

Notes on Section 2.9: Group actions

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2025-06-18

1 Review

We what the notation $\text{End}_{\mathcal{C}}(A)$ and $\text{Aut}_{\mathcal{C}}(A)$ denote for a given category \mathcal{C} and object A in the category.

We know that given a category \mathcal{C} and objects $A, B \in \text{obj}(\mathcal{C})$, then

- $\text{Hom}_{\mathcal{C}}(A, B)$ is the set of all morphisms from $A \rightarrow B$.
- $\text{End}_{\mathcal{C}}(A) = \text{Hom}_{\mathcal{C}}(A, A)$.
- $\text{Aut}_{\mathcal{C}}(A)$ is the set of all isomorphisms $f : A \rightarrow A$, hence $\text{Aut}_{\mathcal{C}}(A) \subseteq \text{End}_{\mathcal{C}}(A)$.
- note that $\text{Aut}_{\mathcal{C}}(A)$ is a group with the operation being composition of groups and the identity being the identity map

2 Action of a group

Defination 2.1: *Action* of a group G on an object A of a category \mathcal{C} is a (group) homomorphism

$$\sigma : G \rightarrow \text{Aut}_{\mathcal{C}}(A)$$

σ is the action of G on A . **For the rest of notes, σ denotes an action homomorphism**

Defination 2.2: $\sigma : G \rightarrow \text{Aut}_{\mathcal{C}}(A)$ is faithful if σ is injective.

3 Actions on sets

We fix a group G .

$\forall A \in \text{obj}(\text{SET})$, we consider functions of the form

$$\rho : G \times A \rightarrow A$$

such that

1. $(\forall a \in A) \rho(e_G, a) = a$
2. $(\forall a \in A) (\forall g, h \in G) \rho(gh, a) = \rho(g, \rho(h, a))$

and we claim that given such a function ρ , we canonically define an action of G on A and every action $\sigma : G \rightarrow \text{Aut}_{\text{SET}}(A)$ defines such a function ρ . Thus we prove the claim

Claim 3.1: Given a set A and a group G , any function $\rho : G \times A \rightarrow A$ such that $\forall a \in A) \rho(e_G, a) = a$ and $(\forall a \in A) (\forall g, h \in G) \rho(gh, a) = \rho(g, \rho(h, a))$, defines an action of G on A . Thus we prove it as follows

Definition 3.2: Given $\rho : G \times A \rightarrow A$, we define $\sigma : G \rightarrow \text{End}_{\text{SET}}(A)$ as $\forall g \in G$

$$\sigma : g \rightarrow \sigma_g$$

where σ_g is a set function such that $\forall a \in A$

$$\sigma_g : a \rightarrow \rho(g, a)$$

It is clear that $\sigma_g \in \text{End}_{\text{SET}}(A)$. Now we show that σ is a group homomorphism.

Lemma 3.3: σ as defined above is a group homomorphism.

Proof: $\forall g, h \in G$ and $\forall a \in A$, we consider

$$\sigma_{gh}(a) = \rho(gh, a) = \rho(g, \rho(h, a)) = \rho(g, \sigma_h(a)) = \sigma_g \circ \sigma_h(a)$$

Since this is true for any $a \in A$, we get that for any $g, h \in G$,

$$\sigma_{gh} = \sigma_g \circ \sigma_h$$

Hence $\sigma(gh) = \sigma(g) \circ \sigma(h)$, therefore, we have shown that σ is a group homomorphism. \square

Lemma 3.4: $\forall g \in G, \sigma_g \in \text{Aut}_{SET}(A)$

Proof: We will show that $\forall g \in G, (\sigma_g)^{-1} = \sigma_{g^{-1}}$, hence

$$\sigma_g \circ \sigma_{g^{-1}} = \sigma_{gg^{-1}} = \sigma_{e_G} = I$$

$$\sigma_{g^{-1}} \circ \sigma_g = \sigma_{g^{-1}g} = \sigma_{e_G} = I$$

Therefore, $\forall g \in G, \sigma_g \in \text{Aut}_{SET}(A)$ \square

Thus we have shown that σ (Defination-3.2), is a group homomorphism from $G \rightarrow \text{Aut}_{SET}(A)$. Thus showing that σ is an action, hence proving the claim (Claim-3.1) \square

Claim 3.5: Given an action $\sigma : G \rightarrow \text{Aut}_{SET}(A)$, it defines a function $\rho : G \times A \rightarrow A$ such that $\forall a \in A$ $\rho(e_G, a) = a$ and $(\forall a \in A) (\forall g, h \in G) \rho(gh, a) = \rho(g, \rho(h, a))$

Definition 3.6: Given an action $\sigma : G \rightarrow \text{Aut}_{SET}(A)$, we define $\rho : G \times A \rightarrow A$ as $\forall a \in A$ and $\forall g \in G$

$$\rho(g, a) := \sigma_g(a)$$

Claim 3.7: ρ as defined (Defination-3.6) follows that $\forall a \in A$ $\rho(e_G, a) = a$ and $(\forall a \in A) (\forall g, h \in G) \rho(gh, a) = \rho(g, \rho(h, a))$

Proof: Consider $a \in A$

$$\rho(e_G, a) = \sigma_{e_G}(a)$$

Since σ is a group homomorphism, it must map the identity e_G in G to the identity in $\text{Aut}_{SET}(A)$, which is the identity map hence

$$\rho(e_G, a) = \sigma_{e_G}(a) = a$$

Thus $\forall a \in A$ $\rho(e_G, a) = a$. Now consider $(\forall a \in A) (\forall g, h \in G)$, then

$$\rho(gh, a) = \sigma_{gh}(a) = \sigma_g \circ \sigma_h(a) = \rho(g, \sigma_h(a)) = \rho(g, \rho(h, a))$$

Thus we have proved that ρ as defined (Defination-3.6) follows that $\forall a \in$

A) $\rho(e_G, a) = a$ and $(\forall a \in A) (\forall g, h \in G) \rho(gh, a) = \rho(g, \rho(h, a))$ \square

Therefore we have proved the claim (Claim-3.5:)

Definition 3.8: $\sigma : G \rightarrow \text{Aut}_{SET}(A)$ is transitive if $\forall a, b \in A. \exists g \in G$ such that

$$\sigma_g(a) = b$$

Definition 3.9: Orbit of $a \in A$ under $\sigma : G \rightarrow \text{Aut}_{SET}(A)$ is the set

$$O_G(a) = \{ga | g \in G\}$$

where ga is the same as $\rho(g, a)$ where ρ is the function defined using σ (Claim-3.1)

Definition 3.10: Given $\sigma : G \rightarrow \text{Aut}_{SET}(A)$, then we define the stabilizer for any $a \in A$ as

$$\text{Stab}_G(a) = \{g \in G | \sigma_g(a) = a\}$$

4 Category G-set

Definition 3.11: Fixing a group G , we define a category as

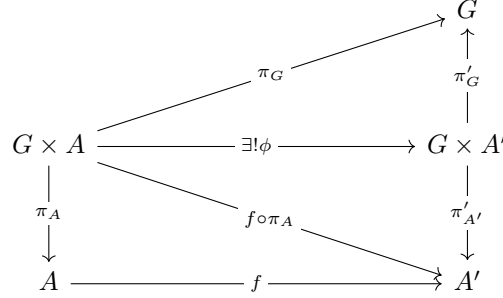
Objects: (ρ, A) where $A \in \text{obj}(SET)$ and $\rho : G \times A \rightarrow A$ as defined in Definition-3.6.

Morphisms: Set functions which are compatible with the actions i.e. functions $f : A \rightarrow A'$ such that

$$\rho'(g, f(a)) = f(\rho(g, a))$$

This category is called G-set.

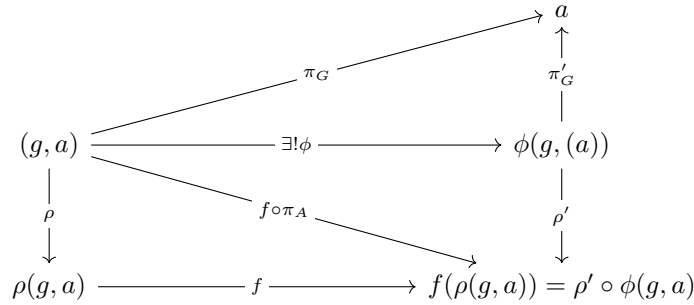
A commutative diagram to explain the concept is (the description follows the diagram)



Description: Since $G \times A$ is a product in the category SET, we know there exist projections π_G and ρ . Similarly, since $G \times A'$ is a product in the category SET, we know there exist projections π'_G and ρ' . We consider f to be any set function from A to A' . Then we trivially have $f \circ \pi_A : G \times A \rightarrow A'$, thus by the universal property of products in set, we get that there exists a unique set function ϕ such that

$$\rho' \circ \phi = f \circ \rho$$

It is to be noted that given any $(g, a) \in G \times A$, then its map becomes as follows



and by the commutative diagram we get that $f(\rho(g, a)) = \rho' \circ \phi(g, a)$. Thus intuitively it makes sense for ϕ to be the function $\text{id} \times f$ i.e. the function that maps $(g, a) \rightarrow (g, f(a))$. Thus we get that $f(\rho(g, a)) = \rho'(g, f(a))$. It is to be recalled that ρ and ρ' are actions of G on A . Thus it makes complete sense and solid motivation to consider morphisms between objects (ρ, A) and (ρ', A') to be the set function $f : A \rightarrow A'$ such that $\rho'(g, f(a)) = f(\rho(g, a))$.