EE 503: Problem Set #6 : Solutions

I. LIPSCHITZ CONTINUOUS PDF

A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *L-Lipschitz continuous* for a given real number $L \geq 0$ if

$$|g(x) - g(y)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}$$

a) Suppose X has a PDF $f_X(x)$ that is L-Lipschitz continuous for some $L > 0$. Fix $\delta > 0$. Prove that for any $a \in \mathbb{R}$ we have

$$|P[X \in [a, a+\delta] - \delta f_X(a)| \leq L\delta^2$$

That is, prove that:

$$\delta f_X(a) - L\delta^2 \leq P[X \in [a, a+\delta]] \leq \delta f_X(a) + L\delta^2$$

b) For X from part (a), explain what is meant by the following sentence: If X has PDF $f_X(x)$ then for sufficiently small $\delta > 0$ we have $P[X \in [a, a+\delta]] \approx \delta f_X(a)$. Explain what happens if $f_X(a) = 23.4$.

Solution:

a) We have

$$P[X \in [a, a+\delta]] = \int_a^{a+\delta} f_X(x)dx = \int_a^{a+\delta} \left[f_X(a) + (f_X(x) - f_X(a))\right] dx = \delta f_X(a) + \underbrace{\int_a^{a+\delta} (f_X(x) - f_X(a))dx}$$

We want to show the term in the underbrace is *neglibible* ($O(\delta^2)$). We have

$$\begin{aligned} P[X \in [a, a+\delta]] &\leq \delta f_X(a) + \int_a^{a+\delta} |f_X(x) - f_X(a)|dx \\ &\leq \delta f_X(a) + \int_a^{a+\delta} L|x - a|dx \\ &\leq \delta f_X(a) + L\int_a^{a+\delta} \delta dx \\ &= \delta f_X(a) + L\delta^2 \end{aligned}$$

where the first inequality holds by Lipschitz continuity of the PDF; the second holds because $|x - a| \leq \delta$ for all $x \in [a, a+\delta]$. Similarly

$$\begin{aligned} P[X \in [a, a+\delta]] &\geq \delta f_X(a) - \int_a^{a+\delta} |f_X(x) - f_X(a)|dx \\ &\geq \delta f_X(a) - \int_a^{a+\delta} L|a - x|dx \\ &\geq \delta f_X(a) - L\int_a^{a+\delta} \delta dx \\ &\geq \delta f_X(a) - L\delta^2 \end{aligned}$$

b) If δ is a very small positive number then δ^2 is much smaller, and so δ^2 is negligible in comparison to δ. If we assume L is not too big, and we remove the negligible term of size at most $L\delta^2$, then we can say $P[X \in [a, a+\delta]] \approx \delta f_X(a)$. If $f_X(a) = 23.4$ then, of course, δ needs to be smaller than $1/23.4$ in order for $\delta f_X(a)$ to be a valid probability. If we want to be more precise then we can say

$$\boxed{\lim_{\delta \to 0^+} \frac{P[X \in [a, a+\delta]]}{\delta} = f_X(a)}$$

II. Theoretical Results on Expectation

(a) Suppose that N is a non-negative integer-valued random variable. There is a nice relationship between the CDF of N and its expectation $E[N]$. Specifically, show that:

$$E[N] = \sum_{i=1}^{\infty} P[N \geq i]$$

(b) Now consider a continuous non-negative random variable X. It can also be shown that:

$$E[X] = \int_0^{\infty} P[X > t]dt$$

Take this as a fact and show that:

$$E[X^n] = \int_0^{\infty} nx^{n-1} P[X > x]dx$$

Hint: start with:

$$E[X^n] = \int_0^{\infty} P[X^n > t]dt$$

Solution:
(a)

$$1 - F_N(i) = P[N > i] = \sum_{j=i+1}^{\infty} P[N = j] \tag{1}$$

$$\sum_{i=0}^{\infty}(1 - F_N(i)) = \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} P[N = j] = \sum_{j=1}^{\infty} jP[N = j] = \sum_{j=0}^{\infty} jP[N = j] = E[N] \tag{2}$$

From (1) and (2): $E[N] = \sum_{i=0}^{\infty}(1 - F_N(i)) = \sum_{i=0}^{\infty} P[N > i] = \sum_{i=1}^{\infty} P[N \geq i]$

Alternative proof: We have

$$N = \sum_{i=1}^{\infty} 1_{\{N \geq i\}}$$

It can be shown that the expectation of a countably infinite sum of nonnegative random variables is equal to the sum of the expectations (a special case of something called the *Fubini-Tonelli theorem*), yielding the result:

$$\mathbb{E}\left[N\right] = \sum_{i=1}^{\infty} \mathbb{E}\left[1_{\{N \geq i\}}\right]$$

(b) $E[X^n] = \int_0^{\infty} P[X^n > t]dt$.
Let $t = x^n \Rightarrow dt = nx^{n-1}dx$
$E[X^n] = \int_0^{\infty} P[X^n > x^n]nx^{n-1}dx$.
Since X is non-negative, $X^n > x^n$ implies that $X > x$. Hence:
$E[X^n] = \int_0^{\infty} nx^{n-1}P[X > x]dx$
Alternative proof: We have

$$X = \int_0^X dt = \int_0^{\infty} 1_{\{X > t\}}dt$$

It can be shown that the expectation of an integral of nonnegative random variables is equal to the integral of expectations (a special case of something called the *Fubini-Tonelli theorem*), yielding the result:

$$\mathbb{E}\left[X\right] = \int_0^{\infty} \mathbb{E}\left[1_{\{X > t\}}\right] dt$$

The result $E[X] = \int_0^{\infty} P[X > t]dt$ can also be viewed as a definition of the expectation for a nonnegative random variable X with *arbitrary distribution* (including distributions that have neither a PDF nor a generalized PDF). This is because all random variables have CDF functions $F_X(x)$ and so $P[X > x] = 1 - F_X(x)$ is always defined. The integral of the nonnegative and non-increasing function $P[X > x]$ is also always defined (possibly being infinity).

III. LAW OF TOTAL CONDITIONING I

Let X be a geometric random variable with success probability p, so that $P[X=k]=p(1-p)^{k-1}$ for all $k\in\{1,2,3,\ldots\}$. Define Y as a random variable that takes values in a finite set $S_Y=\{y_1,y_2,...,y_k\}$. For all $i\in\{1,2,3,\ldots\}$ and $y\in S_Y$, define $p_{Y|\{X=i\}}(y|i)$ as the conditional probability that $Y=y$, given that $X=i$.

a) Verify that $\sum_{i=1}^{\infty}P[X=i]=1$.
b) Compute $\mathbb{E}[Y]$ in terms of $p_{Y|\{X=i\}}(y|i)$.
c) Compute $P[Y=y_1]$ in terms of $p_{Y|\{X=i\}}(y|i)$.
d) Compute $P[X=3|Y=y_1]$ in terms of $p_{Y|\{X=i\}}(y|i)$.
e) Compute $P[X\geq 3|Y=y_1]$ in terms of $p_{Y|\{X=i\}}(y|i)$.

Solution:

a) $\sum_{i=1}^{\infty}p(1-p)^{i-1}=\frac{p}{1-(1-p)}=1$.
b) We get

$$\mathbb{E}[Y]=\sum_{y\in S_Y}yP[Y=y]$$

By the law of total probability

$$P[Y=y]=\sum_{i=1}^{\infty}P[Y=y|X=i]p(1-p)^{i-1}$$

Thus

$$\boxed{\mathbb{E}[Y]=\sum_{y\in S_Y}\sum_{i=1}^{\infty}y\underbrace{P[Y=y|X=i]}_{p_{Y|X=i}(y|i)}p(1-p)^{i-1}}$$

Alternatively, you can use the law of total expectation to directly get the same answer:

$$\mathbb{E}[Y]=\sum_{i=1}^{\infty}\mathbb{E}[Y|X=i]\,p(1-p)^{i-1}=\sum_{i=1}^{\infty}\sum_{y\in S_Y}yp_{Y|X=i}(y|i)p(1-p)^{i-1}$$

c) By the law of total probability

$$P[Y=y_1]=\sum_{i=1}^{\infty}P[Y=y_1|X=i]p(1-p)^{i-1}=\sum_{i=1}^{\infty}p_{Y|X=i}(y_1|i)p(1-p)^{i-1}$$

d) $P[X=3|Y=y_1]=\frac{P[Y=y_1|X=3]p(1-p)^2}{P[Y=y_1]}=\frac{p_{Y|X=3}(y_1|3)p(1-p)^2}{P[Y=y_1]}$ where $P[Y=y_1]$ is computed in part (c).
e) $P[X\geq 3|Y=y_1]=\sum_{i=3}^{\infty}P[X=i|Y=y_1]=\sum_{i=3}^{\infty}\frac{p_{Y|X=i}(y_1|i)p(1-p)^{i-1}}{P[Y=y_1]}$ where $P[Y=y_1]$ is computed in part (c).

IV. LAW OF TOTAL CONDITIONING II

Fix $\lambda>0$ and fix p so that $0<p<1$. We first generate a Poisson random variable X with parameter λ:

$$P[X=n]=\frac{\lambda^n}{n!}e^{-\lambda}\quad \forall n\in\{0,1,2,3,...\}$$

Based on X, we build a new random variable Y as follows:

- If $X=0$ then $Y=0$.
- If $X=1$ then we generate Y equally likely over $\{-3,-2,-1,0,1\}$.
- If $X\in\{2,3,4\}$ then $Y=X$.
- If $X>4$ then we choose Y equally likely over the X-element set $\{1,2,...,X\}$.

a) Compute $P[Y=3]$.
b) Compute $\mathbb{E}[Y]$.
c) Compute $P[X=5|Y=3]$.
d) Compute $\mathbb{E}[X|Y=3]$.
e) Compute $P[X\in\{8,9\}|Y\in\{4,5\}]$.

Solution:

a) We have

$$\begin{aligned}P[Y=3] &= \sum_{i=0}^{\infty} P[Y=3|X=i]\frac{\lambda^i}{i!}e^{-\lambda} \\ &= P[Y=3|X=3]\frac{\lambda^3}{3!}e^{-\lambda} + \sum_{i=5}^{\infty} P[Y=3|X=i]\frac{\lambda^i}{i!}e^{-\lambda} \\ &= \frac{\lambda^3}{3!}e^{-\lambda} + \sum_{i=5}^{\infty}(1/i)\frac{\lambda^i}{i!}e^{-\lambda}\end{aligned}$$

(b) We have

$$\begin{aligned}E[Y] &= \sum_{i=0}^{\infty} E[Y|X=i]P[X=i] \\ &= \sum_{i=0}^{\infty} E[Y|X=i]\frac{\lambda^i}{i!}e^{-\lambda} \\ &= 0 + \lambda e^{-\lambda}\sum_{i=-3}^{1}(\frac{1}{5})i + \sum_{i=2}^{4} i\frac{\lambda^i}{i!}e^{-\lambda} + \sum_{i=5}^{\infty}(\sum_{j=1}^{i}(\frac{1}{i})j)\frac{\lambda^i}{i!}e^{-\lambda}\end{aligned}$$

(c) We have

$$\begin{aligned}P[X=5|Y=3] &= \frac{P[Y=3|X=5]P[X=5]}{P[Y=3]} \\ &= \frac{(\frac{1}{5})\frac{\lambda^5}{5!}e^{-\lambda}}{P[Y=3]}\end{aligned}$$

(d) We have:

$$P[X=k|Y=3] = \begin{cases} 0 & k \in \{0,1,2,4\} \\ \frac{\frac{\lambda^3}{3!}e^{-\lambda}}{P[Y=3]} & k=3 \\ \frac{\frac{1}{k}\frac{\lambda^k}{k!}e^{-\lambda}}{P[Y=3]} & k>4 \end{cases}$$

$$\begin{aligned}\mathbb{E}\left[X|Y=3\right] &= \sum_{k=0}^{\infty} kP[X=k|Y=3] \\ &= 3\frac{\frac{\lambda^3}{3!}e^{-\lambda}}{P[Y=3]} + \sum_{k=5}^{\infty}\frac{\frac{\lambda^k}{k!}e^{-\lambda}}{P[Y=3]}\end{aligned}$$

(e) We have

$$\begin{aligned}P[X\in\{8,9\}|Y\in\{4,5\}] &= P[X=8|Y\in\{4,5\}] + P[X=9|Y\in\{4,5\}] \\ &= \frac{P[Y\in\{4,5\}|X=8]P[X=8]}{P[Y\in\{4,5\}]} + \frac{P[Y\in\{4,5\}|X=9]P[X=9]}{P[Y\in\{4,5\}]} \\ &= \frac{\frac{2}{8}\frac{\lambda^8}{8!}e^{-\lambda}}{P[Y\in\{4,5\}]} + \frac{\frac{2}{9}\frac{\lambda^9}{9!}e^{-\lambda}}{P[Y\in\{4,5\}]}\end{aligned}$$

where:

$$\begin{aligned}P[Y\in\{4,5\}] &= P[Y=4] + P[Y=5] \\ &= \frac{\lambda^4}{4!}e^{-\lambda} + 2\sum_{i=5}^{\infty}\frac{1}{i}\frac{\lambda^i}{i!}e^{-\lambda}\end{aligned}$$

V. Mean and Variance

Let X be Gaussian with mean m and variance σ^2. Define $Y = X/2 - 4$.
a) Compute $\mathbb{E}\left[Y\right]$ and $Var(Y)$.
b) Give the PDF of Y.
c) Compute $\mathbb{E}\left[Y^2 - X\right]$.
Solution:

a) $\mathbb{E}[Y] = \mathbb{E}[X/2-4] = m/2-4$. $Var(Y) = \mathbb{E}\left[[(X/2-4)-(m/2-4)]^2\right] = \mathbb{E}\left[(X-m)^2/4\right] = Var(X)/4 = \sigma^2/4$.
b) Y is Gaussian with mean $m/2-4$ and variance $\sigma^2/4$.
c) $\mathbb{E}\left[Y^2 - X\right] = \mathbb{E}\left[(X/2-4)^2 - X\right] = \mathbb{E}\left[X^2/4 - 4X + 16 - X\right] = \underbrace{\mathbb{E}\left[X^2\right]}_{\sigma^2+m^2}/4 - 5m + 16 = \frac{\sigma^2+m^2}{4} - 5m + 16$.

VI. Book problem 4.54a-b

Solution:
a)

$$\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} g(x) f_X(x) dx \\
&= -a \int_{-\infty}^{-a} f_X(x) dx + \int_{-a}^{a} x f_X(x) dx + a \int_{a}^{\infty} f_X(x) dx \\
&= -a F_X(-a) + \int_{-a}^{a} x f_X(x) dx + a(1 - F_X(a)) \\
\text{Similarly,} \quad \mathbb{E}\left[Y^2\right] &= a^2 F_X(-a) + \int_{-a}^{a} x^2 f_X(x) dx + a^2(1 - F_X(a)) \\
\text{Var}[Y] &= \mathbb{E}\left[Y^2\right] - \mathbb{E}[Y]^2
\end{aligned}$$

b) The random variable X has the Laplacian distribution.

$$f_X(x) = \frac{1}{2} e^{-|x|}$$

Mean of this Laplacian random variable is 0 and variance 2. The integral of this pdf from a point a to ∞ is $1 - F_X(a) = e^{-a}/2$, which is also equal to $F_X(-a)$. And since $f_X(x)$ is symmetric $\int_{-a}^{a} x f_X(x) dx = 0$ which gives us $E[Y] = 0$. For the variance, $E[Y^2] = a^2 e^{-a}\Big|_{a=1} + \int_0^1 x^2 e^{-x} dx = e^{-1} + e^{-x}(x^2+2x+2)\Big|_1^0 = e^{-1} + 2 - 5e^{-1} = 2 - 4e^{-1} = \text{Var}[Y]$.

VII. Linear Transformation

Suppose X is exponential with parameter $\lambda > 0$. Define $Y = 4 - 2X$.
a) Find the PDF of Y. Plot the PDF of X and the PDF of Y.
b) Compute $\mathbb{E}[Y]$ and $Var(Y)$.
Solution:
a) $f_Y(y) = \frac{1}{2} f_X((4-y)/2) = \begin{cases} \frac{\lambda}{2} e^{-\lambda(4-y)/2} & \text{if } y \leq 4 \\ 0 & \text{otherwise} \end{cases}$
b) $\mathbb{E}[Y] = \mathbb{E}[4-2X] = 4 - 2\mathbb{E}[X] = 4 - 2/\lambda$.
$Var(Y) = \mathbb{E}\left[Y^2\right] - (4-2/\lambda)^2$ and $\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[(4-2X)^2\right] = 16 - 16\mathbb{E}[X] + 4\mathbb{E}\left[X^2\right] = 16 - 16/\lambda + 8/\lambda^2$

VIII. Nonlinear transformation

Let X be a continuous random variable with PDF $f_X(x)$. Define $Y = X^2 - 2X$.
a) Compute $\mathbb{E}[Y]$.
b) Compute the PDF of Y.
c) Compute the PDF of Y for the case X is uniform over $[0, 1]$.
Solution:
a) $\mathbb{E}[Y] = \mathbb{E}\left[X^2 - 2X\right] = \int_{-\infty}^{\infty} (x^2 - 2x) f_X(x) dx$
b) $Y = (X-1)^2 - 1 \geq -1$ always and so $f_Y(y) = 0$ if $y \leq -1$(since continuous distribution). Now for $y > -1$ we have:

$$P[Y \leq y] = P[(X-1)^2 - 1 \leq y] = P[|X-1| \leq \sqrt{y+1}] = P[1 - \sqrt{y+1} \leq X \leq 1 + \sqrt{y+1}]$$

So $F_Y(y) = F_X(1 + \sqrt{y+1}) - F_X(1 - \sqrt{y+1})$ So

$$\boxed{f_Y(y) = \frac{f_X(1+\sqrt{y+1})}{2\sqrt{y+1}} + \frac{f_X(1-\sqrt{y+1})}{2\sqrt{y+1}} \quad \forall y > -1}$$

c) We have $f_Y(y) = 0$ if $y \leq -1$. If $y > -1$ we get

$$f_Y(y) = \frac{a(y)}{2\sqrt{y+1}} + \frac{b(y)}{2\sqrt{y+1}}$$

where

$$a(y) = \begin{cases} 1 & \text{if } 1+\sqrt{y+1} \in (0,1) \\ 0 & \text{otherwise} \end{cases}, b(y) = \begin{cases} 1 & \text{if } 1-\sqrt{y+1} \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

That is

$$a(y) = 0, b(y) = \begin{cases} 1 & \text{if } y \in (-1,0) \\ 0 & \text{otherwise} \end{cases}$$

So in fact

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y+1}} & \text{if } y \in (-1,0) \\ 0 & \text{otherwise} \end{cases}}$$

IX. Application of LAW of Total expectation

Imagine that you are at a party and are playing the following game: You roll a pair of six-sided dice. You lose if the sum of the dice is 2 or 3. On the other hand, you win if the sum is greater than 8. If the sum (denoted by i) is any number other than 2, 3, 9, 10, 11, or 12, you keep rolling the dice until the sum is either 10 or i. You lose if the sum is 10, and you win if it's i. Let R be the total number of times you roll the dice in this game.

a) Find the probability distribution for the sum of the dice.

b) Let S be the sum of the dice at the first rolling. Find $\mathbb{E}[R|S=i]$ for all possible values of i.

c) Find E[R].

Solution:

a)

$$P[S=k] = \begin{cases} \frac{1}{36} & k \in \{2,12\} \\ \frac{1}{18} & k \in \{3,11\} \\ \frac{1}{12} & k \in \{4,10\} \\ \frac{1}{9} & k \in \{5,9\} \\ \frac{5}{36} & k \in \{6,8\} \\ \frac{1}{6} & k \in \{7\} \\ 0 & otherwise \end{cases}$$

b) We have

$$P[R=r|S=i] = \begin{cases} 1 & r=1 \\ 0 & otherwise \end{cases} for: i \in \{2,3,9,10,11,12\}$$

Then:

$$\mathbb{E}[R|S=i] = rP[R=r|S=i] = 1, for: i \in \{2,3,9,10,11,12\}$$

We also have

$$P[R=r|S=i] = \begin{cases} [P[S=i]+P[S=10]] \times [1-P[S=i]-P[S=10]]^{r-2} & r \in \{2,3,4,...\} \\ 0 & otherwise \end{cases} for: i \in \{4,5,6,7,8\}$$

Then:

$$\mathbb{E}[R|S=i] = \sum_{r=2}^{\infty} rP[R=r|S=i], for: i \in \{4,5,6,7,8\}$$

$$\mathbb{E}[R|S=i] = \begin{cases} 7 & i=4 \\ \frac{43}{7} & i=5 \\ \frac{11}{2} & i=6 \\ 5 & i=7 \\ \frac{11}{2} & i=8 \end{cases}$$

c) We have:

$$\begin{aligned} E[R] &= \sum_{i=2}^{12} \mathbb{E}[R|S=i]\, P[S=i] \\ &= \frac{2}{36} + \frac{2}{18} + \frac{1}{9} + \frac{1}{12} + 7 \times \frac{1}{12} + \frac{43}{7} \times \frac{1}{9} + \frac{11}{2} \times \frac{5}{36} + 5 \times \frac{1}{6} + \frac{11}{2} \times \frac{5}{36} \\ &= \frac{335}{84} = 3.988095238095238095238095238 \end{aligned}$$

X. Hotel Bookings

A hotel has 300 rooms. It has accepted reservations for 324 rooms. Suppose that from historical data, we know that the probability of *no-shows* is 0.1 (i.e., 10% of the people who book rooms, do not arrive to take the room). Assume no-shows are independent across all 324 reservations. Let X be the number of no-shows.

a) What is the expectation of X?

b) What is the variance of X?

c) What is the probability that the hotel is "over-booked" (i.e., the hotel will not have enough rooms for all those who arrive)?

Solution:

First, we note that $X \sim Binomial(n = 324, p = 0.1)$.

a) $E[X] = np = 324(0.1) = 32.4$

b) $Var(X) = np(1-p) = 324(0.1)(1-0.1) = 29.16$

c) $P[\text{hotel is over-booked}] = P[X < 24] = \sum_{k=0}^{23} \binom{324}{k} (0.1)^k (1-0.1)^{324-k} = 0.0447342552599141493...$