EE 503: Problem Set #2 : Solutions

- Reading: Chapter 2 in Leon-Garcia textbook.
- Submit your homework in D2L by 9pm on the due date.

Facts: Let S be a nonempty set.

- The intersection of two sigma algebras on S is a sigma algebra on S.
- The intersection of an arbitrary (possibly uncountably infinite) number of sigma algebras on S is a sigma algebra on S.
- Let F be a collection of subsets of S. Define $\sigma(F)$ as the smallest sigma algebra on S that includes all sets in F. Formally, $\sigma(F)$ is the intersection of all sigma algebras on S that contain F (there is at least one sigma algebra on S that contains F, namely, $Pow(S)$). It can be shown that $F \subseteq \sigma(F)$ always, and $F = \sigma(F)$ if and only if F is a sigma algebra on S.
- If F and G are collections of subsets of S that satisfy $F \subseteq G$, then $\sigma(F) \subseteq \sigma(G)$.

I. Leon-Garcia textbook 2.2

Solution:

a) The sample space S is given as:
$\{(i,j) : i,j \in \{1,2,3,4,5,6\}\}$

b) The set A is given as:
$\{(i,j) : i,j \in \{1,2,3,4,5,6\}, i \geq j\}$

c) The set B is given as:
$\{(i,j) : i \in \{6\}, j \in \{1,2,3,4,5,6\}\}$

d) Event A is true when the number in the first toss is not less than the number in the second toss. Event B is true when the first toss is 6. So when event B is true (the first toss is 6), we know for sure that the second toss is not greater than 6. So, event B implies event A, but event A does not imply event B, because it can happen that the first toss is greater than the second toss and the first toss is not 6.

e) $A \cap B^c = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4),(5,1),(5,2),(5,3),(5,4),(5,5)\}$
The set $A \cap B^c$ is defined as the first toss is neither less than the second toss nor it is 6.

f) $A \cap C = \{(3,1),(4,2),(5,3),(6,4)\}$

II. Leon-Garcia textbook 2.24

Solution:

a) A occurs and B does not occur is $A \cap B^c$. $A = (A \cap B) \cup (A \cap B^c)$. Since this is a disjoint union, we get: $P[A] = P[A \cap B] + P[A \cap B^c] \Rightarrow P[A \cap B^c] = P[A] - P[A \cap B]$.

b) Exactly one of A or B occurs is the **disjoint union**: $(A \cap B^c) \cup (A^c \cap B)$.
In part (a) we found that $P[A \cap B^c] = P[A] - P[A \cap B]$. In the same way, $P[A^c \cap B] = P[B] - P[A \cap B]$. Hence, $P[(A \cap B^c) \cup (A^c \cap B)] = P[A^c \cap B] + P[A \cap B^c] = P[A] + P[B] - 2P[A \cap B]$

c) Neither A nor B occur is $A^c \cap B^c$.
We know that $P[A^c \cap B^c] = P[(A \cup B)^c] = 1 - P[(A \cup B)]$. Also, we know that $P[(A \cup B)] = P[A] + P[B] - P[A \cap B]$. So, $P[A^c \cap B^c] = 1 - [P[A] + P[B] - P[A \cap B]]$

III. Leon-Garcia textbook 2.43

Solution:

The password length can either be 8,9, or 10. There are 86 total characters, and 24 special characters.

$$\begin{aligned}
&\text{Num with at least one special character} \\
&= \text{(Total num) - (Num with no special characters)} \\
&= (86^8 + 86^9 + 86^{10}) - ((86-24)^8 + (86-24)^9 + (86-24)^{10}) \\
&= 21537422692865611776
\end{aligned}$$

IV. SIGMA ALGEBRA FOR A FINITE SET

Define $S = \{1, 2, 3, 4, 5\}$ and $F = \{\phi, S, \{1, 2\}, \{2, 3, 4\}\}$. Note that $|F| = 4$.

a) Is F a sigma algebra on S?

b) Find $\sigma(F)$ and $|\sigma(F)|$.

Solution:

a) No. $\{1, 2\} \in F$ but $\{1, 2\}^c = \{3, 4, 5\} \notin F$.

b) From part (a) it is clear that we must add the set $\{1, 2\}^c = \{3, 4, 5\}$. Similarly we must add the set $\{2, 3, 4\}^c = \{1, 5\}$. Then, we must add $\{1, 2\} \cup \{1, 5\} = \{1, 2, 5\}$. By repeatedly including sets that we must include (via complement and/or union operations) we find

$$\sigma(F) = \{\phi, S, \{1, 2\}, \{2, 3, 4\}, \{3, 4, 5\}, \{1, 5\}, \{1, 2, 5\}, \{3, 4\}, \{1, 2, 3, 4\}, \{5\}, \{2, 3, 4, 5\}, \{1\}, \{1, 3, 4, 5\}, \{2\}, \{2, 5\}, \{1, 3, 4\}\}$$

So $|\sigma(F)| = 16$. Note that $\sigma(F)$ is the collection of all subsets of S that either include both 3 and 4, or neither 3 nor 4.

Aside: One can verify that the set C of all subsets of S that either include both 3 and 4, or neither 3 nor 4, is indeed a sigma algebra on S: (i) C contains S; (ii) Fix $A \in C$. If A contains both 3 and 4, then A^c has neither and so $A^c \in C$; Else A contains neither 3 nor 4 so that A^c contains both and so $A^c \in C$; (iii) Consider a countable union of sets in C. If at least one of the sets in the union has both 3 and 4 then the resulting union has both 3 and 4 and so is in C. Else, all sets of the union have neither 3 nor 4, so the union has neither 3 nor 4 and so is in C.

V. MASSES

a) We have a sample space $S = \{red, green, blue, pink\}$. We know that

$$P[\{green, blue, pink\}] = 0.9, P[\{red, green\}] = 0.6, P[\{blue, green\}] = 0.5$$

Find $P[\{red\}]$, $P[\{green\}]$, $P[\{blue\}]$, $P[\{pink\}]$.

b) Explain the type difference between the two statements $red \in S$ and $\{red\} \subseteq S$. Specifically, what is the difference between red and $\{red\}$?

c) Consider a general sample space S. Let A be an event. True or False: $P[A] = 0$ if and only if A is the empty set.

Solution:

Let $r = P[\{red\}]$, $g = P[\{green\}]$, $b = P[\{blue\}]$, $p = P[\{pink\}]$. We know

$$\begin{aligned} r + g + b + p &= 1 \\ g + b + p &= 0.9 \\ r + g &= 0.6 \\ b + g &= 0.5 \end{aligned}$$

Solving gives the unique solution $r = 0.1, g = 0.5, b = 0, p = 0.4$.

b)

- Here the object red is a color, while the object $\{red\}$ is a set that contains the single element red. So red and $\{red\}$ are different types of objects. The objects $red, blue, green$ are all *elements* of S. The objects $\{red, blue\}$, $\{blue, green, pink\}$, $\{red\}$ are all *subsets* of S. It would not make sense to say $\{red, blue\} \in S$, and similarly it does not make sense to say $\{red\} \in S$ or $\phi \in S$. On the other hand, it makes sense to say $\{red, blue\} \subseteq S$ and $\{red\} \subseteq S$ and $\phi \subseteq S$.
- The statement $red \in S$ means that red is an element of S. The statement $\{red\} \subseteq S$ means the set $\{red\}$ is a subset of S.

c) False. Part (a) gives a counter-example with $A = \{blue\}$. Clearly $\{blue\}$ is not the empty set, but $P[\{blue\}] = 0$.

VI. PROBABILITY 1 EVENTS

Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of events such that $P[A_i] = 1$ for all $i \in \{1, 2, 3, \ldots\}$.

a) Prove that $P[A_1 A_2] = 1$.

b) Prove that $P[\cap_{i=1}^{n} A_i] = 1$ for all $n \in \{1, 2, 3, \ldots\}$.

c) Prove that $P[\cap_{i=1}^{\infty} A_i] = 1$.

d) Write a sentence about what this means.

Solution:

a) One way to prove this is to use the $P[A \cup B]$ formula together with the fact that all probabilities are less than or equal to 1. With this approach we have

$$\begin{aligned} P[A_1 A_2] &= P[A_1] + P[A_2] - P[A_1 \cup A_2] \\ &\overset{(a)}{=} 2 - P[A_1 \cup A_2] \\ &\overset{(b)}{\geq} 2 - 1 \\ &= 1 \end{aligned}$$

where (a) holds because $P[A_1] = P[A_2] = 1$; (b) holds because $P[A_1 \cup A_2] \leq 1$. So we have proven $P[A_1 A_2] \geq 1$. Of course we also know $P[A_1 A_2] \leq 1$. Therefore, $P[A_1 A_2] = 1$.

a/b/c) We know that $P[A_i^c] = 0$ for all $i \in \{1, 2, 3, \ldots\}$. To show $P[\cap_{i=1}^{\infty} A_i] = 1$ it suffices to show that $P[(\cap_{i=1}^{\infty} A_i)^c] = 0$. We have

$$0 \overset{(a)}{\leq} P[(\cap_{i=1}^{\infty} A_i)^c] \overset{(b)}{=} P[\cup_{i=1}^{\infty} A_i^c] \overset{(c)}{\leq} \sum_{i=1}^{\infty} P[A_i^c] = \sum_{i=1}^{\infty} 0 = 0$$

where (a) uses the fact that probabilities are nonnegative; (b) uses DeMorgan's law; (c) uses the union bound.

d) This means that if we have a countably infinite number of probability-1 events, then the probability that all of them occur together is also 1.

VII. Designing probability experiments

a) Design a probability experiment using balls and bins (and sampling with replacement) with events A, B that satisfy

$$P[A] = 3/4, P[B] = 1/2, P[AB] = 1/4$$

b) Explain how the probability experiment can be implemented in a computer language such as Python. You are encouraged to implement your experiment on Python 3 trinkets by appropriately modifying the following dice roll experiment:

https://trinket.io/python3/09264aea33

Your solution can simply link to your program by using the link generated from the "share" option given on the Python 3 trinkets page.

Solution:

a) There are 4 balls in a jar, labeled $\{1, 2, 3, 4\}$. We pick one ball and look at its label. The sample space is $S = \{1, 2, 3, 4\}$ and all 4 outcomes are equally likely. Define

$$\begin{aligned} A &= \{1, 2, 3\} \\ B &= \{1, 4\} \end{aligned}$$

Then $AB = \{1\}$ and $P[A] = 3/4, P[B] = 2/4 = 1/2, P[AB] = 1/4$.

b) See Python implementation here:

https://trinket.io/python3/a24cfa5e49

VIII. Three events

We have three events A, B, C that satisfy $P[A] = 1/2, P[B] = 1/3, P[C] = 1/4$.

a) Find the largest possible value of $P[ABC]$. Specifically, you should provide a number h and prove that we necessarily have $P[ABC] \leq h$. Then, you should show that h is the best possible upper bound by showing that h is *achievable*. That is, give a specific example of a probability experiment with events A, B, C that satisfy the given requirements and such that $P[ABC] = h$. *To emphasize the concept of balls and bins, your probability experiment should involve sampling with replacement.*

b) Find the smallest possible value of $P[ABC]$. Specifically, you should provide a number b and prove that we necessarily have $P[ABC] \geq b$. Then, you should show that b is the best possible lower bound by giving a specific example of a probability experiment with events A, B, C that satisfy the given requirements and such that $P[ABC] = b$. *To emphasize the concept of balls and bins, your probability experiment should involve sampling with replacement.*

Solution:

a) We claim the best upper bound is $h = 1/4$. Clearly $ABC \subseteq C$ and so $P[ABC] \leq P[C] = 1/4$. To show $1/4$ is achievable, consider the following probability experiment: We have 12 balls in a jar. Label the balls $\{1, ..., 12\}$. We pick a single ball, the outcome being the label of the ball. The sample space is $S = \{1, 2, \ldots, 12\}$ and all 12 outcomes are equally likely. Define

$$\begin{aligned} A &= \{1, 2, 3, 4, 5, 6\} \\ B &= \{1, 2, 3, 4\} \\ C &= \{1, 2, 3\} \end{aligned}$$

Then $P[A] = 6/12 = 1/2$, $P[B] = 4/12 = 1/3$, $P[C] = 3/12 = 1/4$, and $C = ABC$ and so $P[ABC] = P[C] = 1/4$.

b) We claim the best lower bound is $b = 0$. All probabilities are nonnegative and so we know $P[ABC] \geq 0$. To show 0 is achievable, consider the following experiment, again with 12 labeled balls in a jar, pick one ball. The sample space is $S = \{1, 2, \dots, 12\}$ and all 12 outcomes equally likely. Define:

$$\begin{aligned} A &= \{1, 2, 3, 4, 5, 6\} \\ B &= \{1, 2, 3, 4\} \\ C &= \{7, 8, 9\} \end{aligned}$$

Then $P[A] = 6/12 = 1/2$, $P[B] = 4/12 = 1/3$, $P[C] = 3/12 = 1/4$. Also C is disjoint from A and so $P[ABC] = 0$.

IX. Rolling a fair die

Define $S = \{1, 2, 3, 4, 5, 6\}$ and suppose all outcomes are equally likely. Define

$$A = \{1, 2, 3, 4\}, B = \{3, 4, 5\}$$

a) Determine the sets AB, AB^c, A^cB, A^cB^c and their probabilities. Do the probabilities sum to 1?

b) We say that A and B are independent if $P[AB] = P[A]P[B]$. Are A and B independent?

Solution:

a) We have

$$\begin{aligned} AB &= \{3, 4\} \\ AB^c &= \{1, 2\} \\ A^cB &= \{5\} \\ A^cB^c &= \{6\} \end{aligned}$$

We have $P[AB] = 2/6, P[AB^c] = 2/6, P[A^cB] = 1/6, P[A^cB^c] = 1/6$, indeed these probabilities sum to 1.

b) $P[A] = 4/6$, $P[B] = 3/6$, $P[AB] = 2/6 = (4/6)(3/6) = P[A]P[B]$. So A and B are independent.

X. Sampling with replacement

A jar has 50 balls: 25 red, 20 green, 5 blue. We sample 4 times *with* replacement and with all 50^4 outcomes equally likely.

a) What is the probability that all four sampled balls are red?

b) What is the probability that two of the sample balls are red and two are green?

c) What is the probability that the first sample is red and at least one of the samples is blue?

Solution:

a) $P[\text{all red}] = \frac{25^4}{50^4} = \frac{1}{16}$

b) There are $\binom{4}{2}$ ways to choose two locations out of 4 to put the reds. Thus:

$$P[\text{two are red and two are green}] = \frac{\binom{4}{2}(25)^2(20)^2}{50^4} = \frac{6}{25}$$

c) Let R_1 be the event that the first is red. Let B be the event that at least one is blue. Then

$$P[R_1] = P[R_1B] + P[R_1B^c]$$

and so

$$P[R_1B] = P[R_1] - P[R_1B^c] = \frac{25}{50} - \frac{(25)(45)^3}{50^4} = \frac{271}{2000}$$

XI. Sampling without replacement

Repeat the previous problem (50 balls: 25 red, 20 green, 5 blue) for sampling four times *without* replacement.

Solution:

a) $P[\text{all red}] = \frac{(25)(24)(23)(22)}{(50)(49)(48)(47)} = \frac{253}{4606}$.

b) We have

$$P[\text{two red and two green}] = \frac{\binom{4}{2}(25)(24)(20)(19)}{(50)(49)(48)(47)} = \frac{570}{2303}$$

The numerator is the number of ways to choose two red and two green and can be explained as follows:

$$\underbrace{\binom{4}{2}}_{\text{stage 1}} \underbrace{(25)(24)}_{\text{stage 2}} \underbrace{(20)(19)}_{\text{stage 3}}$$

- Stage 1: Choose 2 locations (out of 4) to place red balls.
- Stage 2: Choose one red ball (out of 25) for the first designated red location and then one red ball (from the remaining 24) for the second.
- Stage 3: Choose one green ball (out of 20) for the first designated green location and then one green ball (from the remaining 19) for the second.

c) Let R_1 be the event that the first is red. Let B be the event that at least one is blue. Then

$$P[R_1] = P[R_1 B] + P[R_1 B^c]$$

and so

$$P[R_1 B] = P[R_1] - P[R_1 B^c] = \frac{25}{50} - \frac{(25)(44)(43)(42)}{50(49)(48)(47)} = \frac{185}{1316}$$

where we have used the fact that the number of ways to choose the first ball red and the next three balls not blue is $(25)(44)(43)(42)$, since after we choose the first red (out of 25 options) there are 44 remaining non-blue balls.

XII. Birthday combinatorics

There are 10 people in a room. Assume that each person has a birthday that is independent and equally likely to be one of the 365 days of the year. Define these events:

$$A = \{\text{at least one of the 10 people has a birthday in the set } \{\text{Jan. 1, Jan. 2, Jan. 3, Jan. 4}\}\}$$
$$B = \{\text{at least one of the 10 people has a birthday in the set } \{\text{Dec. 1, Dec. 2, Dec. 3, Dec. 4, Dec. 5}\}\}$$

a) Compute $P[A]$ and $P[B]$.
b) Compute $P[AB]$.
c) We say that A and B are independent if $P[AB] = P[A]P[B]$. Are A and B independent?
Solution:
a) $P[A] = 1 - P[A^c]$. $P[A^c] = P[\text{nobody has a birthday in the first 4 days of year}] = (361/365)^{10}$. So $P[A] = 1 - (361/365)^{10}$. Similarly $P[B] = 1 - (360/365)^{10}$.
b)

$$P[(A \cap B)^c] = P[A^c \cup B^c] = P[A^c] + P[B^c] - P[A^c \cap B^c] = (361/365)^{10} + (360/365)^{10} - (356/365)^{10}$$

So $P[A \cap B] = 1 - (361/365)^{10} - (301/365)^{10} + (356/365)^{10}$.
c) We check if $P[AB] = P[A]P[B]$. Is it true that:

$$(1 - (361/365)^{10})(1 - (360/365)^{10}) = 1 - (361/365)^{10} - (360/365)^{10} + (356/365)^{10}$$

This reduces to

$$(361/365)^{10}(360/365)^{10} = (356/365)^{10}$$

which reduces to

$$(361)(360) = (356)(365)$$

which is not true (for example, the left-hand-side is a multiple of 3 while the right-hand-side is not). So they are not independent. Intuitively, if we know A occurs then it means at least one of the 10 people has a birthday *not* in Dec. 1-Dec. 5, so at most 9 others have a chance to have birthdays there, so it is less likely for B to occur.

XIII. Skat

Skat is a three-player game played with a 32-card pack, containing no cards less than 7 (4 suits: Spades, Hearts, Diamonds, and Clubs. Each suit contains 8 cards: Ace,7,8,9,10,K,Q,J). Each player is dealt 10 cards, and 2 cards are remaining.
a) In how many ways can the cards be dealt?
b) If the deck is well-shuffled, what is the probability that the 'first' player is dealt all the Aces.
c) If the deck is well-shuffled, what is the probability that any player is dealt all the Aces.
d) If the deck is well-shuffled, what is the probability that each player is dealt at least one ace?
Solution:
a) 10 out of 32 cards are dealt to the first player: $\binom{32}{10}$
10 out of 22 cards are dealt to the second player: $\binom{22}{10}$
10 out of 12 cards are dealt to the first player: $\binom{12}{10}$

2 cards are remaining: $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$|S| = \begin{pmatrix} 32 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 22 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} \approx 2.75 \times 10^{15}$

b) {'First player receives 4 Aces'} = A

First player gets 4 Aces $\Rightarrow$ 28 cards remain and are dealt as follows: 6, 10, 10, 2.

$|A| = \frac{28!}{6!10!10!2!} \approx 1.6 \times 10^{13}$

$P(A) = \frac{|A|}{|S|} \approx \frac{1.6 \times 10^{13}}{2.75 \times 10^{15}} \approx 0.58 \times 10^{-2}$

c) {'Any player receives 4 Aces'} = B

$P(B) = 3P(A)$ (because different players receiving 4 Aces are 3 disjoint events) $P(B) \approx 1.75 \times 10^{-2}$

d) {'Each player receives exactly 1 Ace'} = C

{'One player receives 2 Aces and the others receive 1 Ace'} = D

case C: 28 cards are dealt as follows: 9,9,9,1 $\Rightarrow \frac{28!}{9!9!9!1!}$. Since the four Aces can be interchanged, $|C| = \frac{28!}{9!9!9!1!} \times 4!$

case D: We first choose which player gets two aces (3 choices). We then distribute the aces: $\frac{4!}{2!1!1!}$ choices. We then distribute the 28 non-aces: $\frac{28!}{8!9!9!2!}$ choices. So $|D| = \frac{28!}{8!9!9!2!} \times \frac{4!}{2!1!1!} \times 3$

Since C and D are disjoint,

$$
\begin{aligned}
&P(\text{'each player gets at least 1 Ace'}) \\
&= \frac{|C| + |D|}{|S|} = \frac{25}{58} \\
&\approx 0.4310344827586206896551724137931034482758620689655172413793103448
\end{aligned}
$$