CS 323: Numerical Analysis and Computing

MIDTERM #1

Instructions: This is an **open notes** exam, i.e., you are allowed to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are **not permitted** to use laptop computers, cell phones, tablets, or any other hand-held electronic devices.

Name	
NetID	

Part #1	
Part #2	
Part #3	
Part #4	
Part #5	
Part #6	
TOTAL	

1. [24% = 4 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

   (a) Given a non-singular $n \times n$ system of linear equations $Ax = b$, the solution vector x remains unchanged after

   (Circle the TWO most correct answers.)

   i. $\boxed{\text{Permuting the rows of } A \text{ and } b.}$

   ii. Permuting the columns of A.

   iii. $\boxed{\text{Multiplying both sides of the equation from the left by a non-singular } n \times n}$ $\boxed{\text{matrix } M.}$

   (b) Let A be an $m \times n$ matrix. Then the matrix $A^T A$ is always

   (Circle the TWO most correct answers.)

   i. $\boxed{\text{symmetric.}}$

   ii. non-singular.

   iii. positive definite.

   iv. $\boxed{\text{positive semi-definite.}}$

   (c) Suppose that an $n \times n$ matrix A is perfectly well-conditioned, i.e., the condition number $\kappa(A) = 1$ in the L_∞-norm. Which of the following matrices would then necessarily share this same property?

   (Circle the TWO most correct answers.)

   i. $\boxed{cA, \text{ where } c \text{ is any non-zero scalar.}}$

   ii. BA, where B is any non-singular matrix.

   iii. $\boxed{A^{-1}, \text{ the inverse of } A.}$

   iv. DA, where D is a non-singular diagonal matrix.

   (d) What is the condition number of the following matrix using the L_1-norm?

   (Circle the ONE most correct answer.)

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

   i. 2

   ii. $\boxed{3}$

   iii. 1

   iv. 12

2. [18% = 3 questions × 6% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

   (a) Suppose you have already solved the $n \times n$ linear system $Ax = b$ by LU factorization. What is the further cost (order of magnitude will suffice) of solving a new system

   - With the same matrix A, but a different right hand side vector b'?
     Answer: $O(n^2)$.
   - With a different matrix A', but the same right hand side vector b?
     Answer: $O(n^3)$.

   (b) Suppose the second column of the matrix $M_1 A$ during the LU decomposition algorithm is as follows:

$$a_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix}$$

   Specify the elimination matrix M_2 that zeros out the **last two** components of a_2.
   Answer: The matrix M_2 is computed as follows:

$$M_2 = I_{4\times 4} - \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

   (c) If A and B are $n \times n$ matrices, with A non-singular and $c \in \mathbb{R}^n$, how would you efficiently compute the product $A^{-1}Bc$? (A need not be diagonally dominant.)
   Answer: Compute the vector $x = Bc$ using matrix-vector multiplication. Compute the LU decomposition of A. Solve $Ly = x$ using forward substitution, and $Uz = y$ using backward substitution. For sparse matrices, the cost of each of these operations is proportional to the number of non-zero entries in B, L and U.

3. [16%] Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

(a) Show that A is singular.

Answer: The columns of A are linearly dependent because:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$$

(b) If $b = (2, 4, 6)^T$, how many solutions are there to the system $Ax = b$?

Hint: For part (b), you just need to show if $b \in \mathsf{range}(A)$. You are not required to compute the entire solution set. So the answer should be either zero or infinite.

Answer: There are infinite solutions because $b \in \mathsf{range}(A)$. To see this, b can be written as a linear combination of the columns of A, as follows:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

4. [12%] Let A be a rectangular $m \times n$ matrix with linearly independent columns, where $m > n$, and $b \in \mathbb{R}^m$. Show that for every $x \in \mathbb{R}^n$, we have

$$\|Ax - b\|_2^2 = \|A(x - x_0)\|_2^2 + \|Ax_0 - b\|_2^2$$

where $x_0 \in \mathbb{R}^n$ is the *least squares solution* of $Ax = b$.

Hint: Use the identity $\|x\|_2^2 = x^T x$.

Answer: We have

$$\begin{aligned}
\|Ax - b\|_2^2 &= \|A(x - x_0) + (Ax_0 - b)\|_2^2 \\
&= \|[A(x - x_0) + (Ax_0 - b)]^T [A(x - x_0) + (Ax_0 - b)] \\
&= \|A(x - x_0)\|_2^2 + \|Ax_0 - b\|_2^2 + 2(x - x_0)^T A^T (Ax_0 - b) \\
&= \|A(x - x_0)\|_2^2 + \|Ax_0 - b\|_2^2 + 2(x - x_0)^T (A^T A x_0 - A^T b) \\
&= \|A(x - x_0)\|_2^2 + \|Ax_0 - b\|_2^2
\end{aligned}$$

The last equality follows because x_0 satisfies the normal equations.

5. [10%] Prove that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has no LU factorization, i.e., no lower triangular matrix L and upper triangular matrix U exist, such that $A = LU$.

Hint: Assume that there is an LU decomposition

$$\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and arrive at a contradiction!

Answer: Assume that there is an LU decomposition. Then we can write the following equations by equating individual elements on the left and right hand sides:

$$l_{11}u_{11} = 0 \tag{1}$$
$$l_{11}u_{12} = 1 \tag{2}$$
$$l_{21}u_{11} = 1 \tag{3}$$

The equation for the very last element in the second row and second column (a_{22}) is not important, so we have left that out. From equation (2), it follows that l_{11} *cannot* be zero, so equation (1) implies that $u_{11} = 0$. However, in that case, it would be impossible to satisfy equation (3). So we have arrived at a contradiction!

6. [20%] Consider the $n \times n$ linear system $Ax = b$.

   (a) Show that if A is diagonal, the Jacobi method converges after just one iteration.
   Answer: The Jacobi method performs the following iteration:

   $$Dx^{(k+1)} = (L + U)x^{(k)} + b$$

   Since the matrix is diagonal, $A = D$ and $L + U = O$, so after one iteration we have

   $$x^{(1)} = D^{-1}b = A^{-1}b$$

   which is the exact solution.

   (b) Show that if A is lower triangular, the Gauss-Seidel method converges after just one iteration.
   Answer: The Gauss-Seidel method performs the following iteration:

   $$(D - L)x^{(k+1)} = Ux^{(k)} + b$$

   Since A is lower triangular, $U = O$ and $A = D - L$. So after one iteration, we have

   $$x^{(1)} = (D - L)^{-1}b = A^{-1}b$$

   which is the exact solution.

Hint: Decompose $A = D - L - U$ and use the definitions of the Jacobi and Gauss-Seidel methods, as described in the notes.