A Presentation of the Two-Body Problem

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Abstract

This paper will examine the motion of two particles in space according to their mutual gravitational attraction.

1. Introduction

The mechanics of bodies in space has fascinated man for many centuries. The vast universe has intrigued people worldwide to spend infinite numbers of hours to mathematically understand the motion of the cosmos. The specific time of "1525 to 1725" (Kaufmann 46) A.D. was a monumental time frame for mathematical and observational development in understanding the mysteries of space. More specifically, two people in particular changed how people look at motion of planetary bodies. Those two people are Johannes Kepler and Isaac Newton.

Kepler worked under Tycho Brahe for 22 months while making meticulous measurements of planetary orbits. This data aided in Brahe's theory of the Earth being "stationary, with the Sun and Moon revolving around it, while all the other planets revolve around the Sun" (Kaufmann 48). Though the "'Tychonic system"' (Kaufmann 48) failed widespread acceptance, Kepler used Brahe's data of planetary positions after he died to formulate three laws of planetary motion. As told by Kaufmann, "Kepler's first law [states that] every planet travels around the Sun along an elliptical orbit with the Sun at one focus. According to his second law, the line joining the planet and the Sun sweeps out equal areas in equal intervals of time" (Kaufmann 43). Kepler's third law states that, "the square of a planet's sidereal period is proportional to the cube of the length of its orbit's semimajor axis" (Kaufmann 43).

What is so unique about Kepler's laws is that not only are they correct, but also the laws are based on pure observational data.

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Isaac Newton took a different approach with describing the motion of bodies both on Earth and in space. Newton formulated three laws of motion along with a law of physics that relates to the conservation of angular momentum. These are pure mathematical formulations that when juxtaposed with Kepler's observational laws, create a bond that is astronomical.

Newton's laws of motion coincided beautifully with Kepler's three laws of planetary motion. According to Kaufmann, "using his own three laws and Kepler's three laws, Newton succeeded in formulating a general statement describing the nature of the force called gravity that keeps planets in their orbits" (50). This coalescence of these mathematical and observational laws may arguably be the most important scientific discovery of all time.

In 1687 Newton published Philosophae naturalis principia mathematica that presented mathematics of motion, "forces, and gravitation" (Kaufmann 49). From this text the original Two-Body Problem can be read.

The purpose of the Two-Body Problem is to describe and predict the motion of two bodies in space according to their mutual gravitational attraction. Though the specific instance of this existing in real time is very minimal, one cannot restrict the motion of one body purely to one other body without taking the entire system of the universe into consideration, the calculations presented by Newton have been of great support to present day astronomers and scientists. Furthermore, Newton's Two-Body Problem can be used to develop Kepler's three laws of planetary motion.

This paper will present Newton's Two-Body Problem with the mathematics not exceeding first level differential equations. Though the level of mathematical derivation of the problem can be related with very high-end levels of knowledge, this timeless problem is only greater than before due to its many interpretable qualities.

From the derivation of the Two-Body Problem into a simplified One-Body Problem, due to the freedom of choosing coordinate systems, the first and second of Kepler's laws of planetary motion will be presented.

2. Visualizing the Model

The first step with developing the solution to the Two-Body Problem is to present an illustration of the geometry.

Figure 1 shows the basic concept of the Two-Body Problem. The picture begins with an isolated system containing two masses m_1 and m_2 where vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ extend from an arbitrary origin 0 to each mass. The vector $\mathbf{r}$ extends from vector m_2 to m_1. The dimension is three to mimic a realistic scenario.

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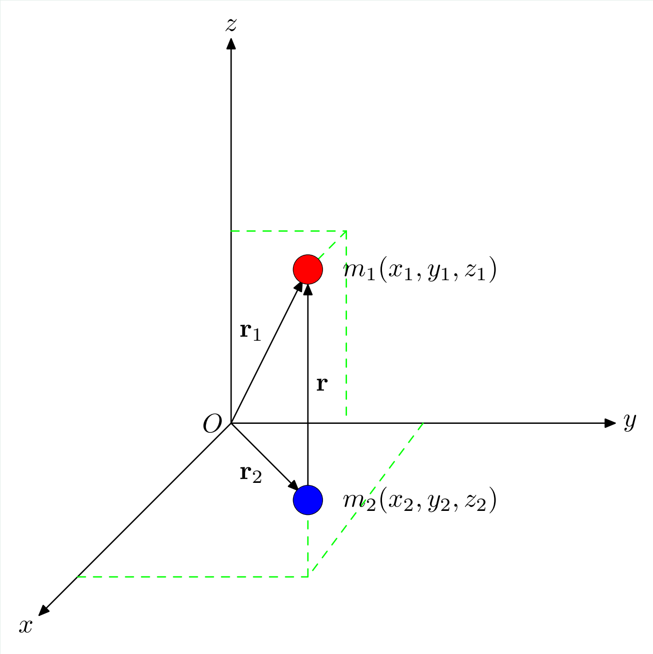

Figure 1: A view of the two bodies in three space.

At this point Newton's laws of motion will be most helpful with formulating the orbit of the two masses as seen from the origin 0.

3. Reducing the Model to Two Space

The dimension of the system can actually be reduced to two space with aid of the cross product. Working with the Two-Body Problem in two space will simplify calculations immensely. The first step with developing a two dimensional model for the Two-Body Problem is to use Newton's Second Law of Motion

$$\mathbf{F} = m\mathbf{a}.$$

We can further define Newton's Second Law to be

$$\mathbf{F} = F(r)\hat{\mathbf{r}},$$

where $F(r)$ is the force function of vector $\mathbf{r}$. Remember, the motion of the two bodies will solely depend on gravitational attraction. Thus, the only force that is acting on either body is gravity. The unit vector $\hat{\mathbf{r}}$ extends in the direction of $\mathbf{r}$. With the substitution of $m_1\mathbf{a}_1$ and $m_2\mathbf{a}_2$ for $\mathbf{F}$ we have the two equations below. The equations are

$$m_1\mathbf{a}_1 = F(r)\hat{\mathbf{r}}$$

and

$$m_2\mathbf{a}_2 = -F(r)\hat{\mathbf{r}}.$$

But, we know that the forces are equal in magnitude but point in opposite directions so we can focus on one force exerted from one mass and know the mathematics will work for the other mass in terms of confining the motion of the mass to two dimensions. We will proceed with m_1.

We know that m_1 and $F(r)$ are scalars from Figure 1 which means that $\mathbf{a}$ and $\hat{\mathbf{r}}$ must be parallel, where $\mathbf{a}$ is acceleration. This knowledge leads to

$$\mathbf{r} \times \mathbf{a} = 0.$$

It is possible with the given information that for vectors exerted from m_1

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}},$$

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where $\mathbf{v}$ is the vector that extends in the direction of motion from m_1. The equation is equal to

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a}$$

which means that

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0} + \mathbf{0}.$$

So we can argue that if

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0}$$

then

$$\mathbf{r} \times \mathbf{v} = \mathbf{h},$$

where $\mathbf{h}$ is any constant vector. Therefore, $\mathbf{r}$ and $\mathbf{v}$ are perpendicular to $\mathbf{h}$ for all time t by the right hand rule of cross product. This means that motion of m_1 is confined to a plane. As defined earlier with the forces of m_1 and m_2 being equal in magnitude but oriented in opposite directions, we can also conclude that motion of m_2 is also confined to the same plane.

Figure 2 illustrates our two dimensional model. The process of deriving a graphable solution of the Two-Body Problem in two space is in the next section.

4. Quantifying the Model

The mathematics begins with Newton's Law of Gravitation

$$F = \frac{Gm_1m_2}{r^2}.$$

Force F will become vector $\mathbf{F}$ with the addition of $\hat{\mathbf{r}}$

$$\mathbf{F} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}.$$

Knowing that

$$r = ||\mathbf{r}_2 - \mathbf{r}_1||$$

can also be represented as

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

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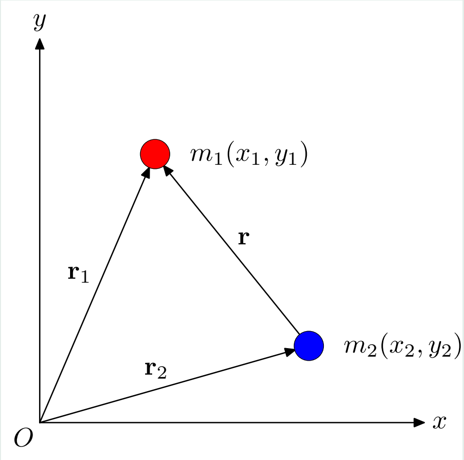

Figure 2: A view of the two bodies in two space.

we now obtain

$$\hat{\mathbf{r}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{r}.$$

Newton's Law of Gravitation can be rewritten as

$$\mathbf{F} = \frac{Gm_1m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1).$$

Combining Newton's second law of motion with his amended law of gravitation results with

$$m\mathbf{a} = \frac{Gm_1m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1).$$

It is possible to express the force exerted by one mass directed toward the other. This is done by breaking down $\mathbf{a}$ into components. The equations are

$$\mathbf{F}_1 = m_1\ddot{\mathbf{r}}_1 = \frac{Gm_1m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

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and

$$\mathbf{F}_2 = m_2\ddot{\mathbf{r}}_2 = -\frac{Gm_1m_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1).$$

Simplification of these two equations will result in four second order differential equations that can be graphed. The minus with $\mathbf{F}_2$ is necessary because the forces exerted by each mass are equal and point in opposite directions. Due to the geometry of Figure 2 $\mathbf{F}_2$ will be written as

$$\mathbf{F}_2 = m_2\ddot{\mathbf{r}}_2 = \frac{Gm_1m_2}{r^3}(\mathbf{r}_1 - \mathbf{r}_2).$$

Simplification of the two equations leaves

$$\ddot{\mathbf{r}}_1 = \frac{Gm_2}{r^3}(\mathbf{r}_2 - \mathbf{r}_1)$$

and

$$\ddot{\mathbf{r}}_2 = \frac{Gm_1}{r^3}(\mathbf{r}_1 - \mathbf{r}_2).$$

To represent r in terms of the x and y coordinates we will insert

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

otherwise known as the distance formula, to result with

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{y}_1\end{bmatrix} = \frac{Gm_2}{(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2})^3}\left(\begin{bmatrix}x_2\\ y_2\end{bmatrix} - \begin{bmatrix}x_1\\ y_1\end{bmatrix}\right)$$

and

$$\begin{bmatrix}\ddot{x}_2\\ \ddot{y}_2\end{bmatrix} = \frac{Gm_1}{(\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2})^3}\left(\begin{bmatrix}x_1\\ y_1\end{bmatrix} - \begin{bmatrix}x_2\\ y_2\end{bmatrix}\right).$$

Here is the solution to the Two-Body Problem in two space. The four second order differential equations describe the motion of m_1 and m_2 referenced from origin 0. With initial conditions of mass, position, and velocity, such pictures like Figure 3 result. The corresponding Matlab code is below Figure 3 for reference.

```
m1=1; m2=1; G=1; x1=0; y1=0; v1x=0.01; v1y=0.01; x2=0; y2=10;
v2x=-0.1; v2y=0.1; tspan=[0,500];
```


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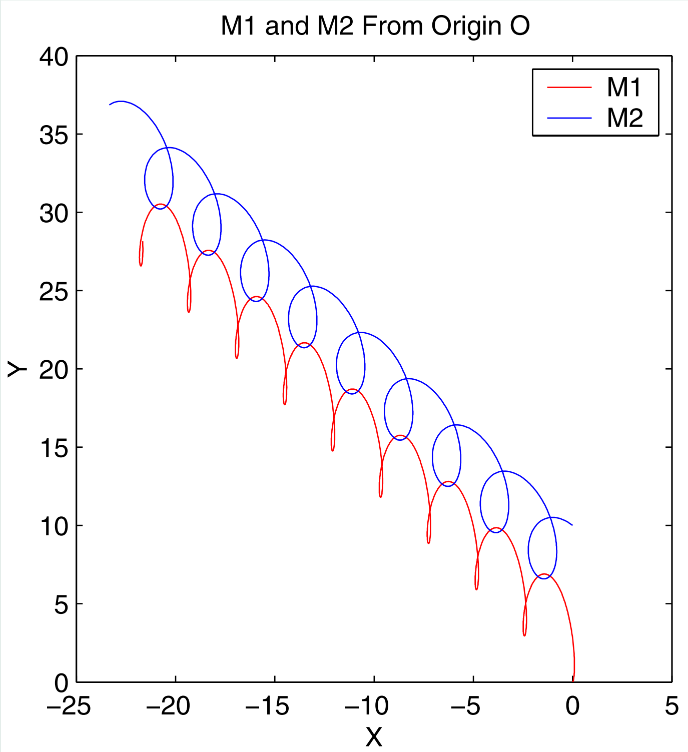

Figure 3: Phase plane portrait of the two bodies as seen from 0.

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```
options=odeset('RelTol',1e-6,...
    'AbsTol',1e-6);

[t,u]=ode45('twobody',tspan,[x1;v1x;y1;v1y;x2;v2x;y2;v2y],options,m1,m2,G);

figure plot(u(:,1),u(:,3),'r') line(u(:,5),u(:,7)) title('M1 and
M2 From Origin O') xlabel('X') ylabel('Y')
legend('M1','M2')

set(gcf,'PaperPosition',[0,0,4,4]) print -depsc2 twobodyA.eps
```

It can be observed that over time the orbits of the two masses as seen in Figure 3 sweep a linear path. We wish to quantify our prediction that motion of two bodies as seen from a third stationary source travel in a line. To mathematically support the prediction we begin with Figure 4. This figure is the same picture as Figure 3 with the exception that the center of mass between the two masses is shown along with vector **R** that extends from origin 0 to the center of mass (CM).

We begin to quantify the path of the two bodies over time with Newton's Second Law of motion

$$\mathbf{F} = m\mathbf{a}.$$

We can represent the equation in terms related to Figure 4 as

$$\Sigma\mathbf{F} = \varphi\ddot{\mathbf{R}}$$

where φ equals the total mass of the system. Now, because the system is closed, we can express that

$$\Sigma\mathbf{F} = \mathbf{0}.$$

But, because

$$\mathbf{0} = \varphi\ddot{\mathbf{R}},$$

we can write

$$\ddot{\mathbf{R}} = 0.$$

Upon integrating the most recent equation we obtain

$$\dot{\mathbf{R}} = \mathbf{A},$$

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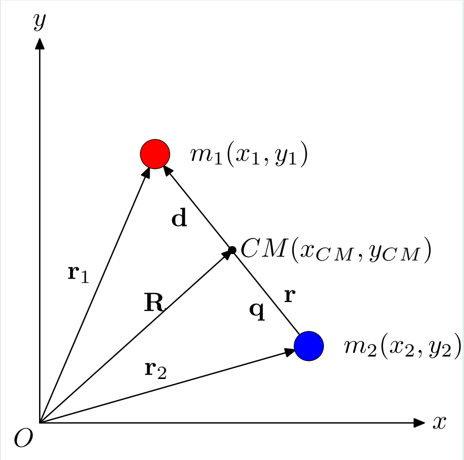

Figure 4: Two dimensional model of the Two-Body Problem with the addition of the center of mass and its corresponding vector $\mathbf{R}$.

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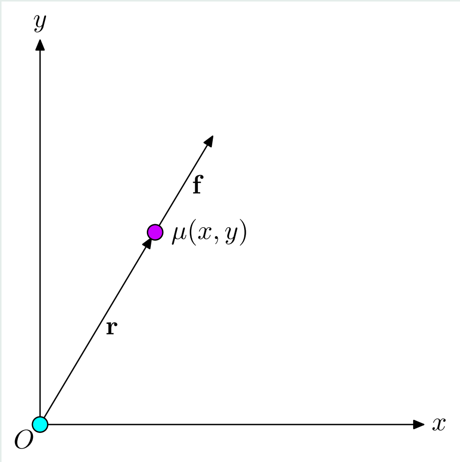

Figure 5: A view of the one-body problem.

where A is any constant. Integrating again obtains

$$\mathbf{R} = \mathbf{A}t + \mathbf{B}.$$

This means that the motion of the two masses as seen from the origin 0 moves in a straight line or does not move over time. The masses orbit in relation to each other, as seen from origin 0, but the result of the two masses moving along in relation to a third stationary source is uninteresting. It would be simpler and more informative to know how one mass moves in relation to another. With the mathematics derived to this point, the Two-Body Problem can proceed to evolve into a One-Body Problem where the motion of one body can be mathematically derived from the other.

5. Solving the One-Body Problem

A visualization of the One-Body Problem is shown in Figure 5. Here μ represents the total reduced mass of the two mass system. One arbitrary body is placed at the origin while the other is at the tip of vector $\mathbf{r}$. Vector $\mathbf{f}$ is the force vector extending from the body at the origin to reduced mass μ.

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The concept behind finding the motion of one body in terms of another is that we are defining the motion as depicted by vector $\mathbf{R}$, as seen in Figure 5. Therefore, vector $\mathbf{R}$ will control the motion of μ.

The equation of μ is defined as

$$\mathbf{R}(m_1 + m_2) = \mathbf{r}_1 m_1 + \mathbf{r}_2 m_2.$$

From our trusty

$$\mathbf{F} = m\mathbf{a}$$

we also define

$$m_1 \ddot{\mathbf{r}}_1 = F(r)\hat{\mathbf{r}}$$

and

$$m_2 \ddot{\mathbf{r}}_2 = -F(r)\hat{\mathbf{r}}.$$

Upon solving for $\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$ we obtain

$$\ddot{\mathbf{r}}_1 = \frac{F(r)\hat{\mathbf{r}}}{m_1}$$

and

$$\ddot{\mathbf{r}}_2 = \frac{F(r)\hat{\mathbf{r}}}{m_2}.$$

Subtracting $\ddot{\mathbf{r}}_2$ from $\ddot{\mathbf{r}}_1$ lends

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{F(r)\hat{\mathbf{r}}}{m_1} - \left(-\frac{F(r)\hat{\mathbf{r}}}{m_2}\right)$$
$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = F(r)\hat{\mathbf{r}}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

But,

$$\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{m_2}{m_1 m_2} + \frac{m_1}{m_1 m_2}$$

and

$$\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{m_2 + m_1}{m_1 m_2}$$

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so the result of $\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2$ is

$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = F(r)\hat{\mathbf{r}}\frac{m_2 + m_1}{m_1 m_2}.$$

Solving for $F(r)\hat{\mathbf{r}}$ we get

$$F(r)\hat{\mathbf{r}} = \left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2).$$

Defining μ to be $\mu = (m_1 m_2)/(m_1 + m_2)$ we obtain

$$\begin{aligned}\mu(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) &= F(r)\hat{\mathbf{r}}\\ \mu(\ddot{\mathbf{r}}) &= F(r)\hat{\mathbf{r}}\end{aligned}$$

where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

The equation for the Center of Mass is

$$m_1(\mathbf{r}_1 - \mathbf{R}) + m_2(\mathbf{r}_2 - \mathbf{R}) = 0.$$

To solve for $\mathbf{R}$ we take the following steps,

$$\begin{aligned}m_1\mathbf{r}_1 - m_1\mathbf{R} + m_2\mathbf{r}_2 - m_2\mathbf{R} &= 0\\ m_1\mathbf{r}_1 + m_2\mathbf{r}_2 &= m_1\mathbf{R} + m_2\mathbf{R}\\ m_1\mathbf{r}_1 + m_2\mathbf{r}_2 &= \mathbf{R}(m_1 + m_2)\\ \mathbf{R} &= \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}\end{aligned}$$

By vector addition we obtain

$$\mathbf{R} + \mathbf{d} = \mathbf{r}_1$$

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and

$$
\begin{aligned}
\mathbf{d} &= \mathbf{r}_1 - \mathbf{R} \\
\mathbf{d} &= \mathbf{r}_1 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \\
\mathbf{d} &= \frac{(m_1\mathbf{r}_1 + m_2\mathbf{r}_1) - (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)}{m_1 + m_2} \\
\mathbf{d} &= \frac{m_2\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2} \\
\mathbf{d} &= \frac{m_2(\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2} \\
\mathbf{d} &= \frac{m_2}{m_1 + m_2}\mathbf{r}.
\end{aligned}
$$

For a reminder of what d and q are see Figure 4. But,

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{d}$$

so

$$\mathbf{r}_1 = \mathbf{R} + \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r}.$$

We want to find where

$$\mathbf{r}_2 = \mathbf{R} + \mathbf{q}.$$

So $\mathbf{q}$ is

$$\mathbf{q} = \mathbf{r}_2 - \mathbf{R}.$$

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Expanding and simplifying $\mathbf{q}$ reveals

$$\mathbf{q} = \mathbf{r}_2 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$
$$\mathbf{q} = \frac{\mathbf{r}_2(m_1 + m_2) - m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$
$$\mathbf{q} = \frac{\mathbf{r}_2 m_1 + \mathbf{r}_2 m_2 - m_1\mathbf{r}_1 - m_2\mathbf{r}_2}{m_1 + m_2}$$
$$\mathbf{q} = \frac{\mathbf{r}_2 m_1 - m_1\mathbf{r}_1}{m_1 + m_2}$$
$$\mathbf{q} = \frac{m_1(\mathbf{r}_2 - \mathbf{r}_1)}{m_1 + m_2}$$
$$\mathbf{q} = \frac{m_1(-\mathbf{r})}{m_1 + m_2}$$
$$\mathbf{q} = -\frac{m_1}{m_1 + m_2}\mathbf{r}.$$

Vector $\mathbf{r}_2$ is simplified by

$$\mathbf{r}_2 = \mathbf{R} + \mathbf{q}$$
$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

The Center of Mass moves with constant velocity so

$$\ddot{\mathbf{R}} = \mathbf{0}.$$

Integrating $\mathbf{R}$ twice in respect to t results with

$$\dot{\mathbf{R}} = v_o$$
$$\mathbf{R} = v_o t + R_0$$

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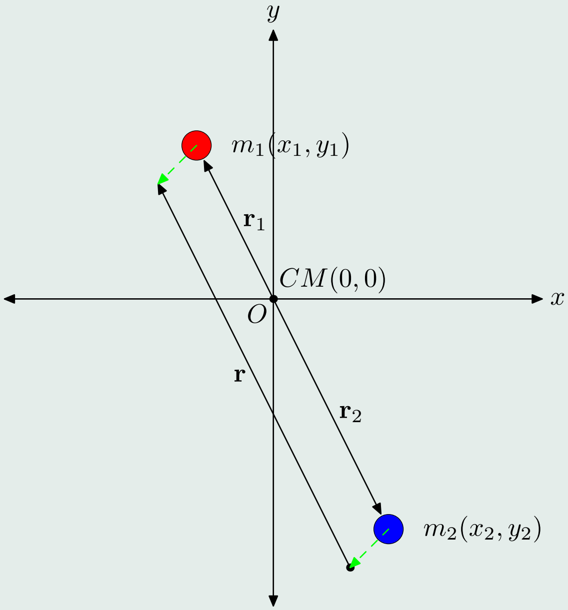

Figure 6: Shifting the center of mass to the origin for simpler calculations. Notice that $\mathbf{r}$ and the vector that contains the center of mass are superimposed upon each other.

We choose initial conditions so that $t = 0$, $v_o = 0$, and $\mathbf{R}_o = 0$. We get $\mathbf{R} \equiv 0$. Note that $\mathbf{R}$ is linear. The origin is at the center of mass. Figure 6 is a visual representation.

We have

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}$$
$$\mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}$$

From definition, μ is equal to

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Now we have a new system with the center of mass at rest located at the origin of the coordinate system. So

$$\mu\ddot{\mathbf{r}} = F(r)\hat{\mathbf{r}}$$

and

$$\ddot{\mathbf{r}} = \frac{F(r)\hat{\mathbf{r}}}{\mu}.$$

Then substituting Newton's law of gravitation in for $F(r)$ leads to

$$\ddot{\mathbf{r}} = \frac{Gm_1m_2}{\mu r^2}\hat{\mathbf{r}}$$
$$\ddot{\mathbf{r}} = \frac{Gm_1m_2}{(m_1m_2)/(m_1+m_2)r^2}\hat{\mathbf{r}}$$
$$\ddot{\mathbf{r}} = \frac{Gm_1m_2(m_1+m_2)}{r^2(m_1m_2)}\hat{\mathbf{r}}$$
$$\ddot{\mathbf{r}} = \frac{G(m_1+m_2)}{r^2}\hat{\mathbf{r}}$$

Reminding the reader that

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{||\mathbf{r}||} = \frac{\mathbf{r}}{r}.$$

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So we proceed as follows

$$\ddot{\mathbf{r}} = \frac{G(m_1+m_2)}{r^2}\frac{\mathbf{r}}{r}$$
$$\ddot{\mathbf{r}} = \frac{G(m_1+m_2)}{r^3}\mathbf{r}.$$

The solution to the One-Body Problem in vector notation is as follows

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \frac{G(m_1+m_2)}{(\sqrt{x^2+y^2})^3}\begin{bmatrix}x\\ y\end{bmatrix}.$$

Figure 7 and Figure 8 are two examples of the three possibilities of orbits of one body in respect to another. The three possibilities of orbits are: ellipse, hyperbola, and parabola. The elliptical orbit was observationally discovered by Kepler. The parabolic and hyperbolic orbits were derived mathematically by Newton. A figure of an elliptical and hyperbolic orbit along with the Matlab code for creating an elliptical orbit are below.

```
m1=81; m2=1; G=.001; x=20; y=0; vx=0; vy=-.05;
tspan=[0,2000];
options=odeset('RelTol',1e-6,...'OutputFcn','odephas2',...'OutputSel',[1,3]);
[t,u]=ode45('twobody2',tspan,[x;vx;y;vy],options,m1,m2,G);
plot(u(:,1),u(:,3)) axis equal grid on
```

There is enough information at this time to solve for Kepler's laws of planetary motion. Kepler's first and second laws will be solved here.

6. Solving for Kepler's First Law

We begin this section with a statement of Kepler's First Law.

Kepler's First Law *The orbit of a planet about the Sun is an ellipse with the Sun at one focus.*

Figure 9 is a visualization of the velocity vector functions that are exerted onto μ in relation to the angle θ and radius r in relation to origin 0. This figure will aid in deriving Kepler's first law. We begin by defining $\mathbf{h}$ as

$$\mathbf{h} = \mathbf{r}\times\mathbf{v}$$

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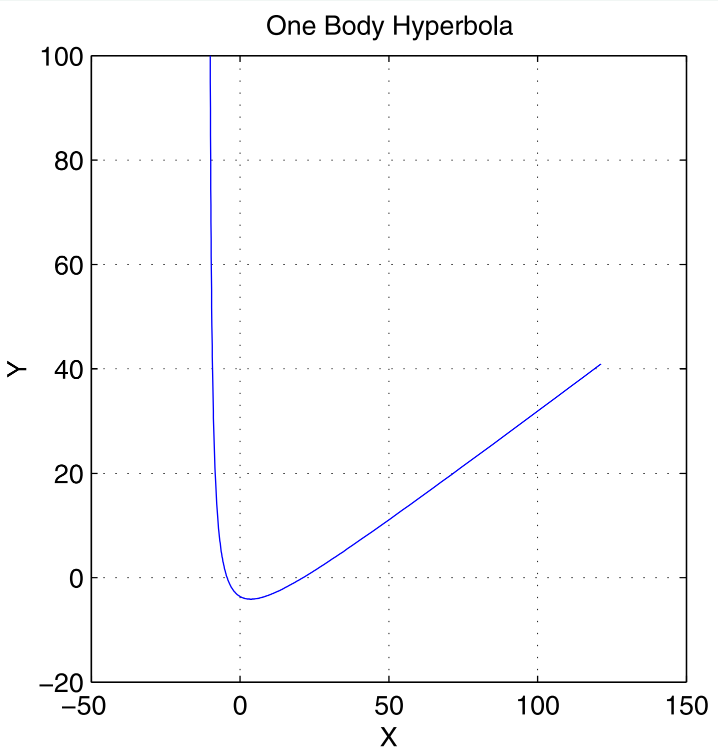

Figure 7: Hyperbolic orbit of one mass seen from another.

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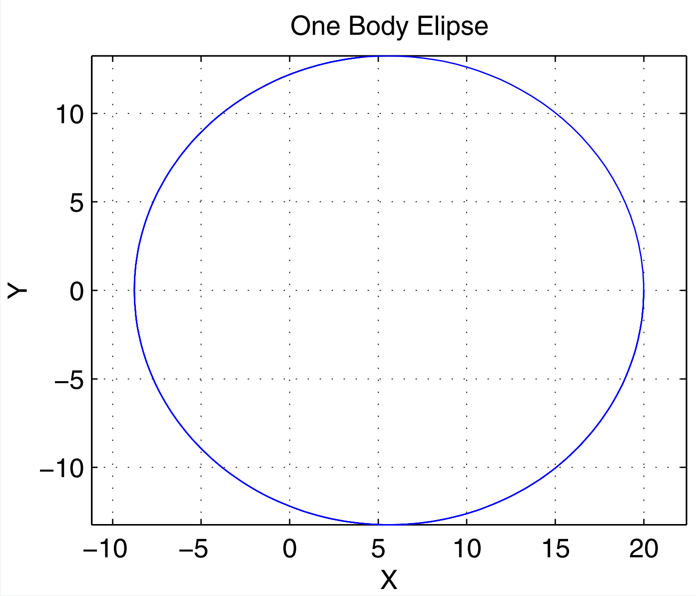

Figure 8: Elliptical orbit of one mass seen from another.

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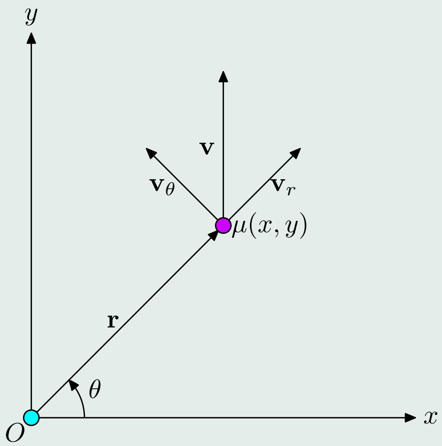

Figure 9: Forces that exert from the mass that is in motion.

where $\mathbf{h}$ is a constant vector that is the result of the cross product of the other two vectors. Simplifying the equation produces

$$\begin{aligned}
\mathbf{h} &= \mathbf{r} \times \dot{\mathbf{r}} \\
\mathbf{h} &= r\hat{\mathbf{r}} \times (\dot{r\hat{r}}) \\
\mathbf{h} &= r\hat{\mathbf{r}} \times (r\dot{\hat{\mathbf{r}}} + \dot{\mathbf{r}}\hat{\mathbf{r}}) \\
\mathbf{h} &= (r\hat{\mathbf{r}} \times r\dot{\hat{\mathbf{r}}}) + (r\hat{\mathbf{r}} \times \dot{\mathbf{r}}\hat{\mathbf{r}}) \\
\mathbf{h} &= r^2(\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}}) + r\dot{\mathbf{r}}(\hat{\mathbf{r}} \times \hat{\mathbf{r}}) \\
\mathbf{h} &= r^2(\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}})
\end{aligned}$$

So,

$$\mathbf{a} \times \mathbf{h} = \mathbf{a} \times (r^2(\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}})).$$

Knowing that $\mathbf{a}$ is equal to

$$\mathbf{a} = -\frac{G(m_1 + m_2)}{r^3}\mathbf{r}$$

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$\mathbf{a} \times \mathbf{h}$ becomes

$$
\begin{aligned}
\mathbf{a} \times \mathbf{h} &= -\frac{G(m_1+m_2)}{r^3}\mathbf{r} \times (r^2(\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}})) \\
\mathbf{a} \times \mathbf{h} &= -\left(\frac{G(m_1+m_2)}{r^2}\frac{\mathbf{r}}{r}\right) \times (r^2(\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}})) \\
\mathbf{a} \times \mathbf{h} &= -(G(m_1+m_2)\hat{\mathbf{r}}) \times (\hat{\mathbf{r}} \times \dot{\hat{\mathbf{r}}}) \\
\mathbf{a} \times \mathbf{h} &= -G(m_1+m_2)\left[(\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}})\hat{\mathbf{r}} - (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\dot{\hat{\mathbf{r}}}\right]
\end{aligned}
$$

But we know that

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = |\hat{\mathbf{r}}|^2 = 1$$

so we can proceed as follows

$$
\begin{aligned}
\frac{d}{dt}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}) &= 0 \\
\dot{\hat{\mathbf{r}}} \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}} &= 0 \\
(\dot{\hat{\mathbf{r}}} \cdot \hat{\mathbf{r}}) &= 0 \\
\dot{\hat{\mathbf{r}}} \cdot \hat{\mathbf{r}} &= 0.
\end{aligned}
$$

Now we have

$$\mathbf{a} \times \mathbf{h} = -G(m_1+m_2)\left[(0)\hat{\mathbf{r}} - (1)\dot{\hat{\mathbf{r}}}\right],$$

where we substituted values from above derivations. So,

$$\mathbf{a} \times \mathbf{h} = G(m_1+m_2)\dot{\hat{\mathbf{r}}}.$$

Replacing $\mathbf{a}$ with $\dot{\mathbf{v}}$ we have

$$\dot{\mathbf{v}} \times \mathbf{h} = G(m_1+m_2)\dot{\hat{\mathbf{r}}}.$$

But,

$$(\mathbf{v} \times \mathbf{h})' = \dot{\mathbf{v}} \times \mathbf{h} + \mathbf{v} \times \dot{\mathbf{h}}$$

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and we also know that $\mathbf{h}$ is constant, so $\dot{\mathbf{h}} = 0$ so we have

$$(\mathbf{v} \times \mathbf{h})' = \dot{\mathbf{v}} \times \mathbf{h}.$$

But,

$$\dot{\mathbf{v}} \times \mathbf{h} = G(m_1 + m_2)\dot{\hat{\mathbf{r}}}$$

so

$$(\mathbf{v} \times \mathbf{h})' = G(m_1 + m_2)\dot{\hat{\mathbf{r}}}.$$

Integrating gives

$$\mathbf{v} \times \mathbf{h} = G(m_1 + m_2)\hat{\mathbf{r}} + \mathbf{C},$$

with constant vector of integration, $\mathbf{C}$. If we take

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})$$

then it is equal to

$$\begin{aligned}
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{r} \cdot (G(m_1 + m_2)\hat{\mathbf{r}} + \mathbf{C}) \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= G(m_1 + m_2)\mathbf{r} \cdot \hat{\mathbf{r}} + \mathbf{r} \cdot \mathbf{C} \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= G(m_1 + m_2)r\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} + |\mathbf{r}||\mathbf{C}| \cos\theta \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= G(m_1 + m_2)r + rg \cos\theta \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= G(m_1 + m_2)r + rg \cos\theta \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= r(G(m_1 + m_2) + g \cos\theta)
\end{aligned}$$

where r and g are magnitudes of $\mathbf{r}$ and $\mathbf{C}$. Solving for r the equation becomes

$$r = \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{G(m_1 + m_2) + g \cdot \cos\theta}.$$

Simplifying and substituting eccentricity e we have

$$r = \frac{1}{G(m_1 + m_2)} \cdot \frac{\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h})}{1 + e \cdot \cos\theta}$$

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where

$$e = \frac{c}{G(m_1 + m_2)}.$$

But,

$$\begin{aligned}
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= \mathbf{h} \times \mathbf{h} \\
\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) &= h^2
\end{aligned}$$

and solving for r gives

$$r = \frac{1}{G(m_1 + m_2)} \cdot \frac{h^2}{1 + e\cos\theta}$$

or

$$r = \frac{h^2}{G(m_1 + m_2)} \cdot \frac{1}{1 + e\cos\theta}.$$

Substituting

$$\frac{e}{c} = \frac{1}{G(m_1 + m_2)}$$

into the equation for $\mathbf{r}$ lends

$$r = \frac{(eh^2/c)}{1 + e\cos\theta}.$$

Then,

$$r = \frac{ed}{1 + e\cos\theta}$$

can be obtained where d equals

$$d = \frac{h^2}{c}.$$

This is the polar equation of the conic section with the focus at the origin and eccentricity e. If $e < 1$ than the orbit of the body will be an ellipse. This is proof that Kepler's first law holds true. Newton went on to express that if $e = 1$ then the orbit is that of a parabola. If $e > 1$ then the orbit is characterized by a hyperbola. Kepler was able to quantify the orbit of an ellipse due to its periodic qualities. As for the parabola and hyperbola, the orbits travel the curve and do not return to the original second body. Therefore, Kepler's first law is derived.

Now Kepler's second law of planetary motion will be derived.

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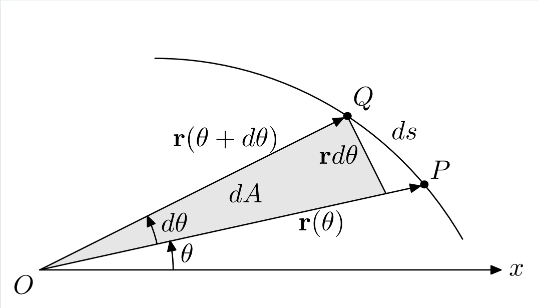

Figure 10: Visual representation of Kepler's second law for one area of an orbit in relation to a stationary source.

7. Solving for Kepler's Second Law

We begin this section with a statement of Kepler's Second Law.

Kepler's Second Law *A line joining a planet and the Sun sweeps out equal areas in equal intervals of time.*

Figure 10 displays the variables in relation to their position along an elliptical orbit of one body in relation to one focus at origin 0.

Before Kepler's second law is derived, we will find angular momentum L that will aid with discovering Kepler's second law. We begin with

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

where $\mathbf{p}$ is linear momentum defined as

$$\mathbf{p} = m\mathbf{v}.$$

So

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v}.$$

We will define

$$\dot{\mathbf{L}} = \tau$$

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where τ is torque. So we can proceed with

$$\tau = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

$$\tau = \frac{d}{dt}(\times m\mathbf{v})$$

$$\tau = \mathbf{r} \times m\frac{d\mathbf{v}}{dt} + \mathbf{v} \times m\mathbf{v}$$

where the previous equation is the derivative of cross product. But,

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{f}$$

and

$$\mathbf{v} \times \mathbf{v} = 0$$

so

$$\tau = \dot{\mathbf{L}} = \mathbf{r} \times \mathbf{f}.$$

We also derive

$$||\mathbf{r} \times \mathbf{f}|| = ||\mathbf{r}|| \cdot ||\mathbf{f}|| \sin\alpha.$$

We have the knowledge that $\alpha = 0^o$ because $\mathbf{r}$ and $\mathbf{f}$ are parallel vectors. We can then say

$$||\mathbf{r} \times \mathbf{f}|| = 0.$$

If

$$\frac{dL}{dt} = 0$$

then L has to be constant by definition of the derivative. We will now find the magnitude of L.

We begin with

$$L = rp$$

$$L = r\mu v_\theta$$

where $p = mv$ which equals $p = \mu v_\theta$.

$$L = r\mu(r\dot{\theta})$$

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$$L = \mu r^2 \dot{\theta}$$

Now onto deriving Kepler's law of areas for planetary motion.

$$dA = \frac{1}{2} r(r d\theta)$$
$$dA = \frac{1}{2} f^2 d\theta$$
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$
$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

But,

$$L = \mu r^2 \dot{\theta}$$

so

$$r^2 \dot{\theta} = \frac{L}{\mu}.$$

And,

$$\frac{dA}{dt} = \frac{1}{2} \frac{L}{\mu}$$

which can be reduced to

$$\frac{dA}{dt} = \frac{L}{2\mu}.$$

But L (angular momentum) is constant and μ (reduced mass) is also always constant so (dA/dt) is constant also. We know that area does not change with respect to time so equal areas are swept out in equal times.

8. Conclusion

The mechanics of space have and still do intrigue people in various ways. With technology improving upon itself at magnificent speeds, one would feel that mysteries would be solved in a flash, but they

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are not. As humanity explores the deeper realms of the universe the data and scientific laws that we know and trust become more and more cloudy. Even though technology and knowledge of outer space improves over time, respect and awe of Kepler and Newton still hold true. They set the path for all scientists and mathematicians to believe that quantitative reasoning does work, no matter how the conclusion is derived.

References

[1] Arnold, David. For his help, support, and Latex knowledge.

[2] Arya, Atam P. **Introduction to Classical Mechanics**. 2nd ed. Prentice-Hall. New Jersey. 1998.

[3] Chandrasekhar, S. **Newton's *Principia* for the Common Reader**. Clarendon Press. Oxford. 1995.

[4] Kaufmann, William J. III and Neil F. Comins. **Discovering the Universe**. 4th ed. W.H. Freeman and Co. New York. 1996.

[5] Pollard, Harry. **Celestial Mechanics**. The Mathematical Association of America. 1976.

[6] Stewart, James. **Calculus: Concepts and Contexts.** Brooks/Cole. Pacific Grove. 1998.

[7] Weisstein, Eric W. "Two-Body Problem". Internet. 16 Apr 2002. http://scienceworld.wolfram.com/physics/Two-BodyProblem.html

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