Z(z)$, 分别求出$R, \Theta, Z$满足的常微分方程, 并写出此时与定解问题相应的固有值问题.

五. (15分)考察初值问题:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_3 u + 3u + f(t,x,y,z),\ (t>0, -\infty < x,y,z < +\infty), \\ u|_{t=0} = \varphi(x,y,z). \end{cases}$$

1. 求出此问题的基本解.

2. 当$f(t,x,y,z)=0$, $\varphi(x,y,z)=e^{-(x^2+y^2+z^2)}$时, 求此问题的解.

六. (15分)已知右半平面区域$S=\{(x,y)|x>0, -\infty<y<+\infty\}$

1. 求出S内Poisson方程第一边值问题的Green函数.

2. 求解定解问题:

$$\begin{cases} u_{xx} + 25u_{yy} = 0,\ (x>0, -\infty < y < +\infty), \\ u|_{x=0} = \varphi(y). \end{cases}$$

七. (5分)求方程: $Z'(\theta) + \cot\theta Z(\theta) + 20Z(\theta) = 0,\ (0<\theta<\frac{\pi}{2})$满足条件$Z(0)=1$的解$Z(\theta)$, 并求$Z(\frac{\pi}{2})$.

2015-2016学年第二学期数理方程B期末试题

一. (12分)求以下固有值问题的固有值和固有函数:

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, \ (0 < x < 16), \\ Y'(0) = 0, \ Y'(16) = 0. \end{cases}$$

二. (16分)利用分离变量法求解定解问题:

$$\begin{cases} u_t = 4u_x x, \ (t > 0, 0 < x < 5), \\ u(t,0) = u(t,5) = 0, \\ u(0,x) = \phi(x). \end{cases}$$

并求$\phi(x) = \delta(x-2)$时此定解问题的解.

三. (14分)考虑初值问题:

$$\begin{cases} u_{tt} = 4u_{xx} + f(t,x), \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2, \ u_t|_{t=0} = \sin 2x. \end{cases}$$

1. 如取$f(t,x) = 0$, 求此初值问题的解.

2. 如取$f(t,x) = t^2x^2$, 求此初值问题相应的解.

四. (14分)求解以下初值问题

$$\begin{cases} u_t = 4u_{xx} + 5u, \ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = \phi(x). \end{cases}$$

并指出当$\phi(x) = e^{-x^2}$时此定解问题的解.

五. (16分)求解以下定解问题:

$$\begin{cases} u_t = u_{rr} + \frac{1}{r}u_r, \ (0 < r < 1), \\ |u(t,0)| < +\infty, \ u(t,1) = 0, \\ u|_{t=0} = \phi(r). \end{cases}$$

并算出$\phi(r) = J_0(ar) + 3J_0(br)$时的解, 其中$0 < a < b$, 且$J_0(a) = J_0(b) = 0$.

六. (14分)已知下半空间$V=\{(x,y,z)|x<0,-\infty<x,y<+\infty)\}$.

1. 求出V内泊松方程第一边值问题的Green函数.

2. 求解定解问题:

$$\begin{cases} 4u_{xx}+u_{yy}+u_{zz}=0,\ (z<0,-\infty<x,y<+\infty), \\ u|_{z=0}=\varphi(x,y). \end{cases}$$

七. (6分)对于三维波动方程

$$u_{tt}=a^2\Delta_3 u,\ (a>0,t>0,-\infty<x,y,z<+\infty)$$

它的形如$u=u(t,r)=T(t)R(r)$的解称为方程的可分离变量的径向对称解, 求方程满足$\lim\limits_{t\to+\infty}u=0$的可分离变量的径向对称解, 这里$r=\sqrt{x^2+y^2+z^2}$.

八. (8分)考虑固有值问题

$$\begin{cases} \frac{d}{dx}[(1-x^2)y']+\lambda y=0,\ (0<x<1), \\ y'(0)=0,\ |y(1)|<+\infty. \end{cases}$$

1. 求此固有值问题的固有值和固有函数.

2. 把$f(x)=2x+1$按此固有值问题所得到的固有函数系展开.

参考公式

1. 直角坐标系: $\Delta_3 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$, 柱坐标系: $\Delta_3 u=\frac{1}{r}r\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}$,
   球坐标系: $r^2\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$.

2. 若ω是$J_\nu(\omega a)=0$的一个正根, 则有模平方$N_{\nu 1}^2=||J_\nu(\omega x)||_1^2=\frac{a^2}{2}J_{\nu+1}^2(\omega a)$.
   若ω是$J'_\nu(\omega a)=0$的一个正根, 则有模平方$N_{\nu 2}^2=||J_\nu(\omega x)||_2^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega^2}\right]J_\nu^2(\omega a)$.

3. 勒让德多项式: $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, $\int_{-1}^1 P_n^2(x)dx=\frac{2}{2n+1}, n=0,1,2,\cdots$,
   母函数: $(1-2xt+t^2)^{-\frac{1}{2}}=\sum\limits_{n=0}^{+\infty}P_n(x)t^n$, 递推公式: $P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x)$.

4. $\displaystyle\frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2t}\right)$

5. 设$G(M;M_0)$是三维Poisson方程第一边值问题

$$\begin{cases} \Delta_3 u=-f(M),\ (M=(x,y,z)\in V), \\ u|S=\phi(M) \end{cases}$$

   对应的Green函数,则

$$u(M_0)=-\iint_S\phi(M)\frac{\partial G}{\partial n}(M;M_0)dS+\iiint_V f(M)G(M;M_0)dM,$$

   其中$M_0=(\xi,\eta,0)$.

2016-2017学年第二学期数理方程B期末试题

一. (10分)求方程$u_x + yu_{xy} = 0$的一般解.

二. (10分)求解一维半无界弦的自由振动问题:

$$\begin{cases} u_{tt} = 9u_{xx},\ (t>0, 0<x<+\infty), \\ u|_{x=0} = 0, \\ u|_{t=0} = x,\ u_t|_{t=0} = 2\sin x. \end{cases}$$

三. (20分)考察一维有界限振动问题:

$$\begin{cases} u_{tt} = u_{xx} + f(t,x),\ (t>0, 0<x<\pi), \\ u|_{x=0} = 0,\ u_x|_{x=\pi} = 0, \\ u|_{t=0} = \sin\frac{3}{2}x,\ u_t|_{t=0} = \sin\frac{x}{2}. \end{cases}$$

1. 当$f(t,x) = 0$时, 求出上述定解问题的解$u_1(x)$.

2. 当$f(t,x) = \sin\frac{x}{2}\sin\omega t,\ (\omega \neq k+\frac{1}{2}, k\in\mathbb{N})$时, 求出上述定解问题的解$u_2(t,x)$.

3. 指出定解问题中方程非齐次项$f(t,x)$, 边界条件和初始条件的物理意义.

四. (15分)求解定解问题:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + u,\ (t>0, 0<x<1), \\ u|_{x=0}\text{有界},\ u_x|_{x=1} = 0, \\ u|_{t=0} = \varphi(x). \end{cases}$$

五. (15分)求解如下泊松方程的边值问题:

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = z,\ (x^2+y^2+z^2<1), \\ u|_{x^2+y^2+z^2=1} = 0. \end{cases}$$

六. (15分)设区域$\Omega = \{(x,y)|y \geq x\}$.

1. 求区域Ω上的Poisson方程Dirichlet边值问题的Green函数.

2. 求解如下Poisson方程的Dirichlet边值问题:

$$\begin{cases} \Delta_2 u = 0,\ ((x,y)\in\Omega), \\ u(x,x) = \phi(x). \end{cases}$$

七. (15分)考察定解问题:

$$\begin{cases} u_t = 4u_{xx} + 3u,\ (-\infty < x < +\infty, t > 0), \\ u(0,x) = \varphi(x). \end{cases}$$

1. 求出上述定解问题相应的基本解.

2. 当$\varphi(x) = x$时, 求解上述定解问题.

参考公式

1. 拉普拉斯算子Δ_3在各个坐标系下的表达形式
$\Delta_3 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$.

2. 二阶欧拉方程: $x^2y'' + pxy' + qy = f(x)$, 在作变量代换$x = e^t$下,可以约化为常系数线性微分方程: $\frac{d^2y}{dt^2} + (p-1)\frac{dy}{dt} + qy = f\left(e^t\right)$.

3. Legendre方程: $\left[(1-x^2)y'\right]' + \lambda y = 0$; n阶Legendre多项式:
$P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
; Legendre多项式的母函数: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} P_n(x)t^n,\ (|t|<1)$;
Legendre多项式的模平方: $||P_n(x)||^2 = \frac{2}{2n+1}$.

4. ν阶Bessel方程: $x^2y'' + xy' + (x^2-\nu^2)y = 0$; ν阶Bessel函数: $J_\nu(x) = \sum_{l=0}^{+\infty}(-1)^k\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$
Bessel函数的母函数: $e^{\frac{x}{2}(\zeta-\zeta^{-1})} = \sum_{n=-\infty}^{+\infty} J_n(x)\zeta^n$; Bessel函数在三类边界条件下的模平方: $N_{\nu 1n}^2 = \frac{a^2}{2}J_{\nu+1}^2(\omega_{1n}a), N_{\nu 2n}^2 = \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_{2n}^2}\right]J_\nu^2(\omega_{2n}a), N_{\nu 3n}^2 = \frac{1}{2}\left[a^2 - \frac{\nu^2}{\omega_{2n}^2} + \frac{a^2\alpha^2}{\beta^2\omega_{3n}^2}\right]J_\nu^2(\omega_{3n}a)$.

5. 傅里叶变换和逆变换: $\mathcal{F}[f](\lambda) = \int_{-\infty}^{+\infty} f(x)e^{i\lambda x}dx$; $\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\lambda)e^{-i\lambda x}d\lambda$; $\mathcal{F}^{-1}[e^{-\lambda^2}] = \frac{1}{2\sqrt{\pi}}e^{-\frac{x^2}{4}}$.

6. 拉普拉斯变换: $L[f(t)] = \int_0^{+\infty} f(t)e^{-pt}, p = \sigma + is$; $L[e^{\alpha t}] = \frac{1}{p-\alpha}$, $L[t^\alpha] = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$, $L[\sin t] = \frac{1}{p^2+1}$, $L[\cos t] = \frac{p}{p^2+1}$, $L\left[\frac{1}{\sqrt{\pi t}}e^{\frac{a^2}{4t}}\right] = \frac{e^{-a\sqrt{p}}}{\sqrt{p}}$.

7. 拉普拉斯方程$\Delta_3 u = \delta(M)$的基本解:
二维, $U(x,y) = -\frac{1}{2\pi}\ln\frac{1}{r},\ r = \sqrt{x^2+y^2}$;
三维, $U(x,y,z) = -\frac{1}{4\pi r},\ r = \sqrt{x^2+y^2+z^2}$.

2018-2019学年第二学期数理方程B期末试题

一. 设有一个均匀圆柱物体, 半径为a, 高为h, 侧面在温度为零的空气中自由冷却. 上底绝热, 下底温度为$g(t,x,y)$, 初始温度为$\varphi(x,y,z)$, 试写出圆柱体内温度所满足的定解问题. (不用求解, 仅列方程)

二. 求解一维无界弦的振动问题

$$\begin{cases} u_{tt} = u_{xx} - 4t + 2x, \ (-\infty < x < +\infty, t > 0), \\ u|_{t=0} = x^2, \ u_t|_{t=0} = \sin 3x. \end{cases}$$

三. 求解固有值问题

$$\begin{cases} y'' + 2y' + \lambda y = 0, \ (0 < x < 9), \\ y(0) = y(9) = 0. \end{cases}$$

四. 求解一维有界弦的振动问题

$$\begin{cases} u_{tt} = u_{xx}, \ (0 < x < 1, t > 0), \\ u|_{x=0} = u|_{x=1} = 1, \\ u|_{t=0} = 0, \ u_t|_{t=0} = 0. \end{cases}$$

五. 求解如下泊松方程的边值问题

$$\begin{cases} \Delta_3 u = 0, \ (x^2 + y^2 < 1, 0 < z < 1), \\ u|_{x^2+y^2=1} = 0, \\ u|_{z=0} = 0, \ u|_{z=1} = 1 - (x^2 + y^2). \end{cases}$$

六. 求解热传导问题

$$\begin{cases} u_t = u_{xx} + u, \ (-\infty < x < +\infty, t > 0), \\ u(0,x) = e^{-x^2}. \end{cases}$$

七. 设平面区域$\Omega = \{(x,y) | x + y > 0\}$,

1. 求出区域Ω的Green函数.

2. 求出区域Ω的定解问题:

$$\begin{cases} \Delta_2 u = 0, \ (x,y) \in \Omega, \\ u(x,-x) = \varphi(x). \end{cases}$$

八. 计算积分

$$\int_{-1}^{1} P_4(x)(1+x+2x^2+3x^3+4x^4)dx$$

参考公式

1. 拉普拉斯算子Δ_3在各个坐标系下的表达形式
   $\Delta_3=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$
2. Legendre方程: $\left[(1-x^2)y'\right]'+\lambda y=0$; n阶Legendre多项式:
   $P_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$
   ; Legendre多项式的母函数: $(1-2xt+t^2)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}P_n(x)t^n,\ (|t|<1)$;
   Legendre多项式的模平方: $||P_n(x)||^2=\frac{2}{2n+1}$.
   Legendre多项式满足的递推公式($n\geq 1$): $(n+1)P_{n+1}(x)-x(2n+1)P_n(x)+nP_{n-1}(x)=0,\ nP_n(x)-xP_n'(x)+P_{n-1}'(x)=0,\ nP_{n-1}(x)-P_n'(x)+xP_{n-1}'(x)=0,\ P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$
3. ν阶Bessel方程: $x^2y''+xy'+(x^2-\nu^2)y=0$; ν阶Bessel函数: $J_\nu(x)=\sum_{l=0}^{+\infty}(-1)^k\frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$
   Bessel函数的母函数: $e^{\frac{x}{2}}(\zeta-\zeta^{-1})=\sum_{n=-\infty}^{+\infty}J_n(x)\zeta^n$; Bessel函数在三类边界条件下的模平方: $N_{\nu 1n}^2=\frac{a^2}{2}J_{\nu+1}^2(\omega_{1n}a),N_{\nu 2n}^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega_{2n}^2}\right]J_\nu^2(\omega_{2n}a),N_{\nu 3n}^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega_{2n}^2}+\frac{a^2\alpha^2}{\beta^2\omega_{3n}^2}\right]J_\nu^2(\omega_{3n}a)$. Bessel函数满足的微分关系和递推公式: $\frac{d}{dx}\left(x^\nu J_\nu(x)\right)=x^\nu J_{\nu-1}(x),\ \frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right)=-\frac{J_{\nu+1}(x)}{x^\nu}$
4. 傅里叶变换和逆变换: $\mathcal{F}[f](\lambda)=\int_{-\infty}^{+\infty}f(x)e^{i\lambda x}dx$; $\mathcal{F}^{-1}[F](x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\lambda)e^{-i\lambda x}d\lambda$; $\mathcal{F}^{-1}[e^{-\lambda^2}]=\frac{1}{2\sqrt{\pi}}e^{-\frac{x^2}{4}}$.
5. 拉普拉斯变换: $L[f(t)]=\int_0^{+\infty}f(t)e^{-pt},p=\sigma+is$; $L[e^{\alpha t}]=\frac{1}{p-\alpha},\ L[t^\alpha]=\frac{\Gamma(\alpha+1)}{p^{\alpha+1}}$
6. 拉普拉斯方程$\Delta_3 u=\delta(M)$的基本解:
   二维, $U(x,y)=-\frac{1}{2\pi}\ln\frac{1}{r},\ r=\sqrt{x^2+y^2}$;
   三维, $U(x,y,z)=-\frac{1}{4\pi r},\ r=\sqrt{x^2+y^2+z^2}$.
7. Green第一公式: $=\iiint_V u\Delta v dV+\iiint_V \nabla u\nabla v dV$
   Green第二公式: $\iint_{\partial V}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS=\iiint_V(u\Delta v-v\Delta u)dV$

2019-2020学年第二学期数理方程B期末试题(毕业年级重修)

一. (18分)求解下列Cauchy问题:

1.

$$\begin{cases} u_{tt} = 4u_{xx},\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2,\ u_t|_{t=0} = \cos 2x. \end{cases}$$

2.

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} = 20, \\ u(0, y) = y^2,\ u(x, 0) = \sin x. \end{cases}$$

二. (18分)求以下固有值问题的固有值和固有函数:

1.

$$\begin{cases} Y''(x) + \lambda Y(x) = 0,\ (0 < x < \pi), \\ Y'(0) = 0,\ Y'(\pi) = 0. \end{cases}$$

2.

$$\begin{cases} x^2 Y''(x) + xY'(x) + \lambda Y(x) = 0,\ (1 < x < b), \\ Y(1) = 0,\ Y'(b) = 0. \end{cases}$$

三. (18分)

1. 求周期边界条件下

$$\begin{cases} u_{tt} = u_{xx},\ (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1),\ u_x(t, 0) = u_x(t, 1) \end{cases}$$

的分离变量解$u = T(t)X(x)$.

2. 求解

$$\begin{cases} u_{tt} = u_{xx},\ (t > 0, 0 < x < 1), \\ u(t, 0) = u(t, 1),\ u_x(t, 0) = u_x(t, 1), \\ u(0, x) = \sin 2\pi x,\ u_t(0, x) = 2\pi \cos 2\pi x. \end{cases}$$

四. (14分)求解

$$\begin{cases} u_t = u_{xx} + u, \ (t > 0, \infty < x < +\infty), \\ u|_{t=0} = \delta(x+1). \end{cases}$$

五. (18分)

1. P_n为n-阶勒让德函数, 写出$P_0(x), P_1(x), P_2(x)$, 并计算积分$\int_{-1}^{1}(20+x)P_2(x)dx$.

2. 求解以下定解问题, 其中(r,θ,ϕ)为球坐标:

$$\begin{cases} \Delta_3 u = 0,\ (r<2), \\ u|_{r=2} = 3\cos 2\theta. \end{cases}$$

六. (14分)已知平面区域$D=\{(x,y)| -\infty < x < +\infty, y<1\}$.

1. 写出D内泊松方程第一边值问题的Green函数所满足的定解问题, 并求出Green函数.

2. 求解定解问题:

$$\begin{cases} u_{xx} + a^2 u_{yy} = 0,\ (-\infty < x < +\infty, y < 1), \\ u|_{y=1} = \varphi(x). \end{cases}$$

参考公式

1. 直角坐标系: $\Delta_3 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$,
   柱坐标系: $\Delta_3 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$,
   球坐标系: $\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$.

2. 若ω是$J_\nu(\omega a)=0$的一个正根, 则有模平方 $N^2_{\nu 1n} = \dfrac{a^2}{2}J^2_{\nu+1}(\omega a)$.
   若ω是$J'_\nu(\omega a)=0$的一个正根, 则有模平方$N^2_{\nu 2n} = \dfrac{1}{2}\left[a^2 - \dfrac{\nu^2}{\omega^2}\right]J^2_\nu(\omega a)$.

3. 勒让德多项式: $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$, $n=0,1,2,3,\cdots$, 母函数: $(1-2xt+t^2)^{-\frac{1}{2}} = \sum\limits_{n=0}^{+\infty}P_n(x)t^n$, $(|t|<1)$, 递推公式: $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$.

4. $\dfrac{1}{\pi}\displaystyle\int_0^{+\infty} e^{-a^2\lambda^2 t}e^{i\lambda x}d\lambda = \dfrac{1}{\pi}\int_0^{+\infty} e^{-a^2\lambda^2 t}\cos\lambda x d\lambda = \dfrac{1}{2a\sqrt{\pi t}}\exp(-\dfrac{x^2}{4a^2 t})$

5. 二维泊松方程基本解为$u = \frac{1}{2\pi}\ln r$, 这里(r,θ)为极坐标.

6. 由平面区域D内Poisson方程第一边值问题的Green函数$G(M;M_0)$, 求得Poisson方程第一边值问题解$u(M)$的公式是:

$$u(M) = \int_S \varphi(M_0)\frac{\partial G}{\partial n}(M;M_0)dS + \iint_D f(M_0)G(M;M_0)dM_0,$$

   其中S是D的边界.

补充题合集

1. 若电报方程

$$u_{xx} = CLu_tt + (CR + LG)u_t + GRu,\ (C, L, R, G\text{为常数})$$

具有形如

$$u(x,t) = A(t)f(x - at)$$

的解(称为阻尼波), 其中f是一个任意函数, 试到处此时C, L, R, G之间应满足的关系.

2. 现有半径为a的半圆形平板, 其表面绝热, 设在板的圆周边界上保持常温u_0, 而在直径边界上保持常温u_1, 求圆板处于稳恒状态(即与时间t无关的状态)时的温度分布.

3. 考虑方程$x^2u_{xx} - y^2u_{yy} = 0$,

   (a) 讨论该方程在实平面$\mathbb{R}^2$上所属得类型;

   (b) 在右半开平面$\{(x,y)|x > 0, -\infty < y < +\infty\}$上将上述方程化为标准形式;

   (c) 请写出上述方程在区域$M = \{(x,y)|1 < x < e, -\infty < y < +\infty\}$上给定边界条件$u(1,y) = u(e,y) = 0$以后构成定解问题, 并且考虑应用分离变量法所得出得Sturm-Liouville问题, 找出所有固有值以及固有函数.

4. 求解定解问题

$$\begin{cases} \frac{\partial u}{\partial t} - 4\frac{\partial u}{\partial x} + f(t,x)u = 0,\ (t > 0, -\infty < x < +\infty), \\ u|_{t=0} = x^2 + \ln x. \end{cases}$$

在

   (a) $f(t,x) = 0$

   (b) $f(t,x) = 1$

   (c) $f(t,x) = t^2x$

时的解.

5. 求解定解问题:

$$\begin{cases} \Delta_3 u = 0,\ (r < 1, 0 < z < 1), \\ u|_{r=0}\text{有界},\ \frac{\partial u}{\partial r}|_{r=1} = 0, \\ u|_{z=0} = 0,\ u|_{z=1} = 3J_0(ar) + 5J_0(br). \end{cases}$$

其中$r = \sqrt{x^2 + y^2}, 0 < a < b$, 且$J_0'(a) = J_0'(b)$.

6. 求解定解问题:

$$\begin{cases} u_t = u_{xx} + 3u + a^2 u_{yy}, \\ u|_{t=0} = \varphi(x,y), \end{cases}$$

并求出$a=0, \varphi(x,y)=\delta(x)$时的解.

7. 已知区域 $V = \{(x,y,z)|x>0, x^2+y^2+z^2<1\}$, 求其Green函数.

8. 求解定解问题:

$$\begin{cases} \Delta_2 u = 0, \\ u|_{r=1} = 0,\ u|_{r=e} = 0, \\ u|_{\theta=0} = 0,\ u|_{\theta=\frac{\pi}{3}} = r. \end{cases}$$

9. 求解定解问题:

$$\begin{cases} u_{tt} = u_{xx} + 2u_x + \delta(t-1, x-2),\ (0<x<3, t>0), \\ u(t,0) = 0,\ u(t,3) = 0, \\ u(0,x) = 1,\ u_t(0,x) = 0. \end{cases}$$

10. 考察定解问题:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + f(t,x),\ (0<x<\pi, t>0), \\ u|_{x=0} = 0, u|_{x=\pi} = 0,\ (t>0), \\ u|_{t=0} = \phi(x), u_t|_{t=0} = \psi(x),\ (0<x<\pi). \end{cases}$$

(a) 当$f(t,x)=0$时, 求此定解问题的解u_1;

(b) 当$f(t,x)=\sin 2x \sin \omega t$(其中$\omega \neq 4$), $\phi(x)=0, \psi(x)=0$时, 求此定解问题的解u_2.

参考答案

2003-2004学年数理方程A期末试题

一. $u(r,\theta)=1+r\cos\theta+r^2\cos 2\theta$.

二. $u(t,x)=-\dfrac{1}{3}xt^3+\dfrac{4}{\pi^4}\displaystyle\sum_{n=1}^{+\infty}\dfrac{(-1)^{n+1}}{n^4}(n\pi t-\sin n\pi t)\sin n\pi x$.

三. $y(x)=x^2-1=\displaystyle\sum_{n=1}^{+\infty}\dfrac{-8}{\omega_0^3 J_1(\omega_n)}J_0(\omega_n x)$.

四. $u(t,x)=U(t,x)*\varphi(x)+\displaystyle\int_0^t U(t-\tau,x)*f(\tau,x)d\tau, U(t,x)=\mathcal{F}^{-1}\left[e^{-\lambda^2+2\lambda i+1t}\right]=\dfrac{e^{-\frac{(x-2t)^2}{4t}}}{2\sqrt{\pi t}}$.

五. 在V内任取一点$M_0(\rho,\theta,\phi)$放置单位点电荷q, 其镜像点电荷分别为

$$M_1(\frac{1}{\rho},\theta,\phi)\ -\frac{1}{\rho}q,\ M_2(\rho,\theta,\pi-\phi)\ -q,\ M_3(\frac{1}{\rho},\theta,\pi-\phi)\ +\frac{1}{\rho}q,$$

基本解为

$$U(M;M_0)=\frac{1}{4\pi}\left[\frac{1}{r(M,M_0)}-\frac{1}{\rho r(M,M_1)}-\frac{1}{r(M,M_2)}+\frac{1}{\rho r(M,M_3)}\right].$$

六. 略.

2003-2004学年第一学期数理方程B期末试题

1. $\dfrac{1}{4}(1-\cos 2t)\sin 2x+6x^2t+2t^3$.

2. $u(r,\theta)=\dfrac{4}{3}P_0(\cos\theta)+\left(\dfrac{2}{67}r^2+\dfrac{128}{201}r^{-3}\right)P_2(\cos\theta)$.

3. $u(t,x)=\displaystyle\sum_{n=1}^{+\infty}C_n e^{-(\omega_0-1)t}J_1(\omega_n x)$,

   其中$C_n=\dfrac{2}{J_1^2(\omega_0)}\displaystyle\int_0^1 x\varphi(x)J_0(\omega_n x)dx$, 而$\omega_n$为方程$J_0(\omega)=0$的第$n$个正根.

4. $u(t,x)=U(t,x)*\varphi(x)+\displaystyle\int_0^t U(t-\tau,x)*f(\tau,x)d\tau,\ U(t,x)=\mathcal{F}^{-1}[\bar{U}(t,\lambda)],\ \bar{U}(t,\lambda)=e^{-(a^2\lambda^2+b\lambda i-c)t}$.

5. $G(x,y;\xi,\eta)=\frac{1}{4\pi}\ln\frac{[(x+\xi)^2+(y-\xi)^2][(x-\xi)^2+(y+\xi)^2]}{[(x-\xi)^2+(y-\xi)^2][(x+\xi)^2+(y+\xi)^2]}$.
$u(x,y)=-\int_L \varphi(x_0,y_0)\frac{\partial G(x,y;\xi,\eta)}{\partial n}dl+\iint_D f(\xi,\eta)G(x,y;\xi,\eta)dA$.

2004-2005学年第二学期数理方程A期末试题

一.

1. $\delta(x-x_0)=\frac{2}{l}\sum_{n=1}^{+\infty}\sin\frac{n\pi}{l}x_0\sin\frac{n\pi}{l}x;\delta'(x-x_0)=\frac{2n\pi}{l^2}\sum_{n=1}^{+\infty}\sin\frac{n\pi}{l}x_0\cos\frac{n\pi}{l}x$.

2. $\mathcal{F}\left[\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2\delta(x,y)\right]=-(\lambda+\mu)^2$.

3. $f(x)=\frac{A}{2\pi}\cos\lambda_0 x$.

4. $G(z;z_0)=\frac{1}{2\pi}\ln\left|\frac{z^3-\bar{z}_0^3}{z^3-z_0^3}\right|$.

5. $\lambda_n=\left(\frac{2ns+1}{2}\pi\right)^2,\ n=0,1,2,\cdots;\ y_n(x)=\sin\frac{2n+1}{2}\pi x,\ ||y_n(x)||^2=\frac{1}{2}$.

二.

1. $u(t,x)=\ln(e^x+t)$.

2. $u(x,y)=-\cos x\cos y+\cos x+4xy$.

3. $u(x,y)=\frac{3}{8}\int_{x-3y}^{x-\frac{1}{3}y}\varphi(\xi)d\xi$.

三.

1. $u(t,x)=\frac{1}{2\sqrt{\pi}}\int_0^t f(\tau,x)*\frac{e^{-\frac{x^2}{4(t-\tau)}-h(t-\tau)}}{\sqrt{t-\tau}}d\tau$.

2. $u(r,\theta)=\left(\frac{r^4}{12}-\frac{a^3+4a^2}{12(a+2)}r^2\right)\cos 2\theta$.

3. $\varphi(r)=A_0+\sum_{n=1}^{+\infty}A_nJ_0(\omega_n r),\ A_0=\frac{2}{b^2}\int_0^b r\varphi(r)dr,\ A_n=\frac{2}{b^2J_0^2(\omega_n b)}\int_0^b r\varphi(r)J_0(\omega_n r)dr$,
其中ω_n为方程$J_1(\omega b)=0$的第n个正根, $n=1,2,\cdots$

2005-2006学年第一学期数理方程B期末试题

一.

1. $e^{-x}f(y)+g(x)+y$, f及g为一阶可微的任意函数.

2. $\lambda_n =^2,\ n=0,1,2,\cdots,\ y_n(x)=\sin\left(n+\dfrac{1}{2}\right)x.$

3. 0.

4. $e^{i\lambda a}$.

5. $f(x)=\dfrac{3}{5}P_1(x)+\dfrac{2}{5}P_3(x).$

二. $u=xt^2$.

三. $u(t,x)=\dfrac{100}{\pi}x+\dfrac{2}{\pi}\sum\limits_{n=0}^{+\infty}(-1)^k e^{-(2k+1)^2t}\sin(2k+1)x.$

四. $u(t,x)=\dfrac{2}{l^2}\sum\limits_{n=1}^{+\infty}\dfrac{\int_0^l xf(x)J_0(\omega_n x)dx}{J_1^2(\omega_n l)}\cos\omega_n l J_0(\omega_n x).$

五.

1. $G(x,y;\xi,\eta)=\dfrac{1}{2\pi}\ln\dfrac{r(M,M_1)r(M,M_2)}{r(M,M_0)r(M,M_3)},\ M_0(\rho,\theta)\ \varepsilon_0,\ M_1(\dfrac{1}{\rho},\theta)\ -\dfrac{\varepsilon_0}{\rho},\ M_2(\rho,-\theta)\ -\varepsilon_0,\ M_3(\dfrac{1}{\rho},-\theta)\ \dfrac{\varepsilon_0}{\rho}.$

2. $u(x,y)=\displaystyle\int_{-1}^{1}\varphi(\xi)\frac{\partial G(x,y;\xi,0)}{\partial\eta}d\xi+\int_{-1}^{1}d\xi\int_0^{\sqrt{1-\xi^2}}f(\xi,\eta)G(x,y;\xi,\eta)d\eta.$

2014-2015学年第二学期数理方程A期末试题

一.

1. $\Delta=(a+b)^2-4ab=(a-b)^2>0,(a\neq b)$, 它为双曲型方程.

2. 代换$\xi=y+ax,\eta=y+bx$.

3. $u(x,y)=f(y+ax)+g(y+bx),\ f,g\in C^2(\mathcal{R}).$

4. $u(x,y)=\psi(\dfrac{y+ax}{a-b})+\varphi(\dfrac{y+bx}{b-a})-\varphi(0).$

二.

1. 特征线方程:$\dfrac{dx}{x}=\dfrac{dy}{2x}$, 解得特征线$y=x^2+c,\ c\in R$.

2. $u(x,y)=x^4-\dfrac{2}{3}x^3+xy-2x^2y+y^2+1.$

三.

1. $u_1(t,x) = \sum_{n=1}^{\infty}(A_n \cos 2nt + B_n \sin 2nt) \sin nx dx,$

   其中$A_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \varphi(x) \sin nx dx,\ B_n = \dfrac{1}{n\pi}\displaystyle\int_0^{\pi} \psi(x) \sin nx dx.$

2. $u_2(t,x) = \dfrac{1}{16-\omega^2}(\sin \omega t - \dfrac{\omega}{4} \sin 4t) \sin 2x,$

   $\lim\limits_{\omega \to 4} u_2(t,x,\omega) = \dfrac{1}{8}(\dfrac{1}{4} \sin 4t - t \cos 4t) \sin 2x.$

四.

1. $u(r,z) = \sum_{n=1}^{+\infty}(A_n \cosh \omega_{1n} z + B_n \sinh \omega_{1n} z J_0(\omega_{1n} r),$

   $A_n = 0,\ B_n = \dfrac{2\int_0^a f(r) J_0(\omega_{1n} r) r dr}{a^2 \sinh \omega_1 n h J_1^2(\omega_{1n} a)}.$

2. $\begin{cases} \Theta'' + \mu^2 \Theta = 0,\ (0 \le \theta \le 2\pi), \\ \Theta(\theta + 2\pi) = \Theta(\theta), \end{cases}$ 和 $\begin{cases} R'' + \frac{1}{r} R' + (\lambda - \frac{\mu^2}{r^2}) R = 0, \\ |R(0)| < +\infty,\ R(a) = 0. \end{cases}$

五.

1. $U(t,x,y,z) = \left(\dfrac{1}{2\sqrt{\pi t}}\right)^3 e^{-\frac{x^2+y^2+z^2}{4t} + 3t}.$

2. $u(t,x,y,z) = \dfrac{1}{(1+4t)^{\frac{2}{3}}} e^{3t - \frac{1}{1+4t}(x^2+y^2+z^2)}.$

六.

1. $G = \dfrac{1}{4\pi} \ln \dfrac{(x+\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y-\eta)^2}.$

2. $u(x,y) = \dfrac{5x}{\pi} \displaystyle\int_{-\infty}^{+\infty} \left(\dfrac{\varphi(\eta)}{(y-\eta)^2 + 25x^2}\right) d\eta.$

七. $Z(\theta) = P_4(\cos\theta),\ Z(\dfrac{\pi}{2} = \dfrac{3}{8}$

(updating)