EE 503: Problem Set #8 : Solutions

I. Conditional Probability

Let X has a pdf of $f_X(x) = 2x$ over (0,1], Y has a Poisson distribution with parameter $\lambda = x$ for $X = x$, and Z has an exponential distribution with parameter $\lambda = x$ for $X = x$.
(a)Find $E[XY]$.
(b)Find $E[XZ]$.
Solution:
(a)

$$\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \times f_{X,Y}(x,y)dydx \\
&= \int_0^1 xf_x(x) \times \sum_{n=0}^{\infty} nP[Y=n|X=x]dx \\
&= \int_0^1 2x^3dx \\
&= 1/2
\end{aligned}$$

(b)

$$\begin{aligned}
E[XZ] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xz \times f_{X,Z}(x,z)dydx \\
&= \int_0^1 xf_x(x) \times \int_0^{\infty} zf_Z(z|X=x)dx \\
&= \int_0^1 2xdx \\
&= 1
\end{aligned}$$

II. Indicator variables

We have three decks of cards, each with 52 distinct cards:
Shuffle deck 1 and lay cards out.
Shuffle deck 2 and lay cards out underneath deck 1.
Shuffle deck 3 and lay cards out underneath deck 2.
(a) Let M be the number of matches where 3 cards are the same. What is E [M]? (0 or 1 match for each stack of cards)
(b) Let N be the number of matches where 2 neighboring cards (one on top of the other) are the same. What is E [N]? (In this case, there can be 0, 1 or 2 matches for each stack of cards)
Solution:
(a)

$$I_i = \begin{cases} 1 & three\ cards\ in\ column\ i\ are\ the\ same \\ 0 & at\ least\ one\ card\ in\ column\ i\ is\ different \end{cases}$$

$$M = \sum_{i=1}^{52} I_i$$

$$
\begin{aligned}
E[M] &= \sum_{i=1}^{52} E[I_i] \\
&= \sum_{i=1}^{52} P[I_i = 1] \\
&= \sum_{i=1}^{52} \frac{1}{52 \times 52} \\
&= \frac{1}{52}
\end{aligned}
$$

(b) For i^th stack of card, the three cards are X_i, Y_i, Z_i. Both solutions would be considered for full credit:
Solution 1: ($E[N_i] = P[X_i = Y_i] + P[Y_i = Z_i]$)

$$
I_{i,j} = \begin{cases} 1 & jth\ card\ in\ column\ i\ is\ the\ same\ as\ the\ previous\ one \\ 0 & jth\ card\ in\ column\ i\ is\ different\ from\ the\ previous\ one \end{cases}
$$

$$
\begin{aligned}
E[N] &= \sum_{i=1}^{52} \sum_{j=2}^{3} E[I_{i,j}] \\
&= \sum_{i=1}^{52} \sum_{j=2}^{3} P[I_{i,j} = 1] \\
&= \sum_{i=1}^{52} (3-1) \frac{1}{52} \\
&= 2
\end{aligned}
$$

The following is also correct (consider there is up to 1 match for each stack)
Solution 2: ($E[N_i] = P[(X_i = Y_i) \cup (Y_i = Z_i)] = P[X_i = Y_i] + P[Y_i = Z_i] - P[X_i = Y_i = Z_i]$)

$$
I_{i,j} = \begin{cases} 1 & jth\ card\ in\ column\ i\ is\ the\ same\ as\ the\ previous\ one \\ 0 & jth\ card\ in\ column\ i\ is\ different\ from\ the\ previous\ one \end{cases}
$$

$$
\begin{aligned}
E[N] &= \sum_{i=1}^{52} \sum_{j=2}^{3} E[I_{i,j}] - E[M] \\
&= \sum_{i=1}^{52} \sum_{j=2}^{3} P[I_{i,j} = 1] - E[M] \\
&= \sum_{i=1}^{52} (3-1) \frac{1}{52} - E[M] \\
&= \frac{103}{52}
\end{aligned}
$$

III. Cauchy-Schwarz inequality

Find the gap in the Cauchy-Schwarz inequality $|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$ (the difference between $|E[XY]|$ and $\sqrt{E[X^2]E[Y^2]}$) for the cases:
(a) $Y = aX$ for some constant a in R, X has an exponential distribution with parameter λ.
(b) (X,Y) are uniformly distributed over a triangle with vertices (0,0), (1,0), (0,1).
(c) (X,Y) is a discrete random vector that takes values equally likely over the three points (0,0), (1,0), (0,1).
Solution:

(a)

$$\begin{aligned}
E[XY] &= E[aX^2] \\
&= a(Var[X] + E[X]^2) \\
&= \frac{2a}{\lambda^2} \\
E[X^2] &= \frac{2}{\lambda^2} \\
E[Y^2] &= \frac{2a^2}{\lambda^2}
\end{aligned}$$

Therefore, $|E[XY]| - \sqrt{E[X^2]E[Y^2]} = 0$
(b)

$$\begin{aligned}
E[XY] &= \int_0^1 \int_0^{1-x} 2xydydx \\
&= \int_0^1 x(1-x)^2 dx \\
&= \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}|_0^1 \\
&= \frac{1}{12} \\
E[Y^2] = E[X^2] &= \int_0^1 \int_0^{1-x} 2x^2 dydx \\
&= \int_0^1 2x^2(1-x)dx \\
&= \frac{2x^3}{3} - \frac{2x^4}{4}|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

Therefore, $|E[XY]| - \sqrt{E[X^2]E[Y^2]} = -\frac{1}{12}$
(c)

$$P[X = 0, Y = 0] = P[X = 1, Y = 0] = P[X = 0, Y = 1] = \frac{1}{3}$$

$E[XY] = 0$, cause at least one of X and Y is 0
$E[X^2] = E[Y^2] = \frac{1}{3} \times 0 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = \frac{1}{3}$
Therefore, $|E[XY]| - \sqrt{E[X^2]E[Y^2]} = -\frac{1}{3}$