第五章相关基础知识整理

矢量微分算子和拉普拉斯算子

算子

$$\nabla = \tfrac{\partial}{\partial x} i + \tfrac{\partial}{\partial y} j + \tfrac{\partial}{\partial z} k$$

称为矢量微分算子，简称矢量算子。在 ∇ 算子的基础上，若函数 $u(x,y,z)$ 和矢量 $\boldsymbol{E}(x,y,z)$ 有连续的一阶偏导数，则可作如下定义。

(1) 梯度：函数 u 的梯度定义为：

$$\nabla u = \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k$$

(2) 散度：矢量 $\boldsymbol{E}$ 的散度定义为：

$$\nabla \cdot \boldsymbol{E} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \right) \cdot \left(E_x i + E_y \boldsymbol{j} + E_2 \boldsymbol{k} \right) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

(3) 旋度: 矢量 $\boldsymbol{E}$ 的旋度定义为：

$$\begin{aligned}\nabla \times \boldsymbol{E} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_y & E_z \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ E_x & E_z \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ E_x & E_y \end{vmatrix} \boldsymbol{k} \\ &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \boldsymbol{k}\end{aligned}$$

拉普拉斯算子表示为

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

它作用于函数 u 给出

$$\nabla^2 u \equiv \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

而作用于矢量 $\boldsymbol{E}$ 给出

$$\nabla^2 \boldsymbol{E} = \left(\nabla^2 E_x \right) \boldsymbol{i} + \left(\nabla^2 E_y \right) \boldsymbol{j} + \left(\nabla^2 E_z \right) \boldsymbol{k}$$

设函数 u, v 和矢量 $\boldsymbol{E}, \boldsymbol{F}$ 都是 (x,y,z) 的函数，如果它们的一阶偏导数是存在的，则存在一些常用的关于 ∇ 和 ∇^2 的公式：

(1) $\nabla(u+v)=\nabla u+\nabla v$

(2) $\nabla\cdot(\boldsymbol{E}+\boldsymbol{F})=\nabla\cdot\boldsymbol{E}+\nabla\cdot\boldsymbol{F}$

(3) $\nabla\times(\boldsymbol{E}+\boldsymbol{F})=\nabla\times\boldsymbol{E}+\nabla\times\boldsymbol{F}$

(4) $\nabla\cdot(uE)=(\nabla u)\cdot\boldsymbol{E}+u(\nabla\cdot\boldsymbol{E})$

(5) $\nabla\times(uE)=(\nabla u)\times E+u(\nabla\times E)$

(6) $\nabla\cdot(\boldsymbol{E}\times\boldsymbol{F})=\boldsymbol{F}\cdot(\nabla\times\boldsymbol{E})-\boldsymbol{E}\cdot(\nabla\times\boldsymbol{F})$

(7) $\nabla\times(\boldsymbol{E}\times\boldsymbol{F})=(\boldsymbol{F}\cdot\nabla)\boldsymbol{E}-\boldsymbol{F}(\nabla\cdot\boldsymbol{E})-(\boldsymbol{E}\cdot\nabla)\boldsymbol{F}+\boldsymbol{E}(\nabla\cdot\boldsymbol{F})$

(8) $\nabla(\boldsymbol{E}\cdot\boldsymbol{F})=(\boldsymbol{F}\cdot\nabla)\boldsymbol{E}+(\boldsymbol{E}\cdot\nabla)\boldsymbol{F}+\boldsymbol{F}\times(\nabla\times\boldsymbol{E})+\boldsymbol{E}\times(\nabla\times\boldsymbol{F})$

(9) $\nabla\times(\nabla u)=0$, 即 u 的梯度的旋度是零

(10) $\nabla\cdot(\nabla\times\boldsymbol{E})=0$, 即 $\boldsymbol{E}$ 的旋度的散度是零

(11) $\nabla\times(\nabla\times\boldsymbol{E})=\nabla(\nabla\cdot\boldsymbol{E})-\nabla^2\boldsymbol{E}$

例 1 证明公式

$$\nabla\cdot(u\boldsymbol{E})=(\nabla u)\cdot\boldsymbol{E}+u(\nabla\cdot\boldsymbol{E})$$

证明

$$\begin{aligned}
\nabla\cdot(u\boldsymbol{E})&=\nabla\cdot(uE_xi+uE_yj+uE_z\boldsymbol{k})\\
&=\frac{\partial}{\partial x}(uE_x)+\frac{\partial}{\partial y}(uE_y)+\frac{\partial}{\partial z}(uE_z)\\
&=\frac{\partial u}{\partial x}E_x+\frac{\partial u}{\partial y}E_y+\frac{\partial u}{\partial z}E_z+u\left(\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right)\\
&=\left(\frac{\partial u}{\partial x}i+\frac{\partial u}{\partial y}j+\frac{\partial u}{\partial z}\boldsymbol{k}\right)\cdot(E_xi+E_yj+E_z\boldsymbol{k})\\
&+u\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right)\cdot(E_xi+E_yj+E_z\boldsymbol{k})\\
&=(\nabla u)\cdot\boldsymbol{E}+u(\nabla\cdot\boldsymbol{E})
\end{aligned}$$

例 2 证明公式

$$\nabla\cdot(\nabla\times\boldsymbol{E})=0$$

证明

$$\begin{aligned}
\nabla\cdot(\nabla\times\boldsymbol{E})&=\nabla\cdot\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\ \frac{\partial}{\partial \boldsymbol{x}}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ E_x&E_y&E_z\end{vmatrix}\\
&=\nabla\cdot\left[\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)j+\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\boldsymbol{k}\right]\\
&=\frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\\
&=\frac{\partial^2E_x}{\partial x\partial y}-\frac{\partial^2E_y}{\partial x\partial z}+\frac{\partial^2E_x}{\partial y\partial z}-\frac{\partial^2E_z}{\partial y\partial x}+\frac{\partial^2E_y}{\partial z\partial x}-\frac{\partial^2E_x}{\partial z\partial y}=0
\end{aligned}$$

例 3 证明公式

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$$

$$\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{E}) =& \nabla \times \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \\
=& \nabla \times \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \boldsymbol{k} \right] \\
=& \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} & \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{vmatrix} \\
=& \left[\frac{\partial}{\partial y} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \right] i \\
&+ \left[\frac{\partial}{\partial z} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \right] j \\
&+ \left[\frac{\partial}{\partial x} \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \right] k \\
=& \left(-\frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2} \right) i + \left(-\frac{\partial^2 E_y}{\partial z^2} - \frac{\partial^2 E_y}{\partial x^2} \right) j + \left(-\frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} \right) k \\
&+ \left(\frac{\partial^2 E_y}{\partial x \partial y} + \frac{\partial^2 E_z}{\partial z \partial x} \right) i + \left(\frac{\partial^2 E_z}{\partial y \partial z} + \frac{\partial^2 E_x}{\partial x \partial y} \right) j + \left(\frac{\partial^2 E_x}{\partial z \partial x} + \frac{\partial^2 E_y}{\partial y \partial z} \right) k \\
=& \left(-\frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\partial^2 E_x}{\partial z^2} \right) i + \left(-\frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_y}{\partial y^2} - \frac{\partial^2 E_y}{\partial z^2} \right) j \\
&+ \left(-\frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} - \frac{\partial^2 E_z}{\partial z^2} \right) \boldsymbol{k} + \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial x \partial y} + \frac{\partial^2 E_z}{\partial z \partial x} \right) i \\
&+ \left(\frac{\partial^2 E_x}{\partial x \partial y} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial y \partial z} \right) j + \left(\frac{\partial^2 E_x}{\partial z \partial x} + \frac{\partial^2 E_y}{\partial y \partial z} + \frac{\partial^2 E_z}{\partial z^2} \right) k \\
&- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (E_x i + E_y j + E_z k) \\
&+ i \frac{\partial}{\partial x} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) + j \frac{\partial}{\partial y} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_v}{\partial y} + \frac{\partial E_z}{\partial z} \right) \\
&+ k \frac{\partial}{\partial z} \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_v}{\partial y} + \frac{\partial E_z}{\partial z} \right) \\
=& -\nabla^2 E + \nabla \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_x}{\partial y} + \frac{\partial E_x}{\partial z} \right) = \nabla(\nabla \cdot E) - \nabla^2 E
\end{aligned}$$