# Problem Set 2 - Quantitative Macroeconomics

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# Question 1) Question 1. Computing Transitions in a Representative Agent Economy

Consider the following closed optimal growth economy populated by a large number of identical infinitely lived households that maximize:

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \tag{1}$$

over consumption and leisure $u(c_t) = ln(c_t)$, subject to:

$$c_t + i_t = y_t \tag{2}$$

$$y_t = k_t^{1-\theta}(zh_t)^{\theta} \tag{3}$$

$$i_t = k_{t+1} - (1-\delta)k_t \tag{4}$$

Set labor share to $\theta = .67$. Also, to start with, $h_t = .31$ for all $t$. Population does not grow.

**(a) Compute the steady-state. Choose $z$ to match an annual capital-output ratio of 4, and an investment-output ratio of .25.**

First, I are going to derive the Euler equation of this problem by taking the FOCs. Rearraging the three constraints (equations 2, 3 and 4) we find the Lagrangean of this problem:

$$\mathcal{L}(c_t, k_{t+1}) = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t ln(c_t) \right\} + \lambda_t[k_t^{1-\theta}(zh_t)^{\theta} - k_{t+1} + (1-\delta)k_t - c_t]$$

So that the FOCs read as follows:

$$\frac{\mathcal{L}(\partial c_t, k_{t+1})}{\partial c_t} = 0 \rightarrow \beta^t \frac{1}{c_t} = \lambda_t \tag{5}$$

$$\begin{aligned} \frac{\mathcal{L}(\partial c_t, k_{t+1})}{\partial k_{t+1}} = 0 &\rightarrow \lambda_{t+1}(1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} - \lambda_t + \lambda_{t+1}(1-\delta) = 0 \\ &\rightarrow \lambda_{t+1}[(1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta)] = \lambda_t \\ &\rightarrow (1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + 1 - \delta = \frac{\lambda_t}{\lambda_{t+1}} \end{aligned} \tag{6}$$

From here (by combining the two FOCs), we can get the Euler equation of this economy, which is:

$$\frac{c_{t+1}}{\beta c_t} = (1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + 1 - \delta \tag{7}$$

So that:

$$c_{t+1} = \beta c_t[(1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + 1 - \delta] \tag{8}$$

And imposing the steady state, we find:

$$\frac{1}{\beta} = (1-\theta)k^{*-\theta}(zh_t)^{\theta} + 1 - \delta = (1-\theta)\left(\frac{zh_t}{k^*}\right)^{\theta} + 1 - \delta \tag{9}$$

Which implies that (after some algebra),

$$k^* = \left(\frac{\beta(1-\theta)}{1-\beta(1-\delta)}\right)^{\frac{1}{\theta}} zh_t \tag{10}$$

In the steady state, we have (after normalizing $y^* = 1$):

$$i^* = k^* - (1-\delta)k^* = k^* - k^* + \delta k^*$$
$$y^* = k^{*1-\theta}(zh_t)^{\theta} = 1$$

Second, we follow by exactly seeing what it means to determining $z$ to match an annual capital-output ratio of 4, and an investment-output ratio of .25 in the steady state (having in mind the normalization we have done). This implies:

$$\frac{k^*}{y^*} = \frac{k^*}{1} = 4 \Rightarrow k^* = 4 \tag{11}$$

And,

$$\frac{i^*}{y^*} = \frac{k^* - (1-\delta)k^*}{1} = \delta k^* = 0.25 \Rightarrow \delta = \frac{0.25}{4} = \frac{1}{16} \tag{12}$$

So that consumption in the steady state is:

$$c^* = y^* - i^* = y^* - \delta k^* = 1 - \frac{0.25}{4}4 = 0.75$$

Now, we can use all this information to find the $z$ we were looking analytically using the production function so that:

$$y^* = k^{*1-\theta}(zh_t)^{\theta} = 1 \Rightarrow z^{\theta} = k^{*\theta-1}(h_t)^{-\theta} \Rightarrow z = k^{*\frac{\theta-1}{\theta}}(h_t)^{-1} = 1.630$$

Finally, I am going to use *Python*, to find the $\beta$ that match this model (having in mind that we have normalized $y^* = 1$), using the Euler Equation. And, we obtain that $\beta = 0.9804$.

Summary of the results:

$$z = 1.6297$$
$$\delta = 0.0625$$
$$\beta = 0.9804$$
$$y^* = 1$$
$$k^* = 4$$
$$c^* = 0.75$$
$$i^* = 0.25$$

**(b) Double permanently the productivity parameter $z$ and solve for the new steady state.**

In this part of the exercise, I am gonna to solve the new steady state by doubling the $z$, so that I am gonna to change every $z$ for $2z$. To solve this part, I am gonna to use the fact that we have already calculated $\beta$ in part a) so that I can use the Euler Equation I have found in part a) (equation 9) to determine the new $k^*$ by putting the new $z$. From, here, I can obtain all the variables needed for the new equilibrium.

Summary of the results:

$$z = 3.2594$$
$$\delta = 0.0625$$
$$\beta = 0.9804$$
$$y^* = 2$$
$$k^* = 8$$
$$c^* = 1.5$$
$$i^* = 0.5$$

After doubling $z$, all the variables of interest (consumption, capital, output and investment) have doubled as well.

**(c) Compute the transition from the first to the second steady state and report the time-path for savings, consumption, labor and output.**

Things I am going to use to define the function in Python (using a "class"):

- Definition of $k^*$, as defined in equation 10.
- Definition of $c^*$, using $k^*$ and the three constraints (equations 2, 3, 4) so that:

$$c^* = k^{*1-\theta}(zh_t)^\theta - \delta k^* \tag{13}$$

- Law of motion of the capital stock:

$$k_{t+1} = i_t + (1-\delta)k_t = k_t^{1-\theta}(zh_t)^\theta - c_t + (1-\delta)k_t \tag{14}$$

- Euler Equation:

$$c_{t+1} = \beta c_t[(1-\theta)k_{t+1}^{-\theta}(zh_t)^\theta + 1 - \delta] \tag{15}$$

- "Forward shooting" algorithm (from Judd (1992)) to determine the transmission paths between different steady states.

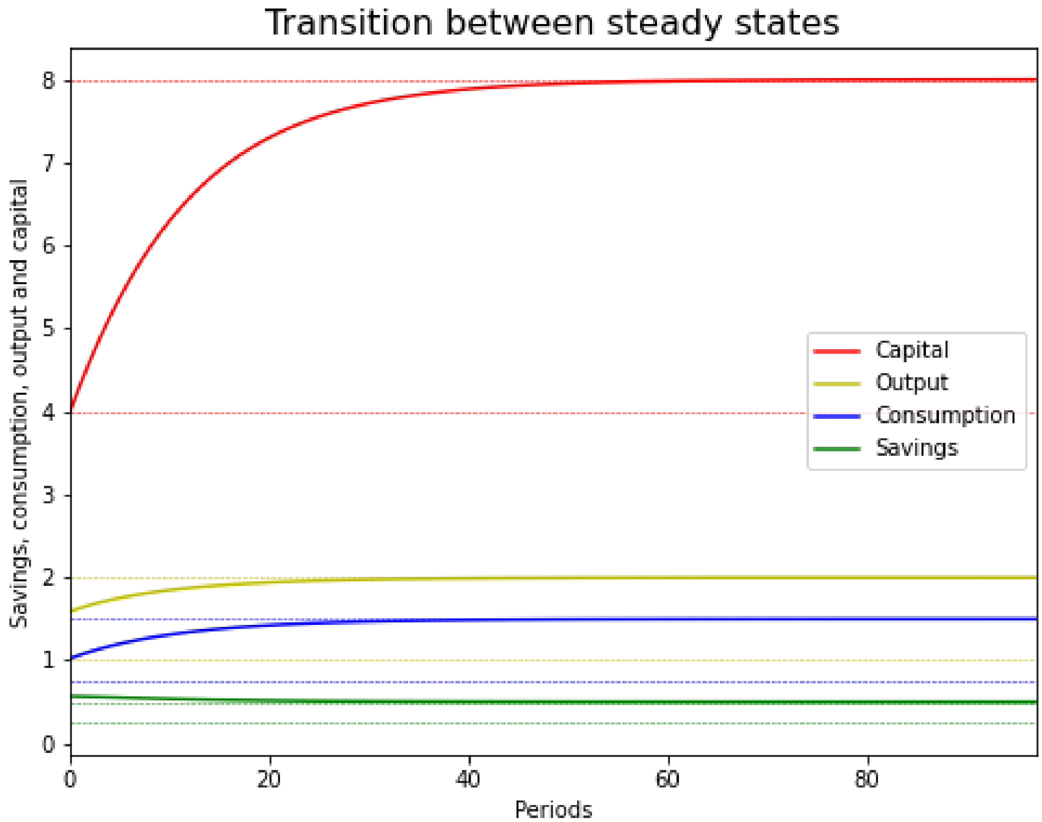


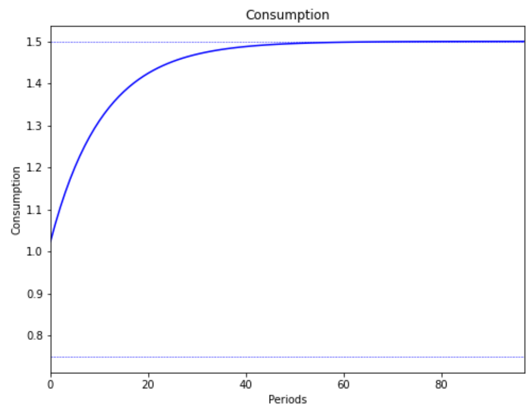

Figure 1: Transition between steady states

The graphical representation of the transition between the two steady states can be seen in figures 1 and 2. It is important to notice that the transition between the two steady states lasts for 98 periods according to the tolerance set in this case.

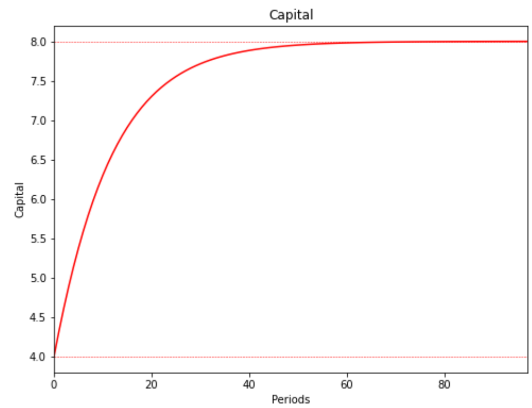
In figure 1 and 2, we can see that the initial effects productivity shock, which results in a higher level of production makes the investment (savings) decision to "overreact" since it goes above the next steady state that it will reach. The jump of the consumption decision is proportional to the one made by the production. From this initial shock the consumption, capital and output increases (proportionally more in the first periods) to the new steady state whereas savings decreases to reach the new steady state.

Figure 2: Transition between steady states

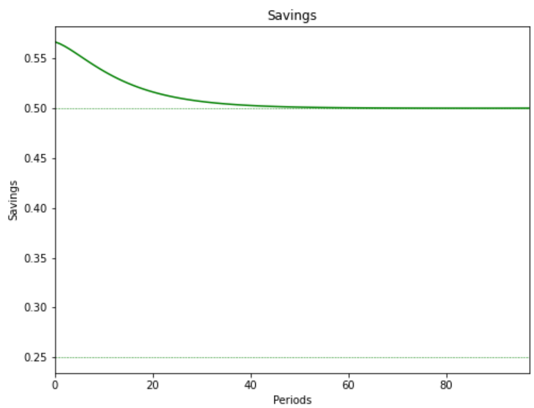
(a) Consumption



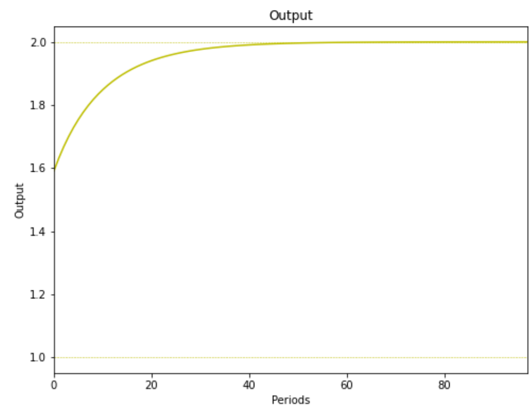

(b) Capital




(c) Savings




(d) Output




**(d) Unexpected shocks. Let the agents believe productivity $z_t$ doubles once and for all periods. However, after 10 periods, surprise the economy by cutting the productivity $z_t$ back to its original value. Compute the transition for savings, consumption, labor and output.**

I have used the same procedure as for part c. In this case, the transition after the productivity shock hits the agent lasts for 106 periods (ten period in the new steady state and 96 periods of transition, two periods less than in part c). The transition can be seen in figures 3 and 4.

As we can see in figures 3 and 4, again (as we already saw in 1), the savings "overreacts" going below the level of the new steady state. after this initial shock, consumption, capital and output decreases (proportionally more in the first periods) to the new steady state whereas savings increases to reach the new steady state.

I have also included a figure with the overall transition of exercises 1.c and 1.d to have

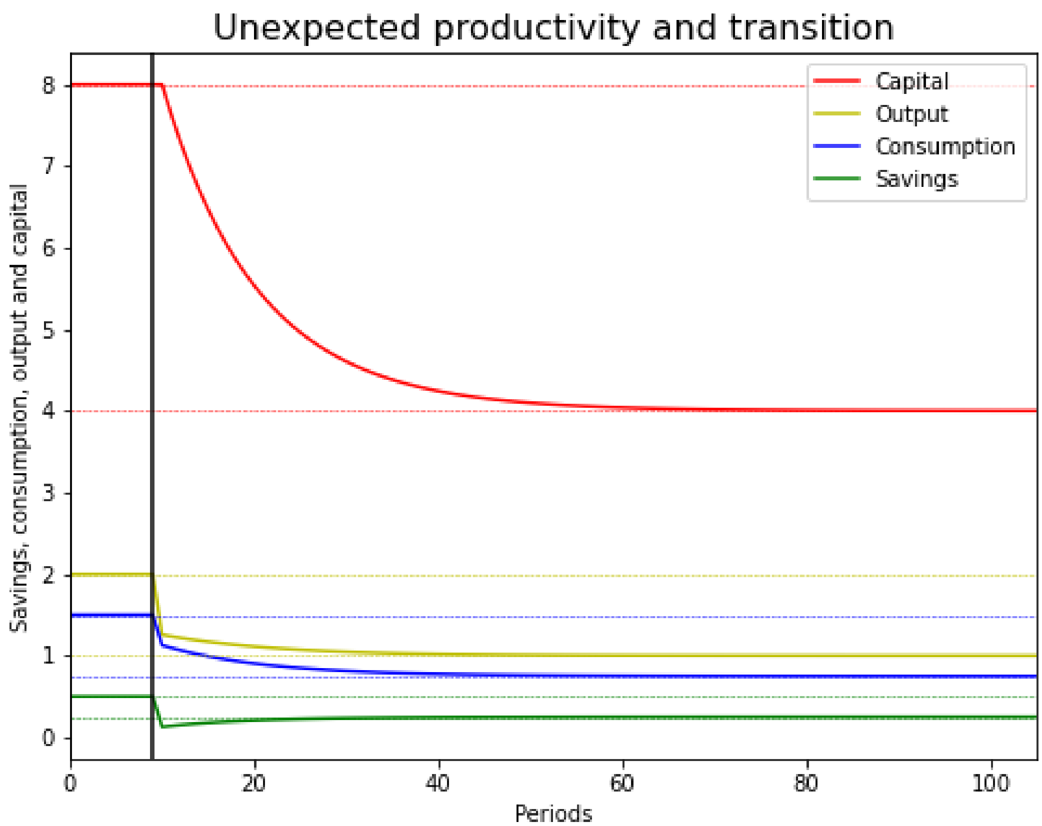


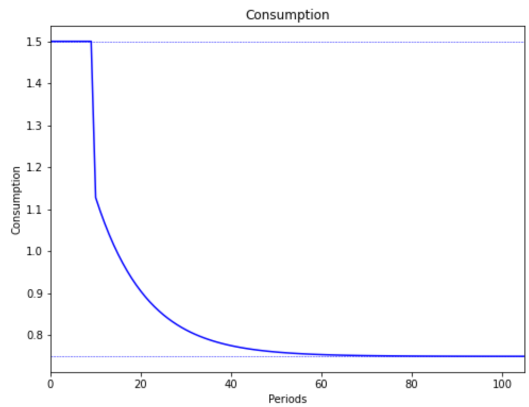

Figure 3: Unexpected productivity shock

a more complete image of the transitions between steady states. This can be seen in figure 5 below. In figure 5, we can see that clearly this overreaction of the savings decision in the first period after the shock, as I have commented before.

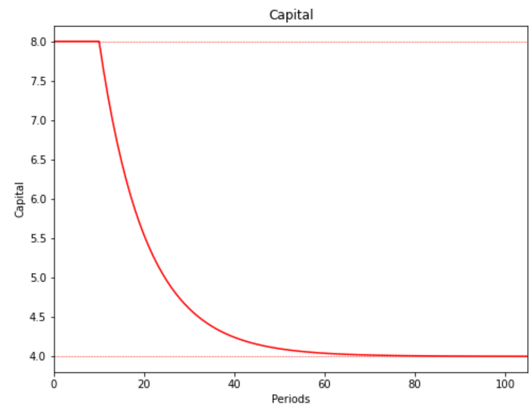
As a practice exercise, I have also computed the following transition. Starting in the first steady state (the one computed in part a), I double the productivity but for only ten periods and then I bring back the economy to the first steady state. This process is also a potential situation compatible with the exercise presented. It lasts for 99 periods (10 of the first transition and 89 after the unexpected shock). The transition can be seen in figure 6 below. In figure 6, we can see a similar pattern as we saw for the other situation exploited for this part. The shock generates a proportionally stronger reaction to the savings decision compared to consumption, as we have seen in the other setup I proposed for this question.

Figure 4: Unexpected productivity shock

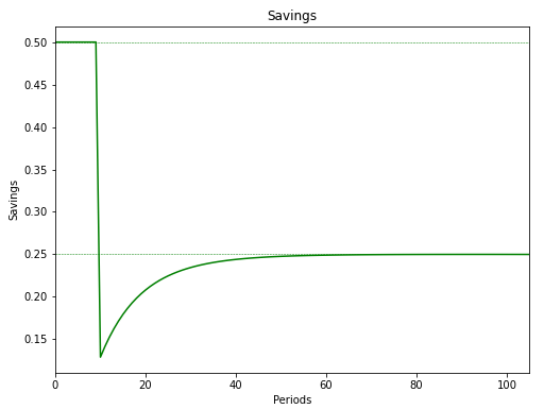
(a) Consumption



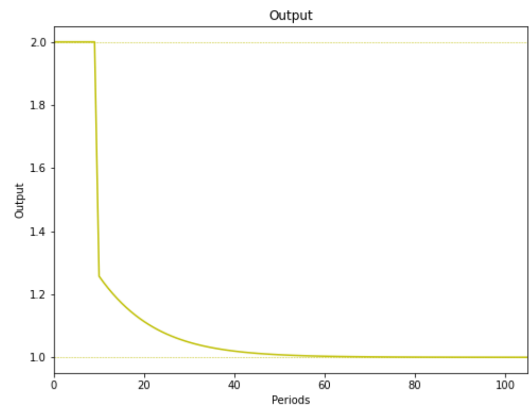

(b) Capital




(c) Savings




(d) Output




**(e) Bonus Question: Labor Choice Allow for elastic labor supply. That is, let preferences be**

$$u(c_t, 1-h_t) = lnc_t - \kappa\frac{h_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \tag{16}$$

**and recompute the transition as posed in Question 1.**

INCOMPLETE ANSWER:

Following the same process as before, I am going to set the Lagrangean to then compute the FOCs and try to bring it to *Python* to the solve the model. It is important to notice that now $h_t$ becomes an endogenous variables as it affects directly the utility of the agent.

Rearraging the three constraints (equations 2, 3 and 4) we find the Lagrangean of this

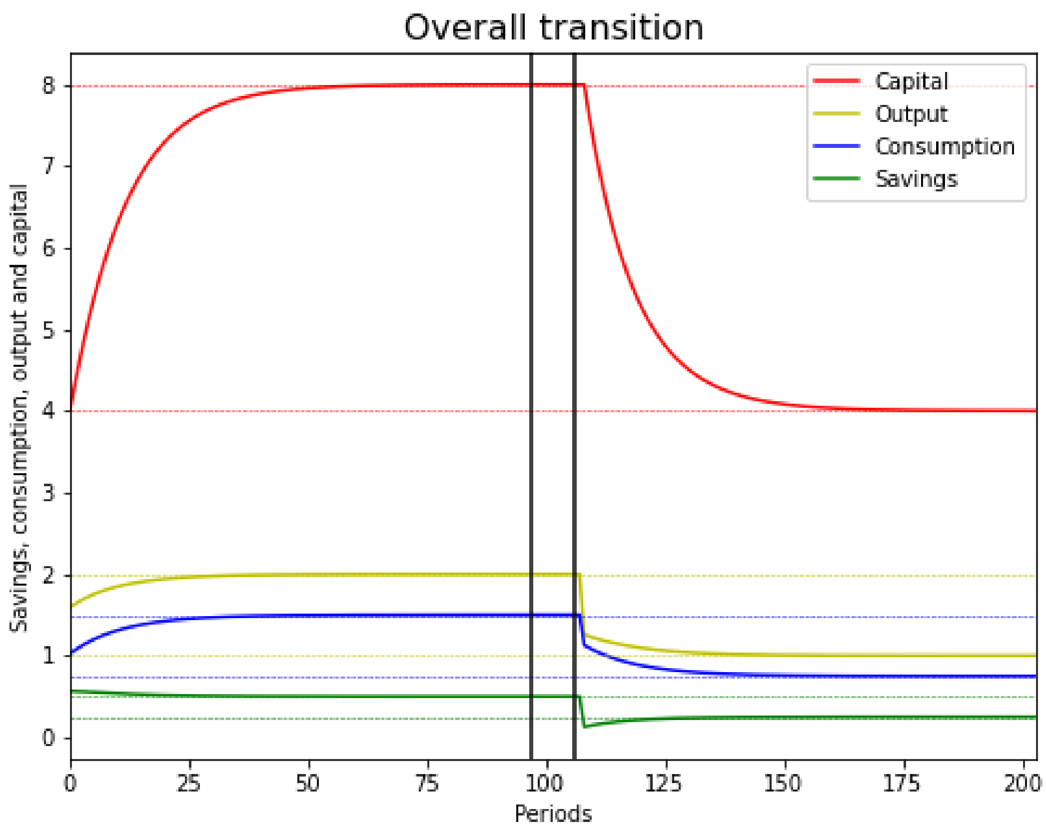



Figure 5: Overall transition

problem:

$$\mathcal{L}(c_t, h_t, k_{t+1}) = \mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left( lnc_t - \kappa \frac{h_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right) \right\} + \lambda_t [k_t^{1-\theta}(zh_t)^{\theta} - k_{t+1} + (1-\delta)k_t - c_t]$$

So that the FOCs read as follows:

$$\frac{\mathcal{L}(\partial c_t, h_t, k_{t+1})}{\partial c_t} = 0 \rightarrow \beta^t \frac{1}{c_t} = \lambda_t \tag{17}$$

$$\frac{\mathcal{L}(\partial c_t, h_t, k_{t+1})}{\partial h_t} = 0 \rightarrow \beta^t \kappa h_t^{\frac{1}{\nu}} = \lambda_t \theta [k_t^{1-\theta} z^{\theta} h_t^{\theta-1}] \tag{18}$$

$$\begin{aligned} \frac{\mathcal{L}(\partial c_t, h_t, k_{t+1})}{\partial k_{t+1}} = 0 &\rightarrow \lambda_{t+1}(1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} - \lambda_t + \lambda_{t+1}(1-\delta) = 0 \\ &\rightarrow \lambda_{t+1}[(1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + (1-\delta)] = \lambda_t \\ &\rightarrow (1-\theta)k_{t+1}^{-\theta}(zh_t)^{\theta} + 1 - \delta = \frac{\lambda_t}{\lambda_{t+1}} \end{aligned} \tag{19}$$

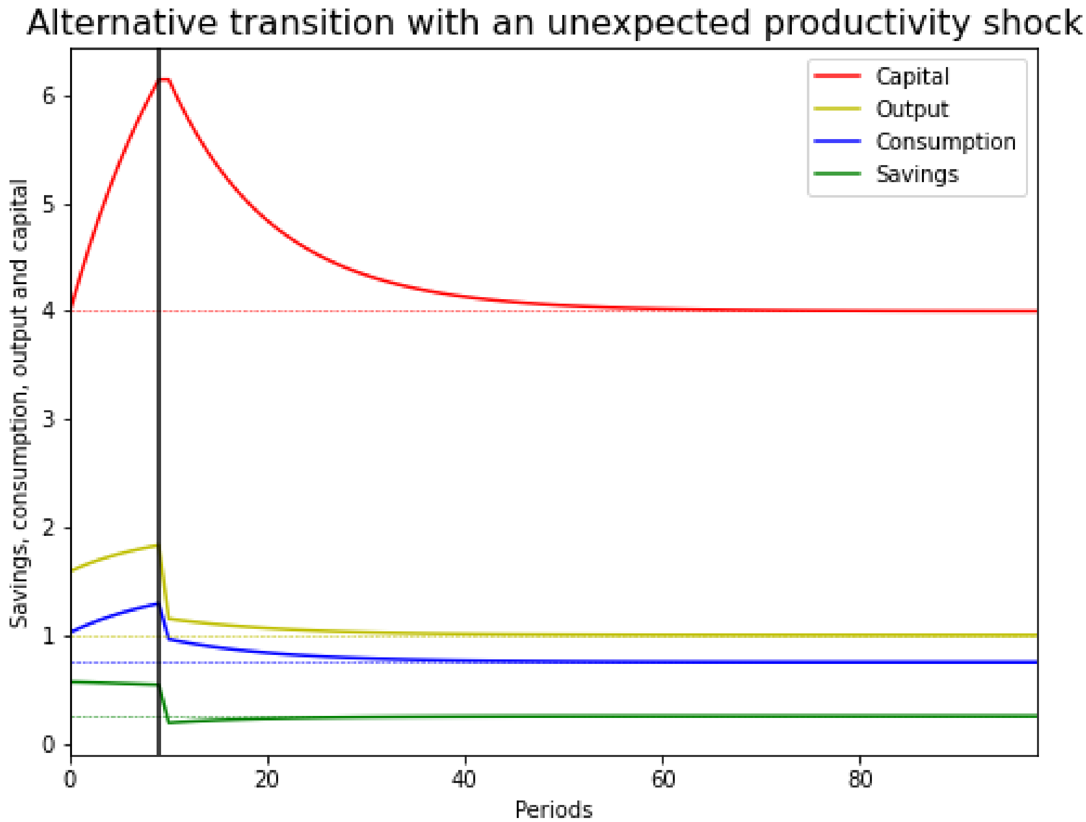



Figure 6: Alternative transition of the unexpected productivity shock

As we have saw in part a of this exercise in equation 10 (after some algebra and imposing the steady state),

$$k^* = \left(\frac{\beta(1-\theta)}{1-\beta(1-\delta)}\right)^{\frac{1}{\theta}} zh^*$$

And combining equations 17 and 18, we find that:

$$\frac{1}{c_t} = \frac{\kappa}{k_t^{1-\theta} z^{\theta}} h_t^{\frac{1}{\nu}-\theta+1} \tag{20}$$

So that in the steady state:

$$\frac{1}{c^*} = \frac{\kappa}{k^{*1-\theta} z^{\theta}} h^{*\frac{1}{\nu}-\theta+1} \tag{21}$$

So that it follows that:

$$h^* = \left(\frac{k^{*1-\theta} z^{\theta}}{\kappa c^*}\right)^{\frac{1}{\frac{1}{\nu}+1-\theta}} \tag{22}$$

# Question 2. Solve the optimal COVID-19 lockdown model posed in the slides.

The economy of this model is the following:

*Economy.* We have an economy with only one sector associated with a pair Human Contact ($HC$) and Telework ($TW$). The economy is populated with a continuum of ex-ante identical individuals normalized to $N = 1$. If employed, then individuals supply hours inelastically — i.e. $h = 1$. Hence, the aggregate hours are identical to the size of employment, $H = hE = E \leq N$.

*Production Technology.* Production in this economy can be done at the workplace or teleworking at home. Working at workplace entails a risk of catching a flu, while the risk of catching a flu working at home is null. The production technology combines work at the workplace and telework as follows:

$$Y = \left( A_f H_f^{\frac{\rho-1}{\rho}} + c(TW) A_f H_{nf}^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}} \tag{23}$$

where $H_f$ denotes the aggregate hours at the workplace and $H_{nf}$ denotes the aggregate hours teleworking. $c(TW) \in [0, 1]$ is a factor that captures a productivity loss associated with teleworking.

*Contagion Process.* Individuals only interact at work. Hence, only employed individuals face contagion risk. The risk occurs only at the workplace, not with telework. The number of infections depends on how much human contact (HC) there is at the workplace. Precisely, we define the (unconditional) infection rate $i$ at which an worker gets infected as:

$$i = \beta(HC) m(H_f) \quad \text{with} \quad m(H_f) = \frac{i_0 H_f}{N} \tag{24}$$

where the odds that an employed individual meets with a contagious individual at work is denoted by the meeting probability $m(H_f) = \frac{i_0 H_f}{N}$, $i_0 \in (0, 1]$ denotes the initial share of infections at work and $\beta(HC) \in [0, 1]$ is the (conditional) infection rate which depends on the extent of human contact. Then, the number of infections is

$$I = i H_f \tag{25}$$

Some of this infections translate into deaths of employed individuals according to rate $1 - \gamma$:

$$D = (1 - \gamma) I \tag{26}$$

where D denotes the total number of deaths due to the flu.

*Panner's problem.* A planner chooses consumption c and hours worked h as to maximize the aggregate welfare of individuals while alive taking into account the number of deaths that this potentially generates. We assume individuals consume before dying. This implies the following planner program:

$$\max_{\{c,h_f,h_{nf}\}_i} \sum_i (c_i - \kappa_f h_f - \kappa_{nf} h_{nf}) - \omega D \tag{27}$$

where $\omega$ denotes how much the planer cares about the number of deaths $D = D_f + D_{nf}$. Inserting the aggregate resource constraint,

$$Y = \sum_i c_i \equiv C \tag{28}$$

and labor market clearing $\sum_{i \in j} h_i = H_j$ for $j = \{f, nf\}$, into the planner's problem (equation 27), we can rewrite the program as:

$$\max_{\{H_j \in [0,N]\}_{j=\{f,nf\}}} Y(H_f, H_{nf}) - \kappa_f H_f - \kappa_{nf} H_{nf} - \omega D \tag{29}$$

subject to

$$H = H_f + H_{nf} \leq N \tag{30}$$

where note that if $\beta(HC) = 0$, $\omega = 0$ or $\gamma = 1$, then there is no additional welfare cost associated with the flu. Further, note that even in a fluless economy, there can be labor that teleworks from home in equilibrium—a feature that is present in the data—due to potential differences in preferences, $\kappa$, that can overcome productivity losses in telework.

**(a) Show your results for a continuum of combinations of the $\beta \in [0, 1]$ parameter (vertical axis) and the $c(TW) \in [0, 1]$ parameter (hztal axis). That is, plot for each pair of $\beta$ and $c(TW)$ the optimal allocations of $H$, $H_f$, $H_{nf}$, $\frac{H_f}{H}$, output, welfare, amount of infections and deaths. Note that if $H = N$ there is no lockdown, so pay attention to the potential non-binding constraint $H < N$. Discusss your results. You may want to use the following parameters: $A_f = A_{nf} = 1$; $\rho = 1.1$, $\kappa_f = \kappa_{nf} = 0.2$, $\omega = 20$, $\gamma = 0.9$, $i_o = 0.2$ and $N = 1$.**

I have solved this creating a grid to evaluate $c(TW)$ and $\beta(HC)$ at each point of the grid (101 points for each of the two parameters). Below I am gonna to present the results for each of the variables that exercise asks us to do. The results can be seen in figure 7. I am gonna to summarize the main conclusions of these figures. The main conclusions we can obtain are:

- To determine the $H_f$ and $H_{nf}$, $c(TW)$ is much more determinant for the social planner compared to $\beta(HC)$. That is, the ability of the population to telework is much more determinant for the social planner compared to the conditional infection rate of the population in this economy with the parameters used given that $c(TW)$ enters directly the production function and the output of this economy is much larger than the cost of labor and the cost for the SP to have more deaths.

- As expected, a higher telework ability (captured by $c(TW)$) is associated with a higher population doing telework.
- $H = 1$ with this parametrization for any point of the grid. The explanation is the same as in the first point (the output is much larger than the cost of labor and the cost for the SP of more deaths).
- Since $H = 1$, the heat map of the ratio $H_f/H$ is the same as the $H_f$ one.
- For the highest levels of telework ability, both the output and the welfare go up by a substantial amount.
- As it could be expected, high levels of the conditional infection rate ($\beta(HC)$) and low levels of telework ability ($c(TW)$) are associated with a higher level of both infections and deaths.

### (b) What happens to your results when you increase (decrease) $\rho$ or $\omega$?

I am gonna to structure this part by summarizing the results by varying the parameters of the $\rho$ and $\omega$. At the end of the document, I have added four figures (one increasing, and one decreasing $\rho$ and one increasing, and one decreasing $\omega$). The main observations are the following:

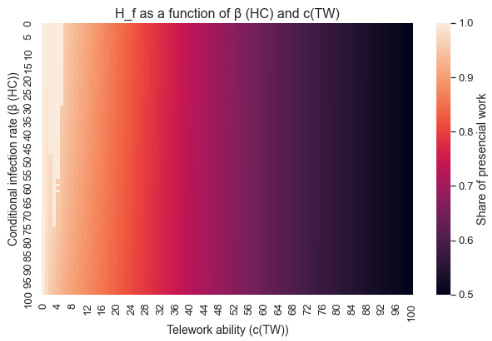
- Increasing $\omega$ implies that the social planner cares more about the agents dying. Therefore, when increasing $\omega$ (example in figure 8), it is remarkable that $H$ is not longer 1 for all the points of the grid (the regions where agents have a lower telework ability and the conditional infection rate is higher). At some points of this region, it even tends to 0 as the SP prefers not to send anybody to work. The output and the welfare maps have a similar form as before. In addition, since $\omega$ is higher, the number of infected people and deaths decreases in this case.
- Decreasing $\omega$ that the SP cares less about the agents dying. The amount of presential works (example in figure 9) increases compared to case when $\omega = 20$. The rest of the figures look very similar to the case when $\omega = 20$. For example, in this case, $H = 1$ in all the heat map for any of the points of the grid.
- Increasing $\rho$ implies increasing the elasticity of substitution $\rho$ and thus, that the social planner and now more easily substitute one input for another (share working at the workplace and share working from home). We can see (example in figure 10) that the amount of the presencial work increases substantially since now the part inside the

production function becomes more relevant. Another remarkable change is that the quantitative amount of output and welfare decreases substantially with the $\rho$ parameter I have proposed.

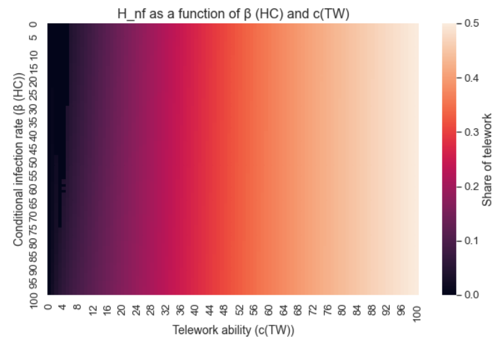
- Decreasing $\rho$ implies decreasing the elasticity of substitution $\rho$ and thus, that the social planner and now less easily substitute one input for another (share working at the workplace and share working from home) . We can see (example in figure 11), the amount of infections and deaths decreases substantially, tending to 0 for almost all the parametrizations. Moreover, the output and the welfare also decreases in a substantial manner compared to the case $\rho = 1.1$. The consequences of this is that the the amount of deaths have a higher effect on the utility function of the SP so that for some levels of the parameters (high $\beta(HC)$ and low telework ($c(TW)$ ability ), $H$ is not 1 by not allowing some part of the population to work (not even teleworking) because of the form of the utility function.

Figure 7: COVID-19 lockdown model as a function of $\beta(HC)$ and $c(TW)$

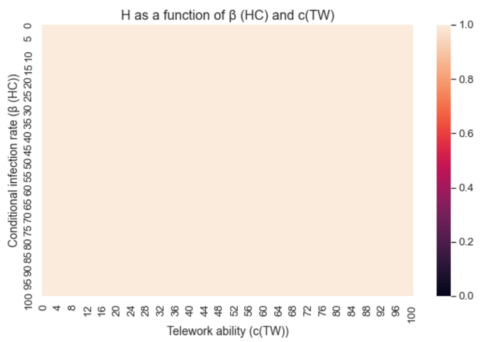
(a) $H_f$



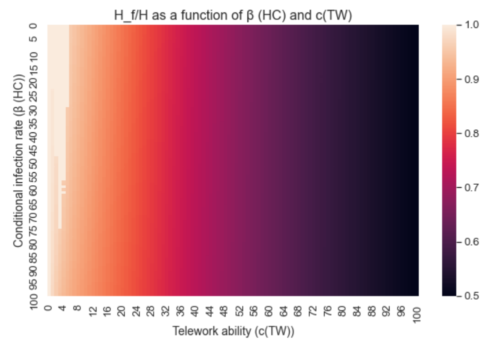

(b) $H_{nf}$



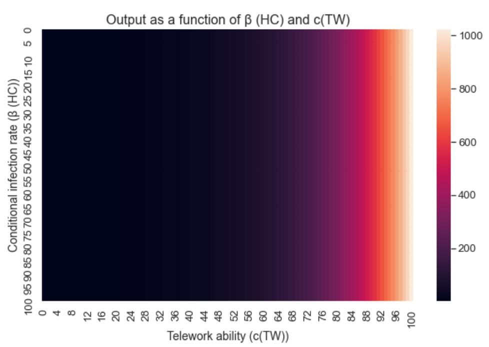

(c) $H$



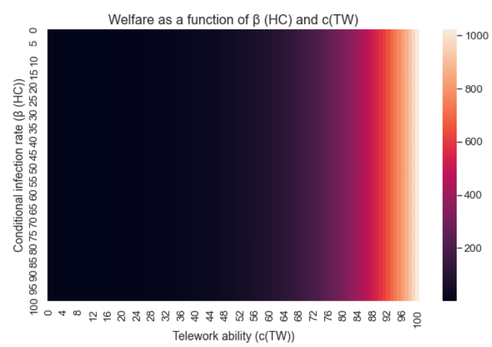

(d) Ratio $H_f/H$



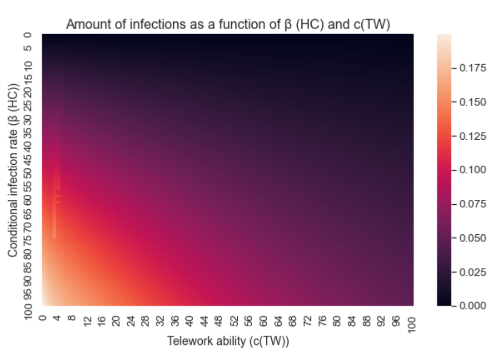

(e) Output




(f) Welfare




(g) Amount of infections




(h) Deaths

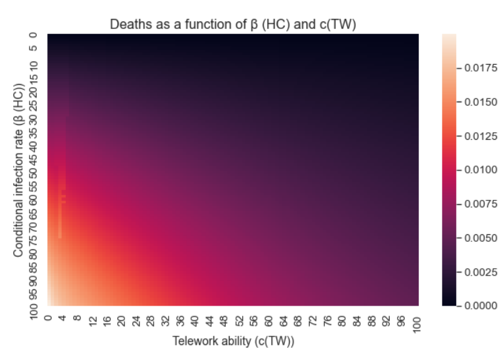
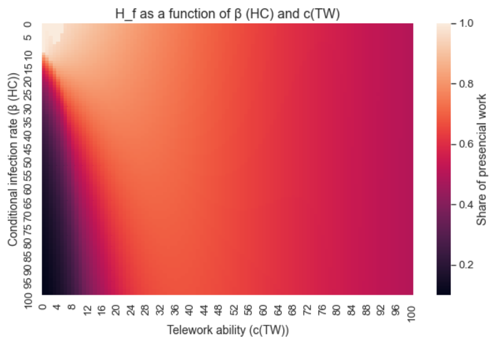
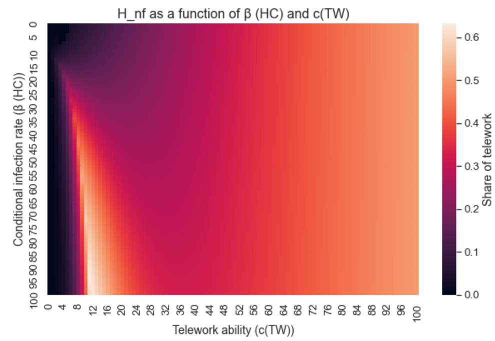


Figure 8: $\omega = 200$

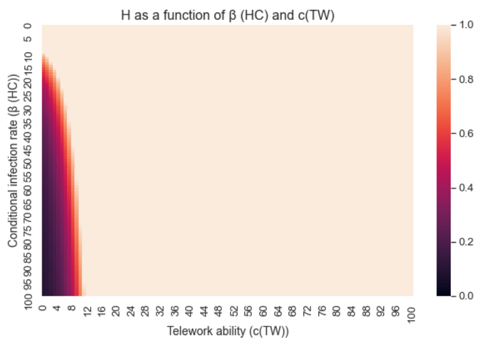
(a) $H_f$



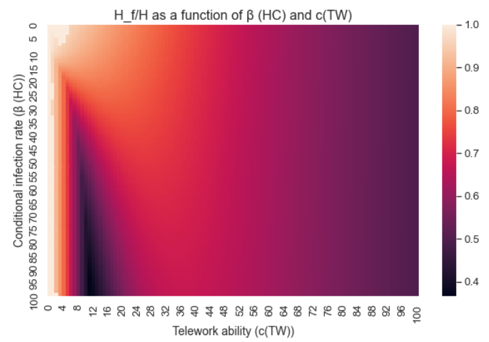

(b) $H_{nf}$



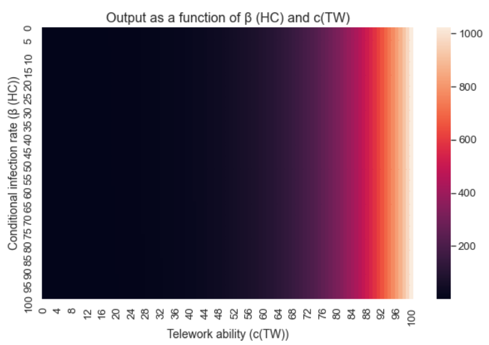

(c) $H$



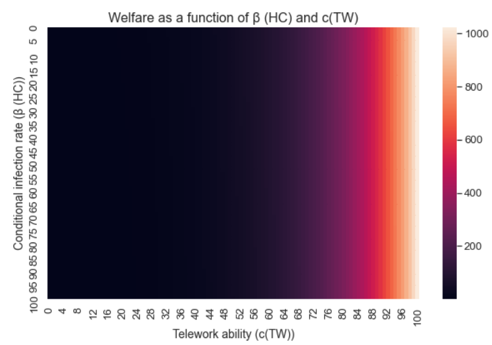

(d) Ratio $H_f/H$



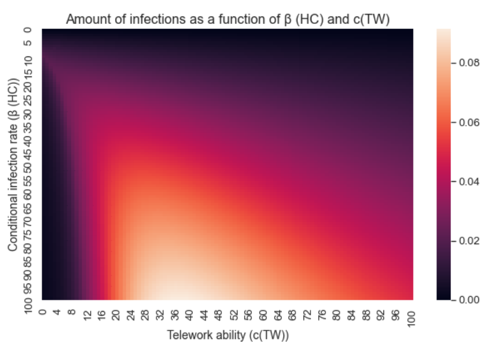

(e) Output




(f) Welfare




(g) Amount of infections




(h) Deaths

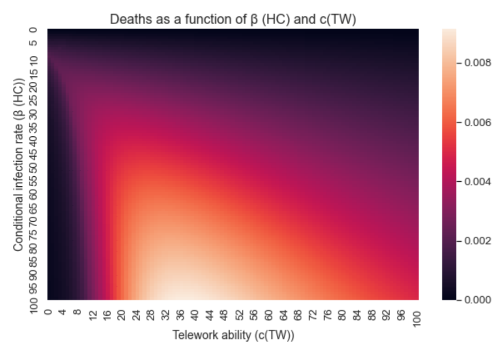
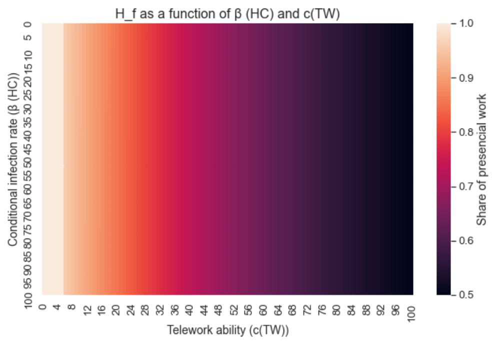
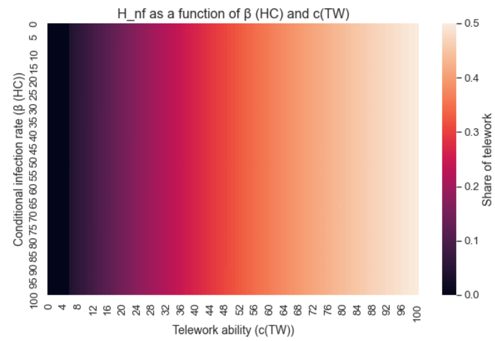


Figure 9: $\omega = 1$

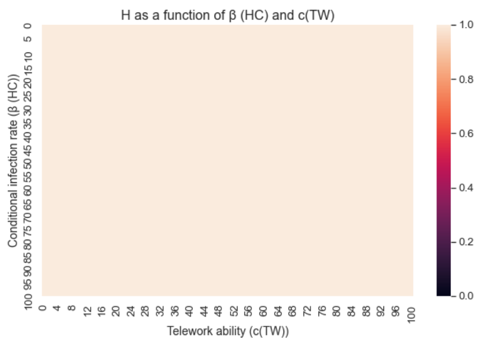
(a) $H_f$



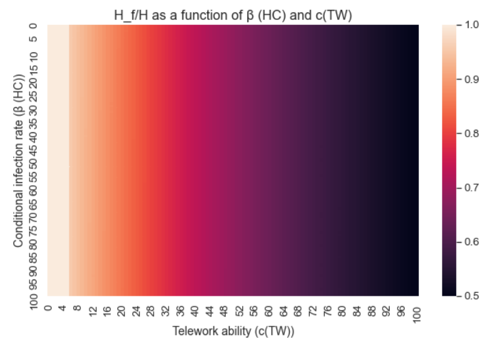

(b) $H_{nf}$



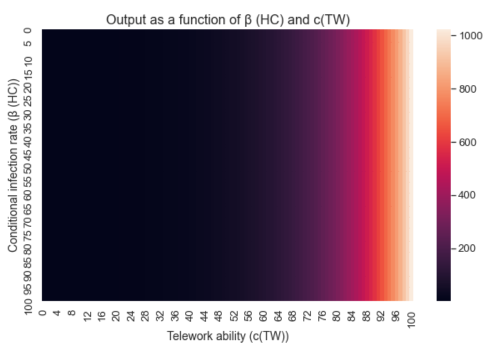

(c) $H$



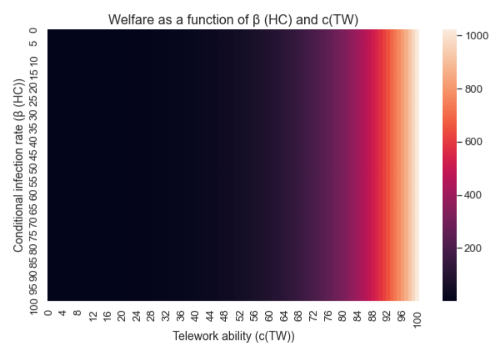

(d) Ratio $H_f/H$



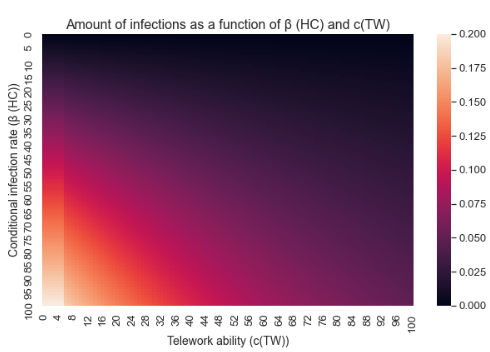

(e) Output




(f) Welfare




(g) Amount of infections




(h) Deaths

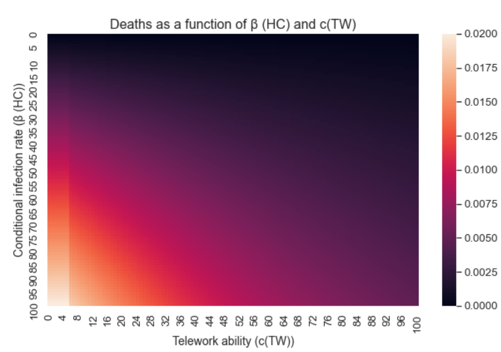
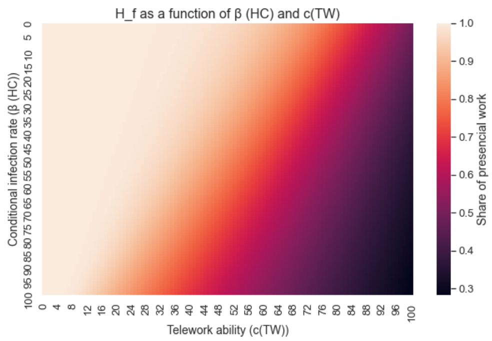
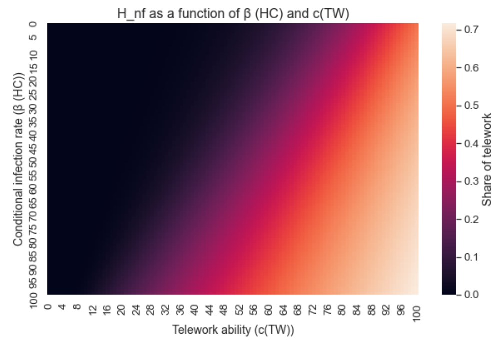


Figure 10: $\rho = 5$

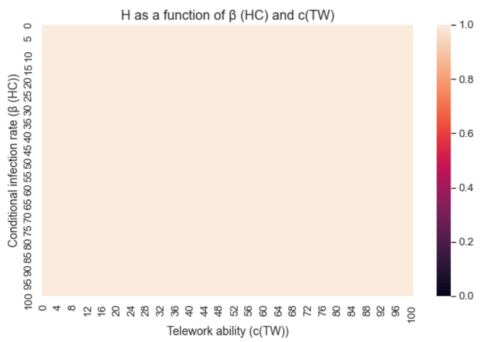
(a) $H_f$



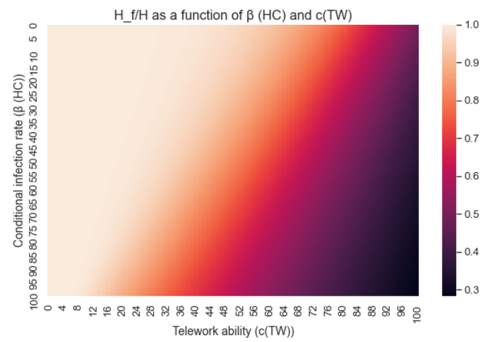

(b) $H_{nf}$



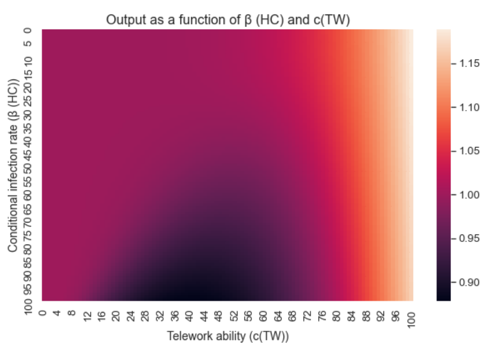

(c) $H$



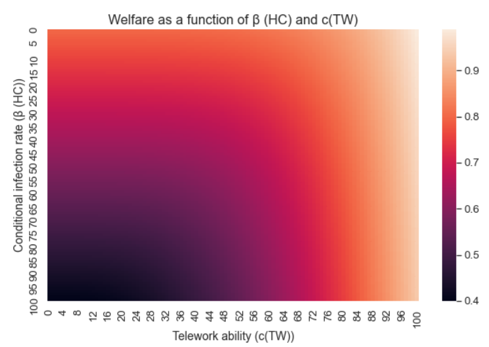

(d) Ratio $H_f/H$



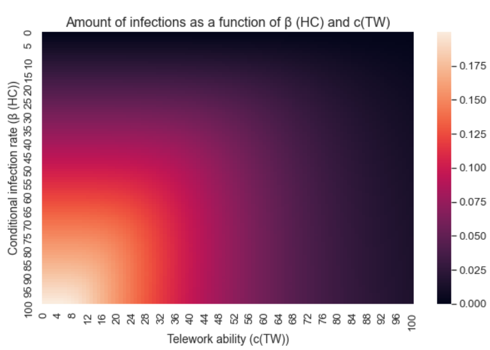

(e) Output




(f) Welfare




(g) Amount of infections




(h) Deaths

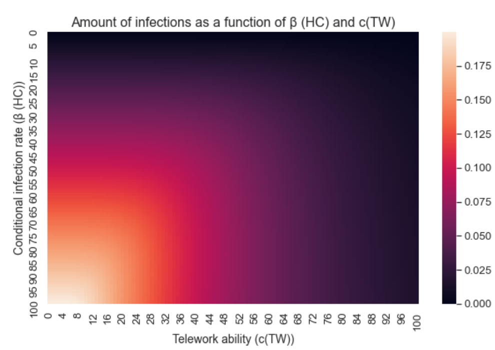
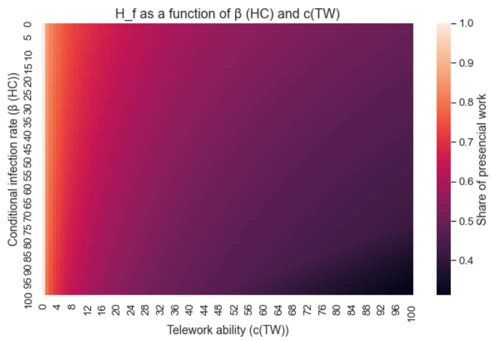
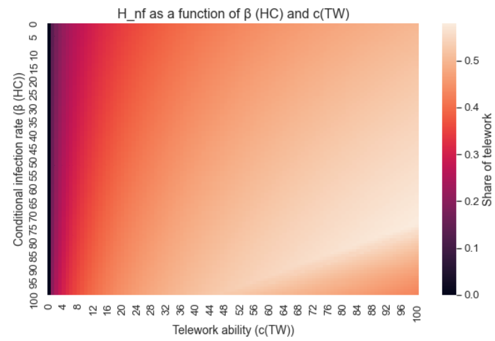


Figure 11: $\rho = 0.4$

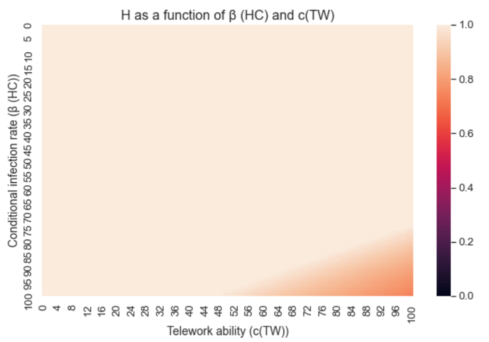
(a) $H_f$



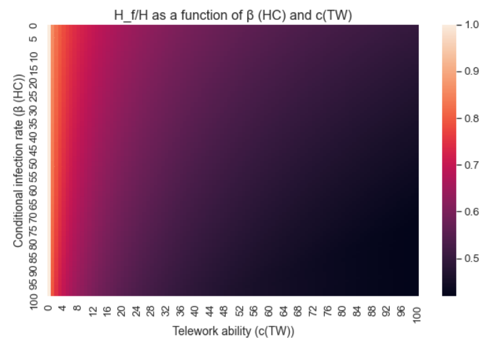

(b) $H_{nf}$



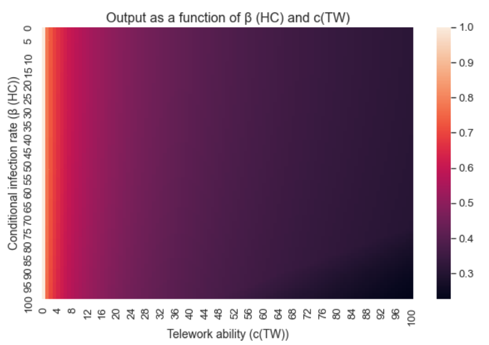

(c) $H$



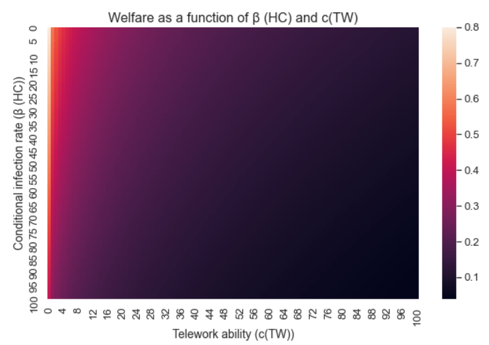

(d) Ratio $H_f/H$



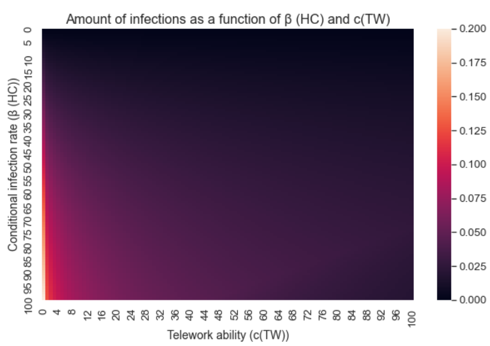

(e) Output




(f) Welfare




(g) Amount of infections




(h) Deaths