Notes on Section 2.9: Group actions

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1 Review

We what the notation $\mathrm{End}_{\mathcal{C}}(A)$ and $\mathrm{Aut}_{\mathcal{C}}(A)$ denote for a given category $\mathcal{C}$ and object A in the category.

We know that given a category $\mathcal{C}$ and objects $A, B \in \mathrm{obj}(\mathcal{C})$, then

- $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is the set of all morphisms from $A \to B$.
- $\mathrm{End}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)$.
- $\mathrm{Aut}_{\mathcal{C}}(A)$ is the set of all isomorphisms $f : A \to A$, hence $\mathrm{Aut}_{\mathcal{C}}(A) \subseteq \mathrm{End}_{\mathcal{C}}(A)$.
- note that $\mathrm{Aut}_{\mathcal{C}}(A)$ is a group with the operation being composition of groups and the identity being the identity map

2 Action of a group

Defination 2.1: *Action* of a group G on an object A of a category C is a (group) homomorphism

$$\sigma : G \to \mathrm{Aut}_C(A)$$

σ is the action of G on A. **For the rest of notes, σ denotes an action homomorphism**

Defination 2.2: $\sigma : G \to \mathrm{Aut}_{\mathcal{C}}(A)$ is faithful if σ is injective.

3 Actions on sets

We fix a group G.

$\forall A \in \text{obj}(\text{SET})$, we consider functions of the form

$$\rho : G \times A \to A$$

such that

1. $(\forall a \in A)\ \rho(e_G, a) = a$
2. $(\forall a \in A)\,(\forall g, h \in G)\,\rho(gh, a) = \rho(g, \rho(h, a))$

and we claim that given such a function ρ, we canonically define an action of G on A and every action $\sigma : G \to \text{Aut}_{SET}(A)$ defines such a function ρ. Thus we prove the claim

Claim 3.1: Given a set A and a group G, any function $\rho : G \times A \to A$ such that $\forall a \in A)\ \rho(e_G, a) = a$ and $(\forall a \in A)\,(\forall g, h \in G)\,\rho(gh, a) = \rho(g, \rho(h, a))$, defines an action of G on A. Thus we prove it as follows

Defination 3.2: Given $\rho : G \times A \to A$, we define $\sigma : G \to \text{End}_{SET}(A)$ as $\forall g \in G$

$$\sigma : g \to \sigma_g$$

where σ_g is a set function such that $\forall a \in A$

$$\sigma_g : a \to \rho(g, a)$$

It is clear that $\sigma_g \in \text{End}_{SET}(A)$. Now we show that σ is a group homomorphism.

Lemma 3.3: σ as defined above is a group homomorphism.

Proof: $\forall g, h \in G$ and $\forall a \in A$, we consider

$$\sigma_{gh}(a) = \rho(gh, a) = \rho(g, \rho(h, a)) = \rho(g, \sigma_h(a)) = \sigma_g \circ \sigma_h(a)$$

Since this is true for any $a \in A$, we get that for any $g, h \in G$,

$$\sigma_{gh} = \sigma_g \circ \sigma_h$$

Hence $\sigma(gh) = \sigma(g) \circ \sigma(h)$, therefore, we have shown that σ is a group homomorphism. $\square$

Lemma 3.4: $\forall g \in G$, $\sigma_g \in \mathrm{Aut}_{SET}(A)$

Proof: We will show that $\forall g \in G$, $(\sigma_g)^{-1} = \sigma_{g^{-1}}$, hence

$$\sigma_g \circ \sigma_{g^{-1}} = \sigma_{gg^{-1}} = \sigma_{e_G} = I$$

$$\sigma_{g^{-1}} \circ \sigma_g = \sigma_{g^{-1}g} = \sigma_{e_G} = I$$

Therefore, $\forall g \in G, \sigma_g \in \mathrm{Aut}_{SET}(A)$ $\square$

Thus we have shown that σ (Defination-3.2), is a group homomorphism from $G \to \mathrm{Aut}_{SET}(A)$. Thus showing that σ is an action, hence proving the claim (Claim-3.1) $\square$

Claim 3.5: Given an action $\sigma : G \to \mathrm{Aut}_{SET}(A)$, it defines a function $\rho : G \times A \to A$ such that $\forall a \in A)$ $\rho(e_G, a) = a$ and $(\forall a \in A)\,(\forall g, h \in G)\,\rho(gh, a) = \rho(g, \rho(h, a))$

Defination 3.6: Given an action $\sigma : G \to \mathrm{Aut}_{SET}(A)$, we define $\rho : G \times A \to A$ as $\forall a \in A$ and $\forall g \in G$

$$\rho(g, a) := \sigma_g(a)$$

Claim 3.7: ρ as defined (Defination-3.6) follows that $\forall a \in A)$ $\rho(e_G, a) = a$ and $(\forall a \in A)\,(\forall g, h \in G)\,\rho(gh, a) = \rho(g, \rho(h, a))$

Proof: Consider $a \in A$

$$\rho(e_G, a) = \sigma_{e_G}(a)$$

Since σ is a group homomorphism, it must map the identity e_G in G to the identity in $\mathrm{Aut}_{SET}(A)$, which is the identity map hence

$$\rho(e_G, a) = \sigma_{e_G}(a) = a$$

Thus $\forall a \in A)$ $\rho(e_G, a) = a$. Now consider $(\forall a \in A)\,(\forall g, h \in G)$, then

$$\rho(gh, a) = \sigma_{gh}(a) = \sigma_g \circ \sigma_h(a) = \rho(g, \sigma_h(a)) = \rho(g, \rho(h, a))$$

Thus we have proved that ρ as defined (Defination-3.6) follows that $\forall a \in$

$A)$ $\rho(e_G, a) = a$ and $(\forall a \in A)\,(\forall g, h \in G)\,\rho(gh, a) = \rho(g, \rho(h, a))$ $\square$

Therefore we have proved the claim (Claim-3.5:)

Defination 3.8: $\sigma : G \to \mathrm{Aut}_{SET}(A)$ is transitive if $\forall a, b \in G.\exists g \in G$ such that

$$\sigma_g(a) = b$$

Defination 3.9: Orbit of $a \in A$ under $\sigma : G \to \mathrm{Aut}_{SET}(A)$ is the set

$$O_G(a) = \{ga | g \in G\}$$

where ga is the same as $\rho(g, a)$ where ρ is teh function defined using σ (Claim-3.1)

Defination 3.10: Given $\sigma : G \to \mathrm{Aut}_{SET}(A)$, then we define the stabelizer for any $a \in A$ as

$$\mathrm{Stab}_G(a) = \{g \in G | \sigma_g(a) = a\}$$

4 Category G-set

Defination 3.11: Fixing a group G, we define a category as

Objects: (ρ, A) where $A \in \mathrm{obj}(SET)$ and $\rho : G \times A \to A$ as defined in Defination-3.6.

Morphisms: Set functions which are compatible with the actions i.e. functions $f : A \to A'$ such that

$$\rho(g, f(a)) = f(\rho(g, a))$$

This category is called G-set.

A commutative diagram to explain the concept is (the description follows the diagram)

$$
\begin{tikzcd}
 & & G \\
G \times A \arrow[rru, "\pi_G"] \arrow[rr, "\exists ! \phi"] \arrow[d, "\pi_A"] \arrow[rrd, "f \circ \pi_A"] & & G \times A' \arrow[u, "\pi'_G"] \arrow[d, "\pi'_{A'}"] \\
A \arrow[rr, "f"] & & A'
\end{tikzcd}
$$

Description: Since $G \times A$ is a product in teh category SET, we know there exists projections π_G and ρ. Simillarly, since $G \times A'$ is a product in teh category SET, we know there exists projections π'_G and ρ'. We consider f to be any set function from A to A'. then we trivially there exists $f \circ \pi_A : G \times A \to A'$, thus by the universal property of products in set, we get that there exists a unique set function ϕ such that

$$\rho' \circ \phi = f \circ \rho$$

It is to be noted that given any $(g, a) \in G \times a$,then it's map becomes as follows

$$
\begin{tikzcd}
 & & g \\
(g, a) \arrow[rru, "\pi_G"] \arrow[rr, "\exists ! \phi"] \arrow[d, "\rho"] \arrow[rrd, "f \circ \pi_A"] & & (g, \phi(a)) \arrow[u, "\pi'_G"] \arrow[d, "\rho'"] \\
\rho(g, a) \arrow[rr, "f"] & & f(\rho(g, a)) = \rho'(g, \phi(a))
\end{tikzcd}
$$

and by the commutative diagram we get that $f(\rho(g, a)) = \rho'(g, \phi(a))$. Hence we an intuition why we consider morphisms to be set functions which are compatible with the actions.