EE 503: Problem Set #3 : Solutions

- Reading: Chapter 2 in Leon-Garcia textbook.
- Submit your homework in D2L by 9pm on the due date.

I. Leon-Garcia textbook 2.46 (ordering a deluxe pizza)

Solution:
The order in which the 4 toppings are selected does not matter so we have sampling without ordering.
If toppings may not be repeated, then we have

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas.}$$

If toppings may be repeated, then we have sampling with replacement and without ordering.
Hence, the number of such arrangements is

$$\binom{14+4}{4} = 3060 \text{ possible deluxe pizzas.}$$

II. Leon-Garcia textbook 2.54 (a collection of 100 items, k of which are defective)

Solution:
a) The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space. We are interested in the outcomes in which m of the chosen items are defective ans $M-m$ are nondefective. The number of ways of choosing m defectives out of k is $\binom{k}{m}$. The number of ways of choosing $M-m$ nondefectives out of $100-k$ is $\binom{100-k}{M-m}$.
Then, the number of ways of choosing m defectives out of k and $M-m$ nondefectives out of $100k$ is $\binom{100-k}{M-m}$ is given as

$$\binom{k}{m}\binom{100-k}{M-m}$$

$$P[m \text{ defectives in } M\text{samples}] = \frac{\text{\# outcomes with } k \text{ defectives}}{\text{Total \# outcomes}} = \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}}.$$

Note that this is called the Hypergeometric distribution.
b)

$$P[\text{A lot is accepted}] = P[m=0 \text{ or } m=1] = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k\binom{100-k}{M-1}}{\binom{100}{M}}$$

III. Leon-Garcia textbook 2.81 (Ternary comm channel)

Solution:
a) By symmetry we have $P[Out=0] = P[Out=1] = P[Out=2] = 1/3$.
b) We have:

$$P[In=0|Out=1] = \frac{P[Out=1|In=0]P[In=0]}{P[Out=1]} = \frac{\epsilon(1/3)}{1/3} = \epsilon$$

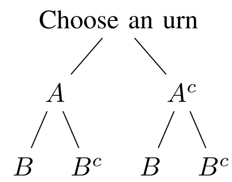
$$P[In=1|Out=1] = \frac{P[Out=1|In=1]P[In=1]}{P[Out=1]} = \frac{(1-\epsilon)(1/3)}{1/3} = 1-\epsilon$$

$$P[In=2|Out=1] = \frac{P[Out=1|In=2]P[In=2]}{P[Out=1]} = \frac{0}{1/3} = 0$$

IV. Leon-Garcia textbook 2.88 (Selecting an urn)

Solution:

We use a tree diagram to show the sequence of events. First we choose an urn, so A or A^c occurs. We then select a ball, so B or B^c occurs:



Now A and B are independent events if

$$P[B|A] = P[B]$$

But

$$P[B|A] = P[B] = P[B|A]P[A] + P[B|A^c]P[A^c] \Rightarrow P[B|A](1 - P[A]) = P[B|A^c]P[A^c] \Rightarrow \boxed{P[B|A] = P[B|A^c]}$$

The probability of B is the same given A or A^c, that is, the probability of B is the same for both urns.

V. Is it true?

Let A, B, C be mutually independent events: Prove true, or produce a counterexample: $(A \cup B)$ and C are independent.

Solution:

True: Suppose A, B, C are mutually independent. Then

$$\begin{aligned}
P[C(A \cup B)] &\overset{(a)}{=} P[CA \cup CB] \\
&= P[CA] + P[CB] - P[(CA)(CB)] \\
&= P[CA] + P[CB] - P[ABC] \\
&\overset{(b)}{=} P[C]P[A] + P[C]P[B] - P[A]P[B]P[C] \\
&= P[C](P[A] + P[B] - P[A]P[B]) \\
&\overset{(c)}{=} P[C](P[A] + P[B] - P[AB]) \\
&= P[C]P[A \cup B]
\end{aligned}$$

where (a) holds by the distributive law; (b) holds by mutual independence of A, B, C; (c) holds becaue A and B are independent. Thus, C and $A \cup B$ are independent.

VI. Playing basketball

We Play one of three teams with equal probability. Our probability of winning against each player is shown below:

- We play team 1: We win with probability $1/2$.
- We play team 2: We win with probability $1/4$.
- We play team 3: We win with probability $1/8$.

Let Win denote the event that we win, and $Loose = Win^c$.

a) Compute $P[Win]$.
b) Compute $P[\text{We play team } 3|Win]$
c) Compute $P[\text{We play either team 1 or } 2|Win]$
d) Compute $P[\text{We play either team 1 or } 2|Loose]$
e) Compute $P[\text{We play team 3 } |Win \cup \{\text{We play team } 2\}]$.

Solution:

a) $P[Win] = P[Win|1](1/3) + P[Win|2](1/3) + P[Win|3](1/3) = (1/3)(1/2 + 1/4 + 1/8) = 7/24$

b) We have

$$\begin{aligned}
P[3|Win] &= \frac{P[Win|3]P[3]}{P[Win]} \\
&= \frac{(1/8)(1/3)}{7/24} \\
&= \frac{1}{7}
\end{aligned}$$

c) We have

$$\begin{aligned} P[1 \cup 2|Win] &= P[1|Win] + P[2|Win] \\ &= \frac{P[Win|1](1/3)}{P[Win]} + \frac{P[Win|2](1/3)}{P[Win]} \\ &= \frac{(1/2)(1/3)}{7/24} + \frac{(1/4)(1/3)}{7/24} \\ &= \frac{6}{7} \end{aligned}$$

Another way is to observe

$$P[1 \cup 2|Win] = 1 - P[3|Win] = 1 - 1/7 = 6/7$$

d) We have

$$\begin{aligned} P[1 \cup 2|Loose] &= P[1|Loose] + P[2|Loose] \\ &= \frac{P[Loose|1](1/3)}{P[Loose]} + \frac{P[Loose|2](1/3)}{P[Loose]} \\ &= \frac{(1/3)(1/2 + 3/4)}{17/24} \\ &= \frac{10}{17} \end{aligned}$$

Alternatively

$$2/3 = P[1 \cup 2] = P[1 \cup 2|Win](7/24) + P[1 \cup 2|Loose](17/24)$$

So $P[1 \cup 2|Loose] = \frac{2/3-(6/7)(7/24)}{17/24} = \frac{16-6}{17} = \frac{10}{17}$

e) Define $D = Win \cup \{\text{Play } 2\} = \{Win\} \cup \{Loose, Play2\}$ which is a disjoint union. So

$$\begin{aligned} P[3|D] &= \frac{P[D|3](1/3)}{P[D]} \\ &= \frac{(1/3)(P[Win|3] + P[Loose, Play2|3])}{P[Win] + P[Play2, Loose]} \\ &= \frac{(1/3)(P[Win|3])}{7/24 + (1/3)(3/4)} \\ &= \frac{(1/3)(1/8)}{7/24 + (1/3)(3/4)} \\ &= \frac{1}{13} \end{aligned}$$

VII. CEREAL, OATMEAL, OR RICE

Suppose that at breakfast time we either eat cereal, oatmeal, or rice. Suppose:

- Having milk and having a side item are independent.
- $P[\{\text{have milk}\}] = 3/4$.
- $P[\{\text{have side dish}\}] = 1/2$.
- Given we only have milk (no side dish): We eat cereal with prob $3/4$ and oatmeal with prob $1/4$.
- Given we have milk and side dish: We eat cereal with prob $1/2$, rice with prob $1/4$, oatmeal with prob $1/4$.
- Given we only have side dish (no milk): We eat oatmeal with prob $1/2$ and rice with prob $1/2$.
- Given we have no milk and no side dish: We eat oatmeal with prob 1.

Let C, O, R be events of eating cereal, oatmeal, rice, respectively. Let M, D be events having milk and having a side dish, respectively. (D stands for "side [D]ish").

a) Compute the probability that we eat cereal.
b) Compute the probability that we eat rice. Then, compute the probability we eat oatmeal.
c) Compute the conditional probability we have a side dish, given we eat cereal.
d) Compute the probability we have milk and eat cereal. Explain this result in relation to part (a).
e) Compute the conditional probability we eat oatmeal, given we only have a side dish (no milk).

Solution:

a) We have

$$P[C] = \underbrace{P[C|M,D^c]}_{3/4}(3/4)(1/2) + \underbrace{P[C|M,D]}_{1/2}(3/4)(1/2) + \underbrace{P[C|M^c,D]}_{0}(1/4)(1/2) + \underbrace{P[C|M^c,D^c]}_{0}(1/4)(1/2) = \frac{15}{32}$$

b) We have

$$P[R] = \underbrace{P[R|M,D^c]}_{0}(3/4)(1/2) + \underbrace{P[R|M,D]}_{1/4}(3/4)(1/2) + \underbrace{P[R|M^c,D]}_{1/2}(1/4)(1/2) + \underbrace{P[R|M^c,D^c]}_{0}(1/4)(1/2) = \frac{5}{32}$$

Then of course

$$P[O] = 1 - \frac{15}{32} - \frac{5}{32} = \frac{12}{32}$$

c) Here are two different ways to do it:

- First method: Using the definition:

$$P[D|C] = \frac{P[D\cap C]}{P[C]} \overset{(a)}{=} \frac{P[D\cap C]}{15/32}$$

where equality (a) holds by part (a). It remains to compute $P[D \cap C]$.

$$\begin{aligned} P[(D\cap C)] &= P[[(D\cap C)\cap M] \cup [(D\cap C)\cap M^c]] \\ &= P[D\cap C\cap M] + P[D\cap C\cap M^c] \\ &= P[C|D,M]P[D,M] + P[C|D,M^c]P[D,M^c] \\ &= (1/2)(1/2)(3/4) + 0 \\ &= 3/16 \end{aligned}$$

Thus

$$\boxed{P[D|C] = \frac{3/16}{15/32} = \frac{2}{5}}$$

- Second method: Using Baye's rule we get:

$$P[D|C] = \frac{P[C|D]P[D]}{P[C]} = \frac{P[C|D](1/2)}{15/32}$$

and it remains to compute $P[C|D]$. Using the law of total probability with pre-conditioning:

$$P[C|D] = \underbrace{P[C|D,M]}_{1/2}\underbrace{P[M|D]}_{3/4} + \underbrace{P[C|D,M^c]}_{0}\underbrace{P[M^c|D]}_{1/4} = 3/8$$

and the answer is the same

$$P[D|C] = \frac{(3/8)(1/2)}{15/32} = 2/5$$

d) We have

$$P[M\cap C] = P[(M\cap C)\cap D] + P[(M\cap C)\cap D^c] = \underbrace{P[C|M,D]}_{1/2}\underbrace{P[M,D]}_{(3/4)(1/2)} + \underbrace{P[C|M,D^c]}_{3/4}\underbrace{P[M,D^c]}_{(3/4)(1/2)} = \frac{15}{32}$$

Alternatively we can observe that $C \subseteq M$ and so $P[C \cap M] = P[C]$, and we know $P[C] = 15/32$ from part (a).

e) This is just given information: $P[O|D,M^c] = 1/2$.

VIII. Lightbulbs and conditioning

We buy one of three types of lightbulbs, types A, B, C, equally likely over each type. The lifetime of a bulb is measured in integer units of days. Each type of bulb has different lifetime properties:

- Type A bulbs: Lifetime L_A is equally likely to be in the set $\{1, 2, ..., 200\}$ days.
- Type B bulbs: Lifetime L_B satisfies a *geometric distribution* $P[L_B = k] = p(1-p)^{k-1}$ for $k \in \{1, 2, 3, ...\}$, for $p = 1/100$.
- Type C bulbs: Lifetime L_C is either 50 or 150, equally likely.

Let L be the lifetime of the bulb we buy.

a) Compute $P[L = 100]$.

b) Compute $P[L \geq 100]$.

c) Compute $P[\text{Type A}|L \geq 100]$.
d) Compute $P[L \geq 100|(A \cup B)]$.
e) Compute $P[\text{Type A}|L = 50]$.

Solution:

a) $P[L = 100] = P[L = 100|A](1/3) + P[L = 100|B](1/3) + P[L = 100|C](1/3) = (1/3)[1/200 + p(1-p)^{99} + 0]$.

b) $P[L \geq 100] = P[L \geq 100|A](1/3) + P[L \geq 100|B](1/3) + P[L \geq 100|C](1/3) = (1/3)[101/200 + \sum_{k=100}^{\infty} p(1-p)^{k-1} + 1/2]$.

c) $P[A|L \geq 100] = \frac{P[L\geq 100|A](1/3)}{P[L\geq 100]} = \frac{(101/200)(1/3)}{P[L\geq 100]}$, where the denominator is in part (b).

d) One method is to do:

$$\begin{aligned} P[L \geq 100|(A \cup B)] &= \frac{P[\{L \geq 100\} \cap (A \cup B)]}{2/3} \\ &= \frac{P[(\{L \geq 100\} \cap A) \cup (\{L \geq 100\} \cap B)]}{2/3} \\ &= \frac{P[\{L \geq 100\} \cap A] + P[\{L \geq 100\} \cap B]}{2/3} \\ &= \frac{P[\{L \geq 100\}|A](1/3) + P[\{L \geq 100\}|B](1/3)}{2/3} \\ &= \frac{101/200 + \sum_{k=100}^{\infty} p(1-p)^{k-1}}{2} \end{aligned}$$

Alternatively, you can do: $P[L \geq 100|(A \cup B)] = \frac{P[(A\cup B)|L\geq 100]P[L\geq 100]}{P[A\cup B]}$. We know $P[L \geq 100]$ from part (b). We know $P[A \cup B] = 2/3$. Finally, since A and B are disjoint we have $P[A \cup B|L \geq 100] = P[A|L \geq 100] + P[B|L \geq 100]$, and

$$P[A|L \geq 100] = \frac{P[L \geq 100|A](1/3)}{P[L \geq 100]}, P[B|L \geq 100] = \frac{P[L \geq 100|B](1/3)}{P[L \geq 100]}$$

where $P[L \geq 100]$ is in part (b), and $P[L \geq 100|A] = 101/200$, $P[L \geq 100|B] = \sum_{k=100}^{\infty} p(1-p)^{k-1}$.

e) $P[A|L = 50] = \frac{P[L=50|A](1/3)}{P[L=50]} = \frac{(1/200)(1/3)}{P[L=50]}$, and:

$$P[L = 50] = (1/3)[P[L = 50|A] + P[L = 50|B] + P[L = 50|C]] = (1/3)[1/200 + p(1-p)^{49} + 1/2]$$

IX. Meteors and students

There are three students. Let A, B, C be the events that student 1 gets hit by a meteor, student 2 gets hit by a meteor, and student 3 gets hit by a meteor, respectively. The students are ordered from least lucky to most lucky, so we assume $P[A] = 1/4, P[B] = 1/8, P[C] = 1/16$. Also assume the events A, B, C are pairwise independent (but not necessarily mutually independent).

a) Find the best possible upper and lower bounds for $P[A \cup B \cup C]$.

b) Explicitly show your upper bound is possible by designing a probability experiment that satisfies all requirements and has $P[ABC]$ equal to your upper bound exactly.

Solution:

By the inclusion-exclusion principle we have

$$\begin{aligned} P[A \cup B \cup C] &= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C] \\ &= (1/4) + (1/8) + (1/16) - (1/4)(1/8) - (1/4)(1/16) - (1/8)(1/16) + P[A \cap B \cap C] \\ &= \frac{49}{128} + P[A \cap B \cap C] \end{aligned}$$

Since $(A \cap B \cap C) \subseteq (B \cap C)$ we have

$$0 \leq P[A \cap B \cap C] \leq P[B \cap C] = (1/8)(1/16) = 1/128$$

Thus

$$\frac{49}{128} \leq P[A \cup B \cup C] \leq \frac{50}{128}$$

To show these are the best possible bounds, we construct example systems that fulfill all probability requirements and pairwise independence requirements, and that also meet the upper and lower bounds with equality.

1) (Lower bound) It suffices to construct a system that fulfills all requirements and that satisfies $P[A \cap B \cap C] = 0$. To this end, define A and B as independent events with $P[A] = 1/4$, $P[B] = 1/8$. Define C as an event that depends on A and B as follows:

$$\begin{aligned} P[C|A, B] &= 0 \\ P[C|A, B^c] &= 1/14 \\ P[C|A^c, B] &= 1/12 \\ P[C|A^c, B^c] &= 5/84 \end{aligned}$$

Then it can be shown

$$\begin{aligned} P[A \cap B \cap C] &= 0 \\ P[C] &= 1/16 \\ P[A \cap C] &= (1/4)(1/16) = P[A]P[C] \\ P[B \cap C] &= (1/8)(1/16) = P[B]P[C] \end{aligned}$$

2) (Upper bound) Here we want a system that fulfills all requirements and has $P[A \cap B \cap C] = 1/128$. Define B, C independent of each other with $P[B] = 1/8$, $P[C] = 1/16$. Define A dependent on B, C via:

$$\begin{aligned} P[A|B, C] &= 1 \\ P[A|B, C^c] &= 1/5 \\ P[A|B^c, C] &= 1/7 \\ P[A|B^c, C^c] &= 9/35 \end{aligned}$$

Then

$$\begin{aligned} P[A \cap B \cap C] &= 1/128 \\ P[A] &= 1/4 \\ P[A \cap B] &= (1/4)(1/8) = P[A]P[B] \\ P[A \cap C] &= (1/4)(1/16) = P[A]P[C] \end{aligned}$$

X. Design and solve your own problem about conditional probability

XI. Randomized data processing part 1 (one disease tester)

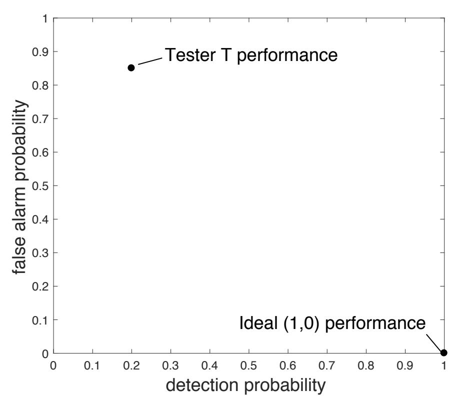


Fig. 1. A 2-d visualization of $(p_{Detect}, p_{FalseAlarm})$ for the original tester T.

We have a simple disease tester T that tests whether or not a person has a disease. However, it gives poor performance. We want to design a randomized data processor that improves the performance. The tester T outputs a (possibly incorrect) binary result $X \in \{0, 1\}$. If $X = 1$ it means the tester thinks the person has the disease. If $X = 0$ it means the tester thinks the person does not have the disease. Let H be the event that the person being tested has the disease. The disease tester has the following detection and false alarm probabilities (see Fig. 1):

$$p_{detect} = P[X = 1|H] = 0.2 \quad , \quad p_{false} = P[X = 1|H^c] = 0.85$$

We want to a design a function f that takes X as input and outputs a random variable $Y \in \{0,1\}$, so $Y = f(X)$. Define the new detection and false alarm probability as:

$$p^{new}_{detect} = P[Y = 1|H] \quad , \quad p^{new}_{false} = P[Y = 1|H^c]$$

a) Suppose $Y = f(X)$ where $f : \{0,1\} \to \{0,1\}$ is a deterministic function. How many deterministic functions are there? Plot the $(p^{new}_{detect}, p^{new}_{false})$ operating points associated with each deterministic function.

b) Assume we can *randomize* the choice of deterministic functions we use: If your part (a) has n functions $f_1(X), \dots, f_n(X)$, then we choose function f_i with some probability p_i, where $p_i \geq 0$ for all $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$. The choice of which f_i to use is made independently of whether or not the person has the disease. Let p^{new}_{detect} and p^{new}_{false} be the detection and false alarm probabilities under this randomized testing model. Design probabilities that maximize p^{new}_{detect} subject to the constraint $p^{new}_{false} \leq 0.13$. Plot your optimized operating point on the same graph as part (a). This is optimal. Is this ethical?

Solution:

a) There are 4 deterministic functions $f : \{0,1\} \to \{0,1\}$, they are:

- $f(0) = 0, f(1) = 0$. This is the function $Y = 0$. So $(p^{new}_{detect}, p^{new}_{false}) = (0,0)$.
- $f(0) = 1, f(1) = 1$. This is the function $Y = 1$. So $(p^{new}_{detect}, p^{new}_{false}) = (1,1)$.
- $f(0) = 0, f(1) = 1$. This is the function $Y = X$. So $(p^{new}_{detect}, p^{new}_{false}) = (.2, .85)$.
- $f(0) = 1, f(1) = 0$. This is the function $Y = 1 - X$. So then $(p^{new}_{detect}, p^{new}_{false}) = (.8, .15)$ because:

$$p^{new}_{detect} = P[Y = 1|H] = P[X = 0|H] = 0.8, p^{new}_{false} = P[Y = 1|H^c] = P[X = 0|H^c] = 0.15$$

These operating points are plotted in Fig. 2.

b)

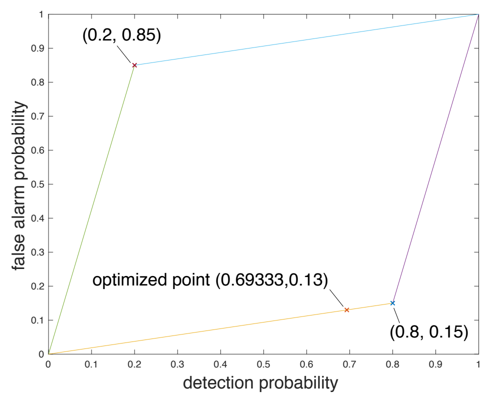


Fig. 2. The corner points are from the 4 deterministic algorithms. Everything in the diamond region can be achieved by randomizing over the 4 deterministic algorithms.

All points inside the diamond region of Fig. 2 form the "convex hull" of the 4 deterministic operating points. These form the $(p^{new}_{detect}, p^{new}_{false})$ points that can be achieved when we consider all probabilistic combinations of the above four deterministic algorithms. Let's just consider the line between the deterministic operating points $(0,0)$ and $(0.8, 0.15)$ by using the deterministic algorithm $Y = 0$ with probability θ and using the algorithm $Y = 1 - X$ with probability $1 - \theta$. We need the false alarm probability to be 0.13 and so

$$p^{new}_{false} = 0\theta + 0.15(1 - \theta) = 0.13 \implies \theta = 1 - 13/15 = 2/15$$

Then $p^{new}_{detect} = 0\theta + (0.8)(1 - \theta) = .8(13/15) \approx 0.69333333$. This yields (see Fig. 2):

$$\boxed{(p^{new}_{detect}, p^{new}_{false}) = (0.69333333, 0.13)}$$