A Bead on a Rotating Hoop

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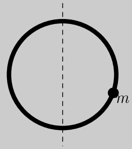
May 16, 2008

Abstract

This paper models a bead moving along a rotating hoop using differential equations. We show how to convert from a second order differential equation to a system of first order differential equations, then graphically analyze the equations without friction and compare it to a model *with* friction. We'll also show how to scale the variables to obtain a simplified dimensionless equation.

1. Introduction and Background

Typically used in first year physics, this model shows an example of a bifurcation in a mechanical system. The model closely resembles a swinging pendulum. The main difference is an additional motion, the rotation of the hoop, where the bead's path along the hoop accounts for the pendulum motion. Refer to figure 1 to help visualize the hoop in 3-space spinning at a constant angular velocity about its vertical axis. The behavior of the bead will vary as it travels along the hoop, the dependent factor being the hoop's angular velocity. As the angular velocity increases, the bead's equilibrium points move up the sides of the hoop. The equation we'll use is autonomous. First, we'll give an example without friction to give a good foundation to the problem, then an example with friction.



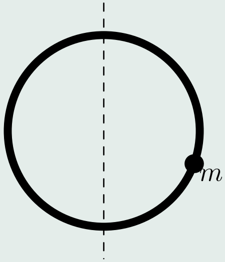


Figure 1: Hoop Diagram

2. Frictionless Bead

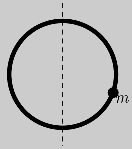
This example represents a conservative system where no energy is dissipated. Here, the bead is forced along a wire hoop by gravitational and centrifugal forces, without friction. The hoop rotates at a constant angular velocity about the vertical axis.

By convention, g equals the gravitational constant near the surface of earth, 9.8 m/s, m, the mass of the bead, r, the distance of the bead from the it's rotational axis, ω, the (constant) angular velocity about the vertical axis, and we'll set ϕ equal to the angle between the bead and the downward vertical direction. In this problem we restrict ϕ to $-\pi < \phi < \pi$. First we'll consider the downward gravitational force mg.

To arrive at the equation we need, we will consider the motion of the bead, using Newton's second law:

$$\sum F = ma \tag{1}$$

Here, we'll define the forces, and obtain the acceleration of the bead. To find the acceleration, recall from previous classes that $\phi = s/r$, where ϕ is the angle created between vertical and the mass, s the arclength, and r, the distance of the mass from it's rotational axis (radius). We solve for s and get $s = r\phi$. Differentiating this we get $ds/dt = rd\phi/dt$, the velocity of the mass. Differentiating this to get acceleration, we find that a is equal to $rd^2\phi/dt^2$. This is also known



Introduction and . . .
Frictionless Bead
Frictionless Examples
Friction Added to the . . .
Friction Examples
Supercritical Pitchfork . . .

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 2 of 16

Go Back

Full Screen

Close

Quit

as the tangential acceleration. Substituting this into the equation we arrive at:

$$\sum F = mr\frac{d^2\phi}{dt^2} \tag{2}$$

Next we define the forces. Referring to Figure 2, we can determine these forces. We know that one is gravity and the other is the centrifugal force, a 'fictitious' force. These components define the motion of the bead To understand why this force is called a fictitious force, think of the last time you went around a corner in your car. You had to use your hands to hold on to the steering wheel as you entered the corner because of Newton's second law, the tendency of you to keep moving in a straight line. The car turned, so you had to grab the steering wheel to turn with the car. In essence, the 'fictitious' force is the bead's inability to "grab the wheel". Back to defining the forces. The gravitational force is mg and the sideways centrifugal force is $m\rho\omega^2$. Breaking down the vector mg into two perpendicular vectors components, we get that the vectors $mg\sin\phi + mg\cos\phi$ are equal to the vector mg. The second force, the $m\rho\omega^2$ vector, is broken down into two perpendicular vector components where the vector $m\rho\omega^2$ is equal to the vectors $m\rho\omega^2\sin\phi + m\rho\omega^2\cos\phi$. These forces are perpendicular to the tangent line at any point. Substituting these forces into the equation we arrive at:

$$-mg\sin\phi + m\rho\omega^2\cos\phi = mr\frac{d^2\phi}{dt^2} \tag{3}$$

Referring to figure 3, we let ρ equal $r\sin\phi$.

$$-mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{d^2\phi}{dt^2} \tag{4}$$

Next we'll make the problem dimensionless by introducing a new variable called τ. We'll set τ equal to t/T, where t is in units of time, and T is a variable to be chosen later. Taking the derivative of τ with respect to time we get $d\tau/dt = 1/T$.

Taking the derivative of ϕ with respect to time we get

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{d\phi}{d\tau}\frac{1}{T} \tag{5}$$

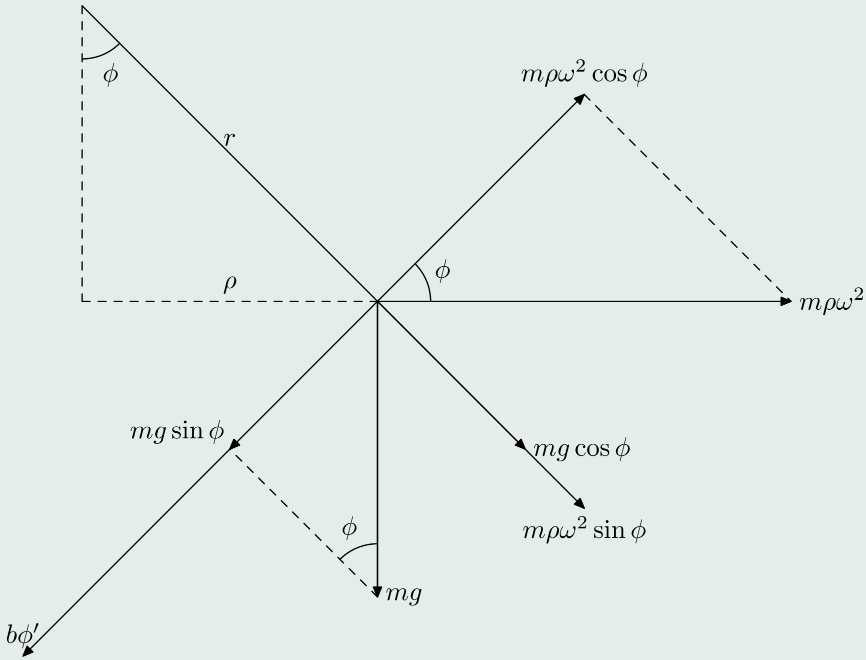


Figure 2: Forces

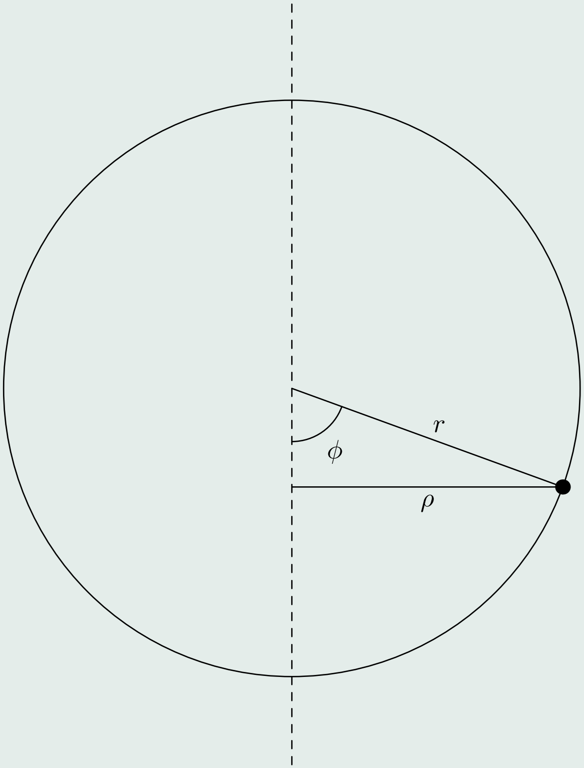


Figure 3: $\rho = r \sin\phi$

Then, take the second derivative of theta with respect to time

$$\frac{d^2\phi}{dt^2} = \frac{d}{dt}\left(\frac{d\phi}{dt}\right) \Rightarrow \frac{d}{dt}\left(\frac{1}{T}\frac{d\phi}{d\tau}\right) \Rightarrow \frac{d}{d\tau}\left(\frac{1}{T}\frac{d\phi}{d\tau}\right)\frac{d\tau}{dt} \Rightarrow \frac{1}{T}\frac{d^2\phi}{d\tau^2}\frac{1}{T} \Rightarrow \frac{1}{T^2}\frac{d^2\phi}{d\tau^2} \tag{6}$$

Substituting these into our equation, we arrive at:

$$-mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{1}{T^2}\frac{d^2\phi}{d\tau^2} \tag{7}$$

Next, divide equation (7) through by mg to get

$$-\sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} \tag{8}$$

To finish making the equation dimensionless, set T equal to b/mg, to arrive at

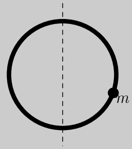
$$-\sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{m^2gr}{b^2}\right)\frac{d^2\phi}{d\tau^2} \tag{9}$$

Simplifying these terms, we set γ equal to $r\omega^2/g$, and ε equal to m^2gr/b^2. The result expresses the original 5 parameters as two more easily analyzed dimensionless parameters. Our result is

$$-\sin\phi + \gamma\sin\phi\cos\phi = \varepsilon\frac{d^2\theta}{d\tau^2} \tag{10}$$

3. Frictionless Examples

Using equation (10), we graph ϕ vs. ϕ'. As we know, $g = 9.8$, and we let the values of $r = 1$, $m = 1$, $b = m^2gr$, and this varies the value of ω. So, ε will always equal 1, and the value of γ will be the term affecting the graph. Figure 4 shows when γ is less than 1. Figure 5, when γ is equal to 1. Figure 6, when γ is just over 1. The final example, figure 7 shows γ is much greater than 1.



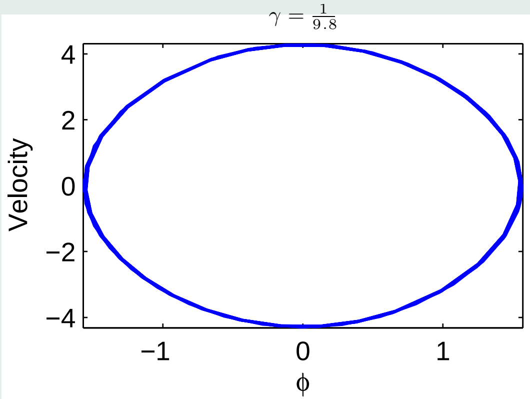
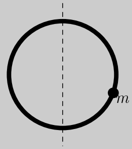


Figure 4: $\gamma < 1$.

In all of these graphs, the bead is in constant motion. The main difference between these graphs is where the bead is moving. In our first example, we see that the bead slides up and down both sides of the hoop. The second example is the same but the velocity of the bead has died down. When we increase the value of γ, the bead is restricted to one side of the hoop, but the area in which it moves gets much smaller as the angular velocity increases. It gets to a point where we can't really see it moving. This is by $\pi/2$, but the bead is moving, even if it is ever so slightly.

In this example, there are two equilibrium solutions, one unstable one of which is at the top of the hoop and the other is stable, but not asymptotically stable, at the bottom as we see in figure 8a. This system has a potential for a third equilibrium solution that would change the bottom to an unstable solution. Figure 8b shows that the third solution would be along the side of the hoop somewhere, and as before this solution would be stable, but not asymptotically stable. Also, because this is a conservative system, the bead will never come to rest.



Introduction and . . .
Frictionless Bead
Frictionless Examples
Friction Added to the . . .
Friction Examples
Supercritical Pitchfork . . .
Home Page
Title Page
Page 7 of 16
Go Back
Full Screen
Close
Quit

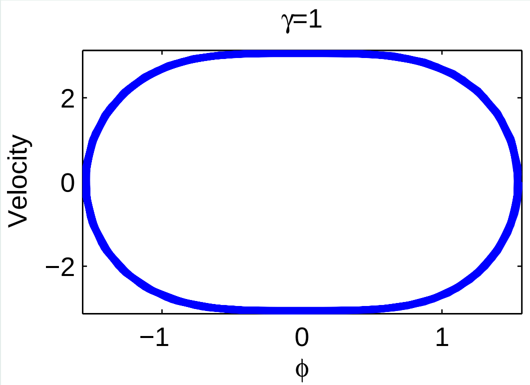


Figure 5: $\gamma = 1$.

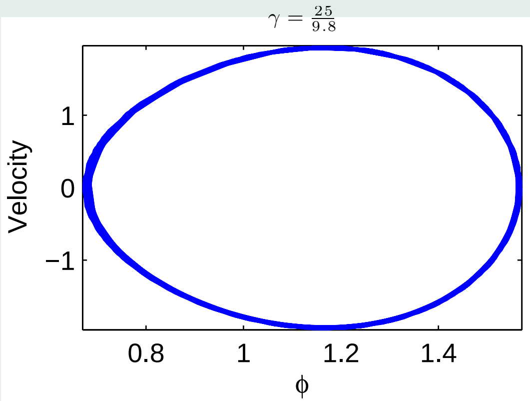
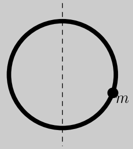


Figure 6: $\gamma > 1$.



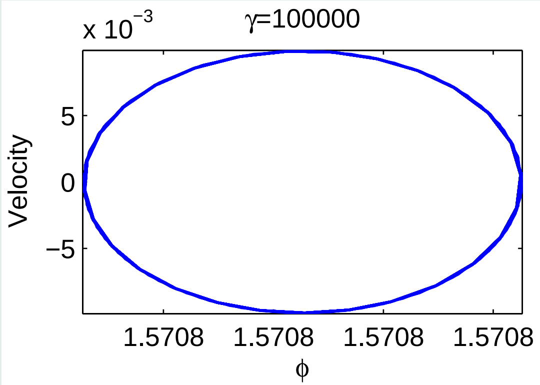
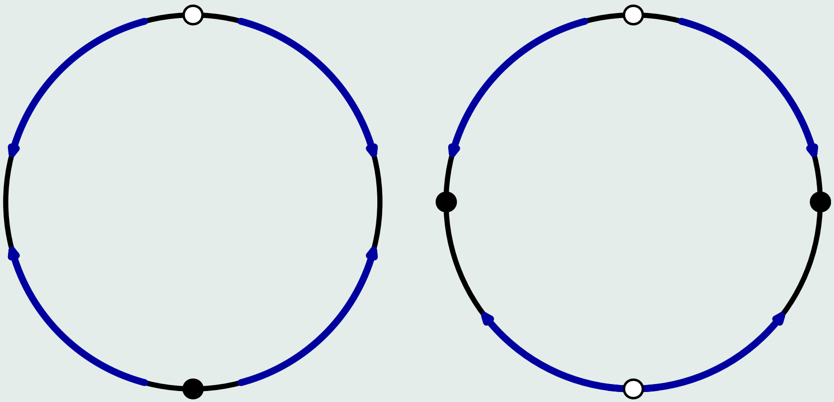


Figure 7: $\gamma \gg 1$.



(a) Two Equilibrium Solutions.

(b) Three Equilibrium Solutions.

Figure 8: Frictionless Equilibrium Solutions.

4. Friction Added to the Equation

Here we show the equation of the bead on a rotating hoop *with* friction. First we assume there is friction, but also lubricant on the hoop to ensure the continuous movement of the bead. Friction is a force that opposes the motion. Also, the bead now moves at a varying speed, due to friction.

For this example we'll use the same values for g, m, r, ω and ϕ as before, and in addition we'll consider friction, $bd\phi/d\tau$. Additionally, we assume that b^2 is going to be much greater than m^2gr. The resulting effect is that ε will always be less than 1 since the denominator is much greater than the numerator. Previously we saw that $d\phi/dt$ was equal to $(1/T)d\phi/d\tau$. We can pick up where we left off with equation (7), this time adding in the friction force.

$$-\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi = mr\frac{1}{T^2}\frac{d^2\theta}{d\tau^2} \tag{11}$$

Again, we divide through by mg to get

$$-\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{r}{gT^2}\right)\frac{d^2\theta}{d\tau^2} \tag{12}$$

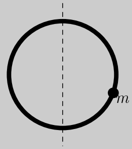
Now checking the units on both sides, they should match up. If we recall that m is in kilograms, r in meters, T in seconds, b is in $(kg)m/s$, g is in m/s^2, and ω is $1/s$. Substituting these into equation (14) we get

$$-\left(\frac{\frac{(kg)(m)}{s}}{(kg)(s)\frac{m}{s^2}}\right) + \left(\frac{m\left(\frac{1}{s}\right)^2}{\frac{m}{s^2}}\right) = \left(\frac{m}{\frac{m}{s^2}s^2}\right) \tag{13}$$

The result shows that the groups in parentheses have no units. Recall, T is equal to b/mg, so now the equation becomes

$$-(1)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi = \left(\frac{m^2gr}{b^2}\right)\frac{d^2\theta}{d\tau^2} \tag{14}$$

Simplifying these terms, we set γ equal to $r\omega^2/g$, and ε equal to m^2gr/b^2. Again converting the original 5 parameters into two more easily analyzed dimensionless parameters.



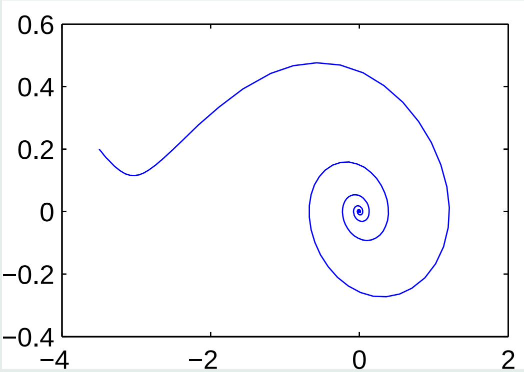


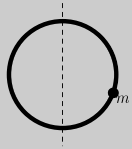
Figure 9: Equilibrium Solution at 0

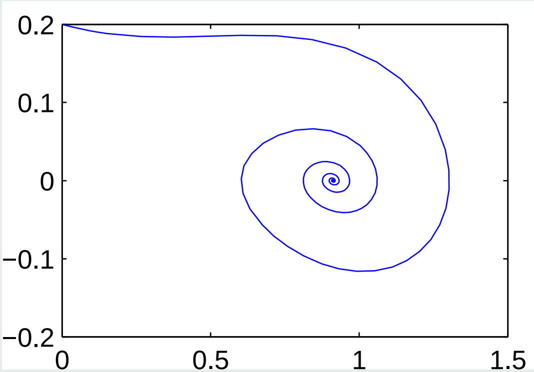
The dimensionless equation we'll use with friction added is

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma \sin(\phi)\cos(\phi) \tag{15}$$

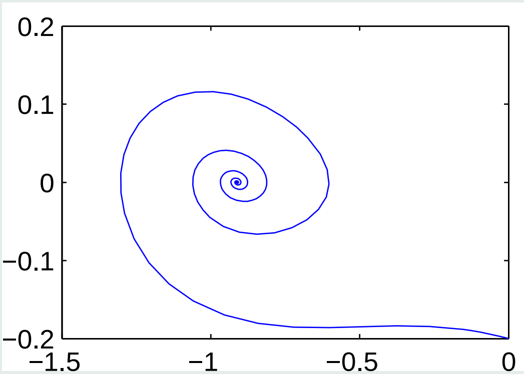
5. Friction Examples

Using equation (15) and the same values for the previous example, we again graph ϕ vs. ϕ'. The value for ω will vary the graph by changing the value of γ. In figure 9, γ is less than 1. As a result, the stable equilibrium solution is at the base of the hoop. The second and third examples occur when γ is much greater than 1, the only differences are the initial conditions. We show two because the equilibrium solution will switch sides of the hoop depending if the initial condition is negative or positive. The graph of the positive solution is shown in figure 10b, the negative solution in figure 10a. In the $\gamma \gg 1$ result, the equilibrium solution has moved from the base of the hoop to either side. Again, if we change the value of ω, the location of the equilibrium solution will change.



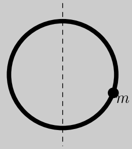


(a) An Equilibrium Solution Between $-\pi/2$ and 0.



(b) An Equilibrium Solution Between 0 and $\pi/2$.

Figure 10: Two of the Four Equilibrium Solutions.



This equation contains the two definite equilibrium solutions seen in Figure 11a. The first equilibrium solution is unstable at the top of the hoop. The other equilibrium solution is asymptotically stable and lies at the base of the hoop. This is where the bead will stop if γ is less than 1.However, an additional equilibrium solution occurs, as the angular velocity increases, inertia overtakes friction, and the bottom equilibrium solution moves up to either side as seen in Figure 11b. This happens when $r\omega^2/g > 1$. Here, the top remains unstable, and the bottom becomes unstable. The third solution slides up and down the sides of the hoop as ω varies..

Next, we'll show how to determine the location of the equilibrium solutions, we consider the first order differential equation.

$$bd\phi/d\tau = -mg\sin(\phi) + mr\omega^2\sin(\phi)\cos(\phi) \tag{16}$$

We set ϕ' equal to 0 and factor

$$mg\sin(\phi)\left(-1 + \frac{r\omega^2}{g}\cos(\phi)\right) = 0 \tag{17}$$

Setting the first part equal to 0

$$mg\sin(\phi) = 0 \tag{18}$$

This happens at 0 and π. If we set the second part equal to 0 we get

$$-1 + \frac{r\omega^2}{g}\cos(\phi) = 0 \tag{19}$$

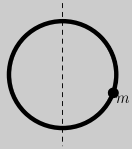
Solving for ϕ we get

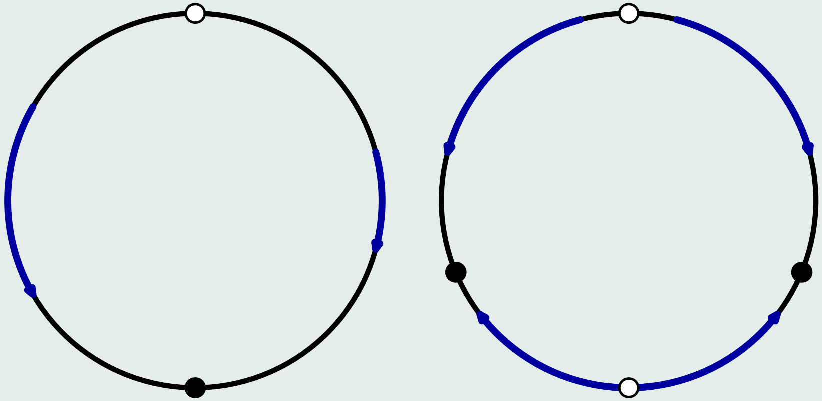
$$\phi = \pm\cos^{-1}\left(\frac{g}{r\omega^2}\right) \tag{20}$$

Recall $\gamma = r\omega^2/g$, subbing in γ we get

$$\phi = \pm\cos^{-1}\left(\frac{1}{\gamma}\right) \tag{21}$$

This creates another equilibrium solution, but it depends on the value of ω.
Referring to figure 11b we see that at 0, and $\pm\pi$ there are unstable equilibrium solutions.





(a) Two Equilibrium Solutions.

(b) Three Equilibrium Solutions (With Friction).

Figure 11: Equilibrium Solutions.

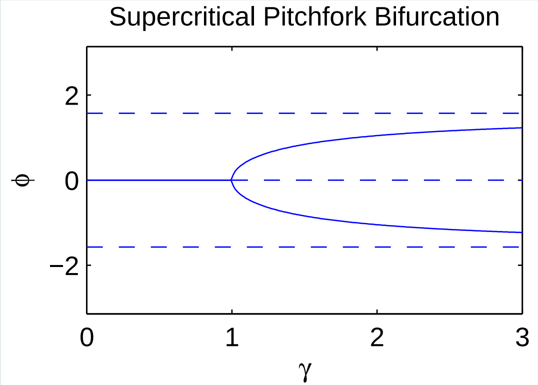
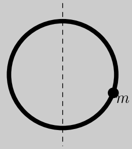


Figure 12: $\pm\cos^{-1}(1/\gamma)$

6. Supercritical Pitchfork Bifurcation

In Figure 12 we see the graph of ϕ vs. γ, a supercritical pitchfork bifurcation. It also shows that the bottom solution becomes unstable when $\gamma > 1$. As the hoop reaches the highest speed possible, the equilibrium solutions approach $-\pi/2$ or $\pi/2$. When $\gamma > 1$ the hoop spins so fast that the bottom equilibrium solution becomes unstable and the bead moves up towards $-\pi/2$ or $\pi/2$. When $\gamma \leq 1$ the hoop is spinning slow enough that the centrifugal force isn't enough to overcome gravity the bead and it comes to rest at the bottom of the hoop.



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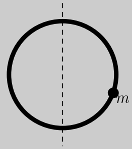

Introduction and . . .
Frictionless Bead
Frictionless Examples
Friction Added to the . . .
Friction Examples
Supercritical Pitchfork . . .

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Page 16 of 16

Go Back

Full Screen

Close

Quit