Some Applied Mathematics: Series, Banking, and Loans

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Interest Bearing Accounts

(a)
$S_0 = 10000$
$S_1 = S_0(1.05) = 10000(1.05) = 10500$
$S_2 = S_1(1.05) = 10500(1.05) = 11025$
$S_3 = S_2(1.05) = 11025(1.05) = 11576.25$
$S_4 = S_3(1.05) = 11576.25(1.05) = 12115.06$
$S_5 = S_4(1.05) = 112115.06(1.05) = 12762.82$
$S_n = S_{n-1}(1.05)$

(b)
From part (a) we have
$S_n = S_{n-1}(1.05)$
$S_n = 10000(1.05)^n$
$S_n = 10000(1 + 0.05)^n$
Expand $(1 + 0.05)^n$ using Binomial Theorem

$$S_n = 10000 \sum_{k=0}^{n} \binom{n}{k} 1^{k-n} + 0.05^n$$

Remove 1^{k-n} term

$$S_n = 10000 \sum_{k=0}^{n} \binom{n}{k} 0.05^n$$

(c)
From part (a) $S_n = S_{n-1}(1.05)$

Plugging in S_{n-1}
$S_n = S_{n-2}(1.05)(1.05)$
$S_n = S_{n-2}(1.05)^2$

Continuing yields
$S_n = S_{n-3}(1.05)^3$
$S_n = S_{n-4}(1.05)^4$
$S_n = S_{n-n}(1.05)^n$
$S_n = S_0(1.05)^n$
$S_n = 10000(1.05)^n$

Loans

To determine a series we examine a case where the loan is paid off in n payments
The amount left after first payment is given below

$$L_0(1 + \frac{r}{12}) - m$$

Where m represents the monthly payment
For the second payment, we multiply what is left by the monthly interest rate and subtract the monthly payment m again

$$(L_0(1 + \frac{r}{12}) - m)(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

Doing the same for the third payment yields

$$(L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m)(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^3 - m(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

By examining the expressions one notices an emerging pattern

$$L_0(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

$$L_0(1 + \frac{r}{12})^3 - m(1 + \frac{r}{12})^2 - m(1 + \frac{r}{12}) - m$$

Therefore we can write a summation to represent this pattern and extend the expression to the nth payment

$$L_0(1 + \frac{r}{12})^n - \sum_{k=0}^{n-1} m(1 + \frac{r}{12})^k$$

The series represented here is geometric with common ratio $(1+\frac{r}{12})$ so we can write an expression for the sum

$$L_0(1+\frac{r}{12})^n - \sum_{k=0}^{n-1} m(1+\frac{r}{12})^k$$

$$L_0(1+\frac{r}{12})^n - \frac{m(1-(1+\frac{r}{12})^n)}{1-(1+\frac{r}{12})}$$

$$L_0(1+\frac{r}{12})^n - \frac{m(1-(1+\frac{r}{12})^n)}{-\frac{r}{12}}$$

After n payments, it should be fully paid off so we can say that

$$0 = L_0(1+\frac{r}{12})^n - \frac{m(1-(1+\frac{r}{12})^n)}{-\frac{r}{12}}$$

$$\frac{m(1-(1+\frac{r}{12})^n)}{-\frac{r}{12}} = L_0(1+\frac{r}{12})^n$$

$$m = \frac{-\frac{r}{12}L_0(1+\frac{r}{12})^n}{1-(1+\frac{r}{12})^n}$$

$$m = \frac{\frac{r}{12}L_0(1+\frac{r}{12})^n}{(1+\frac{r}{12})^n - 1}$$

Fractional Reserve Banking

Suppose the Federal Reserve loans out 1000 dollars to a customer, and the customer decides to deposit the same amount of money that he just owned into the same bank. It is legally required for the Federal Reserve to keep at least 10% of that deposited money, and the rest can be used by the bank for reinvestment or other purposes.
For this case,
Bank lent out 1000, and 1000 is put back in, with 100 saved and 900 spendable for other means.
Now let's say the bank takes that 900 and reinvests it by dealing it out as a loan to another customer. That customer then puts their money from the loan back into the bank.
Now the bank is holding a total of 1000 + 900 for other people,even though it started out with only 1000 dollars.
If this continues, with the bank continuing to loan out the 90% of the money that they are not required to keep, the total amount of money that the bank is holding for others proceeds like so
$S_0 = 1000$
$S_1 = 1000 + 0.9 * 1000 = 1000 + 900 = 1900$
$S_2 = 1000 + 0.9 * 1000 + 0.9^2 * 1000 = 1000 + 900 + 810 = 2710$
$S_3 = 1000 + 0.9 * 1000 + 0.9^2 * 1000 + 1000 * 0.9^3 = 1000 + 900 + 810 + 729 = 3439$
$S_n = \sum_{k=0}^{n} 1000(0.9)^k$