EE 503: Problem Set #4 : Solutions

- Reading: Chapter 2,3.1-3.2,4.1 in Leon-Garcia textbook.
- Submit your homework in D2L by 9pm on the due date.

I. Leon-Garcia Book Problem 2.97 (Clarified version)

Blocks of 100 bits are transmitted over a binary communications channel. Any individual bit has a probability of error $p = 10^{-2}$.

(a) If any transmitted block has 1 or fewer errors, then the receiver accepts the block. Find the probability that the block is accepted after the first transmission.

(b) If any transmitted block has more than 1 error, it is transmitted again as many times as needed until it is accepted. The process ends when the block is accepted. Find the probability that a total of M transmissions (including the first) are required until the block is accepted.

Solution:

NOTE:
Binomial Random Variable: The probability of exactly k successes in n independent Bernoulli trials, with the same probability of success p in each trial, is

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

where X is the number of successes.
Geometric Random Variable: Consider a sequence of independent Bernoulli trials with the same probability of success p in each trial. The probability that exactly k trials are required until the occurrence of the first success is

$$P(X = k) = (1-p)^{k-1} p \tag{2}$$

i.e., failures in all the first $k-1$ trials and success in the k^{th} trial, where X is the number of trials to (and including) the first occurrence of success.

(a) In each block, there are 100 bits and the probability of error in any bit is $p = 10^{-2}$. If there is no or one bit error, the block is accepted. Then the probability of acceptance of a block is

$$\begin{aligned}
p_a &= P(\text{no bit error or one bit error in 100 bits}) \\
&= P(\text{no bit error in 100 bits}) + P(\text{one bit error in 100 bits}) \\
&= \binom{100}{0} p^0 (1-p)^{100-0} + \binom{100}{1} p^1 (1-p)^{100-1} \\
&= 0.99^{100} + 0.99^{99} \approx 0.7358
\end{aligned}$$

(b) From part (a), the probability of acceptance of any block is $p_a = 0.7358$, then the probability that exactly M transmissions are required until the block is accepted is

$$(1-p_a)^{M-1} p_a = (0.2642)^{M-1}(0.7358) = (0.2642)^M (2.78),$$

i.e., failures in all the first $M-1$ transmissions and success in the M^{th} transmission.

II. Non-fair Coin

We have two coins: the first is fair and the second has two heads. We pick one of the coins at random, we toss it twice and heads shows both times. Find the probability that the fair coin was picked.

Solution:

Denote by C_1 and C_2 the events Coin 1 was picked and Coin 2 was picked, respectively. Then

$$\begin{aligned}
&P(hh|C_1) = \frac{1}{4} \text{ and } P(hh|C_2) = 1 \\
&\Rightarrow P(hh) = P(hh|C_1)P(C_1) + P(hh|C_2)P(C_2) = \frac{5}{8}.
\end{aligned}$$

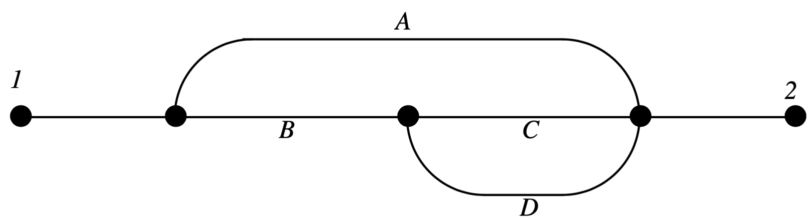
According to Baye's Rule

$$P(C_1|hh) = \frac{P(hh|C_1)P(C_1)}{P(hh)} = \frac{1}{5}.$$

In other words, if two heads are observed, it is four times more likely that C_2 was the coin chosen rather than C_1.

III. NETWORK TRANSMISSION

See the network below. All unlabeled links never fail. The probability that links A, B, C, D transmit are a, b, c, d respectively. All links behave independently.



(a) Find a set expression for the event that transmission exists between points 1 and 2.
(b) Find the probability of transmission between 1 and 2.
(c) Given that a transmission exists between 1 and 2, find the probability that link B is transmitting.

Solution:
(a) There are three paths from point 1 to point 2: A, BC, and BD, then the event that transmission exists can be expressed as union of the three events: $T = A \cup BC \cup BD$.

(b) By applying inclusion-exclusion principle,

$$\begin{aligned} P(T) &= P(A \cup BC \cup BD) \\ &= P(A) + P(BC) + P(BD) - P(ABC) - P(ABD) - P(BCD) + P(ABCD). \end{aligned}$$

Since all links behave independently,

$$P(T) = a + bc + bd - abc - abd - bcd + abcd.$$

(c)

$$P(B|T) = \frac{P(BT)}{P(T)} = \frac{P(AB \cup BC \cup BD)}{P(T)}$$

$$\begin{aligned} P(AB \cup BC \cup BD) &= P(AB) + P(BC) + P(BD) - P(ABC) - P(ABD) - P(BCD) + P(ABCD) \\ &= ab + bc + bd - abc - abd - bcd + abcd \end{aligned}$$

Then,

$$P(B|T) = \frac{ab + bc + bd - abc - abd - bcd + abcd}{a + bc + bd - abc - abd - bcd + abcd}.$$

IV. DISEASE TESTING

Suppose we have two disease testers with two outputs $Y_1 \in \{0, 1\}$ and $Y_2 \in \{0, 1\}$ (where 1 plays the role of "positive" and 0 plays the role of "negative"). Let H and H^c be the events of having and not having the disease. Assume:

- Tester 1: ($p_{detect} = 0.85, p_{falsealarm} = 0.15$). Thus, $P[Y_1 = 1|H] = 0.85, P[Y_1 = 1|H^c] = 0.15$.
- Tester 2: ($p_{detect} = 0.75, p_{falsealarm} = 0.05$). Thus, $P[Y_2 = 1|H] = 0.75, P[Y_2 = 1|H^c] = 0.05$.

Suppose that when a person visits the medical clinic, a biased coin is flipped (independent of whether or not the person has the disease) to determine which test the person will take. Let T_1 be the event that the person takes test 1, and T_1^c the event that the person takes test 2. Assume that T_1 and H are independent, and $P[T_1] = \theta$ for some $\theta \in [0, 1]$. Define Z by

$$Z = \begin{cases} 1 & \text{if the person takes test 1 and } Y_1 = 1\text{, or the person takes test 2 and } Y_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the new detection and false alarm probabilities $p_{Detect} = P[Z = 1|H]$ and $p_{FalseAlaram} = P[Z = 1|H^c]$ as a function of θ.

(b) Choose $\theta \in [0, 1]$ to maximize p_{detect} subject to the constraint that $p_{FalseAlarm} \leq 0.1$.

Solution:
(a) We have

$$\begin{aligned} p_{Detect} &= P[Z = 1|H] \\ &= P[Z = 1|H, T_1]P[T_1|H] + P[Z = 1|H, T_1^c]P[T_1^c|H] \\ &\overset{(a)}{=} P[Z = 1|H, T_1]P[T_1] + P[Z = 1|H, T_1^c]P[T_1^c] \\ &= P[Y_1 = 1|H, T_1]P[T_1] + P[Y_2 = 1|H, T_1^c]P[T_1^c] \\ &= (0.85)\theta + (0.75)(1 - \theta) \\ &= 0.75 + 0.1\theta \end{aligned}$$

where (a) holds because T_1 and H are independent. Similarly

$$\begin{aligned} p_{FalseAlarm} &= P[Z = 1|H^c] \\ &= P[Z = 1|H^c, T_1]P[T_1|H^c] + P[Z = 1|H^c, T_1^c]P[T_1^c|H^c] \\ &= P[Z = 1|H^c, T_1]P[T_1] + P[Z = 1|H^c, T_1^c]P[T_1^c] \\ &= P[Y_1 = 1|H^c, T_1]P[T_1] + P[Y_2 = 1|H^c, T_1^c]P[T_1^c] \\ &= (0.15)\theta + (0.05)(1 - \theta) \\ &= 0.05 + 0.1\theta \end{aligned}$$

(b) From (a), both p_{Detect} and $p_{FalseAlarm}$ increase with θ. So we choose the largest θ so that the constraint is satisfied:

$$\begin{aligned} p_{FalseAlarm} &\leq 0.1 \\ 0.05 + 0.1\theta &\leq 0.1 \\ \theta &\leq 0.5 \end{aligned}$$

p_{Detect} is maximized with $\theta^* = 0.5$. The resulting optimal values are:

$$p^*_{FalseAlarm} = 0.1, \quad p^*_{Detect} = 0.8$$

V. Conditional probability with two random variables

We flip 3 coins with all 8 outcomes equally likely: $S = \{(H, H, H), \ldots, (T, T, T)\}$. Define the following *random variables*

$$X = \begin{cases} 1 & \text{, if first and second flip are both Heads} \\ 0 & \text{, else} \end{cases}$$
$$Y = \begin{cases} 1 & \text{, if first and third flip are both Heads} \\ 0 & \text{, else} \end{cases}$$

Define events $A = \{X = 1\}, B = \{Y = 1\}, C = \{\text{first flip is Heads}\}$
a) Compute $P[A], P[B], P[AB]$. Are A and B independent?
b) Compute $P[A|C], P[B|C], P[AB|C]$. Are A and B conditionally independent given C?
Solution:
a) We have

$$\begin{aligned} P[A] &= P[X = 1] = P[\{(H, H, H), (H, H, T)\}] = 2/8 = 1/4 \\ P[B] &= P[Y = 1] = P[\{(H, H, H), (H, T, H)\}] = 2/8 = 1/4 \\ P[AB] &= P[\{(H, H, H)\}] = 1/8 \end{aligned}$$

So $P[AB] = 1/8 \neq 1/16 = P[A]P[B]$, so A, B are not independent.
b) We have $A = \{X = 1\} \subseteq C$ and $B = \{Y = 1\} \subseteq C$ so

$$\begin{aligned} P[A|C] &= \frac{P[AC]}{P[C]} = \frac{P[A]}{1/2} = 1/2 \\ P[B|C] &= \frac{P[BC]}{P[C]} = \frac{P[B]}{1/2} = 1/2 \\ P[AB|C] &= \frac{P[ABC]}{P[C]} = \frac{P[\{(H, H, H)\}]}{1/2} = 1/4 \end{aligned}$$

So $P[AB|C] = 1/4 = (1/2)^2 = P[A|C]P[B|C]$. So A, B are conditionally independent given C.

VI. Borel sigma algebra

Define C as the set of all intervals of $\mathbb{R}$ of the type $(-\infty, x]$ for some $x \in \mathbb{R}$. Define $B = \sigma(C)$ as the *Borel sigma algebra on* $\mathbb{R}$. A subset of $\mathbb{R}$ is said to be *Borel measurable* if it is in the sigma algebra B. For this problem, recall that since B is a sigma algebra, the finite or countable union of sets in B must be in B; the finite or countable intersection of sets in B must be in B; the complement of any set in B must be in B.

a) Fix real numbers a, b such that $a < b$. Show that the interval $(a, b]$ is in B.

b) Fix $x \in \mathbb{R}$. Show that the single point set $\{x\}$ is in B.

c) Fix real numbers a, b such that $a < b$. Show that all intervals $(a, b), (a, b], [a, b), [a, b]$ are all in B.

Aside: Subsets of $\mathbb{R}$ that are Borel measurable are said to have a well defined length *(lengths of 0 and ∞ are allowed). All subsets of $\mathbb{R}$ of practical interest are Borel measurable. Using an axiom called the* axiom of choice *it can be shown that there are sets that are not Borel measurable. There are many of them. In fact, the cardinality of the set of all subsets of $\mathbb{R}$ that are* not *Borel measurable is strictly larger than the cardinality of the set of all Borel measurable subsets of $\mathbb{R}$. However, sets that are not Borel measurable cannot be explicitly constructed. There is no danger of "accidentally" stumbling across a set that is not Borel measurable when working on practical problems.*

Solution:

a) We know $(-\infty, b]$ and $(-\infty, a]$ are both in the sigma algebra B. It follows that $(-\infty, a]^c = (a, \infty)$ is in B. Hence

$$(a, b] = (-\infty, b] \cap (a, \infty)$$

This is the intersection of two sets in B and so it is in B.

b) We have $\{x\} = \cap_{i=1}^{\infty}(x - 1/i, x]$. This is a countable intersection of sets in the sigma algebra B, so it is also in B.

c) We already know $(a, b] \in B$. We also know the single point sets $\{a\}$ and $\{b\}$ are in B, and so their complements $\{a\}^c$ and $\{b\}^c$ must be in B. Thus

1) $[a, b] = \{a\} \cup (a, b]$ is the union of two sets in B so must be in B.
2) $(a, b) = (a, b] \cap \{b\}^c$ is the intersection of two sets in B so must be in B.
3) $[a, b) = (a, b) \cup \{a\}$ is the union of two sets in B so must be in B.

VII. Uniformly distributed over $[0, 1]$

Let $S = [0, 1]$. Let F be the sigma algebra of all Borel measurable subsets of $[0, 1]$. We define a measure $P : F \to \mathbb{R}$ by first specifying the measure on certain types of sets: Define

$$P[[0, x]] = x \quad \forall x \in [0, 1]$$

In particular, $P[\{0\}] = P[[0, 0]] = 0$. The measure with this property is called the *uniform measure over the interval* $[0, 1]$. We want to infer the value of $P[A]$ for additional interesting Borel measurable sets $A \subseteq [0, 1]$.

a) Compute $P[(a, b]]$ for $0 \leq a < b \leq 1$.

b) Compute $P[\{x\}]$ for $x \in [0, 1]$.

c) Compute $P[(a, b)], P[(a, b]], P[[a, b)], P[[a, b]]$ for $0 \leq a < b \leq 1$.

d) Let A be the set of all rational numbers in $[0, 1]$. Let A^c be the set of all irrational numbers in $[0, 1]$. Compute $P[A]$ and $P[A^c]$.

Solution:

a) We have the disjoint union $[0, a] \cup (a, b] = [0, b]$, and so by Axiom 3 we know $P[[0, a]] + P[(a, b]] = P[[0, b]]$. That is, $a + P[(a, b]] = b$ and so $P[(a, b]] = b - a$.

b) If $x = 0$ we already know $P[\{x\}] = P[\{0\}] = 0$. Suppose $x \in (0, 1]$. We already know $P[\{x\}] \geq 0$ by Axiom 1. It suffices to show $P[\{x\}] \leq 0$. For any $y \in [0, x)$ we have

$$\{x\} \subseteq (y, x]$$

thus

$$P[\{x\}] \leq P[(y, x]] = x - y$$

This is true for all $y \in [0, x)$. Taking a limit as $y \to x^-$ gives

$$P[\{x\}] \leq \lim_{y \to x^-}(x - y) = 0$$

c) We already know $P[(a, b]] = b - a$. Intuitively, we know these three other intervals also have probability $b - a$ since any single point has probability mass 0. Formally:

To compute $P[(a, b)]$, we have the disjoint union $(a, b) \cup \{b\} = (a, b]$ and so $P[(a, b)] + P[\{b\}] = P[(a, b]]$. Thus, $P[(a, b)] = P[(a, b]] - 0 = b - a$.

To compute $P[[a, b)]$ we have the disjoint union

$$[a, b) = (a, b) \cup \{a\} \implies P[[a, b)] = P[(a, b)] + P[\{a\}] = P[(a, b)] + 0 = b - a$$

To compute $P[[a,b]]$ we have the disjoint union

$$[a,b] = (a,b] \cup \{a\} \implies P[[a,b]] = P[(a,b]] + P[\{a\}] = P[(a,b]] + 0 = b - a$$

d) The set A of all rational numbers in $[0,1]$ is countably infinite. Thus, $A = \cup_{r\in A}\{r\}$ is a countable union of disjoint events, and so by Axiom 3b:

$$P[A] = \sum_{r\in A} P[\{r\}] = \sum_{r\in A} 0 = 0$$

It follows that $P[A^c] = 1$. Thus, the probability that the outcome is irrational is 1. We say that the outcome will *almost surely be irrational*. It is not *surely* irrational because rational outcomes are in the sample space.

VIII. Limiting event

Let's consider some mutually independent biased coin flips. We denote $A_i = \{\text{Flip i is heads}\}$ and $\{A_i\}_{i=1}^{\infty}$ are mutually independent with

$$P[A_i] = \frac{1}{(i+1)^2} \forall i \in \{1,2,3,...\}.$$

(a) Find the probability that all flips are heads, namely, $P[\bigcap_{i=1}^{\infty} A_i]$.
(b) Find the probability that at least one flip is heads, namely, $P[\bigcup_{i=1}^{\infty} A_i]$.
Solution:
(a) By the properties of mutually independent events, we have

$$P[\bigcap_{i=1}^{\infty} A_i] = \prod_{i=1}^{\infty} P[A_i] = \prod_{i=1}^{\infty} \frac{1}{(i+1)^2} = 0.$$

(b) First, we can write

$$P[\bigcup_{i=1}^{\infty} A_i] = 1 - P[\left(\bigcup_{i=1}^{\infty} A_i\right)^c].$$

By De Morgan's laws, we have

$$P[\left(\bigcup_{i=1}^{\infty} A_i\right)^c] = P[\bigcap_{i=1}^{\infty} A_i^c].$$

$\{A_i\}_{i=1}^{\infty}$ are mutually independent. So, $\{A_i^c\}_{i=1}^{\infty}$ are also mutually independent. We can write

$$P[\bigcap_{i=1}^{\infty} A_i^c] = \prod_{i=1}^{\infty} P[A_i^c] = \prod_{i=1}^{\infty}\left(1 - \frac{1}{(i+1)^2}\right) = \prod_{i=1}^{\infty} \frac{i(i+2)}{(i+1)^2} = \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdots = \lim_{i\to\infty} \frac{i+2}{2(i+1)} = \lim_{i\to\infty}\left(\frac{1}{2} + \frac{1}{2(i+1)}\right) = \frac{1}{2}$$

Finally, we have

$$P[\bigcup_{i=1}^{\infty} A_i] = 1 - P[\left(\bigcup_{i=1}^{\infty} A_i\right)^c] = 1 - \frac{1}{2} = \frac{1}{2}.$$

IX. Leon-Garcia Book Problem 3.1 (Carlos and Michael)

(Replace the initial problem statement in the text with the clarified statement as follows:) Carlos and Michael **each** flip a fair coin **twice**. X is a random variable equal to the **maximum** of the number of heads obtained. Answer the questions (a), (b) and (c) as stated.

Solution:
Each person flips a coin twice and X is defined to be the maximum number of heads obtained in these two independent experiments.
(a) Let (k_1, k_2) be an outcome of the experiment, where k_1 is the number of heads obtained by Carlos and k_2 be the number of heads obtained by Michael. Then, the sample space is given by

$$\begin{aligned} S = \{&(0,0),(0,1),(0,2),\\ &(1,0),(1,1),(1,2),\\ &(2,0),(2,1),(2,2)\}. \end{aligned}$$

Note that the outcomes in S are not equally likely; for example, $(0,0)$ is equivalent to $\{(TT,TT)\}$, $(1,2)$ is equivalent to $\{(HT,HH),(TH,HH)\}$, and $(1,1)$ is equivalent to $\{(HT,HT),(TH,HT),(HT,TH),(TH,TH)\}$. Since both guys flip a fair coin $P(H)=0.5$, then

$$\begin{aligned}
P((k_1,k_2)) &= P(k_1 \text{ successes in 2 trials for Carlos } \& \; k_2 \text{ successes in 2 trials for Michael})\\
&= P(k_1 \text{ successes in 2 trials for Carlos }) \; P(k_2 \text{ successes in 2 trials for Michael})\\
&= \binom{2}{k_1}(0.5)^{k_1}(1-0.5)^{2-k_1} \; \binom{2}{k_2}(0.5)^{k_2}(1-0.5)^{2-k_2}\\
&= \frac{1}{16}\binom{2}{k_1}\binom{2}{k_2}
\end{aligned}$$

The following tables show the sample space and the probabilities corresponding to each outcome.

$$\underbrace{\begin{bmatrix}(0,0) & (0,1), & (0,2)\\ (1,0) & (1,1), & (1,2)\\ (2,0) & (2,1), & (2,2)\end{bmatrix}}_{\text{Sample space } \mathcal{S}} \rightarrow \underbrace{\begin{bmatrix}\frac{1}{16} & \frac{1}{8}, & \frac{1}{16}\\ \frac{1}{8} & \frac{1}{4}, & \frac{1}{8}\\ \frac{1}{16} & \frac{1}{8}, & \frac{1}{16}\end{bmatrix}}_{\text{Corresponding probabilities}}$$

(b) The random variable X maps the outcome (k_1,k_2) to $\max\{k_1,k_2\}$, then the range of the random variable is $\mathcal{S}_X = \{0,1,2\}$.

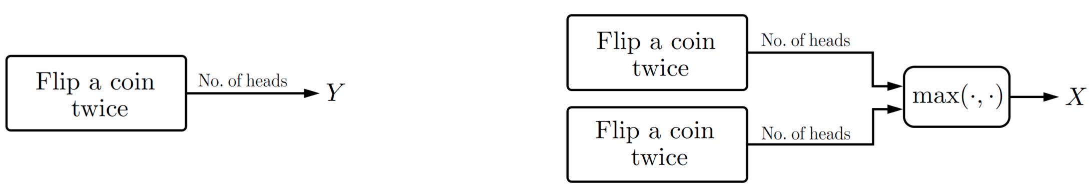
(c)

$$\begin{aligned}
&p_X(0) = P(X=0) = P((0,0)) = 1,\\
&p_X(1) = P(X=1) = P((0,1) \text{ or } (1,0) \text{ or } (1,1)) = P((0,1)) + P((1,0)) + P((1,1)) = \frac{1}{8}+\frac{1}{8}+\frac{1}{4} = \frac{1}{2},\\
&p_X(2) = P(X=2) = 1 - P(X=0 \text{ or } X=1) = 1 - (\frac{1}{16}+\frac{1}{2}) = \frac{7}{16}.
\end{aligned}$$

X. Leon-Garcia Book Problem 3.11 (Carlos and Michael again)

Solution:

(a) Random variable X is defined in the previous question, where two persons each flip a fair coin twice and X is the maximum number of heads. Random variable Y is defined to be the number of heads when a person flip a fair coin twice.



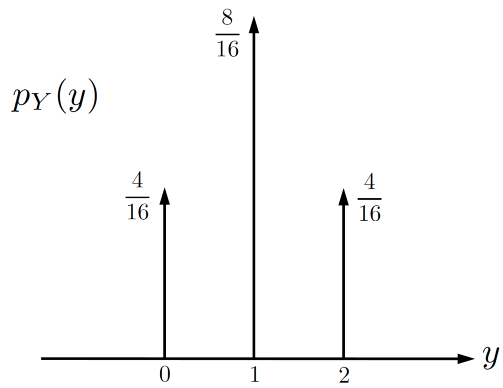
$$\begin{aligned}
&p_Y(0) = P(Y=0) = P(\text{no success in 2 trials}) = \binom{2}{0}(0.5)^0(1-0.5)^{2-0} = \frac{1}{4} \quad (= P(\{TT\}),\\
&p_Y(1) = P(Y=1) = P(\text{one success in 2 trials}) = \binom{2}{1}(0.5)^1(1-0.5)^{2-1} = \frac{1}{2} \quad (= P(\{HT,TH\}),\\
&p_Y(2) = P(Y=2) = P(\text{two success in 2 trials}) = \binom{2}{2}(0.5)^2(1-0.5)^{2-2} = \frac{1}{4} \quad (= P(\{HH\}).
\end{aligned}$$

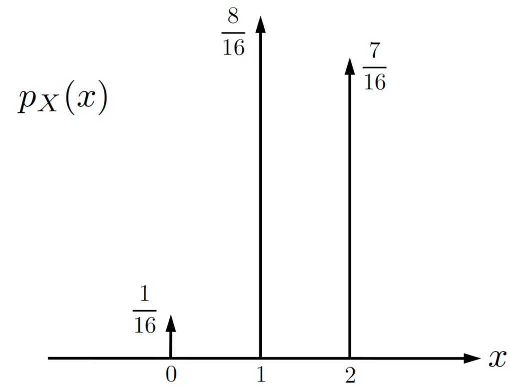
Figures below show the graph of pmf of random variables Y and X. It can be observed that the $\max$ function shifts the pmf to higher values of number of heads.

(b) If Carlos uses a biased coin with $P(H)=0.75$, then we have

$$\begin{aligned}
P((k_1,k_2)) &= \binom{2}{k_1}(0.75)^{k_1}(1-0.75)^{2-k_1} \; \binom{2}{k_2}(0.5)^{k_2}(1-0.5)^{2-k_2}\\
&= \frac{3^{k_1}}{64}\binom{2}{k_1}\binom{2}{k_2}
\end{aligned}$$

The following tables show the sample space and the probabilities corresponding to each outcome.





$$\underbrace{\begin{bmatrix} (0,0) & (0,1), & (0,2) \\ (1,0) & (1,1), & (1,2) \\ (2,0) & (2,1), & (2,2) \end{bmatrix}}_{\text{Sample space } \mathcal{S}} \rightarrow \underbrace{\begin{bmatrix} \frac{1}{64} & \frac{1}{32}, & \frac{1}{64} \\ \frac{3}{32} & \frac{3}{16}, & \frac{3}{32} \\ \frac{9}{64} & \frac{9}{32}, & \frac{9}{64} \end{bmatrix}}_{\text{Corresponding probabilities}}$$

Then, the pmf is given by

$$P(X=0) = P((0,0)) = \frac{1}{64},$$
$$P(X=1) = P((0,1) \text{ or } (1,0) \text{ or } (1,1)) = P((0,1)) + P((1,0)) + P((1,1)) = \frac{1}{32} + \frac{3}{32} + \frac{3}{16} = \frac{5}{16},$$
$$P(X=2) = 1 - P(X=0 \text{ or } X=1) = 1 - \left(\frac{1}{64} + \frac{5}{16}\right) = \frac{43}{64}.$$

XI. Properties of cumulative distribution function

Let's consider the experiment that we toss a coin until we see head for the first time. Let's denote the random variable X as the number of tosses we made till we see the first head. Assume the coin land heads with probability $1/3$ at each toss.
(a) Compute $P[X = k]$ for $k = 1, 2, ...$ (Hint: make sure they sum up to one.)
(b) Find the cumulative distribution function (CDF) $F_X(x)$ of X.
(c) Verify that the function you find in (b) is indeed a valid CDF.
<u>Solution</u>:
(a) $X = k$ is the event that we see the first head at k^{th} toss which means that we see tails in the first $(k-1)^{th}$ toss. So,

$$P[X=k] = \left(1 - \frac{1}{3}\right)^{k-1} \frac{1}{3} = \left(\frac{2}{3}\right)^{k-1} \frac{1}{3},\ k = 1, 2, ...$$
$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \frac{1}{3} = \frac{1}{1 - 2/3} \times \frac{1}{3} = 1$$

(b) It's easy to see that $P[X \le x] = 0$ for $x < 1$. When $x \ge 1$,

$$P[X \le x] = \sum_{k=1}^{\lfloor x \rfloor} P[X=k] = \sum_{k=1}^{\lfloor x \rfloor} \left(\frac{2}{3}\right)^{k-1} \frac{1}{3} = 1 - \left(\frac{2}{3}\right)^{\lfloor x \rfloor}$$

where $\lfloor x \rfloor$ denote the largest integer that is smaller than or equal to x. So the CDF is

$$F_X(x) = \begin{cases} 1 - \left(\frac{2}{3}\right)^{\lfloor x \rfloor}, & x \ge 1 \\ 0, & x < 1 \end{cases} \tag{3}$$

(c) It is right continuous and monotonically increase from zero to one.

$$\lim_{x \to -\infty} F_X(x) = 0 \tag{4}$$
$$\lim_{x \to +\infty} F_X(x) = 1 - 0 = 1 \tag{5}$$

XII. CDF TRANSFORMATIONS

Suppose $X \in \mathbb{R}$ has a continuous CDF $F_X(x)$. Define $Y = -2X^3 + 1$, $Z = 2\mathrm{e}^X - 1$ and $W = X^2$.
(a) Find the CDF of Y, namely, $F_Y(y)$ for all $y \in \mathbb{R}$, in terms of $F_X(x)$.
(b) Find the CDF of Z, namely, $F_Z(z)$ for all $z \in \mathbb{R}$, in terms of $F_X(x)$.
(c) Find the CDF of W, namely, $F_W(w)$ for all $w \in \mathbb{R}$, in terms of $F_X(x)$.
Solution:
(a) When $R_X = \mathbb{R}$, then $R_Y = \mathbb{R}$. We can write

$$F_Y(y) = P[Y \leq y] = P[-2X^3 + 1 \leq y] = P[X \geq \sqrt[3]{\frac{1-y}{2}}] = 1 - P[X < \sqrt[3]{\frac{1-y}{2}}] = 1 - F_X(\sqrt[3]{\frac{1-y}{2}}),$$

where $y \in \mathbb{R}$.
(b) When $R_X = \mathbb{R}$, then $R_Z = (-1, \infty)$. When $z \leq -1$, we have

$$F_Z(z) = P[Z \leq z] = 0.$$

When $z > -1$, we can write

$$F_Z(z) = P[Z \leq z] = P[2\mathrm{e}^X - 1 \leq z] = P[X \leq \ln \frac{1+z}{2}] = F_X(\ln \frac{1+z}{2}).$$

Finally, we write

$$F_Z(z) = \begin{cases} F_X(\ln \frac{1+z}{2}), & z > -1 \\ 0, & \text{otherwise} \end{cases}$$

(c) When $R_X = \mathbb{R}$, then $R_Y = [0, \infty)$. When $w < 0$,

$$F_W(x) = P[W \leq w] = 0$$

When $w \geq 0$,

$$F_W(w) = P[X^2 \leq w] = P[-\sqrt{w} \leq X \leq \sqrt{w}] = F_X(\sqrt{w}) - F_X(-\sqrt{w}), \quad \text{for } w \geq 0.$$

Therefore,

$$F_W(w) = \begin{cases} F_X(\sqrt{w}) - F_X(-\sqrt{w}), & w \geq 0 \\ 0, & \text{otherwise} \end{cases}$$