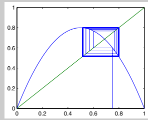

A Brief Overview of Bifurcating Iterators and Chaos.

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Abstract

A brief overview of the bifurcating iterator $f(x) = ax(1-x)$ and an introduction to the chaos it possesses.

1. The Bifurcating Iterator

First let's consider the basic logistic equation.

$$dP/dt = aP(1 - P/k)$$

And now with k, the carrying capacity, equal to one.

$$dP/dt = aP(1 - P)$$

But this equation can be written as an iterator, where $0 < x_0 < 1$.

$$x_{n+1} = ax_n(1 - x_n)$$

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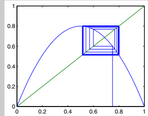
Here we will examine the long term behavior of the iterator with the parameters $a = 2, x_0 = 0.1$.

$$
\begin{aligned}
x_1 &= ax_0(1 - x_0) = 2(0.1)(1 - 0.1) = 0.18000 \\
x_2 &= ax_1(1 - x_1) = 2(0.18)(1 - 0.18) = 0.29520 \\
x_3 &= ax_2(1 - x_2) = 2(0.2952)(1 - 0.2952) = 0.41611... \\
x_4 &= ax_3(1 - x_3) = 2(0.41611)(1 - 0.41611) = 0.48592... \\
x_5 &= ax_4(1 - x_4) = 2(0.48592)(1 - 0.48592) = 0.49960... \\
x_6 &= ax_5(1 - x_5) = 2(0.49960)(1 - 0.49960) = 0.49999... \\
x_7 &= ax_6(1 - x_6) = 2(0.49999)(1 - 0.49999) = 0.5000 \\
x_8 &= ax_7(1 - x_7) = 2(0.5000)(1 - 0.5000) = 0.5000 \\
x_9 &= ax_8(1 - x_8) = 2(0.5000)(1 - 0.5000) = 0.5000 \\
x_{10} &= ax_9(1 - x_9) = 2(0.5000)(1 - 0.5000) = 0.5000
\end{aligned}
$$

All the solutions here are called the orbit of x_n. The orbit of x_n with the parameter $a = 2$ with the initial condition $x_0 = 0.1$. Figure 1 is the Time Series portrait of the same iteration, but this time iterated twenty times. There is another way that this iteration can be looked at, with a web diagram as shown in Figure 2.

This is an example of a single steady solution. In fact, all solutions to the iterator $x_{n+1} = ax_n(1 - x_n)$ are single steady solutions whenever $1 \leq a < 3$. However, for values of $a \geq 3$ the function experiences period doubling bifurcation.

Here, let's examine side by side the Time Series portraits and Web diagrams for both cases: $a < 3$ and $a \geq 3$ in Figures 3, 4, 5 and 6. In Figures 3 and 4, we notice that $f_{2.75}(x)$ takes longer to stabilize than when $a = 2$, but quickly becomes stable on the value $x_n = 0.636363....$ The greater length of time it requires to steady is made evident within the Web portrait, Figure 4, where rather than moving directly to the final state solution, the spider spins a whirling web towards the center, reflecting the oscillating mechanism of the Time Series for the same value of a. In Figures 5 and 6 we see an iteration with a pair of steady solutions: $x_n = 0.5130...$ and $0.7994...$ This is to be expected for $3.2 > 3.0$. This



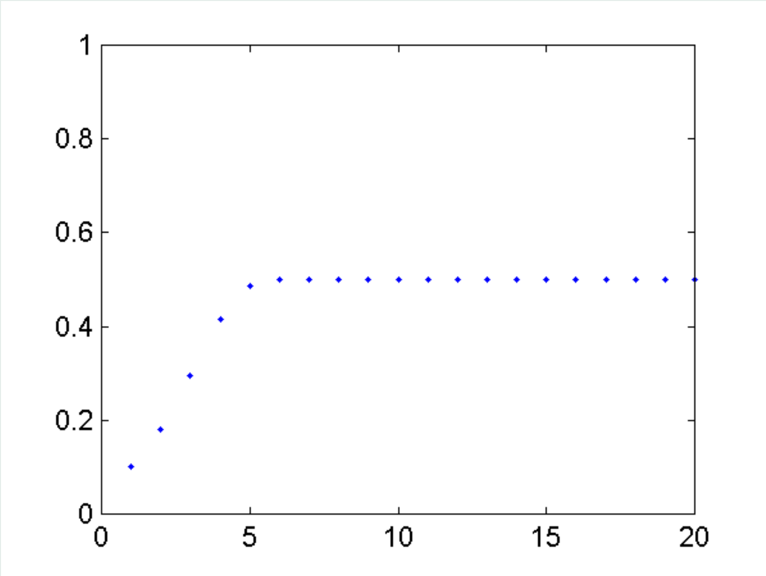
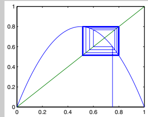


Figure 1: The Time Series portrait of the first twenty iterations of $f_2(x)$ for $x_0 = 0.1$.



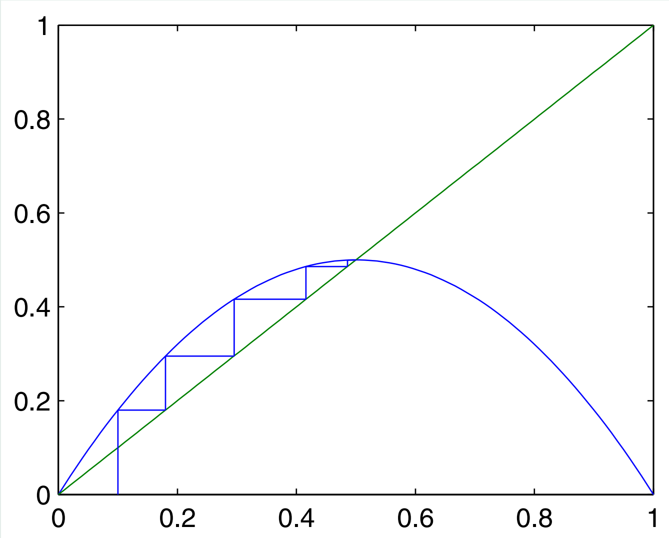
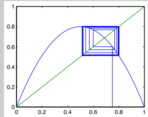
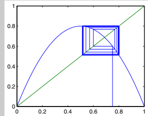


Figure 2: Web diagram of the iterator $f_2(x)$, where $x_0 = 0.1$.



iteration, for $a = 3.2$, is the first example we have seen so far of bifurcating. The Web diagram in Figure 6 can be characterized by a spider attempting to whirl its way towards the center, but being caught in a repetitive path, a rut dictated by the composition of the web's environment (the parabola around which it whirls). The first line in the web diagrams is drawn vertically from an initial value x_0 to the value $f(x_0)$ lying on the parabola. The next line runs horizontally from the value on the parabola, $f(x_0)$, towards the point on the line $y = x$. This visual image helps to create a more intuitive understanding of the mechanics of iteration. The next line moves vertically from the point on the line $y = x$, landing on the value $f_a^2(x_0)$ that exists on the parabola. This process cycles on as our spider spins its web and the values $x_n = f_a^{n+1}(x_0) = ax_{n-1}(1 - x_{n-1})$ become discovered by increasing the number of iterations.

Remember the web found in Figure 6 is a representation of the iterator when $a = 3.2$, but with the specific initial condition $x_0 = 0.3$. Now let's view a Web portrait of the same value for a, but this time with a different initial condition: $x_0 = 0.75$. The result is shown in Figure 7. We can see that the spider gets caught into "practically the same" rut as shown in Figure 6, but the path way to the final state differs greatly. Viewing the orbits of both iterations we notice they stabilize to the "same" pair of values $x_n = 0.5130...$ and $x_n = 0.7994....$



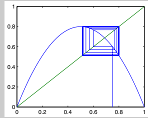
Comparative orbitals:

$$
\begin{array}{ll}
x_0 = 0.3 & x_0 = 0.75 \\
x_1 = 0.672 & x_1 = 0.6 \\
x_2 = 0.7053... & x_2 = 0.768 \\
x_3 = 0.6650... & x_3 = 0.5701... \\
x_4 = 0.7127... & x_4 = 0.7842... \\
\quad . & \quad . \\
\quad . & \quad . \\
\quad . & \quad . \\
x_{50} = 0.7994... & x_{50} = 0.7994... \\
x_{51} = 0.5130... & x_{51} = 0.5130... \\
x_{52} = 0.7994... & x_{52} = 0.7994... \\
x_{53} = 0.5130... & x_{53} = 0.5130...
\end{array}
$$

Yes the spiders do end up in basically identical final states, and when examined over a long enough time scale this will appear to be true. However, by existence and uniqueness this could not be farther from the truth. The path set by the initial condition forever resonates within the path of the spider's orbiting, and no two differing spiders can ever exist in the same location. This sensitivity to initial conditions will be defined and discussed later on.

2. The Final State Portrait and the Feigenbaum Constant

At this point it becomes practical for a moment to consider the iterating function from a different vantage point. We will set aside our concerns with the orbital motions of x_n.



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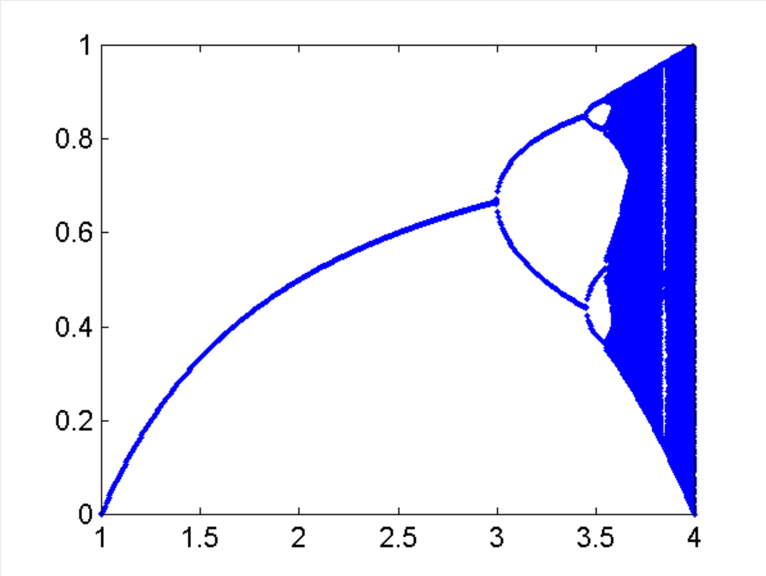
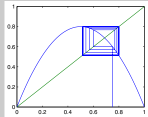


Figure 3: Time Series portrait $f_{2.75}x_n$ here stabilizes at 0.636363....



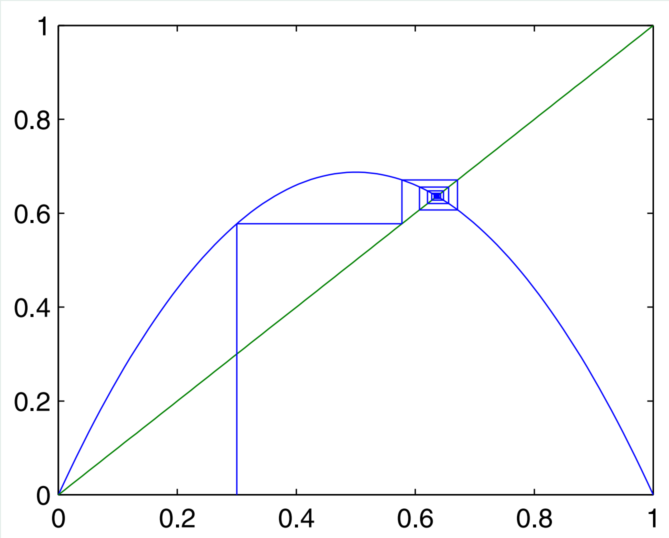
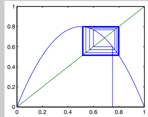


Figure 4: Here, path to steady solution is not direct as when $a = 2$, however stabilization occurs relatively quickly when $a = 2.75$ on to the value $x_n = 0.636363\ldots$.



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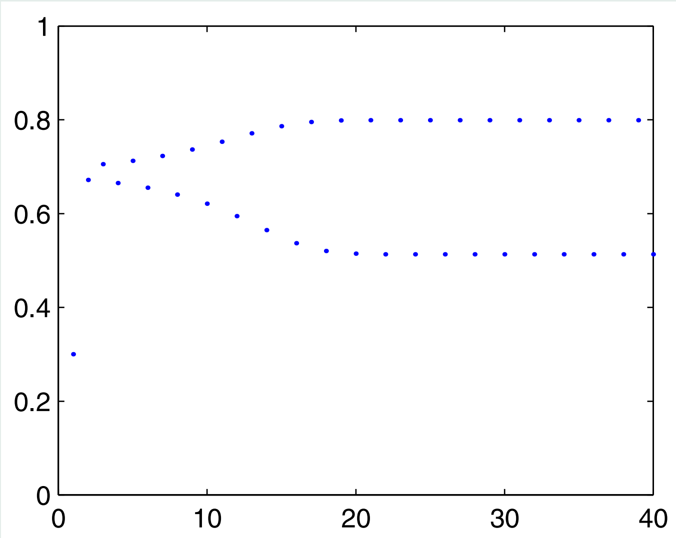
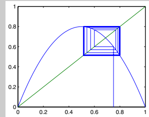


Figure 5: When $a = 3.2, x_n$ stabilizes on two values, $x_n = 0.5130...$ and $0.7994....$ This double solution is a result of the first event of bifurcation.



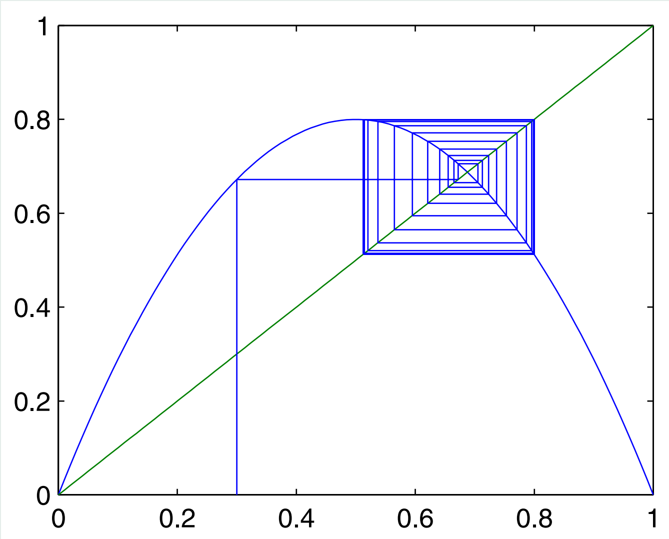
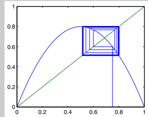


Figure 6: The Web diagram for $a = 3.2$; this behavior is analogous to a spider spinning a web, but getting caught in a rut for which he cannot escape.



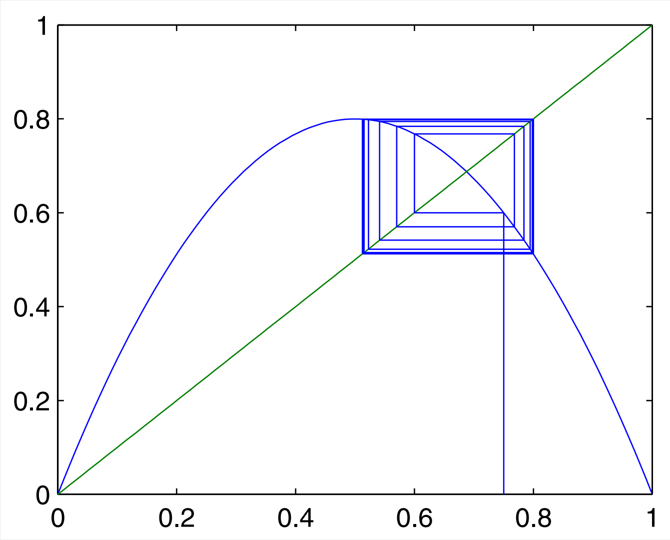
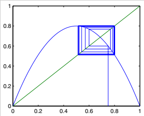


Figure 7: The path to stability changes as the initial condition changes. Differences like these resonate throughout the orbitals of x forever.



Instead we will move our attention to the *final states* of x_n as the values of a change discretely. Here the function,

$$f_a(x) = ax_n(1 - x_n)$$

will be thought of as a vector function of a,

$$\vec{x}(a) = a\vec{x}(1 - \vec{x}),$$

where the vector $\vec{x}$ is the collection of final state solutions of the original iterating function for differing values of a. In other words, the function $f_a(x)$ is a "sub-function" of $\vec{x}(a)$, such that

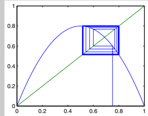
$$\vec{x}(a) = a(f_a^{n+1}(x_0))(1 - (f_a^{n+1}(x_0))),$$

for $n = 0, 1, 2, 3, 4, \ldots$. One method for gathering the Final State vectors would be to iterate the function several times and then discard the primary iterations that oscillate about dramatically varying values and keep the values that occur periodically with minimal variation (here, quantity *does* mean quality — as interactions increase so to does delicateness). The Final State portrait plots the vector values against the dependent variable a.

2.1. Final State

First let's look at a Final State portrait of $x(a)$ for values of $1 \leq a < 3$ in Figure 8. Remembering that on this interval of a the Final State settles to one value $x(a)$. Made obvious in Figure 9, at $x(3)$ the first event of bifurcation insues.

When $a \approx 3.4494$ another bifurcation occurs, and on the interval that immediately follows there exists four solutions that $x(a)$ settles to. Notice in Figure 10 that yet another bifurication occurs at $a \approx 3.544090$, and then another at $a \approx 3.564407$, and then yet again at $a \approx 3.568759$. The values of a where further bifurcation occurs exist for this iterator at constantly increasing values of a. Each bifurcation produces a pair of new stable points. Thusly the number of stable values of $x(a)$ doubles from bifurcation interval to bifurcation interval to bifurcation interval. The number of stable points at $x(a)$ equals 2^b, where b is the number of bifurcations having taken place.



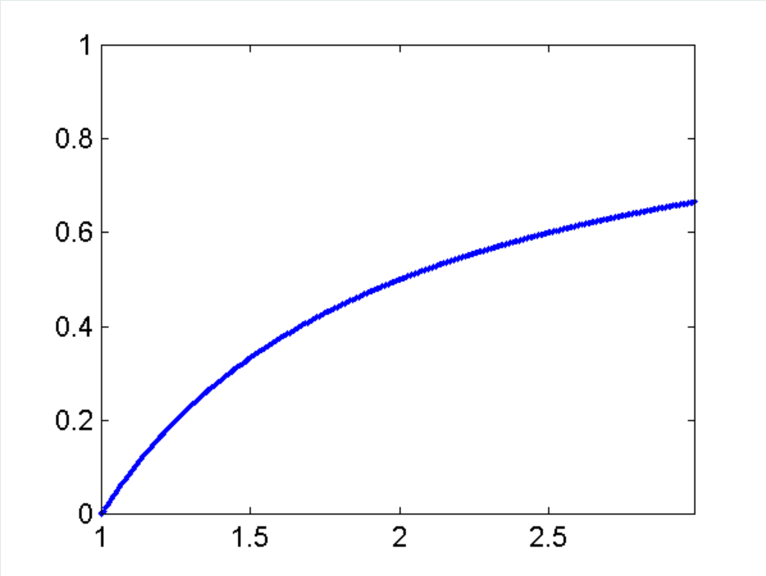


Figure 8: This is a graph of the fixed points $p_a = (a-1)/a$.

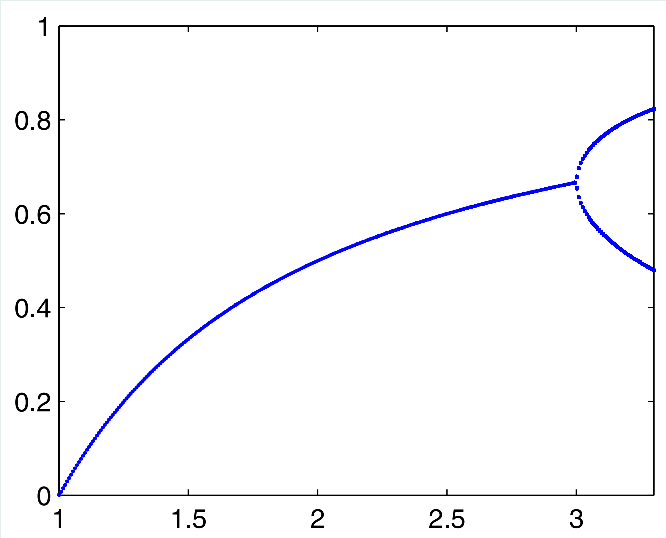
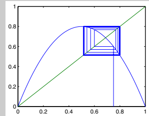


Figure 9: Bifurcation occurs for the first time at $x(3)$. We can see that on the interval $3 \leq a < 3.3$, $x(a)$ has a pair of steady solutions.



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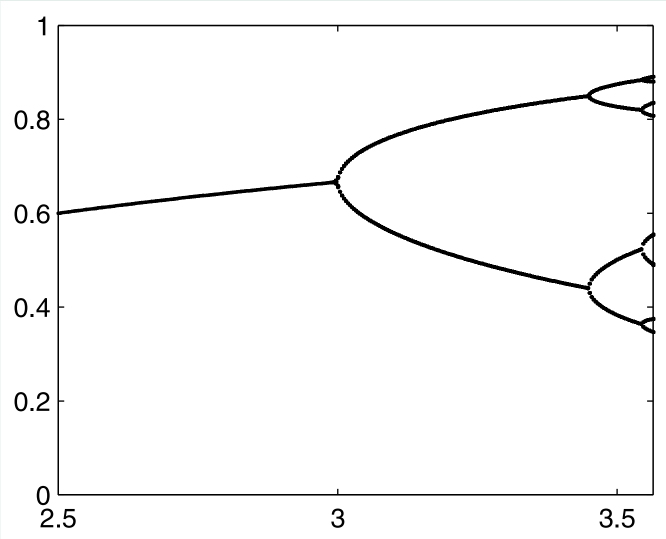


Figure 10: A plot of a few of the primary bifurcations. We will be tallying the difference between immediately following intervals.

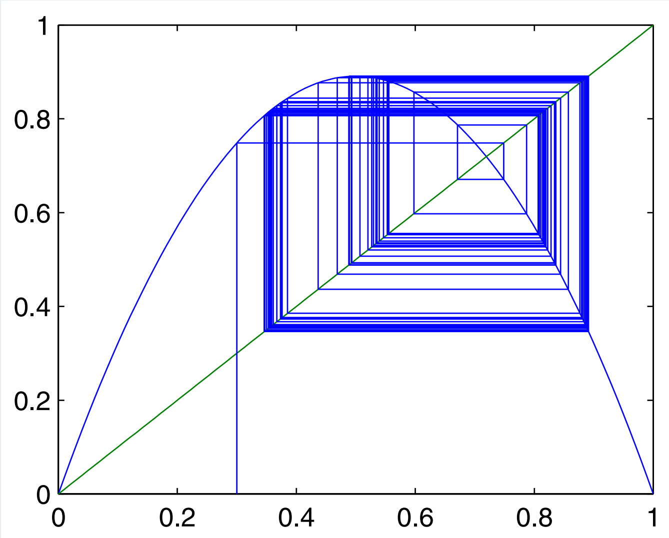
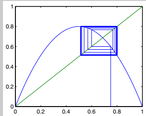


Figure 11: Here our little spider's path complexifies, but still he is caught in a rut of stable solutions.



2.2. The Feigenbaum Constant

When the distance between bifurcation intervals is considered from one to the next, the ratio between them approaches a constant value.

$$
\begin{aligned}
b_2 - b_1 &= 3.449489... - 3.0 & &\approx 0.44949 = d_1 \\
b_3 - b_2 &= 3.544090... - 3.449489... & &\approx 0.04611 = d_2 \\
b_4 - b_3 &= 3.564407... - 3.54409... & &\approx 0.020316 = d_3 \\
b_5 - b_4 &= 3.568759... - 3.564407... & &\approx 0.004352 = d_4 \\
b_6 - b_5 &= 3.569692... - 3.568759... & &\approx 0.000932 = d_5 \\
b_7 - b_6 &= 3.569891... - 3.569692... & &\approx 0.000199 = d_6
\end{aligned}
$$

Now let's calculate the ratio between one interval to the next, such that $d_k/d_{k+1} = \delta_k$.

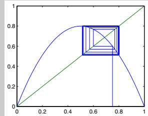
$$
\begin{aligned}
d_1/d_2 &= 4.7514... \\
d_2/d_3 &= 4.6562... \\
d_3/d_4 &= 4.6682... \\
d_4/d_5 &= 4.6687... \\
d_5/d_6 &= 4.6693...
\end{aligned}
$$

From the above values calculated for d_k/d_{k+1} it seems as though the ratio is approaching a constant value. This could not be more true, in fact:

$$\lim_{n\to\infty} d_n/d_{n+1} = 4.6692016091029... = \delta,$$

The Feigenbaum constant δ is considered universal. Universality implies that the constant makes predictions for not only one, but an entire family of functions. This family is a class of quadratic iterators subject to the following guide lines.

1. f must be a smooth function from $[0, 1]$.



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2. f has a maximum value at x_{max} which is quadratic such that

$$f^{''}(x_{max}) \neq 0.$$

3. f is monotone on $[0, x_{max}]$ and on $[x_{max}, 1]$.

4. f has a negative Schwarian derivative. This means $S_f(x) < 0$ for $x \epsilon [0, 1]$ where,

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Allow us now to view a couple of other functions that fall within the above guidelines.

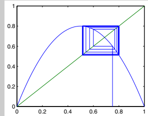
$$f_a(x) = ax^2 \sin(x\pi)$$

$$f_c(x) = x^2 + c$$

2.3. Self Similarity

The Final State diagram on interval $[1, s_\infty]$ exhibits the most curious property of self-similarity(s_∞ is a discrete irrational number, to be discussed in more details later). This feature is both quantitative and qualitative. The qualitative feature of self-similarity is best understood whence visualized by comparative sections of the final state diagram: Figures 12, 13, 14, and 15. The distances of the horizontal lines of the smaller boxes in Figures 12, 13,and 14 are scaled by a number that approaches the Feigenbaum constant as this process of magnification continues.

Self-similarity is intuitively recognized though qualitative, for the Feigenbaum constant is approached, but never reached, for this reason, what it is that we observe is self-similarity, not "self-sameness". Theoretically this qualitative Phenomenon continues infinitely onward, this is implied by the term self-similarity. This also implies further that the path of our spider, as a increases, becomes infinitely complex while retaining a finite



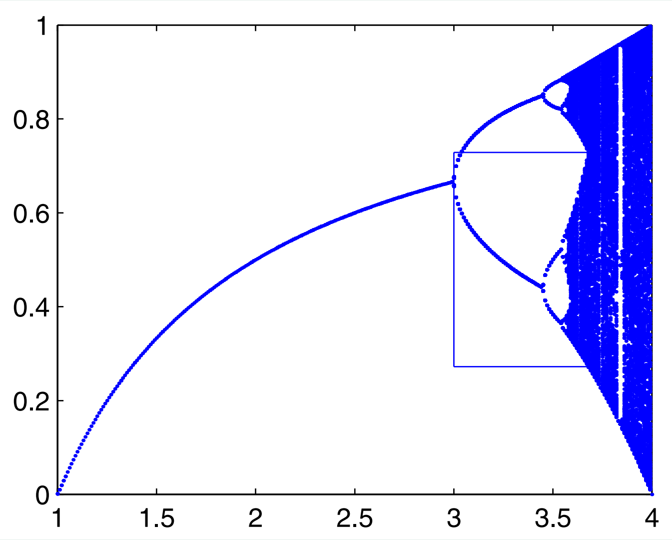
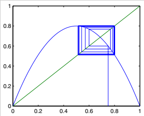


Figure 12: Final state, to demonstrate self-similarity, each subsequent slide is a δ times smaller than the slide before.



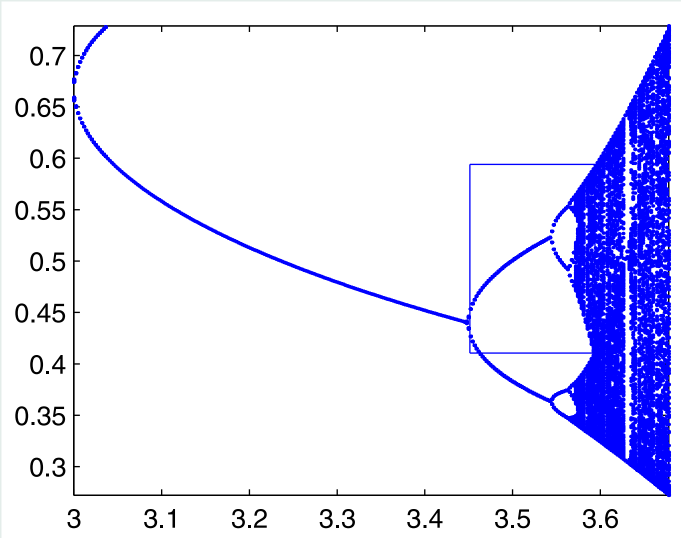
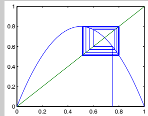


Figure 13: Final state enlarged from the boxed off region in Figure 12.



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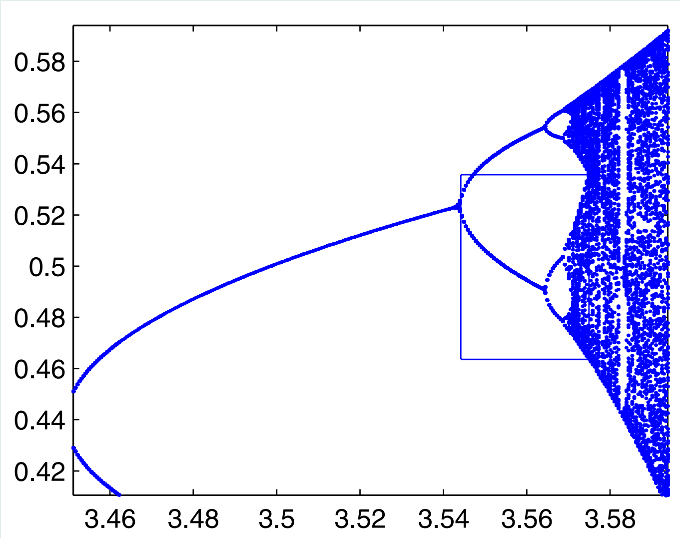
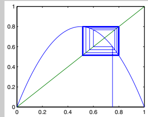


Figure 14: Final state enlarged from the boxed off region in Figure 13.



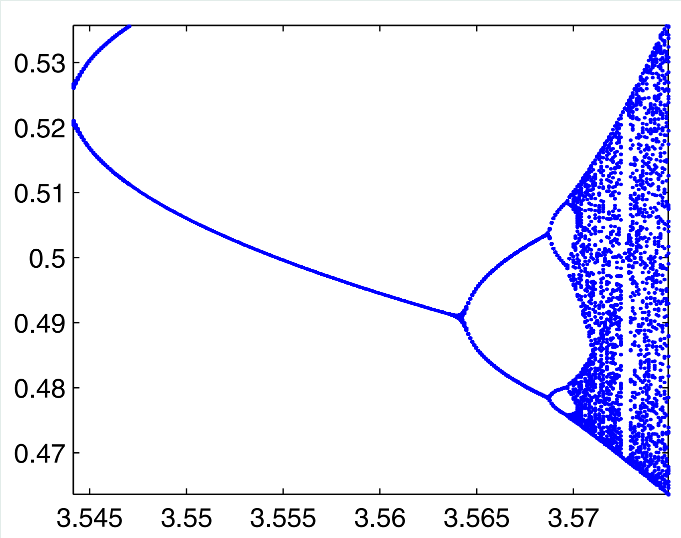


Figure 15: Final state enlarged from the boxed off region in Figure 14

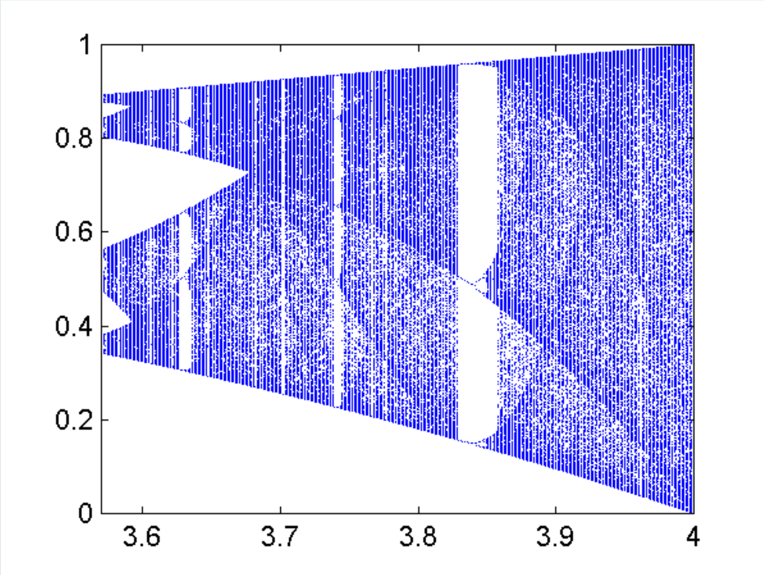
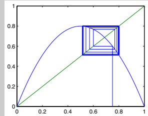


Figure 16: Final state of the chaotic region, .

number of fixed orbitals as self-similarity continues forward for infinite generations. This contradiction is a signpost indicating the end of the reign of the period-doubling tree and the entrance into the world of chaos. At s_∞, the final stated diagram enters a region that is both profoundly different to and profoundly similar to the situation for parameter values less than s_∞. This region exists between s_∞ and 4: Figure 16.



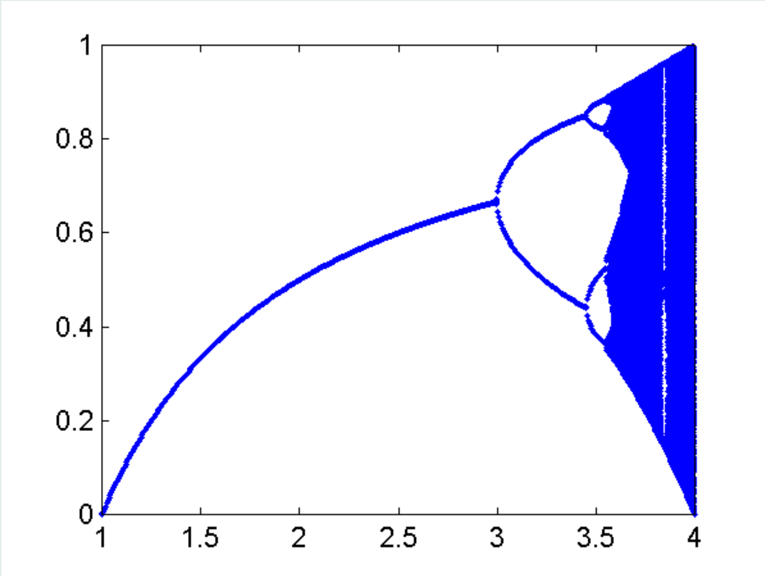
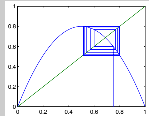


Figure 17: Final state, site of the impossible race.

3. Entrance into Chaos

3.1. The Complete Final State

The complete Final State, Figure 17, is analogous to the story of the race that could never be finished. The story about a man who could instantly run half the distance to the finish line, and in the next moment he would half the remaining distance, moment to moment he would half and half, but never would he reach the finish line. The Final State diagram



is a portrait similar to this scenario, but instead of leaving half of the race to be run this function bifurcates, each time leaving $1/\delta$ the distance yet to be run. The finish line here, like the one in the story, ought never to be reached, but still the function presses forward. Not only is the finish line met, but it is surpassed. The finish line is the Feigenbaum point(s_∞), and the track beyond the finish line exists amidst chaos.

Still we have only considered part of the final state diagram. The behavior of the orbitals abruptly destabilizes at $a = s_\infty = 3.5699456...$, the Feigenbaum point. On the interval $1 \leq a < s_\infty$, values of x are periodic, implying that a finite and repeating orbital values. As a increases the moments of bifurcation(doubling of stable solutions) are increasingly frequent, but still a repetitive path for our spider exists such that a rut exists though its path has become super complex, as seen in Figure 11 though the mechanism of iterating stable solutions remains the same.

3.1.1. Changing our direction of motion for a moment

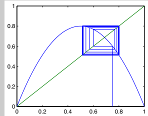
In our previous discussion we moved from order to the boarder of chaos. Now we will begin working in the opposite direction, namely from $a = 4$ to $a = s_\infty$. At $a = 4$ of $f_a(x) = ax(1-x)$, $f(x)$ spans the entire interval zero to one and chaos is observed in the entire unit interval: as seen in Figure 18. This ought to bring about the following question,"What is chaos?"

3.2. "What is Chaos?"

Well, as of now a firm definition of chaos(as used here in this article) is still under dispute with in the scientific community, but for the time being will satiate ourselves with the definition proposed by Robert Devaney, one of the current stars of modern chaos research.

We will first need to define three terms: a dense set, a dynamic transitive system, and sensitivity to initial condition.

1. The notion of a dense set. Suppose **V** is a set and that **U** is a subset of **V**(meaning that all numbers inside **U** exist as well in **V**, but this does not necessitate that all



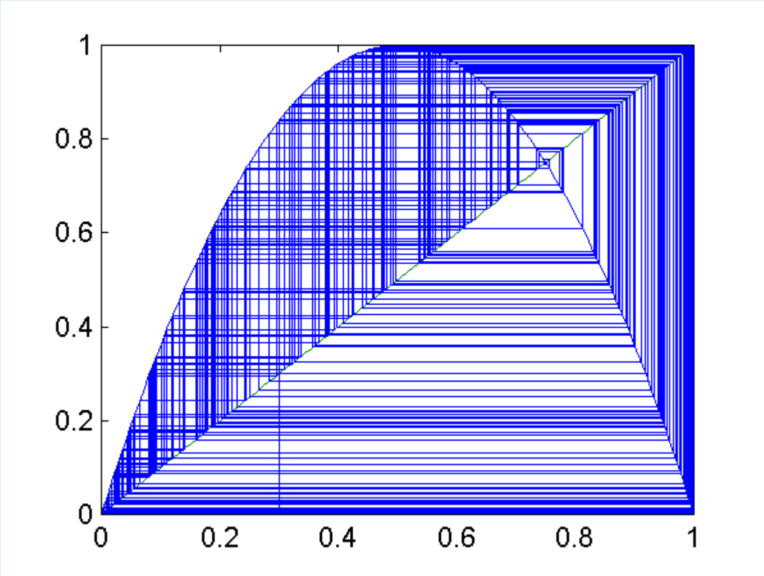
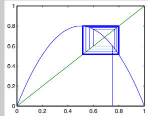


Figure 18: The values of x span the entire set of real numbers from zero to one.



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the numbers within $\mathbf{V}$ are in $\mathbf{U}$). It is claimed that wherever $\mathbf{U}$ is *dense* in $\mathbf{V}$, that for any arbitrary point $\mathbf{v}\epsilon\mathbf{V}$ there is a point $\mathbf{u}\epsilon\mathbf{U}$ arbitrarily close to $\mathbf{v}$.

2. A transitive dynamic system: A dynamic system is considered *transitive* if for any pair of points $\mathbf{v}$ and $\mathbf{u}$ and any $\epsilon > 0$ there will exists a third point $\mathbf{z}$ within ϵ of $\mathbf{v}$ whose value comes within ϵ of $\mathbf{u}$.

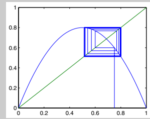
   Therefore for any arbitrary value $\mathbf{v}$ within $\mathbf{V}$ there exists a value $\mathbf{u}$ within $\mathbf{U}$ which exists arbitrarily close to $\mathbf{v}$. Clearly a dynamic system which has *dense* orbits could be said to be a *transitive* system.

3. Sensitivity to initial condition: A dynamic system F depends *sensitively* on initial conditions if there is a $\beta > 0$ such that for any $\mathbf{v}$ and any $\epsilon > 0$ there is a $\mathbf{u}$ within ϵ of $\mathbf{v}$ and a k such that the distance between $f^k(\mathbf{v})$ and $f^k(\mathbf{u})$ is at least β . Therefore $f^k(v) - f^k(u) = \rho \geq \beta > 0., \mathbf{v} \neq \mathbf{u}$. As $\mathbf{u}$ approaches $\mathbf{v}$ the size of β shrinks, but so long as $\mathbf{v}\neq\mathbf{u}$ there exists a space of size β that functions to keep $f^k(v)$ and $f^k(u)$ unique solutions. Here we are reminded of our spider traveling about its path. We said before that no two spiders could exist in the same position. Now, redundant but profound, by Uniqueness and existence, if any two spiders were to occupy the same position those two would be one spider. The above definition of sensitivity sets limits to how close one spider can be from another spider without those spiders being the same spider(same initial condition).

3.3. Chaos, the complete definition

A function is considered chaotic on an interval if on that interval:

1. Periodic points for f are *dense*.
2. f is *transitive*.
3. f depends *sensitively* on initial conditions.



Bifurcating quadratic iterators obeying the Feigenbaum constant and s_∞ so to0 obey the constraints of a chaotic dynamic system.

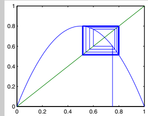
3.4. Passage from Chaos to Order

Now that we have defined our function as chaotic, let us focus the passage from chaos to order. Recall the final state diagram of the chaotic region (Figure 16). Also recall that $f_a(x)$, when $a = 4$, refers to a band of values spanning the entire unit interval $[0, 1]$. So for sake of continuity with our definitions, let's consider this spanning set of values to be the set $\mathbf{V}$. As a decreases the band slowly narrows. Let us consider this narrowed band $\mathbf{U}$ such that $\mathbf{U}$ is a subset dense on $\mathbf{V}$. Before we examined the phenomenon of bifurcation, now we will examine a second similar phenomenon which we will call bicondensation. In the chaotic region of the final state, an abrupt condensation of the dense set $\mathbf{U}$ into two separate spanning sets, within $\mathbf{V}$, is noticed. Moving backwards from $a = 4$ to s_∞ the first bicondensation occurs at $a = c_1 = 3.6785....$ The second and third instances of bicondensation pass at $a = 3.5923$ and $a = 3.5749$ respectively as visualized in Figure 20. Figure 19 displays a time series portrait of chaotic movements between discrete bands with $a = c_1 - \epsilon = 3.65$. The void between the pair of bands is a direct result of the first bicondensation at c_1. At $a = c_1 + \epsilon$ there exists one dense band, $\mathbf{U}(a)$, however at $a = c_1 - \epsilon$ there exists two dense bands $\mathbf{U}(a)$. Bicondensation implies band splitting. Figure 21 demonstrates this event when $a = c_3 - \epsilon$. Visible in this figure is the existence of six bands; this is to be expected, for bands double with each bicondensation and $3 \times 2 = 6$.

3.4.1. Resurgence of the Feigenbaum Constant

Also as found in orderly region, by tallying the quotients of the distances of subsequent intervals of bicondensations and taking the limit such that:

$$\lim_{k\to\infty} \frac{Ic_k}{Ic_{k+1}} = 4.669201609... = \delta$$



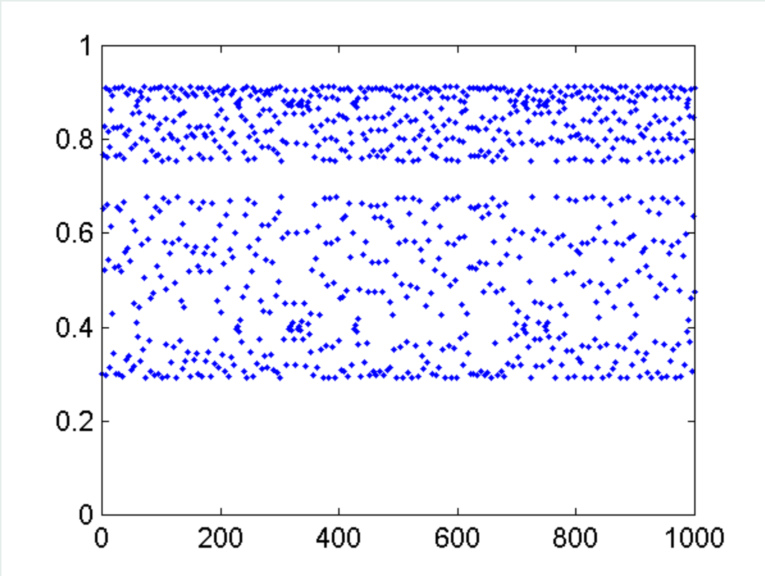
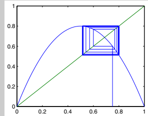


Figure 19: The orbit of x oscillates between spanning bands of dense sets.Within each band the dynamics of chaos exists. Here $a = 3.65$, which is a value of a just slightly lower than $a = c_1$.



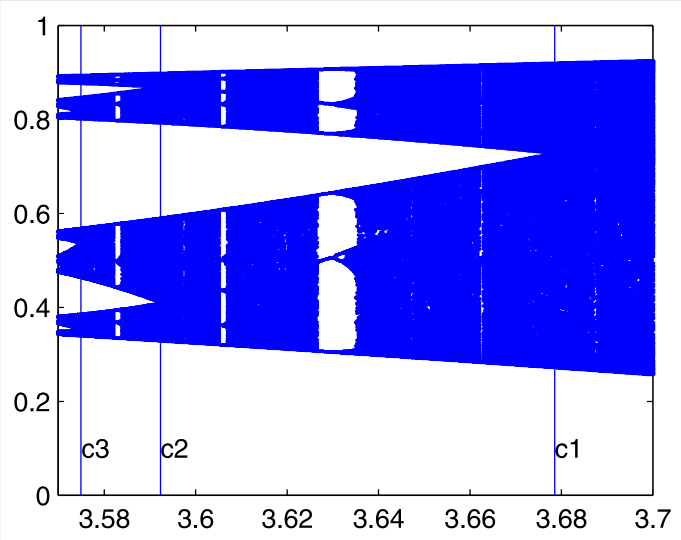
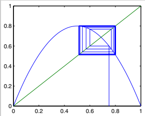


Figure 20: In this figure the discrete values of a where bicondensations fall are marked with vertical lines, .



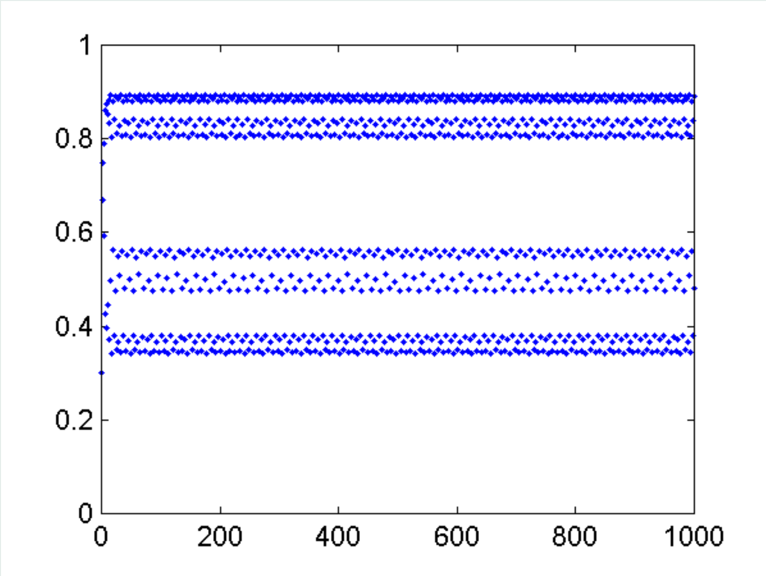
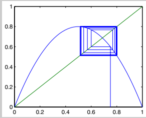


Figure 21: Here $a = c_3 - \epsilon = 3.57$, where the bands have just split for the third time. Note the existence of $3 \times 2 = 6$ discrete bands.



3.5. Self-Similarity in the Final State of the Chaotic Region

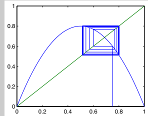
Another similarity between the orderly and chaotic region is Self- similarity. The pair of Figures 22 and 23 demonstrate Self-similarity such that the box within the first figure lies on the frame of the next figure. As self similarity implies this phenomenon continues for an infinite number of generations.

Self-similarity amidst the bicondensating region implies that an infinite number of bicondensating events takes place as the function moves (in reverse) from $a = 4$ to $a = s_\infty$. This in turn implies the existence of an infinite number of spanning discrete bands at $a = s_\infty$ being that $c_\infty = s_\infty$.

But still the behavior and mechanisms of chaos and bicondensation are without suffice explanation. Before we used the analogy of a spider, but now, here, amidst chaos, the path of our spider is all too chaotic. His time series portraits reveal that he is no longer stuck in his ruts and is free from pattern and monotone existence, but his freedom is confined now to specific regions in space, where the locations of his orbitals were unconfined. When our friend blew threw the sign post at s_∞, he exchanged unconfined order for restricted freedom.

3.6. Chaotic parabola and the Mechanics of Chaos

Let's revisit Web diagrams, but this time within chaos. Figures 24 and 18 visualizes the parabola for $a = 4$. The parabola at this value of a is called the generic or chaotic parabola; chaotic parabolas are characterized by their fitting perfectly into a square for which the diagonal of this square is the line $y = x$. Since here at $a = 4$ the values of x span the entire unit interval $[0, 1]$ looking at full iteration would be redundant for this Figure 18 is only iterated 250 times, but still it is obvious that the function lies on the brink of saturation. Like in the Webs we observed before, the geometry of the Web's frame is responsible for the path taken by our spider. Dense solutions of this nature occur in all chaotic parabolas. But chaos exists for more values the just $a = 4$. To investigate the existence of chaos in



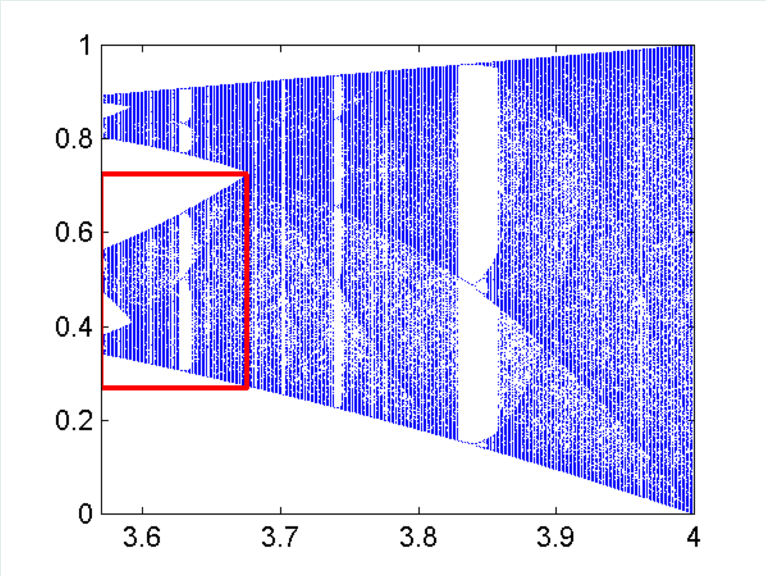


Figure 22: Here Self-similarity is witnessed again, but in the chaotic region, thus the frame of Figure 23 is the box within this figure.

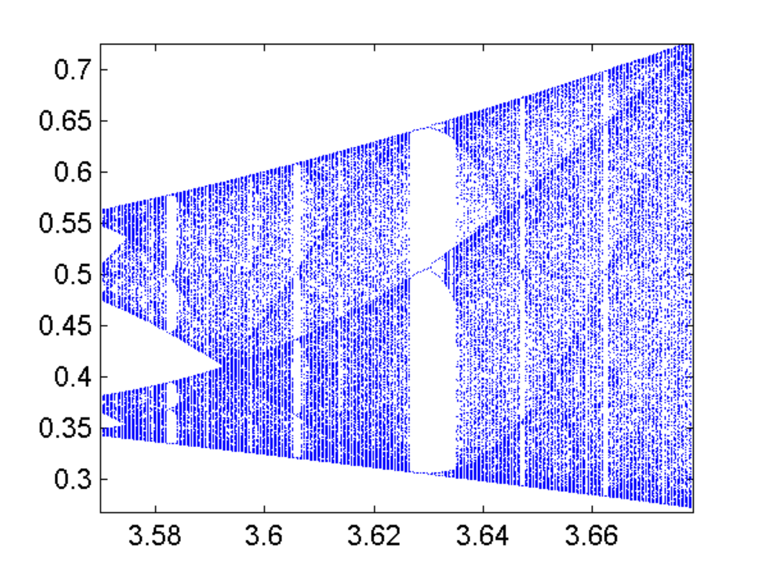
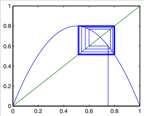


Figure 23: Here Self-similarity is observed as this graph is an enlargement of the boxed off region in Figure 22.



the chaotic region, let's investigate the second iteration of our function at $a = c_1$:

$$f_a(x) = ax(1-x)$$

$$f_a^2(x) = a(ax(1-x))(1-(ax(1-x)))$$
$$f_a^2(x) = -a^3x^4 + 2a^3x^3 - (a^2+a^3)x^2 + a^2x$$

The first and second iterations of $f_{c_1}(x)$ are displayed in Figures 25(a) and (b). Quickly one will notice that the first iteration at $a = c_1$ is not a chaotic parabola, and that the second iteration is not a parabola at all. But the figure of the second iteration contains square boxes about a pair of chaotic parabolas. Regardless of size, however, the mechanics within these smaller parabolas are retained(Self-similarity) implying chaotic behavior. It is correct to assume a third chaotic parabola will be born within the Web frame of $f_{c_2}^4$, moreover that the birth of the $(n+1)th$ chaotic parabola will be evident in the Web frames of $f_{c_{n-1}}^{2^n}$ suggesting the number of potential chaotic parabolas approaches infinity as n approaches infinity. Take a moment to look at Figure 19 again. This is a Time series portrait of our iterator where the parameter a is just below $a = c_1$. Here we can imagine how the pair of Chaotic parabolas migrate away from one another thus slowly creating a void lending further rational towards the mechanics of bicondensation and the self-similar event within chaos.

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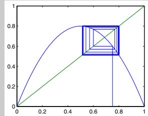
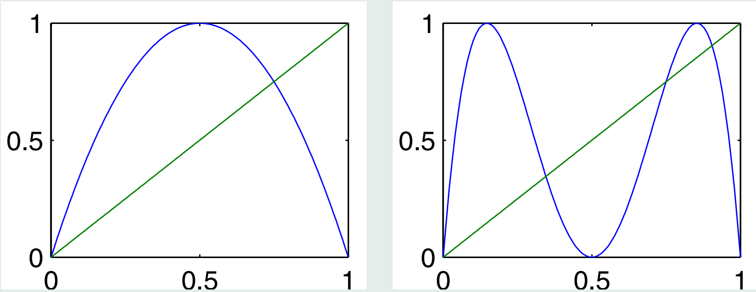
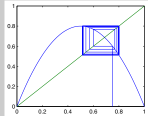




(a) Chaotic parabola. (b) Second iteration of $f_4^2(x)$.

Figure 24:



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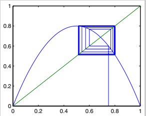
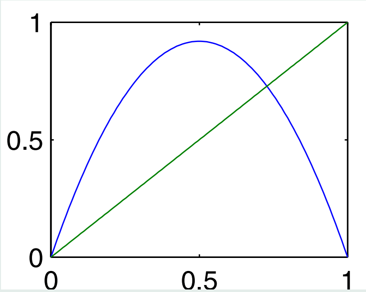
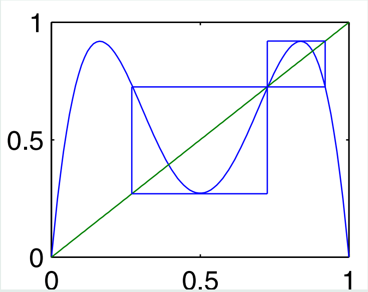




(a) When $a = c_1$, this is not a chaotic parabola during the first iteration.



(b) But for the second iteration, two small chaotic parabolas are born.

Figure 25: