Quiz I: Marking Scheme

1. Find the dimension and a basis of the set of solutions of [4]

$$\begin{aligned} x + 2y &= 0 \\ y - z &= 0 \\ x + y + z &= 0 \end{aligned}$$

Soln: We have $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[E_{31}(-1)]{} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[E_{32}(1)]{} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$ [2]

Thus we have $x + 2y = 0,\ y - z = 0$. Hence $S = \{(-2t, t, t) \mid t \in \mathbb{R}\}$ is the set of all solutions. Hence $dim(S) = 1$ and $\{(-2, 1, 1)\}$ is a basis of S. [2]

2. Let $V = \mathbb{R}^2$. Then for which of the followings, V forms a vector space over $\mathbb{R}$? If not then justify your answer. [2+2]

   (a) for $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, addition is defined as: $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$, scalar multiplication is defined as: $c.(a_1, a_2) = (a_1, 0)$.

   (b) for $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, addition is defined as: $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$, scalar multiplication is defined as: $c.(a_1, a_2) = (0, 0)$ if $c = 0$ otherwise $(ca_1, \frac{a_2}{c})$.

Soln: (a) Property: $1_F.v = v\ \forall v \in V$. [1]

But for $c = 1$ we get $1.(a_1, a_2) = (a_1, 0) \neq (a_1, a_2)$. So V is not a vector space. [1]

(b) Property: $v_1 + v_2 = v_2 + v_1\ \forall v_1, v_2 \in V$. [1]

But we see that $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \neq (b_1 + 2a_1, b_2 + 3a_2) = (b_1, b_2) + (a_1, a_2)$. So V is not a vector space. [1]

3. The vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent if and only if $ad - bc \neq 0$. [3]

Soln: $(\Rightarrow)$ Since (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent, $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$. So we can assume that either $a \neq 0$ or $b \neq 0$ or $c \neq 0$ or $d \neq 0$. [1]

Suppose $ad - bc = 0$. So $d = \dfrac{bc}{a}$ (if $a \neq 0$).Hence $(c, d) = (c, \dfrac{bc}{a}) = \dfrac{c}{a}(a, b)$ i. e. (a, b) and (c, d) in $\mathbb{R}^2$ are linearly dependent which is a contradiction. Thus $ad - bc \neq 0$. [1]

$(\Leftarrow)$ If (a, b) and (c, d) in $\mathbb{R}^2$ are linearly dependent, then $(c, d) = k(a, b)$, for some $k \in \mathbb{R}$. Thus $ad - bc = a(kb) - b(ka) = 0$ which is a contradiction. Hence the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent. [1]

4. Find the dimension of following spaces:

(a) $W = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_k = 0,\ if k\ is\ even\}$. [1]

Soln: $dim(W) = \begin{cases} m & \text{if } n = 2m,\ for\ some\ m \in \mathbb{N} \\ m+1 & \text{if } n = 2m+1,\ for\ some\ m \in \mathbb{N} \end{cases}$ [1]

(b) $W = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid x_k{}'s\ are\ all\ equal\ for\ k\ even\}$. [1]

Soln: $dim(W) = \begin{cases} m+1 & \text{if } n = 2m,\ for\ some\ m \in \mathbb{N} \\ m+2 & \text{if } n = 2m+1,\ for\ some\ m \in \mathbb{N} \end{cases}$ [1]

(c) $W = \{A = (a_{ij}) \in M_{n\times n} \mid trace(A) = 0\ i.e.\ a_{11} + a_{22} + \cdots + a_{nn} = 0\}$. [1]

Soln: $dim(W) = n^2 - 1$ [1]

5. Let V be a 3 dimensional vector space over $\mathbb{R}$ with W_1 and W_2 its subspaces of dimension 2 and 1 respectively. Suppose $W_1 \cap W_2 \neq \{0\}$. Then find the dimension of $W_1 \cap W_2$. [2]

Soln: We know that

$dim(W_1) + dim(W_2) - dim(V) \leq dim(W_1 \cap W_2) \leq min(dim(W_1), dim(W_2))$ [1].

$\Rightarrow 0 \leq dim(W_1 \cap W_2) \leq 1$.

Since $W_1 \cap W_2 \neq \{0\}$, $dim(W_1 \cap W_2) = 1$. [1]

6. Let V be a vector space of all 2×2 matrices over $\mathbb{R}$. Define: [4]

$W_1 = \{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in V\ : a, b, c \in \mathbb{R}\}$

$W_2 = \{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in V\ : a, b \in \mathbb{R}\}$

Then find the dimensions of W_1, W_2, $W + W_2$, $W_1 \cap W_2$.

Soln: $\begin{pmatrix} a & b \\ c & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and

$\alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, implies $\alpha = \beta = \gamma = 0$. Therefore the set

$S_1 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ is a basis for W_1. Hence $dim(W_1) = 3$. [1]

$\begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, implies $\alpha = \beta = 0$. Therefore the set $S_2 = \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ is basis for W_2. Hence $dim(W_2) = 2$. [1]

$W_1 \cap W_2 = \{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in V\ : a \in \mathbb{R}\}$. Then $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the set $S_3 = \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ is singleton with non-zero element so it is linearly independent. Therefore S_3 is a basis for $W_1 \cap W_2$. Hence $dim(W_1 \cap W_2) = 1$. [1]

$dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = dim(W_1 + W_2)$, implies $dim(W_1 + W_2) = 3 + 2 - 1 = 4$. [1]