Spruce Budworm

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Logistic Equation

Logistic Equation

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right)$$

Adding Predation

Logistic Equation with Predation

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - p(N)$$

Predation

Ludwig's Suggested Form for $p(N)$

$$p(N) = \frac{BN^2}{A^2 + N^2}$$

Predation

Graph of $BN^2/(A^2 + N^2)$

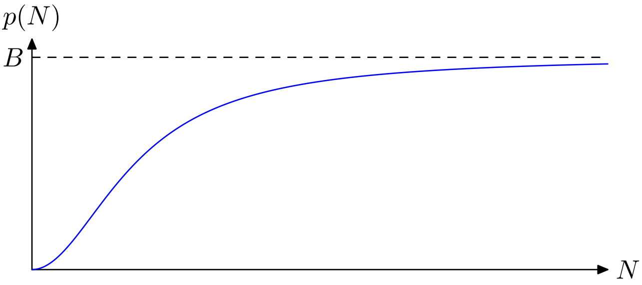

Figure 1: Behavior of predation as budworm population increases

Budworm Population

Budworm Population is Goverened by

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}$$

Saturation

Differentiate $p(N)$ to see where function is increasing

$$p(N) = \frac{BN^2}{A^2 + N^2}$$

$$p'(N) = \frac{(A^2 + N^2)(2BN) - (BN^2)(2N)}{(A^2 + N^2)^2}$$

$$p'(N) = \frac{2A^2BN}{(A^2 + N^2)^2}$$

Saturation

Differentiate $p'(N)$ to check for concavity

$$p'(N) = \frac{2A^2BN}{(A^2 + N^2)^2}$$

$$p''(N) = \frac{(A^2 + N^2)^2(2A^2B) - (2A^2BN)[2(A^2 + N^2)2N]}{(A^2 + N^2)^4}$$

$$p''(N) = \frac{2A^2B(A^2 - 3N^2)}{(A^2 + N^2)^3}$$

Roots of Equation

$$A^2 - 3N^2 = 0$$

$$N = \pm\sqrt{\frac{1}{3}A^2}$$

$$N_c = \sqrt{\frac{A}{3}}$$

Critical value

Threshold value is at N_c.

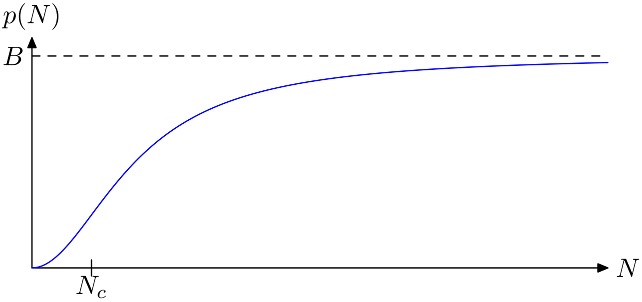

Figure 2: The population value N_c is an approximate threshold value. For $N < N_c$ predation is small, while for $N > N_c$ it is switched on.

Scaling

Convert to nondimensional terms.

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}$$

Scaling

Introduction of nondimensional terms.

$$u = \frac{N}{A}, r = \frac{Ar_B}{B},\ q = \frac{K_B}{A},\ \tau = \frac{Bt}{A}$$

With these substitutions,

$$\frac{dN}{dt} = r_B N \left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}$$

$$\frac{du}{d\tau} = ru \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}.$$

Steady States

Finding equilibrium points

$$0 = u\left[r\left(1-\frac{u}{q}\right)-\frac{u}{1+u^2}\right]$$

Either $u = 0$ or

$$r\left(1-\frac{u}{q}\right)=\frac{u}{1+u^2}.$$

Graph of r and q

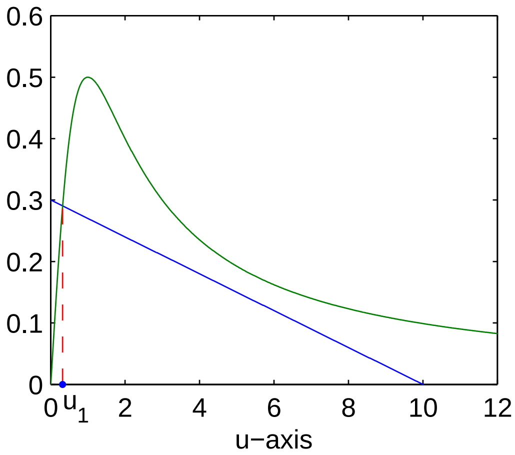

Figure 3: There is an asymptotically stable equilibrium point at u_1.

Graph of r and q

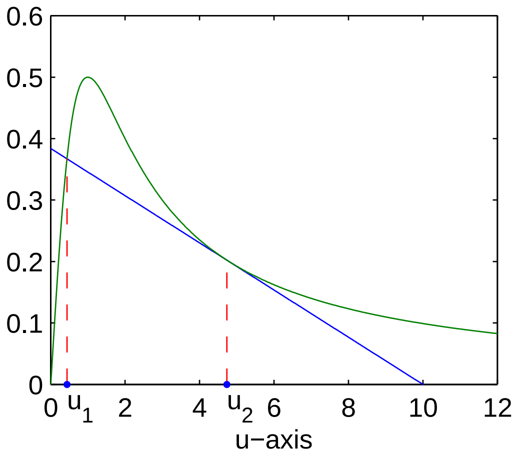

Figure 4: There is a additional semi-stable equilibrium point at u_2.

Graph of r and q

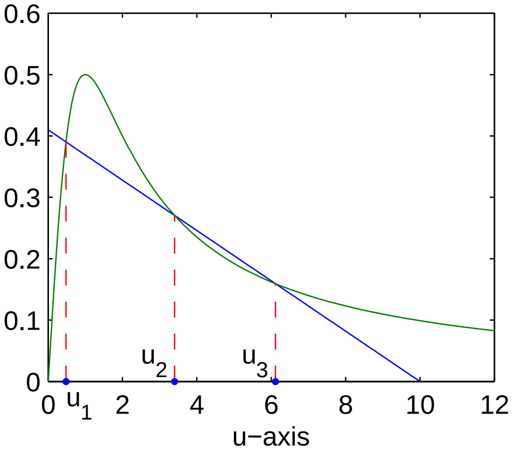

Figure 5: Three equilibrium points.

Graph of r and q

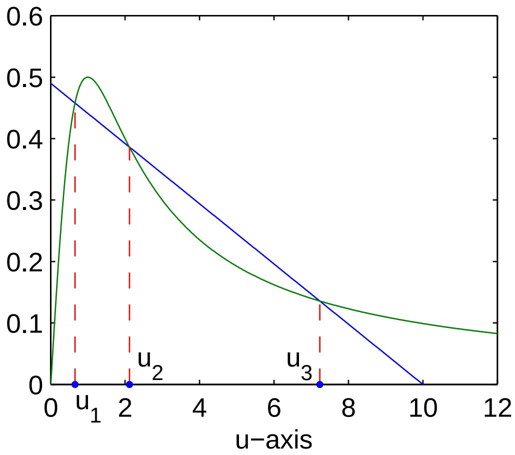

Figure 6: As r increases u_1 and u_2 move closer together.

Graph of r and q

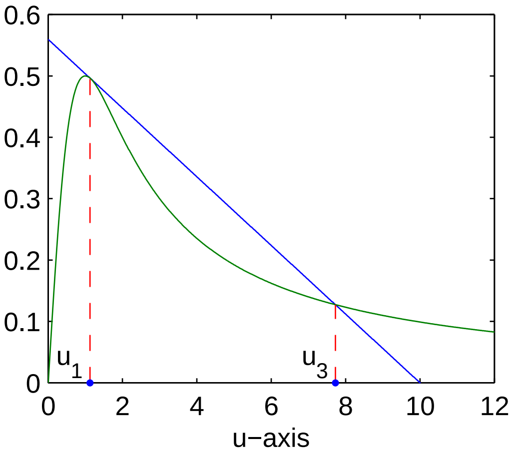

Figure 7: Increasing r u_1 and u_2 coalesce into one semi stable equilibrium point.

Graph of r and q

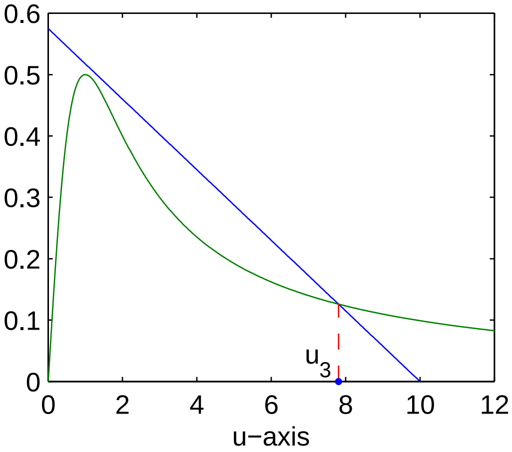

Figure 8: Increasing r we are back to one stable equilibrium point at u_3.

Hysteresis

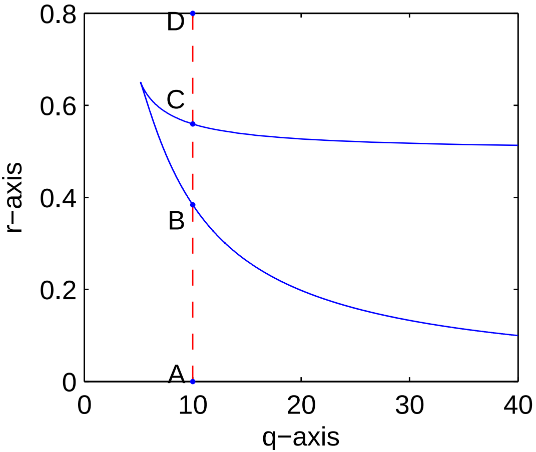

Figure 9: Path of r Along $ABCD$

Hysteresis

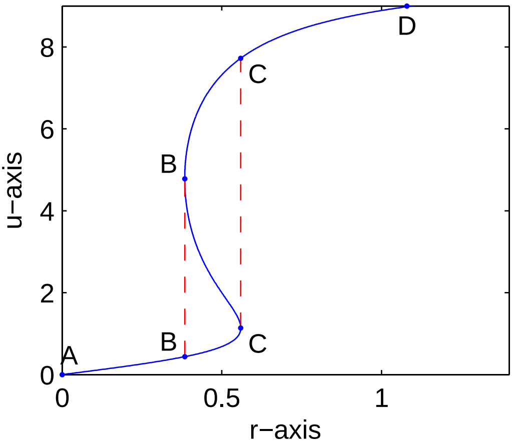

Figure 10: Path of r Along $ABCD$