

例 1.12 随机和的矩母函数. 记 X_1, X_2, \dots 为一串独立同分布的随机变量, N 为非负整数值随机变量且与 X 序列相独立. Y 为随机和 $\sum_{i=1}^N X_i$. 求 Y 的矩母函数 $g_Y(t)$.

解 为求 g_Y , 先算条件期望

$$\begin{aligned} E[e^{tY} \mid N = n] &= E\left[\exp\left\{t \sum_{i=1}^N X_i\right\} \mid N = n\right] \\ &= E\left[\exp\left\{t \sum_{i=1}^n X_i\right\} \mid N = n\right] \\ &= E\left[\exp\left\{t \sum_{i=1}^n X_i\right\}\right] = [g_X(t)]^n. \end{aligned}$$

于是有 $g_Y(t) = E[\exp\{tY\}] = E\{E[\exp\{tY\} \mid N]\} = E[(g_X(t))^N]$. 对 $g_Y(t)$ 关于 t 求导即有

$$\begin{aligned} g'_Y(t) &= E[N(g_X(t))^{N-1} g'_X(t)], \\ g''_Y(t) &= E[N(N-1)(g_X(t))^{N-2} (g'_X(t))^2 + N(g_X(t))^{N-1} g''_X(t)]. \end{aligned}$$

将 $t = 0$ 代入上面两式得

$$\begin{aligned} EY &= E[NE(X)] = EN \cdot EX, \\ EY^2 &= EN \cdot \text{Var}X + EN^2 \cdot E^2X, \\ \text{Var}Y &= EN \cdot \text{Var}X + E^2X \cdot \text{Var}N. \end{aligned} \tag{1.20}$$