I. a) experiments $\{X_i\}_{i=1}^{100}$ are i.i.d Bernoulli (0.5). $E[X_i]=\frac{1}{2}$. $\sigma^2=\frac{1}{4}$

$P[40 \le \sum_{i=1}^{100} X_i \le 60] = P[\frac{40-50}{\sqrt{n\sigma^2}} \le \frac{1}{\sqrt{n\sigma^2}}\sum_{i=1}^{100}(X_i-\frac{1}{2}) \le \frac{1}{\sqrt{n\sigma^2}}(60-50)] = P[-2 \le G_n \le 2]$

$\approx 1-2Q(2)$

$P[50 \le \sum_{i=1}^{100} X_i \le 55] = P[0 \le G_n \le 1] \approx \frac{1}{2} - Q(1)$

b) $P[400 \le \sum_{i=1}^{1000} X_i \le 600] = P[\frac{-100}{5\sqrt{10}} \le G_n \le \frac{100}{5\sqrt{10}}] \approx 1-2Q(2\sqrt{10})$

$P[500 \le \sum_{i=1}^{1000} X_i \le 550] = P[0 \le \frac{1}{\sqrt{n\sigma^2}}(550-500)] \approx \frac{1}{2} - Q(\sqrt{10})$

II. experiments $\{X_i\}_{i=1}^{16}$ are i.i.d. $E[X_i]=36 \Rightarrow \sigma^2 = 6^4$

$P[\sum_{i=1}^{16} X_i \le 600] = P[\frac{1}{\sqrt{n\sigma^2}}\sum_{i=1}^{16}(X_i-36) \le \frac{1}{\sqrt{n\sigma^2}}(600-36\times 16)] = P[G_n \le \frac{1}{6}] = 1-Q(\frac{1}{6})$

III. $m=\lambda t \Rightarrow \sigma^2=\lambda t$. $P[|\frac{N(t)}{t}-\lambda| \ge \varepsilon] = P[|N(t)-t\lambda| \ge t\varepsilon] \le \frac{\lambda}{\varepsilon t}$

IV. (7.20): $E[M_n]=0$, $Var[M_n]=\frac{1}{n}$. $\therefore P[|M_n|<\varepsilon] \ge 1-\frac{1}{n^2\varepsilon^2}$

For $E[X_i]=0$, $Var[X_i]=1$, $P[|\frac{1}{n}\sum_{i=1}^{n} X_i|<\varepsilon] = P[-\varepsilon \le \frac{1}{n}\sum_{i=1}^{n} X_i \le \varepsilon] = P[-\varepsilon\cdot\sqrt{\frac{n}{\sigma^2}} \le G_n \le \varepsilon\cdot\sqrt{\frac{n}{\sigma^2}}]$

$n=16 \Rightarrow P[|M_n|<\varepsilon] = 1-2Q(4\varepsilon)$; $n=81 \Rightarrow P[|M_n<\varepsilon|] = 1-2Q(9\varepsilon)$ $\quad = 1-2Q(\varepsilon\sqrt{n})$

I, II, III, IV

V. $\mu_Y = A^T\mu_X + B = (2\ -1\ 2)\begin{pmatrix}5\\-5\\6\end{pmatrix} + 5 = 32$. $K_Y = A^T K_X A = (2\ -1\ 2)\begin{pmatrix}5&2&-1\\2&5&0\\-1&0&4\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix} = 25$.

VI. a) $\sum\sum_{x\in S_X, y\in S_Y} P[X=x, Y=y] = 10k = 1$ $\quad k=0.1$

b) ~~When $Y=-1$, value of X that maximize $P[X=x|Y=-1]P[Y=-1] = -1$;~~

When $Y=0$, value of X that maximize $P[X=x|Y=0]P[Y=0] = -1$;

When $Y=1$, value of X that maximize $P[X=x|Y=1]P[Y=1] = -1$. $\quad \therefore MAP(X) = -1$

c) When $Y=-1$, value of X that maximize $P[\ |Y=-1] = -1$

When $Y=0$, value of X that maximize $P[X=x|Y=0] = -1$

When $Y=1$, value of X that maximize $P[X=x|Y=0] = -1$. $\quad \therefore ML(X) = -1$

VIII. ~~When $\hat{X}\in[a,b]$, $|\hat{X}-X| = |\tilde{X}-X|$; when $\hat{X}\notin[a,b]$ $\because X\in[a,b]$ $\therefore |\hat{X}-X| > |\tilde{X}-X|$~~,

$\therefore |\hat{X}-X| \ge |\tilde{X}-X|$, $\forall \hat{X}\in R$. $\quad \therefore E[(\hat{X}-X)^2] \ge E[(\tilde{X}-X)^2]$

$\therefore MSE(\tilde{X}) \le MSE(\hat{X})$.

V, VI, VIII

VII. a). $f_X(x) = \int_0^{1-x} 2\,dy = 2(1-x)\cdot 1\{x\in[0,1]\}$. $E[X] = \int_0^1 x\cdot 2(1-x)\,dx = \frac{1}{3}$.

$E[X^2] = \int_0^1 x^2\, 2(1-x)\,dx = \frac{1}{6}$. $MSE(\hat{X}_c) = E[(c-X)^2] = c^2 - 2cE[X] + E[X^2]$

$\frac{d}{dc} MSE(\hat{X}_c) = 2c - 2E[X] = 0 \Rightarrow c = E[X] = \frac{1}{3}$. $MSE(\hat{X}_c) = \frac{1}{18}$

b) $f_Y(y) = \int_0^{1-y} 2\,dx = 2(1-y)$. $E[Y] = \int_0^1 y\cdot 2(1-y)\,dy = \frac{1}{3}$ $E[Y^2] = \frac{1}{6}$

$E[XY] = 2\int_0^1 x \int_0^{1-x} y\,dy\,dx = \frac{1}{12}$ $MSE(\hat{X}_{HL}) = E[(aY-X)^2] = a^2E[Y^2] - 2aE[XY] + E[X^2]$

$\frac{d}{da} MSE(\hat{X}_{HL}) = 2aE[Y^2] - 2E[XY] = 0 \Rightarrow a = \frac{1}{2}$ $MSE(\hat{X}_{HL}) = \frac{1}{8}$ estimator: $\hat{X}_{HL} = \frac{1}{2}Y$

c) $MSE(\hat{X}_{HL}) = E[(aY+b-X)^2] = a^2E[Y^2] + 2abE[Y] + b^2 - 2aE[XY] - 2bE[X] + E[X^2]$

$$\left.\begin{array}{l} \frac{d}{da} MSE(\hat{X}_{HL}) = 2aE[Y^2] + 2bE[Y] - 2E[XY] = 0 \\ \frac{d}{db} MSE(\hat{X}_{HL}) = 2aE[Y] + 2b - 2E[X] = 0 \end{array}\right\} \Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = \frac{1}{2} \end{cases} \quad MSE(\hat{X}_{HL}) = \frac{1}{24} \quad \hat{X}_{HL} = -\frac{1}{2}Y + \frac{1}{2}$$

VII

IX. a). $MSE(\hat{X}_1(Y)) = E[(a(\rho X+N) - X)^2] = (a\rho-1)^2E[X^2] + 2a(a\rho-1)E[XN] + a^2E[N^2]$

$\frac{d}{da} MSE(\hat{X}_1) = (\frac{98}{75} + 2\sigma^2)a - \frac{8}{3} = 0$ $a = \frac{4\rho}{4\rho^2+3\sigma^2}$ $\alpha = \frac{4}{3}(\rho a-1)^2 + a^2\sigma^2$

$MSE = \frac{4\sigma^2}{4\rho^2+3\sigma^2}$

b) $E[X|Y=y] = \dfrac{\int_{-2}^{2} x\cdot \exp\left(\frac{-(y-0.7x)^2}{2\sigma^2}\right)dx}{\int_{-2}^{2} \exp\left(\frac{-(y-0.7t)^2}{2\sigma^2}\right)dt}$

d) values of MSE $\hat{X}_1$, $\hat{X}_2$ and $\hat{X}_3$ are roughly the same, especially for large σ^2 value.

X. a) $MSE(\hat{X}_1(Y)) = E[(a(\rho X^3+N) - X)^2]$ $E[X^6] = \frac{1}{7}$ $E[N^2] = \sigma^2$ $E[X^4] = \frac{1}{5}$ $E[X^2] = \frac{1}{3}$

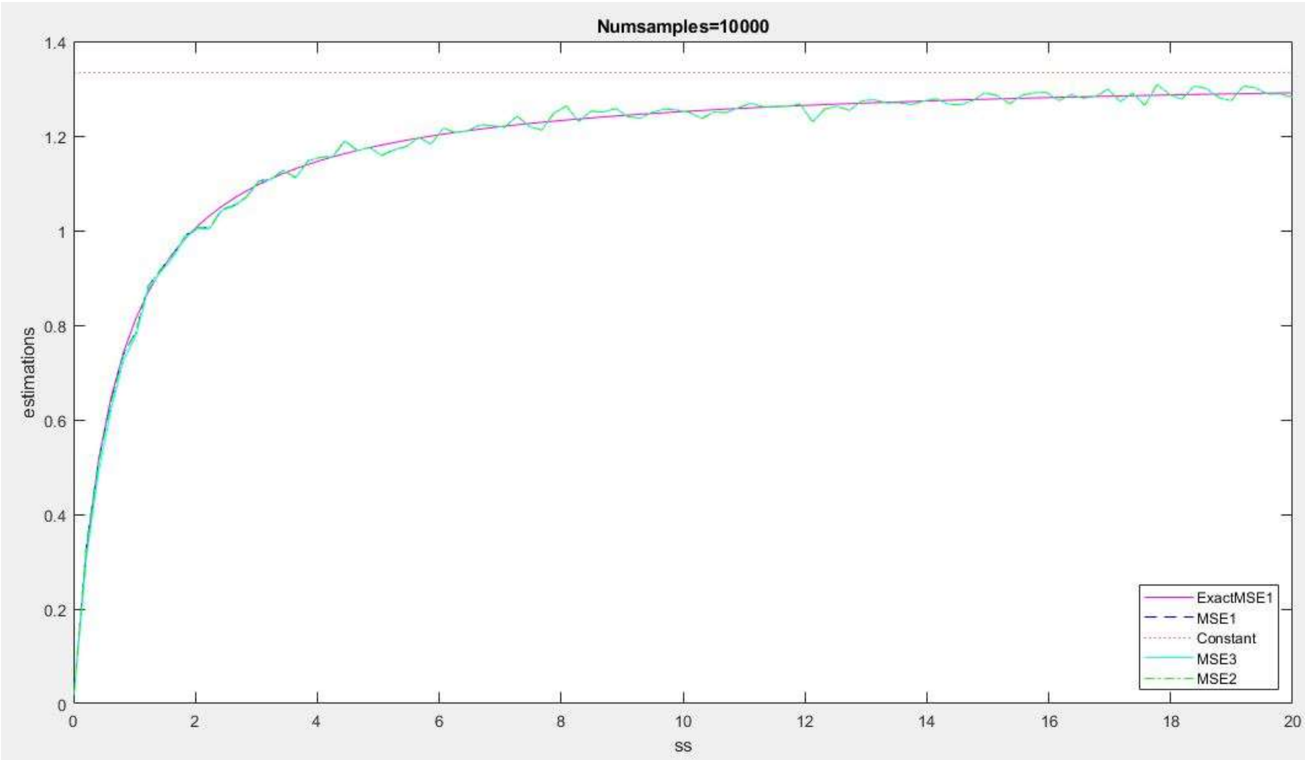
$= a^2\rho^2E[X^6] + 2\rho a^2E[X^3N] + a^2E[N^2] - 2\rho aE[X^4] - 2aE[XN] + E[X^2]$

$a = \frac{7\rho}{5(\rho^2+7\sigma^2)}$ $= a^2(\frac{\rho^2}{7} + \sigma^2) - \frac{2}{5}\rho a + \frac{1}{3}$ $\frac{d}{da} MSE(\hat{X}_1) = 2a(\frac{\rho^2}{7} + \sigma^2) - \frac{2}{5}\rho = 0$

$MSE = \frac{4\rho^2 + 175\sigma^2}{25(\rho^2+7\sigma^2)}$

b) $E[X|Y=y] = \dfrac{\int_{-1}^{1} x\cdot \exp\left(-\frac{(y-\rho x^3)^2}{2\sigma^2}\right)dx}{\int_{-1}^{1} \exp\left(-\frac{(y-\rho t^3)^2}{2\sigma^2}\right)dt}$

IX-a) b) d), X-a) b)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ss
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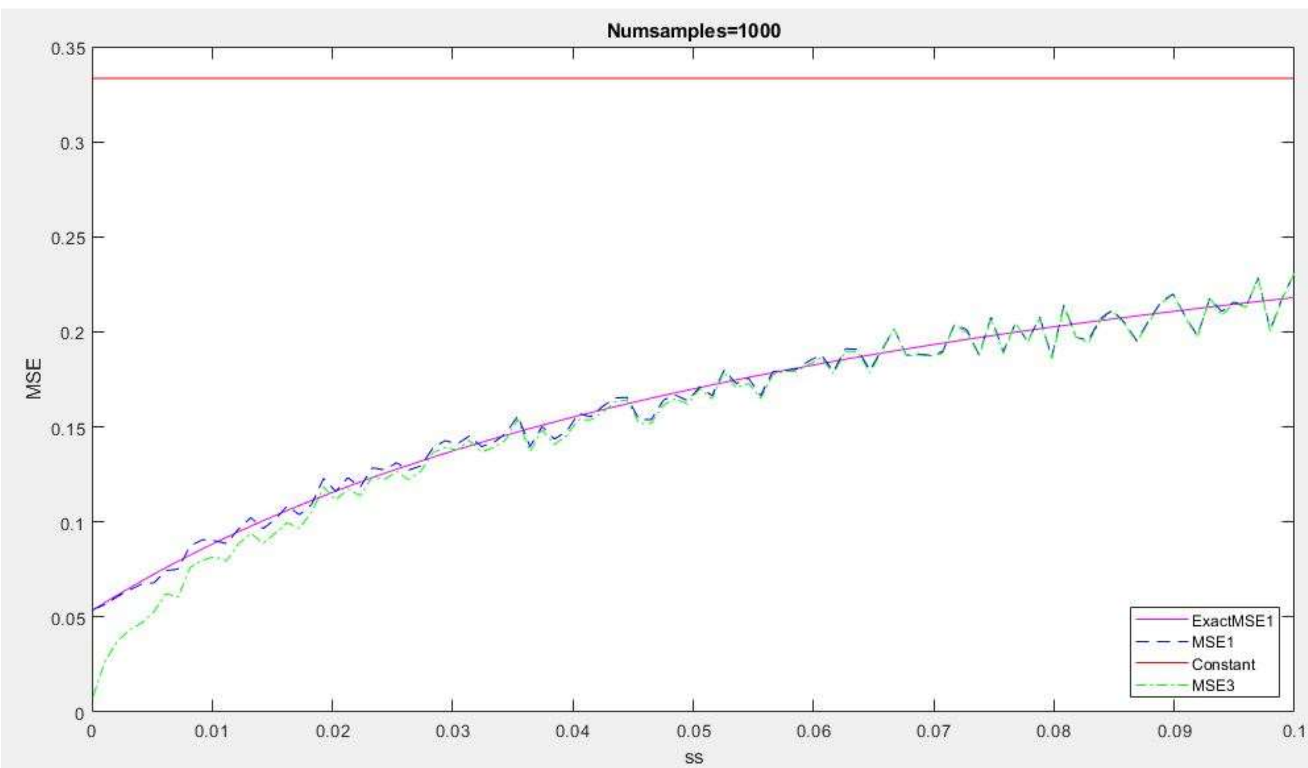
MSE1
Constant
MSE3
MSE2

IX-c) d)


Numsamples=1000
MSE
ss
ExactMSE1
MSE1
Constant
MSE3

X-c)