Forced Oscillations and Nuclear Magnetic Resonance

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Abstract

When analyzing forced oscillations and nuclear magnetic resonance, both math and physics play a significant role. Thus, during this analysis we will be referring to an experiment that is easily done in the laboratory. A pair of Helmholtz coils is set up. By definition, when current is run through the coils, a magnetic field is produced. This magnetic field is uniform because the coils are separated by a distance that is equal to the radius of the coils. Within the Helmholtz coils, another coil is placed in such a fashion as to produce a magnetic field that is perpendicular to that of the Helmholtz coils. Inside the inner coil a compass rests, acting like a magnetic dipole, and it attempts to adjust itself to the two perpendicular fields, as shown in Figure 1. This attempt causes an oscillatory deflection of the compass needle. By varying the frequency, the resonant state for the compass needle can be found.

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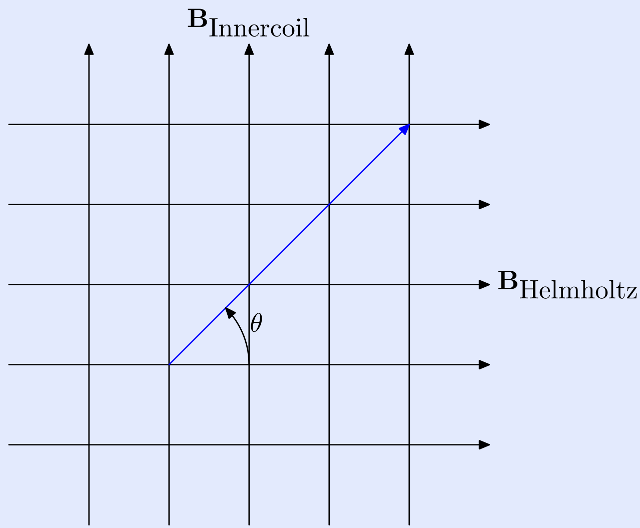

Figure 1: Diagram of the magnetic field due to both coils, along with the angle associated with the angular displacement of the compass needle from equilibrium

1. The Physics of Forced Oscillations and Magnetic Resonance

Before starting the analysis, it is most helpful to review a few basic physics relationships. We begin with Newton's second law for rotational motion,

$$\sum \tau = I\theta'' \tag{1}$$

where the sum of the torques is equal to the product of I, the rotational inertia of the compass needle, and θ'' the angular acceleration. We define θ to be the angular displacement of the compass needle from equilibrium.

There are three torques acting on the compass needle. One torque is magnetic, due to the Helmholtz coils. There is another magnetic torque due to the inner coils. Lastly, there is the damping torque due to the friction between the compass needle and the holding pin. Now, the left hand side of Newton's second law can be rewritten as:

$$\tau_{\text{net}} = \tau_{\text{driving field}} + \tau_{\text{restoring}} + \tau_{\text{damping}} \tag{2}$$

The torque due to the driving field is

$$\tau_{\text{driving field}} = \mathrm{F}\cos\omega\mathrm{t}, \tag{3}$$

where F is the amplitude of the driving field and ω is the driving field frequency, and t represents time. Equation 3 holds true because the driving field torque is caused by a sinusoidal wave input, which in the experiment was generated by a power amplifying device. This sinusoidal input is modeled by the cosine function.

The restoring torque can be expressed as

$$\tau_{\text{restoring}} = \mu \mathrm{B} \theta. \tag{4}$$

where μ is the magnetic dipole moment of the compass needle, and B is the magnetic induction due to the Helmholtz coils. By definition the restoring torque is the cross product of μ and B, which can be rewritten as

$$\tau_{\text{restoring}} = \mu \times \mathrm{B} = |\mu||\mathrm{B}| \sin\theta. \tag{5}$$

Making use of the small angle approximation, $\sin\theta \approx \theta$, Equation 5 becomes

$$\tau_{\text{restoring}} = \mu \mathrm{B} \theta. \tag{6}$$

The damping torque is defined to be

$$\tau_{\text{damping}} = \mu \beta \theta'. \tag{7}$$

Based on previous research, the assumption that

$$\tau_{\text{damping}} \propto \mu \theta' \tag{8}$$

can be made. Thus, the damping torque is dependent on the angular velocity and magnetic dipole moment. In Equation 7, β represents the constant of proportionality, which is also known as the damping constant.

Now Newton's second law can be rewritten in the form of a second order differential equation.

$$\mathrm{I}\theta'' = \mathrm{F} \cos\omega \mathrm{t} - \mu \mathrm{B} \theta - \mu \beta \theta' \tag{9}$$

In the above equation, the quantity $\mu B\theta$ is negative because the restoring torque is opposite the displacement. In other words, it tries to compensate for the positive change caused by the driving field torque, $F\cos\omega t$. The term $\mu\beta\theta'$ is negative because it is a damping torque, which always opposes motion. A clear correspondance can be seen between Newton's second law and the second order differential equation.

2. Mathematical Analysis

The first thing is to divide the differential Equation 9 through by I, the rotational inertia. This yields

$$\theta'' = \frac{F}{I}\cos\omega t - \frac{\mu B}{I}\theta - \frac{\mu\beta}{I}\theta'. \tag{10}$$

Next, to simplify the equation, we will make the following two substitutions:

$$\omega_1^2 = \frac{\mu\beta}{I} \tag{11}$$

$$\omega_0^2 = \frac{\mu B}{I} \tag{12}$$

Substituting Equation 11 and Equation 12 into Equation 10 and rearranging gives,

$$\theta'' + \omega_1^2\theta' + \omega_0^2\theta = \frac{F}{I}\cos\omega t. \tag{13}$$

The above equation can be solved by first assuming a particular solution,

$$\theta_p = A\cos\omega t + B\sin\omega t \tag{14}$$

Since we are dealing with an oscillatory function, it makes sense to assume the most general solution involving the two oscillatory functions, sine and cosine.

Taking the first and second derivatives gives,

$$\theta_p' = -A\omega\sin\omega t + B\omega\cos\omega t, \tag{15}$$

$$\theta_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t. \tag{16}$$

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Next we substitute the above derivatives along with the assumed particular solution into Equation 13 and this yields:

$$\begin{aligned}&-A\omega^2\cos\omega t - B\omega^2\sin\omega t + \omega_1^2(-A\omega\sin\omega t + B\omega\cos\omega t)\\&+\omega_0^2(A\cos\omega t + B\sin\omega t) = \frac{F}{I}\cos\omega t\end{aligned} \tag{17}$$

Combining like terms

$$\begin{aligned}&\cos\omega t(-A\omega^2 + B\omega\omega_1^2 + \omega_0^2 A)\\&+\sin\omega t(-B\omega^2 - A\omega\omega_1^2 + B\omega_0^2) = \frac{F}{I}\cos\omega t + 0\sin\omega t,\end{aligned} \tag{18}$$

allows us to equate the coefficients of Equation 18 in order to solve for A and B.

$$-A\omega^2 + B\omega\omega_1^2 + \omega_0^2 A = \frac{F}{I} \tag{19}$$

$$-B\omega^2 - A\omega\omega_1^2 + B\omega_0^2 = 0 \tag{20}$$

Cramer's rule can be used to solve for A,

$$A = \frac{\begin{vmatrix} F/I & \omega\omega_1^2 \\ 0 & \omega_0^2 - \omega^2 \end{vmatrix}}{\begin{vmatrix} \omega_0^2 - \omega^2 & \omega\omega_1^2 \\ -\omega\omega_1^2 & \omega_0^2 - \omega^2 \end{vmatrix}} = \frac{(\omega_0^2 - \omega^2)F/I}{(\omega_0^2 - \omega^2)^2 + \omega^2\omega_1^4},$$

and B,

$$B = \frac{\begin{vmatrix} \omega_0^2 - \omega^2 & F/I \\ -\omega\omega_1^2 & 0 \end{vmatrix}}{\begin{vmatrix} \omega_0^2 - \omega^2 & \omega\omega_1^2 \\ -\omega\omega_1^2 & \omega_0^2 - \omega^2 \end{vmatrix}} = \frac{(F/I)\omega\omega_1^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\omega_1^4}$$

To simplify things, let D equal the denominator of A and B.

$$D = (\omega_0^2 - \omega^2)^2 + \omega^2\omega_1^4 \tag{21}$$

Substituting A and B back into our assumed particular solution, Equation 14, yields,

$$\theta_p = \frac{(\omega_0^2 - \omega^2)}{D}\frac{F}{I}\cos\omega t + \frac{\omega\omega_1^2}{D}\frac{F}{I}\sin\omega t \tag{22}$$

Next we use the following trigonometric identity,

$$y\sin\theta + x\cos\theta = \sqrt{y^2 + x^2}\cos(\theta - \delta) \tag{23}$$

to simplify our particular solution. Between Equation 22 and Equation 23 we see the following correspondences.

$$\begin{aligned}
\theta &= \omega t \\
y &= \left(\frac{F}{I}\right)\frac{\omega\omega_1^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\omega_1^4} \\
x &= \left(\frac{F}{I}\right)\frac{\omega_0^2 - \omega^2}{\omega_0^2 - {\omega^2}^2 + \omega^2\omega_1^4}
\end{aligned}$$

Referring to Figure 2 we can see that the following are true,

$$\begin{aligned}
\delta &= \tan^{-1}\frac{y}{x} \\
R &= \sqrt{x^2 + y^2}
\end{aligned}$$

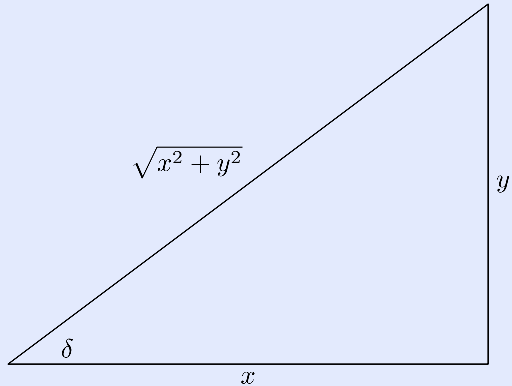

Figure 2: Using right triangle trigonometry.

Using the above conclusions we can rewrite our particular solution from Equation 22 in terms of cosine only,

$$\theta_p = R\cos(\omega t - \delta) \tag{24}$$

where R is equal to:

$$R = \sqrt{\left(\frac{F\omega\omega_1^2/I}{(\omega_0^2-\omega^2)^2+\omega^2\omega_1^4}\right)^2 + \left(\frac{F(\omega_0^2-\omega^2)/I}{(\omega_0^2-\omega^2)^2+\omega^2\omega_1^4}\right)^2} \tag{25}$$

We can rewrite this complex looking equation, by performing some simple algebra, not shown here.

$$R = \frac{F}{\sqrt{I^2(\omega_0^2-\omega^2)+I^2\omega^2\omega_1^4}} \tag{26}$$

Furthermore, we want to substitute $\mu\beta/I$ back in for ω_1^2, in order to have an equation containing the damping constant, β .

$$I^2\omega^2\omega_1^4 = I^2\omega^2\left(\frac{\mu\beta}{I}\right)^2 = \omega^2\mu^2\beta^2 \tag{27}$$

Next we want to simplify what is under the radical in Equation 26 by setting it equal to z,

$$z = I^2\omega_0^2 - \omega^{2^2} + \omega^2\mu^2\beta^2 \tag{28}$$

We now want to substitute z into Equation 26,

$$R = \frac{F}{\sqrt{z}} \tag{29}$$

When we substitute R into our particular solution, Equation 24, we get the following result,

$$\theta_p = \frac{F}{\sqrt{z}}\cos(\omega t - \delta) \tag{30}$$

This is a useful form of the particular solution because it allows for the experimental data to be used directly in the equation. The ratio in front of the cosine function in Equation 30 corresponds to the amplitude of the deflection of the compass needle.

Our analysis deals with oscillatory functions, and with any kind of wave motion, the oscillatory energy is directly proportional to the square of the amplitude of the motion. Thus the following is true,

$$E_{\mathrm{osc}} \propto \left(\frac{F}{\sqrt{z}}\right)^2 \tag{31}$$

Therefore, the maximum energy of the oscillations will correspond to a minimum value of z, due to the fact that F, the amplitude of the driving field, is fixed,

$$E_{\text{max}} \Leftrightarrow z_{\text{min}} \tag{32}$$

In order to get a minimum value of z, the derivative must be taken and set equal to zero,

$$\frac{dz}{d\omega} = 0 \tag{33}$$

Thus

$$\begin{aligned}
2I^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega\mu^2\beta^2 &= 0 \\
-4I^2\omega(\omega_0^2 - \omega^2) + 2\omega\mu^2\beta^2 &= 0 \\
-4I^2\omega\omega_0^2 + 4I^2\omega^3 + 2\omega\mu^2\beta^2 &= 0 \\
2\omega(-2I^2\omega_0^2 + 2I^2\omega^2 + \mu^2\beta^2) &= 0 \\
-2I^2(\omega_0^2 - \omega^2) + \mu^2\beta^2 &= 0 \\
2I^2(\omega^2 - \omega_0^2) + \mu^2\beta^2 &= 0 \\
2I^2(\omega^2 - \omega_0^2) + (\mu\beta)^2 &= 0
\end{aligned}$$

Solve the last equation for ω^2.

$$\omega^2 = \omega_0^2 - \frac{(\mu\beta)^2}{2I^2} \tag{34}$$

By definition, we know that the magnetic field due to the Helmholtz coils is given by

$$B = \frac{8\mu_0 N i}{\sqrt{125}\pi^2 r}, \tag{35}$$

where N is the number of turns of coil wire, and r is the radius of the coils. We know that the angular velocity is equal to

$$\omega = 2\pi f. \tag{36}$$

If we make the above substitution for ω in Equation 34, we get the following statement.

$$4\pi^2 f^2 = \omega_0^2 - \frac{\mu^2\beta^2}{2I^2} \tag{37}$$

Substituting $\frac{\mu B}{I}$ back in for ω_0^2 yields

$$4\pi^2 f^2 = \frac{\mu B}{I} - \frac{\mu^2\beta^2}{2I^2}. \tag{38}$$

We solve for the frequency squared, f^2.

$$f^2 = \frac{\mu B}{4\pi^2 I} - \frac{\mu^2\beta^2}{8\pi^2 I^2} \tag{39}$$

Using Equation 35 in place of B yields

$$f^2 = \frac{2\mu_0 N}{\sqrt{125}\pi^2 r}\left(\frac{\mu}{I}\right) i - \left(\frac{\mu}{I}\right)^2 \frac{\beta^2}{8\pi^2}. \tag{40}$$

3. The Experimental Data and Results

In this experiment we varied the current to the Helmholtz coils, thus changing the magnetic field. For each magnetic field, we found the frequency of the driving field of the inner coils, which corresponded to the maximum deflection of the compass needle. This condition is known as *resonance*. After performing the experiment, we were able to use our data to generate a graph of frequency squared versus current. This graph is shown in Figure 3. We can see that the relationship is linear, having form

$$f^2 = Ai - B, \tag{41}$$

where A and B correspond to the coefficients in Equation 40. We can conclude from our model that the slope of the graph, shown in Figure 3, is directly proportional to μ/I, the magnetic dipole moment-rotational inertia ratio, and the y-intercept is directly proportional to β, the damping constant. At this point we can go ahead and find the magnetic dipole moment-rotational inertia ratio

$$\frac{\mu}{I} = 3.66 \times 10^6 \mathrm{A/kg}, \tag{42}$$

and the damping constant

$$\beta = 2.03 \times 10^{-5} \mathrm{kg/C}. \tag{43}$$

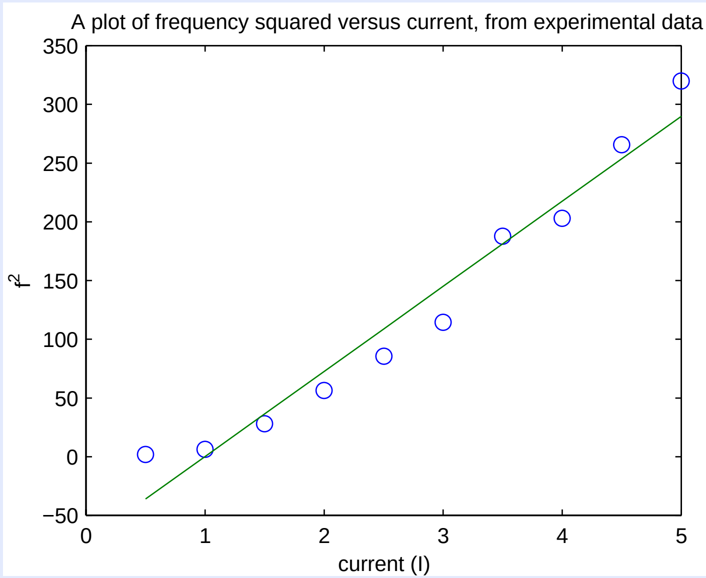

Figure 3: The linear relationship between frequency squared and current

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4. Visual Data and Behavior Analysis

The notable behavior among resonant solutions is that of maximum amplitude. Figure 4 shows a resonant solution to the second order differential equation, with current set to one ampere. As can be seen, the amplitude has a magnitude of about 8000 units. The compared non-resonant solution, seen in Figure 5, has a signficantly smaller amplitude of about 5000 units. This behavior holds true if the current supplied is increased, as Figure 6 and Figure 7 show.

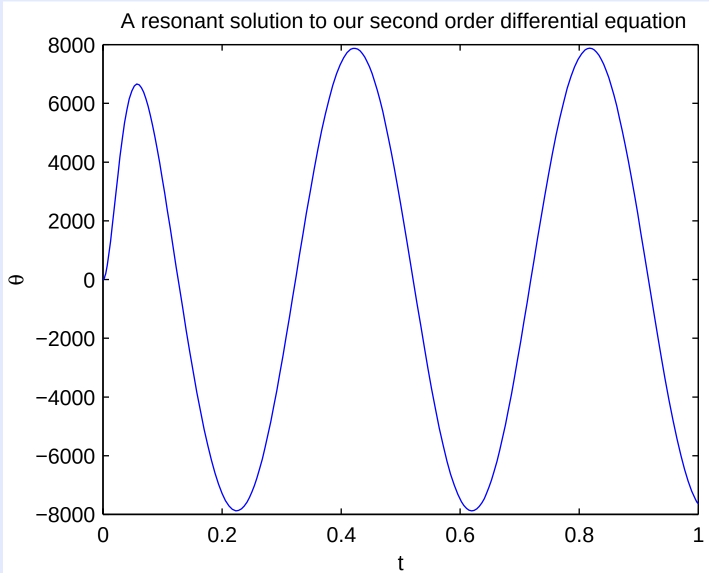

Figure 4: Current of 1 ampere, and frequency set at 2.53Hz

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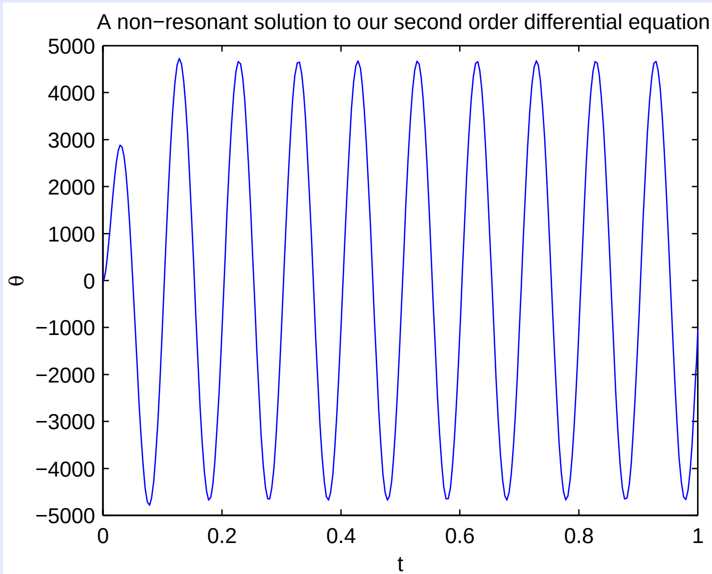

Figure 5: Current of 1 ampere, and frequency set at 10Hz

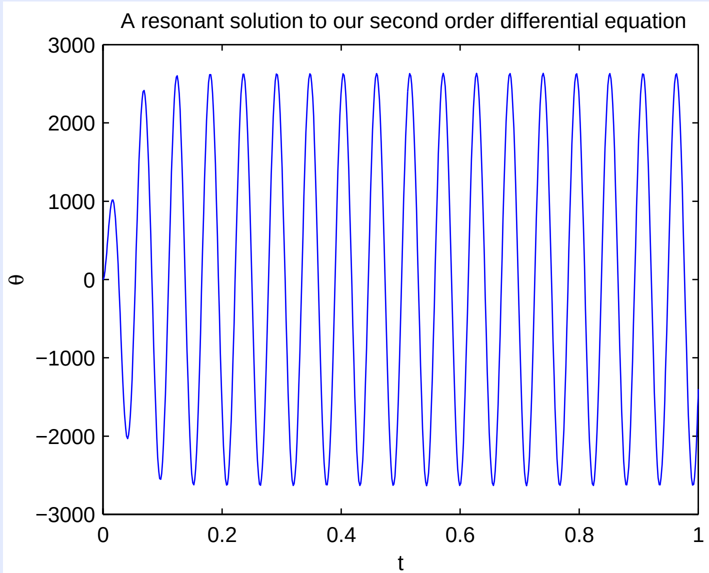

Figure 6: Current of 5 amperes, and frequency set at 17.88Hz

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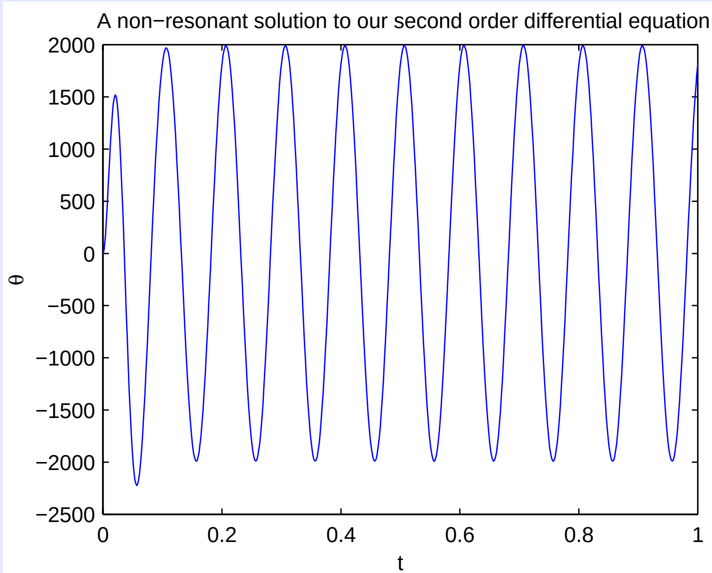

Figure 7: Current of 5 amperes, and frequency set at 10Hz

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5. Applications of Nuclear Magnetic Resonance

One of the many applications of the analysis described in this paper is that of Nuclear Magnetic Resonance Spectroscopy. This chemical analysis method operates on the same model as described. A chemical is injected into a NMR machine, which outputs data in the form of a graph, like the one seen in Figure 8. From this graph chemists can identify different compounds by analyzing the peaks and their numbers. On a molecular level, the nuclei of the unknown compound behave as if spinning about an axis. Since they are charged, they will act like tiny magnets and interact with an externally applied magnetic field. This is analogous to the compass needle trying to adjust itself to the magnetic field of the coils. If the nuclei are placed in a non-magnetic environment, their nuclear spins are oriented randomly in space. But, if there is a strong external magnetic field, they will orient themselves so that their own tiny magnetic field will align with the external field. If these nuclei are irradiated with electromagnetic radiation of proper frequency, energy absorption occurs and the nucleus is said to be in resonance with the applied radiation. This is similar to adjusting the frequency for each magnetic field in our analytical experiment in order to find the resonant condition. NMR spectroscopy is widely used in the chemistry field and aids in the identification of many organic and non-organic compounds.

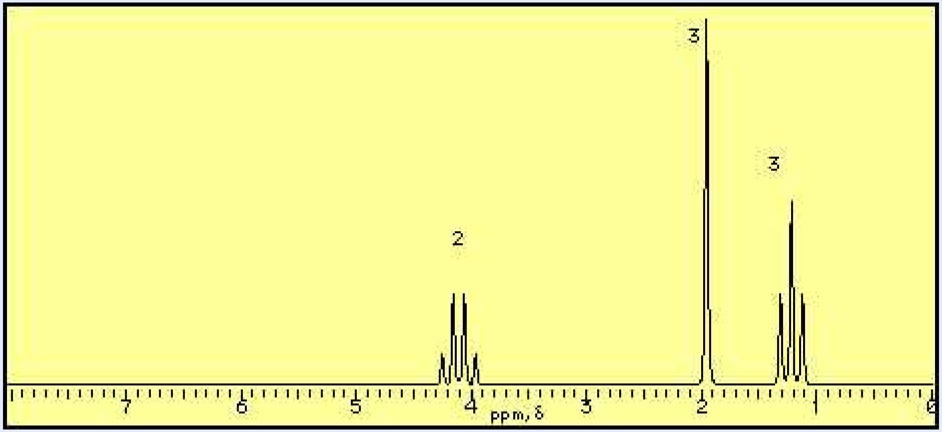

Figure 8: The NMR for ethyl acetate, $C_4H_8O_2$