CS 323: Numerical Analysis and Computing

MIDTERM #1

Instructions: This is an **open notes** exam, i.e., you are allowed to consult any textbook, your class notes, homeworks, or any of the handouts from us. You are **not permitted** to use laptop computers, cell phones, tablets, or any other hand-held electronic devices.

Name	
NetID	

Part #1	
Part #2	
Part #3	
Part #4	
Part #5	
TOTAL	

1. [24% = 4 questions × 6% each] MULTIPLE CHOICE SECTION. Circle or underline the correct answer (or answers). No justification is required for your answer(s).

   (a) Which of the following statements regarding the cost of methods for solving an $n \times n$ linear system $Ax = b$ are true?

      i. The cost of computing the LU factorization is generally proportional to n^2.

      ii. The cost of backward substitution on a dense upper triangular matrix is generally proportional to n^2.

      iii. If a matrix A has no more than 3 non-zero entries per row, the cost of each iteration of the Jacobi method is proportional to n.

   (b) If an $n \times n$ matrix A is poorly conditioned (i.e., it has a very large condition number), then

      i. Solving $Ax = b$ would be difficult with LU decomposition or Gaussian elimination, but iterative methods (Jacobi, Gauss-Seidel) would not have a problem.

      ii. Solving $Ax = b$ accurately with iterative methods (Jacobi, Gauss-Seidel) would be difficult, but LU decomposition with pivoting would not have a problem.

      iii. Solving $Ax = b$ accurately will be challenging regardless of the method we use.

   (c) Consider the rectangular $m \times n$ matrix A (with $m > n$), and the vector $b \in \mathbb{R}^m$. If x is the *least squares solution* to $Ax \approx b$, can we say that x is an actual solution to $Ax = b$?

      i. Yes, in fact $Ax = b$ has many solutions and the *least squares solution* is the one with the smallest L_2-norm of the residual vector $\|r\|_2$.

      ii. No, the system $Ax = b$ will generally not have a solution. What we call the *least squares solution* is the vector x with the smallest L_2-norm of the error vector $\|x - x_{\mathsf{exact}}\|_2$.

      iii. No, the system $Ax = b$ will generally not have a solution. What we call the *least squares solution* is the vector x with the smallest L_2-norm of the residual vector $\|b - Ax\|_2$.

   (d) Which of the following methods can be used for solving the system $Ax = b$, where A is a square $n \times n$ matrix?

      i. LU factorization with full pivoting.

      ii. System of normal equations.

      iii. Gauss-Seidel method.

      iv. Jacobi method.

2. [18% = 3 questions $\times$ 6% each] SHORT ANSWER SECTION. Answer each of the following questions in no more than 2-3 sentences.

   (a) Consider the following matrix A whose LU factorization we wish to compute using Gaussian elimination:

$$A = \begin{bmatrix} 4 & -8 & 1 \\ 6 & 5 & 7 \\ 0 & -10 & -3 \end{bmatrix}$$

   What will be the initial pivot element if (no explanation required)

   - No pivoting is used?
   - Partial pivoting is used?
   - Full pivoting is used?

   (b) State one defining property of a *singular* matrix A. Suppose that the linear system $Ax = b$ has two distinct solutions x and y. Use the property you gave to prove that A must be singular.

   (c) Mention one advantage of the Gauss-Seidel algorithm over the Jacobi algorithm and one disadvantage.

3. [14%] Consider the five points:

$$\begin{aligned}(x_1, y_1) &= (-3, -1)\\ (x_2, y_2) &= (-2, 1)\\ (x_3, y_3) &= (0, 2)\\ (x_4, y_4) &= (1, 3)\\ (x_5, y_5) &= (3, 2)\end{aligned}$$

(a) We want to determine a straight line $y = c_1x + c_0$ that approximates these points as closely as possible, in the least squares sense. Write a least squares system $Ax \approx b$ which can be used to determine the coefficients c_1 and c_0.

(b) Solve this least squares system, using the method of normal equations.

4. [18%] The general form of an iterative method for solving the system $Ax = b$ has the form

$$x^{(k)} = Tx^{(k-1)} + c$$

where the matrix T and the vector c are such that the equation $x = Tx + c$ is equivalent to the original system $Ax = b$.

(a) If $x^\star$ is the *exact* solution of the system $Ax = b$, show that

$$x^{(k)} - x^\star = T\left(x^{(k-1)} - x^\star\right)$$

(b) If $r^{(k)} = b - Ax^{(k)}$ is the residual vector after the k^{th} iteration of the method, show that

$$r^{(k)} = ATA^{-1}r^{(k-1)}$$

Hint: Use the identity $r^{(k)} = -Ae^{(k)}$, or equivalently $e^{(k)} = -A^{-1}r^{(k)}$. Here, $e^{(k)} = x^{(k)} - x^\star$ is the *error vector* after the k^{th} iteration.

(c) Show that

$$r^{(k)} = AT^kA^{-1}r^{(0)}$$

5. [26%] Consider the elimination matrix $M_k = I - m_k e_k^T$ and its inverse $L_k = I + m_k e_k^T$ used in the LU decomposition process, where

$$m_k = (0, \ldots, 0, m_{k+1}^{(k)}, \ldots, m_n^{(k)})^T$$

and e_k is the kth column of the identity matrix. Let $P^{(ij)}$ be the permutation matrix that results from swapping the i-th and j-th rows (or columns) of the identity matrix.

(a) [6%] Show that if $i, j > k$ then $L_k P^{(ij)} = P^{(ij)}(I + P^{(ij)} m_k e_k^T)$.

(b) [10%] Recall that the matrix L resulting from performing Gaussian elimination with partial pivoting is given by

$$L = P_1 L_1 \ldots P_{n-1} L_{n-1}$$

where the permutation matrix P_i permutes row i with some row i' where $i < i'$. Show that L can be rewritten as

$$L = P_1 \ldots P_{n-1} L_1^P \ldots L_{n-1}^P$$

where $L_k^P = I + (P_{n-1} \ldots P_{k+1} m_k) e_k^T$.

(c) [10%] Show that $L_1^P \ldots L_{n-1}^P$ is lower triangular.