Application 10.3
Damping and Resonance Investigations

Here we outline *Maple*, *Mathematica*, and MATLAB investigations of the behavior of the mass-spring-dashpot system

$$m x'' + c x' + k x = F(t), \quad x(0) = x'(0) = 0 \tag{1}$$

with parameter values $m = 25$, $c = 10$, and $k = 226$ in response to a variety of possible external forces:

1. $F(t) \equiv 0$

This is the case of free damped oscillations, similar to those illustrated in Fig. 5.4.12 in the text.

2. $F(t) \equiv 901 \cos 3t$

With this periodic external force you should see a steady periodic oscillation with an exponentially damped transient motion (as illustrated in Fig. 5.6.13 in the text).

3. $F(t) \equiv 900 e^{-t/5} \cos 3t$

Now the periodic external force is exponentially damped, and the transform $X(s)$ involves a repeated quadratic factor that signals the presence of a resonance phenomenon. The response $x(t)$ is a constant multiple of that shown in Fig. 10.3.5 in the text.

4. $F(t) \equiv 900 t e^{-t/5} \cos 3t$

We have inserted a *t*-factor to make it a bit more interesting. The response $x(t)$ is plotted in Fig. 10.3.6 in the text.

5. $F(t) \equiv 16200 t^3 e^{-t/5} \cos 3t$

Now you'll find that the transform $X(s)$ involves the *fifth* power of a quadratic factor, and its inverse transform by manual methods would be impossibly tedious.

To see the advantage of using Laplace transforms, you might set up the appropriate differential equation **de** for case 5 and take a look at the result of the commands

```
dsolve({de, x(0)=0,D(x)(0)=0}, x(t));    (Maple)
```

```
DSolve[{de, x[0]==0,x'[0]==0},x[t],t]      (Mathematica)
dsolve(de, 'x(0)=0, Dx(0)=0')              (MATLAB)
```

Of course you can substitute you own favorite mass-spring-dashpot parameters for those used above. However, it will simplify the calculations if you choose m, c, and k so that

$$m r^2 + c r + k \;=\; (p r + a)^2 + b^2 \tag{2}$$

where p, a, and b are integers. One way is to select the latter integers first, then use (2) to determine m, c, and k.

Using *Maple*

First we define the mass-spring dashpot parameters

```
m := 25:    c := 10:    k := 226:
```

and the external force function

```
F := 900*t*exp(-t/5)*cos(3*t);
```

for case 4. Then our differential equation is defined by

```
de := m*diff(x(t),t$2) + c*diff(x(t),t) + k*x(t) = F;
```

and the initial conditions are given by

```
inits := {x(0)=0, D(x)(0)=0}:
```

Now we apply the Laplace transform to this equation, solve for the transform $X(s)$ of $x(t)$, and substitute the initial conditions.

```
with(inttrans):
DE := laplace(de, t,s);
X(s) := solve(DE, laplace(x(t), t,s));
X(s) := subs(inits, X(s));
```

At this point the command `factor(denom(X))` shows that

$$X(s) \;=\; \frac{22500(25s^2+10s-224)}{(25s^2+10s+226)^3}\,.$$

The cubed quadratic factor would be difficult to handle manually, but Maple readily calculates the inverse transform

```
x(t) := invlaplace(X(s) ,s,t);
```

$$x(t) := 3e^{\left(-\frac{1}{5}t\right)}t^2\sin(3t)+t\,e^{\left(-\frac{1}{5}t\right)}\cos(3t)-\frac{1}{3}e^{\left(-\frac{1}{5}t\right)}\sin(3t)$$

Let's collect the coefficients

```
A := t:
B := 3*t^2-1/3:
```

Then our solution has the form

$$x(t) = C(t)\cos(3t-\alpha)$$

where the time-varying amplitude function for these damped oscillations is defined by

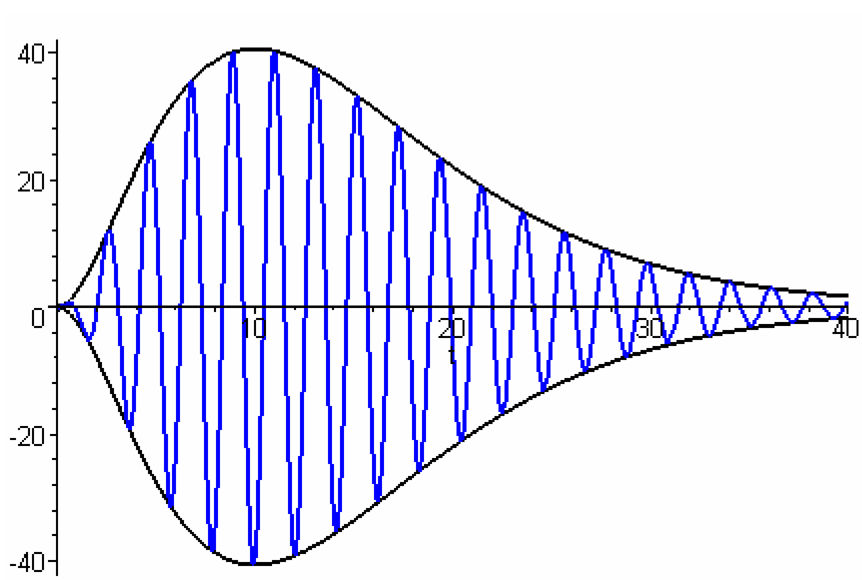
```
C(t) := sqrt(A^2 + B^2)*exp(-t/5);
```

$$C(t) := \frac{1}{3}\sqrt{-9t^2+81t^4+1}\,e^{\left(-\frac{1}{5}t\right)}$$

Finally, the command

```
plot([x(t), C(t), -C(t)], t=0..40);
```

produces the plot shown below. The resonance resulting (in effect) from the repeated quadratic factor visible in the Laplace transform of the solution $x(t)$ consists of a temporary buildup before the oscillations are damped out.



Using *Mathematica*

First we define the mass-spring dashpot parameters

```
m = 25;    c = 10;    k = 226;
```

and the external force function

```
F = 900 Exp[-t/5] Cos[3t];
```

for case 3. Then our differential equation is defined by

```
de =   m x''[t] + c x'[t] + k x[t] == F
```

and the initial conditions are given by

```
inits = {x[0]->0,  x'[0]->0};
```

Now we apply the Laplace transform to this equation, solve for the transform $X(s)$ of $x(t)$, and substitute the initial conditions.

```
Needs["Calculus`LaplaceTransform`"]
DE = LaplaceTransform[ de, t, s ]
X =  Solve[DE, LaplaceTransform[x[t],t,s]]
X =  X // Last // Last // Last
X =  X /. inits // Simplify
```

$$\frac{4500(5s+1)}{(25s^2+10s+226)^2}$$

The repeated quadratic factor visible here in the denominator of the Laplace transform $X(s)$ signals a resonance phenomenon. We now inverse transform to get the solution

```
x = InverseLaplaceTransform[X, s, t] // Expand
```

$$6e^{-t/5}\,t\sin(3t)$$

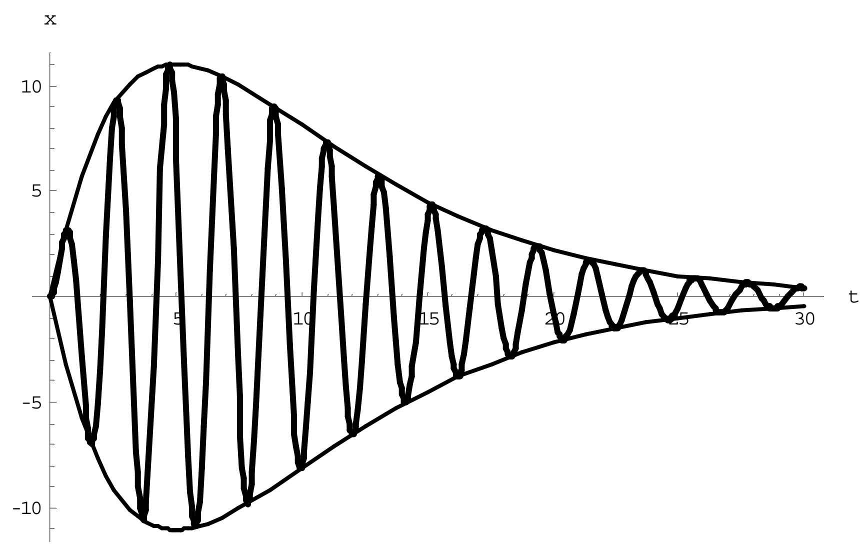
Thus we have a damped oscillation with the time-varying amplitude function

```
c = 6 t Exp[-t/5];
```

When we plot the solution curve and envelope curves,

```
Plot[ {x, c, -c}, {t,0,30}];
```

we see that the "resonance" consists in the buildup in the amplitude of the forced oscillations before the damping prevails.



Using MATLAB

First we define the mass-spring dashpot parameters

```
m = 25;      c = 10;      k = 226;
```

and the differential equation

```
syms s t x X
x = sym('x(t)');
de = m*diff(x,t,2) + c*diff(x,t) +k*x -
               16200*t^3*exp(-t/5)*cos(3*t)
```

corresponding to case 5.

Now we apply the Laplace transform to this equation, substitute the initial conditions $x(0) = x'(0) = 0$, and solve for the transform $X(s)$ of $x(t)$.

```
DE = laplace(de);
DE = subs(DE,{'x(0)','D(x)(0)'},{0,0});
DE = subs(DE,'laplace(x(t),t,s)','X')
```

```
X = solve(DE,X);
pretty(X)
```

```
                    2         4         3
          -33600 s  + 625 s  + 500 s  - 13480 s + 49276
60750000 -------------------------------------------
                                 2         5
                       (226 + 25 s  + 10 s)
```

The quintic quadratic factor visible in the denominator of the Laplace transform here would be pretty discouraging if we were working manually, but MATLAB readily finds the inverse transform

```
x = ilaplace(X,s,t)
x =
27*exp(-1/5*t)*t^4*sin(3*t) + 18*t^3*exp(-1/5*t)*cos(3*t)
- 9*exp(-1/5*t)*t^2*sin(3*t) - 3*exp(-1/5*t)*t*cos(3*t)
+ exp(-1/5*t)*sin(3*t)
```

where we see the coefficients

$$A = 18t^3 - 3 \quad \text{and} \quad B = 27t^4 - 9t^2 + 1$$

of the damped $\cos(3t)$ and $\sin(3t)$ terms, respectively. Then our solution has the damped oscillatory form

$$x(t) = C(t)\exp(-t/5)\cos(3t - \alpha)$$

where the time-varying coefficient $C(t)$ is defined by $C = \sqrt{A^2 + B^2}$.

We can therefore proceed to plot this damped oscillation with the commands

```
ezplot(x,0,50)
axis([0 50 -10^5 10^5])
hold on
t = 0 : 0.2 : 60;
A = 18*t.^3 - 3;
B = 27*t.^4 - 9*t.^2 + 1;
C = sqrt(A.^2 + B.^2);
plot(t,C.*exp(-t/5),'k')
plot(t,-C.*exp(-t/5),'k')
```

which produce the lovely plot shown below. The resonance resulting (in effect) from the repeated quadratic factor visible in the Laplace transform of the solution $x(t)$ consists of a temporary buildup before the oscillations are damped out. Note the exceptional "flatness" of the solution curve at the origin, resulting from the t^3-factor in the external force function, and the consequent high multiplicity of the repeated quadratic.