I. a) $\left| P[X\in[a,a+\delta]] - \delta f_X(a) \right| \le L\delta^2$

$\Rightarrow -L\delta^2 \le P[X\in[a,a+\delta]] - \delta f_X(a) \le L\delta^2 \Rightarrow \delta f_X(a) - L\delta^2 \le P[X\in[a,a+\delta]] \le \delta f_X(a) + L\delta^2$

b). It means we can use $\delta f_X(a)$ to estimate $P[X\in[a,a+\delta]]$. When $f_X(a)=23.4$, certainly such rough equation only applies with sufficiently small $\delta$ or $P[X\in[a,a+\delta]]$ will lose actual meaning.

II. a) $E(N) = \sum_{i=0}^{\infty} iP(N=i) = \sum_{i=1}^{\infty} iP(N=i) = \sum_{i=1}^{\infty}\sum_{n=i}^{\infty} P(N=n) = \sum_{i=1}^{\infty} P[N\ge i]$

b) let $t=x^n$. $E(X^n) = \int_0^{+\infty} P[X^n > x^n]\,dx^n = \int_0^{+\infty} nx^{n-1} P[X^n > x^n]$ since $X\ge 0$, $X^n > x^n \Rightarrow X > x$.

$\therefore E[X^n] = \int_0^{+\infty} nx^{n-1} P[X>x]\,dx$

III. a) $\sum_{i=1}^{\infty} P(X=i) = p\cdot\sum_{i=0}^{\infty}(1-p)^i = p\cdot 1/p = 1$

b) $E(Y) = \sum_{y\in S_Y} y P(Y=y) = \sum_{y\in S_Y} y\left(\sum_{i=1}^{\infty} P_{Y|\{X=i\}}(y|i)\,P(X=i)\right)$

c) $P(Y=y_1) = \sum_{i=1}^{\infty} P_{Y|\{X=i\}}(y_1|i)\,P(X=i)$

d) $P[X=3|Y=y_1] = \dfrac{P[Y=y_1|X=3]\,P[X=3]}{P[Y=y_1]} = \dfrac{P_{Y|\{X=3\}}(y_1|3)\cdot p(1-p)^2}{\sum_{i=1}^{\infty} P_{Y|\{X=i\}}(y_1|i)\,p(1-p)^{i-1}}$

e) $P[X\ge 3|Y=y_1] = \sum_{n=3}^{\infty} P[X=n|Y=y_1] = \dfrac{\sum_{n=3}^{\infty}\left(P[Y=y_1|X=n]\,P[X=n]\right)}{\sum_{i=1}^{\infty}\left(P_{Y|\{X=i\}}(y_1|i)\,p(1-p)^{i-1}\right)}$

$= \dfrac{\sum_{n=3}^{\infty}\left(P_{Y|\{X=n\}}(y_1|n)\cdot p(1-p)^{n-1}\right)}{\sum_{i=1}^{\infty}\left(P_{Y|\{X=i\}}(y_1|i)\cdot p(1-p)^{i-1}\right)}$

I, II, III

V. (a). $E(Y) = E\left(\frac{X}{2}-4\right) = \int_{-\infty}^{+\infty}\left(\frac{x}{2}-4\right)f_X(x)\,dx = \frac{m}{2}-4$

$Var(Y) = E(Y^2) - [E(Y)]^2$ $\quad E(Y^2) = E\left(\frac{X^2}{4}-4X+16\right) = \frac{\sigma^2+m^2}{4} - \frac{m}{4} + 16$ $\quad\therefore Var(Y) = \frac{\sigma^2}{4}$

(b) $f_Y(y) = \dfrac{f_X(2y+8)}{\frac{1}{2}} = 2f_X(2y+8) = 2\cdot\dfrac{e^{-(2y+8-m)^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}$, $y\in\mathbb{R}$.

(c) $E[Y^2-X] = E\left[\frac{X^2}{4}-5X+16\right] = \int_{-\infty}^{+\infty}\left(\frac{x^2}{4}-5x+16\right)f_X(x)\,dx = \frac{\sigma^2}{4}+\frac{m^2}{4}-5m+16$.

VI. (a) $E(Y) = \int_{-\infty}^{+\infty} g(x)f(x)\,dx = -a\int_{-\infty}^{-a} f_X(x)\,dx + \int_{-a}^{a} x f_X(x)\,dx + a\int_a^{+\infty} f_X(x)\,dx$

$Var(Y) = E(Y^2) - [E(Y)]^2 = a^2\int_{-\infty}^{-a} f_X(x)\,dx + \int_{-a}^{a} x^2 f_X(x)\,dx + a^2\int_a^{+\infty} f_X(x)\,dx$

$- \left[-a\int_{-\infty}^{-a} f_X(x)\,dx + \int_{-a}^{a} x f_X(x)\,dx + a\int_a^{+\infty} f_X(x)\,dx\right]^2$

(b) $f_X(x) = \frac{1}{2}e^{-|x|}$ $\quad E(Y) = \int_{-1}^{1}\frac{1}{2}xe^{-|x|}\,dx = \frac{1}{2}\int_{-1}^{0} xe^{x}\,dx + \frac{1}{2}\int_0^1 xe^{-x}\,dx = 0$

$Var(Y) = \int_{-\infty}^{-1}\frac{1}{2}e^{x}\,dx + \int_{-1}^{0} x^2\frac{1}{2}e^{x}\,dx + \int_0^1 x^2\frac{1}{2}e^{-x}\,dx + \int_1^{+\infty}\frac{1}{2}e^{-x}\,dx = 4-10e^{-1}+e^{-1} = 4-9e^{-1}$

V, VI

IV. a) $P[Y=3] = \sum_{n=0}^{\infty} P[Y=3 \mid X=n]\, P[X=n] = \sum_{n=2} \frac{\lambda^3}{3!} e^{-\lambda} + \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n}$

b) $\cancel{E[Y] = \sum_{n=3}^{\infty} y\, P[Y=y]\ \ P[Y=1] = \sum_{n=1}^{\infty} P[Y=1 \mid X=n]\, P[X=n] = \frac{1}{5}\lambda e^{-\lambda} + \sum_{i=5}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{1}{i}}$

$P[Y=y],\ y \in \{2,3,4\} = \frac{\lambda^y}{y!} e^{-\lambda} + \sum_{i=5}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{1}{i}$ ; $P[Y=y],\ y>4 = \sum_{i=5}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \cdot \frac{1}{i}$

$\therefore E[Y] = (\cancel{\lambda} + \lambda^2 + \frac{\lambda^3}{2} + \frac{\lambda^4}{6}) e^{-\lambda} + \sum_{m=1}^{\infty} (m \cdot \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n})$

c) $P[X=5 \mid Y=3] = \frac{P[Y=3 \mid X=5]\, P[X=5]}{P[Y=3]} = (\frac{1}{5} \cdot \frac{\lambda^5}{5!} e^{-\lambda}) / (\frac{\lambda^3}{3!} e^{-\lambda} + \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n})$

d) $E[X \mid Y=3] = \sum_{n=0}^{\infty} n \cdot P[X=n \mid Y=3] = \sum_{i=0}^{\infty} n \cdot \frac{P[Y=3 \mid X=n]\, P[X=n]}{P[Y=3]}$

$= (3 \cdot \frac{\lambda^3}{3!} e^{-\lambda} + \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda}) / (\frac{\lambda^3}{3!} e^{-\lambda} + \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n})$

e) $P[X \in [8,9] \mid Y \in [4,5]] = P[X=8 \mid Y=4] + P[X=9 \mid Y=4] + P[X=8 \mid Y=5] + P[X=9 \mid Y=5]$

$= (a) \frac{P[Y=4 \mid X=8]\, P[X=8]}{P[Y=4]} + (b) \frac{P[Y=4 \mid X=9]\, P[X=9]}{P[Y=4]} + (c) \frac{P[Y=5 \mid X=8]\, P[X=8]}{P[Y=5]} + (d) \frac{P[Y=5 \mid X=9]\, P[X=9]}{P[Y=5]}$

$P[Y=4] = \sum_{n=0}^{\infty} P[Y=4 \mid X=n]\, P[X=n] = \frac{\lambda^4}{4!} e^{-\lambda} + \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n}$ $\quad (a) = \frac{1}{8} \cdot \frac{\lambda^8}{8!} e^{-\lambda} / P[Y=4]$

$P[Y=5] = \sum_{n=0}^{\infty} P[Y=5 \mid X=n]\, P[X=n] = \sum_{n=5}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \cdot \frac{1}{n}$ $\quad (b) = \frac{1}{9} \cdot \frac{\lambda^9}{9!} e^{-\lambda} / P[Y=4]$

$(c) = \frac{1}{8} \cdot \frac{\lambda^8}{8!} e^{-\lambda} / P[Y=5]$ $\quad (d) = \frac{1}{9} \cdot \frac{\lambda^9}{9!} e^{-\lambda} / P[Y=5]$

IV

IX. a) Let X be sum of the dice. $P[X=6] = P[X=8] = \frac{5}{36}$ $\quad P[X=7] = \frac{6}{36}$

$P[X=2] = P[X=12] = \frac{1}{36}$ $\quad P[X=3] = P[X=11] = \frac{2}{36}$ $\quad P[X=4] = P[X=10] = \frac{3}{36}$ $\quad P[X=5] = P[X=9] = \frac{4}{36}$

b) $E[R \mid S=2] = \sum_{n=1}^{\infty} n P[R=n \mid S=2] = \frac{1}{36}$ $\quad E[R \mid S=3] = \frac{2}{36}$ $\quad E[R \mid S=9] = \frac{4}{36}$ $\quad E[R \mid S=10] = \frac{3}{36}$

$E[R \mid S=11] = \frac{2}{36}$ $\quad E[R \mid S=12] = \frac{1}{36}$

$E[R \mid S=4] = \sum_{n=1}^{\infty} n P[R=n \mid S=4] = \sum_{n=2}^{\infty} n \cdot (\frac{3}{36})^2 \cdot (\frac{30}{36})^{n-2}$

$\sum_{n=2}^{\infty} n k^{n-2} = \sum_{n=2}^{\infty} (n-1) k^{n-2}\ (a) + \sum_{n=2}^{\infty} k^{n-2}\ (b)$ $\quad (b)\ \sum_{n=2}^{\infty} k^{n-2} = \frac{1}{1-k}$

(a). Let $f'(k) = \sum_{n=2}^{\infty} (n-1) k^{n-2} \Rightarrow f(k) = \sum_{n=2}^{\infty} k^{n-1} = \frac{k}{1-k} \Rightarrow f'(k) = \frac{1-2k}{(1-k)^2}$

$\therefore \sum_{n=2}^{\infty} n k^{n-2} = \frac{2-3k}{(1-k)^2}$, converge. $\quad E[R \mid S=7] = \sum_{n=2}^{\infty} n (\frac{6}{36})^2 (\frac{27}{36})^{n-2} = \frac{5}{9}$

$E[R \mid S=5] = \sum_{n=2}^{\infty} n (\frac{4}{36})^2 (\frac{29}{36})^{n-2} = \frac{172}{441}$ $\quad E[R \mid S=6] = E[R \mid S=8] = \sum_{n=2}^{\infty} n (\frac{5}{36})^2 (\frac{28}{36})^{n-2} = \frac{275}{576}$

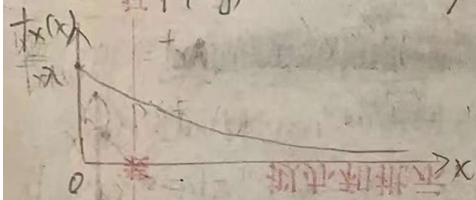
c) $E[R] = \sum_{n=2}^{12} n E[R \mid S=n] = \frac{219061}{14112} \approx 15.523$

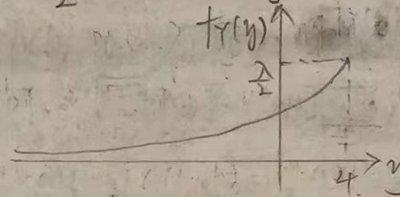
X. a) $E[X] = \sum_{k=0}^{324} k P[X=k] = k \binom{324}{k} (\frac{1}{10})^k (\frac{9}{10})^{324-k} = 32.4$ $\quad$ b) $Var(X) = 324 \cdot 0.1 \cdot 0.9 = 29.16$

c) $P[X<24] = \sum_{k=0}^{23} P[X=k] = \sum_{k=0}^{23} \binom{324}{k} (\frac{1}{10})^k (\frac{9}{10})^{324-k}$

IX, X

VII. a) $f_Y(y) = f_X\left(\frac{y-4}{-2}\right)/2 = \frac{\lambda}{2} e^{(y-4)/2\lambda},\ y \le 4$




b) $E(Y) = -2E[X] + 4 = -\frac{2}{\lambda} + 4$  $Var(Y) = 4Var(X) = \frac{4}{\lambda^2}$

VIII b) $h(x) = x^2 - 2x$  $h'(x) = 2x - 2$  $|h'(x)| = 2|x-1|$  $Y = X^2 - 2X \Rightarrow X = 1 \pm \sqrt{y+1}$

$$f_Y(y) = \begin{cases} \frac{f_{X_1}(1+\sqrt{y+1})}{|2x_1 - 2|} + \frac{f_{X_2}(1-\sqrt{y+1})}{|2x_2 - 2|} = \frac{f_X(1+\sqrt{y+1}) + f_X(1-\sqrt{y+1})}{2\sqrt{y+1}}, & y > -1 \\ 0, & y < -1 \end{cases}$$

a). $E(Y) = E(X^2 - 2X) = E(X^2) - 2E(X)$

c) $f_Y(y) = \begin{cases} 0, & y \in [-\infty, -1) \cup (0, +\infty) \\ \frac{1}{2\sqrt{y+1}}, & y \in (-1, 0] \end{cases}$

VII, VIII