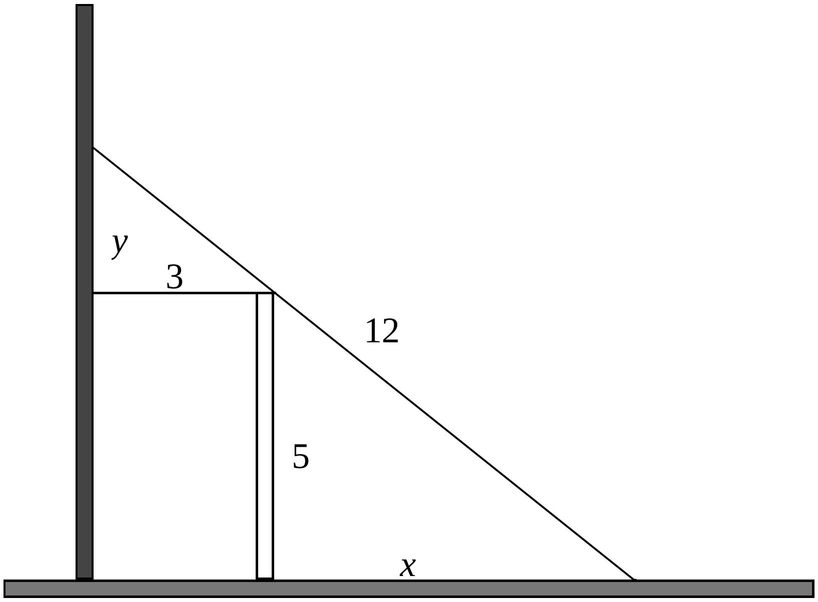
1.3 Project
A Leaning Ladder



The figure shows a 12-ft ladder that leans across a 5-ft fence and just touches a high wall located 3 feet behind the fence. You are to find the distance x from the foot of this ladder to the bottom of the fence.

Note that x is the base of a right triangle with height 5. The smaller right triangle at the upper left in the figure has base 3 and unknown height y. We can get started by setting up some equations involving the two unknowns x and y. First, the equality of base/height ratios for these two similar right triangles yields the equation

$$\frac{y}{3} = \frac{5}{x}, \quad \text{so} \quad y = \frac{15}{x}. \tag{1}$$

Then the Pythagorean relation for the largest of the three right triangles in the figure yields the equation

$$(x+3)^2 + (y+5)^2 = 12^2. \tag{2}$$

If we substitute $y = 15/x$ from (1) into (2) and simplify, we get in turn the equations

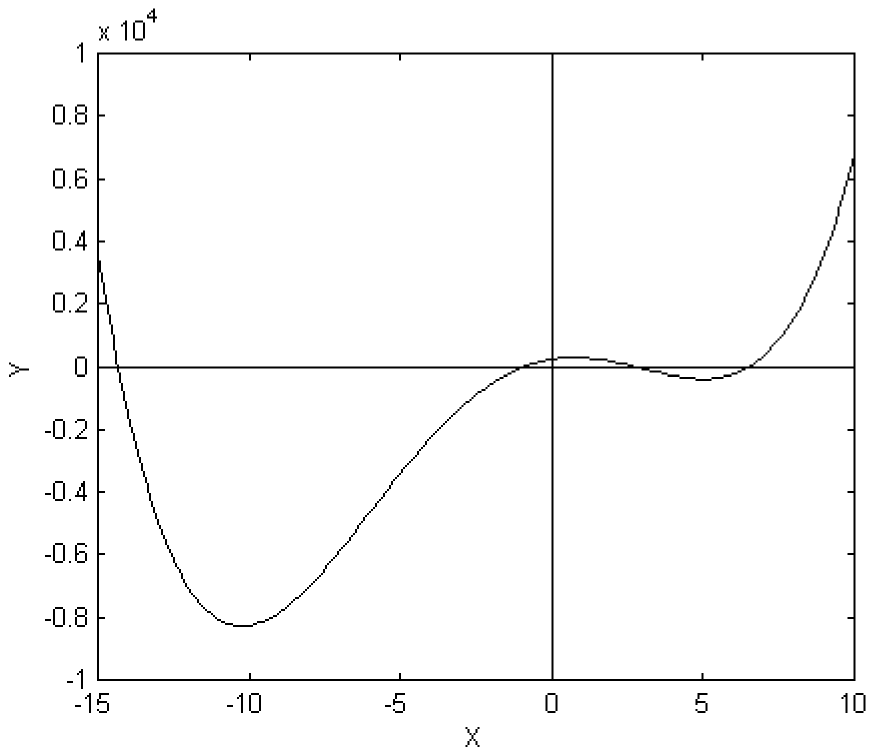
$$(x+3)^2 + \left(\frac{15}{x}+5\right)^2 = 144,$$

$$\left(x^2+6x+9\right)+\left(\frac{225}{x^2}+\frac{150}{x}+25\right) = 144,$$

$$x^4 + 6x^3 - 110x^2 + 150x + 225 = 0, \qquad (3)$$

the final quartic (4th degree) equation resulting when we multiply by x^2 (to clear fractions) and then collect coefficients of like terms. Our derivation of (3) shows that, **if** the (positive) number x denotes the physical dimension labeled in the ladder figure above, **then** x must satisfy this quartic equation.

The graph $y = f(x)$ for $-15 < x < 10$ in the figure below indicates that this equation has **four** real solutions — two of them negative and two of them positive. But only the two positive solutions — one of them apparently near $x = 3$ and the other near $x = 7$ — represent actual physical possibilities for the placement of the foot of the ladder. (Why?)



Use the graphical method of successive magnifications to zoom in on each of the **four** real solutions (accurate to two rounded decimal places) of Eq. (3). Then determine whether **both** of the two positive solutions correspond to a physically possible position of the leaning ladder as shown in the figure? Is there more than one answer? (Suggestion: For each value of x, verify that it and $y = 15/x$ satisfy equation (2). What does this prove?)

For your own personal ladder problem, let p and q denote the last two nonzero digits in your student ID number (with p the larger), and begin instead with a ladder of length $L = 3p$ feet. Suppose the fence is p ft tall and stands q ft from the high wall.

The sections below illustrate the use of graphing calculators and computer algebra systems to apply the method of magnification and other available numerical methods for the solution of equations.

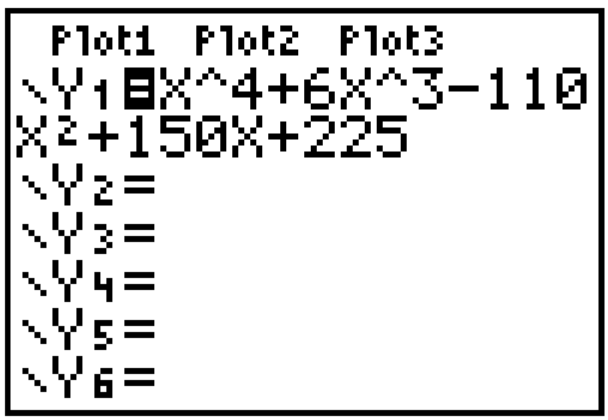
Using a Graphing Calculator

To investigate the placement of the given 12-foot ladder we need to solve the quartic equation

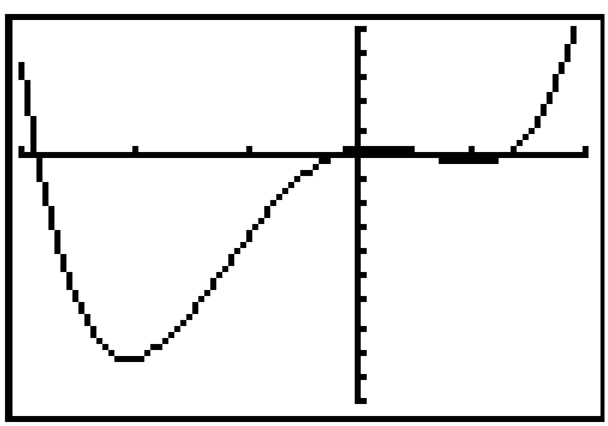
$$x^4 + 6x^3 - 110x^2 + 150x + 225 \ = \ 0.$$

Of course you must derive your own leaning ladder equation (as described on the initial page for this project) in order to investigate a different ladder-fence-wall situation.

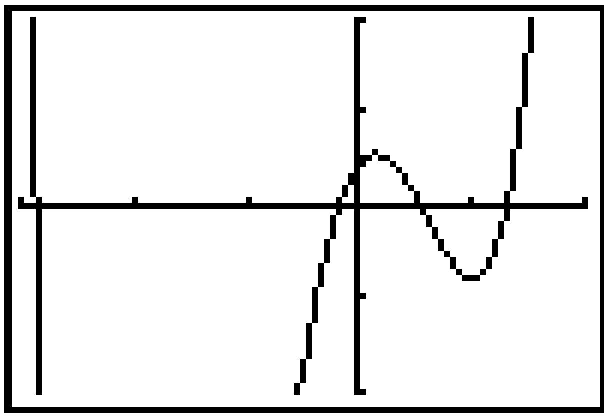
In order to use a TI-83 calculator (for instance) to solve our equation, we first define the function $f(x)$. The following calculator screen shows this function defined in the **Y=** menu.



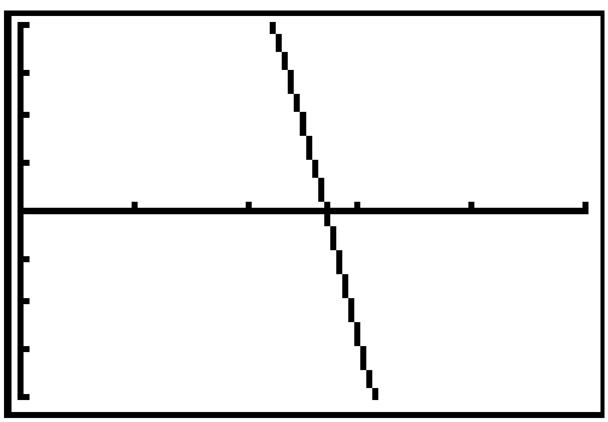
When we plot the graph $y = f(x)$ after entering **`Xmin = -15, Xmax = 10, Ymin = -10000, Ymax = 5000`** in the **`WINDOW`** screen, we see the following graph:



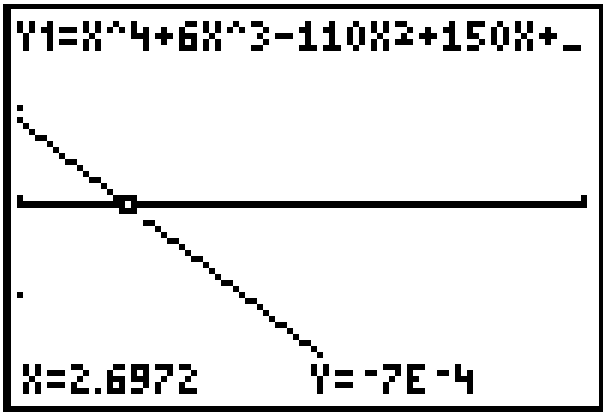
Here it's a bit hard to make out the *x*-intercepts, so we magnify in the *y*-direction only by going back to the **`WINDOW`** screen and entering **`Ymin = -1000, Ymax = 1000.`** This gives the picture



where the *x*-intercepts are more clearly visible. We see that our equation has four real solutions — two of them negative and two of them positive. But only the two positive solutions — one of them apparently near $x = 3$ and the other near $x = 7$ — represent actual physical possibilities for the placement of the foot of the ladder. We can focus on either of these by the method of repeated magnification of the graph. Zooming in on the solution between $x = 0$ and $x = 5$, for instance, we get first with **`Xmin = 0, Xmax = 5, Ymin = -100, Ymax = 100`** the figure

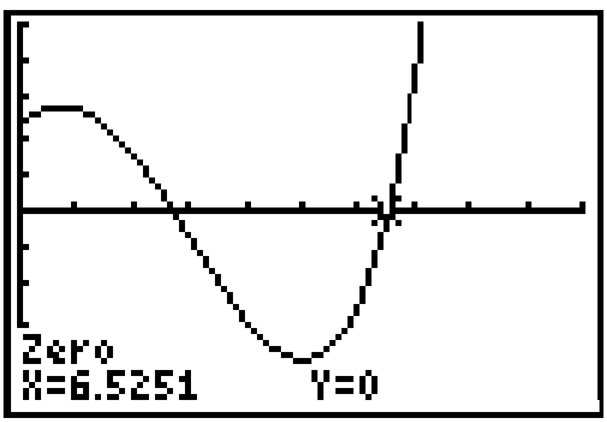


where we see a solution between $x = 2$ and $x = 3$. After several more zooms we finally get with **`Xmin = 2.697, Xmax = 2.698, Ymin = -0.1, Ymax = 0.1`** the figure



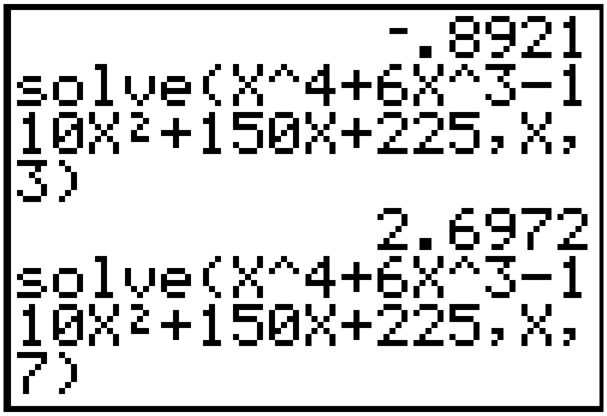
when we **`Trace`** the graph. Thus the smaller of the two positive solutions of our quartic equation is $x = 2.6972$ accurate to 4 decimal places.

Exercise: Find in similar fashion the other 3 solutions accurate to 4 decimal places. Zooming is time-consuming if you want more than a couple of decimal places of accuracy, so we can use the TI-calculator's built-in graphical root-finder. Suppose we graph our function once again, now with the window `with` **`Xmin = 0, Xmax = 10, Ymin = -500, Ymax = 500`** in order to see the two positive roots. The figure



then results if we press **`2nd CALC zero`** and locate the right-hand *x*-intercept (for instance) with the cursor. So $x = 6.5251$ is the second positive solution.

It doesn't get much easier than this! But if you really want to make the calculator do ALL the work, you can use the **`solve`** function that is entered from the **`CATALOG`** menu.



***Exercise*:** Set up your own leaning ladder equation and find all four of its solutions by at least two of these calculator methods.

Using Maple

To investigate the placement of the given 12-foot ladder we need to solve the equation

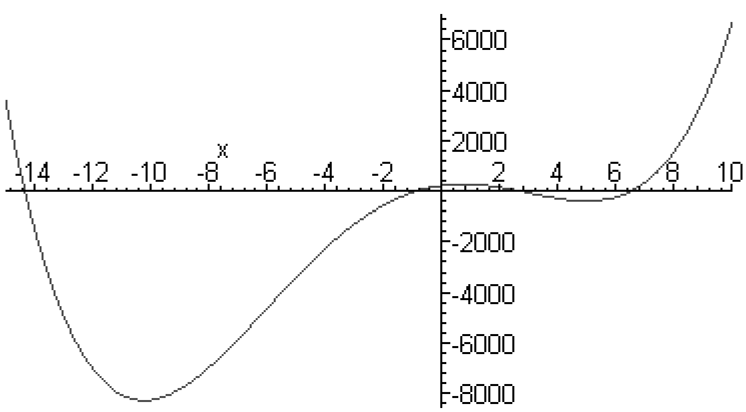
$$x^4 + 6x^3 - 110x^2 + 150x + 225 = 0.$$

Of course you must derive your own equation (as described on the initial page for this project) in order to investigate a different ladder-fence-wall situation.

In order to graph the function on the left-hand side in our quartic equation, we need only execute the plot command

```
plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=-15..10);
```

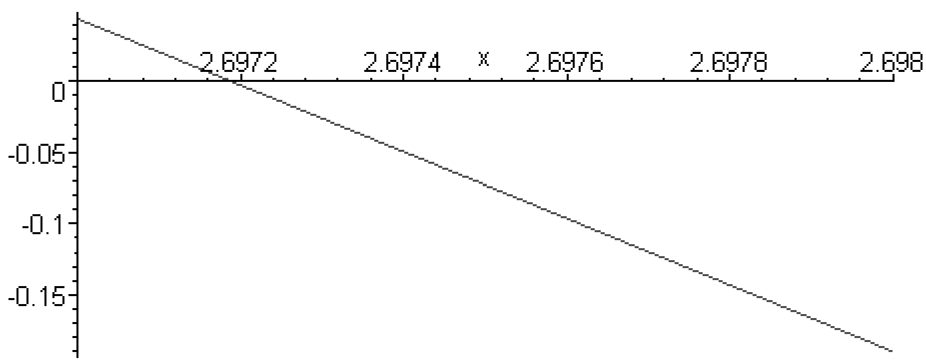
specifying the interval from $x = -15$ to $x = 10$.



Looking at the x-intercepts of the graph, we see that this equation has four real solutions — two of them negative and two of them positive. But only the two positive solutions — one of them apparently near $x = 3$ and the other near $x = 7$ — represent actual physical possibilities for the placement of the foot of the ladder. We can focus on either of these by the method of repeated magnification of the graph. Zooming in on the solution between $x = 0$ and $x = 5$, for instance, we execute the successive commands

```
plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=0..5);

plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=2..3);

plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=2.6..2.7);

plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=2.69..2.70);

plot(x^4 + 6*x^3 - 110*x^2 + 150*x + 225, x=2.697..2.698);
```

and see finally the following figure which shows that $x = 2.6972$ accurate to 4 decimal places.



Exercise: Find in this way the other three solutions of our quartic equation.

Of course, if you want to take the easy route, Maple can do the whole job at once:

```
fsolve(x^4 + 6*x^3 - 110*x^2 + 150*x + 225 = 0, x);

            -14.3301, -.892144, 2.69719, 6.52508
```

Thus we see our 4 roots again. The two positive solutions indicate that we can place the foot of the ladder either about 2 ft 8.4 in or about 6 ft 6.3 in from the base of the fence. [The **f** in **fsolve** is for floating-point (that is, numerical, solution). With **solve** you'd get impossibly complicated symbolic solutions.]

Apply each of these methods to investigate your own ladder/fence/wall situation. The Acrobat Text Tool can be used to select the Maple commands displayed above and copy/paste them into a Maple worksheet for execution one at a time.

Using Mathematica

To investigate the placement of the given 12-foot ladder we need to solve the equation

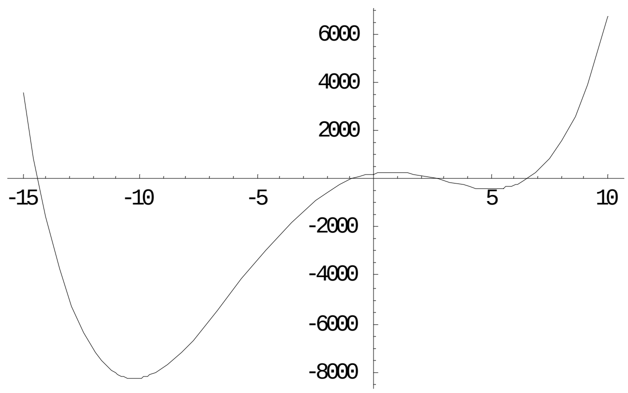
$$x^4 + 6x^3 - 110x^2 + 150x + 225 = 0.$$

Of course you must derive your own equation (as described on the initial page for this project) in order to investigate a different ladder-fence-wall situation.

In order to graph the function on the left-hand side in our quartic equation, we need only execute the plot command

```
Plot[x^4 + 6 x^3 - 110 x^2 + 150 x + 225, {x, -15, 10}]
```

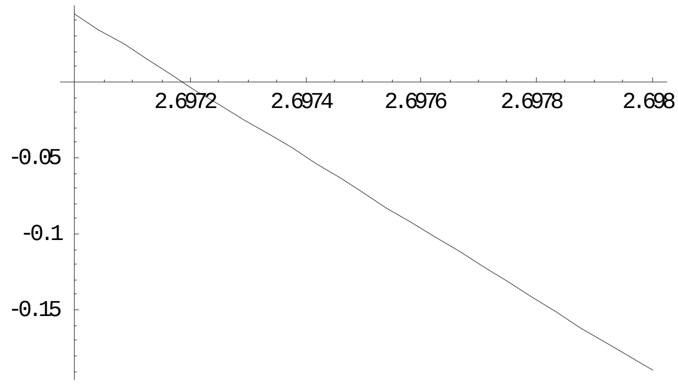
specifying the interval from $x = -15$ to $x = 10$.



Looking at the x-intercepts of the graph, we see that this equation has four real solutions two of them negative and two of them positive. But only the two positive solutions -- one of them apparently near $x = 3$ and the other near $x = 7$ -- represent actual physical possibilities for the placement of the foot of the ladder. We can focus on either of these by the method of repeated magnification of the graph. Zooming in on the solution between $x = 0$ and $x = 5$, for instance, we execute the successive commands

```
Plot[x^4 + 6x^3 - 110x^2 + 150x + 225, {x, 0, 5}];
Plot[x^4 + 6x^3 - 110x^2 + 150x + 225, {x, 2, 3}];
Plot[x^4 + 6x^3 - 110x^2 + 150x + 225, {x, 2.6, 2.7}];
Plot[x^4 + 6x^3 - 110x^2 + 150x + 225, {x, 2.69, 2.70}];
Plot[x^4 + 6x^3 - 110x^2 + 150x + 225, {x, 2.697, 2.698}];
```

and see finally the following figure which shows that $x = 2.6972$ accurate to 4 decimal places.



Exercise: Find in this way the other three solutions of our quartic equation.

Of course, if you want to take the easy route, Mathematica can do the whole job at once:

```
NSolve[x^4 + 6x^3 - 110x^2 + 150x + 225 == 0, x]
```

```
{{x -> -14.3301}, {x -> -0.89214}, {x -> 2.69719},
  {x -> 6.52508}}
```

Thus we see our 4 roots again. The two positive solutions indicate that we can place the foot of the ladder either about 2 ft 8.4 in or about 6 ft 6.3 in from the base of the fence. [The **N** in **NSolve** is for numerical solution. With **Solve** you'd get impossibly complicated symbolic solutions.]

Apply each of these methods to investigate your own ladder/fence/wall situation. The Acrobat Text Tool can be used to select the Mathematica commands displayed above and copy/paste them into a Mathematica notebook for execution one at a time.

Using MATLAB

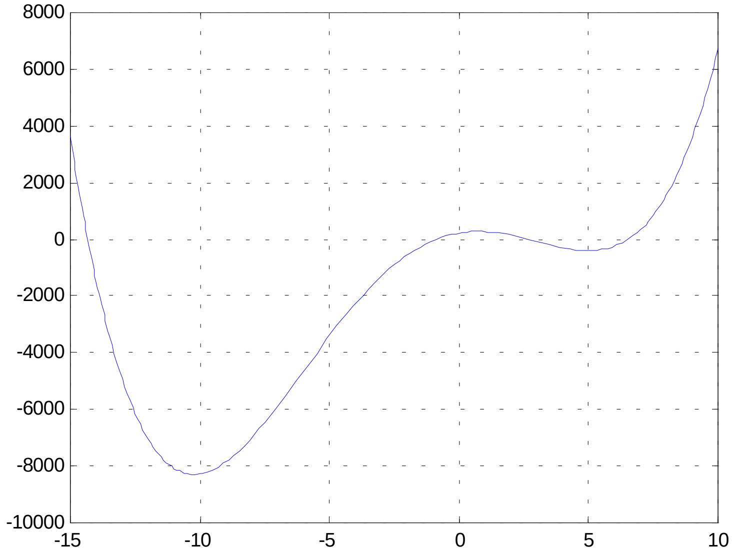
To investigate the placement of the given 12-foot ladder we need to solve the equation

$$x^4 + 6x^3 - 110x^2 + 150x + 225 = 0.$$

Of course you must derive your own equation (as described on the initial page for this project) in order to investigate a different ladder-fence-wall situation.

In order to graph the function on the left-hand side in our quartic equation, we need only execute the plot command

```
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',...
       [-15 10 -10000 10000]), grid on
```

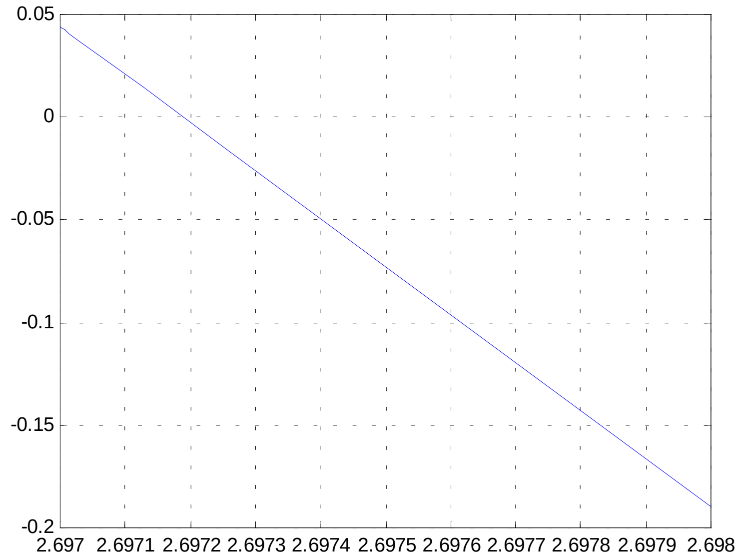


Looking at the *x*-intercepts of the graph, we see that this equation has four real solutions — two of them negative and two of them positive. But only the two positive solutions — one of them apparently near $x = 3$ and the other near $x = 7$ — represent actual physical possibilities for the placement of the foot of the ladder. We can focus on either of these by the method of repeated magnification of the graph. Zooming in on the solution between $x = 0$ and $x = 5$, for instance, we execute the successive commands

```
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',[0 5]), grid on
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',[2 3]), grid on
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',[2.6 2.7]), grid on
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',[2.69 2.70]), grid on
fplot('x.^4+6*x.^3-110*x.^2+150*x+225',[2.697 2.698])
grid on
```

and see finally the following figure shown at the top of the next page; it shows that $x = 2.6972$ accurate to 4 decimal places.

Exercise: Find the other 3 roots in this manner.



We can use MATLAB's **`fzero`** function to approximate all 4 solutions numerically, glancing at our graph to provide an initial guess for each.

```
fzero('x.^4+6*x.^3-110*x.^2+150*x+225',-15)
ans = -14.3301

fzero('x.^4+6*x.^3-110*x.^2+150*x+225',-1)
ans = -0.8921

fzero('x.^4+6*x.^3-110*x.^2+150*x+225',3)
ans = 2.6972

fzero('x.^4+6*x.^3-110*x.^2+150*x+225',7)
ans = 6.5251
```

MATLAB also includes a special polynomial-solver roots that finds simultaneously all the roots of a polynomial whose "coefficient vector" is input. The coefficients of our polynomial $x^4 + 6x^3 - 110x^2 + 150x + 225$ are given (in descending order) by the coefficient vector **`[1 6 -110 150 225]`**, so the appropriate command is

```
roots([1 6 -110 150 225])
ans =
-14.3301
6.5251
2.6972
-0.8921
```

Thus we see our 4 roots again. The two positive solutions indicate that we can place the foot of the ladder either about 2 ft 8.4 in or about 6 ft 6.3 in from the base of the fence.

Apply each of these methods to investigate your own ladder/fence/wall situation. The Acrobat Text Tool can be used to select the MATLAB commands displayed above and copy/paste them into a MATLAB command window for execution one at a time.