# Automata Theory, Computability and Complexity

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## Course Staff

- **Instructor:** Mridul Aanjaneya
  Office Hours: 2:00PM - 4:00PM, Gates 206 (Mon).
- **Course Assistant:** Sean Kandel
  Office Hours: 6:00PM - 8:00PM, Gates 372 (Tue/Thu).
- **Textbook:** Michael T. Sipser, *Introduction to the Theory of Computation* (second edition).
- **Website:** http://www.stanford.edu/class/cs154/
- **Discussion:** Lore (http://www.lore.com/)
  **Access code:** R4ML4Z



## Course Requirements

- Homework
- Midterm
- Final (optional)

**Note:**

- Final grade will be computed from Homework (60%) and best of Midterm/Final (40%).
- Projected grade will be out one week before the Final, taking Homework (60%) and Midterm (40%).
  - Can skip Final if satisfied with grade.
- Can collaborate on homeworks, but separate write-up per person and must mention collaborators.
- 4 homeworks, 2 free late days, use them however you want.
  - No late days for final homework.



## Why Study Automata?

- Regular expressions:
  - are used in many systems, e.g., UNIX.
  - DTD's describe XML tags with RE like format.
- Finite automata model protocols, electronic circuits.
  - Theory is used in model-checking.
- Context free languages:
  - are used as syntax descriptors for PL's, parsers (YACC).
  - have an important role in describing natural languages.
  - find use in procedural modeling.

## Course Outline

- Deterministic/Nondeterministic finite automata.
- Regular expressions.
- Decision/closure properties of regular languages.
- Context-free languages. Parse trees. Normal forms.
- Pushdown automata. Equivalence of CFG's and PDA's.
- Pumping lemma for CFL's. Properties of CFL's.
- Enumerations, Turing machines.
- Undecidable problems. NP-completeness.
- Satisfiability. Cook's theorem.



## Automata, Computability and Complexity

**Main Question:**
What are the fundamental capabilities and limitations of computers?

- Dates back to the 1930s.
- Different interpretations in all three areas.



## Complexity

**Question**
Why are some problems hard and others easy?

- **Subset sum:** Given a set of integers, does any non-empty subset of them add up to zero?

- So far only a classification according to hardness.
- Can help alter the root of difficulty, settle for an approximate solution, some problems are hard only in the worst case.



## Computability

**Question**
Which problems are solvable?

- **Halting problem:** Given an arbitrary computer program, decide if it finishes or continues running forever.

- Computability and complexity are closely related.

## Automata

- Deals with different models of computation.




## Preliminaries: Mathematical Induction



## Preliminaries: Mathematical Induction

**Example**

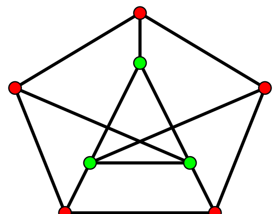
Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2$ is $\frac{n(n+1)(2n+1)}{6}$.

- **Basis (n=1):** $\frac{1(1+1)(2\cdot1+1)}{6} = \frac{2\cdot3}{6} = 1 = 1^2$.
- **Induction:** Let $\mathcal{P}(n) := 1^2 + \ldots + n^2$ is $\frac{n(n+1)(2n+1)}{6}$.
  $\mathcal{P}(n+1) = 1^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$
  $= (n+1)\left\{\frac{n(2n+1)+6n+6}{6}\right\} = (n+1)\left\{\frac{2n^2+7n+6}{6}\right\}$
  $= (n+1)\left\{\frac{2n^2+4n+3n+6}{6}\right\} = \frac{(n+1)(n+2)(2n+3)}{6}$
- Since $\mathcal{P}(1)$ is true and $\mathcal{P}(n) \Rightarrow \mathcal{P}(n+1)$, we conclude that $\mathcal{P}(n)$ is true for all $n \geq 1$.



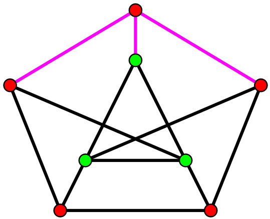
## Preliminaries: Graphs

- Undirected graph



## Preliminaries: Graphs

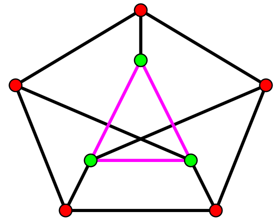
- Degree of a vertex





## Preliminaries: Graphs

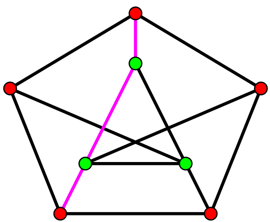
- Subgraph





## Preliminaries: Graphs

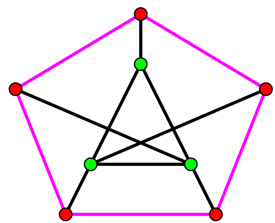
- Path





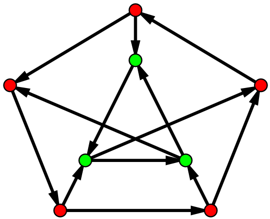
## Preliminaries: Graphs

- Cycle



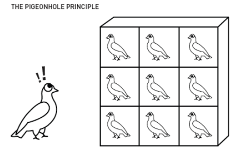
## Preliminaries: Graphs

- Directed graph





## Preliminaries: Pigeonhole Principle

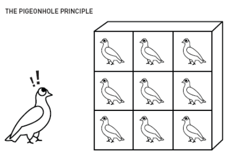



**Pigeonhole Principle**

If there are $n+1$ pigeons and $n$ pigeonholes, then some pigeonhole must contain at least two pigeons.



## Preliminaries: Pigeonhole Principle




**Generalized Pigeonhole Principle**

If there are $kn+1$ pigeons and $n$ pigeonholes, then some pigeonhole must contain at least $k+1$ pigeons.



## Preliminaries: Pigeonhole Principle

**Example 1**

Given twelve integers, show that two of them can be chosen whose difference is divisible by 11.

**Example 2**

Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and all the apples in each crate are of the same sort. Show that among these crates there are at least nine containing the same sort of apple.

**Example 3**

Show that in any group of five people, there are two who have an identical number of friends within the group.

## Alphabet

- An alphabet is any finite set of symbols.
  - Some examples include: ASCII, {0,1} (binary), {a,b,c}.
- The set of strings over an alphabet $\Sigma$ is the set of lists, each element of which is a member of $\Sigma$.
  - **Note:** Strings are shown with no commas, e.g., abc, 01101.
- $\Sigma^*$ denotes this set of strings.
- $\varepsilon$ stands for the empty string (string of length 0).

**Example:**

- For the alphabet $\Sigma = \{0, 1\}$, the set of strings in $\Sigma^*$ is:
  - {$\varepsilon$,0,1,00,01,10,11,000,001,010,011,100,101,110,111,...}
- For the alphabet $\Sigma = \{a, b, c\}$, the set of strings in $\Sigma^*$ is:
  - {$\varepsilon$,a,b,c,aa,ab,ac,ba,bb,bc,ca,cb,cc,...}



## Alphabet

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**Example:**

- For the alphabet $\Sigma = \{0, 1\}$, the set of strings in $\Sigma^*$ is:
  - {$\varepsilon$,0,1,00,01,10,11,000,001,010,011,100,101,110,111,...}
- **Subtlety:** 0 as a string, and 0 as an alphabet look the same.
  - Context determines the type.



## Languages

- A language is a subset of $\Sigma^*$ for some alphabet $\Sigma$.
- **Example 1:** The set of strings over {0,1} with no two consecutive 1's.
  - {$\varepsilon$,0,1,00,01,10,000,001,010,100,101,0000,0001,0010, 0100,0101,1000,1001,1010,...}
- **Example 2:** The set of strings over {a,b,c} with unequal number of a's and b's.
  - {a,b,aa,ac,bb,bc,ca,cb,aaa,bbb,aab,baa,aba,aac,caa, aca,bbc,cbb,bcb,bba,abb,bab,cca,ccb,cac,cbc,bcc,acc,...}

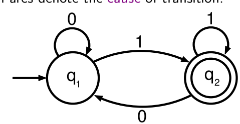


## Finite Automata

- Informally, finite automata are finite collections of states with transition rules for going from one state to another.
- There is a start state and (one or more) accept states.

**Representation:** Simplest representation is often a graph.

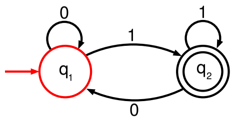
- Nodes denote states, and arcs indicate state transitions.
- Labels on arcs denote the cause of transition.




- The above automaton only accepts binary strings ending in 1.

## An example

- 01100011100101






## An example

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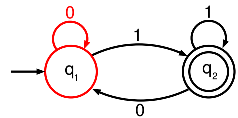
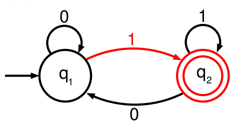





## An example

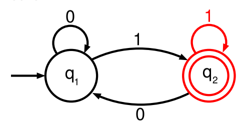
- 01100011100101





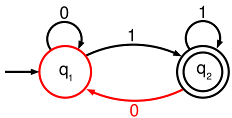

## An example

- 01100011100101



## An example

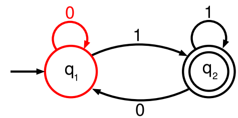
- 01100011100101






## An example

- 01100011100101






## An example

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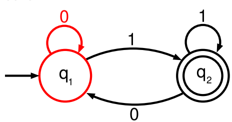
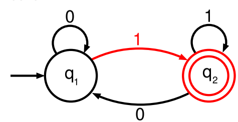




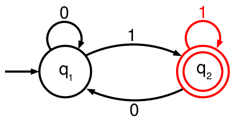

## An example

- 01100011100101



# An example

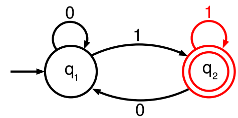
- 01100011100101






# An example

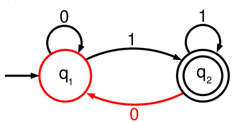
- 01100011100101






# An example

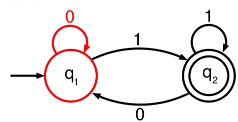
- 01100011100101





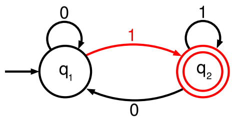

# An example

- 01100011100101



## An example

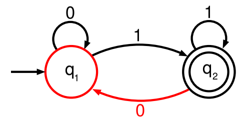
- 01100011100101






## An example

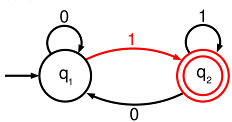
- 01100011100101






## An example

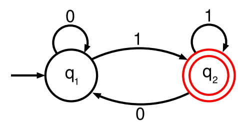
- 01100011100101





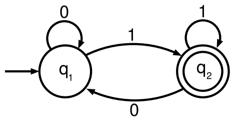

## An example

- Accept!



## An example

- **Exercise:** Verify that 000111010110 is rejected.






## Deterministic Finite Automata

**Definition**

A DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ consisting of:

- A finite set of states $Q$,
- A set of input alphabets $\Sigma$,
- A transition function $\delta : Q \times \Sigma \to Q$,
- A start state $q_0$, and
- A set of accept states $F \subseteq Q$.

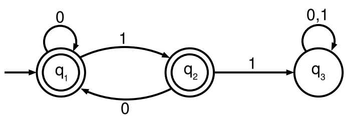
The transition function $\delta$:

- Takes two arguments, a state q and an alphabet a.
- $\delta$(q,a) = the state the DFA goes to when it is in state q and the alphabet a is received.



## Graph representation of DFA's

- Nodes correspond to states.
- Arcs represent transition function.
  - Arc from state p to state q labeled by all those input symbols that have transitions from p to q.
- Incoming arrow from outside denotes start state.
- Accept states indicated by double circles.

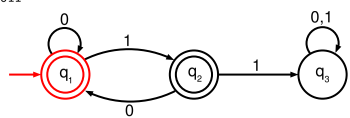



- Accepts all binary strings without two consecutive 1's.



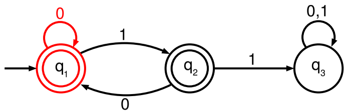
## Example

- 011



## Example

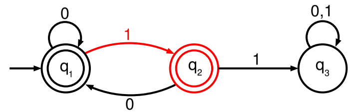
- 011






## Example

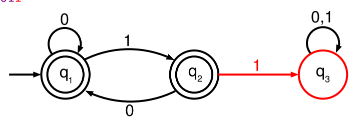
- 011






## Example

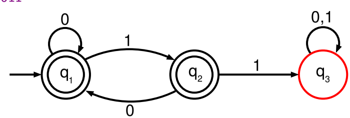
- 011






## Example

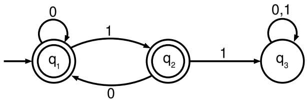
- 011



## Transition table for DFA's

| | | 0 | 1 |
|---|---|---|---|
| $\rightarrow$ | $q_1^*$ | $q_1$ | $q_2$ |
| | $q_2^*$ | $q_1$ | $q_3$ |
| | $q_3$ | $q_3$ | $q_3$ |

- A row for each state, a column for each alphabet.
- Accept states are starred.
- Arrow for the start state.






## Extended transition function

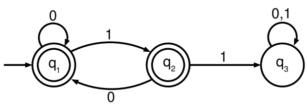
- Effect of a string on a DFA can be described by extending $\delta$ to a state and a string.
- Induction on length of the input string.
- Basis: $\delta(q,\varepsilon)=q$
- Induction: $\delta(q,wa)=\delta(\delta(q,w),a)$
  - w is a string, a is an input alphabet.
  - **Convention:** w,x,y,... are strings, a,b,c,... are alphabets.
- For a DFA, extended $\delta$ is computed for state q and inputs $a_1a_2\ldots a_n$ by following a path starting at q and selecting arcs with labels $a_1,a_2,\ldots,a_n$ in turn.



## Example using transition table

| | | 0 | 1 |
|---|---|---|---|
| $\rightarrow$ | $q_1^*$ | $q_1$ | $q_2$ |
| | $q_2^*$ | $q_1$ | $q_3$ |
| | $q_3$ | $q_3$ | $q_3$ |

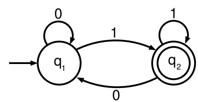
$$\begin{aligned} & \delta(q_2,011) = \delta(\delta(q_2,01),1) = \delta(\delta(\delta(q_2,0),1),1) \\ = \; & \delta(\delta(q_1,1),1) = \delta(q_2,1) = q_3 \end{aligned}$$






## Language of a DFA

- Automata of all kinds define languages.
- If A is an automaton, L(A) is its language.
- For a DFA A, L(A) is the set of strings labeling paths from the start state to an accept state.
- Formally, L(A) = set of strings w such that $\delta(q_0,w)$ is in F.
- **Example:** 01100011100101 is in L(A), where A is:

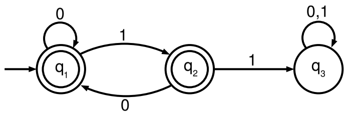


## Proofs of set equivalence

**Question**

How to prove that two different sets are in fact the same?

- Here, $T$ is the set of strings of 0's and 1's with no consecutive 1's and $L = \{w \mid M \text{ accepts } w\}$, where $M$ is:



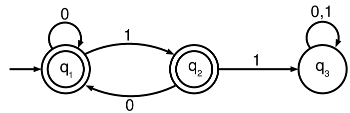



## Proofs of set equivalence

**Question**

How to prove that two different sets are in fact the same?

- In general, to show $T = L$, we need to prove two parts:
  - If $w$ is in $L$, then $w$ is in $T$, i.e., $L \subseteq T$.
  - If $w$ is in $T$, then $w$ is in $L$, i.e., $T \subseteq L$.






## Part 1: $L \subseteq T$

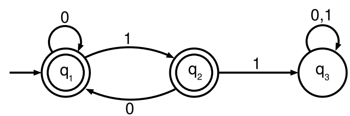
- **To prove:** If $w \in L$, then $w$ has no consecutive 1's.
- Proof is induction on length of $w$.
- **Important trick:** Expand the inductive hypothesis to be more detailed than you need.



## Part 1: $L \subseteq T$ (Inductive Hypothesis)

1. If $\delta(q_1,w)=q_1$, then $w$ has no consecutive 1's and does not end in 1.
2. If $\delta(q_1,w)=q_2$, then $w$ has no consecutive 1's and ends in a single 1.

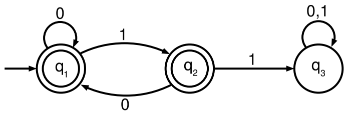
- **Basis:** $|w| = 0$, i.e., $w = \varepsilon$.
  1. holds as $\varepsilon$ has no 1's at all.
  2. holds vacuously, since $\delta(q_1,\varepsilon)$ is not $q_2$.
- **Note:** $p \Rightarrow q$ is always true, if $p$ is false.



## Part 1: $L \subseteq T$

1. If $\delta(q_1,w)=q_1$, then w has no consecutive 1's and does not end in 1.
2. If $\delta(q_1,w)=q_2$, then w has no consecutive 1's and ends in a single 1.

- **Inductive step:** Assume **IH** is true for strings shorter than w, where $|w| \geq 1$.
- Because w is not empty, we can write $w = xa$, where a is the last alphabet of w and x is the string that precedes.
- **IH** is true for x.

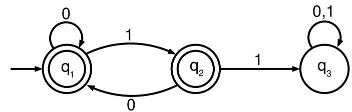





## Part 1: $L \subseteq T$

1. If $\delta(q_1,w)=q_1$, then w has no consecutive 1's and does not end in 1.
2. If $\delta(q_1,w)=q_2$, then w has no consecutive 1's and ends in a single 1.

- **Inductive step:** Assume **IH** is true for strings shorter than w, where $|w| \geq 1$.
  1. for w is $\delta(q_1,w) = q_1$.
- Since $\delta(q_1,w) = q_1$, $\delta(q_1,x)$ must be $q_1$ or $q_2$ and a must be 0.



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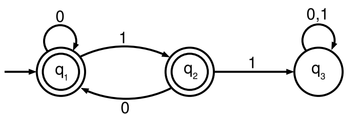


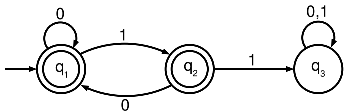



## Part 2: $T \subseteq L$

- **To prove:** If w has no consecutive 1's, then $w \in L$.
- We prove the contrapositive.
- **Note:** $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$.
- **To prove (contrapositive):** If $w \notin L$, then w has no 11's.

## Part 2: T ⊆ L

- Simple induction on length of w.
- The only way w is not accepted is if it gets to $q_3$.
- The only way to get to $q_3$ is if w = x1y.
- If $\delta(q_1,x) = q_2$, then surely x = z1 (from Part 1: L ⊆ T).
- ⇒ w = z11y!






## Regular languages

**Definition**

A DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts $w$ if there exists a sequence of states $r_0, r_1, \ldots, r_n$ in $Q$ with three conditions:

- $r_0 = q_0$
- $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, \ldots, n-1$, and
- $r_n \in F$

- Condition 1 says that M starts in the start state $q_0$.
- Condition 2 says that M follows $\delta$ between two states.
- Condition 3 says that last state is an accept state.
- We say that M recognizes L if L = {w | M accepts w}.



## Regular languages

- A language L is regular if it is the language of some DFA.
  - **Note:** the DFA must accept only the strings in L.
- Some languages are not regular (more later).
  - Intuitively, DFA's are memoryless.



## A regular language

**Example**

Let L = {w | w ∈ $\{0,1\}^*$ and w, viewed as a binary integer, is divisible by 5.}