5.26. a) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 k(x+y)\,dx\,dy = k\int_0^1 (\frac{1}{2}+y)\,dy = k = 1$;

b) $F_{X,Y}(x,y) = \begin{cases} \int_{-\infty}^{x}\int_{-\infty}^{y}(x+y)\,dy\,dx = \int_0^x (xy+\frac{1}{2}y^2)\,dx = \frac{1}{2}(x^2y+xy^2), & x\in[0,1] \text{ or } y\in[0,1] \\ 0, & x\in(-\infty,0) \text{ or } y\in(-\infty,0); \\ 1, & x,y\in(1,+\infty). \end{cases}$

c) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^1 (x+y)\,dy = (\frac{1}{2}+x)\cdot \mathbb{1}\{x\in[0,1]\}$, $f_Y(y) = (\frac{1}{2}+y)\cdot\mathbb{1}\{y\in[0,1]\}$

5.80. a) $f_Y(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{x+y}{\frac{1}{2}+x}\cdot\mathbb{1}\{x\in[0,1]\}$.

b) $P[Y>X|x] = 1-P[Y\le X|x] = 1-F_Y(x)$ $\quad F_Y(y) = \begin{cases} 0, & y\in(-\infty,0) \\ 1, & y\in(1,+\infty) \\ \int_{-\infty}^{y} f_Y(y)\,dy = \frac{1}{2}(y+y^2), & y\in[0,1]. \end{cases}$

$\therefore P[Y>X|x] = \begin{cases} 1, & x\in(-\infty,0); \\ 1-\frac{1}{2}(x+x^2), & x\in[0,1], \\ 0, & x\in(1,+\infty) \end{cases}$

c) $P[Y>X] = \int_{-\infty}^{+\infty} P[Y>X|x]f_X(x)\,dx = \int_0^1 [1-\frac{1}{2}(x+x^2)](x+\frac{1}{2})\,dx = \int_0^1 (\frac{1}{2}+\frac{3}{4}x-\frac{3}{4}x^2-\frac{1}{2}x^3)\,dx = \frac{1}{2}$

d) $E[Y|X=x] = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)\,dy = \int_0^1 y\cdot\dfrac{x+y}{\frac{1}{2}+x}\,dy = \dfrac{1}{\frac{1}{2}+x}\int_0^1 (xy+y^2)\,dy = \dfrac{\frac{x}{2}+\frac{1}{3}}{x+\frac{1}{2}}$

I

II. $F_Z(z) = P[\frac{X}{X+Y}<z] = \int_{-\infty}^{+\infty} P[\frac{X}{X+Y}<z|Y=y]f_Y(y)\,dy = \int_{-\infty}^{+\infty} P[X<\frac{zy}{1-z}]f_Y(y)\,dy$

$= \int_0^{+\infty} F_X(\frac{zy}{1-z})f_Y(y)\,dy = \int_0^{+\infty}(1-e^{-\lambda\frac{zy}{1-z}})\lambda e^{-\lambda y}\,dy = \begin{cases} 0, & z\in(-\infty,0) \\ z, & z\in[0,1] \\ 1, & z\in(1,+\infty) \end{cases}$

$f_Z(z) = \mathbb{1}\{z\in[0,1]\}$

III. a) $\because$ X & Y are i.i.d $\therefore f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{8\pi}\exp[-(x^2+y^2)/8]$

b) $P[X^2+Y^2\le 1] = \int_0^1 2\pi r\cdot\frac{1}{8\pi}e^{-\frac{r^2}{8}}\,dr = (u=\frac{r^2}{8})\int_0^{\frac{1}{8}} e^{-u}\,du = 1-e^{-\frac{1}{8}}$

c) $F_Z(z) = P[X^2+Y^2\le z] = \int_0^z 2\pi r\cdot\frac{1}{8\pi}e^{-\frac{r^2}{8}}\,dr = \begin{cases} (1-e^{-\frac{z}{8}}), & z\ge 0 \\ 0, & z<0 \end{cases}$ Half-Gaussian distribution

IV. a) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{+\infty} f_{X|Y}(x|y)\cdot f_Y(y)\,dy = \frac{1}{\sqrt{2\pi}}\int_0^1 \exp(-\frac{(x-y)^2}{2})\,dy$.

b) $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x\int_0^1 \exp(-\frac{(x-y)^2}{2})\,dy\,dx$

$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^2\int_0^1 \exp(-\frac{(x-y)^2}{2})\,dy\,dx$ $\quad Var(X) = E[X^2]-E[X]^2$

II, III, IV

V. a) $f_{X,Y}(x,y) = \frac{\partial}{\partial x \partial y} F_{X,Y}(x,y) = \alpha\beta e^{-\alpha x} e^{-\beta y} \cdot 1\{x \geq 0, y \geq 0\}$

b) $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^{+\infty} \alpha\beta e^{-\alpha x} e^{-\beta y}\,dy = -\alpha e^{-\alpha x} \cdot 1\{x \geq 0\}$. $f_Y(y) = -\beta e^{-\beta y} \cdot 1\{y \geq 0\}$

c) $f_X(x) f_Y(y) = f_{X,Y}(x,y)$, ∴ X & Y are independent.

VI. a) $F_Z(z) = P[X+Y \leq z] = \int_{-\infty}^{+\infty} P[Y \leq z-X \mid X=x] f_X(x)\,dx = \int_{-\infty}^{+\infty} F_Y(z-x) f_X(x)\,dx$

$f_Z(z) = \int_{-\infty}^{+\infty} \frac{d}{dz} F_Y(z-x) f_X(x)\,dx = \int_0^2 f_Y(z-x) f_X(x)\,dx$

Denote the triangular region as A, $f_{X,Y}(x,y) = c \cdot 1\{(x,y) \in A\}$, $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y) =$ area of $A = 1$

∴ $c = 1$. $f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dy = \int_0^{\frac{1}{2}x} dy = \frac{1}{2}x$, $f_Y(y) = \int_{2y}^{2} dx = 1-2y$

∴ $f_Z(z) = \int_0^2 [1-2(z-x)] \cdot \frac{1}{2}x\,dx = \frac{1}{2}\int_0^2 (1-2zx+2x^2)\,dx = \frac{22}{3} - 4z$

b) $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^2 x \cdot \frac{1}{2}x\,dx = \frac{4}{3}$ $E[Y] = \int_{-\infty}^{+\infty} y f_Y(y)\,dy = \int_0^1 y(1-2y)\,dy = \frac{1}{3}$

$E[XY] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\,dx\,dy = \int_0^2 x \int_0^{\frac{1}{2}x} y\,dy\,dx = \int_0^2 \frac{x^3}{8}\,dx = \frac{2}{3}$

c) $Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{9}$

V, VI

VII. a) $\int_0^{+\infty}\int_0^{y} c e^{-\frac{x+y}{2}}\,dx\,dy = c\int_0^{+\infty} e^{-\frac{y}{2}} \int_0^{y} e^{-\frac{x}{2}}\,dx\,dy = -2c\int_0^{+\infty} e^{-y}\,dy = 2c = 1$ $c = \frac{1}{2}$

b) $\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} z-y \\ y \end{bmatrix}$, $|J(x,y)| = 1$ ∴ $f_{Z,Y}(z,y) = \frac{1}{2} e^{-\frac{z}{2}} \cdot 1\{z > 0\}$

$f_Z(z) = \int_{-\infty}^{+\infty} f_{Z,Y}(z,y)\,dy = \int_0^{\frac{1}{2}z} \frac{1}{2} e^{-\frac{z}{2}}\,dy = \frac{1}{4} z e^{-\frac{z}{2}}$

$\Phi_Z(\omega) = E[j\omega Z] = \int_0^{+\infty} e^{j\omega z} \cdot \frac{1}{4} z e^{-\frac{z}{2}}\,dz = \frac{1}{4(j\omega - \frac{1}{2})^2} \int_0^{+\infty} (j\omega - \frac{1}{2}) z e^{(j\omega - \frac{1}{2})z}\,d(j\omega - \frac{1}{2})z$

$= \frac{1}{4(j\omega - \frac{1}{2})^2} = \frac{1}{(1-2j\omega)^2}$

c) $\frac{1}{4}(\frac{1}{2} - j\omega)^{-2} = \frac{1}{4}\frac{d}{d\omega}(\frac{1}{2} - j\omega)^{-1} \leftrightarrow f_Z(z) = \frac{1}{4} z e^{-\frac{z}{2}} u(z)$

VII