Application 11.3

Automating the Frobenius Series Method

Here we illustrate the use of a computer algebra system to apply the method of Frobenius. In the paragraphs that follow, we consider the differential equation

$$2x^2 y'' + 3x\, y' - (x^2+1)y = 0 \tag{1}$$

of Example 4 in Section 11.3 of the text, where we found the two indicial roots $r_1 = \frac{1}{2}$ and $r_2 = -1$. We carry through the formal Frobenius method starting with the larger indicial root r_1, and you can then apply the same process to derive the second Frobenius series solution (found manually in the text) corresponding to r_2.

In the following examples, use this method to derive Frobenius series solutions that can be checked against the given known general solutions.

1. $x\, y'' - y' + 4x^3\, y = 0$, $\quad y(x) = A\cos x^2 + B\sin x^2$

2. $x\, y'' - 2\, y' + 9x^5\, y = 0$, $\quad y(x) = A\cos x^3 + B\sin x^3$

3 $4x\, y'' + 2\, y' + y = 0$, $\quad y(x) = A\cos\sqrt{x} + B\sin\sqrt{x}$

4. $x\, y'' + 2\, y' + x\, y = 0$, $\quad y(x) = \dfrac{1}{x}\left(A\cos x + B\sin x\right)$

5. $4x\, y'' + 6\, y' + y = 0$, $\quad y(x) = \dfrac{1}{\sqrt{x}}\left(A\cos\sqrt{x} + B\sin\sqrt{x}\right)$

6. $x\, y'' + x\, y' + (4x^4 - 1)\, y = 0$, $\quad y(x) = \dfrac{1}{x}\left(A\cos x^2 + B\sin x^2\right)$

7. $x\, y'' + 3\, y' + 4x^3\, y = 0$, $\quad y(x) = \dfrac{1}{x^2}\left(A\cos x^2 + B\sin x^2\right)$

8. $x^2\, y'' + x^2\, y' - 2\, y = 0$, $\quad y(x) = \dfrac{1}{x}\left[A\,(2-x) + B\,(2+x)\,e^{-x}\right]$

The next three problems involve the arctangent series

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

9. $(x+x^3)\, y'' + (2+4x^2)\, y' - 2x\, y = 0$, $\quad y(x) = \frac{1}{x}\left(A + B \tan^{-1} x\right)$

10. $(2x+2x^2)\, y'' + (3+5x)\, y' + y = 0$, $\quad y(x) = \frac{1}{\sqrt{x}}\left(A + B \tan^{-1} \sqrt{x}\right)$

11. $(x+x^5)\, y'' + (3+7x^4)\, y' + 8x^3\, y = 0$, $\quad y(x) = \frac{1}{x^2}\left(A + B \tan^{-1} x^2\right)$

Using *Maple*

Beginning with the indicial root

```
r := 1/2:
```

we first write the initial k+1 terms of a proposed Frobenius series solution:

```
k := 6:
a := array(0..k):
y := x^(1/2)*sum( a[n]*x^(n), n = 0..k);
```

$$y := \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6\right)$$

Then we substitute this series (partial sum) into the left-hand side of Equation (1),

```
de := 2*x^2*diff(y,x$2) + 3*x*diff(y,x) - (x^2+1)*y:
eq1 := simplify(de);
```

$$deq1 := -x^{\left(\frac{3}{2}\right)}\left(-5a_1 - 14a_2 x - 27a_3 x^2 - 44a_4 x^3 - 65a_5 x^4 - 90a_6 x^5 + a_0 x + x^6 a_5 + x^7 a_6 + x^2 a_1 + x^3 a_2 + x^4 a_3 + x^5 a_4\right)$$

Noting the $x^{3/2}$ factor, we multiply by $x^{-3/2}$ and then collect coefficients of like powers of x.

```
eq2 := collect( x^(-3/2)*eq1, x);
```

$$deq2 := -x^7 a_6 - x^6 a_5 + (-a_4 + 90a_6)x^5 + (-a_3 + 65a_5)x^4 + (-a_2 + 44a_4)x^3 + (-a_1 + 27a_3)x^2 + (14a_2 - a_0)x + 5a_1$$

Next we set up the equations the successive coefficients must satisfy. We do this by defining an array and then filling the elements of this array by equating (in turn) each of the series coefficients to zero.

```
eqs := array(0..5):
for n from 0 to 5 do
    eqs[n] := coeff(eq2,x,n) = 0:
    od:
coeffEqs := convert(eqs, set);
```

$$coeffEqs := \{5\,a_1 = 0, -a_2 + 44\,a_4 = 0, -a_3 + 65\,a_5 = 0,\\ 90\,a_6 - a_4 = 0, 14\,a_2 - a_0 = 0, -a_1 + 27\,a_3 = 0\}$$

We have here a collection of six linear equations relating the seven coefficients a_0 through a_6. Hence we should be able to solve for the successive coefficients

```
succCoeffs := convert([seq(a[n], n=1..6)], set);
```

$$succCoeffs := \{a_1, a_2, a_3, a_4, a_5, a_6\}$$

in terms of a_0:

```
ourCoeffs := solve(coeffEqs, succCoeffs);
```

$$ourCoeffs := \left\{a_1 = 0, a_3 = 0, a_5 = 0, a_2 = \frac{1}{14}a_0, a_4 = \frac{1}{616}a_0, a_6 = \frac{1}{55440}a_0\right\}$$

Finally we substitute all these coefficients back into the original series.

```
partSoln := subs(ourCoeffs, y);
```

$$partSoln := \sqrt{x}\left(a_0 + \frac{1}{14}a_0\,x^2 + \frac{1}{616}a_0\,x^4 + \frac{1}{55440}a_0\,x^6\right)$$

Note that (after factoring out a_0) this result agrees with the first particular solution

$$y_1(x) = a_0 x^{1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{616} + \frac{x^6}{55440} + \cdots\right)$$

found in the text.

Using *Mathematica*

Beginning with the indicial root

```
r = 1/2;
```

we first write the initial k+1 terms of a proposed Frobenius series solution:

```
k = 6;

y = x^r (Sum[ a[n] x^n, {n,0,k} ] + O[x]^(k+1))
```

$$a(0)\sqrt{x}+a(1)\,x^{3/2}+a(2)\,x^{5/2}+a(3)\,x^{7/2}+a(4)\,x^{9/2}+a(5)\,x^{11/2}+ \\ a(6)\,x^{13/2}+O(x^{15/2})$$

Then we substitute this series into the differential equation in (1),

```
deq =
2x^2 D[y, x,x] + 3x D[y, x] - (x^2 + 1)y == 0
```

$$5a(1)\,x^{3/2}+(14\,a(2)-a(0))\,x^{5/2}+(27\,a(3)-a(1))\,x^{7/2}+ \\ (44\,a(4)-a(2))x^{9/2}+(65\,a(5)-a(3))\,x^{11/2}+ \\ (90\,a(6)-a(4))x^{13/2}+O(x^{15/2})==0$$

Mathematica has automatically collected like powers for us, and we can use the **`LogicalExpand`** command to extract the equations that the successive coefficients satisfy.

```
coeffEqns  =  LogicalExpand[ deq ]
```

$$5a(1)==0\ \wedge\ 14\,a(2)-a(0)==0\wedge\ 27\,a(3)-a(1)==0\ \wedge \\ 44\,a(4)-a(2)==0\ \wedge\ 65\,a(5)-a(3)==0\ \wedge\ 90\,a(6)-a(4)==0$$

We have here a collection of six linear equations relating the seven coefficients $a_0 = a(0)$ through $a_6 = a(6)$. Hence we should be able to solve for the successive coefficients

```
succCoeffs  =  Table[ a[n], {n, 1, 6} ]
```

$$\{a(1), a(2), a(3), a(4), a(5), a(6)\}$$

in terms of $a(0)$:

```
ourCoeffs  =  Solve[coeffEqns, succCoeffs]
```

$$\left\{\left\{a(1)\to 0, a(2)\to \frac{a(0)}{14}, a(3)\to 0, a(4)\to \frac{a(0)}{616}, a(5)\to 0, a(6)\to \frac{a(0)}{55440}\right\}\right\}$$

Finally we substitute all these coefficients back into the original series.

```
partSoln = y  /.  ourCoeffs  //  First
```

$$a(0)\sqrt{x}+\frac{1}{14}a(0)\,x^{5/2}+\frac{1}{616}a(0)\,x^{9/2}+\frac{a(0)\,x^{13/2}}{55440}+O(x^{15/2})$$

Note that (after factoring out $a_0 x^{1/2}$) this result agrees with the first particular solution

$$y_1(x) \;=\; a_0 x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55440}+\cdots\right).$$

found in the text.

Using MATLAB

Beginning with the indicial root $r_1=\frac{1}{2}$, we first write the initial seven terms of a proposed Frobenius series solution:

```
syms x y a0 a1 a2 a3 a4 a5 a6
y =
x^(1/2)*(a0+a1*x+a2*x^2+a3*x^3+a4*x^4+a5*x^5+a6*x^6);
pretty(y)
 1/2                      2        3        4        5        6
x    (a0 + a1 x + a2 x  + a3 x  + a4 x  + a5 x  + a6 x )
```

Then we substitute this series (partial sum) into the left-hand side of Equation (1):

```
de = 2*x^2*diff(y,2) + 3*x*diff(y) - (x^2 + 1)*y;
simple(de)
ans =
-x^(3/2)*(x^6*a5+x^7*a6+x*a0+x^4*a3+x^3*a2+x^2*a1
          +x^5*a4-65*a5*x^4-44*a4*x^3-27*a3*x^2
          -14*a2*x-5*a1-90*a6*x^5)
```

Noting the $x^{3/2}$ factor, we multiply by $x^{-3/2}$ and then collect coefficients of like powers of x.

```
de = collect(simplify(x^(-3/2)*de))
de =
-x^7*a6-x^6*a5+ (90*a6-a4)*x^5 + (-a3+65*a5)*x^4
 + (-a2+44*a4)*x^3 + (-a1+27*a3)*x^2
 + (14*a2-a0)*x + 5*a1
```

Next we set up the equations the original coefficients in our Frobenius series must satisfy. By cutting and pasting it is a simple matter to pick out the successive coefficients in **de** and equate each of them to zero.

```
eq1 = 5*a1;
eq2 = 14*a2-a0;
eq3 = -a1+27*a3;
eq4 = -a2+44*a4;
eq5 = -a3+65*a5;
eq6 = 90*a6-a4;
```

We have here a collection of six linear equations relating the seven coefficients **a0** through **a6**. Hence we should be able to solve for the successive coefficients **a1** through **a6** in terms of **a0**:

```
soln =
solve(eq1,eq2,eq3,eq4,eq5,eq6, a1,a2,a3,a4,a5,a6);
coeffs =
[soln.a1,soln.a2,soln.a3,soln.a4,soln.a5,soln.a6]
coeffs =
[ 0,  1/14*a0,  0,  1/616*a0,  0,  1/55440*a0]
```

Finally we substitute all these coefficients back into the original series **y**:

```
y = subs(y, [a1,a2,a3,a4,a5,a6], coeffs);
pretty(y)
 1/2                  2               4                 6
x    (a0 + 1/14 a0 x  + 1/616 a0 x  + 1/55440 a0 x )
```

Note that (after factoring out a_0) this result agrees with the first particular solution

$$y_1(x) = a_0 x^{1/2}\left(1+\frac{x^2}{14}+\frac{x^4}{616}+\frac{x^6}{55440}+\cdots\right).$$

found in the text.