EE 503: Problem Set #10 : Solutions

- Reading: Chapter 2 in Leon-Garcia textbook.
- Submit your homework in D2L by 9pm on the due date.

I. MAP DETECTION

Let X and N be independent random variables. Suppose $X \in \{-1, 0, 1\}$ with probabilities $P[X = -1], P[X = 0], P[X = 1]$. Suppose $N \sim N(0, \sigma^2)$ and $Y = X + N$.

a) Find the CDF and PDF $F_Y(y)$ and $f_Y(y)$.

b) Find the conditional PDF $f_{Y|X=i}(y)$ for all $i \in \{-1, 0, 1\}$.

c) Suppose $\sigma^2 = 1$ and $P[X = 1] = 0.5, P[X = 0] = P[X = -1] = 0.25$. We observe $Y = y$. Find optimal thresholds α and β such that the MAP detector chooses

$$\hat{X} = \begin{cases} 1 & \text{if } y \geq \beta \\ 0 & \text{if } \alpha \leq y < \beta \\ -1 & \text{if } y < \alpha \end{cases}$$

d) Compute the error probability $P[Error] = P[\hat{X} \neq X]$.

Solution:

a)

$$F_Y(y) = F_N(y+1)P[X = -1] + F_N(y)P[X = 0] + F_N(y-1)P[X = 1]$$
$$f_Y(y) = f_N(y+1)P[X = -1] + f_N(y)P[X = 0] + f_N(y-1)P[X = 1]$$

where $f_N(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/(2\sigma^2)}$.

b) We have for all $y \in \mathbb{R}$:

$$f_{Y|X=-1}(y) = f_N(y+1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y+1)^2/(2\sigma^2)}$$
$$f_{Y|X=0}(y) = f_N(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y)^2/(2\sigma^2)}$$
$$f_{Y|X=1}(y) = f_N(y-1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-1)^2/(2\sigma^2)}$$

c) We choose $\hat{X} = 1$ if:

$$(0.5)e^{-(y-1)^2/(2\sigma^2)} \geq (0.25)e^{-y^2/(2\sigma^2)}$$
$$(0.5)e^{-(y-1)^2/(2\sigma^2)} \geq (0.25)e^{-(y+1)^2/(2\sigma^2)}$$

where $\sigma^2 = 1$. This happens when

$$\exp\left((\frac{1}{2})(y^2 - (y-1)^2)\right) \geq 1/2$$
$$\exp\left((\frac{1}{2})((y+1)^2 - (y-1)^2)\right) \geq 1/2$$

Equivalently

$$2y - 1 \geq 2\log(1/2)$$
$$4y \geq 2\log(1/2)$$

So we need

$$y \geq -0.193147180 = \beta$$

We choose $\hat{X} = 0$ if

$$(0.25)e^{-y^2/(2\sigma^2)} \geq (0.5)e^{-(y-1)^2/(2\sigma^2)}$$
$$(0.25)e^{-y^2/(2\sigma^2)} \geq (0.25)e^{-(y+1)^2/(2\sigma^2)}$$

where $\sigma^2 = 1$. This happens when

$$\exp\left((\frac{1}{2})((y-1)^2 - y^2)\right) \geq 2$$
$$\exp\left((\frac{1}{2})((y+1)^2 - y^2)\right) \geq 1$$

Equivalently

$$-2y + 1 \geq 2\log(2)$$
$$2y + 1 \geq 0$$

So we need

$$\alpha = -1/2 \leq y \leq 1/2 - \log(2) = \beta$$

Overall we have $\alpha = -1/2$, $\beta = 1/2 + \log(1/2) = -0.193147180$

d) If $y < \alpha$ we get

$$\begin{aligned} P[Error|Y=y] &= P[X \neq -1|Y=y] \\ &= P[X=0|Y=y] + P[X=1|Y=y] \\ &= \frac{f_{Y|X=0}(y)(0.25)}{f_Y(y)} + \frac{f_{Y|X=1}(y)(0.5)}{f_Y(y)} \\ &= \frac{f_N(y)(0.25)}{f_Y(y)} + \frac{f_N(y-1)(0.5)}{f_Y(y)} \end{aligned}$$

If $\alpha < y < \beta$ we get

$$\begin{aligned} P[Error|Y=y] &= P[X \neq 0|Y=y] \\ &= P[X=-1|Y=y] + P[X=1|Y=y] \\ &= \frac{f_{Y|X=-1}(y)(0.25)}{f_Y(y)} + \frac{f_{Y|X=1}(y)(0.5)}{f_Y(y)} \\ &= \frac{f_N(y+1)(0.25)}{f_Y(y)} + \frac{f_N(y-1)(0.5)}{f_Y(y)} \end{aligned}$$

If $y > \beta$ we get

$$\begin{aligned} P[Error|Y=y] &= P[X \neq 1|Y=y] \\ &= P[X=-1|Y=y] + P[X=0|Y=y] \\ &= \frac{f_{Y|X=-1}(y)(0.25)}{f_Y(y)} + \frac{f_{Y|X=0}(y)(0.25)}{f_Y(y)} \\ &= \frac{f_N(y+1)(0.25)}{f_Y(y)} + \frac{f_N(y)(0.25)}{f_Y(y)} \end{aligned}$$

Thus

$$\begin{aligned} P[Error] &= \int_{y=-\infty}^{\infty} P[Error|Y=y] f_Y(y) dy \\ &= \int_{-\infty}^{\alpha} \left[(0.25)f_N(y) + (0.5)f_N(y-1)\right] dy \\ &+ \int_{\alpha}^{\beta} \left[(0.25)f_N(y+1) + (0.5)f_N(y-1)\right] dy \\ &+ \int_{\beta}^{\infty} \left[(0.25)f_N(y+1) + (0.25)f_N(y)\right] dy \\ &= 0.0771344 + 0.0334036 \\ &+ 0.0246655 + 0.0247993 \\ &+ 0.0524689 + 0.144145 \\ &= 0.3566167 \end{aligned}$$

II. Gaussian, $\chi^2(n)$, and Student's t distribution

Let X and Y be independent with $X \sim N(0,1)$ and $Y \sim \chi^2(n)$. It can be shown that $\mathbb{E}[Y] = n$ and $Var(Y) = 2n$. The Q function is defined by $Q(x) = P[X > x]$. Define $T = \frac{X}{\sqrt{Y/n}}$. For this problem, you need to provide numerical values using either lookup tables or numerical solvers.

a) Find $P[X > 0]$, $P[X > 0.2]$, and $P[5X + 3 \leq 2]$.
b) For $n = 5$ find $P[Y \in [0,1]]$.
c) For $n \in \{2, 5, 10, 20, 50, 100\}$ find $P[T > 0.2]$. Compare with $P[X > 0.2]$.
d) Find $Var(Y/n)$.
e) Define $W_1 = X + Y$ and $W_2 = X - 2Y$. Find the covariance matrix K_W for the vector (W_1, W_2).

Solution:

a) We have

$$\begin{aligned} P[X > 0] &= Q(0) = 1/2 \\ P[X > 0.2] &= Q(0.2) = 0.42074 \\ P[5X + 3 \leq 2] &= P[X \leq -0.2] = P[X > 0.2] = 0.42074 \end{aligned}$$

b) $P[Y \in [0,1]] = F_Y(1) - F_Y(0)$
d) $Var(Y/n) = (1/n^2)Var(Y) = (1/n^2)2n = 2/n$.
e) We have

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

We have $K_W = AK_XA^\top$ where

$$K_X = \begin{bmatrix} 1 & 0 \\ 0 & 2n \end{bmatrix}$$

So

$$K_W = AK_XA^\top = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2n & -4n \end{bmatrix} = \begin{bmatrix} 1+2n & 1-4n \\ 1-4n & 1+8n \end{bmatrix}$$

III. Another Computation of $f_{X|Y}(x|y)$

Let $Y = RX$ where R, X are i.i.d Gaussian $N(0, \sigma^2)$.

a) Find $f_{X|Y}(x|y)$ by using $F_{Y|X=x}(y) = P[Y \leq y|X = x]$. Remember to treat cases $x > 0$ and $x < 0$ separately.
b) Find $f_{X|Y}(x|y)$ by using the PDF transformation for $(X, R) \to (X, Y)$ to find $f_{X,Y}(x, y)$.

Solution:

a) For $x > 0$ we get

$$F_{Y|X=x}(y|x) = P[XR \leq y|X = x] = P[R \leq y/x|X = x] = P[R \leq y/x] = F_R(y/x) \tag{1}$$

so

$$f_{Y|X}(y|x) = d/dyF_R(y/x) = (1/x)f_R(y/x) = \frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(y/x)^2/(2\sigma^2)} \tag{2}$$

The case is similar for $x < 0$, with the exception that the denominator is $|x|$. So then

$$f_{Y|X}(y|x) = \frac{1}{|x|\sqrt{2\pi\sigma^2}}e^{-(y/x)^2/(2\sigma^2)} \quad \forall x \neq 0 \tag{3}$$

and

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \frac{\frac{1}{|x|(2\pi\sigma^2)}\exp(-(y/x)^2/(2\sigma^2))\exp(-x^2/(2\sigma^2))}{f_Y(y)} \tag{4}$$

where $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx$ and $f_X(x)$ is given and we have already solved for $f_{Y|X}(y|x)$.

b) Doing the PDF transformation $(U, V) = (X, Y) = (X, RX)$ gives a Jacobian of $|J| = |x|$, indeed

$$J = \begin{bmatrix} dx/dx & dx/dr \\ dy/dx & dy/dr \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ r & x \end{bmatrix} \implies |J| = |x| \tag{5}$$

and so

$$f_{X,Y}(x,y) = \frac{f_{X,R}(x, y/x)}{|x|} = \frac{f_X(x) f_R(y/x)}{|x|} = \frac{\frac{1}{2\pi\sigma^2} \exp(-x^2/(2\sigma^2)) \exp(-(y/x)^2/(2\sigma^2))}{|x|} \tag{6}$$

and so we get the same solution as part (a):

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{|x|(2\pi\sigma^2)} \exp(-(y/x)^2/(2\sigma^2)) \exp(-x^2/(2\sigma^2))}{f_Y(y)} \tag{7}$$

where $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$ and we have already solved for $f_{X,Y}(x,y)$.
Assuming $x \neq 0$, you can notice that

$$f_{X|Y}(x|y) = \frac{1}{c(y)} \cdot \frac{1}{|x|} \exp\left(\frac{-1}{2\sigma^2} \left[(y/x)^2 + x^2 \right] \right) \tag{8}$$

where for each fixed y, the value $c(y)$ makes the conditional PDF (given $Y = y$) integrate to 1, and so we must have

$$c(y) = \int_{-\infty}^{\infty} \frac{1}{|x|} \exp\left(\frac{-1}{2\sigma^2} \left[(y/x)^2 + x^2 \right] \right) dx \tag{9}$$

IV. Linear Combination of Gaussian

Let X, Y, Z be mutually independent Gaussian random variables. Assume X has mean 2 and variance 4, Y has mean -1 and variance 1, Z has mean 5 and variance $1/4$. Let $W = 2Z - X + 3Y/2$. Compute the PDF $f_W(w)$ for all $w \in \mathbb{R}$. Use the $Q()$ function to find $P[W > -10]$.

Solution:
$W \sim \mathcal{N}(2\mu_Z - \mu_X + 3\mu_Y/2, 4\sigma_Z^2 + \sigma_X^2 + 9\sigma_Y^2/4)$
$\mu_W = 10 - 2 - 3/2 = 6.5$
$\sigma_W^2 = 1 + 4 + 9/4 = 29/4$
$P[W > -10] = Q(\frac{-10-6.5}{\sqrt{29/4}})$

V. Characteristic Functions

a) Let X be a Bernoulli random variable with parameter p. Find the characteristic function of X.

b) Let X be a binomial random variable with parameter p. Find the characteristic function of X and comment on the relation between the Bernoulli and binomial random variables.

c) Let X be a Poisson random variable with parameter λ. Find the characteristic function of X.

Solution:

a)

$$\phi_X(x) = \mathbb{E}(e^{i\omega X}) \tag{10}$$
$$= (1-p)e^{i\omega 0} + pe^{i\omega 1} \tag{11}$$
$$= 1 - p + pe^{i\omega} \tag{12}$$

b)

$$\phi_X(x) = \mathbb{E}(e^{i\omega X}) \tag{13}$$
$$= \sum_{k=0}^{n} \binom{n}{k} e^{i\omega k} p^k (1-p)^{n-k} \tag{14}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (e^{i\omega} p)^k (1-p)^{n-k} \tag{15}$$
$$= (1 - p + pe^{i\omega})^n \tag{16}$$

Using the binomial theorem. We notice that the characteristic function of a binomial random variable is the product of n Bernoulli characteristic functions. So a binomial random variable is a sum of n i.i.d Bernoulli random variables c)

$$\phi_X(x) = \mathbb{E}(e^{i\omega X}) \tag{17}$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{i\omega k} \tag{18}$$
$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{i\omega})^k}{k!} \tag{19}$$
$$= e^{-\lambda} e^{\lambda e^{i\omega}} \tag{20}$$
$$= e^{\lambda(e^{i\omega}-1)} \tag{21}$$

VI. Another PDF transformation Problem

Suppose that X follows a $\mathcal{U}[-3, 2]$ distribution. Let $Y = 3X^2 - 1/2$. Find the PDF of Y.
_Solution:_For $-2 \leq x \leq 2$ we have $-1/2 \leq y \leq 11.5$

$$g(x) = 3x^2 - 1/2 \tag{22}$$
$$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{|g'(x_i)|} \tag{23}$$
$$x_{1,2} = \pm\sqrt{(1/3)(y + 1/2)} \tag{24}$$
$$f_Y(y) = \sum_{i=1}^{2} \frac{1/5}{|6x_i|} \tag{25}$$
$$f_Y(y) = \frac{2/5}{6(\sqrt{(1/3)(y + 1/2)})} \tag{26}$$

For $-3 \leq x \leq -2$ we have $11.5 \leq y \leq 26.5$

$$g(x) = 3x^2 - 1/2 \tag{27}$$
$$f_Y(y) = \sum_{i=1}^{2} \frac{f_X(x_i)}{|g'(x_i)|} \tag{28}$$
$$x_1 = -\sqrt{(1/3)(y + 1/2)} \tag{29}$$
$$f_Y(y) = \frac{1/5}{|6x_i|} \tag{30}$$
$$f_Y(y) = \frac{1/5}{6(\sqrt{(1/3)(y + 1/2)})} \tag{31}$$

VII. Book Problem 5.35

For part (c), your answer should depend on the value of p. Find out the values of p for which $X = 1$ is more likely and the values of p for which $X = -1$ is more likely.
_Solution:_a) For $j = -1$

$$P[X = j, Y \leq y] = P[X = -1, N - 1 \leq y] = P[X = -1, N \leq y + 1] = (1 - p)P[N \leq y + 1] \tag{32}$$
$$= (1 - p)\Phi(\frac{y + 1}{0.5}) \tag{33}$$

For $j = 1$

$$P[X = j, Y \leq y] = P[X = 1, N - 1 \leq y] = P[X = 1, N \leq y - 1] = pP[N \leq y - 1] \tag{34}$$
$$= p\Phi(\frac{y - 1}{0.5}) \tag{35}$$

b)

$$P[X=1]=p, \quad P[X=-1]=1-p \tag{36}$$
$$F_Y(y)=(1-p)P[Y\le y, X=-1]+pP[Y\le y, X=1] \tag{37}$$
$$=(1-p)P[N-1\le y]+pP[N+1\le y] \tag{38}$$
$$=\int_{-\infty}^{y}\frac{(1-p)e^{-\frac{(t+1)^2}{2(0.25)}}}{0.5\sqrt{2\pi}}dt+\int_{-\infty}^{y}\frac{pe^{-\frac{(t-1)^2}{2(0.25)}}}{0.5\sqrt{2\pi}}dt \tag{39}$$
$$f_Y(y)=\frac{d}{dy}F_Y(y) \tag{40}$$
$$=(1-p)e^{-2(y+1)^2}\sqrt{2/\pi}+pe^{-2(y-1)^2}\sqrt{2/\pi} \tag{41}$$

c) Test for $X=1$:

$$P[X=1|Y>0]=\frac{P[X=1,Y>0]}{P[Y>0]} \tag{42}$$
$$=\frac{P[X=1,N>-1]}{(1-p)P[N>1]+pP[N>-1]} \tag{43}$$
$$=\frac{pQ(-1/0.5)}{(1-p)Q(1/0.5)+pQ(-1/0.5)} \tag{44}$$
$$=\frac{0.9772p}{0.0228+0.9544p} \tag{45}$$

$P[X=1|Y>0]>1/2$ when $0.0228<p\le 1$
and $<1/2$ when $0\le p\le 0.0228$.
Therefore $X=1$ more likely when $p\in[0.0228,1]$

VIII. Book Problem 5.101

Solution:

$$f_{X,Y}(x,y)=\frac{1}{2\pi}e^{-(\frac{x^2+y^2}{2})} \tag{46}$$
$$Z=X/Y \tag{47}$$
$$f_Z(z)=\int_{-\infty}^{\infty}|y|f_{X,Y}(zy,y)dy \tag{48}$$
$$=(1/2\pi)\int_{-\infty}^{\infty}|y|e^{-(\frac{(zy)^2+y^2}{2})}dy \tag{49}$$
$$=(1/\pi)\int_{0}^{\infty}ye^{-y^2(\frac{z^2+1}{2})}dy \tag{50}$$

But $\int_0^\infty ye^{-ay^2}dy=[(-1/2a)e^{-ay^2}]_0^\infty=1/2a$
So $f_Z(z)=\frac{1}{\pi(1+z^2)}$. Z is a Cauchy RV with $\alpha=1$

IX. Book Problem 4.92 (a) and (b)

*Solution:*a)

$$f_X(x)=4x(1-x^2), \quad 0\le x\le 1 \tag{51}$$
$$Y=\pi X^2, \quad \frac{dy}{dx}=2\pi x_1, \quad x_1=\sqrt{y/\pi} \tag{52}$$

Only the positive root since x non-negative

$$f_Y(y)=\frac{f_X(x_1)}{|2\pi x_1|} \tag{53}$$
$$=\frac{4\sqrt{\frac{y}{x}}(1-\frac{y}{\pi})}{|2\pi\sqrt{\frac{b}{\pi}}|} \tag{54}$$
$$=\frac{2}{\pi}(1-\frac{y}{\pi}), \quad 0<y<\pi \tag{55}$$

b)

$$Y = \frac{4}{3}\pi X^3 \tag{56}$$

$$x_1 = (\frac{3}{4\pi}y)^{1/3}, \quad \frac{dy}{dx} = 4\pi x^2 \tag{57}$$

$$f_Y(y) = \frac{f_X(x_1)}{|4\pi x_1^2|} = \frac{4x_1(1-x_1)^2}{4\pi x_1^2} = \frac{1-x_1^2}{\pi x_1} \tag{58}$$

$$= \frac{1-(\frac{3}{4\pi}y)^{2/3}}{\pi(\frac{3}{4\pi}y)^{1/3}}, \quad 0 < y < \frac{4}{3}\pi \tag{59}$$