EE 503: Problem Set #7 : Solutions

It can be shown that: $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$; $\sum_{k=0}^{n} k^2\binom{n}{k} = 2^{n-2}n(n+1)$; $\sum_{k=0}^{n} x^k\binom{n}{k} = (x+1)^n$.

I. Expectation

Suppose a fair coin is tossed n times.

(a) Each coin toss costs d dollars and the reward in obtaining X heads is $aX^2 + bX$. Find the expected value of the net reward.

(b) Suppose that the reward in obtaining X heads is a^X, where $a > 0$. Find the expected value of the reward.

Solution:

(a) The expected value of the reward is obtained as

$$\mathbb{E}[aX^2 + bX] = a\ \mathbb{E}[X^2] + b\ \mathbb{E}[X]$$

The probability of obtaining X heads in tossing the fair coin n times is obtained as

$$P[X = k] = \binom{n}{k}(\frac{1}{2})^{n-k}\left(\frac{1}{2}\right)^k = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

Then, we can find the following terms

$$\mathbb{E}[X] = \sum_{k=0}^{n} kP[X = k] = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} k\binom{n}{k} = \frac{n}{2}.$$

$$\mathbb{E}[X^2] = \sum_{k=0}^{n} k^2P[X = k] = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} k^2\binom{n}{k} = \frac{n(n+1)}{4}.$$

Hence, the expected value of the reward is given as

$$\mathbb{E}[aX^2 + bX] = \frac{n}{2}\left(b + a\frac{n+1}{2}\right)$$

So, the expected net reward is given by

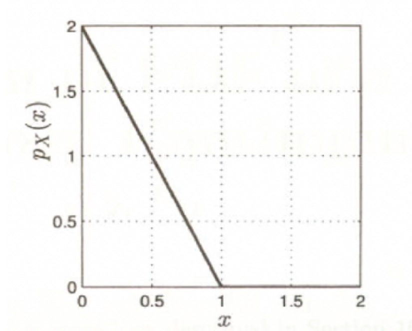
$$\frac{n}{2}\left(b + a\frac{n+1}{2}\right) - nd.$$

(b) The expected value of the reward is given by

$$\mathbb{E}[a^X] = \sum_{k=0}^{n} a^kP[X = k] = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} a^k\binom{n}{k} = \left(\frac{a+1}{2}\right)^n.$$

II. Nonlinear Transformation

Find a bijective transformation so that $X = g(U)$, where $U \sim \mathcal{U}[0, 1]$ has the PDF $p_X(x)$ as shown in the figure below



Solution:

We can observe that the PDF of X is given by

$$p_X(x) = \begin{cases} 2-2x & ; \quad 0 \le x \le 1 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

Hence the CDF of X is expressed as

$$F_X(x) = \begin{cases} 0 & ; \quad x < 0 \\ 2x - x^2 & ; \quad 0 \le x < 1 \\ 1 & ; \quad x \ge 1 \end{cases}$$

The CDF of the random variable U is given by

$$F_U(u) = \begin{cases} 0 & ; \quad u < 0 \\ u & ; \quad 0 \le u < 1 \\ 1 & ; \quad u \ge 1 \end{cases}$$

Then, $F_X(x) = P[X \le x] = P[g(U) \le x] = P[U \le g^{-1}(x)] = F_U(g^{-1}(x)) = g^{-1}(x) \quad \text{as} \quad ; \quad 0 \le g^{-1}(x) \le 1$.
So, $F_X(x) = g^{-1}(x) = u \Rightarrow 2x - 2x^2 - u = 0 \Rightarrow x_1 = 1 - \sqrt{1-u} \in [0,1]$ or $x_2 = 1 + \sqrt{1-u} \in [1,2]$. Therefore,

$$X = 1 - \sqrt{1-U}.$$

III. Conditioning on random vectors

Let X and Y be independent random variables with $P[X = k] = p(1-p)^{k-1}$ and $P[Y = k] = g_k$ for all $k \in \{1, 2, 3, ...\}$, where values p and $\{g_k\}_{k=1}^{\infty}$ are given and satisfy $0 < p < 1$, $g_k \ge 0$, and $\sum_{k=1}^{\infty} g_k = 1$. We observe (X, Y) and build a new random variable Z as follows:

- If $X \le 3$ then Z is chosen uniformly over the discrete set of integers $\{1, 2, 3, ..., 100\}$.
- If $X > 3$ then Z is exponentially distributed with rate $\lambda = XY$.

a) Compute $P[Z > 10]$.
b) Compute $\mathbb{E}[Z]$.
Solution:
a) We have by the law of total probability:

$$\begin{aligned} P[Z > 10] &= \sum_{k=1}^{3} P[Z > 10 | X = k] p(1-p)^{k-1} + \sum_{k=4}^{\infty} \sum_{i=1}^{\infty} P[Z > 10 | X = k, Y = i] p(1-p)^{k-1} g_i \\ &= \sum_{k=1}^{3} \frac{90}{100} p(1-p)^{k-1} + \sum_{k=4}^{\infty} \sum_{i=1}^{\infty} e^{-10ik} p(1-p)^{k-1} g_i \end{aligned}$$

b) We have by the law of total expectation:

$$\begin{aligned} \mathbb{E}[Z] &= \sum_{k=1}^{3} \mathbb{E}[Z | X = k] p(1-p)^{k-1} + \sum_{k=4}^{\infty} \sum_{i=1}^{\infty} \mathbb{E}[Z | X = k, Y = i] p(1-p)^{k-1} g_i \\ &= \sum_{k=1}^{3} \frac{1+100}{2} p(1-p)^{k-1} + \sum_{k=4}^{\infty} \sum_{i=1}^{\infty} \frac{1}{ik} p(1-p)^{k-1} g_i \end{aligned}$$

IV. Independence in a table

Let X and Y be independent random variables with:

- $P[X = 0] = .4, P[X = 1] = .3, P[X = 2] = .3$.
- $P[Y = 0] = .5, P[Y = 1] = .4, P[Y = 2] = .1$.

a) Fill in the table for the joint PMF function $P[X = x, Y = x]$.

PMF	X=0	X=1	X=2
Y=0			
Y=1			
Y=2			

b) Compute $P[(X, Y) \in \{(0, 1), (0, 2), (1, 1)\}$.
c) Compute $P[X + Y = 2]$.
d) Compute $\mathbb{E}[X], \mathbb{E}[Y], \mathbb{E}[XY]$ and $\mathbb{E}\left[X^2 Y\right]$.

e) Compute $\mathbb{E}\left[2^{XY}\right]$.
f) Compute $\mathbb{E}\left[X^2|XY=2\right]$.
Solution:
a) The table is:

PMF	X=0	X=1	X=2
Y=0	.2	.15	.15
Y=1	.16	.12	.12
Y=2	.04	.03	.03

b) $P[(X,Y) \in \{(0,1),(0,2),(1,1)\} = .16 + .04 + .12$.
c) $P[X+Y=2] = .04 + .12 + .15$.
d) $\mathbb{E}[X] = .9$, $\mathbb{E}[Y] = .6$, $\mathbb{E}[XY] = .54$, $\mathbb{E}\left[X^2Y\right] = 0.9$.
e) $\mathbb{E}\left[2^{XY}\right] = 1(.2 + .15 + .15 + .16 + .04) + 2(.12) + 2^2(.12) + 2^2(.03) + 2^4(.03)$.
f) We have:

$$\begin{aligned}\mathbb{E}\left[X^2|XY=2\right] &= 0P[X=0|XY=2] + 1P[X=1|XY=2] + 4P[X=2|XY=2] \\ &= \frac{P[X=1,Y=2]}{P[X=1,Y=2]+P[X=2,Y=1]} + \frac{4P[X=2,Y=1]}{P[X=1,Y=2]+P[X=2,Y=1]}\end{aligned}$$

V. Design a random vector

Design a discrete random vector (X,Y) such that all of the following hold: (i) $X \in \{0,1,2\}$ and $Y \in \{0,1,2\}$, (ii) X and Y are *not* independent, (iii) $P[X=2] = 1/2$, (iv) The events $\{X=2\}$ and $\{Y=1\}$ are independent.
Solution:
Why not make the example as easy as possible? Suppose (X,Y) have joint mass function:

$$\begin{aligned}&P[X=0,Y=0]=1/2\\&P[X=0,Y=1]=0\\&P[X=0,Y=2]=0\\&P[X=1,Y=0]=0\\&P[X=1,Y=1]=0\\&P[X=1,Y=2]=0\\&P[X=2,Y=0]=0\\&P[X=2,Y=1]=0\\&P[X=2,Y=2]=1/2\end{aligned}$$

Then indeed $P[X=2] = P[Y=2] = 1/2$, but

$$P[X=2,Y=2] = 1/2 \neq 1/4 = P[X=2]P[Y=2]$$

so X and Y are not independent. However the desired criteria are satisfied:

- $X, Y \in \{0,1,2\}$.
- X and Y are not independent.
- $P[X=2] = 1/2$.
- $P[Y=1] = 0$ and so $\{Y=1\}$ is independent of every event, including $\{X=2\}$.

VI. Covariance

Let X and Y be independent random variables each is uniformly distributed over $[0,1]$ i.e., each with distribution $\mathcal{U}(0;1)$. We define the random variables U and V as $U = \min(X;Y)$ and $V = \max(X;Y)$.
(a) Find $\mathbb{E}[U]$ and $\mathbb{E}[V]$.
(b) Find $cov[U,V]$.
Solution:
(a) The CDF of the random variable U is given as

$$F_U(u) = 1 - P(X \geq u)P(Y \geq u) = 1 - (1-u)^2 \quad ; \quad u \in [0,1]$$

Hence,

$$f_U(u) = \begin{cases} -2u+2 & \text{if } u \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\mathbb{E}[U] = \frac{1}{3}$$

The CDF of the random variable V is given as

$$F_V(v) = P(X \le v)P(Y \le v) = v^2 \quad ; \quad v \in [0,1]$$

Hence,

$$f_V(v) = \begin{cases} 2v & \text{if } v \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Thus,

$$\mathbb{E}[V] = \frac{2}{3}$$

(b) $UV = XY$, so

$$\mathbb{E}[UV] = \mathbb{E}[XY] = \frac{1}{4}$$

Therefore,

$$cov[U,V] = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = \frac{1}{4} - \frac{1}{3} \times \frac{2}{3} = \frac{1}{36}$$

VII. Conditional Expectation I

Suppose X and Y are random variables and Y is discrete.
Suppose $\mathbb{E}[X|Y=y] = E[X]$ for all $y \in S_Y$.
(a) Show that X and Y are uncorrelated.
(b) Give an example to show that X and Y need not be independent.
Solution:
a) Given $\mathbb{E}[X|Y=y] = \mathbb{E}[X]$ for all $y \in S_Y$, we can write

$$\begin{aligned} cov[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \sum_{y \in S_Y} \mathbb{E}[XY|Y=y]P[Y=y] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \sum_{y \in S_Y} y\mathbb{E}[X|Y=y]P[Y=y] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \sum_{y \in S_Y} y\mathbb{E}[X]P[Y=y] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[X] \sum_{y \in S_Y} yP[Y=y] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[X]\mathbb{E}[Y] = 0. \end{aligned}$$

Hence X and Y are uncorrelated.

b) We can design the following example: we define Y to be a Bernoulli random variable with parameter $p = \frac{1}{2}$. So, $P[Y=0] = \frac{1}{2}$ and $P[Y=1] = \frac{1}{2}$. We observe the outcome of the of the random variable Y and then we define a new random variable X according to the following way. If $Y=1$, we assign the value $\frac{1}{2}$ to X. If $Y=0$, X is a Bernoulli random variable with parameter $\frac{1}{2}$, (i.e., $P[X=0] = P[X=1] = \frac{1}{2}$).
Based on the described experiment, we have the following table summurizing the joint PMF:

PMF	X=0	X=$\frac{1}{2}$	X=1
Y=0	$\frac{1}{4}$	0	$\frac{1}{4}$
Y=1	0	$\frac{1}{2}$	0

We can see that:

$$\mathbb{E}[X|Y=0] = \mathbb{E}[X|Y=1] = \frac{1}{2} = \mathbb{E}[X]$$

However,

$$P[X=0, Y=1] = 0 \neq \frac{1}{8} = \frac{1}{4} \times \frac{1}{2} = P[X=0] \times P[Y=1]$$

So, X and Y are not independent.

VIII. Conditional Expectation II

Let X be a random variable that follows Bernoulli with parameter $p = \frac{2}{3}$ and suppose that given $X = i$, Y follows Poisson distribution with parameter $3(i+1)$. Find $\mathbb{E}[(X+1)Y^2]$.

Solution:

$$\begin{aligned}\mathbb{E}[(X+1)Y^2] &= \mathbb{E}[(X+1)Y^2|X=0]P[X=0] + \mathbb{E}[(X+1)Y^2|X=1]P[X=1]\\ &= \mathbb{E}[Y^2|X=0]P[X=0] + 2\times\mathbb{E}[Y^2|X=1]P[X=1]\\ &= \frac{1}{3}\mathbb{E}[Y^2|X=0] + \frac{4}{3}\times\mathbb{E}[Y^2|X=1]\\ &= \frac{1}{3}\times(3^2+3) + \frac{4}{3}\times(6^2+6) = 60.\end{aligned}$$

For a Poisson random variable X with parameter λ, we have $\mathbb{E}[X^2] = var(X) + (\mathbb{E}[X])^2 = \lambda^2 + \lambda$.

IX. Book Problem 5.11

Solution:

(i)

$$P(X=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\} \qquad ; \qquad P(Y=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\}$$

$$P(X>0) = \frac{1}{3} \qquad ; \qquad P(X \geq Y) = \frac{1}{2} \qquad ; \qquad P(X=-Y) = \frac{1}{6}$$

(ii)

$$P(X=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\} \qquad ; \qquad P(Y=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\}$$

$$P(X>0) = \frac{1}{3} \qquad ; \qquad P(X \geq Y) = \frac{2}{3} \qquad ; \qquad P(X=-Y) = \frac{1}{3}$$

(iii)

$$P(X=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\} \qquad ; \qquad P(Y=i) = \frac{1}{3} \;\; ; \;\; i \in \{-1,0,1\}$$

$$P(X>0) = \frac{1}{3} \qquad ; \qquad P(X \geq Y) = 1 \qquad ; \qquad P(X=-Y) = \frac{1}{3}$$

X. Decorrelating Random Variables

If X and Y have a covariance of $cov[X,Y]$, we can transform them to a new pair of random variables whose covariance is zero. To do so we let

$$W = X$$

$$Z = aX + Y$$

a) Express $cov[X, X+Y]$ in terms of $var[X]$ and $cov[X,Y]$.

b) Using your result from part a), find the value of a such that $cov[W,Z] = 0$. This process is called decorrelating the random variables.

Solution:

a) $cov[X, X+Y] = var[X] + cov[X,Y]$.

b) We have $cov[W,Z] = a \times var[X] + cov[X,Y]$. So, $cov[W,Z] = 0$ implies $a = -\frac{cov[X,Y]}{varX]}$.