EE 503: Problem Set #1 : Solutions

- Reading: Notes on "Sets, Infinity, and Mappings":
https://viterbi-web.usc.edu/~mjneely/docs/infinity.pdf
- Submit your homework in D2L by 9pm on the due date.

I. Infinity notes Exercise 1

Solution:
a) $b = 0.22020...$ and so $b \in [0, 1)$.
b) No. The decimal expansion of b has only 0s and 2s, it does not have an infinite tail of 9s.
c) Claim: $b \notin \{x_1, x_2, x_3, \ldots\}$. Proof: For every positive integer j we know that $b_j \neq a_{j,j}$. So for every positive integer j, the unique decimal expansion of b differs from the unique decimal expansion of x_j, and so $b \neq x_j$.

II. Infinity notes Exercise 3

Solution:
a) Observe that $0 \leq 0.3333... \leq y \leq 0.5555... < 1$ and so $y \in [0, 1)$. Note that the decimal expansion of y is $y = 0.b_1b_2b_3\ldots$ and this does not have an infinite tail of 9s. By construction we have $b_k \neq a_{k,k}$ for all positive integers k, so the kth digit of the unique decimal expansion of y differs from the kth digit of the unique decimal expansion of x_k. So $y \neq x_k$ for all positive integers k. So y is not in the list $\{x_1, x_2, x_3, \ldots\}$. Since this list includes all rational numbers in $[0, 1)$, and $y \in [0, 1)$, we know that y is irrational.
b) Suppose z is rational (we reach a contradiction). Then the sequence $\{a_{1,1}, a_{2,2}, a_{3,3}, \ldots\}$ must be eventually repeating, which means the sequence $\{b_1, b_2, b_3, \ldots\}$ must be eventually repeating, which means y is rational, contradicting part (a).

III. Infinity notes Exercise 4

Solution:
a) We have $H = \{2, 5, 6, 7\}$. There is no i for which $L_i = H$.
b) No, it is impossible. Suppose there is an i for which $L_i = H$ (we reach a contradiction).
Case 1: Suppose $i \in L_i$. Then student i is not humble so $i \notin H$. But $H = L_i$, so $i \notin L_i$, a contradiction.
Case 2: Suppose $i \notin L_i$. Then student i is humble so $i \in H$. But $H = L_i$ so $i \in L_i$, a contradiction.

IV. Infinity notes Exercise 5

Solution:

$$\begin{aligned}
[a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\
(a, \infty) &= \{x \in \mathbb{R} : a < x\}
\end{aligned}$$

V. Infinity notes Exercise 10

Solution:
$S = \{(H, x) : x \in [0, 1]\} \cup \{(T, blue), (T, green), (T, red)\}$. We have $|S| =$ uncountably infinite because there are uncountably many objects (H, x) such that $x \in [0, 1]$.

VI. Infinity notes Exercise 11

$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. $|S| =$ uncountably infinite because the uncountably many points $(x, 0)$ such that $-1 \leq x \leq 1$ all are in S.
Alternatively, we could describe S in terms of radius and angle:

$$S = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$$

VII. Infinity notes Exercise 14

Solution:
a) $S = \cup_{i=0}^{\infty} A_i$ where we define

$$A_0 = \{(x_1, x_2, x_3, \dots) : x_i \geq 0 \quad \forall i \in \{1, 2, 3, \dots\}\}$$
$$A_1 = (-\infty, 0)$$

and for each $i \in \{2, 3, 4, \dots\}$ we define

$$A_i = \{(x_1, \dots, x_i) \in \mathbb{R}^i : x_j \geq 0 \quad \forall j \in \{1, \dots, i-1\}, x_i < 0\}$$

We have $|S|$ is uncountably infinite because S contains A_1, and the set A_1 is the set of all negative real numbers, which is uncountably infinite.

b) $S = \{(-1), (1, -1), (1, 1, -1), (1, 1, 1, -1), \dots\} \cup (1, 1, 1, 1, \dots)$ We have $|S|$ is countably infinite.

c) $S = \{(-1), (1, -1), (1, 1, -1), (1, 1, 1, -1), \dots\}$ and so $|S|$ is again countably infinite.

VIII. Infinity notes Exercise 16

Solution:
a) Define the injective function $f : B \to A$ by

$$f(3, 4) = red, f(3.3, 9.786) = green$$

b) Define the surjective function $f : A \to B$ by

$$f(red) = (3, 4), f(green) = (3, 4), f(blue) = (3.3, 9.786)$$

c) Define the bijective function $f : B \to C$ by

$$f(3, 4) = (cat, 5), f(3.3, 9.786) = (dog, 8 : 00pm, blue)$$

IX. Infinity notes Exercise 20

Solution:
a) Yes: Suppose (x, y) and (a, b) are points in $\mathbb{N}^2$ such that

$$h(x, y) = h(a, b)$$

We want to show that $(x, y) = (a, b)$. We know

$$f(x, xy + 1) = f(a, ab + 1)$$

Since f is injective we obtain

$$(x, xy + 1) = (a, ab + 1)$$

Thus $x = a$ and $xy + 1 = ab + 1$. It follows that $xy = ab$ and since $x = a$ we get $xy = xb$. Since $x \in \mathbb{N}$ we know $x \neq 0$. Dividing the equation $xy = xb$ by x gives $y = b$. Thus $(x, y) = (a, b)$.

b) No: We know that $f(1, 1) \in \mathbb{N}$. Suppose there is an $(x, y) \in \mathbb{N}^2$ such that $h(x, y) = f(1, 1)$ (we reach a contradiction). Then

$$f(x, xy + 1) = f(1, 1)$$

and since f is injective we obtain

$$(x, xy + 1) = (1, 1)$$

It follows that $x = 1$ and $xy + 1 = 1$, so $xy = 0$. So $y = 0$. This contradicts the fact that $y \in \mathbb{N}$.

X. Infinity notes Exercise 21

Solution:

- h is not injective: This is because $h(1, 3) = h(3, 2)$. Indeed:

$$h(1, 3) = f(7, 28)$$
$$h(3, 2) = f(7, 28)$$

- h is not surjective. We know $f(1, 1) \in \mathbb{N}$. Suppose there is an $(x, y) \in \mathbb{N}^2$ such that $h(x, y) = f(1, 1)$ (we reach a contradiction). Then

$$f(x + 2y, 4x + 8y) = f(1, 1)$$

  Since f is injective it follows that $(x + 2y, 4x + 8y) = (1, 1)$. So $x + 2y = 1$ and $4x + 8y = 1$. But multiplying $x + 2y = 1$ by 4 gives the contradiction $4x + 8y = 4 \neq 1$.

XI. Infinity notes Exercise 40

Solution:
a) $f(4,1) = 10$, $f(4,2) = 14$
b)

- $h(1,1,1) = f(f(1,1),1) = f(1,1) = 1$
- $h(1,2,1) = f(f(1,2),1) = f(2,1) = 3$
- $h(3,2,2) = f(f(3,2),2) = f(9,2) = f(10,1) - 1 = 55 - 1 = 54$

c) $h^{-1}(9) = (x,y,z) \Rightarrow h(x,y,z) = 9 = f(f(x,y),z)$. Since $f(3,2) = 9$ (From Fig. 1), $f(f(x,y),z) = 9$, and f is bijective, then, $f(x,y) = 3$ and $z = 2$. Similarly, $f(2,1) = 3$ (From Fig. 1), $f(x,y) = 3$, and f is bijective, then, $(x,y) = (2,1)$. Hence, $(x,y,z) = (2,1,2)$

XII. Infinity notes Exercise 41

Solution:
a) Suppose (a,b,c) and (x,y,z) are in $\mathbb{N}^3$ and $h(a,b,c) = h(x,y,z)$. We want to show $(a,b,c) = (x,y,z)$. We know

$$f(f(a,b),c) = f(f(x,y),z)$$

Since f is injective this implies

$$(f(a,b),c) = (f(x,y),z)$$

and so $c = z$ and $f(a,b) = f(x,y)$. Again since f is injective we know $(a,b) = (x,y)$, that is, $a = x$ and $b = y$.

b) Fix $y \in \mathbb{N}$. Since f is surjective there is an $(a,b) \in \mathbb{N}^2$ such that $f(a,b) = y$. Again since f is surjective there is a $(c,d) \in \mathbb{N}^2$ such that $f(c,d) = a$. Thus

$$h(c,d,b) = f(f(c,d),b) = f(a,b) = y$$

c) Statement: Suppose $f : A \times A \to A$ is bijective. Define $h : A \times A \times A \to A$ by $h(a,b,c) = f(f(a,b),c)$. Then h is both injective and surjective.

Proof: The injective argument is the same as (a) with the exception that we start by assuming (a,b,c) and (x,y,z) are in A^3. The surjective argument is the same as (b) with the exception that we start by fixing $y \in A$ (we also choose $(a,b) \in A^2$ and $(c,d) \in A^2$).

Notation: Note that $A^2 = A \times A$ and $A^3 = A \times A \times A$.