Chapter 4

Vector Spaces

Application 4.4
Solution Spaces and Null Spaces

Suppose that we're asked to solve the homogeneous system

$$\begin{aligned} 15x_1 + 16x_2 + 10x_3 + 11x_4 + 19x_5 &= 0 \\ 14x_1 + 15x_2 + \ \ 9x_3 + 10x_4 + 17x_5 &= 0 \\ 17x_1 + 18x_2 + 12x_3 + 13x_4 + 23x_5 &= 0 \\ 19x_1 + 20x_2 + 14x_3 + 15x_4 + 27x_5 &= 0 \end{aligned} \tag{1}$$

of 4 equations in 5 unknowns. Obviously, it has at least the trivial solution $(0,0,0,0,0)$. So the real question is *how many* independent solutions the system has. The approach outlined in the text is to reduce the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 15 & 16 & 10 & 11 & 19 \\ 14 & 15 & 9 & 10 & 17 \\ 17 & 18 & 12 & 13 & 23 \\ 19 & 20 & 14 & 15 & 27 \end{bmatrix} \tag{2}$$

to the echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 6 & 5 & 13 \\ 0 & 1 & -5 & -4 & -11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Then the system $\mathbf{Ax} = \mathbf{0}$ in (1) has the same solution vectors as $\mathbf{Ex} = \mathbf{0}$, so the two all-zero rows here indicate that two of the equations in (1) must be redundant. Indeed, because of the two leading entries in (3) it is obvious now that we can select $x_3 = r$, $x_4 = s$, and $x_5 = t$ arbitrarily and solve for x_1 and x_2. This gives

$$x_1 = -6r - 5s - 13t, \qquad x_2 = 5r + 4s + 11t,$$

so the solution vector

$$\mathbf{x} = (-6r-5s-13t, 5r+4s+11t, r, s, t) = r\begin{bmatrix} -6 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -13 \\ 11 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

is a linear combination of the linearly independent solution vectors

$$\mathbf{v}_1 = (-6, -5, 1, 0, 0), \quad \mathbf{v}_2 = (-5, -4, 0, 1, 0), \text{ and } \mathbf{v}_3 = (-13, 11, 0, 0, 1). \tag{5}$$

Thus the solution space W of our original homogeneous system in (1) has basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and dimension 3. This solution space is often called the **null space** of the coefficient matrix $\mathbf{A}$.

Of course, we can use a computer algebra system (as in Project 3.3) to find the echelon form $\mathbf{E}$ in (3). Moreover, we illustrate below *Maple*, *Mathematica*, and MATLAB functions that provide directly a basis for the null space of a given matrix $\mathbf{A}$, and hence for the solution space of the corresponding homogeneous linear system. You can use these commands to solve Problems 15–26 in Section 4.4 of the text. The following problems may be a bit more interesting numerically.

1.
$$\begin{aligned} 13x_1 + 8x_2 + 7x_3 + 8x_4 &= 0 \\ 8x_1 + 5x_2 + 6x_3 + 7x_4 &= 0 \\ 23x_1 + 14x_2 + 9x_3 + 10x_4 &= 0 \\ 28x_1 + 17x_2 + 10x_3 + 11x_4 &= 0 \end{aligned}$$

2.
$$\begin{aligned} 5x_1 + 10x_2 + 9x_3 + 8x_4 &= 0 \\ 7x_1 + 14x_2 + 10x_3 + 9x_4 &= 0 \\ x_1 + 2x_2 + 7x_3 + 6x_4 &= 0 \\ -x_1 - 2x_2 + 6x_3 + 5x_4 &= 0 \end{aligned}$$

3.
$$\begin{aligned} 11x_1 + 12x_2 + 9x_3 + 8x_4 + 13x_5 &= 0 \\ 10x_1 + 11x_2 + 10x_3 + 9x_4 + 7x_5 &= 0 \\ 13x_1 + 14x_2 + 7x_3 + 6x_4 + 25x_5 &= 0 \\ 14x_1 + 15x_2 + 6x_3 + 5x_4 + 31x_5 &= 0 \end{aligned}$$

4. $$\begin{aligned} 15x_1 + 9x_2 + 14x_3 + 8x_4 + 23x_5 &= 0 \\ 16x_1 + 10x_2 + 15x_3 + 9x_4 + 27x_5 &= 0 \\ 13x_1 + 7x_2 + 12x_3 + 6x_4 + 15x_5 &= 0 \\ 12x_1 + 6x_2 + 11x_3 + 5x_4 + 11x_5 &= 0 \end{aligned}$$

5. $$\begin{aligned} 15x_1 + 9x_2 + 14x_3 + 8x_4 + 23x_5 &= 0 \\ 16x_1 + 10x_2 + 15x_3 + 9x_4 + 27x_5 &= 0 \\ 23x_1 + 31x_2 + 13x_3 + 29x_4 + 19x_5 &= 0 \\ 36x_1 + 38x_2 + 25x_3 + 35x_4 + 34x_5 &= 0 \\ 34x_1 + 56x_2 + 15x_3 + 53x_4 + 27x_5 &= 0 \end{aligned}$$

Using *Maple*

First we enter the 4×5 coefficient matrix in (2):

```
with(linalg):
A := array( [[15, 16, 10, 11, 19],
             [14, 15,  9, 10, 17],
             [17, 18, 12, 13, 23],
             [19, 20, 14, 15, 27]] ):
```

Then the reduced echelon form of **A** is given by

```
E := rref(A);
```

$$E := \begin{bmatrix} 1 & 0 & 6 & 5 & 13 \\ 0 & 1 & -5 & -4 & -11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and a basis for the null space W of **A** is given by

```
W := nullspace(A);
```

$$W := \{[1, 0, 4, -5, 0], [0, 0, -3, 1, 1], [0, 1, 5, -6, 0]\}$$

Thus a basis for W consists of the three vectors

```
u1 := W[1];  u2 := W[2];  u3 := W[3];
```

$$u1 := [1, 0, 4, -5, 0]$$
$$u2 := [0, 0, -3, 1, 1]$$
$$u3 := [0, 1, 5, -6, 0]$$

Hence the solution space of our linear system is 3-dimensional. However, the basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ here differ from the three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ displayed in (5) that come directly from the echelon matrix $\mathbf{E}$. How would you verify that these are bases for the same 3-dimensional subspace of $\mathbf{R}^5$? Can you see that it would suffice to find a 3×3 matrix $\mathbf{C}$ such that

$$\mathbf{C}\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix}?$$

How would you find such a matrix $\mathbf{C}$? (*Suggestion*: Consider just the first 3 columns of the two 3×5 matrices here.)

Using *Mathematica*

First we enter the 4×5 coefficient matrix in (2):

```
A = {{322, -163, 231, -455, 889},
     {107, -181, 428, -571, 445},
     {351, -144, 421, -936, 848},
     {111, -709, 484,  625, 421}};
```

Then the reduced echelon form of $\mathbf{A}$ is given by

```
R = RowReduce[A]
```

$$\begin{pmatrix} 1 & 0 & 6 & 5 & 13 \\ 0 & 1 & -5 & -4 & -11 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and a basis for the null space W of $\mathbf{A}$ is given by

```
W = NullSpace[A]
```

$$\begin{pmatrix} -13 & 11 & 0 & 0 & 1 \\ -5 & 4 & 0 & 1 & 0 \\ -6 & 5 & 1 & 0 & 0 \end{pmatrix}$$

The three row vectors here are (respectively) the basis vectors $\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1$ displayed in (5), which we extracted directly from the echelon matrix $\mathbf{R}$. Thus we see that the solution space of our linear system is 3-dimensional.

Using MATLAB

First we enter the 4×5 coefficient matrix in (2):

```
A = [15  16  10  11  19
     14  15   9  10  17
     17  18  12  13  23
     19  20  14  15  27];
```

Then the reduced echelon form of $\mathbf{A}$ is given by

```
E = rref(A)
E =
     1     0     6     5    13
     0     1    -5    -4   -11
     0     0     0     0     0
     0     0     0     0     0
```

A basis for the null space W of $\mathbf{A}$ is given by

```
null(A)
ans =
    0.6909   -0.3170    0.1367
   -0.5263    0.2222   -0.3191
    0.3310    0.8540    0.0721
   -0.3629   -0.0854    0.8572
   -0.0663   -0.3369   -0.3735
```

This result doesn't look especially palatable at first glance; the three column vectors here constitute an "orthonormal basis" for W. That is, they are mutually orthogonal unit vectors in $\mathbf{R}^5$ that span W. However, the command

```
null(A,'r')
ans =
    -6    -5   -13
     5     4    11
     1     0     0
     0     1     0
     0     0     1
```

produces the "rational basis" for W that is derived from the reduced echelon form. Indeed, we see here the three column vectors displayed in (5).