Special instructions: You may work either individually or in a group of exactly two persons. Write your solutions out on paper, photocopy them and hand in both the original and the copy to SSO by 14:00 on Friday, 22nd November, 2019. If you are working in a group, one script should be submitted for the group, with both names at the top; both members of the group will then receive the same mark. Clearly write your name(s), student ID number(s) and the words "36111-cwk2-S-exerciseB" on the front (cover) sheet and staple all sheets of each copy together. Submitted solutions must be entirely the effort of the persons whose names appear at the top of the submission.

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Advanced Algorithms I: 36111-cwk2-S-exerciseB

Time: This should take you a few hours

Please answer all questions.

The marks for this exercise WILL count towards your final mark for Comp36111.

The use of electronic calculators is <u>not</u> recommended.

Section A

1. The problem 3D-MATCH is defined as follows:

> 3D-MATCH
> Given: Three disjoint sets U, X and Y such that $|U| = |X| = |Y|$, and a set of triples $M \subseteq U \times X \times Y$.
> Return: Yes if there exists a subset $Z \subseteq M$ such that each element of $U \cup X \cup Y$ is in exactly one triple in Z; No otherwise.

Show that 3D-MATCH is in NPTIME. (4 marks)

2. Let n and m be positive integers. For each i ($1 \leq i \leq n$), let $U_i = \{u_{i,j}, \bar{u}_{i,j} \mid 1 \leq j \leq m\}$ be a collection of $2m$ elements and $A_i = \{a_{i,j} \mid 1 \leq j \leq m\}$ and $B_i = \{b_{i,j} \mid 1 \leq j \leq m\}$ collections of m elements, with all sets disjoint. Let $U = U_1 \cup \cdots \cup U_n$. Fixing i for the moment, we seek a set of triples $T_i \subseteq U_i \times A_i \times B_i$ with the following properties:

   (i) there exists a subset $Z^+ \subseteq T_i$ such that each of the elements $a_{i,j}$, $b_{i,j}$, $u_{i,j}$ ($1 \leq j \leq m$) is contained in exactly one triple in Z^+, while none of the elements $\bar{u}_{i,j}$ ($1 \leq j \leq m$) is contained in any triple of Z^+;

   (ii) there exists a subset $Z^- \subseteq T_i$ such that that each of the elements $a_{i,j}$, $b_{i,j}$, $\bar{u}_{i,j}$ ($1 \leq j \leq m$) is contained in exactly one triple in Z^-, while the none of the elements $u_{i,j}$ ($1 \leq j \leq m$) is contained in any triple of Z^-;

   (iii) for any subset $Z \subseteq T_i$ such that each of the elements $a_{i,j}$, $b_{i,j}$ ($1 \leq j \leq m$) is contained in exactly one triple of Z, either $Z = Z^+$ or $Z = Z^-$.

   Define such a T_i. You may find it helpful to draw a diagram. (4 marks)

3. Now let $C = \{c_j \mid 1 \leq j \leq m\}$ and $D = \{d_j \mid 1 \leq j \leq m\}$ be collections of m distinct elements, disjoint from each other, and from all the sets defined before. And let $\Gamma = \{\gamma_j \mid 1 \leq j \leq m\}$ be a collection of clauses involving propositional letters $p_1, \ldots, p_n$. Fixing j for the moment ($1 \leq j \leq m$) and recalling the definition $U = U_1 \cup \cdots \cup U_n$, define $S_j \subseteq U \times C \times D$ to be the set of triples

$$\{(u_{i,j}, c_j, d_j) \mid p_i \text{ occurs in } \gamma_j\} \cup \{(\bar{u}_{i,j}, c_j, d_j) \mid \neg p_i \text{ occurs in } \gamma_j\}$$

   Given your answer to Question 2, show that, if $Z \subseteq (T_1 \cup \cdots \cup T_n \cup S_1 \cup \cdots \cup S_m)$ is a set of triples such that, for all i ($1 \leq i \leq n$) and all j ($\leq j \leq m$) each of the elements $a_{i,j}$, $b_{i,j}$, c_j, d_j lies in exactly one triple in Z, and none of the elements $u_{i,j}$ or $\bar{u}_{i,j}$ lies in more than one of the triples of Z, then the clause set Γ is satisfiable. Explain how the satisfying assignment is derived from Z, and calculate how many objects $u_{i,j}$ or $\bar{u}_{i,j}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) are *not* contained in any triple of Z. (4 marks)

4. Now let $G = \{g_k \mid 1 \leq k \leq m(n-1)\}$ and $H = \{h_k \mid 1 \leq k \leq m(n-1)\}$ be distinct collections of $m(n-1)$ elements, disjoint from each other, and from all the sets defined before. Define a set of triples $R \subseteq U \times G \times H$ such that, if Γ is satisfiable, then there is a collection of triples Z from $T_1 \cup \cdots \cup T_n \cup S_1 \cup \cdots \cup S_m \cup R$ such that every element $u_{i,j}$, $\bar{u}_{i,j}$, $a_{i,j}$, $b_{i,j}$ c_j, d_j, g_k and h_k (for i, j, k in the above ranges) lies in exactly one of the tuples of Z. (4 marks)

5. Hence, give a careful proof that the problem 3D-MATCH is NPTIME-complete. (4 marks)

END OF EXAMINATION