Linear Algebra

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At first, let's prove, that $(x,\ y)$ can represent in the form of linear combination x, y:
Auxiliary lemmas:

1. $\forall a, b, c, d \in \mathbb{Z},\ c = a + bd \Rightarrow (a,\ b) = (c,\ b)$
   Prove:
   - $d_1 = (a,\ b),\ d_2 = (c,\ b)$
   - $(a \vdots d_1) \land (b \vdots d_1) \Rightarrow$
     $c = d_1(a_0 + b_0 d),\ a_0, b_0 \in \mathbb{Z} \Rightarrow c \vdots d_1 \Rightarrow d_2 \vdots d_1$
   - $(a + bd \vdots d_2) \land (b \vdots d_2) \Rightarrow bd \vdots d_2 \Rightarrow a \vdots d_2 \Rightarrow d_1 \vdots d_2$
   - $d_1 \vdots d_2,\ d_2 \vdots d_1 \Rightarrow d_1 = d_2$
2. Euclid's Algorithm:
   Without detracting from the generality, suppose that $a > b, a \neq b$, then:
   - $a = k_0 b + r_0 \Rightarrow (a,\ b) = (b,\ r_0),\ r_0 < b$

   Let $a = r_{-2}, b = r_{-1}$, then:
   $r_{-2} = k_0 r_{-1} + r_0$
   $r_{-1} = k_1 r_0 + r_1$
   $\dots$
   $r_{n-2} = k_n r_{n-1} + r_n$
   $r_{n-1} = k_{n+1} r_n$
   $(a,\ b) = r_n$
   We know, that
   $r_{-2} > r_{-1} > r_0 > \dots > r_n \to \forall r_i, r_j, i < j \Rightarrow r_i < r_j$.
   Consequently Euclid's algorithm is finite.

Now prove, that $\forall a, b \in \mathbb{Z},\ \exists x, y \in \mathbb{Z} : ax + by = (a,\, b)$ We build induction in reverse way. The base is trivial:

$$0 \times r_{n-1} + 1 \times r_n = r_n$$

We know, that $r_m = r_{m-2} - k_m r_{m-1}$. Let's say that
$r_{m-1}x_m + r_m y_m = r_n$.
$m \to m-1$:

$$r_{m-1}x_m + (r_{m-2} - k_m r_{m-1})y_m = r_n$$

$$r_{m-2}y_m + r_{m-1}(x_m - k_m y_m) = r_n$$

Q.E.D.

№6

$17x + 336y = 1$
$17x \equiv 1(\text{mod } 336)$, x – inverse number.
$(336,\ 17) \to (17,\ 13) \to (13,\ 4) \to (4,\ 1)$
$(336,\ 17) = 1$
Upper number is coefficient of linear combination:
$(4^0,\ 1^1) \to (13^1,\ 4^{-3}) \to (17^{-3}, 13^4) \to (336^4, 17^{-79})$
$17 \times (-79) \equiv 1(mod\ 336)$
$-79 \notin 0..335 \to -79 + 336 = 257$
Answer: $257 + 336t, t \in \mathbb{Z}$.

№7

a) $91x \equiv 154(\text{mod } 112)$
$91x \equiv 42(\text{mod } 112)$
$91x + 112y = 42$
$(112^{-4},\ 91^5) \to (91^1,\ 21^{-4}) \to (21^0,\ 7^1)$
$42 \,\vdots\, 7 \Rightarrow 13x_0 + 16y_0 = 6 \mid :6$
Let $u = x_0/6, v = y_0/6 \to 13u + 16v = 1$
$(16^{-4},\ 13^5) \to (13^1, 3^{-4}) \to (3^0, 1^1)$
$u = 5, v = -4 \to x_0 = 30, y_0 = -24$

$$\begin{cases} x = 30 - 16t, \\ y = 13t - 24, \quad t \in \mathbb{Z} \end{cases}$$

Answer: $x = 30 - 16t,\ t \in \mathbb{Z}$.

$\overline{a} = \{a + x\ :\ x \in I\}$

$\overline{b} = \{b + x\ :\ x \in I\}$

$$\begin{cases} \overline{a} + \overline{b} = \{a + b + x_i + x_j : x_i, x_j \in I\} \\ \forall a, b \in I, a + b \in I \end{cases} \Rightarrow\ \overline{a}+\overline{b} = \{a+b+x : x \in I\}$$

$\overline{a + b} = \{a + b + x : x \in I\}$

$\overline{a} + \overline{b} = \overline{a + b}$ q.e.d.