A Bead on a Rotating Hoop

Ryan Seng and Michael Meeks

May 15, 2008

Table of Contents

Introduction & Background

The behavior of the bead will vary as it travels along the hoop, the dependent factor being the hoop's angular velocity.

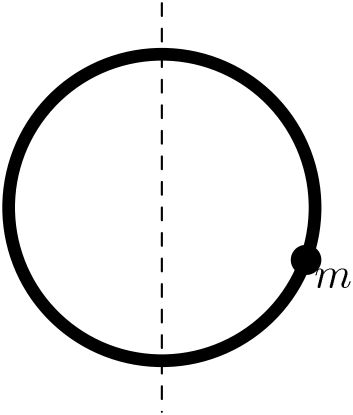
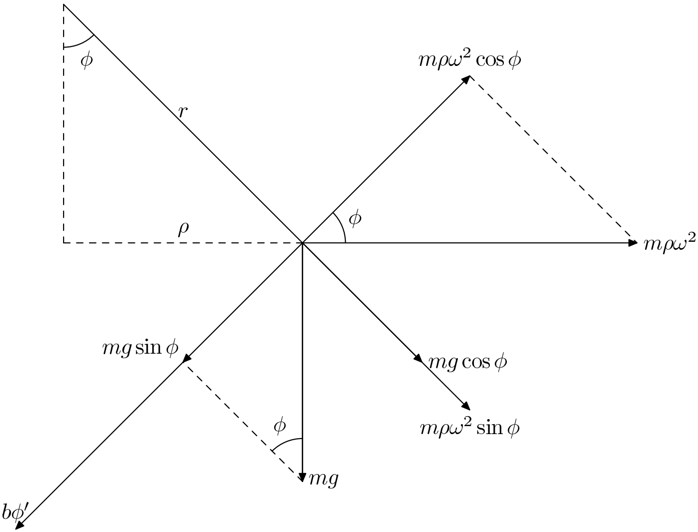


Figure: Hoop Diagram

Forces Diagram



Frictionless Introduction

- Acceleration
- Forces
  - 'Fictitious' Force - Centrifugal Force -
  - Gravitational Force -

Frictionless Calculations

Begin with Newton's Equation

$$\sum F = ma$$

Substitute in Forces and Acceleration

$$-mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{d^2\phi}{dt^2}$$

Making Equation Dimensionless

Introduce new variable τ, $\tau = t/T$. $d\tau/dt = 1/T$.
Taking the derivative of ϕ with respect to time we get

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{d\phi}{d\tau}\frac{1}{T}$$

We then take the second derivative of ϕ with respect to time

$$\frac{d^2\phi}{dt^2} = \frac{d}{dt}\left(\frac{d\phi}{dt}\right) \Rightarrow \frac{d}{dt}\left(\frac{1}{T}\frac{d\phi}{d\tau}\right) \Rightarrow$$

$$\frac{d}{d\tau}\left(\frac{1}{T}\frac{d\phi}{d\tau}\right)\frac{d\tau}{dt} \Rightarrow \frac{1}{T}\frac{d^2\phi}{d\tau^2}\frac{1}{T} \Rightarrow \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$$

Frictionless Calculations Cont.

Replace $\frac{d^2\phi}{dt^2}$ with $\frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$

$$-mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{1}{T^2}\frac{d^2\phi}{d\tau^2} \tag{1}$$

Divide through by mg and let T equal b/mg

$$-\sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{m^2gr}{b^2}\right)\frac{d^2\phi}{d\tau^2}.$$

Introduce γ and ε

$$-\sin\phi + \gamma\sin\phi\cos\phi = \varepsilon\frac{d^2\phi}{d\tau^2}.$$

Frictionless Equilibrium Solutions

Two equilibrium solutions

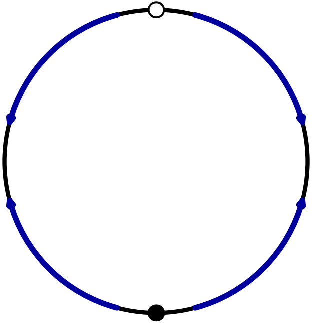


Figure: Two Equilibrium Solutions.

Frictionless Equilibrium Solutions

Potential for three equilibrium solutions

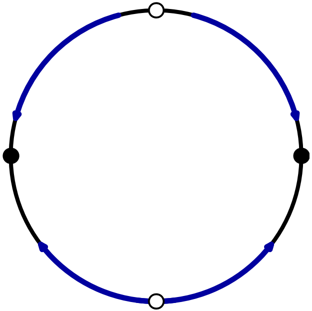
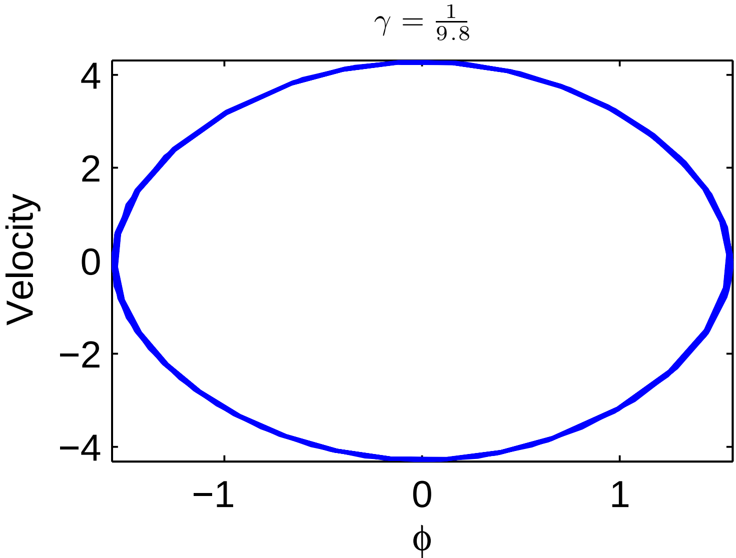
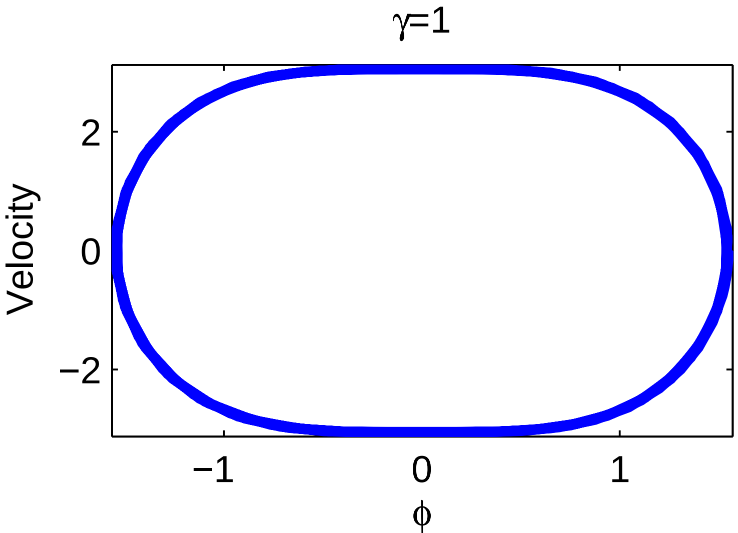


Figure: Three Equilibrium Solutions.

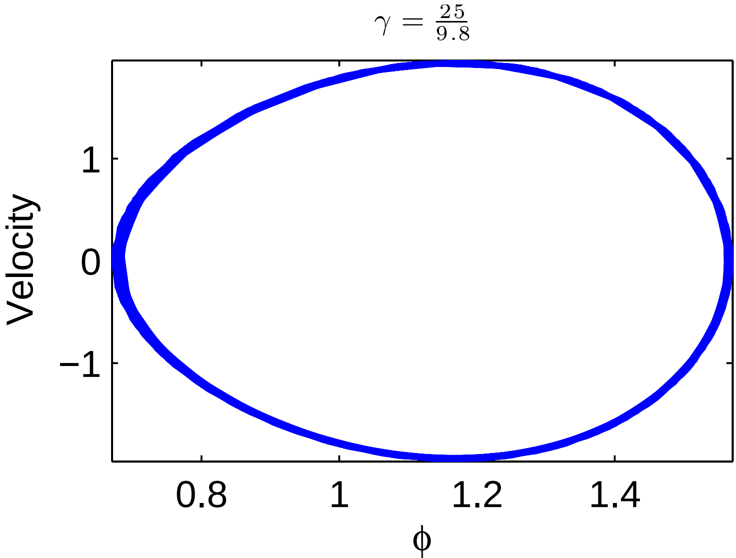
Frictionless Examples



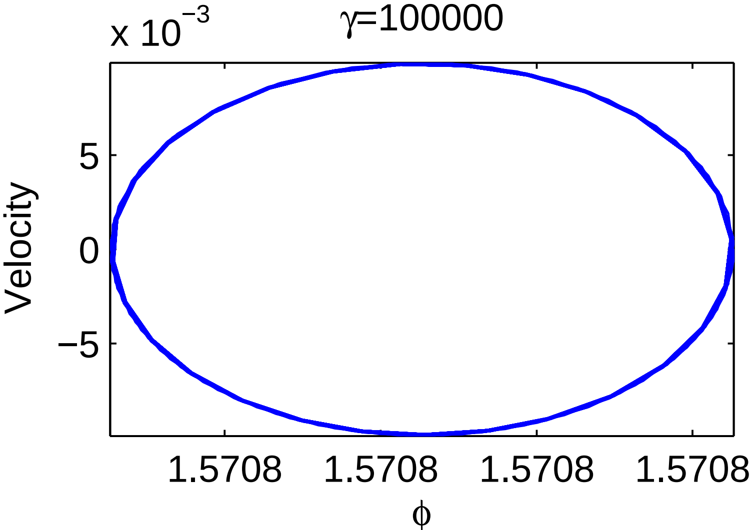
Frictionless Examples Cont.



Frictionless Examples Cont.



Frictionless Examples Cont.



Friction Introduction

- Acceleration
- Forces
  - 'Fictitious' Force - Centrifugal Force -
  - Gravitational Force -
- Friction

Friction Calculations

Pick up where we left off on Equation (**??**)

$$-mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{1}{T^2}\frac{d^2\theta}{d\tau^2}$$

Introduce friction

$$-\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{1}{T^2}\frac{d^2\theta}{d\tau^2}$$

Friction Calculations Cont.

Divide through by mg and sub in b/mg for T

$$-(1)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{m^2 gr}{b^2}\right)\frac{d^2\theta}{d\tau^2}$$

Introduce γ and ε again

$$\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma\sin(\phi)\cos(\phi).$$

Friction Equilibrium Solutions

To find the Equilibrium Solutions we deal with the first ODE.

$$bd\phi/dt = -mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi)$$

We then set ϕ' equal to 0 and factor

$$mg\sin(\phi)\left(-1 + \frac{r\omega^2}{g}\cos(\phi)\right) = 0$$

Setting the first part equal to 0

$$mg\sin(\phi) = 0$$

This happens at 0 and π

Friction Equilibrium Solutions Cont.

Setting the second part equal to 0

$$-1 + \frac{r\omega^2}{g}\cos(\phi) = 0$$

Solving for ϕ we get

$$\phi = \pm\cos^{-1}\left(\frac{g}{r\omega^2}\right)$$

Subbing γ back in we get

$$\phi = \pm\cos^{-1}\left(\frac{1}{\gamma}\right)$$

This creates another equilibrium solution, but it is not in a specific position

Friction Equilibrium Solutions Cont.

Two obvious Equilibrium Solutions

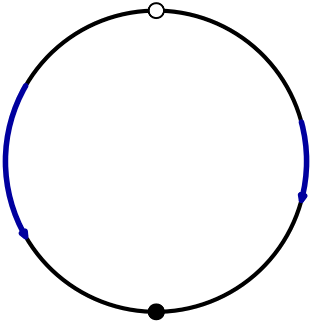


Figure: Two Equilibrium Solutions.

Friction Equilibrium Solutions Cont.

Potential for Three Equilibrium Solutions

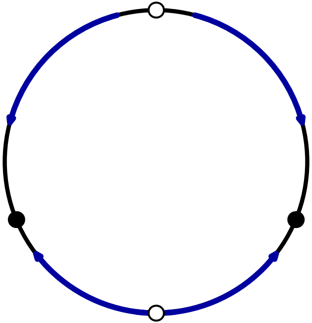
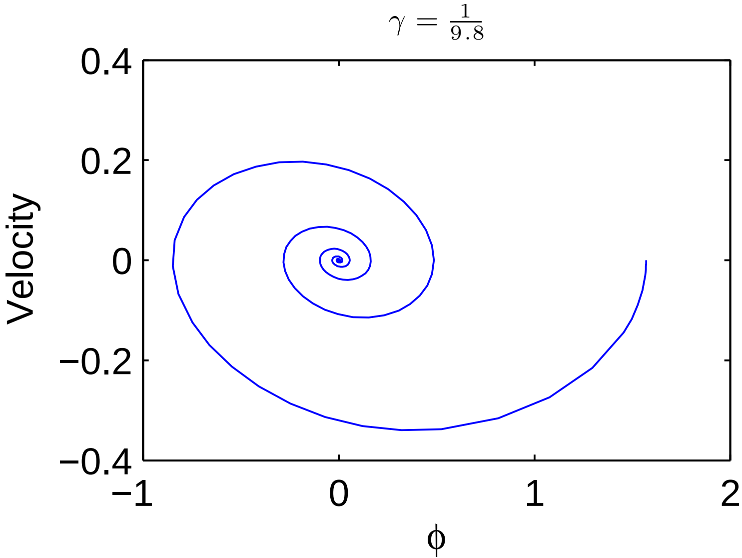
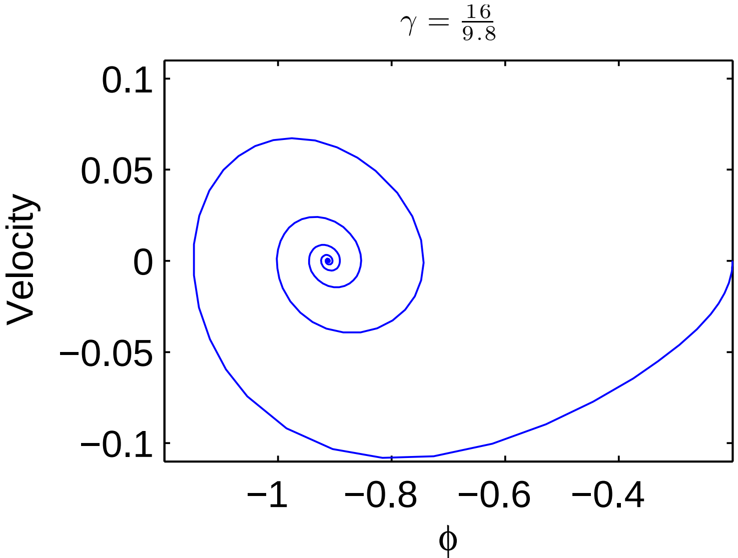


Figure: Three Equilibrium Solutions.

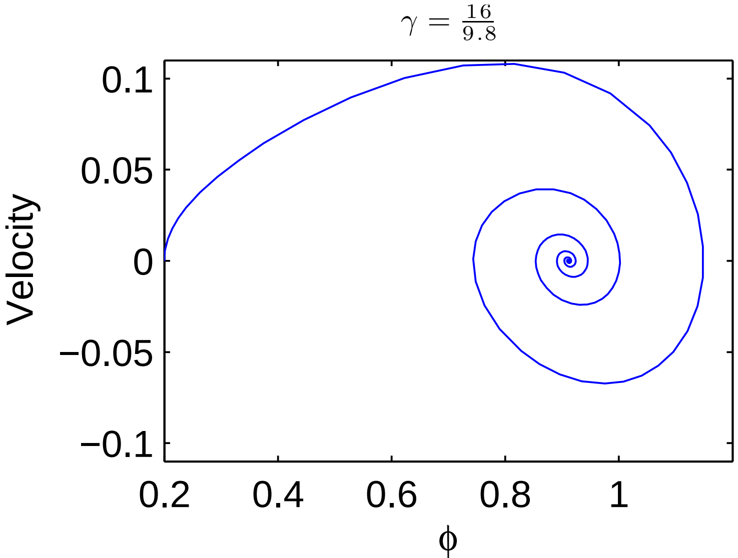
Friction Examples



Friction Examples Cont.



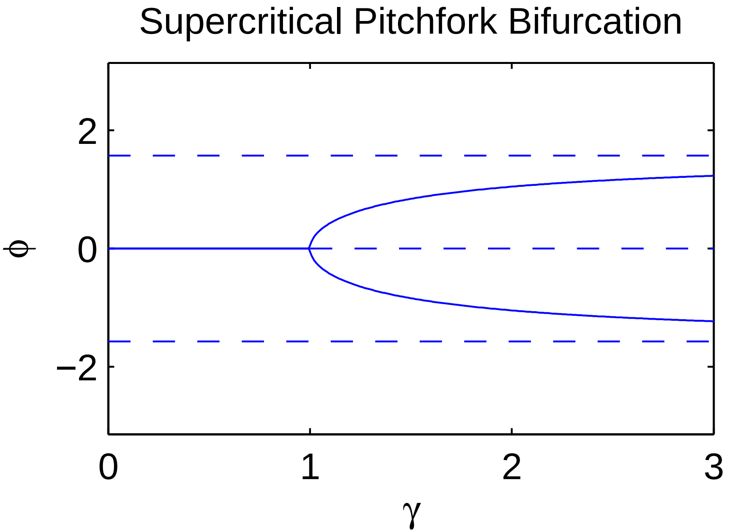
Friction Examples Cont.



Supercritical Pitchfork Bifurcation

- $\gamma \leq 1$, Equilibrium Solution at 0
- $\gamma > 1$, Equilibrium Solution $0 < \phi < \pi/2$
- $\phi = \pm \cos^{-1}\left(\frac{1}{\gamma}\right)$

Supercritical Pitchfork Bifurcation Cont.



References

- Arnold, David. Department of Mathematics. College of the Redwoods. Spring 2008.
- Brizard, Alain J. "Lagrangian Mechanics." Department of Physics. Saint Michael's College. March 15, 2008
- Bundschuh, R. "Therotical Mechanics." Department of Physics. Ohio State University. Spring 2004. March 15, 2008
- Frederic Moisy. "Supercritical bifurcation of a spinning hoop." American Journal of Physics 71.10 (2003): 999-1004. Research Library. ProQuest. College of the Redwoods Library, Eureka, CA. 25 Mar. 2008
  `http://www.proquest.com/`

References Cont.

- Rosales, Rodolfo R. "Bead moving along a thin, rigid, wire." Department of mathematics. Massachusetts Inst. of Technonlogy, Cambridge, Massachusetts, MA. October 17, 2004. March 15, 2008.
- Strogatz, Steven H. *Nonlinear Dynamics and Chaos.* "3.5 Overdamped Bead on a Rotating Hoop" 1994. Perseus Books Publishing.