2014-2015方程 打分建议

[6分] 一、方法(一):

2分 $\frac{\partial u}{\partial y}=\int\frac{\partial^2 u}{\partial x\partial y}dx=\frac{x^3}{3}y+\varphi(y)$

2分 $u=\int\frac{\partial u}{\partial y}dy=\int(\frac{x^3}{3}y+\varphi(y))dy$

$=\frac{x^3}{3}\frac{y^2}{2}+\int\varphi(y)dy+g(x)$

2分 $u=\frac{x^3y^2}{6}+f(y)+g(x)$

方法(二):

2分 特解 $u^*=\frac{x^3y^2}{6}$

2分 齐次方程通解 $w=f(y)+g(x)$

2分 $u=u^*+w=\frac{x^3y^2}{6}+f(y)+g(x)$

[6分] 二(1)

2分 边条左II右I 由S-L定理, 固有值>0

故 $X=A\cos\left(\sqrt{\lambda}x\right)+B\sin\left(\sqrt{\lambda}x\right)$

$X'(0)=-A\cdot 0+B\cos(0)=0,\ B=0$

$X(5)=A\cos\left(\sqrt{\lambda}5\right)=0$

2分 $\lambda_n=\left[\frac{(n-\frac{1}{2})\pi}{5}\right]^2,\ n=1,2,3\cdots$

2分 $X_n=cos\left[(n-\frac{1}{2})\pi\frac{x}{5}\right]$

*若结果错,据边条等推导酌情给,总分≤ 3

[6分] (2)

2分 方程$y''+\frac{1-x}{x}y'+\frac{\lambda}{x}y=0$ 同乘$e^{\int\frac{1-x}{x}dx}$

2分 $e^{\int\frac{1-x}{x}dx}=e^{\ln x-x}=xe^{-x}$

2分 $(xe^{-x}y')'+\lambda e^{-x}y=0$

[12分] 三(1)

2分 ①分离 $u=T(t)X(x)$

$T''(t)+\lambda a^2T(t)=0$

$X''(x)+\lambda X(x)=0$

边条分离 $X(0)=0,\ X'(\pi)=0$

②解固有值问题

2分 $\lambda_n=\left[(n-\frac{1}{2})\frac{\pi}{\pi}\right]^2,\ n=1,2,3\cdots$

$X_n=\sin\left[(n-\frac{1}{2})x\right]$

2分 解常微分方程 $T''(t)+\lambda_n a^2T(t)=0$

$T_n=C_n\cos(n-\frac{1}{2})at+D_n\sin(n-\frac{1}{2})at$

特解 $T_n(t)X_n(x)\cdots$

2分 ③叠加 $u(t,x)=\sum\limits_{n=1}^{\infty}[C_n\cos(n-\frac{1}{2})at$

$+D_n\sin(n-\frac{1}{2})at]\sin\left[(n-\frac{1}{2})x\right]$

4分 定系数 $u|_{t=0}=\sum\limits_{n=1}^{\infty}C_n\sin\left[(n-\frac{1}{2})x\right]=\varphi$

$u_t|_{t=0}=\sum\limits_{n=1}^{\infty}(n-\frac{1}{2})aD_n\sin[n-\frac{1}{2})x]=\psi$

$C_n=\frac{2}{\pi}\int_0^{\pi}\varphi(x)\sin[(n-\frac{1}{2})x]dx$

$D_n=\frac{1}{(n-\frac{1}{2})a}\frac{2}{\pi}\int_0^{\pi}\psi(x)\sin[(n-\frac{1}{2})x]dx$

*注意基,内积,模^2; 基若错全错 步骤分≤ 6

[8分] 三(2)

2分 $t=0,\ \sum\limits_{n=1}^{\infty}C_n\sin\left[(n-\frac{1}{2})x\right]=\sin(\frac{1}{2}x)$

$C_1=1,\ C_n=0(n>1)$

4分 $D_n=\frac{1}{(n-\frac{1}{2})a}\frac{2}{\pi}\int_0^{\pi}\delta(x)\sin[(n-\frac{1}{2})x]dx$

$=\frac{2}{\pi}\frac{\sin[3(n-\frac{1}{2})]}{(n-\frac{1}{2})a}$

2分 $u=\cos(\frac{at}{2})\sin(\frac{x}{2})$

$+\sum\limits_{n=1}^{\infty}\frac{4}{\pi}\frac{\sin(\frac{6n-3}{2})}{(2n-1)a}\sin(\frac{2n-1}{2}at)\sin(\frac{2n-1}{2}x)$

[12分] 四、$u=v+w$

4分 设$v=Ax+B$把边条齐次化

$v|_{x=0}=B=0,\ v|_{x=1}=A\cdot 1=1,\ v=x$

$w|_{x=0}=w|_{x=1}=0$ I类

1分 $w_t-4w_{xx}=u_t-4u_{xx}-(v_t-4v_{xx})=0$

1分 $w|_{t=0}=u|_{t=0}-v|_{t=0}=\varphi+x-x=\varphi$

4分 对w分离变量, 经①②③可得

$w=\sum\limits_{n=1}^{\infty}A_ne^{-4(n\pi)^2t}\sin(n\pi x)$

$A_n=2\int_0^1\varphi(x)\sin(n\pi x)dx$

2分 $u=v+w=x+\sum\limits_{n=1}^{\infty}A_ne^{-4(n\pi)^2t}\sin n\pi x$

*若结果错,据推导酌情给,总分≤ 6

[14分] 五、

2分 ①分离 $u=R(r)\Theta(\theta)$

$\left\{\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right\}u=0$

$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\lambda$

$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-\lambda$

$x=\cos\theta\to[(1-x^2)y']'+\lambda y=0$

②解固有值问题(Legendre方程)

$\lambda_n=n(n+1),\ n=0,1,2\cdots$

$\Theta_n=P_n(\cos\theta)$

解Euler方程 $R_n=A_nr^n+B_nr^{-(n+1)}$

2分 ③叠加: 轴对称情形下解为

$u(t,x)=\sum\limits_{n=0}^{\infty}[A_nr^n+B_nr^{-(n+1)}]P_n(\cos\theta)$

2分 球外: 自然边条无穷远有界 *漏A_0扣分

$u(t,x)=A_0+\sum\limits_{n=0}^{\infty}[B_nr^{-(n+1)}]P_n(\cos\theta)$

2分 定系数 $u|_{r=+\infty}=A_0=0$

4分 $u|_{r=R}=\frac{B_0}{R}P_0+\frac{B_1}{R^2}P_1+\frac{B_2}{R^3}P_2(\cos\theta)+\cdots$

$=1-\cos^2\theta$

偶$\to B_1=0$; 正交性: 与$P_n,\ n>2$无关;

$B_2=-\frac{2}{3}R^3,\ B_0=\frac{2}{3}R$

2分 $u=\frac{2}{3}\frac{R}{r}-\frac{1}{3}\frac{R^3}{r^3}(3\cos^2\theta-1)$

[14分] 六、

2分 ①分离 $u = R(r)Z(z)$ *轴对称指出$\nu = 0$

$r^2R'' + rR' + \lambda r^2 R = 0$ Bessel

$Z'' - \lambda Z = 0$

边条分离 $R'(a) = J_0'(\omega r) = 0$ II类

1分 ②解固有值问题(Bessel方程)$\lambda = \omega^2$

有界解$R(r) = J_0(\omega r)$

2分 设ω_n是$J_0'(\omega r) = 0$的非负根, $n = 0, 1\cdots$

*也可写$J_1(\omega r) = 0$的非负根; 漏$n = 0$扣分

1分 则固有值$\lambda_n = \omega_n^2$, 固有函数$J_0(\omega_n r)$

1分 解其余问题 $\lambda_0 = 0$, $Z_0 = C_0 + D_0 z$

1分 $n = 1, 2\cdots$时, $Z_n = C_n ch\omega_n z + D_n sh\omega_n z$

*也可用$C_n e^{\omega_n z} + D_n e^{-\omega_n z}$

2分 ③叠加: $u(r, z) = C_0 + D_0 z + \sum_{n=1}^{\infty}\cdots$

4分 定系数: 见课本P288 $f_2 = 0$

$N_{02n}^2 = \frac{a^2}{2}J_0^2(\omega_n a)$, $J_0(0) = 1$, $N_{020}^2 = \frac{a^2}{2}$

[12分] 七、

$\Delta_3 G = -\delta(M - M_0)$, $G|_S = 0$

3分 设M_0处 $(+\xi, +\eta, \zeta)$ 放置+电荷ε_0, 则

M_1坐标 $(+\eta, +\xi, \zeta)$ 镜像-电荷

M_2坐标 $(+\eta, -\xi, \zeta)$ 镜像+

M_3坐标 $(-\xi, +\eta, \zeta)$ 镜像-

M_4坐标 $(-\xi, -\eta, \zeta)$ 镜像+

M_5坐标 $(-\eta, -\xi, \zeta)$ 镜像-

M_6坐标 $(-\eta, +\xi, \zeta)$ 镜像+

M_7坐标 $(+\xi, -\eta, \zeta)$ 镜像-

2分 考虑点源在$M(x, y, z)$处的电势, 约定

$r(M, M_0) = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

$r(M, M_1) = \sqrt{(x-\eta)^2 + (y-\xi)^2 + (z-\zeta)^2}$

$r(M, M_2) = \sqrt{(x-\eta)^2 + (y+\xi)^2 + (z-\zeta)^2}$

$r(M, M_3) = \sqrt{(x+\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$

$r(M, M_4) = \sqrt{(x+\xi)^2 + (y+\eta)^2 + (z-\zeta)^2}$

$r(M, M_5) = \sqrt{(x+\eta)^2 + (y+\xi)^2 + (z-\zeta)^2}$

$r(M, M_6) = \sqrt{(x+\eta)^2 + (y-\xi)^2 + (z-\zeta)^2}$

$r(M, M_7) = \sqrt{(x-\xi)^2 + (y+\eta)^2 + (z-\zeta)^2}$

3分 $G = U_0 + U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + U_7$

$= \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)} + \frac{1}{4\pi r(M,M_2)} - \frac{1}{4\pi r(M,M_3)}$

$+ \frac{1}{4\pi r(M,M_4)} - \frac{1}{4\pi r(M,M_5)} + \frac{1}{4\pi r(M,M_6)} - \frac{1}{4\pi r(M,M_7)}$

2分 验证 $G|_{y=0} = (U_7 + U_0)|_{y=0}$

$+(U_1 + U_2) + (U_3 + U_4) + (U_5 + U_6)|_{y=0}$

$= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-\xi)^2 + (-\eta)^2 + (z-\zeta)^2}} - \frac{1}{\sqrt{(x-\xi)^2 + (\eta)^2 + (z-\zeta)^2}}\right)\ldots$

$= (0) + (同理为0) + (0) + (0) = 0$

2分 $G|_{y=x} = (U_0 + U_1)|_{y=x}$

$+(U_2 + U_3) + (U_4 + U_5) + (U_6 + U_7)|_{y=x}$

$= 0$

[10分] 八、方法(一):

2分 变换$u = e^{-t}v$

4分 $u_t = e^{-t}v_t - e^{-t}v$

$u_{tt} = e^{-t}v_{tt} - 2e^{-t}v_t + e^{-t}v$

$u_{tt} + 2u_t + u = e^{-t}v_{tt} - 2e^{-t}v_t + e^{-t}v$

$+2e^{-t}v_t - 2e^{-t}v + e^{-t}v = e^{-t}v_{tt}$

$v_{tt} = 4v_{xx}$

*此方法过简,步骤不全应严格扣分以公平

2分 $v = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$

$= \frac{1}{4}\int_{x-2t}^{x+2t}\psi(\xi)d\xi$

2分 $u = e^{-t}\frac{1}{4}\int_{x-2t}^{x+2t}\psi(\xi)d\xi$

方法(二):

2分 ①FT $\bar{u}(t, \lambda) = F[u(t, x)]$

$4(-i\lambda)^2\bar{u} = \frac{d^2\bar{u}}{dt} + 2\frac{d\bar{u}}{dt} + \bar{u}$

$\bar{u}|_{t=0} = 0$

$\bar{u}_t|_{t=0} = \bar{\psi}$, 约定$\bar{\psi}(\lambda) = F[\psi(x)]$

3分 ② 解特征方程得$k = \frac{-2 \pm \sqrt{4 - 4(1 + 4\lambda^2)}}{2}$

通解 $\bar{u} = Ae^{(-1+i2\lambda)t} + Be^{(-1-i2\lambda)t}$

定解条件 $\bar{u}|_{t=0} = A + B = 0$

$\bar{u}_t|_{t=0} = A(-1 + i2\lambda) + B(-1 - i2\lambda) = \bar{\psi}$

$A = -B = \frac{\bar{\psi}}{4i\lambda}$

定解 $\bar{u} = e^{-t}\left[\frac{1}{4}\frac{\bar{\psi}}{i\lambda}e^{i2\lambda t} + \frac{1}{4}\frac{\bar{\psi}}{-i\lambda}e^{-i2\lambda t}\right]$

③$F^{-1}T$

3分 $F^{-1}\left[\frac{\bar{\psi}}{-i\lambda}\right] = \int_{-\infty}^{x}\psi(\xi)d\xi$

$F^{-1}\left[\frac{1}{4}\frac{\bar{\psi}}{-i\lambda}e^{-i2\lambda t}\right] = \frac{1}{4}\int_{-\infty}^{x+2t}\psi(\xi)d\xi$

$F^{-1}\left[\frac{1}{4}\frac{\bar{\psi}}{i\lambda}e^{i2\lambda t}\right] = -\frac{1}{4}\int_{-\infty}^{x-2t}\psi(\xi)d\xi$

2分 $u(t, x) = F^{-1}[\bar{u}] = e^{-t}\frac{1}{4}\int_{x-2t}^{x+2t}\psi(\xi)d\xi$

$\frac{\bar{\psi}}{i\lambda}e^{i2t\lambda} - \frac{\bar{\psi}}{i\lambda}e^{-i2\lambda t}$

$= \frac{\bar{\psi}}{i\lambda}\,2i\sin 2\lambda t$

$= 2\frac{\bar{\psi}}{\lambda}\sin 2\lambda t$

$\frac{\bar{\psi}}{2\lambda}\sin 2\lambda t$