Some Applied Mathematics: Series, Banking, and Loans

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Interest Bearing Accounts

(a)
$S_0 = 10000$
$S_1 = S_0(1.05) = 10000(1.05) = 10500$
$S_2 = S_1(1.05) = 10500(1.05) = 11025$
$S_3 = S_2(1.05) = 11025(1.05) = 11576.25$
$S_4 = S_3(1.05) = 11576.25(1.05) = 12115.06$
$S_5 = S_4(1.05) = 112115.06(1.05) = 12762.82$
$S_n = S_{n-1}(1.05)$

(b)
From part (a) we have
$S_n = S_{n-1}(1.05)$
$S_n = 10000(1.05)^n$
$S_n = 10000(1 + 0.05)^n$
Expand $(1 + 0.05)^n$ using Binomial Theorem

$$S_n = 10000 \sum_{k=0}^{n} \binom{n}{k} 1^{k-n} + 0.05^n$$

Remove 1^{k-n} term

$$S_n = 10000 \sum_{k=0}^{n} \binom{n}{k} 0.05^n$$

(c)
From part (a) $S_n = S_{n-1}(1.05)$

Plugging in S_{n-1}
$S_n = S_{n-2}(1.05)(1.05)$
$S_n = S_{n-2}(1.05)^2$

Continuing yields
$S_n = S_{n-3}(1.05)^3$
$S_n = S_{n-4}(1.05)^4$
$S_n = S_{n-n}(1.05)^n$
$S_n = S_0(1.05)^n$
$S_n = 10000(1.05)^n$

Loans

To determine a series we examine a case where the loan is paid off in n payments
The amount left after first payment is given below

$$L_0(1 + \frac{r}{12}) - m$$

Where m represents the monthly payment
For the second payment, we multiply what is left by the monthly interest rate and subtract the monthly payment m again

$$(L_0(1 + \frac{r}{12}) - m)(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

Doing the same for the third payment yields

$$(L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m)(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^3 - m(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

By examining the expressions one notices an emerging pattern

$$L_0(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^3 - m(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

Therefore we can write a summation to represent this pattern and extend the expression to the nth payment

$$L_0(1 + \frac{r}{12})^n - \sum_{k=0}^{n-1} m(1 + \frac{r}{12})^k$$

The series represented here is geometric with common ratio $(1 + \frac{r}{12})$ so we can write an expression for the sum

$$L_0(1 + \frac{r}{12})^n - \sum_{k=0}^{n-1} m(1 + \frac{r}{12})^k$$

$$L_0(1 + \frac{r}{12})^n - \frac{m(1 - (1 + \frac{r}{12})^n)}{1 - (1 + \frac{r}{12})}$$

$$L_0(1 + \frac{r}{12})^n - \frac{m(1 - (1 + \frac{r}{12})^n)}{-\frac{r}{12}}$$

After n payments, it should be fully paid off so we can say that

$$0 = L_0(1 + \frac{r}{12})^n - \frac{m(1 - (1 + \frac{r}{12})^n)}{-\frac{r}{12}}$$

$$\frac{m(1 - (1 + \frac{r}{12})^n)}{-\frac{r}{12}} = L_0(1 + \frac{r}{12})^n$$

$$m = \frac{-\frac{r}{12} L_0(1 + \frac{r}{12})^n}{1 - (1 + \frac{r}{12})^n}$$

$$m = \frac{\frac{r}{12} L_0(1 + \frac{r}{12})^n}{(1 + \frac{r}{12})^n - 1}$$

Fractional Reserve Banking