# Max–Min and Saddles

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Fall, 1997

**Abstract**

In this activity you will use the *Matlab* software to find maximum, minimum, and saddle points of a function.

**Prerequisites.** Partial differentiation. Familiarity with Matlab's **meshgrid**, **mesh**, **contour**, and its element wise operators (**.*,./,.^**). The Symbolic Toolbox is used as an aid in finding critical points, but readers can complete the activity without it.

## 1 The Problem

Define $f : R^2 \to R$ by $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$. Note that this function, being a polynomial, is continuous on all of $R^2$.

### 1.1 The Critical Points

You must find all points $(x, y)$ that make both first partials equal to zero.

$$\begin{aligned} f_x(x,y) &= 3x^2 + 6x \\ f_y(x,y) &= 3y^2 - 6y \end{aligned}$$

or, equivalently,

$$\begin{aligned} f_x(x,y) &= 3x(x+2) \\ f_y(x,y) &= 3y(y-2) \end{aligned}$$

If you examine each of these partials, you will note that both $f_x(x,y) = 0$ and $f_y(x,y) = 0$ for each of the following *critical points*: $\{(0,0),(0,2),(-2,0),(-2,2)\}$.

If you have the *Symbolic Toolbox* installed on your system, then you can let *Matlab* do this work for you.

```
>> syms x y
>> f=x^3+y^3+3*x^2-3*y^2-8

f =

x^3+y^3+3*x^2-3*y^2-8
```

Taking the partial derivatives is easy with the *Symbolic Toolbox*[1].

```
>> fx=diff(f,x)

fx =

3*x^2+6*x

>> fy=diff(f,y)

fy =

3*y^2-6*y
```

You can use the `solve` command from the *Symbolic Toolbox* to find where these first partials are simultaneously equal to zero[2]. The following command returns the solution of the system

$$\begin{aligned} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{aligned}$$

in a structure variable `S`.

```
>> S=solve(fx,fy)

S =

    x: [4x1 sym]
    y: [4x1 sym]
```

To examine the fields of `S`, enter the command

```
>> [S.x,S.y]

ans =

[  0,  0]
[ -2,  0]
[  0,  2]
[ -2,  2]
```

This last result indicates that each point in the set $\{(0,0), (0,2), (-2,0), (-2,2)\}$ is a critical point of the function $f$.

[1] If you are using Matlab 4, enter the following commands:

```
>> f='x^3+y^3+3*x^2-3*y^2-8'
>> fx=diff(f,'x')
>> fy=diff(f,'y')
```

[2] Most solvers interpret `solve(fx)` to mean `solve(fx=0)`.

### 1.2 The Second Derivative Test

The following table summarizes how to apply the second derivative test to each critical point.

| $f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$ | $f_{xx}(a,b)$ | Classification |
|---|---|---|
| negative | | Saddle point |
| positive | positive | Local minimum |
| positive | negative | Local maximum |
| zero | | Test fails |

To apply the test, first calculate each second partial of the function $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

$$\begin{aligned} f_{xx}(x,y) &= 6x+6 \\ f_{yy}(x,y) &= 6y-6 \\ f_{xy}(x,y) &= 0 \end{aligned}$$

Now set up the second derivative test at each critical point in tabular form.

| $(x,y)$ | $f_{xx}(x,y)$ | $f_{xx}(x,y)f_{yy}(x,y) - f_{xy}^2(x,y)$ | Classification |
|---|---|---|---|
| $(0,0)$ | 6 | $(6)(-6) - (0)^2 = -36$ | Saddle |
| $(-2,0)$ | $-6$ | $(-6)(-6) - (0)^2 = 36$ | Local max |
| $(-2,2)$ | $-6$ | $(-6)(6) - (0)^2 = -36$ | Saddle |
| $(0,2)$ | 6 | $(6)(6) - (0)^2 = 36$ | Local min |

### 1.3 Visualizing the Results

*Matlab* provides two powerful ways of visualizing your results: the mesh surface and the contour map. You've seen these in previous activities but their use in examining extrema is particularly powerful. The following *Matlab* commands should produce an image similar to that shown in Figure 1.

```
>> [x,y]=meshgrid(-3:.2:3);
>> z=x.^3+y.^3+3*x.^2-3*y.^2-8;
>> mesh(x,y,z)
>> xlabel('x-axis')
>> ylabel('y-axis')
```

If you examine the image in Figure 1 closely, you can see that the function has a local minimum and local maximum at $(0,2)$ and $(-2,0)$, respectively. You can also see saddle points at $(0,0)$ and $(-2,2)$. Note that none of the extrema in Figure 1 are obvious. You really have to carefully inspect the surface before they come to view. It is for this reason that we sometimes find the contour map much more helpful in identifying extrema. The following *Matlab* commands should produce an image similar to that in Figure 2.

```
>> contour(x,y,z)
```

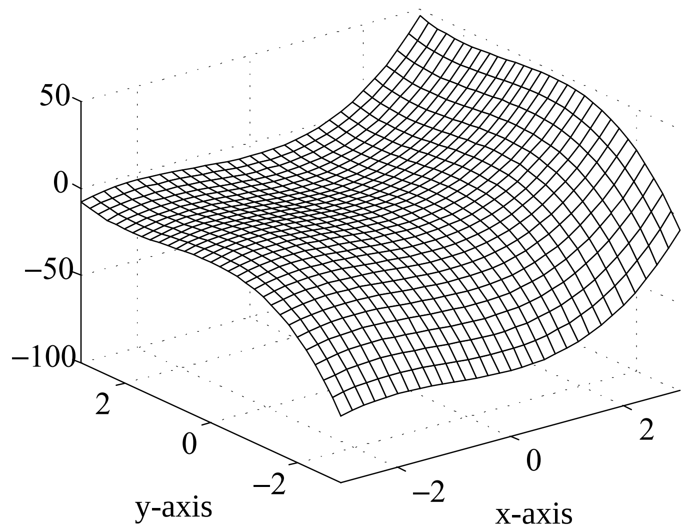


Figure 1: The surface defined by $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

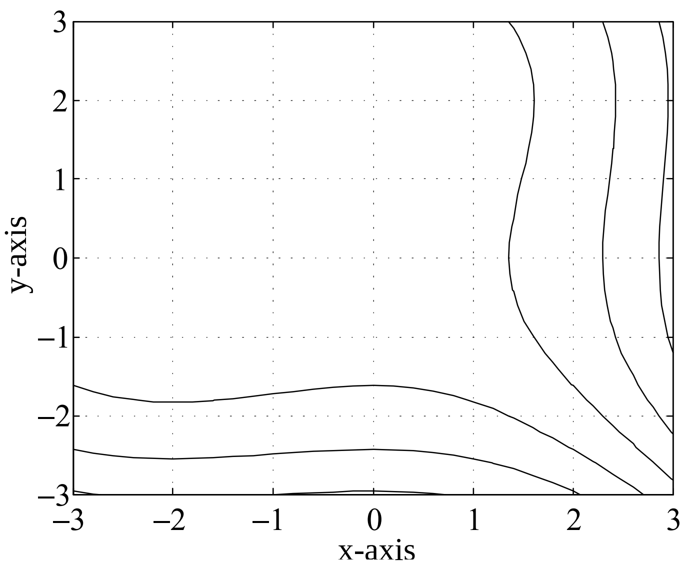


Figure 2: Level curves of $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

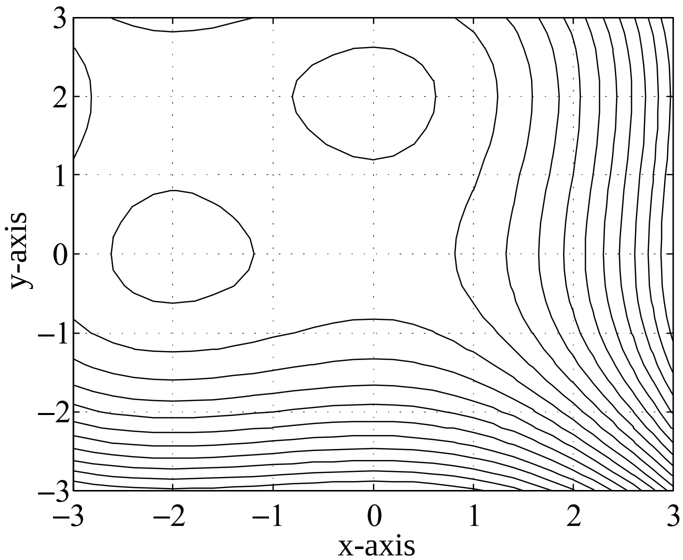


Figure 3: Additional level curves of $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$.

The image in Figure 2 is not particularly helpful. More level curves are needed if we are to classify the extrema at all critical points. Recall, however, that it is easy to draw more than the default number of level curves[3]. The following *Matlab* commands should produce an image similar to that in Figure 3.

```
>> contour(x,y,z,20)
```

If you think of the contour map in Figure 3 as a topographical map of a landscape, then the circular regions in the upper left hand corner could indicate valleys or hilltops. Note that these seem to be centered at the critical points $(-2, 0)$ and $(0, 2)$.

You can easily request *Matlab* to draw level curves at selected heights. After some experimentation, we found that the following *Matlab* commands produce an image similar to that in Figure 4.[4]

```
>> [c,h]=contour(x,y,z,-14:-4);
```

[3]Type `help contour` at the *Matlab* prompt for more information on *Matlab's* `contour` command.

[4]Recall that the command `-14:-4` produces the vector `[-14,-13,-12,...,-4]`. Therefore, the command `contour(x,y,z,-14:-4)` produces level curves at the heights $-14, -13, -12, \ldots, -4$. Also, you might want to try `clabel(c,h,'manual')` to get a better hand-eye feel for the elevations of the contours.

If you are using Matlab 4, you will need to use the following commands:

```
>>c=contour(x,y,z,-14:4);
>>clabel(c)
```

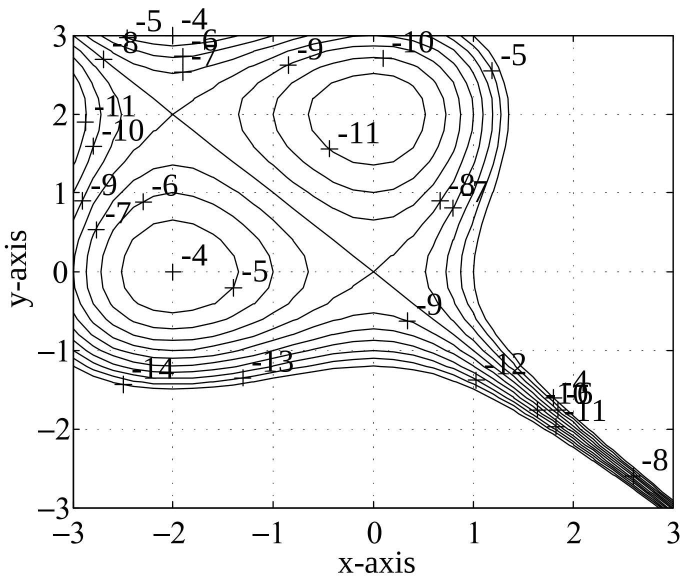


Figure 4: Level curves of $f$ at heights $-14, -13, -12, \ldots, -4$.

```
>> clabel(c,h)
```

We can now make the following observations:

- As you approach the point $(-2, 0)$ from any direction, note that you are walking uphill, crossing contours at heights $-7, -6, -5$, and $-4$. Consequently, there is a local maximum at $(-2, 0)$.
- As you approach the point $(0, 2)$ from any direction, note that you are walking downhill, crossing contours at heights $-9, -10$, and $-11$. Consequently, there is a local minimum at $(0, 2)$.
- If you are at the point $(0, 0)$ and you proceed north or south, the heights of the contours are decreasing. If you proceed east or west, the heights of the contours are increasing. Consequently, a saddle point exists at $(0, 0)$.
- If you are at the point $(-2, 2)$ and you proceed north or south, the heights of the contours are increasing. If you proceed east or west, the heights of the contours are decreasing. Consequently, a saddle point exists at $(-2, 2)$.

### 1.4 Homework

Define $f : R^2 \to R$ by $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$. Do a **complete** analysis of the extrema of this function by using each and every analytical and *Matlab* technique presented in this example. Include printouts of the mesh and the contours with your presentation.