二 I a) $F_Y(y) = P[X+N \le y] = P[N \le y - X]$. $F_{Y|X=1}(y) = F_N(y-1)$; $F_{Y|X=-1}(y) = F_N(y+1)$

$F_{Y|X=0}(y) = F_N(y)$; $F_Y(y) = F_N(y-1)P[X=1] + F_N(y)P[X=0] + F_N(y+1)P[X=-1]$

$f_Y(y) = f_N(y-1)P[X=1] + f_N(y)P[X=0] + f_N(y+1)P[X=-1]$

b) $f_{Y|X=1}(y) = f_N(y-1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-1)^2}{2\sigma^2}}$ $f_{Y|X=-1} = f_N(y+1) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y+1)^2}{2\sigma^2}}$

$f_{Y|X=0}(y) = f_N(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$

c) $P[X=1|Y=y] = \frac{0.5\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-1)^2}{2}}}{f_Y(y)}$ $P[X=-1|Y=y] = \frac{0.25\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(y+1)^2}{2}}}{f_Y(y)}$ $P[X=0|Y=y] = \frac{0.25\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}{f_Y(y)}$

$\hat{X} = 1 \Rightarrow \frac{P[X=1|Y=y]}{P[X=-1|Y=y]} > 1 \Rightarrow y > -\frac{1}{2}\ln 2$; $\frac{P[X=1|Y=y]}{P[X=0|Y=y]} > 1 \Rightarrow y > \frac{1}{2} - \ln 2$; $\therefore \alpha = -\frac{1}{2}$, $\beta = \frac{1}{2} - \ln 2$

$\hat{X} = 0 \Rightarrow \frac{P[X=0|Y=y]}{P[X=1|Y=y]} > 1 \Rightarrow y < \frac{1}{2} - \ln 2$, $\frac{P[X=0|Y=y]}{P[X=-1|Y=y]} > 1 \Rightarrow y > -\frac{1}{2}$

$\hat{X} = -1 \Rightarrow \frac{P[X=-1|Y=y]}{P[X=1|Y=y]} > 1 \Rightarrow y < -\frac{1}{2}\ln 2$; $\frac{P[X=-1|Y=y]}{P[X=0|Y=y]} > 1 \Rightarrow y < -\frac{1}{2}$

I a)-c)

d) $P[\text{Error}] = 1 - P[\hat{X} = X = 1] - P[\hat{X} = X = 0] - P[\hat{X} = X = -1]$

$P[\hat{X} = X = 1] = P\left[\frac{P[X=1|Y=y]}{P[X=0|Y=y]} > 1, \frac{P[X=1|Y=y]}{P[X=-1|Y=y]} > 1 \middle| X=1\right] =$

$= P\left[y > \max\left\{\frac{2\sigma^2 \ln\left(\frac{P[X=0]}{P[X=1]}\right)+1}{2}, \frac{\sigma^2 \ln\left(\frac{P[X=-1]}{P[X=1]}\right)}{2}\right\} \middle| X=1\right]$

$P[\hat{X} = X = 0] = P\left[\frac{P[X=0|Y=y]}{P[X=1|Y=y]} > 1, \frac{P[X=0|Y=y]}{P[X=-1|Y=y]} > 1 \middle| X=0\right]$

$P\left[\frac{2\sigma^2 \ln\left(\frac{P[X=-1]}{P[X=0]}\right)-1}{2} < y < \frac{2\sigma^2 \ln\left(\frac{P[X=0]}{P[X=1]}\right)+1}{2} \middle| X=0\right]$

$P[\hat{X} = X = -1] = P\left[\frac{P[X=-1|Y=y]}{P[X=0|Y=y]} > 1, \frac{P[X=-1|Y=y]}{P[X=-1|Y=y]} > 1 \middle| X=-1\right]$

$= P\left[y < \min\left\{\frac{2\sigma^2 \ln\left(\frac{P[X=-1]}{P[X=0]}\right)-1}{2}, \frac{\sigma^2 \ln\left(\frac{P[X=-1]}{P[X=1]}\right)}{2}\right\} \middle| X=-1\right]$

I d)

II. a) $P[X>0] = \frac{1}{2}$, $P[X>0.2] = 0.421$, $P[5X+3 \le 2] = P[X \le -0.2] = 0.421$

b) ~~$P[Y \in [0,1]] = 0.03743$~~

c) $P[T>0.2 \mid n=2] = 0.43$; $P[T>0.2 \mid n=5] = 0.4247$; $P[T>0.2 \mid n=10] = 0.4227$;

$P[T>0.2 \mid n=20] = 0.4218$; $P[T>0.2 \mid n=50] = 0.4211$; $P[T>0.2 \mid n=100] = 0.4209$

as n gets larger, $P[T>0.2 \mid n]$ converge to $P[X>0.2]$.

d) $Var(\frac{Y}{n}) = Var(\sum_{i=1}^{n} \frac{X_i^2}{n}) = \sum_{i=1}^{n} Var(\frac{X_i^2}{n}) = \frac{Var(Y)}{n^2} = \frac{2}{n}$

e) $K_W = \begin{pmatrix} Cov(W_1, W_1), & Cov(W_1, W_2) \\ Cov(W_2, W_1), & Cov(W_2, W_2) \end{pmatrix}$ $\quad Cov(W_1, W_1) = Var(W_1) = Var(X) + Var(Y) = 2n+1$

$Cov(W_2, W_2) = Var(W_2) = Var(X) + 4Var(Y) = 8n+1$

$Cov(W_1, W_2) = Cov(X+Y, X-2Y) = Cov(X,X) - Cov(X,2Y) + Cov(X,Y) - Cov(Y,2Y)$

$= Var(X) - Cov(Y,Y) - Var(Y) = 1-2n$. $\quad \therefore K_W = \begin{pmatrix} 2n+1, & 1-2n \\ 1-2n, & 8n+1 \end{pmatrix}$

IV. $E[W] = 2E[Z] - E[X] + \frac{3}{2}E[Y] = 6.5$; $Var(W) = 4Var(Z) + Var(X) + \frac{9}{4}Var(Y) = 7.25$

$f_W(w) = \frac{1}{\sqrt{2\pi} \cdot \frac{\sqrt{17}}{2}} \cdot \exp\left(\frac{-4(x-6.5)^2}{17}\right)$ $\quad P[W > -10] = 1$

II, IV

III. (1) a) $f_{X|Y}(x|y) = \frac{d}{dx} F_{X|Y}(x|y) = \frac{d}{dx} \frac{F_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{d}{dx} \frac{P[Rx \le y] f_X(x)}{f_Y(y)}$

$= \frac{d}{dx} \frac{F_R(\frac{y}{x}) f_X(x)}{f_Y(y)} = \frac{f_R(\frac{y}{x}) f_X(x) \cdot \frac{1}{x}}{f_Y(y)}$, $\quad f_Y(y) = \frac{d}{dy} \int_{x \in S_x} F_{Y|X}(y|x) f_X(x) dx$

$x>0$. $f_Y(y) = \frac{d}{dy} \int_0^{+\infty} F_R(\frac{y}{x}) f_X(x) dx = \int_0^{+\infty} f_R(\frac{y}{x}) f_X(x) \frac{1}{x} dx$.

$x<0$. $f_Y(y) = \frac{d}{dy} \int_{-\infty}^{0} (1 - F_R(\frac{y}{x})) f_X(x) dx = -\int_{-\infty}^{0} f_R(\frac{y}{x}) f_X(x) \frac{1}{x} dx$

b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} x \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ r \end{pmatrix} = \begin{pmatrix} x \\ \frac{y}{x} \end{pmatrix} \Rightarrow |J| = |x|$. $\therefore f_{X,Y}(x,y) = \frac{f_{X,R}(x, \frac{y}{x})}{|x|} = f_X(x) f_R(\frac{y}{x}) \cdot |\frac{1}{x}|$

$f_Y(y) = \int_{x \in S_x} f_X(x) f_R(\frac{y}{x}) |\frac{1}{x}| dx$. $\quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.

$x>0$. $f_Y(y) = \int_0^{+\infty} f_R(\frac{y}{x}) f_X(x) \frac{1}{x} dx$. $\quad f_{X|Y}(x|y) = \frac{f_R(\frac{y}{x}) f_X(x) \cdot \frac{1}{x}}{f_Y(y)}$.

$x<0$. $f_Y(y) = \int_{-\infty}^{0} f_R(\frac{y}{x}) f_X(x) \frac{1}{x} dx$. $\quad f_{X|Y}(x|y) = \frac{-f_R(\frac{y}{x}) f_X(x) \frac{1}{x}}{f_Y(y)}$.

III

V. a). $\phi_W(x) = E(e^{jwX}) = 1-p+e^{jw}p$

b) let X_i be the i-th Bernoulli experiment, $P[X_i=1]=p$, $\therefore X=\sum_{k=1}^{n} X_k$

$$\phi_W(x) = E(e^{jwX}) = E(e^{jw\sum_{k=1}^{n} X_k}) = E(\prod_{k=1}^{n} e^{jwX_k}) = \prod_{k=1}^{n} E(e^{jwX_k}) = (1-p+e^{jw}p)^n$$

c). $\phi_W(x) = E(e^{jwX}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{jwk} = e^{jw\lambda-\lambda}$

VI. $y = 3x^2 - \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{y+\frac{1}{2}}{3}}$, $|y'| = |6x|$ $\therefore f_Y(y) = \frac{f_X(\sqrt{\frac{y+\frac{1}{2}}{3}}) + f_X(-\sqrt{\frac{y+\frac{1}{2}}{3}})}{\sqrt{12(y+\frac{1}{2})}} \cdot 1\{y \geq \frac{1}{2}\}$

when $y \geq \frac{1}{2}$, $\sqrt{12(y+\frac{1}{2})} > 0$.

for $y \in [0.5, 26.5]$, $f_X(-\sqrt{\frac{y+\frac{1}{2}}{3}}) = \frac{1}{5}$; for $y \in [0.5, 11.5]$, $f_X(\sqrt{\frac{y+\frac{1}{2}}{3}}) = \frac{1}{5}$

$$\therefore f_Y(y) = \frac{2}{5\sqrt{12(y+\frac{1}{2})}} \cdot 1\{y \in [0.5, 11.5]\} + \frac{1}{5\sqrt{12(y+\frac{1}{2})}} \cdot 1\{y \in (11.5, 26.5]\}$$

IX. a) $\int_0^1 cx(1-x^2)\,dx = 1 \Rightarrow c = 4 \Rightarrow f_X(x) = 4x(1-x^2) \cdot 1\{x \in [0,1]\}$

$$f_Y(y) = \frac{f_X(x)}{|y'(x)|} = \frac{4x(1-x^2)}{|2\pi x|} = \frac{4\sqrt{\frac{y}{\pi}}(1-\frac{y}{\pi})}{2\pi\sqrt{\frac{y}{\pi}}} = \frac{2}{\pi}(1-\frac{y}{\pi}) \cdot 1\{y \in [0, \pi]\}$$

b).

$$f_Y(y) = \frac{f_X(x)}{|\frac{dy}{dx}|} = \frac{4x(1-x^2)}{|4\pi x^2|} = \frac{4(\frac{3}{4\pi}y)^{\frac{1}{3}}(1-(\frac{3}{4\pi}y)^{\frac{2}{3}})}{|4\pi(\frac{3}{4\pi}y)^{\frac{2}{3}}|} = \frac{1-(\frac{3}{4\pi}y)^{\frac{2}{3}}}{\pi(\frac{3}{4\pi}y)^{\frac{1}{3}}} \cdot 1\{y \in [0, \frac{3}{4}\pi]\}$$

V, VI, IX

VII. a) $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ N \end{pmatrix}$ $|J| = 1$. $\begin{pmatrix} X \\ N \end{pmatrix} = \begin{pmatrix} X \\ Y-X \end{pmatrix}$ $\therefore f_{X,Y}(x,y) = \dfrac{f_{X,N}(x, y-x)}{|J|}$

$f_{X,Y}(x,y) = f_X(x) f_N(y-x) = P[X=x] \cdot \frac{2}{\sqrt{2\pi}} \cdot \exp[-2(y-x)^2]$

$P[X=j, Y\le y] = P[Y\le y \mid X=j]\, P[X=j] = P[N \le y-j]\, P[X=j] = F_N(y-j) \cdot P[X=j]$

b) $f_X(x) = \int_{-\infty}^{+\infty} P[X=x] \frac{2}{\sqrt{2\pi}} \exp[-2(y-x)^2]\, dy$

$f_Y(y) = \sum_{x\in S_X} P[X=x] \frac{2}{\sqrt{2\pi}} \exp[-2(y-x)^2] = (1-p) \cdot \frac{2}{\sqrt{2\pi}} \exp[-2(y+1)^2] + p \cdot \frac{2}{\sqrt{2\pi}} \exp[-2(y-1)^2]$

c) $F_Y(y) = \sum_{j\in S_X} P[Y\le y \mid X=j]\, P[X=j] = (1-p) F_N(y+1) + p F_N(y-1)$

$P[Y>0] = 1 - F_Y(0) = 1-(1-p)F_N(1) - pF_N(-1)$

$P[X=1 \mid Y>0] = \dfrac{P[Y>0 \mid X=1]\, P[X=1]}{P[Y>0]} = \dfrac{p\cdot(1-F_N(-1))}{P[Y>0]}$

$P[X=-1 \mid Y>0] = \dfrac{P[Y>0 \mid X=-1]\, P[X=-1]}{P[Y>0]} = \dfrac{(1-p)(1-F_N(1))}{P[Y>0]}$

$P[X=1 \mid Y>0] > P[X=-1 \mid Y>0] \Rightarrow p(1-F_N(-1)) > (1-p)(1-F_N(1)) \Rightarrow p > \dfrac{1}{1+\dfrac{1-F_N(-1)}{1-F_N(1)}}$

$P[X=1 \mid Y>0] < P[X=-1 \mid Y>0] \Rightarrow p < \dfrac{1}{1+\dfrac{1-F_N(-1)}{1-F_N(1)}}$

VII

VIII. $P[Z\le z] = P\left[\frac{X}{Y} \le z\right]$

① $Y>0 \Rightarrow P[X \le Yz] = \int_0^{+\infty} F_X(yz) \cdot f_Y(y)\, dy$ $f_Z(z) = \int_0^{+\infty} f_X(yz) \cdot y f_Y(y)\, dy$

$= \int_0^{+\infty} \frac{1}{2\pi} e^{-\frac{z^2+1}{2} y^2} y\, dy = \dfrac{1}{2\pi(z^2+1)}$

② $Y<0$ $= P[X > Yz] = 1 - P[X \le Yz] = \int_{-\infty}^{0} (1-F_X(yz)) f_Y(y)\, dy = -\int_{-\infty}^{0} \frac{1}{2\pi} e^{-\frac{z^2+1}{2} y^2} y\, dy$

$= \dfrac{1}{2\pi(z^2+1)}$

$f_Z(z) = f_{Z|Y}(z \mid Y<0) + f_{Z|Y}(z \mid Y>0) = \dfrac{1}{\pi(z^2+1)}$ $\therefore$ Z is Cauchy Random Variable

VIII