Special instructions: You may work either individually or in a group of exactly two persons. Write your solutions out on paper, photocopy them and hand in both the original and the copy to SSO by 14:00 on Friday, 4th October, 2019. If you are working in a group, one script should be submitted for the group, with both names at the top; both members of the group will then receive the same mark. Clearly write your name(s), student ID number(s) and the words "36111-cwk1-F-Formulating Arguments" on the front (cover) sheet and staple all sheets together. Submitted solutions must be entirely the effort of the persons whose names appear at the top of the submission.

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Advanced Algorithms I: 36111-cwk1-F-Formulating Arguments

Time: This should take you a few hours

Please answer all questions.

The marks for this exercise will NOT count towards your final mark for Comp36111.

The use of electronic calculators is not recommended.

Section A

1. Show that every finite graph has an even number of vertices with odd degree.

   [Hint: If $G = (V,E)$ is a graph, consider the bi-partite graph $H = (V,E,I)$ with vertices $V \cup E$ and edges $I = \{(v,e) \mid v \text{ incident on } e\}$. Now count the number of edges I.]

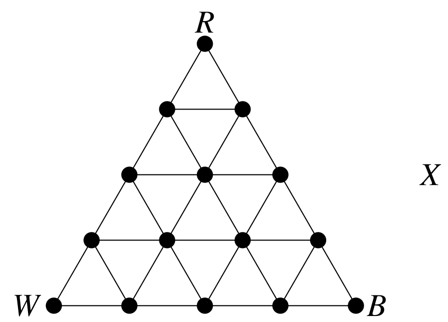
   (5 marks)

2. A *k-colouring* of a graph $G = (V,E)$ is a function $f : V \to \{0,\ldots,k-1\}$ such that $(u,v) \in E$ implies $f(u) \neq f(v)$. Notice that a graph is bi-partite just in case it admits a 2-colouring.

   Show that any graph in which the maximum degree of any vertex is d admits a $(d+1)$-colouring. Show also that this bound cannot be improved.

   (5 marks)

3. Consider a large triangle divided into smaller triangles (n of them on each side) as shown.



   Say that a *proper colouring* of such an arrangement is an assignment of the colours *red*, *white* and *blue*, to the nodes (small black circles) such that all nodes on the edge (R,W) are coloured red or white, all nodes on the edge (W,B) are coloured white or blue, and all nodes on the edge (B,R) are coloured blue or red. It follows, of course, that R is coloured red, W white, and B blue. (Note that this not be a 3-colouring in the sense of question 2: there is no requirement that neighbouring nodes be differently coloured.)

   Say that a *zone* is any of the small triangles or the region X exterior to the large triangle. Supposing that the nodes have been properly coloured, define the graph $G = (V,E)$ where V is the set of zones, and $(z,z') \in E$ just in case the boundaries of z and z' share an edge of the original arrangement featuring one blue node and one white node.

   a) What can we say about the degree (in the graph-theoretic sense) of the zone (i.e. vertex) X?

   (2 marks)

b) List the possible degrees (in the graph-theoretic sense) of all the zones (i.e. vertices) in V other than X, and describe the conditions, in terms of the colours of their vertices, for such a zone to have this degree.

(5 marks)

c) Conclude that there is at least one small triangle which has vertices coloured red, white and blue.

(3 marks)