Reversibility and Markov chains
Product form solutions

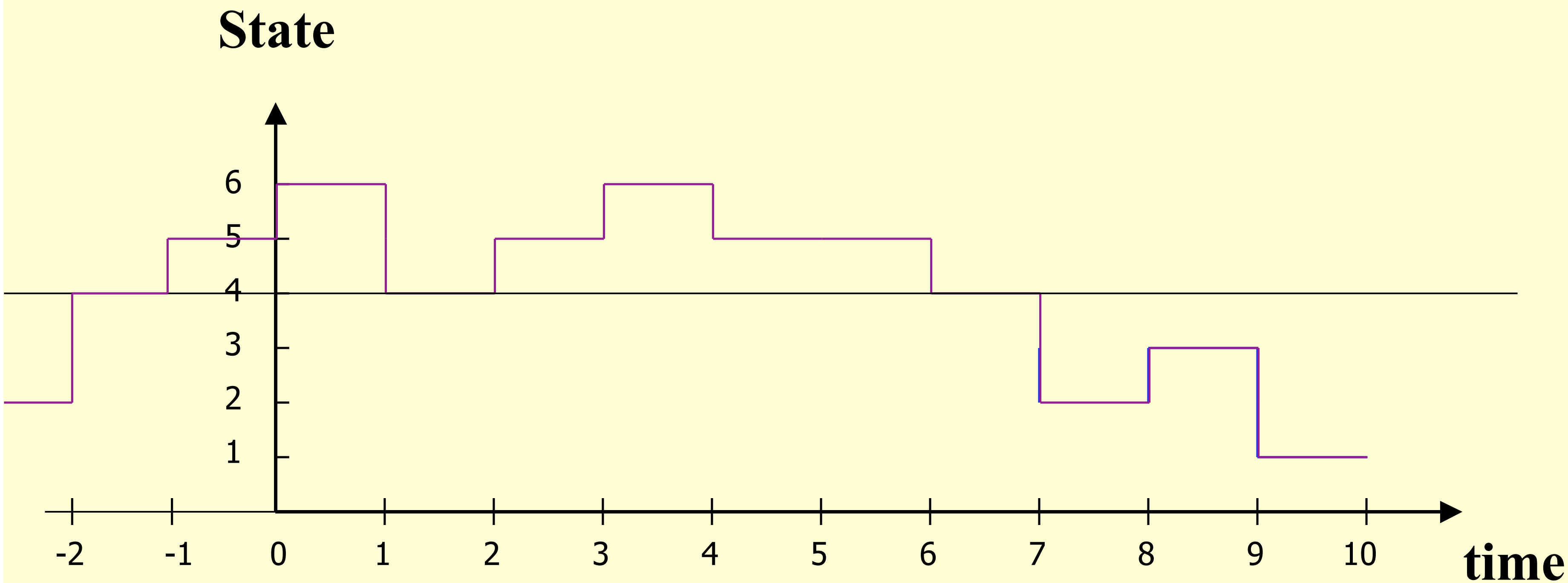
Goals : compositional approach

- queuing networks,
- stochastic automata networks,
- process algebra, stochastic Petri nets,

Methodology :

- independent behaviour,
- local balance equations,
- computation of the normalisation constant

Backward in time, reversibility



$$\pi_i = \lim \frac{1}{n} \sum_{k=1}^{n} 1_{X_k = i} = \lim \frac{1}{n} \sum_{k=n}^{1} 1_{X_{n-k+1} = i}$$

Inverse process

$\{X_t\}_{t\in T}$ Discrete Markov process
Stationary, steady-state $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$

$\{X_{\tau - t}\}_{t\in T}$ Reversed process at time τ
Discrete Markov process
Stationary, steady-state $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$

$$q^r_{i,j}dt = P\left(X_{t-dt} = j \middle| X_t = i\right) = \frac{P\left(X_{t-dt} = j, X_t = i\right)}{P\left(X_t = i\right)}$$

$$= \frac{P\left(X_t = i, X_{t-dt} = j\right)}{P\left(X_{t-dt} = j\right)} \cdot \frac{P\left(X_{t-dt} = j\right)}{P\left(X_t = i\right)} = q_{j,i}\frac{\pi_j}{\pi_i}dt \cdot$$

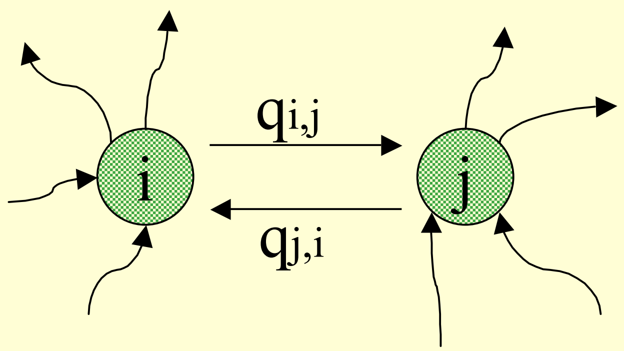
Inverse process (2)

Local balance equation $\pi_i q^r_{i,j} = \pi_j q_{j,i}$

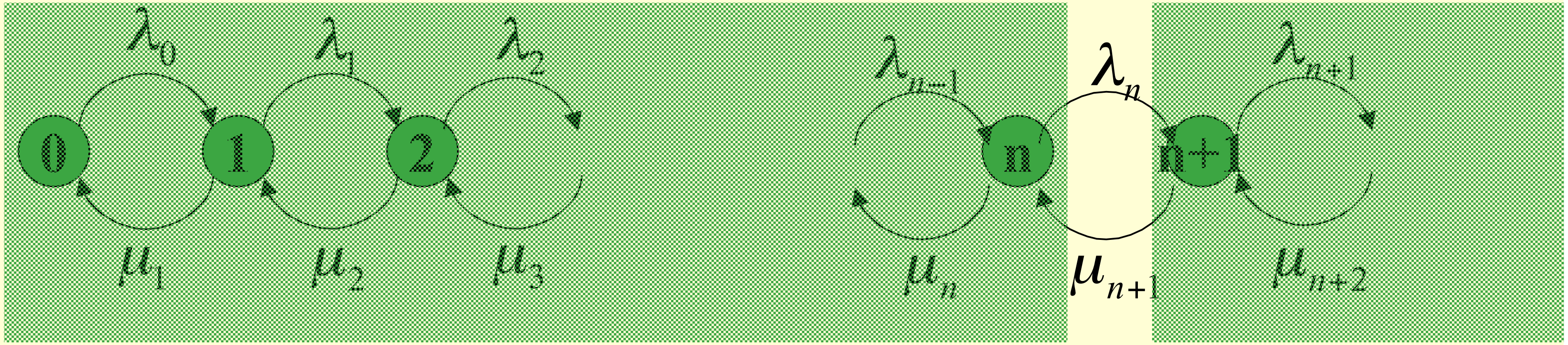
$\{X_t\}_{t\in T}$ **Reversible iff** $\{X_t\}_{t\in T} \approx \{X_{\tau-t}\}_{t\in T}$

$q^r_{i,j} = q_{i,j}$

$\Rightarrow \pi_i q_{i,j} = \pi_j q_{j,i}$



Birth and death process (1)



$$\pi_0 \lambda_0 = \pi_1 \mu_1;$$

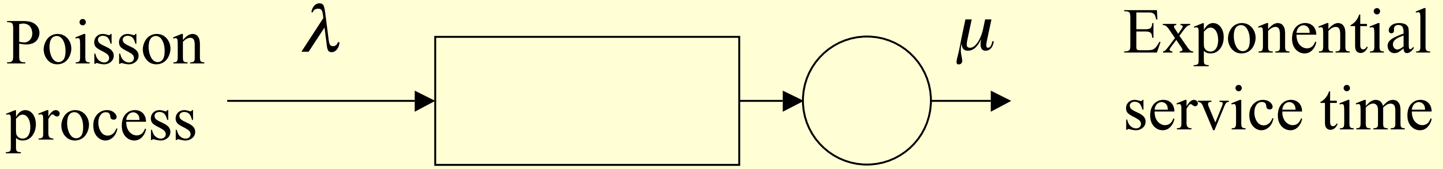
$$\pi_i (\lambda_i + \mu_i) = \pi_{i+1} \mu_{i+1} + \pi_{i-1} . \lambda_{i-1}$$

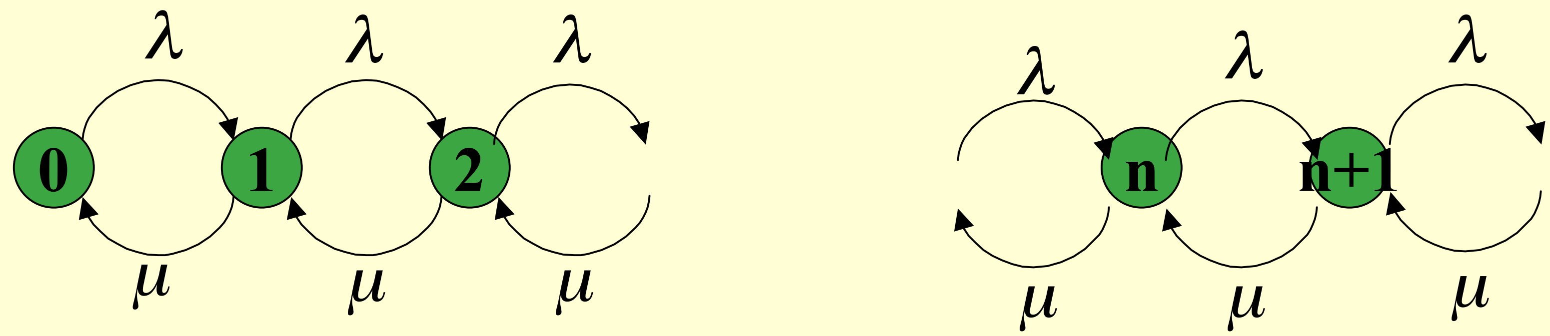
$$\Rightarrow \quad \pi_n \lambda_n = \pi_{n+1} \mu_{n+1}$$

$$\boxed{\pi_n = \pi_0 \frac{\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \cdot \mu_2 \cdots \mu_n}}$$

Reversible !

Burke theorem (1)





$$\text{Stability} \Leftrightarrow \quad (\rho < 1)$$

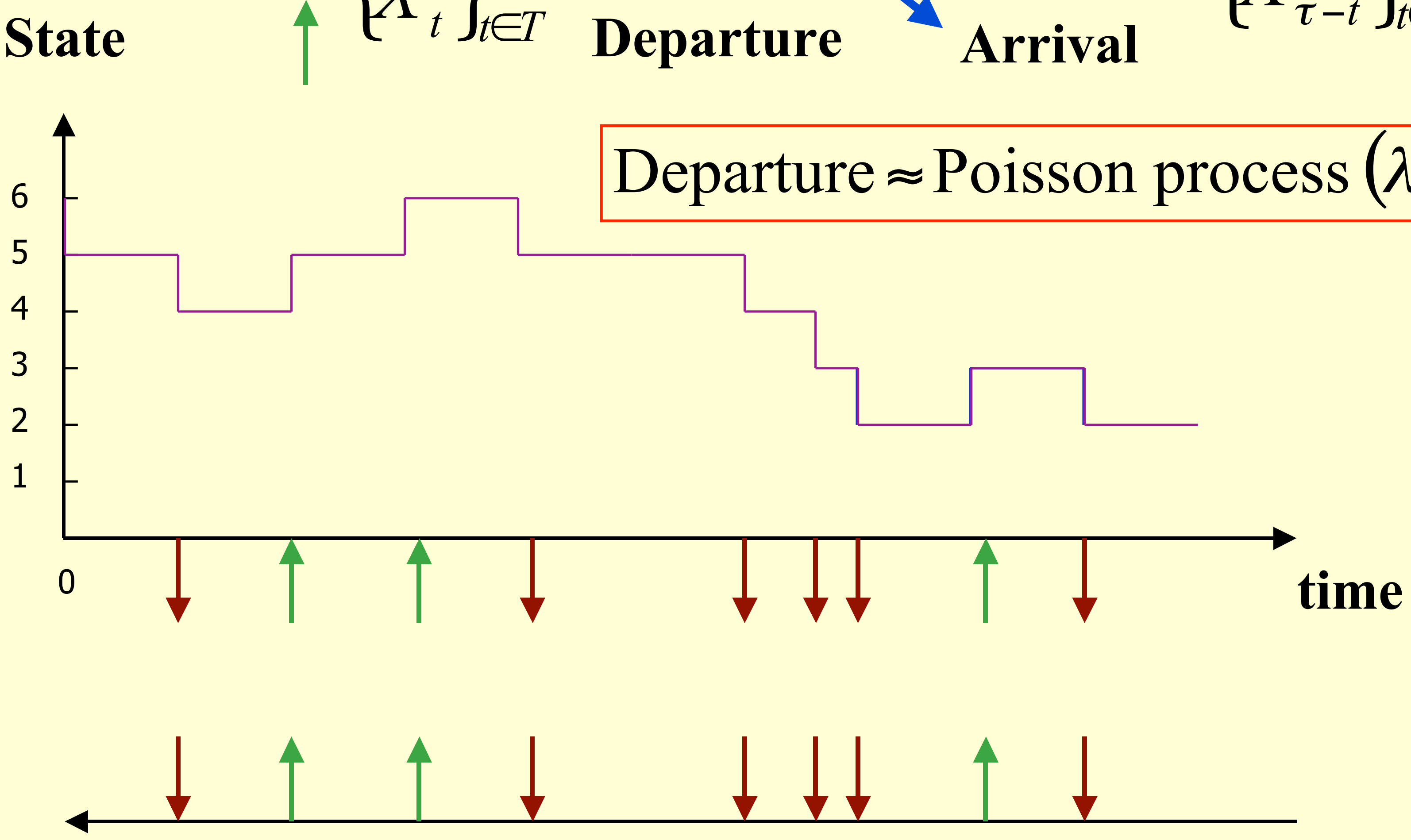
$$\boxed{\pi_n = \pi_0 \left(\frac{\lambda}{\mu}\right)^n = (1-\rho)\rho^n}$$

Burke theorem (2)

$\{X_t\}_{t\in T}$ — Arrival, Departure

$\{X_{\tau-t}\}_{t\in T}$ — Departure, Arrival

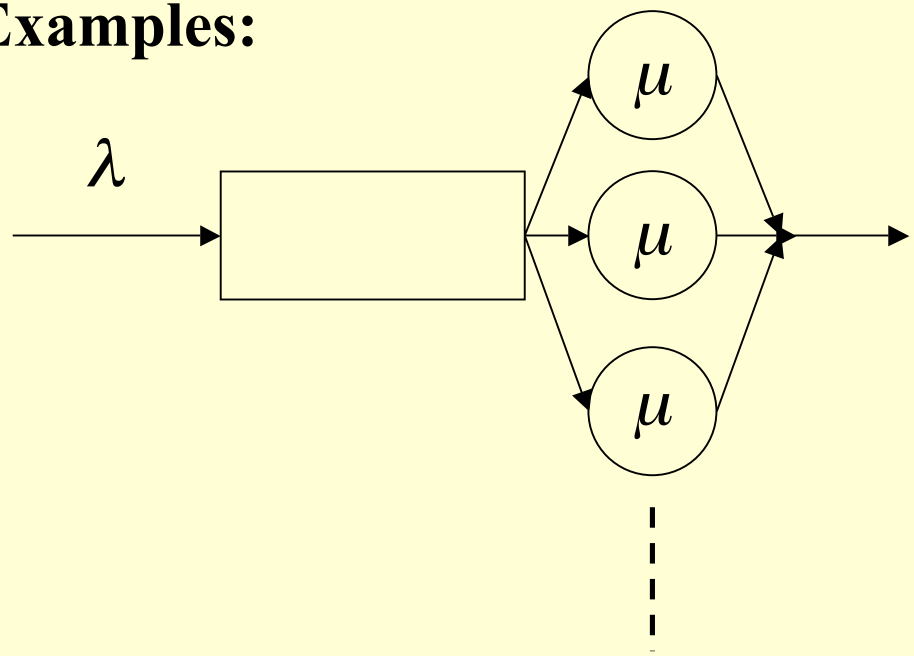
$$\boxed{\text{Departure} \approx \text{Poisson process}\,(\lambda)}$$



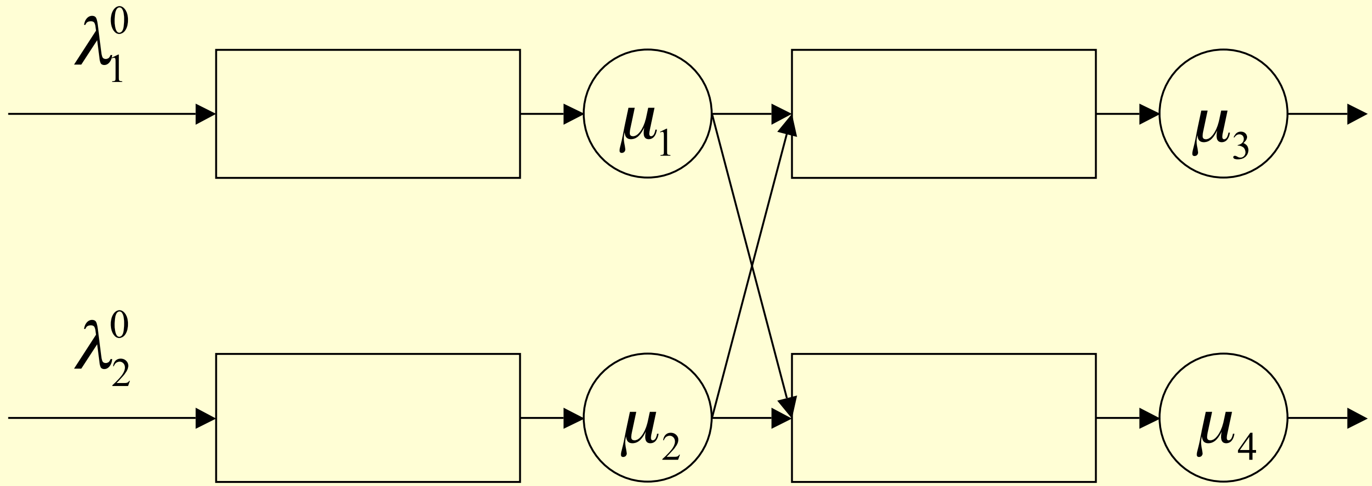
Reversed time

General behavior of output queues

- Output process of M/M/. Queues is Poisson
- Departure before time t independent of X_t
- Examples:



Feed-forward networks



$$R = \begin{pmatrix} 0 & 0 & r_{1,3} & r_{1,4} \\ 0 & 0 & r_{2,3} & r_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

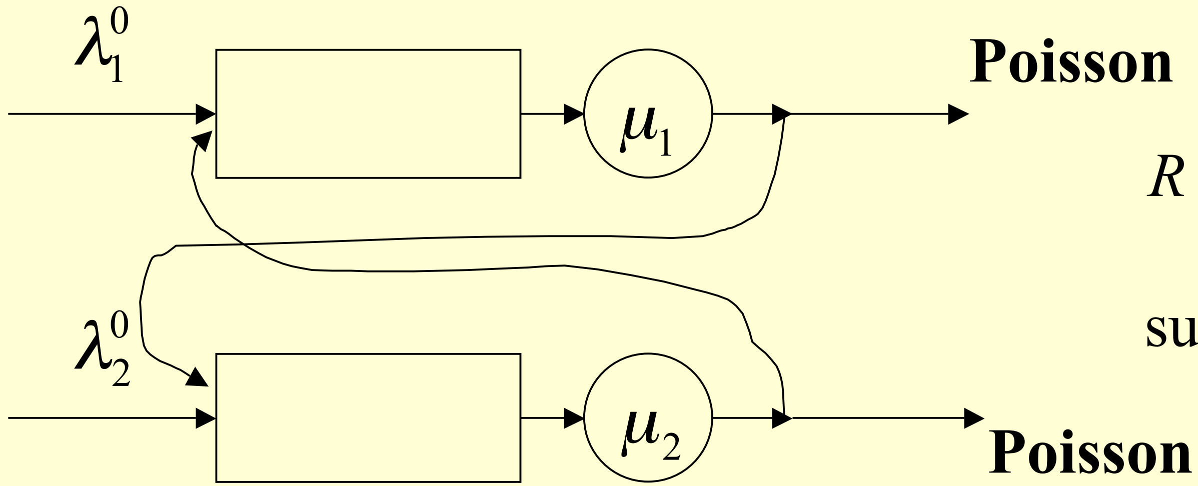
$$\rho_1 = \frac{\lambda_1^0}{\mu_1} \qquad \lambda_1 = \lambda_1^0 \qquad \rho_3 = \frac{\lambda_1 r_{1,3} + \lambda_2 r_{2,3}}{\mu_3}$$

$$\rho_2 = \frac{\lambda_2^0}{\mu_2} \qquad \lambda_2 = \lambda_2^0 \qquad \rho_4 = \frac{\lambda_1 r_{1,4} + \lambda_2 r_{2,4}}{\mu_4}$$

Stable iff $\forall\, \mathrm{i}\, (\rho_\mathrm{i} < 1)$

$$\boxed{\pi(n_1, n_2, n_3, n_4) = \prod_{i=1}^{4} (1 - \rho_i) \rho_i^{\,n_i}}$$

Jackson networks



$$R = \begin{pmatrix} 0 & r_{1,2} \\ r_{2,1} & 0 \end{pmatrix}$$

sub - stochastic

Flow equations

$$\lambda_1 = \lambda_1^0 + \lambda_2 r_{2,1};$$

$$\lambda_2 = \lambda_2^0 + \lambda_1 r_{1,2}.$$

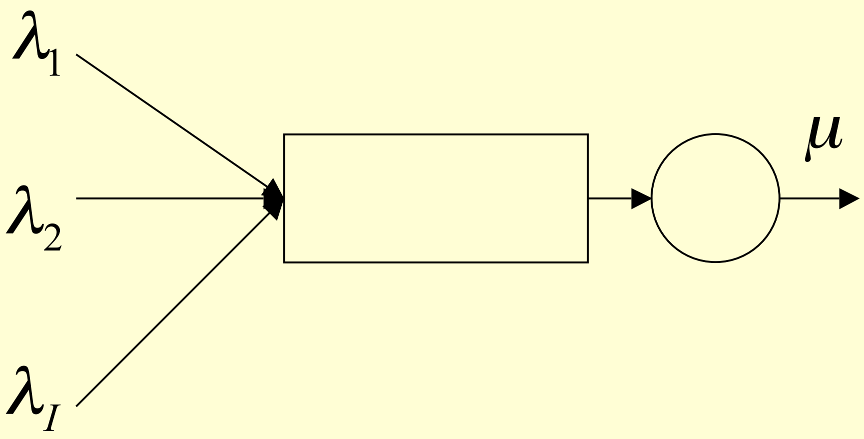
Stable iff $\forall\, \mathrm{i}\ (\rho_{\mathrm{i}} < 1)$

Unique solution of the system

$$\Lambda = \Lambda^0 + \Lambda R$$

$$\boxed{\pi(n_1, n_2) = \prod_{i=1}^{2} (1 - \rho_i)\rho_i^{\,n_i}}$$

Multiplexing



Processor sharing policy

$+e_i$ rate λ_i

$-e_i$ rate $\mu . \frac{n_i}{\sum n_j}$

Reversible

$$\boxed{\pi(n_1,\ldots,n_I) = (1-\rho)\frac{n!}{n_1!\ldots n_I!}\rho_1^{\,n_1}\ldots\rho_I^{\,n_I}}$$

Flow i : M/M/1 : $\lambda_i, \mu - \lambda + \lambda_i$

Generalization : Product form solution

Multiple server queuing networks

Controlling global input rate

Deterministic routing strategies (Kelly networks)

Classes, general services (BCMP networks)

Negative customers

...

Robustness of queuing models

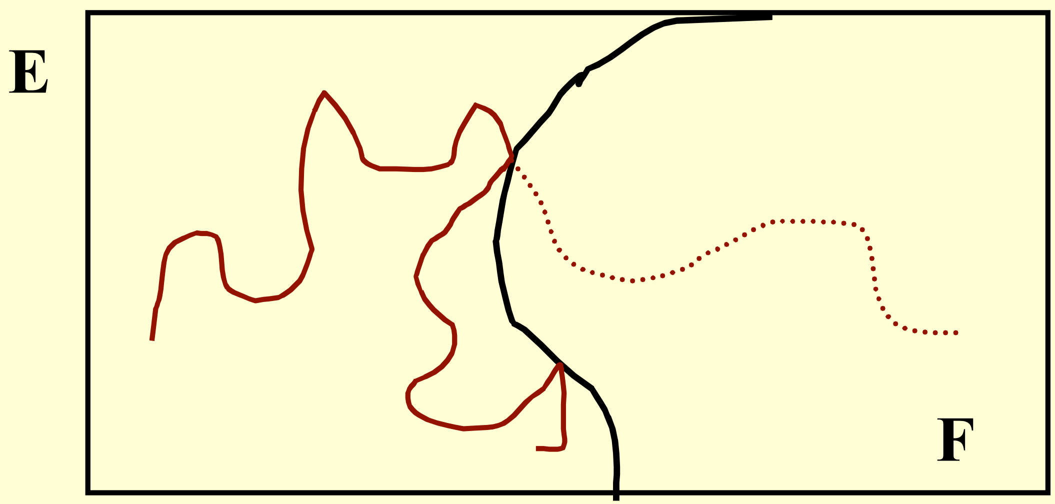
Truncation argument (1)

$\{X_t\}_{t \in T}$ is reversible

State space E

F = forbidden state space $E = F \cup F^c$

$\pi = (\pi_1, \ldots, \pi_n)$ steady state probability



Truncation argument (2)

$\left\{ X_t \big|_{\mathrm{F^c}} \right\}_{i \in T}$ is also reversible

$\mathrm{F^c}$ = state space

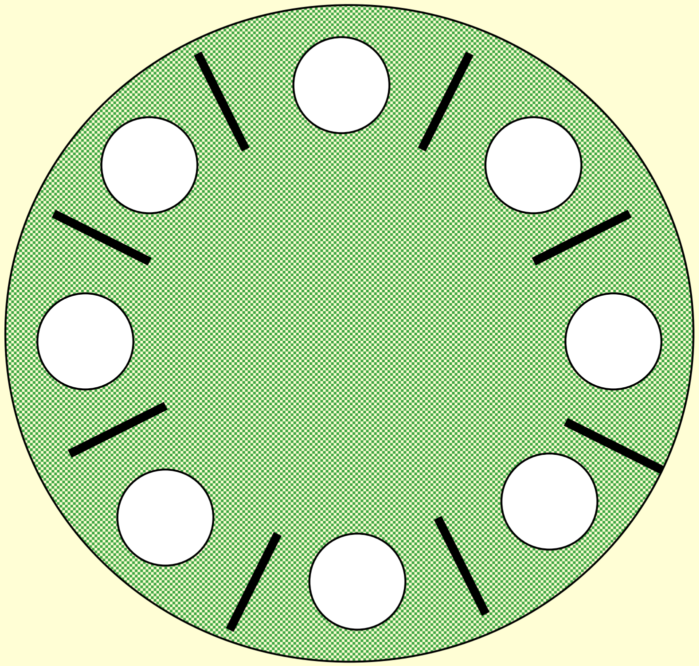
$\pi^c = \left(\pi_1^c, \ldots, \pi_n^c \right)$ trace of steady state probability

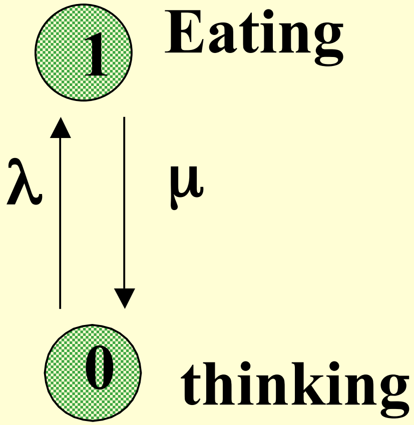
$$\pi_i^c = \begin{cases} 0 \text{ if } i \in \mathrm{F} \\ \mathrm{G}\pi_{\mathrm{i}} \text{ if } i \notin \mathrm{F} \end{cases}$$

G normalization constant $= \left(\sum_{i \notin \mathrm{F}} \pi_i \right)^{-1}$

Truncation example

Dining philosophers

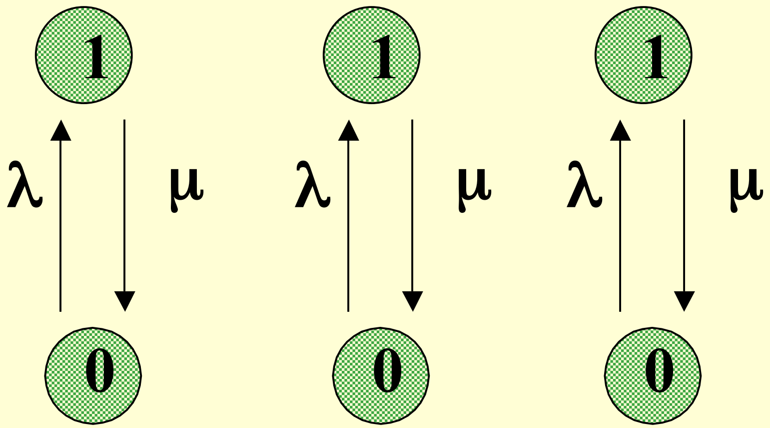


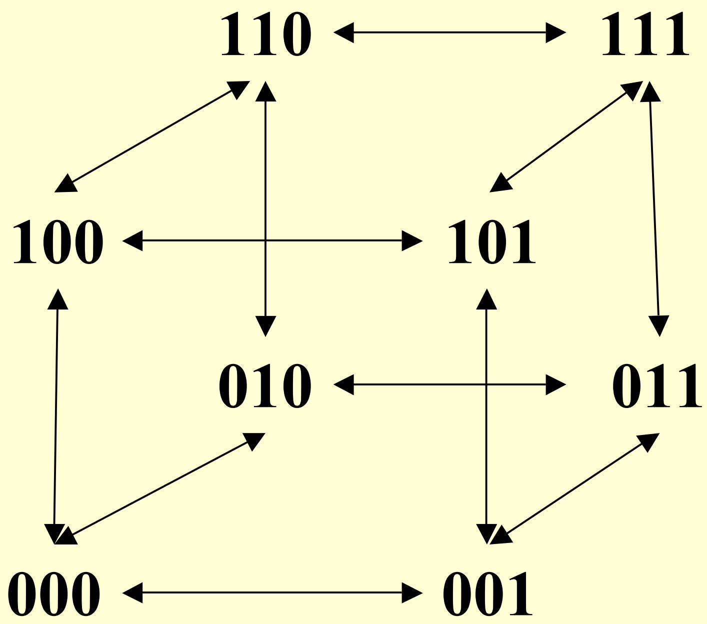


$$\pi_0 = \frac{\mu}{\lambda + \mu}$$

$$\pi_1 = \frac{\lambda}{\lambda + \mu}$$

Truncation example (2)





Independent, no constraints

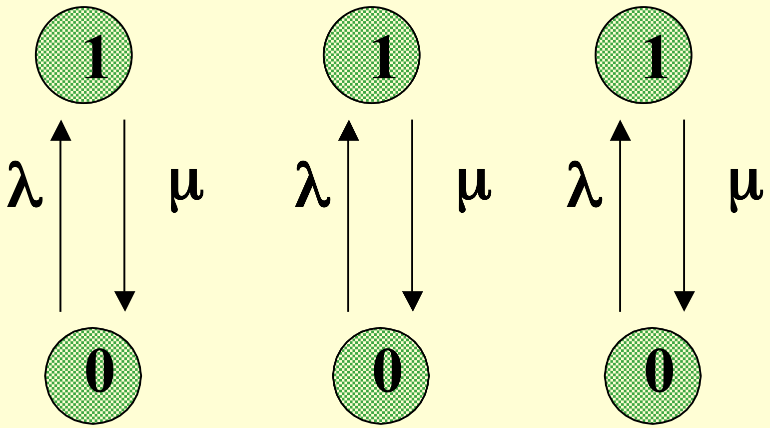
reversible

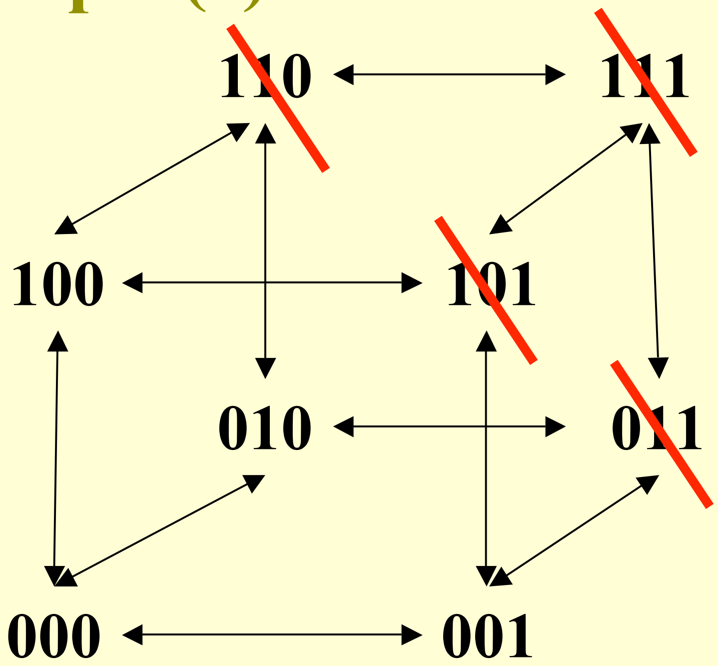
$$\pi(a_1,\ldots,a_n) = \pi_0^{\sum(1-a_i)} \pi_1^{\sum a_i}$$

$$\pi_0 = \frac{\mu}{\lambda+\mu}$$

$$\pi_1 = \frac{\lambda}{\lambda+\mu}$$

Truncation example (3)





$$\pi^c(a_1,\ldots,a_n) = G\pi_0^{\sum(1-a_i)}\pi_1^{\sum a_i}$$

$$G = \left(\pi(0,0,0)+\pi(1,0,0)+\pi(0,1,0)+\pi(0,0,1)\right)^{-1}$$