The Gradient

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Abstract

Matlab is used to explore properties of the gradient.

Prerequisites. A basic understanding of the level curve concept. It is recommended that you practice drawing contours in Matlab before attempting this project. You should also know how to use Matlab's `meshgrid` and `mesh` commands. Knowledge of Matlab's element wise operators `(.*, ./, .^)` is required.

1 Properties of the Gradient

First, let's begin with the definition of the gradient vector.

Definition 1 *Let $f : R^2 \to R$. Then the **gradient vector** is defined as follows:*

$$\begin{aligned} \nabla f(x,y) &= f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} \\ & \quad \textit{or} \\ \nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \end{aligned}$$

Example 1 *Define $f : R^2 \to R$ by $f(x,y) = x^2 + y^2$. Find $\nabla f(1,-1)$.*

Using the definition of the gradient we make the following argument:

$$\begin{aligned} \nabla f(x,y) &= f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} \\ \nabla f(x,y) &= 2x\mathbf{i} + 2y\mathbf{j} \\ \nabla f(1,-1) &= 2(1)\mathbf{i} + 2(-1)\mathbf{j} \\ \nabla f(1,-1) &= 2\mathbf{i} - 2\mathbf{j} \end{aligned}$$

Therefore, the gradient is a function which maps each point of R^2 to a vector. If we attach the tail of the vector $2\mathbf{i} - 2\mathbf{j}$ to the point $(1,-1)$, we obtain the image in Figure 1.

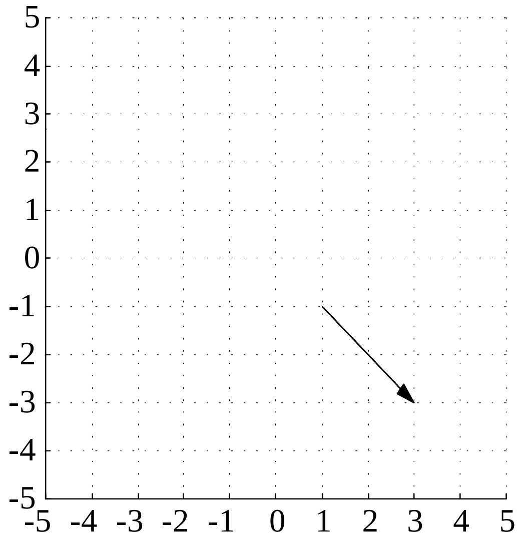

Figure 1. Attach the gradient vector to the point $(1, -1)$.

Next, if we compute the gradient vector at each point of the grid in Figure 1, then attach the tails of the computed vectors to their associated points, we can create what is known as a **vector field**. This would be very tedious to do by hand but *Matlab* can handle this with ease.

First, create a grid with *Matlab's* `meshgrid` command.

```
>> x=-5:5;
>> y=-5:5;
>> [X,Y]=meshgrid(x,y);
```

Next, read the helpfile for the `quiver` command.

```
>> help quiver

 QUIVER Quiver (or velocity) plot.
  QUIVER(X,Y,U,V) plots the velocity vectors with components
  (u,v) at the points (x,y).  The matrices X,Y,U,V must all be
  the same size and contain the corresponding position and
  velocity components (X and Y can also be vectors to specify a
  uniform grid).  QUIVER automatically scales the velocity
  vectors to fit within the grid.

  QUIVER(U,V) plots the velocity vectors at equally spaced
  points in the x-y plane.

  QUIVER(X,Y,S) or QUIVER(X,Y,U,V,S,...) automatically scales
  the velocity vectors to fit within the grid and then multiplies
  them by S.  Use S=0 to plot the velocity vectors without the
  automatic scaling.
```

The idea is as follows. The gradient vectors which we wish to plot on our grid are given by the equation

$$\nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$$

which has the form

$$\nabla f(x,y) = U(x,y)\mathbf{i} + V(x,y)\mathbf{j}$$

where $U(x,y) = 2x$ and $V(x,y) = 2y$. The following *Matlab* commands should now be clear.

```
>> U=2*X;
>> V=2*Y;
```

Finally, the following commands should produce an image similar to that in Figure 2.

```
>> quiver(X,Y,U,V)
>> grid
```

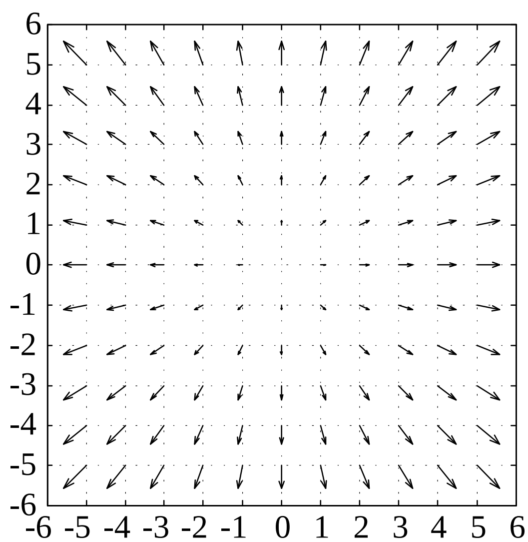

Figure 2. The vector field $\nabla f(x,y) = 2x\mathbf{i} + 2y\mathbf{j}$.

Note that each of the vectors are scaled in Figure 2. The vectors do not have the correct lengths but they do have the correct *relative* lengths. To see a gradient field with vectors that are not scaled, try entering `quiver(X,Y,U,V,0)` at the *Matlab* prompt.

1.1 The Gradient and Contour Plots

Finally, let's superimpose some level curves of the function $f(x, y) = x^2 + y^2$ on our plot of the gradient field in Figure 2. First, to avoid crowding our image with too much detail, remove the grid with the following command.

```
>> grid off
```

We've already created our mesh so we can superimpose the contours with the following commands.

```
>> hold on
>> Z=X.^2+Y.^2;
>> [c,h]=contour(X,Y,Z);
>> clabel(c,h)
```

These commands should produce an image similar to that in Figure 3.

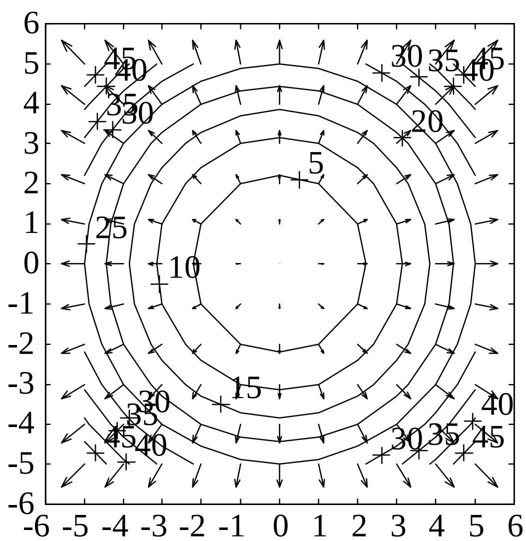

Figure 3. Superimpose the level curves of $f(x, y) = x^2 + y^2$.

1.2 Observations

Take a moment to absorb the implications of Figure 3. Several things seem apparent.

- The gradient vectors appear to be **orthogonal (perpendicular)** to the level curves.
- The gradient vectors point in the direction of steepest ascent.
- The length of the gradient vector increases as the rate of ascent increases.

Example 2 *Sketch the gradient field for* $f(x,y) = e^{-x^2-y^2}$ *over the region* $\{(x,y) : -1 \le x \le 1,\ -1 \le y \le 1\}$. *Superimpose the level curves of the function as in Example 1. Do the observations hold for this example as well?*

Solution First, set up a grid.

```
>> x=-1:.25:1;
>> y=-1:.25:1;
>> [X,Y]=meshgrid(x,y);
```

Next, the gradient is computed as follows:

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \\ \nabla f &= -2xe^{-x^2-y^2}\mathbf{i} - 2ye^{-x^2-y^2}\mathbf{j} \end{aligned}$$

Therefore, $\nabla f(x,y) = U(x,y)\mathbf{i} + V(x,y)\mathbf{j}$, where $U(x,y) = -2xe^{-x^2-y^2}$ and $V(x,y) = -2ye^{-x^2-y^2}$.

```
>> U=-2*X.*exp(-X.^2-Y.^2);
>> V=-2*Y.*exp(-X.^2-Y.^2);
```

Finally, the following *Matlab* commands will produce an image similar to that in Figure 4.

```
>> quiver(X,Y,U,V)
>> hold
>> Z=exp(-X.^2-Y.^2);
>> [c,h]=contour(X,Y,Z);
>> clabel(c,h)
```

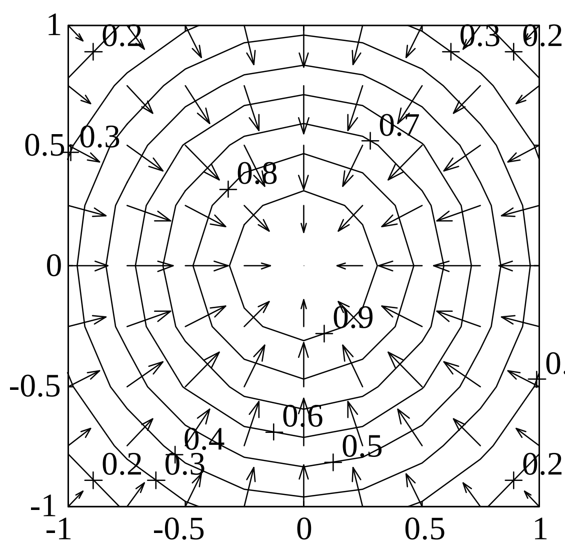

Figure 5. Contours and gradient field of $f(x,y) = e^{-x^2-y^2}$.

Note the following:

- The gradient vectors are orthogonal to the level curves of f.
- The gradient vectors point in the direction of steepest ascent, which is toward the origin in this example.
- The gradient vectors are longest in the areas of steepest ascent, then shorten near the origin, indicating a maximum function value occurs at the origin.

1.3 Homework

- Create a gradient field superimposed on a labelled [1]contour plot for each of the following functions (over the indicated domain) using the techniques of Examples 1 and 2.
- Use the `mesh` command to create a surface plot for each function over the indicated domain. Explain how your surface plot agrees with your gradient plot.

1. $f(x, y) = 4 - x^2 - y^2$ over $\{(x, y) : -2 \leq x \leq 2,\ -2 \leq y \leq 2\}$
2. $f(x, y) = 2 - |x| - |y|$ over $\{(x, y) : -2 \leq x \leq 2,\ -2 \leq y \leq 2\}$

   Here is a little hint for the second problem:

$$\begin{aligned}
D_x\,|x| &= D_x\sqrt{x^2} \\
&= D_x(x^2)^{1/2} \\
&= \frac{1}{2}(x^2)^{-1/2}(2x) \\
&= \frac{x}{(x^2)^{1/2}} \\
&= \frac{x}{\sqrt{x^2}} \\
&= \frac{x}{|x|}
\end{aligned}$$

[1] You might also want to try the `clabel` in each of the following forms:

- `clabel(c,h)`
- `clabel(c,h,'manual')`