Application 8.2
Automated Variation of Parameters

According to Eq. (28) in Section 8.2 of the text, the variation-of-parameters formula

$$\mathbf{x}(t) \;=\; e^{\mathbf{A}t}\left(\mathbf{x}_0 + \int_0^t e^{-\mathbf{A}s}\,\mathbf{f}(s)\,ds\right) \tag{1}$$

provides the solution to the nonhomogeneous initial value problem

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x} + \mathbf{f}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0\,. \tag{2}$$

The formula in (1) constitutes a mechanical algorithm that encourages the use of a computer algebra system. In the sections that follow we illustrate this approach by applying *Maple*, *Mathematica*, and MATLAB to derive the solution

$$\mathbf{x}(t) \;=\; \frac{1}{14}\begin{bmatrix} (6+28t-7t^2)e^{-2t} + 92\,e^{5t} \\ (-4+14t+21t^2)e^{-2t} + 46\,e^{5t} \end{bmatrix} \tag{3}$$

of the initial value problem

$$\mathbf{x}' \;=\; \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}\mathbf{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t\,e^{-2t}, \qquad \mathbf{x}(0) \;=\; \begin{bmatrix} 7 \\ 3 \end{bmatrix} \tag{4}$$

in Example 4 in the text. You can try this approach with Problems 17–34 in Section 8.2.

Using *Maple*

First we define the coefficient matrix

```
with(linalg):
A := matrix(2,2, [4,  2,
                  3, -1] ):
```

the initial vector

```
x0 := matrix(2,1, [7,
                   3]):
```

and the nonhomogeneous term

```
f := t -> matrix(2,1, [-15*t*exp(-2*t),
                        -4*t*exp(-2*t)]);
```

for the initial value problem in (4). The special matrix exponential function

```
expA := t-> exponential(A,t);
```

will simplify the notation a bit. We can now build up the solution in stages. First, the integrand matrix in the variation-of-parameters formula (1) is given by

```
integrand := evalm(expA(-s) &* f(s));
```

Next, *Maple*'s **map** function applies the integral function **int** element-wise to this matrix:

```
integral :=
simplify(map(int, integrand, s=0..t));
```

$$integral := \begin{bmatrix} -\frac{1}{2}t^2 + 2t\,e^{(-7t)} + \frac{2}{7}e^{(-7t)} - \frac{2}{7} \\ t\,e^{(-7t)} + \frac{1}{7}e^{(-7t)} + \frac{3}{2}t^2 - \frac{1}{7} \end{bmatrix}$$

Finally, the variation-of-parameters solution in (1) takes the form

```
solution :=
simplify(evalm(expA(t) &* (x0 + integral)));
```

$$solution := \begin{bmatrix} \frac{3}{7}e^{(-2t)} - \frac{1}{2}e^{(-2t)}t^2 + \frac{46}{7}e^{(5t)} + 2\,e^{(-2t)}\,t \\ \frac{23}{7}e^{(5t)} + e^{(-2t)}\,t - \frac{2}{7}e^{(-2t)} + \frac{3}{2}e^{(-2t)}\,t^2 \end{bmatrix}$$

which evidently is equivalent to the solution in Eq. (3) that was found manually in the text.

Using *Mathematica*

First we define the coefficient matrix

```
A = {{4,  2},
     {3, -1}};
```

the initial vector

```
x0 = {{7},
      {3}};
```

and the nonhomogeneous term

```
f[t_] := {{(-15*t)/E^(2*t)},
          {(-4*t)/E^(2*t)}}
```

for the initial value problem in (4). The special matrix exponential function

```
exp[A_] := MatrixExp[A]
```

will simplify the notation a bit. We can now build up the solution in stages. First, the integral matrix in the variation-of-parameters formula (1) is given by

```
integral =
Integrate[exp[-A*s] . f[s], {s, 0, t}] // Simplify
```

$$\begin{pmatrix} -\dfrac{t^2}{2} + 2\,e^{-7t}\,t + \dfrac{2}{7}\left(-1 + e^{-7t}\right) \\ \dfrac{3t^2}{2} + e^{-7t}\,t + \dfrac{1}{7}\left(-1 + e^{-7t}\right) \end{pmatrix}$$

Then the variation-of-parameters solution in (1) takes the form

```
solution :=
simplify(evalm(expA(t) &* (x0 + integral)));
```

$$\begin{pmatrix} \frac{1}{14}e^{-2t}\left(-7t^2 + 28t + 92\,e^{7t} + 6\right) \\ \frac{1}{14}e^{-2t}\left(21t^2 + 14t + 46\,e^{7t} - 4\right) \end{pmatrix}$$

which evidently is equivalent to the solution in Eq. (3) that was found manually in the text.

Using MATLAB

First we define the coefficient matrix

```
A = [4   2
     3  -1]
```

the initial vector

```
x0 = [7
      3]
```

and the nonhomogeneous function

```
syms s t
f = s*exp(-2*s)*[-15; -4]
f =
[ -15*s*exp(-2*s)]
[  -4*s*exp(-2*s)]
```

(as a function of *s*) for the initial value problem in (4). We can now build up the solution in stages. First, the integrand matrix in the variation-of-parameters formula (1) is given by

```
integrand = expm(-A*s)*f;
```

Next, the integral in (1) is given by

```
integral = int(integrand, s, 0,t);
```

Finally, the variation-of-parameters solution in (1) takes the form

```
solution = expm(A*t)*(x0 + integral);
solution = simple(solution)
solution =
[3/7/exp(t)^2-1/2/exp(t)^2*t^2+46/7*exp(t)^5+2/exp(t)^2*t]
[23/7*exp(t)^5+1/exp(t)^2*t-2/7/exp(t)^2+3/2/exp(t)^2*t^2]
```

When we pretty-print this solution,

```
pretty(solution)

[                       2                               ]
[        1             t                    5         t   ]
[3/7 ------- - 1/2 ------- + 46/7 exp(t)   + 2 -------]
[           2              2                            2]
[     exp(t)         exp(t)                       exp(t) ]
[                                                        ]
[                                                   2    ]
[             5       t              1             t     ]
[ 23/7 exp(t)   + ------- - 2/7 ------- + 3/2 ------- ]
[                        2              2             2 ]
[                  exp(t)         exp(t)        exp(t)  ]
```

we see that it is equivalent to the solution in Eq. (3) that was found manually in the text.