Chapter 6

Eigenvalues and Eigenvectors

Application 6.2
Diagonalization of Matrices

Below we illustrate *Maple*, *Mathematica*, and MATLAB functions that enable us to find immediately the eigenvalues and corresponding eigenvectors

$$\begin{aligned} \lambda_1 &= 14, & \mathbf{v}_1 &= (2,0,3) \\ \lambda_2 &= 31, & \mathbf{v}_2 &= (1,-1,4) \\ \lambda_3 &= 48, & \mathbf{v}_3 &= (6,2,3) \end{aligned} \tag{1}$$

of the matrix

$$\mathbf{A} = \begin{bmatrix} 371 & -612 & -238 \\ 51 & -54 & -34 \\ 357 & -663 & -224 \end{bmatrix}. \tag{2}$$

We can then verify directly that the eigenvector matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & -1 & 2 \\ 3 & 4 & 3 \end{bmatrix} \tag{3}$$

provides the diagonalization

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 1 & 6 \\ 0 & -1 & 2 \\ 3 & 4 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 371 & -612 & -238 \\ 51 & -54 & -34 \\ 357 & -663 & -224 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 \\ 0 & -1 & 2 \\ 3 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 48 \end{bmatrix},$$

which shows explicitly that $\mathbf{A}$ is similar to its diagonal eigenvalue matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Textbook examples of eigenvalues and eigenvectors typically involve only matrices with carefully selected small-integer entries that are amenable to manual computation. The following problems may take greater advantage of automatic computation facilities. In each problem, first find the eigenvalues and eigenvectors of **A**, and then verify as above that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

1. $\mathbf{A} = \begin{bmatrix} 255 & -310 \\ 200 & -243 \end{bmatrix}$

2. $\mathbf{A} = \begin{bmatrix} 447 & -391 \\ 450 & -392 \end{bmatrix}$

3. $\mathbf{A} = \begin{bmatrix} 2899 & -2664 \\ 3120 & -2867 \end{bmatrix}$

4. $\mathbf{A} = \begin{bmatrix} 40255 & -41000 \\ 39396 & -40125 \end{bmatrix}$

5. $\mathbf{A} = \begin{bmatrix} 252 & -476 & -1309 \\ 154 & -294 & -847 \\ -14 & 28 & 91 \end{bmatrix}$

6. $\mathbf{A} = \begin{bmatrix} 33769 & -28203 & 2638 \\ 27141 & -21487 & 2442 \\ -107466 & 103122 & -4522 \end{bmatrix}$

7. $\mathbf{A} = \begin{bmatrix} 101 & 174 & 94 & -76 \\ -139 & -158 & -43 & 145 \\ 271 & 345 & 98 & -277 \\ -27 & 27 & 51 & 58 \end{bmatrix}$

8. $\mathbf{A} = \begin{bmatrix} 387 & -300 & 260 & -588 & 200 \\ 82 & 9 & 44 & -194 & 68 \\ -140 & 120 & -67 & 250 & -80 \\ 840 & -720 & 624 & -1389 & 480 \\ 2188 & -1812 & 1600 & -3812 & 1319 \end{bmatrix}$

Using *Maple*

First we enter our matrix **A**:

```
with(linalg):
A := array( [[371, -612, -238],
             [ 51,  -54,  -34],
             [357, -663, -224]] );
```

Then the eigenvalues of **A** are given by

```
eigenvalues(A);
```

$$31,\ 48,\ 14$$

and the complete eigensystem by

```
eigs := [eigenvectors(A)];
```

$$eigs := \left[\left[48,\ 1,\ \left\{\left[3, 1, \frac{3}{2}\right]\right\}\right], \left[14,\ 1,\ \left\{\left[1, 0, \frac{3}{2}\right]\right\}\right], \left[31,\ 1,\ \left\{\left[-1, 1, -4\right]\right\}\right]\right]$$

We see here the three eigenvalues, each of multiplicity 1, with their corresponding eigenvectors (after multiplication by convenient factors to clear fractions)

```
v1 := scalarmul(eigs[1][3][1],2);
```

$$v1 := [6, 2, 3]$$

```
v2 := scalarmul(eigs[2][3][1],2);
```

$$v2 := [2, 0, 3]$$

```
v3 := scalarmul(eigs[3][3][1],-1);
```

$$v3 := [1, -1, 4]$$

Hence the eigenvector matrix $\mathbf{P} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is defined by

```
P := transpose(array([[6, 2, 3],
                      [2, 0, 3],
                      [1, -1, 4]]));
```

$$P := \begin{bmatrix} 6 & 2 & 1 \\ 2 & 0 & -1 \\ 3 & 3 & 4 \end{bmatrix}$$

and the desired diagonalization of $\mathbf{A}$ is given by

```
D := multiply(inverse(P),A,P);
```

$$D := \begin{bmatrix} 48 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 31 \end{bmatrix}$$

Using *Mathematica*

First we enter our matrix $\mathbf{A}$:

```
A = {{371, -612, -238},
     { 51,  -54,  -34},
     {357, -663, -224}};
```

Then the eigenvalues of **A** are given by

```
Eigenvalues[A]
```

{14, 31, 48}

and the complete eigensystem by

```
eigs = Eigensystem[A]
```

$$\begin{pmatrix} 14 & 31 & 48 \\ \{2, 0, 3\} & \{1, -1, 4\} & \{6, 2, 3\} \end{pmatrix}$$

We see here the three eigenvalues with their corresponding eigenvectors

```
v1 = eigs[[2,1]]
v2 = eigs[[2,2]]
v3 = eigs[[2,3]]
```

{2, 0, 3}

{1, –1, 4}

{6, 2, 3}

Hence the eigenvector matrix $\mathbf{P} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is defined by

```
P = Transpose[{v1, v2, v3}]
```

$$\begin{pmatrix} 2 & 1 & 6 \\ 0 & -1 & 2 \\ 3 & 4 & 3 \end{pmatrix}$$

and the desired diagonalization of **A** is given by

```
D = Inverse[P].A.P
```

$$\begin{pmatrix} 14 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 48 \end{pmatrix}$$

Using MATLAB

First we enter our matrix **A**:

```
A = [371  -612  -238
      51   -54   -34
     357  -663  -224];
```

Then the eigenvalues of **A** are given by

```
eig(A)
ans =
   14.0000
   48.0000
   31.0000
```

and the complete eigensystem by

```
[P,D] = eig(A)
P =
    0.5547    0.8571    0.2357
   -0.0000    0.2857   -0.2357
    0.8321    0.4286    0.9428
D =
   14.0000         0         0
         0   48.0000         0
         0         0   31.0000
```

Here we see both the eigenvector matrix $\mathbf{P} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ and the diagonal eigenvalue matrix **D**. The matrix **P** looks a bit unpleasant because MATLAB returns *unit* eigenvectors, but we can clean up its appearance by dividing the column eigenvectors by apparent common factors:

```
P(:,1) = P(:,1)/(P(1,1)/2);
P(:,2) = P(:,2)/(P(2,2)/2);
P(:,3) = P(:,3)/P(1,3)
P =
    2.0000    6.0000    1.0000
   -0.0000    2.0000   -1.0000
    3.0000    3.0000    4.0000
```

Finally the desired diagonalization of **A** is given by

```
inv(P)*A*P
ans =
     14.0000   -0.0000    0.0000
     -0.0000   48.0000   -0.0000
      0.0000    0.0000   31.0000
```