RLC Circuit

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Abstract

In my differential equation project I will explore the behavior of a RLC circuit. The second order differential equation that models the circuit will exhibit different behaviors depending on the chosen values for the parameters.

1 Introduction

RLC circuits can be very complex and can be modeled in many various ways. The RLC circuit that will be discussed is a simpler example in which a battery is allowed to charge up a capacitor and then is removed from the circuit. The equation is written in terms of v (voltage) and and it's behavior is determined by it's parameters R,L and C. No wonder they call it a RLC circuit! The circuit it modeled by the following homogenous differential equation

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0 \quad v(0) = v_o \quad v'(0) = w_o$$

where R is the resistance (in ohms), L the inductance (in henries), and C the capacitance (in farads).

As with most parameters, R,L,and C are all positive coefficients. Further simplification of the differential equation leads to the following and more familiar form.

$$L\frac{d^2v}{dt^2} + R\frac{dv}{dt} + \frac{1}{C}v = 0$$

Since the equation is second order it must be written a system of equations.

$$\begin{aligned}\frac{dv}{dt} &= w \\ \frac{dw}{dt} &= -\frac{1}{LC}v - \frac{R}{L}w\end{aligned}$$

In matrix form.

$$\frac{dY}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} Y, \qquad \text{where } Y = \begin{bmatrix} v \\ w \end{bmatrix} \tag{1}$$

Solving this system of equations will produce solutions $v(t)$ and $w(t)$. However, in this case, the only solution of interest is the $v(t)$. The $v(t)$ solution will determine the voltage of the circuit as a function of time, which is of particular interest in the study of a RLC circuit. The $w(t)$ solution determines the velocity of the voltage as a function of time. Although somewhat interesting, the velocity of voltage does not give any real information about the behavior of the circuit.

1.1 Circuit Are Like Springs?

Believe it or not, but the behaviors of a RLC circuit are much like that of a linear spring model. Both have three parameters, $R, L,$and C for the circuit and $m, k_d,$and k_s for the spring, and both display similar behavior for their homogenous differential equations. Also, the parameters of both equations came be compared very closely in how they effect the behavior.

$$L\frac{d^2v}{dt^2} + R\frac{dv}{dt} + \frac{1}{C}v = 0 \tag{1}$$

$$m\frac{d^2y}{dt^2} + k_d\frac{dy}{dt} + k_s y = 0 \tag{2}$$

Circuit Element	Spring Element
Inductance L	Mass m
Resistance R	Damping coefficient k_d
Capacitance C	Reciprocal of spring constant k_s
Voltage v	Displacement y

1.2 Finding the Solution of the Homogenous Equation

Finding the solution to a homogenous second order differential equation can be made very simple by using The Method of Undetermined Coefficients: a.k.a the "lucky guess method". The general solution can then be interpreted to classify the overall behavior of the system.

$$\begin{aligned}
L\frac{d^2v}{dt^2} + R\frac{dv}{dt} + \frac{1}{C}v &= 0 \\
Let\ v &= e^{\lambda t} \\
L\lambda^2 e^{\lambda t} + R\lambda e^{\lambda t} + \frac{1}{C}e^{\lambda t} &= 0 \\
L\lambda^2 + R\lambda + \frac{1}{C} &= 0 \\
\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} &= 0
\end{aligned}$$

Now having the form $ax^2 + bx + c$,the eigenvalues are easily found by using the quadratic formula.

$$\lambda = \frac{\frac{-R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \left(\frac{4}{LC}\right)}}{2}$$

Which simplifies to

$$\lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} \tag{2}$$

This solution for lamda now allows for some generalizing of the solutions to the original differential equation.(1)

The solution depends on whether the determinate of the characteristic polynomial $\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)$ is positive, negative, or zero. These three cases can be summarized in the following manner.

1.2.1 Case 1: $\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) > 0$

In this case, both eigenvalues will be real negative coefficients and the equilibrium point will be a sink. This case is know as overdamped. Hey, that sounds like the linear spring model !?

1.2.2 Case 2: $\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) < 0$

In this case the determinate is less than zero, therefore producing complex eigenvalues having the form $a + bi$. Recall in equation 3a that the a term is $\frac{-R}{2L}$, therefore the equilibrium point will be a spiral sink. This implies that the voltage will oscillate as it decays to zero. Oscillating until reaching an equilibrium point, this sure sounds like a spring to me.

1.2.3 Case 3: $\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) = 0$

Having the determinate equal zero produces the degenerate situation in which the solutions tries to oscillate but can't. This system is considered critically damped and is characterized by the existence of repeating eignenvalues.

1.3 Demonstrating Behavior

As stated in the previous example, only three different cases can occur depending on the values of R, L, and C. To demonstrate the behavior of these cases, it is best to first choose values for two of the parameters and adjust values of the third. This will simplify the work and avoid the time consuming process of selecting parameter values that demonstrate a specific behavior. For all three cases, $L = .003$ and $C = 1 \times 10^{-10}$. R will be determined at a later time.

1.3.1 Case 1

$$\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) > 0 \tag{3}$$

By using the values for L and C, R can now be determined. Solving for R will produce a range of values for which the circuit will remain overdamped.

$$\begin{aligned} \left(\frac{R}{2L}\right)^2 &> \left(\frac{1}{LC}\right) \\ \left|\left(\frac{R}{2L}\right)\right| &> \sqrt{\left(\frac{1}{LC}\right)} \end{aligned}$$

Since R, L,and C are positive, the absolute value can be eliminated and R can be solved for.

$$R > 2L\left(\frac{1}{LC}\right) \tag{4}$$

Substituting in L and C

$$\begin{aligned} R &> 2(.003)\left(\sqrt{\frac{1}{(.003)(1 \times 10^{-10})}}\right) \\ R &> 10954.45 \end{aligned}$$

This means that as long as the resistance (R) is greater than 10954.45 the circuit remains overdamped.

A value of R can now be chosen and the general solution can be found. A value of 11000 will be chosen for R Also, at this point, initial conditions will need to be chosen.. For all three cases $v(0) = 3 \quad v'(0) = 0$. This means that the initial voltage will be 3 and the initial velocity of the voltage will be 0.

$$\lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

$$\begin{aligned} \lambda_1 &= -1.6667 \times 10^6 \\ \lambda_1 &= -2 \times 10^6 \end{aligned}$$

Which produces the general solution

$$Y(t) = C_1 e^{(-1.6667 \times 10^6)t} + C_1 e^{(-2 \times 10^6)t} \tag{5}$$

Using the initial conditions, the values for C_1 and C_2 can be determined and the exact solution can be found. For this, the matrix form (1) serves as an easy way for finding C_1 and C_2.

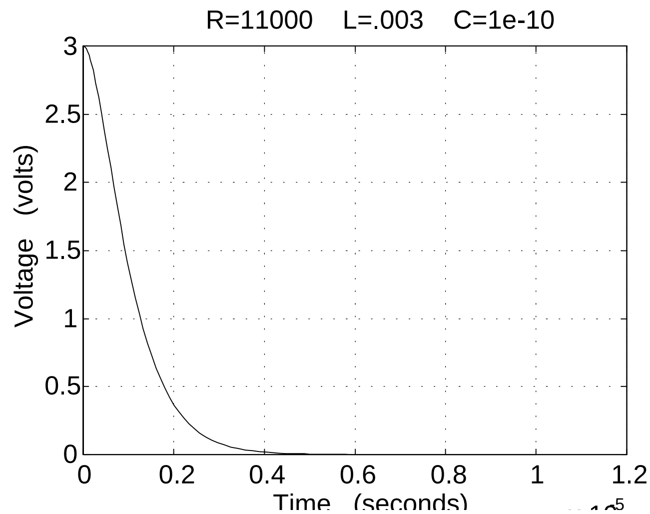

Figure 1:

$$
\begin{aligned}
Y(t) &= C_1 e^{(-1.6667\times 10^6)t} + C_1 e^{(-2\times 10^6)t} \\
Y'(t) &= (-1.6667\times 10^6) C_1 e^{(-1.6667\times 10^6)t} + (-2\times 10^6) C_1 e^{(-2\times 10^6)t}
\end{aligned}
$$

$$
\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1.6667\times 10^6 & -2\times 10^6 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}
$$

Set up an augmented matrix.

$$
\begin{bmatrix} 1 & 1 & 3 \\ -1.6667\times 10^6 & -2\times 10^6 & 0 \end{bmatrix} \tag{6}
$$

Row reducing the augmented matrix will solve for C_1 and C_2

$$
\begin{matrix} C_1 \\ C_2 \end{matrix} = \begin{matrix} 1.6363 \\ 1.3677 \end{matrix}
$$

$$
Y(t) = 1.6363 e^{(-1.6667\times 10^6)t} + 1.3677 e^{(-2\times 10^6)t} \tag{7}
$$

Figure 1.,the graph of $Y(t)$, will show that as time passes the voltage of the circuit will decay without oscillation. Figure 2.,the phase portrait, also demonstrates that the circuit is overdamped and the voltage is not oscillating..

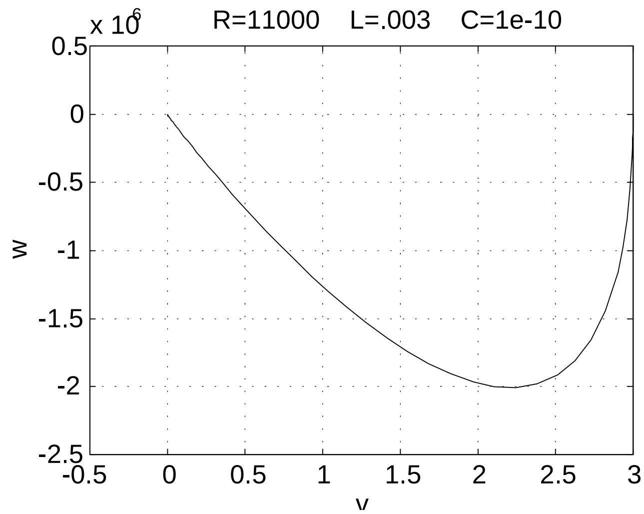

Figure 2:

1.3.2 Case2

$$\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) < 0$$

Once again $L = .003$ and $C = 1 \times 10^{-10}$and R will be determined at a later time.

$$\begin{aligned}\left(\frac{R}{2L}\right)^2 &< \left(\frac{1}{LC}\right)\\ \left|\left(\frac{R}{2L}\right)\right| &< \sqrt{\left(\frac{1}{LC}\right)}\end{aligned}$$

Again, because $R, L,$and C are positive, the absolute value can be eliminated and R can be solved for.

$$R < 2L\left(\frac{1}{LC}\right)$$

As in the first case, solving for R will produce a range of values for which the circuit will remain underdamped.

$$R < 10954.45$$

This means that as long as the resistance (R) is less than 10954.45 the circuit remains underdamped.

A value of R can now be chosen and the general solution can be found. A value of 300 will be chosen for R and $v(0) = 3 \quad v'(0) = 0$.

$$\lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

$$\begin{aligned} \lambda_1 &= -5 \times 10^4 + i1.825 \\ \lambda_1 &= -5 \times 10^4 - i1.825 \end{aligned}$$

Which gives the general solution

$$Y(t) = e^{(-5\times10^4 + i1.825)t}$$

Because of the imaginary eigenvalue, more simplification will be needed.

$$\begin{aligned} Y(t) &= e^{(-5\times10^4 + i1.825)t} \\ Y(t) &= e^{-5\times10^4\ t} e^{i1.825t} \\ Y(t) &= C_1 e^{-5\times10^4\ t}\cos(1.825t) + C_2 e^{-5\times10^4\ t}\sin(1.825t) \end{aligned}$$

Using the matrix form (1)

$$\begin{aligned} Y(t) &= C_1 e^{-5\times10^4\ t}\cos(1.825t) + C_2 e^{-5\times10^4\ t}\sin(1.825t) \\ Y'(t) &= C_1\left[(-5\times10^4)e^{(-5\times104)t}\cos(1.825t) - 1.825e^{(-5\times10^4)t}\sin(1.825t)\right] \\ &\quad + C_2\left[(-5\times10^4)e^{(-5\times104)t}\sin(1.825t) + 1.825e^{(-5\times10^4)t}\cos(1.825t)\right] \end{aligned}$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (-5\times10^4) & 1.825 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Set up an augmented matrix.

$$\begin{bmatrix} 1 & 0 & 3 \\ -5\times10^4 & 1.825 & 0 \end{bmatrix} \tag{8}$$

Row reducing the augmented matrix will solve for C_1 and C_2.

$$\begin{matrix} C_1 \\ C_2 \end{matrix} = \begin{matrix} 3 \\ 82192 \end{matrix}$$

$$Y(t) = 3e^{-5\times10^4\ t}\cos(1.825t) + 82192e^{-5\times10^4\ t}\sin(1.825t) \tag{9}$$

This time figure 3.will show that as time passes the voltage of the circuit will decay with oscillation. Notice the very small amount of time it takes for these oscillations to occur. Also notice the similarities to that of the linear spring model.Here figure. 4, the phase portrait, demonstrates that the circuit is underdamped and the voltage is oscillating..

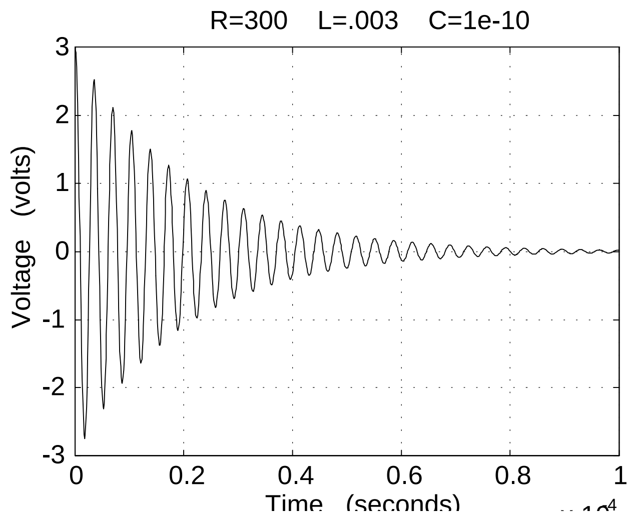

Figure 3:

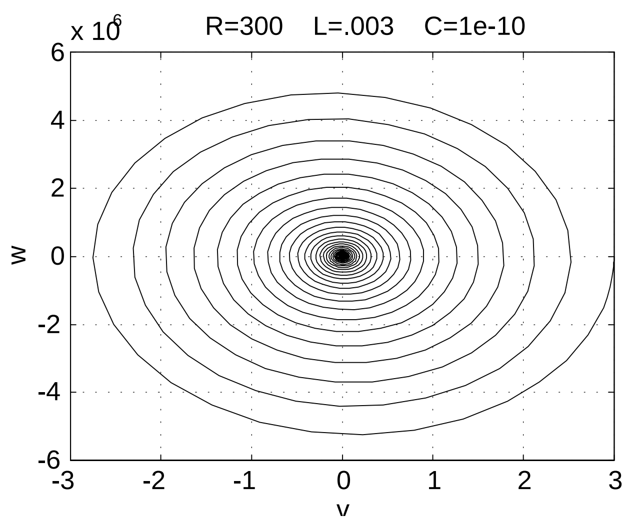

Figure 4:

1.3.3 Case 3

$$\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) = 0 \tag{10}$$

From the previous cases, it can be seen that when $R = 10954.45$ the above equation will be zero and there will be a repeating eigenvalue.

$$\begin{aligned}
\lambda &= -\frac{R}{2L} \pm 0 \\
\lambda_1 &= 1.8257 \times 10^7 \\
\lambda_1 &= 1.8257 \times 10^7
\end{aligned}$$

Repeating eigenvalues will cause the degenerate case to occur in which the circuit will be critically damped. The general solution is

$$Y(t) = C_1 e^{\ 1.8257\times10^7 t} + C_2 t e^{1.8257\times10^7\ t} \tag{11}$$

Using the matrix form (1)

$$\begin{aligned}
Y(t) &= C_1 e^{\ 1.8257\times10^7 t} + C_2 t e^{1.8257\times10^7\ t} \\
Y'(t) &= (1.8257\times10^7) C_1 e^{(1.8257\times10^7)t} + C_2 \left[e^{(1.8257\times10^7)t} + (1.8257\times10^7) t e^{(1.8257\times10^7)t}\right]
\end{aligned}$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (1.8257\times10^7) & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

Set up an augmented matrix.

$$\begin{bmatrix} 1 & 0 & 3 \\ (1.8257\times10^7) & 1 & 0 \end{bmatrix} \tag{12}$$

Again, row reducing the augmented will solve for C_1 and C_2.

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5.4771\times10^7 \end{bmatrix}$$

$$Y(t) = 3e^{(1.8257\times10^7)t} + (5.4771\times10^7) t e^{(1.8257\times10^7)t} \tag{13}$$

Figure 5.shows that as time passes the voltage of the circuit will decay to zero, but notice that it takes longer than the over damped case. This is because the circuit is critically damped and the voltage is trying to oscillate.

Figure 6., the phase portrait, demonstrates how the voltage is trying to oscillate but can't. This is indicative of a critically damped system.

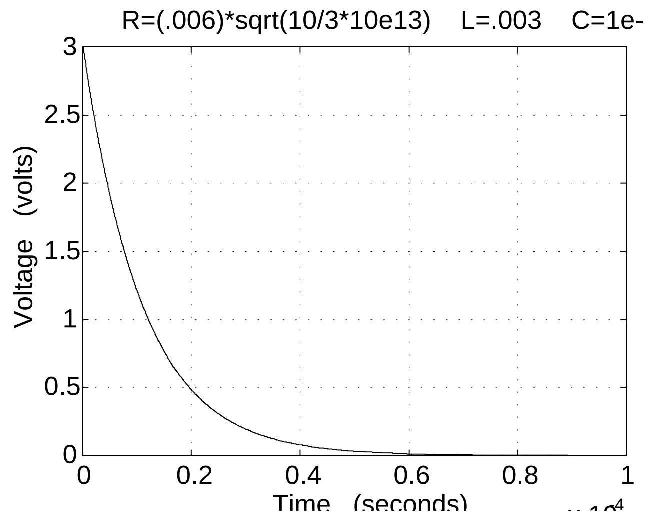

Figure 5:

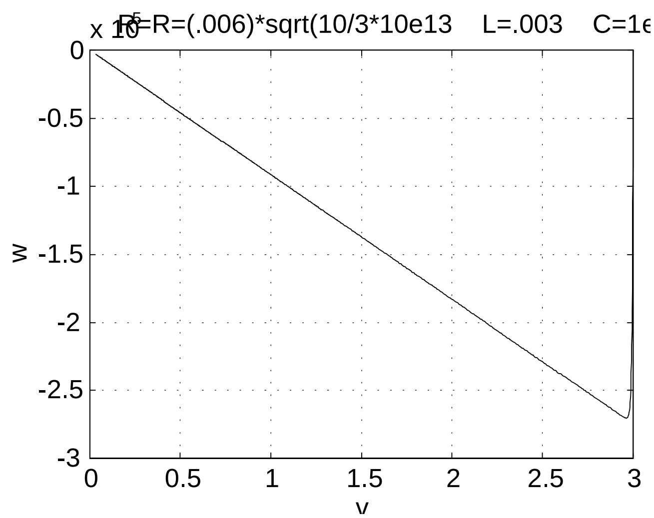

Figure 6:

1.4 Conclusion

As one can see, an RLC circuit can behave in different ways depending on the parameters R, L, and C.Even slight changes can result in completely different behaviors. Using this model and applying some basic differential equation mathematics, any of the three behaviors can induced. By carefully selecting R, L, and C value, the circuit is at your mercy!