# DETERMINATION OF A POSITION IN THREE DIMENSIONS USING TRILATERATION AND APPROXIMATE DISTANCES

by

William S. Murphy Jr.

and

Willy Hereman

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## Abstract

Trilateration positioning systems can be used in applications where the use of other positioning systems is impractical. Trilateration calculations use distance measurements to determine the three dimensional coordinates of unknown positions. These calculations facilitate the implementation of fully automated real time positioning systems by eliminating the need to measure angles. The distance measurements available for use in fully automated systems are frequently only approximations. Fairly accurate positions can be calculated with these approximate distances by using various iterative least squares solution procedures. This paper discusses and illustrates the mathematical solution of an ill-conditioned positioning problem that was developed for Thunder Basin Coal Company (TBCC) in Wright, Wyoming.

# 1 Introduction

Thunder Basin Coal Company (TBCC) is developing a fully automated system to electronically determine the position of equipment in an open pit mine on a real time basis. They need an automated system because their current manual surveying techniques are too labor intensive and too slow. TBCC determined that the Global Positioning System cannot currently provide elevations that are accurate enough for their applications in a cost effective manner. One of the alternative positioning systems they proposed uses a system of radio beacons to measure the approximate distances between the equipment in the mine, and the known fixed positions of the beacons on the rim of the mine. The electronics firm that TBCC contracted to develop the radio beacons argued that a mathematical solution to this three dimensional positioning problem could not be developed. Instead, the electronics firm proposed that TBCC contract with them to develop an additional piece of equipment that uses an oscillating, rotating laser that could be used along with the radio beacons to determine the elevation of the equipment in the mine. TBCC contacted the Department of Mathematical and Computer Sciences at the Colorado School of Mines in December 1990, to determine if a mathematical solution to this three dimensional positioning problem exists; and if so, whether or not a programmable fast algorithm could be designed to implement the solution. This paper is a presentation of the mathematical solution procedures that were developed to solve this positioning problem.

TBCC's proposed automated positioning system does not measure angles. The lack of measured angles precludes the use of conventional surveying procedures to calculate the coordinates of the equipment in the mine. The only information that is measured in TBCC's positioning problem are the approximate slope distances between the equipment in the mine and several distance-measuring radio beacons. These beacons are at known fixed coordinates on the rim of the mine. The class of problems that is used to describe calculations which involve only distance measurements in position determination is trilateration. Since TBCC's measured distances are not exact, the most successful solution techniques use iterative trilateration procedures to calculate the best approximation to the exact coordinates.

The solution of trilateration problems with approximate distances is necessitated by the lack of fully automated accurate omni-directional distance measuring equipment. Although exact distance measurements are available through the use of man operated laser ranging equipment, the goal of implementing a fully automated system that could be utilized with any number of vehicles on a real time basis precludes the use of these directional lasers. Instead, TBCC has to settle for less accurate distance measuring equipment which uses an omni-directional signal to measure distances without human intervention. The usefulness of these iterative trilateration procedures should not diminish as technological advances increase the accuracy of fully automated distance measuring equipment. These iterative procedures are capable of calculating the exact position when used with exact data, and are more robust than non-iterative solution techniques which potentially could be used with exact data.

The magnitude of the distances that are involved in this trilateration positioning system are small enough to permit the use of a local orthographic coordinate system. This eliminates ellipsoidal distance and angular reductions from the problem because calculations that are completed entirely in a three dimensional orthographic coordinate system do not require any ellipsoidal distance or angle reductions [1].

The usefulness of the iterative mathematical solution algorithms for the three dimensional trilateration positioning problem is not restricted to mining applications that use radio signals to measure distances. These calculations can be utilized in any system that involves distance measurements. The distance measurements used in these calculations could also be obtained from any practical method to include radar, lasers, or manual measurement procedures. These solution procedures could be used to improve the accuracy of existing trilateration applications, or they could be implemented in new applications where alternative methods of determining positions in a timely, accurate, and cost effective manner are not currently available. A few of the possible applications include dredging operations, precision farming, underwater positioning, construction related surveying inside large building shells, robotics applications, and navigational systems.

# 2 Problem Statement and Mathematical Notation

TBCC specified that the calculated positions be accurate within a tolerance of five feet, provided that the calculations are performed with distance measurements that each have a maximum allowable error of plus or minus six inches. Actual data from TBCC's mine in Wyoming is used to evaluate the suitability of the various solution procedures. This data includes eight fixed position radio beacons that are installed with the aid of conventional surveying techniques to determine their three dimensional coordinates. These fixed position beacons use radio signal timing data to measure the approximate distances between themselves and a mobile radio beacon that is permanently mounted on the equipment in the mine. The coordinates of the fixed position beacons and the approximate distance measurements are the only variables in the calculations which are used to determine the coordinates of the unknown location.

The limited number of fixed position radio beacons combine with the physical dimensions of the mine to produce a positioning problem that is ill-conditioned. In this three dimensional problem the magnitude of the $x$ and $y$ components of the distance measurements are so much larger in magnitude than the $z$ components that some of the matrices used in the solution procedures are very close to being singular. A noticeable effect of this ill-conditioning is that the elevation component of the calculated position is very sensitive to the errors in the distance measurements.

The positioning algorithms used to solve this problem were written in both the syntax of MACSYMA [2, 3], a symbolic manipulation program, and in the C programming language. Comparison of the results from the two independent programs helped identify programming and roundoff errors. Additionally, symbolic MACSYMA calculations were used to gain insight into the effects of the distance measurement errors on the position of the unknown point.

The mathematical notation and symbols used for the variables in this paper are given in Figure 1. The fixed-position beacons are labeled $B_i$ where $i$ refers to the $i$th beacon. The three dimensional coordinates of the beacons, ($x_i$, $y_i$, and $z_i$), stand for the actual easting, northing, and elevation values of the beacons. For notational convenience we write $B_i(x_i, y_i, z_i)$. The approximate measured distance between the $i$th fixed position beacon and the mobile radio beacon that is permanently mounted on the equipment in the mine is denoted as $r_i$. For the exact distances we use $\hat{r}_i$ instead of $r_i$. The individual three dimensional coordinates of the equipment in the mine, which are the unknowns that are solved for in this paper, are denoted by $x$, $y$, and $z$. The position of the equipment in the mine is then simply represented as $P(x, y, x)$.

# 3 Test Data

The coordinates of the fixed position beacons that TBCC provided for use in evaluating and testing the various solution procedures are found in Table 1. These beacon coordinates represent actual locations where beacons can be placed around the rim of the mine. The origin of this coordinate system is a point in western Wyoming that was arbitrarily selected by TBCC.

TBCC also provided approximate distance measurements from these fixed position beacons to several equipment positions in the mine. They produced these approximate distance measurements by using the following procedure. First, they arbitrarily selected the coordinates of a piece of equipment in the mine. They then calculated the exact distances or radii, $\hat{r}_i$, by using the coordinates of the beacons in Table 1, the coordinates of the equipment in

Figure 1: Illustration of Mathematical Notation and Symbols

Table 1: Coordinates of Fixed Position Radio Beacons

| $B_i(x_i, y_i, z_i)$ | $x_i$ | $y_i$ | $z_i$ |
|---|---|---|---|
| $B_1$ | 475060 | 1096300 | 4670 |
| $B_2$ | 481500 | 1094900 | 4694 |
| $B_3$ | 482230 | 1088430 | 4831 |
| $B_4$ | 478050 | 1087810 | 4775 |
| $B_5$ | 471430 | 1088580 | 4752 |
| $B_6$ | 468720 | 1091240 | 4803 |
| $B_7$ | 467400 | 1093980 | 4705 |
| $B_8$ | 468730 | 1097340 | 4747 |

the mine, and the distance formula

$$\hat{r}_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \qquad (i = 1, 2, ..., n), \tag{1}$$

where $i$ denotes the beacon number, and $n$ is the total number of beacons. The approximate distances, $r_i$, are obtained by adding errors $\Delta_i$ to the exact distances, $\hat{r}_i$.

TBCC used a heuristic method to evaluate the effectiveness of the proposed solution algorithms by requiring us to calculate unknown positions with TBCC's approximate distance measurements. TBCC then compared the calculated positions with the true positions to determine the effectiveness of the algorithms. After TBCC made this comparison, they provided us with the coordinates of the exact positions and directed us to proceed with the development and refinement procedures.

We developed a simulation program that incorporates the distance approximation procedures from TBCC's heuristic while at the same time providing a comprehensive and systematic procedure to evaluate and compare the different solution algorithms. The simulation program is based on a three dimensional rectangular grid of 1000 equally spaced points that is shown in Figure 2. The random error generator for the distance approximation procedure was used in a manner that ensured the distance errors corresponding with each of the points in the grid was the same for each simulation run regardless of which solution algorithm was used. These errors along with their application to three characteristic test points are given in Table 2. The distances for the test points in this table are given only to illustrate the application of the standard errors to test positions, and are only accurate to three decimal places. In practice, the exact distances used by the simulation program are precise to the number of decimal places that are given by (3). Likewise, the approximate distances and error terms have more than three decimal places of accuracy in practice.

The missing values to the right of the decimal places in Table 2 have very little impact on the accuracy of the calculated positions. However, the distances in this table are not accurate enough to be used to reproduce the values shown in (7) and the many of the other equations in Section 7.

The simulation program calculates the coordinates for each of the 1000 points using both exact and approximate distances. It then compares the calculated coordinates with the true coordinates. The relative effectiveness of each of the solution algorithms is determined by statistically analyzing the differences between the calculated positions and the true positions. The results of this analysis also provide insight into the effects of the fixed beacon positioning pattern on the accuracy of the calculated positions.

Table 2: Distances and Standard Errors for Three Test Points (Accurate to Three Decimal Places)

| | | Descriptions and Coordinates of Three Test Points | | | | | |
|---|---|---|---|---|---|---|---|
| | | Outside Perimeter of Beacons Near the Top of the Mine $P_1(480000, 1093000, 4668)$ | | Inside Perimeter of Beacons Near the Bottom of the Mine $P_2(480000, 1093000, 4525)$ | | Outside Perimeter of Beacons Near the Bottom of the Mine $P_3(480000, 1095500, 4525)$ | |
| | Distance Errors $\Delta_i$ | Exact Distances | Modified Distances | Exact Distances | Modified Distances | Exact Distances | Modified Distances |
| $B_1$ | -0.458 | 5940.893 | 5940.381 | 5942.607 | 5942.149 | 5006.458 | 5006.000 |
| $B_2$ | 0.173 | 2420.883 | 2421.056 | 2426.635 | 2426.808 | 1624.635 | 1624.808 |
| $B_3$ | 0.317 | 5087.666 | 5087.983 | 5094.254 | 5094.571 | 7419.664 | 7419.981 |
| $B_4$ | -0.191 | 5545.271 | 5545.080 | 5549.874 | 5549.683 | 7937.320 | 7937.129 |
| $B_5$ | 0.468 | 9643.044 | 9643.512 | 9645.353 | 9645.821 | 11047.380 | 11017.848 |
| $B_6$ | 0.141 | 11417.270 | 11417.411 | 11419.870 | 11420.011 | 12060.820 | 12060.961 |
| $B_7$ | 0.329 | 12638.110 | 12638.439 | 12639.330 | 12639.659 | 12692.620 | 12692.949 |
| $B_8$ | -0.390 | 12077.030 | 12076.640 | 12078.820 | 12078.430 | 11421.370 | 11420.980 |

Figure 2: Illustration of Test Grid and Fixed Position Beacon Locations

# 4 Solution Techniques

The obvious approach in solving this positioning problem is to treat the coordinates of the equipment in the mine $P(x, y, z)$ as the point of intersection of several spheres, whose centers are the locations of the $n$ beacons $B_i(x_i, y_i, z_i)$ for $i = 1, 2, ..., n$. The exact distances between the beacons and the equipment in the mine, $r_i$, are the radii of the individual spheres. The equation for any of these spheres is

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = {r_i}^2. \tag{2}$$

The point of intersection of the surfaces of $n$ of these spheres is obtained by letting $i = 1, 2, ..., n$, and solving the resulting $n$ nonlinear equations simultaneously to eliminate two coordinates. This solution technique is not feasible because it produces a nonlinear equation of high degree. Furthermore, since the equations are quadratic, many cases for the signs would have to be considered.

Linearizing the system of equations geometrically converts the problem into one of finding the point of intersection of several planes. When the exact distances from four beacons are available, the solution of the linear system of equations is completely determined. There are three equations, three unknowns, and exactly one solution. Consequently, the theoretical minimum number of beacons is four. When approximate distances are used, the position that is calculated by the direct solution of the linear equations is no longer acceptable. The sophistication needed when working with approximate distances is dealt with in the linear least squares, and nonlinear least squares solution techniques.

# 5 Linearized System of Equations

The solution of the linear system $\tilde{\mathbf{A}}\vec{x} = \vec{b}$ is an improvement over solving for the intersection of spheres. However, it is unacceptable because it does not determine the locations within a tolerance of five feet when used with approximate distances. The linear system which is developed below (10) can be used with exact distances and four arbitrarily selected beacons to accurately calculate an unknown location by determining the point of intersection of three planes. However, the straightforward solution of any three equations of the linear system (10) will produce unacceptable results when approximate distances are used.

The constraints are the equations of the spheres with radii $r_i$,

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = {r_i}^2 \qquad (i = 1, 2, ..., n). \tag{3}$$

The $j^{\text{th}}$ constraint is used as a *linearizing* tool. Adding and subtracting $x_j$, $y_j$ and $z_j$ in (3) gives

$$(x - x_j + x_j - x_i)^2 + (y - y_j + y_j - y_i)^2 + (z - z_j + z_j - z_i)^2 = {r_i}^2 \tag{4}$$

with $(i = 1, 2, ..., j - 1, j + 1, ..., n)$.

Expanding and regrouping the terms, leads to

$$\begin{aligned}
&(x - x_j)(x_i - x_j) + (y - y_j)(y_i - y_j) + (z - z_j)(z_i - z_j) \\
&= \frac{1}{2}[(x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 - {r_i}^2 + (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2] \\
&= \frac{1}{2}[{r_j}^2 - {r_i}^2 + d_{ij}^2] = b_{ij},
\end{aligned} \tag{5}$$

where

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} \tag{6}$$

is the distance between beacons $B_i$ and $Bj$.

Since it does not matter which constraint is used as a linearizing tool, arbitrarily select the first constraint ($j = 1$). This is analogous to selecting the first beacon. Since $i = 2, 3, ..., n$, this leads to a linear system of $(n - 1)$ equations in 3 unknowns:

$$(x - x_1)(x_2 - x_1) + (y - y_1)(y_2 - y_1) + (z - z_1)(z_2 - z_1) = \frac{1}{2}[{r_1}^2 - {r_2}^2 + d_{21}^2] = b_{21} \tag{7}$$

$$(x - x_1)(x_3 - x_1) + (y - y_1)(y_3 - y_1) + (z - z_1)(z_3 - z_1) = \frac{1}{2}[{r_1}^2 - {r_3}^2 + d_{31}^2] = b_{31} \tag{8}$$

$$\vdots$$

$$(x - x_1)(x_n - x_1) + (y - y_1)(y_n - y_1) + (z - z_1)(z_n - z_1) = \frac{1}{2}[{r_1}^2 - {r_n}^2 + d_{n1}^2] = b_{n1}. \tag{9}$$

This linear system is easily written in matrix form

$$\mathbf{A}\vec{x} = \vec{b}, \tag{10}$$

with

$$\mathbf{A} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ \vdots & \vdots & \vdots \\ x_n - x_1 & y_n - y_1 & z_n - z_1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x - x_1 \\ y - y_1 \\ z - z_1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_{21} \\ b_{31} \\ \vdots \\ b_{n1} \end{pmatrix}. \tag{11}$$

The linear system (10) has $(n-1)$ equations in three unknowns. Therefore, theoretically only four beacons ($n = 4$) are needed to determine the unique position of a piece of equipment in the mine; provided no more than two beacons are co-linear.

# 6 Linear Least Squares

The coordinates of positions obtained by applying the linear least squares method to the linear system of equations (10) are generally more accurate than the coordinates obtained by solving four equations from the linearized system of equations (10) directly. However, the accuracy of the coordinates calculated with linear least squares method are unacceptable because they are not within a tolerance of five feet when used with approximate distances.

Since the distances $r_i$ are only approximate, the problem requires the determination of $\vec{x}$ such that $\mathbf{A}\vec{x} \approx \vec{b}$. Minimizing the sum of the squares of the residuals,

$$S = \vec{r}^{\,T}\vec{r} = (\vec{b} - \mathbf{A}\vec{x})^T(\vec{b} - \mathbf{A}\vec{x}), \tag{12}$$

leads to the *normal* equation [13]

$$\mathbf{A}^T\mathbf{A}\vec{x} = \mathbf{A}^T\vec{b}. \tag{13}$$

There are several methods to solve (13) for $\vec{x}$. The condition number of $\mathbf{A}^T\mathbf{A}$ determines which method is best.

If $\mathbf{A}^T\mathbf{A}$ is non-singular and well-conditioned then

$$\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}. \tag{14}$$

If $\mathbf{A}^T\mathbf{A}$ is singular or poorly conditioned then the normalized QR-decomposition of $\mathbf{A}$ is generally used [13]. In this method $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthonormal matrix and $\mathbf{R}$ is upper-triangular matrix. The solution for $\vec{x}$ in the normalized QR-decomposition is then found from

$$\mathbf{R}\,\vec{x} = \mathbf{Q}^T\vec{b} \tag{15}$$

by back substitution when $\mathbf{A}$ is full rank.

It may happen that the matrix $\mathbf{A}^T\mathbf{A}$ is close to singular even when the original matrix $\mathbf{A}$ was not close to singular. For situations like that, $\mathbf{QR}$ decomposition may overcome the problem. If not, singular value decomposition (SVD) can be used to solve the least squares problem fairly accurately.

We compute the position of point $P_2(480000, 1093000, 4525)$ using (14). These results are in Table 3.

# 7 Singular Value Decomposition (SVD)

In terms of the pseudo-inverse, the optimal solution $\vec{x}_0$ to the problem of minimizing $||\mathbf{A}\vec{x} - \vec{b}||_2$ is given by $\vec{x}_0 = \mathbf{A}^+\vec{b}$. The pseudo-inverse [13] $\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^H$ involves the unitary matrices $\mathbf{U}, \mathbf{V}$ occurring in the SVD of $\mathbf{A}$, this is $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$. The matrix $\mathbf{\Sigma}^+$ is obtained from the "diagonal" matrix $\mathbf{\Sigma}$ as follows: The $p \times q$ matrix $\mathbf{\Sigma}$ has entries $< \mathbf{\Sigma} >_{ij} = 0$ if $i \neq j$ and $< \mathbf{\Sigma} >_{ij} = \sigma_i \geq 0$ for $1 \leq i \leq k$ and $k+1 \leq i \leq \min\{p, q\}$. The numbers $\sigma_i$ are called the *singular values*. The matrix $\mathbf{\Sigma}^+$ is then the $q \times p$ matrix whose nonzero entries are $< \mathbf{\Sigma}^+ >_{ii} = \frac{1}{\sigma_i}$, for $1 \leq i \leq k$.

To detect degeneracy of the matrix $\mathbf{A}$ one computes the ratio $\sigma_1/\sigma_n$, where $\sigma_1$ is the largest singular value and $\sigma_n$ is the smallest singular value when $\mathbf{A}$ is full rank. The ratio $\sigma_1/\sigma_n$ may be regarded as a condition number of the matrix $\mathbf{A}$.

The smallest singular value, $\sigma_n$, is the distance in the 2-norm from $\mathbf{A}$ to the nearest singular matrix. The fact that $\sigma_1/\sigma_n$ is small may be considered as a condition of near-singularity of $\mathbf{A}$ [6] [7].

Let us now apply the above ideas to the matrix

$$\mathbf{A} = \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_5 - z_1 \\ x_5 - x_1 & y_5 - y_1 & z_5 - z_1 \\ x_6 - x_1 & y_6 - y_1 & z_6 - z_1 \\ x_7 - x_1 & y_7 - y_1 & z_7 - z_1 \\ x_8 - x_1 & y_8 - y_1 & z_8 - z_1 \end{pmatrix} = \begin{pmatrix} 6440 & -1400 & 24 \\ 7170 & -7870 & 161 \\ 2990 & -8490 & 105 \\ -3630 & -7720 & 82 \\ -6340 & -5060 & 133 \\ -7660 & -2320 & 35 \\ -6330 & 1040 & 77 \end{pmatrix}, \tag{16}$$

which corresponds to the beacon positions given in Table 1.

To analyze the inaccuracy that is introduced by the differences in magnitude of the coordinates, we perform a singular value decomposition on matrix $\mathbf{A}$ in (16) using MATLAB [12].

Matrix $\mathbf{A}$ has singular values $\sigma_1 = 16259, \sigma_2 = 14741$, $\sigma_3 = 118$. Hence,

$$\sigma_1/\sigma_3 = 16259/118 = 137.788, \tag{17}$$

which confirms that the entries in the third column of A are about 100 times smaller than the entries in the first column of $\mathbf{A}$.

Furthermore,

$$\mathbf{A} = \mathbf{U\Sigma V^H} \tag{18}$$

with

$$\mathbf{U} = \begin{pmatrix} 0.3901 & -0.1216 & 0.1425 & 0.4442 & 0.4634 & 0.5609 & 0.2933 \\ 0.6169 & 0.2425 & 0.4632 & -0.0262 & -0.4278 & 0.0860 & -0.3937 \\ 0.4079 & 0.4129 & -0.1722 & -0.5263 & 0.1164 & -0.1285 & 0.5713 \\ 0.0263 & 0.5780 & -0.3926 & 0.6619 & -0.1840 & -0.1913 & 0.0495 \\ -0.1977 & 0.5053 & 0.3365 & -0.0662 & 0.6641 & -0.1501 & -0.3528 \\ -0.3487 & 0.3833 & -0.1568 & -0.2424 & -0.2092 & 0.7731 & -0.0822 \\ -0.3736 & 0.1398 & 0.6660 & 0.1537 & -0.2647 & -0.0681 & 0.5467 \end{pmatrix}, \tag{19}$$

$$\mathbf{V} = \begin{pmatrix} 0.8825 & -0.4703 & 0.0023 \\ -0.4702 & -0.8824 & 0.0156 \\ 0.0053 & 0.0148 & 0.9999 \end{pmatrix}, \tag{20}$$

$$\mathbf{\Sigma} = \begin{pmatrix} 16259 & 0 & 0 \\ 0 & 14741 & 0 \\ 0 & 0 & 118 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Let us continue with point $P_2(480000, 1093000, 4525)$ and

$$\vec{b} = \begin{pmatrix} 36426980 \\ 61363150 \\ 42770710 \\ 7524660 \\ -14645310 \\ -30196370 \\ -34711440 \end{pmatrix}, \tag{22}$$

where these components are computed via (7) through (9), with approximate distances that are determined by using the procedure discussed in Section 3.

Since neither of the singular values is very close to zero, we compute the true pseudo-inverse of $\mathbf{A}$, namely

$$\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^\mathbf{H} \tag{23}$$

with

$$\mathbf{\Sigma}^+ = \begin{pmatrix} 0.00006150 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.00006784 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0085 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

We obtain

$$\mathbf{A}^+ = \frac{1}{100000} \begin{pmatrix} 2.505 & 2.574 & 0.896 & -1.701 & -2.685 & -3.115 & -2.474 \\ -0.400 & -3.236 & -3.652 & -3.536 & -2.453 & -1.286 & 0.244 \\ 0.001 & 0.045 & 0.055 & 0.059 & 0.044 & 0.027 & 0.002 \end{pmatrix}. \tag{25}$$

The optimal solution is then given by

$$\vec{x_0} = \mathbf{A}^+\vec{b} = \begin{pmatrix} 4940.2 \\ -3299.1 \\ -131.8 \end{pmatrix}, \tag{26}$$

hence, the actual position is given by

$$\vec{x_0} + \vec{B_1} = \begin{pmatrix} 4940.2 \\ -3299.1 \\ -131.8 \end{pmatrix} + \begin{pmatrix} 475060 \\ 1096300 \\ 4670 \end{pmatrix} = \begin{pmatrix} 480000.2 \\ 1093000.9 \\ 4538.2 \end{pmatrix}. \tag{27}$$

This result is not better than what could be obtained via $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$.
Replacing $1/\sigma_3$ by 0 in $\mathbf{\Sigma}^+$ does not improve matters much.

From the SVD of $\mathbf{A}$ we learn that $\Sigma_1$ and $\Sigma_2$ are of order $10^{-5}$, whereas $\Sigma_3$ is of order $10^{-3}$ Although all of the $\Sigma$'s are sufficiently large to be in machine precision, their difference in magnitude of order 100 causes the $s$ coordinate of the unknown position to be inaccurate. A more precise iterative algorithm to compute the unknown position is given in Section 9.

# 8 Symbolic Analysis of Error Propagation in the Linear Least Squares Method

To analyze the effects of the errors on the distance measurements we perform a symbolic calculation with MACSYMA [2]. In this calculation we use the beacon locations in (1), and the point $P_2(480000, 1093000, 4525)$. Using these numerical values we solve for $x$, $y$, and $z$ in terms of $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$, $\Delta_5$, $\Delta_6$, $\Delta_7$, and $\Delta_8$; where $\Delta_1$ represents the symbolic error on the distance associated with the first beacon, etc.

Using this method we get the theoretical errors on the coordinates of $P_2$ :

$$\begin{aligned}
\delta x \;=\; & -0.00000596589(\Delta_8)^2 - 0.144122(\Delta_8) - 0.00001708401(\Delta_7)^2 \\
& -0.431861(\Delta_7) - 0.00001019150(\Delta_6)^2 - 0.232771(\Delta_6) \\
& -0.00001228196(\Delta_5)^2 - 0.236928(\Delta_5) + 0.00000282727(\Delta_4)^2 \\
& +0.031382(\Delta_4) + 0.00001732629(\Delta_3)^2 + 0.176529(\Delta_3) \\
& +0.00001389623(\Delta_2)^2 + 0.067442(\Delta_2) + 0.00001147358(\Delta_1)^2 \\
& +0.136366(\Delta_1), \qquad (28)
\end{aligned}$$

$$\begin{aligned}
\delta y \;=\; & 0.00004504562(\Delta_8)^2 + 1.088196(\Delta_8) - 0.00001674421(\Delta_7)^2 \\
& -0.423271(\Delta_7) + 0.00000987568(\Delta_6)^2 + 0.225558(\Delta_6) \\
& -0.00004351283(\Delta_5)^2 - 0.839393(\Delta_5) - 0.00002958805(\Delta_4)^2 \\
& -0.328420(\Delta_4) + 0.00001429731(\Delta_3)^2 + 0.145668(\Delta_3) \\
& +0.00000737499(\Delta_2)^2 + 0.035793(\Delta_2) + 0.00001325149(\Delta_1)^2
\end{aligned}$$

$$+0.157496(\Delta_1), \tag{29}$$

and

$$\begin{aligned}\delta z \quad = \quad & 0.002815000(\Delta_8)^2 + 68.00692(\Delta_8) - 0.00066240(\Delta_7)^2 \\ & -16.74474(\Delta_7) + 0.001422000(\Delta_6)^2 + 32.48654(\Delta_6) \\ & -0.00165900(\Delta_5)^2 - 32.00419(\Delta_5) - 0.000727510(\Delta_4)^2 \\ & -8.075191(\Delta_4) + 0.00195800(\Delta_3)^2 + 19.94801(\Delta_3) \\ & +0.000602234(\Delta_2)^2 + 2.922807(\Delta_2) - 0.00374900(\Delta_1)^2 \\ & -44.55374(\Delta_1) + 0.984375. \end{aligned} \tag{30}$$

To get an overall estimate of how accurate the distances need to be for this specific test point we perform the following calculation. Assume that

$$\Delta_1 = \Delta_2 = \cdots = \Delta_8 = \Delta. \tag{31}$$

Starting with the elevation, add the absolute values of the coefficients of the linear terms in $\Delta_1$ through $\Delta_8$ in (30). Similarly, add the absolute values of the coefficients of the quadratic terms in $\Delta_1$ through $\Delta_8$. Requiring an accuracy of five feet on the elevation of the equipment in the mine, leads to

$$(224.741)\Delta + (0.01359)\Delta^2 < 5 \text{ feet}. \tag{32}$$

Solving for $\Delta$ gives $\Delta \approx 0.022$ feet. This means that the error on the distances should be less than 0.022 feet in order to calculate the elevation within a tolerance of five feet. Similar calculations for the $x$ and $y$ coordinates requires that the error on the distances should be less than 3.432 and 1.542 feet respectively.

If we use all of the decimal places of precision that are given by the random number generator discussed in Section 3, we can evaluate (28), (29), and (30). Substituting the errors for $\Delta_1$ through $\Delta_8$, gives the results in Table 3. This table also lists the results for the same data using the linear least squares method directly. The errors generated by both methods are equal, as expected.

Table 3: Comparison of the Locations Calculated Directly using Linear Least Squares, and the Locations Calculated by Substituting Known Errors into the Symbolic Linear Least Squares Solution

| Test Data | Test Coordinates | Symbolic Equations Calculated Position | Symbolic Equations Errors | Linear Least Squares Calculated Position | Linear Least Squares Errors |
|---|---|---|---|---|---|
| Point Inside Mine | 480000 | 480000.250 | 0.250 | 480000.250 | 0.250 |
| Near the Bottom | 1093000 | 1093001.875 | 0.875 | 1093000.875 | 0.875 |
| $P_2(x, y, z)$ | 4525 | 4538.375 | 13.375 | 4538.375 | 13.375 |

# 9 Nonlinear Least Squares

The nonlinear least squares method gives the most accurate results of all methods developed and examined (until now) when approximate distances are involved in the calculations. It is acceptable for use by TBCC because the coordinates of the equipment are calculated within the required tolerance of five feet for both exact and approximate distances. The use of this method should be restricted to situations where the equipment in the mine is inside the perimeter of the beacons, and below the more or less common plane of the beacons. The elevation restriction is required because of a calculation which involves a square root requiring us to choose between a positive or negative sign in developing the equations. The positioning restriction is required because there is a significant loss of accuracy when the unknown position is outside the perimeter of the beacons. This method will provide results if these restrictions are violated. The accuracy of the solution decreases as the elevation of the equipment increases, and as the equipment moves farther outside the perimeter of the beacons.

The sum of the squares of the errors on the distances is minimized in this least squares method. Recall that $r_i$ denotes the *approximate* distance between the equipment in the mine, and the $i^{\text{th}}$ beacon; and that $\hat{r}_i$ stands for the *exact* distance, i.e.

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = \hat{r}_i{}^2. \tag{33}$$

To minimize the sum of the squares of the errors on the distances, one must minimize the function

$$F(x, y, z) = \sum_{i=1}^{n} (\hat{r_i} - r_i)^2 = \sum_{i=1}^{n} f_i(x, y, z)^2, \tag{34}$$

with

$$f_i(x, y, z) = \hat{r_i} - r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} - r_i. \tag{35}$$

Minimizing the sum of the square errors is a fairly common problem in applied mathematics for which various algorithms are available [9]. Numerous different approaches can be taken, from simple to very complicated [11]. The Newton iteration was selected from among those available to find the 'optimal' solution $P(x, y, z)$. A 'good' initial guess for $(\tilde{x}, \tilde{y}, \tilde{z})$ is obtained from the linear least squares method developed in Section 6.

The only case considered is the case for which $F_{\min} > 0$ and therefore $n > 3$. Differentiating (34) with respect to $x$ yields

$$\frac{\partial F}{\partial x} = 2 \sum_{i=1}^{n} f_i \frac{\partial f_i}{\partial x}. \tag{36}$$

The formulae for the partials with respect to $y$ and $z$ are similar. Introducing the vectors $\vec{f}, \vec{g}$ and the Jacobian matrix $\mathbf{J}$, leads to

$$\vec{g} = 2\mathbf{J}^T \vec{f}, \tag{37}$$

where

$$\mathbf{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial z} \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix}. \tag{38}$$

Using the vector $\vec{R}$

$$\vec{R} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{39}$$

Newton iteration gives

$$\vec{R}_{\{k+1\}} = \vec{R}_{\{k\}} - (\mathbf{J}_{\{k\}}^T \mathbf{J}_{\{k\}})^{-1} \mathbf{J}_{\{k\}}^T \vec{f}_{\{k\}}, \tag{40}$$

where $\vec{R_{\{k\}}}$ denotes the $k$th approximate solution. The subscript $\{k\}$ in $\mathbf{J}$ and $\vec{f}$ means that these quantities are evaluated at $\vec{R_{\{k\}}}$. Obviously $\vec{R}_{\{1\}} = (\tilde{x}, \tilde{y}, \tilde{z})^T$.

Using the explicit form of the function $f_i(x, y, z)$ leads to

$$\mathbf{J}^T\mathbf{J} = \begin{pmatrix} \sum_{i=1}^{n} \frac{(x-x_i)^2}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(x-x_i)(y-y_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(x-x_i)(z-z_i)}{(f_i+r_i)^2} \\ \sum_{i=1}^{n} \frac{(x-x_i)(y-y_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)^2}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)(z-z_i)}{(f_i+r_i)^2} \\ \sum_{i=1}^{n} \frac{(x-x_i)(z-z_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(y-y_i)(z-z_i)}{(f_i+r_i)^2} & \sum_{i=1}^{n} \frac{(z-z_i)^2}{(f_i+r_i)^2} \end{pmatrix}, \tag{41}$$

and

$$\mathbf{J}^T\vec{f} = \begin{pmatrix} \sum_{i=1}^{n} \frac{(x-x_i)f_i}{(f_i+r_i)} \\ \sum_{i=1}^{n} \frac{(y-y_i)f_i}{(f_i+r_i)} \\ \sum_{i=1}^{n} \frac{(z-z_i)f_i}{(f_i+r_i)} \end{pmatrix}. \tag{42}$$

In practice this type of iteration works fast, in particular when the matrix $\mathbf{J}^T\mathbf{J}$ is augmented by a diagonal matrix which effectively biases the search direction towards that of *steepest decent*. Levenberg and Marquardt [7] developed this improvement. As the solution is approached such modifications can be expected to have a decreasing effect.

# 10 Results

The most effective approach is the nonlinear least squares method which calculates the locations within the required five foot tolerance for both exact and approximate distances. Table 4 summarizes the errors produced by the various calculation methods for the three test points.

The data in this table shows that the linearized equations provide the least accurate position calculations, while the nonlinear least squares solution algorithm provides the most accurate positions.

Table 5 summarizes the number of positions from the 1000 point grid that are not within the five foot tolerance for the three solution algorithms. There is a significant reduction in the number of points that are outside the five foot tolerance when the nonlinear least

Table 4: Comparison of Errors for Three Test Points Using Various Calculation Procedures

| Position and Description | | Solution Method: Linearized Equations, Errors Using Exact Distances | Linearized Equations, Errors Using Modified Distances | Linear Least Squares, Errors Using Exact Distances | Linear Least Squares, Errors Using Modified Distances | Nonlinear Least Squares, Errors Using Exact Distances | Nonlinear Least Squares, Errors Using Modified Distances |
|---|---|---|---|---|---|---|---|
| $P_1$, Inside Perimeter, Top | $x$ | 0.000 | -0.469 | 0.000 | 0.219 | 0.000 | -0.062 |
| | $y$ | 0.000 | -0.125 | 0.000 | 0.875 | 0.000 | 0.125 |
| | $z$ | 0.562 | -10.750 | -0.313 | 13.437 | 0.014 | -4.101 |
| $P_2$, Inside Perimeter, Bottom | $x$ | 0.000 | -0.469 | 0.000 | 0.250 | 0.000 | -0.063 |
| | $y$ | 0.000 | -0.125 | 0.000 | 0.875 | 0.000 | 0.125 |
| | $z$ | 0.562 | -11.000 | -1.000 | 13.375 | -0.010 | -1.514 |
| $P_3$, Outside Perimeter, Bottom | $x$ | 0.000 | -0.375 | 0.000 | 0.219 | 0.000 | -0.062 |
| | $y$ | 0.000 | -0.500 | 0.000 | 0.875 | 0.000 | 0.375 |
| | $z$ | 0.219 | -35.813 | -0.039 | 14.039 | 0.000 | 1.271 |

squares method is used instead of the other two methods. Analysis of these tables and the details of the individual points that are out of tolerance indicates that the accuracy of the calculated position is severely degraded when the elevation of the equipment is above or near the elevation of the lowest beacon. The accuracy is also degraded when the equipment is located outside the perimeter of the fixed position beacons.

Using the simulation program and the grid to analyze other beacon placement patterns reveals that the accuracy of the calculated positions is also dependent upon the relative positions of the beacons. Thus, in addition to being used to compare various solution algorithms, the 1000 point grid simulation program can also be used as a calibration program to ensure that the placement of the fixed position beacons does not adversely affect the accuracy of the calculated positions.

81 points are not within the required tolerance for the nonlinear least squares method because the $z$ coordinates were off more than 5 feet. 41 of these points are at an elevation that is 2 feet below the lowest beacon, 23 are at an elevation that is 68.7 feet below the lowest beacon, and 13 points are at an elevation that is 135.4 feet below the lowest beacon.

Table 5: Summary of Locations Out of Tolerance Calculated with Various Solution Techniques

| Method | Number of Calculated Locations Out or 1000 That Are Not Within a Tolerance of 5.0 Feet | |
|---|---|---|
| | Exact Distances | Approximate Distances |
| Linearized Equations * | 0 | 919 |
| Linear Least Squares | 0 | 856 |
| Nonlinear Least Squares | 0 | 81 |

* This is a smaller test area than for the other cases.

The remaining 4 points at lower elevations are all outside the perimeter of the beacons. Approximately 50% of the 81 positions that are outside the 5 foot tolerance are outside the perimeter of the beacons even though this area composes only about 20% of the test grid.

# 11 Implementation

TBCC validated the nonlinear least squares trilateration positioning algorithm and is satisfied with the accuracy of the calculated positions. They began implementing a user friendly automated system based on these calculations in 1991. This system included touch screens computer terminals which were mounted on the equipment for use by operators who require positioning information. When an operator pushes a video image on the touch screen the positioning system is activated. The coordinates of the equipment are calculated by the nonlinear least squares algorithm, and is transmitted back to the touch screen. TBCC is innovative in making this information user friendly by programming the positioning system to look up information from a data base, perform calculations, and to provide feedback to the equipment operators on their touch screens. An example of the type of information that is provided to the equipment operators in the open pit mine is a comparison of the actual coordinates with the design coordinates, and a message that is based on the corresponding interpretation of the data that tells the operator how much deeper to dig. The touch screen computer terminals and the positioning system communicate with each other via radio modems.

Unfortunately, the electronics firm that TBCC contracted with to develop the electronic distance measuring equipment failed to deliver on their contract. Consequently, TBCC was forced to stop implementing the system until they can obtain fairly accurate fully automated distance measuring equipment at a reasonable price.

The actual code for the nonlinear least squares algorithm is implemented in BorlandC++, and is available from the authors.

# 12 Conclusion

Of the methods we proposed, the nonlinear least squares method gives the most accurate position calculations for TBCC's three-dimensional trilateration positioning problem that is designed for use with approximate distances and a physical configuration which (unfortunately) leads to a problem that is poorly conditioned. The nonlinear least squares solution procedure calculates the exact position when exact distances are known, and reasonably accurate answers when approximate distances are known. These results satisfy TBCC's needs. Although restricted to applications where the elevation of the unknown position is below the elevation of the lowest beacon, this method will provide results if this restriction is violated. The accuracy of the solution is degraded when the elevation of the unknown position is close to or above the elevation of the lowest known position, and when the unknown position is outside the perimeter of the beacons.

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