EE 503: Problem Set #5 : Solutions

Facts: Let V and S be nonempty sets.

1) If C is a collection of subsets of V then $C \subseteq \sigma(C)$ with equality if and only if C is a sigma algebra on V.
2) If C and D are collections of subsets of V that satisfy $C \subseteq D$, then $\sigma(C) \subseteq \sigma(D)$.
3) Let $X : S \to \mathbb{R}$ be a function. For all sets $A \subseteq \mathbb{R}$ define the *inverse image of A with respect to the function X*, denoted $X^{-1}(A)$, as the following set:
$$X^{-1}(A) = \{\omega \in S : X(\omega) \in A\}$$
It can be shown that if $A \subseteq \mathbb{R}$ then $X^{-1}(A)^c = X^{-1}(A^c)$; If $\{A_i\}_{i=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}$ then
$$X^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} X^{-1}(A_i)$$

I. Jumps in the CDF

Fix a probability triplet (S, F, P). Let X be a random variable. Fix $a \in \mathbb{R}$. Recall that $P[X = a]$ is equal to the height of the jump in the CDF $F_X(x)$ at the point a. That is, $P[X = a] = F_X(a) - F_X(a^-)$. This problem proves it.

a) Use the definition of $A_n \searrow A$ to prove that $\left\{X \in \left(a - \frac{1}{n}, a\right]\right\} \searrow \{X = a\}$.

b) By continuity of probability together with part (a), we conclude that $\lim_{n\to\infty} P[X \in (a - \frac{1}{n}, a]] = P[X = a]$. Use this to conclude the result about the jump in the CDF at point a.

c) Use the fact $P[X = a] = F_X(a) - F_X(a^-)$ to prove that if the CDF is continuous at all points $x \in \mathbb{R}$ then $P[X = x] = 0$ for all $x \in \mathbb{R}$.

Solution:

a) There are two properties to show:

1) Shrinking sets: Fix n as a positive integer. Since $(a - \frac{1}{n}, a] \supseteq (a - \frac{1}{n+1}, a]$ we know that
$$\{X \in (a - \tfrac{1}{n}, a]\} \supseteq \{X \in (a - \tfrac{1}{n+1}, a]\}$$
Indeed, any $\omega \in S$ that satisfies $X(\omega) \in (a - \frac{1}{n+1}, a]$ must also satisfy $X(\omega) \in (a - \frac{1}{n}, a]$.
2) Infinite intersection: We want to show
$$\cap_{n=1}^{\infty}\{X \in (a - \tfrac{1}{n}, a]\} = \{X = a\}$$
Suppose $\omega \in \{X = a\}$. Then $X(\omega) = a$ and so $X(\omega) \in (a - \frac{1}{n}, a]$ for all positive integers n, so $\omega \in \cap_{n=1}^{\infty}\{X \in (a - \frac{1}{n}, a]$. Now suppose $\omega \in \cap_{n=1}^{\infty}\{X \in (a - \frac{1}{n}, a]\}$. Then $X(\omega) \in (a - \frac{1}{n}, a]$ for all positive integers n. It follows that $X(\omega) \leq a$. Now suppose $X(\omega) < a$ (we reach a contradiction). Then there must be a positive integer n such that $X(\omega) < a - \frac{1}{n} < a$ and so $X(\omega) \notin (a - \frac{1}{n}, a]$, a contradiction. So $X(\omega) \geq a$ and $X(\omega) \leq a$, it must be that $X(\omega) = a$. So $\omega \in \{X = a\}$.

b) By continuity of probability, since $\{X \in (a - \frac{1}{n}, a]\} \searrow \{X = a\}$ we conclude that
$$P[X = a] = \lim_{n\to\infty} P[X \in (a - \tfrac{1}{n}, a]] = \lim_{n\to\infty}(F_X(a) - F_X(a - \tfrac{1}{n})) = F_X(a) - F_X(a^-)$$
where we have used the fact $P[X \in (a - \frac{1}{n}, a]] = F_X(a) - F_X(a - \frac{1}{n})$.

c) If $F_X(x)$ is continuous at all x then there are no discontinuous jumps, meaning the height of the jump at any point x is 0, so $P[X = x] = 0$.

II. Borel Measurable functions

Let (S, F, P) be a probability triplet: S is the sample space; F is a sigma algebra on S; $P : F \to \mathbb{R}$ is a probability measure. Define C as the set of all intervals in $\mathbb{R}$ of the type $(-\infty, x]$ for some $x \in \mathbb{R}$. The set $\sigma(C)$ is called the *Borel sigma algebra on* $\mathbb{R}$. A subset of $\mathbb{R}$ is said to be a *Borel measurable set* if it is in $\sigma(C)$. A function $X : S \to \mathbb{R}$ is said to be a *Borel measurable function* if:[1]
$$X^{-1}(A) \in F \quad \forall A \in C \tag{1}$$
We want to show that if $X : S \to \mathbb{R}$ is a Borel measurable function then
$$X^{-1}(A) \in F \quad \forall A \in \sigma(C) \tag{2}$$
In particular, (2) implies that if $A \in \sigma(C)$ then $\{X \in A\}$ is an *event* (and hence has a well defined probability $P[X \in A]$). That is, a function $X : S \to \mathbb{R}$ with the property that $\{X \leq x\}$ is an event for all $x \in \mathbb{R}$ must also have the property that $\{X \in A\}$ is an event for all Borel measurable subsets $A \subseteq \mathbb{R}$.

[1]A Borel measurable function $X : S \to \mathbb{R}$ is called a *random variable*.

For the tasks below, assume that $X : S \to \mathbb{R}$ satisfies (1).

a) Define H as the following set of subsets of $\mathbb{R}$:

$$H = \{A \subseteq \mathbb{R} : X^{-1}(A) \in F\}$$

Show that $C \subseteq H$. It immediately follows (by Fact 2) that $\sigma(C) \subseteq \sigma(H)$.

b) Argue that H is a sigma algebra on $\mathbb{R}$. It immediately follows (by Fact 1) that $\sigma(H) = H$.

c) Argue that $\sigma(C) \subseteq H$. Explain why this yields our desired conclusion (2).

Solution:

a) Fix $A \in C$. By property (1) we know $X^{-1}(A) \in F$. Then $A \in H$ by definition of H.

b) We show the three defining properties of a sigma algebra on $\mathbb{R}$ are satisfied:

1) Clearly $\mathbb{R} \in H$ because $X^{-1}(\mathbb{R}) = S \in F$ (since F is a sigma algebra on S so $S \in F$).
2) Suppose $A \in H$. Then $X^{-1}(A) \in F$ and so $X^{-1}(A)^c \in F$ (because F is a sigma algebra). But Fact 3 ensures $X^{-1}(A)^c = X^{-1}(A^c)$, so $X^{-1}(A^c) \in F$, which means (by definition of H) that $A^c \in H$.
3) Suppose $C_1, C_2, ...$ are a countable sequence of sets in H. Fix i as a positive integer. Since $C_i \in H$ we know (by definition of H) that $X^{-1}(C_i) \in F$. This holds for all positive integers i. So $\cup_{i=1}^{\infty} X^{-1}(C_i)$ is a countable union of sets in the sigma algebra F so it must also be in F. That is, $\cup_{i=1}^{\infty} X^{-1}(C_i) \in F$. But Fact 3 ensures $\cup_{i=1}^{\infty} X^{-1}(C_i) = X^{-1}(\cup_{i=1}^{\infty} C_i)$. So $X^{-1}(\cup_{i=1}^{\infty} C_i) \in F$. Thus $\cup_{i=1}^{\infty} C_i \in H$ (by definition of H).

c) Part (a) implies $\sigma(C) \subseteq \sigma(H)$ and part (b) implies $\sigma(H) = H$. Thus $\sigma(C) \subseteq H$. It means that if $A \in \sigma(C)$ then $A \in H$ and so (by definition of H) we must have $X^{-1}(A) \in F$.

III. CDFs and Expectations

Let X be a random variable defined by the pdf $f_X(x) = x^2[u(x) - u(x-1)] + a\delta(x-2)$, where $u(x)$ is the unit step function which is equal to 1 when $x \geq 0$ and 0 otherwise, and $\delta(x)$ is the impulse function (Dirac delta function) which is the (generalized) derivative of $u(x)$.

a) Find a, $\mathbb{E}[X]$ and $\text{Var}[X]$

b) Define the event $W = \{X \geq 0.5\}$. Find $P[X \leq x | W]$ for all $x \in \mathbb{R}$. We shall define $F_{X|W}(x) = P[X \leq x|W]$ the *conditional CDF given the event* W. Find a nonnegative function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\int_{-\infty}^{x} f(t)dt = F_{X|W}(x) \quad \forall x \in \mathbb{R}$$

We can call this a *conditional PDF* given W. Find the set of all $x \in \mathbb{R}$ over which your function f satisfies $f(x) > 0$. This is often called the "support" of the function.

Solution:

a) The pdf must integrate to 1:

$$\int_{-\infty}^{\infty} f_X(x) \mathrm{dx} = 1$$
$$\int_{-\infty}^{\infty} \left(x^2[u(x) - u(x-1)] + a\delta(x-2)\right) \mathrm{dx} = 1$$

The $u(x) - u(x-1)$ term is what is called a box function, it starts from $x = 0$ and ends at $x = 1$ and takes the value 1 in between. So we have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \left(x^2[u(x) - u(x-1)] + a\delta(x-2)\right) \mathrm{dx} \\
&= \int_{0}^{1} x^2 dx + a \int_{-\infty}^{\infty} \delta(x-2)dx \\
&= 1/3 + a
\end{aligned}$$

and so

$$\boxed{a = 2/3}$$

Further

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} \left(x^3[u(x)-u(x-1)] + ax\delta(x-2)\right)\mathrm{dx} \\
&= \int_{0}^{1} x^3 dx + a\int_{-\infty}^{\infty} x\delta(x-2)dx \\
&\overset{(a)}{=} 1/4 + a\int_{-\infty}^{\infty} 2\delta(x-2)dx \\
&= 1/4 + 2a \\
&= 1/4 + 4/3 \\
&= 19/12
\end{aligned}
$$

where (a) holds because $g(x)\delta(x-2) = g(2)\delta(x-2)$ for continuous functions g.

Finally

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_{0}^{\infty} x^2 f_X(x)\mathrm{dx} \\
&= \int_{-\infty}^{\infty} x^4[u(x)-u(x-1)]dx + a\int_{-\infty}^{\infty} x^2\delta(x-2)dx \\
&= \int_{0}^{1} x^4\mathrm{dx} + a\int_{-\infty}^{\infty} 2^2\delta(x-2)dx \\
&= 1/5 + 4a \\
&= 1/5 + 8/3 \\
&= 43/15
\end{aligned}
$$

$$
\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{259}{720}
$$

b)

$$
\begin{aligned}
F_{X|W}(x) &= \frac{P[\{X \le x\} \cap \{X \ge 0.5\}]}{P[\{X \ge 0.5\}]} \\
P[\{X \ge 0.5\}] &= \int_{0.5}^{\infty} f_X(x)\mathrm{dx} = \frac{23}{24} > 0 \\
F_{X|W}(x) &= \begin{cases} \frac{P[0.5 \le X \le x]}{P[\{X \ge 0.5\}]} & x \ge 0.5 \\ 0 & x < 0.5 \end{cases} \\
&= \begin{cases} \frac{F_X(x) - F_X(0.5)}{23/24} & x \ge 0.5 \\ 0 & x < 0.5 \end{cases} \\
F_X(0.5) &= 1/24 \\
F_X(x) &= \frac{x^3}{3},\ 0 \le x < 1 \\
F_X(x) &= 1/3,\ 1 \le x < 2 \\
F_X(x) &= 1, x \ge 2
\end{aligned}
$$

So

$$
F_{X|W}(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ \frac{\frac{x^3}{3} - \frac{1}{24}}{23/24} & \text{if } 0.5 \le x < 1 \\ 7/23 & \text{if } 1 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}
$$

This has a discontinuous jump of height $16/23$ at $x = 2$ and so

$$
f_{X|W}(x) = F'_{X|W}(x) = \frac{16}{23}\delta(x-2) + g(x)
$$

where

$$g(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ \frac{x^2}{23/24} & \text{if } 0.5 \leq x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$