（2）球对称的三维波动方程的初始问题

$$\begin{cases} u_{tt} = a^2 \Delta_3 u, \\ u\big|_{t=0} = \varphi(r), \\ u_t\big|_{t=0} = \psi(r); \end{cases}$$

[**提示：**利用球坐标可将方程化为

$$u_{tt} = a^2\left(u_{rr} + \frac{2}{r}u_r\right),$$

再令 $v=ru$,就可化为弦振动方程.]

通解：$u = tx^2 + C$　　∴ $C = x^2$　　∴ 特解 $u_0 = tx^2 + x^2$

(2)

$u_{tt} = a^2 \Delta_3 u = a^2\left(u_{rr} + \frac{2}{r}u_r\right)$　　$u = \frac{v}{r}$

$\frac{1}{r}v_{tt} = a^2\left(\frac{1}{r}\cdot\frac{\partial^2 v}{\partial r^2} - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{2}{r^3}v + \frac{2}{r^2}\frac{\partial v}{\partial r} - \frac{2}{r^3}v\right)$

即 $v_{tt} = a^2\left(\frac{\partial^2 v}{\partial r^2}\right)\cancel{\frac{2}{r}\frac{\partial v}{\partial r} + \frac{2}{r^2}v)}$

通解为 $v = f(\cancel{x}-at) + g(\cancel{x}+at)$

∴ $u = \frac{1}{r}f(\cancel{x}-at) + \frac{1}{r}g(\cancel{x}+at)$　　又 $\begin{cases} u\big|_{t=0} = \varphi(r) \\ u_t\big|_{t=0} = \psi(r) \end{cases}$

(3)

∴ $\begin{cases} \frac{1}{r}(f(r)+g(r)) = \varphi(r) \\ -\frac{a}{r}f'(r) + \frac{a}{r}g'(r) = \psi(r) \end{cases} \Rightarrow \begin{cases} g'(r) = \frac{1}{2}\left(\varphi(r) + r\varphi'(r) + \frac{r}{a}\cdot\psi(r)\right) \\ f'(r) = \frac{1}{2}\left(\varphi(r) + r\varphi'(r) - \frac{r}{a}\psi(r)\right) \end{cases}$

$\cancel{故}= \begin{cases} g(r) = \frac{1}{2}\left(r\cancel{\int}\varphi(r)\cancel{dr} + \int\frac{r}{a}\psi(r)dr\right) \\ f(r) = \frac{1}{2}\left(r\cancel{\int}\varphi(r)\cancel{dr} + \int\frac{r}{a}\psi(r)dr\right) \end{cases}$

∴ 特解：$u_0 = \frac{1}{2r}\left((r-at)\varphi(r-at) + (r+at)\varphi(r+at)\right) + \frac{1}{2ar}\int_{r-at}^{r+at} s\psi(s)\,ds$

(3)

$\cancel{\Delta_3 u = 0}$ 据Hint. 当 $x_0^2 + y_0^2 + z_0^2 > 1$ 时 $u = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{-\frac{1}{2}}$

满足 $\Delta_3 u = 0$ $(x^2+y^2+z^2<1)$, 故只需找到一个合适的 (x_0, y_0, z_0) 满足边界条件.

$$\Rightarrow u_{tt} = a^2 \Delta_3 u = a^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

$$= a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$$

令 $u = \frac{v}{r}$，则

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2} v$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2} - \frac{2}{r^2}\frac{\partial v}{\partial r} + \frac{2}{r^3} v$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial^2 v}{\partial t^2}$$

$\Rightarrow u_{tt} = a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right)$ 化为 $v_{tt} = a^2 v_{rr}$

为使 $u|_{r=0}$ 有限，应有 $v(t,0) = 0$

问题化为
$$\begin{cases} v_{tt} = a^2 v_{rr}, & t>0,\ r>0 \\ v(t,0) = 0 \\ v(0,r) = r\varphi(r), & v_t(0,r) = r\psi(r) \end{cases}$$

作奇延拓，令

$$\phi(r) = \begin{cases} r\varphi(r), & r\geq 0 \\ r\varphi(-r), & r<0 \end{cases}, \quad \Psi(r) = \begin{cases} r\psi(r), & r\geq 0 \\ r\psi(-r), & r<0 \end{cases}$$

由 d'Alembert 公式：

$$v(t,r)=\frac{1}{2}\left[\phi(r+at)+\phi(r-at)\right]+\frac{1}{2a}\int_{r-at}^{r+at}\Psi(\xi)\,d\xi$$

$$=\begin{cases}\frac{1}{2}\left[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(\xi)\,d\xi, & t\leqslant\frac{r}{a} \\ \frac{1}{2}\left[(r+at)\varphi(-r-at)+(r-at)\varphi(at-r)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(-\xi)\,d\xi, & t>\frac{r}{a}\end{cases}$$

取 $r\geqslant 0$ 的部分有

$$u(t,r)=\frac{v}{r}$$

$$=\begin{cases}\frac{1}{r}\Big\{\frac{1}{2}\left[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(\xi)\,d\xi\Big\}, & r\geqslant at \\ \frac{1}{r}\Big\{\frac{1}{2}\left[(r+at)\varphi(-r-at)+(r-at)\varphi(at-r)\right] \\ \quad +\frac{1}{2a}\int_{r-at}^{r+at}\xi\psi(-\xi)\,d\xi\Big\}, & 0\leqslant r<at\end{cases}$$