# Rigid Body Rotations

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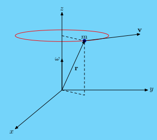
**Abstract**

The goal of this paper is to provide a fundamental understanding of the concepts and mathematics behind rotational rigid body dynamics. Along the way we will consider concepts such as angular momentum, rotational kinetic energy, rotating coordinate systems and the inertia tensor. We will then apply what we have learned to the case of an object rotating in the absence of a net force, and provide a detailed description of its motion. In doing so, we will provide a numerical solution and analysis of the resulting differential equations governing it's motion.

## A Point Particle and Fundamental Quantities

We begin by defining angular momentum, $\mathbf{L}$, and rotational kinetic energy, $K$, of a point particle $P$ with mass $m$. It's position is defined by the vector $\mathbf{r}$, which rotates with $P$. At any instant the linear velocity of $P$ is measured to be $\mathbf{v}$. For convenience, we let the z-axis be the axis of rotation. The direction of $\boldsymbol{\omega}$ that uniquely defines this rotation is then also along the z-axis. It points in the positive direction as determined by the right hand rule. This is illustrated in Figure 1.

In order to derive expressions for the particle's angular momentum and rotational kinetic energy, we need to relate $\mathbf{v}$ to $\boldsymbol{\omega}$. Notice that at any instant, $\mathbf{v}$ is perpendicular to



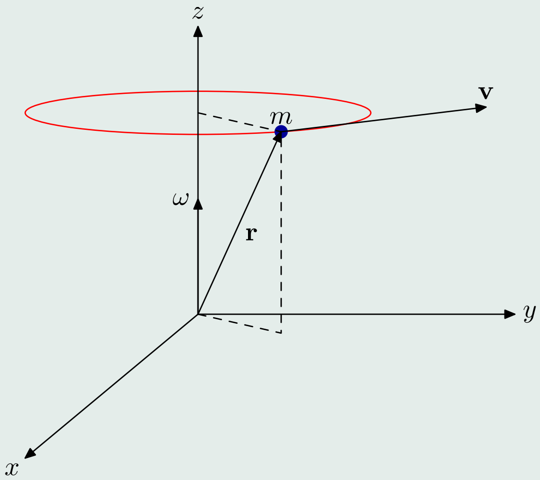



Figure 1: A Point Particle.

the plane formed by $\boldsymbol{\omega}$ and $\mathbf{r}$. Along with the fact that $\mathbf{v}$ is proportional to $\mathbf{r}$, we have the cross product relationship

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}.$$

Also, recall from basic physics that the particle's linear momentum, $\mathbf{p} = m\mathbf{v}$, and it's kinetic energy, $K = \frac{1}{2}mv^2$.

# Rigid Bodies

We can treat a rigid body as a system of particles that all rotate with the same angular velocity. The angular velocity vector, $\boldsymbol{\omega}$, of a rigid body whose center of mass stays constant is directed along its axis of rotation.

## Angular Momentum

Since the linear momentum $\mathbf{p}$ of a point particle is $m\mathbf{v}$, it follows that the particle's angular momentum $\mathbf{L}$ is defined as

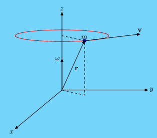
$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$$

The total angular momentum of a rigid body is then the vector sum of the individual angular momenta of its particles.

$$\mathbf{L} = \sum_{i=1} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{1}$$

Note that each particle has a unique position vector $\mathbf{r}_i$, but $\boldsymbol{\omega}$ remains constant throughout the sum. We can now make use of the identity for a triple cross product.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$$



Using this identity to compute the cross product in equation (1) we have

$$\mathbf{L} = \sum_{i=1} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = m_i r_i^2 \boldsymbol{\omega} - m_i \mathbf{r}_i (\mathbf{r}_i \cdot \boldsymbol{\omega})$$

If we let $(x_i, y_i, z_i)$ be the coordinates if the $i^{th}$ particle, then we have

$$\mathbf{L} = \sum_{i=1} m_i (x_i^2 + y_i^2 + z_i^2)(\omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{\kappa}}) - m_i (x_i \hat{\boldsymbol{\imath}} + y_i \hat{\boldsymbol{\jmath}} + z_i \hat{\boldsymbol{\kappa}})(\omega_x x_i + \omega_y y_i + \omega_z z_i)$$

$$\begin{aligned} \mathbf{L} = \textstyle\sum_{i=1} m_i [&(x_i^2 + y_i^2 + z_i^2)\omega_x \hat{\boldsymbol{\imath}} + (x_i^2 + y_i^2 + z_i^2)\omega_y \hat{\boldsymbol{\jmath}} + (x_i^2 + y_i^2 + z_i^2)\omega_z \hat{\boldsymbol{\kappa}} - \\ &(\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \hat{\boldsymbol{\imath}} - (\omega_x x_i + \omega_y y_i + \omega_z z_i) y_i \hat{\boldsymbol{\jmath}} - (\omega_x x_i + \omega_y y_i + \omega_z z_i) z_i \hat{\boldsymbol{\kappa}}] \end{aligned}$$

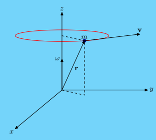
Grouping like components:

$$\begin{aligned} \mathbf{L} = \textstyle\sum_{i=1} m_i [(x_i^2 + y_i^2 + z_i^2)\omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i] \hat{\boldsymbol{\imath}} + \\ \sum_{i=1} m_i [(x_i^2 + y_i^2 + z_i^2)\omega_y - (\omega_x x_i + \omega_y y_i + \omega_z z_i) y_i] \hat{\boldsymbol{\jmath}} + \\ \sum_{i=1} m_i [(x_i^2 + y_i^2 + z_i^2)\omega_z - (\omega_x x_i + \omega_y y_i + \omega_z z_i) z_i] \hat{\boldsymbol{\kappa}} \end{aligned}$$

But noting that $\mathbf{L} = L_x \hat{\boldsymbol{\imath}} + L_y \hat{\boldsymbol{\jmath}} + L_z \hat{\boldsymbol{\kappa}}$

$$\begin{aligned} L_x \hat{\boldsymbol{\imath}} &= \sum_{i=1} m_i [(x_i^2 + y_i^2 + z_i^2)\omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i] \hat{\boldsymbol{\imath}} \\ &= \sum_{i=1} m_i [\omega_x (y_i^2 + z_i^2) - \omega_y y_i x_i - \omega_z z_i x_i] \hat{\boldsymbol{\imath}} \end{aligned}$$

$$L_x \hat{\boldsymbol{\imath}} = \left[ \omega_x \sum_{i=1} m_i (y_i^2 + z_i^2) - \omega_y \sum_{i=1} m_i y_i x_i - \omega_z \sum_{i=1} m_i x_i z_i \right] \hat{\boldsymbol{\imath}} \tag{2}$$



Similarly:

$$L_y\hat{\boldsymbol{\jmath}} = \left[\omega_y \sum_{i=1} m_i(x_i^2 + z_i^2) - \omega_x \sum_{i=1} m_i x_i y_i - \omega_z \sum_{i=1} m_i y_i z_i\right]\hat{\boldsymbol{\jmath}} \tag{3}$$

and

$$L_z\hat{\boldsymbol{\kappa}} = \left[\omega_z \sum_{i=1} m_i(x_i^2 + y_i^2) - \omega_x \sum_{i=1} m_i z_i x_i - \omega_y \sum_{i=1} m_i y_i z_i\right]\hat{\boldsymbol{\kappa}} \tag{4}$$

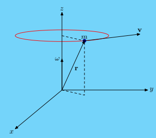
## Moments of Inertia

In physics, the individual sums in equations (2), (3) and (4) have a fundamental importance. The three sums involving sums of the coordinates of the particles are known as moments of inertia.

$$\sum_{i=1} m_i(y_i^2 + z_i^2) = \text{moment of inertia about the } x\text{-axis} = I_x$$
$$\sum_{i=1} m_i(x_i^2 + z_i^2) = \text{moment of inertia about the } y\text{-axis} = I_y$$
$$\sum_{i=1} m_i(x_i^2 + y_i^2) = \text{moment of inertia about the } z\text{-axis} = I_z$$

The moment of inertia about one of the axes is a measure of an object's resistance to rotations about that axis. This quantity is analogous to an object's mass with respect to translational motion.

Now notice that the other three sums repeat. That is, we are left with three other distinct quantities involving products of coordinates. Hence, we call them products of



inertia and are denoted using the following notation.

$$\sum_{i=1} m_i x_i y_i = xy\text{-product of inertia} = I_{xy}$$
$$\sum_{i=1} m_i x_i z_i = xz\text{-product of inertia} = I_{xz}.$$
$$\sum_{i=1} m_i y_i z_i = yz\text{-product of inertia} = I_{yz}.$$

Substituting these into equations (2), (3), and (4), and then adding to obtain an expression for the total angular momentum gives:

$$\mathbf{L} = (\omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz})\hat{\boldsymbol{\imath}} + (-\omega_x I_{xy} + \omega_y I_y - \omega_z I_{yz})\hat{\boldsymbol{\jmath}} + (-\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_z)\hat{\boldsymbol{\kappa}} \quad (5)$$

## Rotational Kinetic Energy

The rotational form of kinetic energy is obtained by again using the relationship $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{v}$$

And, similarly to finding the total angular momentum of a rigid object, we obtain the total kinetic energy by summing up the individual kinetic energies of its constituent particles.

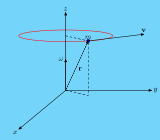
$$K = \frac{1}{2}\sum_{i=1} m_i(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\mathbf{v}_i$$

Noting the triple scalar product identity

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

we have

$$K = \frac{1}{2}\sum_{i=1} m_i\boldsymbol{\omega}\cdot(\mathbf{r}_i\times\mathbf{v}_i) = \frac{1}{2}\boldsymbol{\omega}\cdot\sum_{i=1} m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i) = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$$



# The Inertia Tensor

As we've seen, the expressions for angular momentum and kinetic energy for an object rotating in three dimensions are complex and cumbersome. But we can write these quantities in a more compact form if we introduce what is known as the inertia tensor. Recalling equation (5)

$$\mathbf{L} = (\omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz})\hat{\boldsymbol{\imath}} + (-\omega_x I_{xy} + \omega_y I_y - \omega_z I_{yz})\hat{\boldsymbol{\jmath}} + (-\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_z)\hat{\boldsymbol{\kappa}}$$

and writing this expression in components we have

$$\begin{aligned} L_x &= \omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz} \\ L_y &= -\omega_x I_{xy} + \omega_y I_y - \omega_z I_{yz} \\ L_z &= -\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_z \end{aligned}$$

This system of equations can be written in matrix form as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or

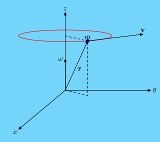
$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

where $\mathbf{I}$ is the $3 \times 3$ matrix known as the inertia tensor. But, it is important to note that while the inertia tensor takes the form of a matrix, it is really an operator. If we feed $\mathbf{I}$ and angular velocity vector $\boldsymbol{\omega}$, it will conveniently spit out the object's angular momentum, $\mathbf{L}$.

Since we have already defined an object's kinetic energy in terms of $\mathbf{L}$, we can now also express $K$ in terms of $\mathbf{I}$.

$$K = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}\boldsymbol{\omega}) \tag{6}$$

This will be useful later on.



## Principal Axes and Diagonalization

We can simplify the inertia tensor by finding an object's principal axes. Every rigid body has three orthogonal principal axes. As we will see later, rotations purely about one of these axes results in a constant angular velocity. But the goal of this section is to show that when we choose our coordinate axes to coincide with an object's principal axes, the object's products of inertia become zero. The inertia tensor then contains only the object's moments of inertia along its main diagonal. This process is called diagonalizing the inertia tensor or simply *diagonalization*. This greatly simplifies computations involving $\mathbf{I}$.

For an arbitrarily shaped object, the mathematical procedure for this is complex, but when an object exhibits symmetry, we can use symmetry arguments. Take, for example, a symmetrical cone. We begin by letting the $z$-axis be the body symmetry axis. For convenience, we let the vertex of the cone reside at the origin.

Every point $(x,y,z)$on the cone in Figure 2 has a corresponding point $(-x,-y,z)$ which is a reflection across the z-axis. And in the sum of the products of the $x$ and $z$ or $y$ and $z$ coordinates gives zero. Therefore, in the sum of the $xz$ and $yz$ products of inertia

$$I_{xz} = \sum_{i=1} m_i x_i z_i = 0$$

$$I_{yz} = \sum_{i=1} m_i y_i z_i = 0$$

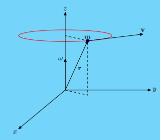
But we also need to show that the $xy$ product of inertia is zero. Looking down on the $z$-axis at the cone, we must now consider the reflections of our point $(x,y,z)$ in all four quadrants. These points are shown in Figure 3

Taking the sum of the four $xy$ products we get

$$(xy) + (-xy) + (-xy) + (xy) = 2xy - 2xy = 0$$

Therefore, in the total sum we can conclude that

$$I_{xy} = \sum_{i=1} m_i x_i y_i = 0$$



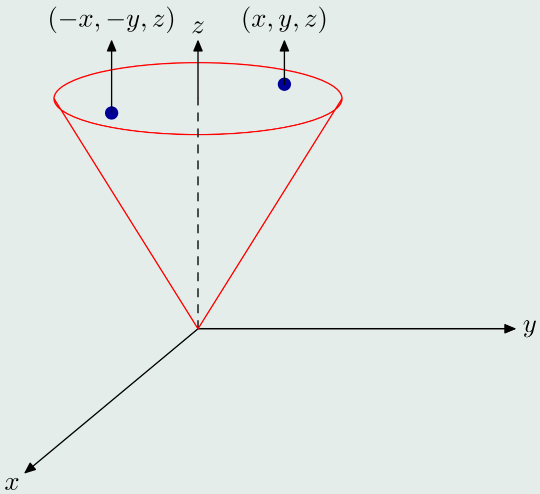
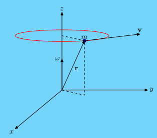



Figure 2: A symmetrical cone



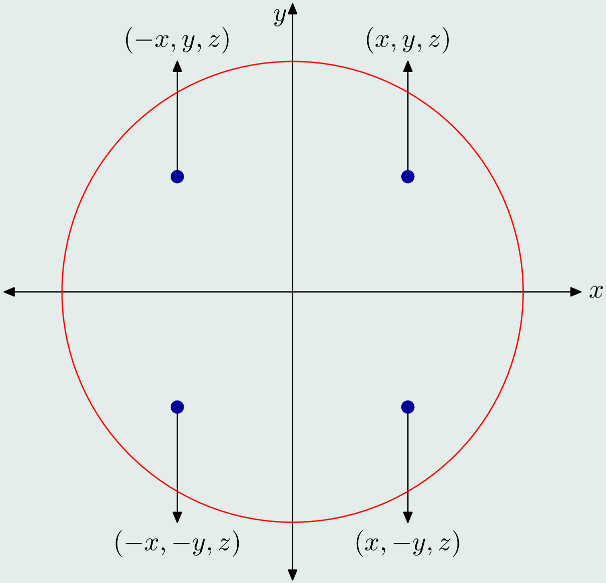
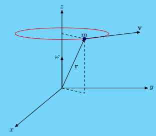



Figure 3: A symmetrical cone



With this coordinate system orientation, the inertia tensor becomes

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

If we now re-examine the object's rotational kinetic energy you can see how this process greatly simplifies our analysis, which yields a concise expression for $K$. Expanding equation (6)

$$K = \frac{1}{2}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \cdot \left[\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}\begin{pmatrix} \omega_x I_x \\ \omega_y I_y \\ \omega_z I_z \end{pmatrix}$$

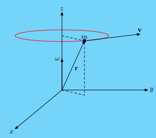
$$K = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) \tag{7}$$

# Rotating Coordinate Systems

As we will see, in order to complete our analysis, we will need to introduce a rotating coordinate system that stays fixed with our rotating body. We will let the primed set of axes represent the fixed, or inertial, coordinate system, and the unprimed axes to be the rotating coordinate system, as shown. Our goal now is to be able to relate vectors as measured in one coordinate system to the same vectors as measured from the other. Imagine that the point $P$ in Figure 4 rotates with the unprimed axes. Its velocity vector $\mathbf{v}$ is perpendicular to its position vector $\mathbf{r}$. Since the origins of the coordinate system coincide, the vector $\mathbf{r}$ must be the same as measured from either coordinate system.

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'$$

At first glance this seems strange. This is because the individual components of $\mathbf{r}$ will be different in the two coordinate systems, but the sum of these components will yield the same vector $\mathbf{r}$.



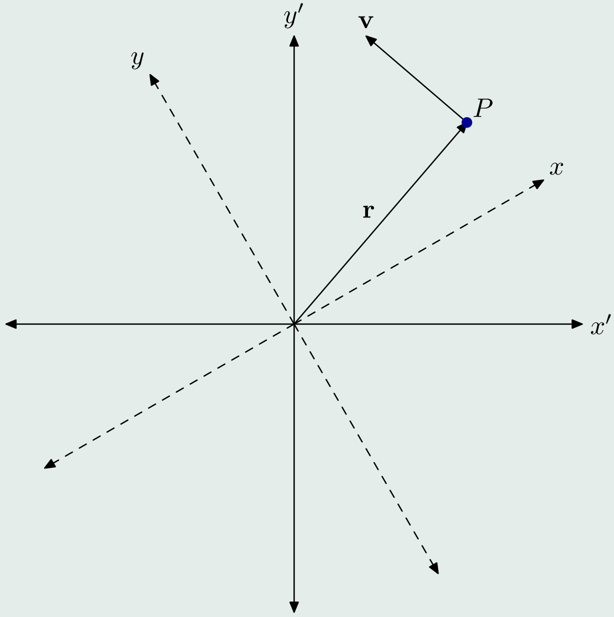



Figure 4: The above view

Our goal now is to relate how $\mathbf{r}$ changes as measured from the fixed frame of reference to its rate of change as measured from the rotating frame of reference. In doing so, we will consider the most general case. That is we will allow $\mathbf{r}$ to be changing with respect to *both* coordinate systems. So, we take $\mathbf{r}$ as measured from the unprimed, or rotating, coordinate system.

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

Next, we take the time derivative of $\mathbf{r}$ from the rotating frame of reference.

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} = \left(\frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}}\right) \tag{8}$$

We then take the same time derivative from the *fixed* frame of reference. But now, $x$,$y$,$\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$ are all changing in time. Therefore, we must apply the product rule.

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \frac{dx}{dt}\hat{\boldsymbol{\imath}} + x\frac{d\hat{\boldsymbol{\imath}}}{dt} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}$$

Grouping:

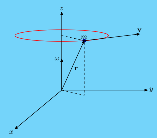
$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}}\right) + \left(x\frac{d\hat{\boldsymbol{\imath}}}{dt} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}\right)$$

Substituting equation (8)

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \left(x\frac{d\hat{\boldsymbol{\imath}}}{dt} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}\right) \tag{9}$$

So we are left with evaluating the right hand term of equation (9). To evaluate the time derivatives of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, recall that $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$. We can treat the rotating unit vectors as position vectors and infer that the time derivatives are

$$\frac{d\hat{\boldsymbol{\imath}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{\imath}} \qquad \text{and} \qquad \frac{d\hat{\boldsymbol{\jmath}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{\jmath}}$$



Substituting these relationships into equation (9)

$$\begin{aligned}
\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + [x(\boldsymbol{\omega}\times\hat{\boldsymbol{\imath}}) + y(\boldsymbol{\omega}\times\hat{\boldsymbol{\jmath}})] \\
&= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) \\
&= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{r}
\end{aligned}$$

Although we have derived this relationship for a position vector, the relationship applies to any vector. This gives us the general operator on any vector $\mathbf{A}$

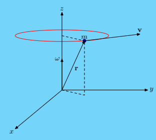
$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega}\times\mathbf{A} \tag{10}$$

# Euler's Equations

We now have all the necessary tools to derive Euler's equations. We start with Newton's second law in rotational form, which states that the net torque on an object is equal to its rate of change of angular momentum. That is

$$\sum\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}$$

This statement is only valid when applied from an inertial reference frame. But notice that in this scenario, the orientation of an object's mass with respect to some fixed coordinate system may change, if the object's center of mass is changing in time. Therefore the inertia tensor $\mathbf{I}$ is also changing in time. The difficulty is that we do not know how to take the derivative of a changing inertia tensor. In order to overcome this difficulty, we must adopt



a rotating coordinate system. Using the operator in equation (10)

$$\boldsymbol{\tau} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt} = \left(\frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

For convenience we denote the net torque simply as $\boldsymbol{\tau}$. $\mathbf{I}$ is now constant and can be taken outside the derivative.

$$\boldsymbol{\tau} = \mathbf{I}\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

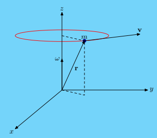
An interesting simplification can be made if we consider the relationship between $\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}}$ and $\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}}$. Again using our operator

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \boldsymbol{\omega}$$

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}}$$

Therefore, we need not distinguish which frame of reference we are in when we take the time derivative of $\boldsymbol{\omega}$, and we now have

$$\boldsymbol{\tau} = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) \tag{11}$$



Or, in matrix form, using the diagonalized inertia tensor we have

$$
\begin{aligned}
\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} &= \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \dot\omega_x \\ \dot\omega_y \\ \dot\omega_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\
&= \begin{pmatrix} \dot\omega_x I_x \\ \dot\omega_y I_y \\ \dot\omega_z I_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \omega_x I_x \\ \omega_y I_y \\ \omega_z I_z \end{pmatrix} \\
&= \begin{pmatrix} \dot\omega_x I_x \\ \dot\omega_y I_y \\ \dot\omega_z I_z \end{pmatrix} + \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{\kappa}} \\ \omega_x & \omega_y & \omega_z \\ \omega_x I_x & \omega_y I_y & \omega_z I_z \end{vmatrix} \\
&= \begin{pmatrix} \dot\omega_x I_x \\ \dot\omega_y I_y \\ \dot\omega_z I_z \end{pmatrix} + \begin{pmatrix} \omega_y\omega_z I_z - \omega_y\omega_z I_y \\ -\omega_x\omega_z I_z + \omega_x\omega_z I_x \\ \omega_x\omega_y I_y - \omega_x\omega_y I_x \end{pmatrix}
\end{aligned}
$$

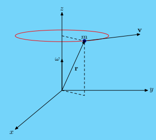
Or, in components

$$
\begin{aligned}
\tau_x &= \omega_y\omega_z I_z - \omega_y\omega_z I_y + \dot\omega_x I_x \\
\tau_y &= \omega_x\omega_z I_x - \omega_x\omega_z I_z + \dot\omega_y I_y \\
\tau_z &= \omega_x\omega_y I_y - \omega_x\omega_y I_x + \dot\omega_z I_z
\end{aligned}
$$

These three equations are known as Euler's equations of motion for a rigid body.

## Force Free Rotations and Precession

We now move on to examine the case when the net torque on an object is zero. This has many applications including an object freely falling in a gravitational field. (Assuming



that the field is uniform.) Setting $\boldsymbol{\tau} = 0$ in Euler's equations

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_x\omega_y$$

This is a system of three nonlinear first order differential equations easily solved using a numerical solver such as Matlab's ODE45. However, let's first consider the nature of such rotations qualitatively, and consider different initial conditions.

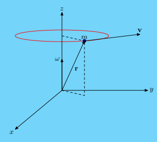
## Initial Conditions and Equilibrium Solutions

The equilibrium points are obtained in the usual manner, by setting $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z = 0$. Solving the resulting system of equations simultaneously yields all solutions of the form

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

where $a$, $b$, and $c$ are any real numbers. That is, we have all points that lie on the coordinate axes. Let's think about this result.

If our initial condition lies on an axis, that means that $\boldsymbol{\omega}$, initially, is directed along that axis. This corresponds to a rotation purely about that axis. But we know that solutions that start at equilibrium points stay constant for all time. And recall that our coordinate axes correspond to the object's principal axes. Therefore, we can conclude that if an object begins rotating purely about one of its principal axes, its angular velocity will remain constant forever. Otherwise, as we will see, $\boldsymbol{\omega}$ will change.



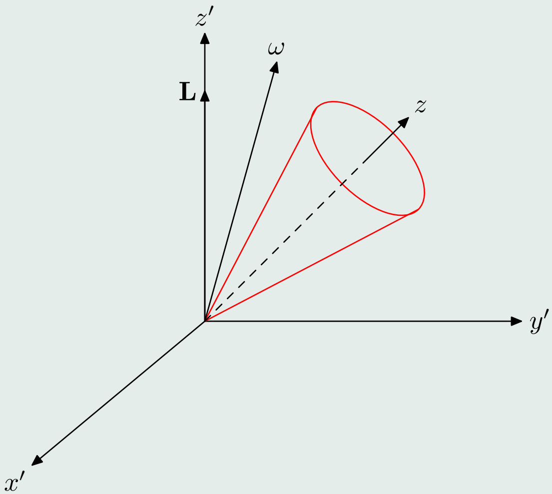



Figure 5: An arbitrary rotation

So, now let's consider the case when $\boldsymbol{\omega}$ is in some arbitrary direction. This physical situation is depicted in Figure 5. In this situation, the cone is rotating about the $z$ axis. Meanwhile, its center of mass is orbiting the $z'$ axis. So, we essentially have two rotations superimposed upon one another, and the cone's total angular velocity vector, $\boldsymbol{\omega}$, is the sum of the components along the $z$ and $z'$ axes, due to the individual rotations.

Let's now consider how $\boldsymbol{\omega}$ can change without the influence of a torque. Recalling, that from an inertial frame of reference

$$\sum \boldsymbol{\tau} = 0 = \frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}$$

and we see that $\mathbf{L}$ must remain constant. However, since $\mathbf{I}$ is changing in time, $\boldsymbol{\omega}$ *must* also change. In fact, $\boldsymbol{\omega}$ will do whatever it needs to, in order to keep $\mathbf{L}$ constant. This change in $\boldsymbol{\omega}$ in the absence of a net torque is called *precession*. But let's look again at equation (11), which is our equation for torque from a rotating frame of reference. Setting $\boldsymbol{\tau} = 0$

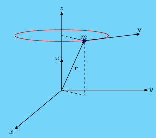
$$0 = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

In this case, $\mathbf{I}$ is now constant in time. However, the additional right hand term, or *pseudo force* is what causes $\boldsymbol{\omega}$ to precess. And, an interesting note is that two observers in the different reference frames see a different cause for the precession of $\boldsymbol{\omega}$. However, an analysis from either reference frame will yield the same physical situation.

## Kinetic Energy and the Ellipsoid

Since we are considering force free motion, we expect, from the principle of energy conservation, that the object's rotational kinetic energy will remain constant. This is verified by taking the time derivative of $K$. Recalling equation (7) and differentiating, we have

$$K = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$



$$\frac{dK}{dt} = \frac{1}{2}(2I_x\omega_x\dot{\omega}_x + 2I_y\omega_y\dot{\omega}_y + 2I_z\omega_z\dot{\omega}_z)$$

Substituting in the differential equations for $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$

$$\begin{aligned}
\frac{dK}{dt} &= \frac{1}{2}\left(2I_x\omega_x\frac{I_y - I_z}{I_x}\omega_y\omega_z + 2I_y\omega_y\frac{I_z - I_x}{I_y}\omega_x\omega_z + 2I_z\omega_z\frac{I_x - I_y}{I_z}\omega_x\omega_y\right)\\
\frac{dK}{dt} &= \omega_x\omega_y\omega_z\left[(I_y - I_z) + (I_z - I_x) + (I_x - I_y)\right]\\
&= 0
\end{aligned}$$

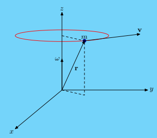
So, since $K$ is constant, our expression for rotational kinetic energy becomes the equation of an ellipsoid, whose surface represents values of $\boldsymbol{\omega}$ with the same energy. That is, the surface of the ellipsoid represents a surface of constant energy in the 3-dimensional phase space of $\boldsymbol{\omega}$. This is called an *inertial ellipsoid*. And since $K$ is constant for all time, solutions that start on the surface of the ellipsoid, must remain on the surface for all time. Otherwise, the rotating object would either gain or lose energy.

Furthermore, we already know that equilibrium points lie on the coordinate axes. Therefore, for a given energy, the equilibrium points of the system are precisely where the coordinate axes intersect the surface of the ellipsoid.

## Example

If we let the moments of inertia about the $x$, $y$, and $z$ axes equal 2, 1, and 3 respectively, we have the given system of differential equations.

$$\begin{aligned}
\dot{\omega}_x &= -\omega_y\omega_z\\
\dot{\omega}_y &= \omega_x\omega_z\\
\dot{\omega}_z &= \frac{1}{3}\omega_x\omega_y
\end{aligned}$$



And our equation for kinetic energy becomes

$$K = \frac{1}{2}(2\omega_x^2 + \omega_y^2 + 3\omega_z^2)$$

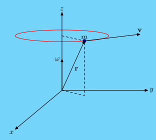
If we let $K = 12$, we obtain the equation of an inertial ellipsoid

$$24 = 2\omega_x^2 + \omega_y^2 + 3\omega_z^2$$

If we now pick off any initial condition that satisfies this equation of the ellipsoid, the resulting solution trajectory must remain on the surface of the ellipsoid for all time. We now use Matlab's ODE45 routine to plot several of these solution trajectories. Figure 6 shows these solutions and the approximate locations of the six equilibrium points. Notice that there are four stable equilibrium points that are centers. Two lie where the $z$ axis intersects the ellipsoid, and two lie where the $y$ axis intersects the ellipsoid. However, the other two equilibrium points are unstable, saddle points. These points are seen repelling solutions that approach the two points where the $x$ axis intersects the ellipsoid.

Now let's imagine that we have an object rotating at the origin, whose axis of rotation is purely about the $z$ axis. $\boldsymbol{\omega}$, then, is along the $z$ axis and points directly at the stable equilibrium point. If the object is slightly perturbed off this axis, $\boldsymbol{\omega}$ precesses slightly and traces out nearly a perfect circle. However, if it is perturbed more, you can see the dips that begin to show up in the path that $\boldsymbol{\omega}$ traces. These dips correspond to a slight wobble in the object's precession. This phenomenon is called *nutation*. The same motion occurs when the axis of rotation is the $y$ axis. However, if the axis of rotation is the $x$ axis and $\boldsymbol{\omega}$ points at an unstable equilibrium point, the story is quite different. If the object is barely tapped, $\boldsymbol{\omega}$ will virtually reverse its direction and oscillate between the two positions in a rather peculiar manner.

So, the numerical solution is consistent with our qualitative analysis. And along with Euler's equations and our physical interpretation, we have a complete description of force free rotations of rigid bodies.



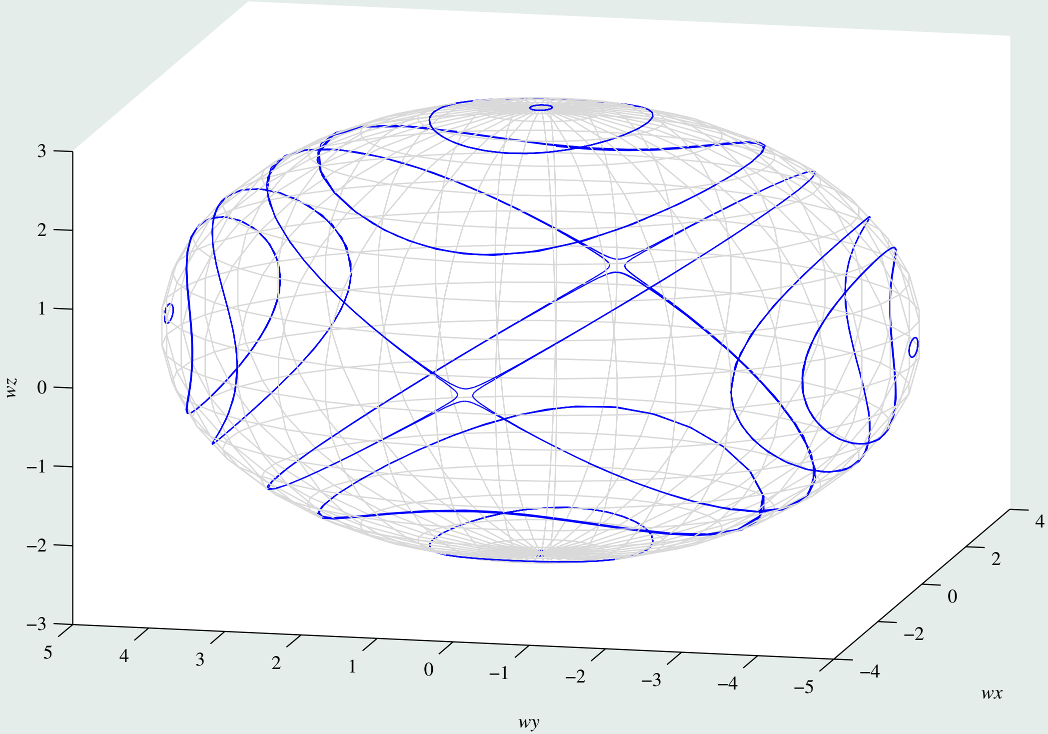
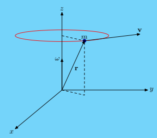
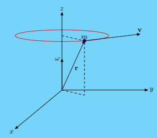



Figure 6: An inertial ellipsoid



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# References


[1] Marion Thornton. *Classical Dynamics.*

[2] James Polking and David Arnold. *Ordinary Differential Equations with Boundary Value Problems.*

[3] Gilbert Strang. *Introduction to Linear Algebra.*