EE 503: Problem Set #9 : Solutions

- Reading: Chapter 5.1-5.8 in Leon-Garcia textbook.
- Submit your homework in D2L by 9pm on the due date.

I. Leon-Garcia Book Problem 5.26abc + 5.80abcd (Joint PDFs of Two RVs)

Solution:
(5.26a)

$$\begin{aligned}
1 &= k\int_0^1\int_0^1 (x+y)\,dx\,dy = k\int_0^1\left[\frac{x^2}{2}+xy\right]_0^1 dy \\
&= k\int_0^1\left(\frac{1}{2}+y\right)dy = k\left[\frac{1}{2}y+\frac{y^2}{2}\right]_0^1 = k \\
\therefore k &= 1
\end{aligned}$$

(5.26b)

$$F_{X,Y}(x,y)=\begin{cases}\frac{xy(x+y)}{2}, & 0\le x\le 1,\ 0\le y\le 1\\ \frac{y(y+1)}{2}, & x>1,\ 0\le y\le 1\\ \frac{x(x+1)}{2}, & 0\le x\le 1,\ y>1\\ 1, & x>1,\ y>1\\ 0, & \text{otherwise}\end{cases}$$

(5.26c)

$$\begin{aligned}
F_X(x) &= \lim_{y\to\infty} F_{XY}(x,y) = F_{XY}(x,1), \quad 0<x<1\\
\Rightarrow f_X(x) &= \frac{d}{dx}F_X(x) = x+\frac{1}{2}
\end{aligned}$$

Similarly,

$$f_Y(y) = y+\frac{1}{2}$$

(5.80a)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}, \quad 0<y<1$$

(5.80b)

$$\begin{aligned}
P[Y>X|x] &= \int_x^1 \frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}}\int_x^1 (x+y)\,dy\\
&= \frac{1}{x+\frac{1}{2}}\left[xy+\frac{y^2}{2}\right]_{y=x}^1\\
&= \frac{1}{x+\frac{1}{2}}\left[x+\frac{1}{2}-x^2-\frac{x^2}{2}\right]\\
&= \frac{1}{x+\frac{1}{2}}\left[x+\frac{1}{2}-\frac{3}{2}x^2\right].
\end{aligned}$$

(5.80c)

$$\begin{aligned}P[Y > X] &= \int_0^1 P[Y > X|x] f_X(x)\,dx \\ &= \int_0^1 \frac{x + \frac{1}{2} - \frac{3}{2}x^2}{x + \frac{1}{2}} (x + \tfrac{1}{2})\,dx \\ &= \int_0^1 (x + \tfrac{1}{2} - \tfrac{3}{2}x^2)\,dx \\ &= \frac{1}{2}\end{aligned}$$

(5.80d)

$$\begin{aligned}\mathbb{E}[Y|X = x] &= \int_0^1 y \frac{x+y}{x+\frac{1}{2}}\,dy = \frac{1}{x+\frac{1}{2}} \int_0^1 (xy + y^2)\,dy \\ &= \frac{\frac{1}{2}x + \frac{1}{3}}{x + \frac{1}{2}}.\end{aligned}$$

II. Independent Exponential Random Variables

If X and Y are i.i.d. exponential random variables, find the CDF of $Z = \frac{X}{X+Y}$, and hence its PDF.
<u>Solution</u>:

$$\begin{aligned}f_{X,Y}(x,y) &= \lambda^2 e^{-\lambda(x+y)}, \quad x, y \geq 0. \quad Z = \frac{X}{X+Y} \in [0,1]. \\ F_Z(z) &= P(Z \leq z) = P\left(\frac{Y}{X} \geq \frac{1}{z} - 1\right) = \int_0^\infty \int_{\frac{x}{z}-x}^\infty f_{X,Y}(x,y)\,dy\,dx = z. \\ f_Z(z) &= \frac{dF_Z(z)}{dz} = 1.\end{aligned}$$

III. Independent Gaussian Random Variables

Let X and Y be independent identically distributed Gaussian random variables with $\mu = 0$ and $\sigma^2 = 4$.

(a) Find the joint PDF of X and Y.

<u>Solution</u>:
The marginal PDFs of X and of Y are identical:

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{8}\right).$$

Since X and Y are independent,

$$\begin{aligned}f_{X,Y}(x,y) &= f_X(x) f_Y(y) \\ &= \frac{1}{8\pi} \exp\left(-\frac{x^2+y^2}{8}\right).\end{aligned}$$

(b) Using the joint PDF of part (a), find $P[X^2 + Y^2 \leq 1]$.

<u>Solution</u>:
It will be easier to integrate in polar coordinates, i.e.

$$\iint_A f(x,y)\,dx\,dy = \iint_A f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

We need to find the volume under $f_{X,Y}(x,y)$ within the unit disk, i.e.

$$\begin{aligned}
P[X^2+Y^2 \le 1] &= \int_{-\pi}^{\pi}\int_0^1 f_{X,Y}(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta \\
&= \int_{-\pi}^{\pi}\int_0^1 \frac{r}{8\pi}\exp\left(-\frac{r^2}{8}\right) dr\, d\theta \\
&= \int_{-\pi}^{\pi}\frac{1}{2\pi}\left[e^{-r^2/8}\right]_1^0 d\theta \\
&= \int_{-\pi}^{\pi}\frac{1}{2\pi}\left(1-e^{-1/8}\right) d\theta \\
&= 1-e^{-1/8}.
\end{aligned}$$

(c) *If we define $Z = X^2+Y^2$, find the CDF of Z. What is the name of this distribution?*

<u>Solution</u>:
Now we need to find $P[X^2+Y^2 \le z], z \ge 0$, which is the volume under the joint PDF over the disc of radius $\sqrt{z}$ centered on the origin.

$$\begin{aligned}
F_Z(z) &= P[X^2+Y^2 \le z] \\
&= \int_{-\pi}^{\pi}\int_0^{\sqrt{z}} \frac{r}{8\pi}\exp\left(-\frac{r^2}{8}\right) dr\, d\theta \\
&= \int_{-\pi}^{\pi}\frac{1}{2\pi}\left[e^{-r^2/8}\right]_{\sqrt{z}}^0 d\theta \\
&= 1-e^{-z/8}, \quad z \ge 0.
\end{aligned}$$

This is the CDF of an exponential random variable with $\lambda = 1/8$.

IV. Conditional PDF and Iterated Expectations

The conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-y)^2}{2}\right),$$

and Y is a uniform random variable in $[0,1]$.

(a) *Find the marginal PDF of X. (You can keep your answer in integral form)*

<u>Solution</u>:
The marginal PDF of X is

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\, dy \\
&= \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}\, dy
\end{aligned}$$

(b) *Find $\mathbb{E}[X]$ and Var(X).*
<u>Solution</u>:
It should be clear from the Gaussian form of the conditional PDF that $\mathbb{E}[X|Y] = Y$, and hence by iterated expectations,

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[Y] = 0.5$$

Similarly, we have

$$\begin{aligned}
&\text{Var}(X|Y) = \mathbb{E}[X^2|Y] - \mathbb{E}^2[X|Y] = 1 \\
\Rightarrow\ &\mathbb{E}[X^2|Y] = 1 + Y^2
\end{aligned}$$

and therefore,

$$\mathbb{E}[X^2] = \mathbb{E}[\mathbb{E}[X^2|Y]] = \mathbb{E}[1+Y^2] = 1 + \int_0^1 y^2\,dy = \frac{4}{3}.$$

Finally, $\text{Var}(X) = 4/3 - 0.5^2 = 13/12$.

V. Two Random Variables and Independence

The joint CDF of X and Y is

$$F_{X,Y} = \begin{cases} (1-e^{-\alpha x})(1-e^{-\beta x}), & x \geq 0, y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

(a) *Find the joint PDF of X and Y.*

Solution:
The joint PDF is

$$f_{X,Y}(x,y) = \frac{\partial F_{X,Y}(x,y)}{\partial x \partial y} = \alpha e^{-\alpha x} \beta e^{-\beta y}, \quad x \geq 0, y \geq 0.$$

(b) *Find the marginal PDF's of X and of Y.*

Solution:

$$f_X(x) = \int_0^\infty \alpha e^{-\alpha x} \beta e^{-\beta y}\,dy = \alpha e^{-\alpha x}, \quad x \geq 0$$
$$f_Y(y) = \beta e^{-\beta y}, \quad y \geq 0$$

(c) *Are X and Y independent?*

Solution:
Yes, because $f_{X,Y}(x,y) = f_X(x) f_Y(y)$.

VI. Sum of Random Variables

X and Y are jointly uniform in the triangular region.

$$\{0 < x < 2\} \cap \{0 < y < 1\} \cap \{2y < x\}.$$

(a) *Find the PDF of $Z = X + Y$.*

Solution:

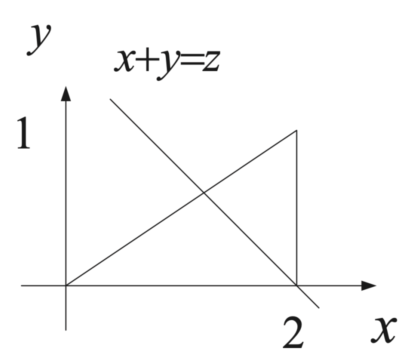

Fig. 1. Triangular region $\{0 < x < 2\} \cap \{0 < y < 1\} \cap \{2y < x\}$.

The region of (x, y) is given in Fig. 1.

$$F_Z(z) = P(X+Y \le z) = \begin{cases} \int_0^{\frac{z}{3}} \int_{2y}^{z-y} dx\, dy = \frac{z^2}{6} & 0 \le z \le 2 \\ 1 - \int_{\frac{2z}{3}}^{2} \int_{z-x}^{\frac{x}{2}} dy\, dx = -\frac{z^2}{3} + 2z - 2, & 2 < z \le 3 \\ 1, & z > 3 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{z}{3}, & 0 \le z \le 2 \\ -\frac{2z}{3} + 2, & 2 < z \le 3 \\ 0, & \text{otherwise} \end{cases}$$

(b) *Find* $\mathbb{E}[X]$, $\mathbb{E}[Y]$ *and* $\mathbb{E}[XY]$.

Solution:

$$\mathbb{E}[XY] = \int_0^2 \int_0^{\frac{x}{2}} xy\, dy\, dx = 1/2$$
$$\mathbb{E}[X] = \int_0^2 \int_0^{\frac{x}{2}} x\, dy\, dx = 4/3$$
$$\mathbb{E}[Y] = \int_0^2 \int_0^{\frac{x}{2}} y\, dy\, dx = 1/3$$

(c) *Find the covariance between* X *and* Y.

Solution:

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{1}{18}$$

VII. Characteristic Function

Let X *and* Y *have the joint PDF*

$$f_{X,Y}(x, y) = \begin{cases} ce^{-(x+y)/2}, & 0 < x \le y < \infty \\ 0, & \text{otherwise} \end{cases}$$

where c *is a constant.*

(a) *Find the constant* c.

Solution:
The total volume under the joint PDF must be unity, i.e.

$$\int_0^{\infty} \int_x^{\infty} ce^{-(x+y)/2}\, dy\, dx = 1$$
$$\int_0^{\infty} 2ce^{-(x+y)/2} \Big|_{\infty}^{x} dx = 1$$
$$2c \int_0^{\infty} e^{-x}\, dx = 1$$
$$2c = 1$$
$$c = 0.5.$$

(b) *Show that the characteristic function of* $Z = X + Y$ *is*

$$\phi_Z(u) = \frac{1}{(1 - 2ju)^2}, \quad (j = \sqrt{-1})$$

Solution:
By definition, $\phi_Z(u) = \mathbb{E}[e^{ju(X+Y)}]$, and therefore

$$\begin{aligned}
\phi_Z(u) &= \int_0^\infty \int_x^\infty \frac{1}{2} e^{-(x+y)/2} e^{ju(x+y)}\, dy\, dx \\
&= \frac{1}{2ju-1} \int_0^\infty \left. e^{(ju-0.5)(x+y)} \right|_x^\infty dx \\
&= \frac{1}{1-2ju} \int_0^\infty e^{(2ju-1)x}\, dx \\
&= \frac{1}{(1-2ju)^2}.
\end{aligned}$$

(c) *By using the Fourier transform property*

$$jt\, x(t) \longleftrightarrow \frac{d}{d\omega} X(\omega)$$

where $X(\omega)$ is the Fourier transform of $x(t)$, find the PDF of Z. Note that

$$e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{\alpha - j\omega}.$$

Solution:
Using the given property, we have

$$\begin{aligned}
\frac{1}{0.5-ju} &\longleftrightarrow e^{-0.5t} u(t) \\
\Rightarrow \frac{d}{du} \frac{1}{0.5-ju} &\longleftrightarrow jt\, e^{-0.5t} u(t) \\
\Rightarrow \frac{j}{(0.5-ju)^2} &\longleftrightarrow jt\, e^{-0.5t} u(t) \\
\Rightarrow \frac{1}{(0.5-ju)^2} &\longleftrightarrow t\, e^{-0.5t} u(t) \\
\Rightarrow \frac{1}{(1-2ju)^2} &\longleftrightarrow \frac{1}{4} t\, e^{-0.5t} u(t)
\end{aligned}$$

Therefore the PDF of Z is

$$f_Z(z) = \frac{1}{4} z\, e^{-z/2} u(z).$$