Computers, Lies and the Fishing Season

Liz Arnold

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Abstract

The purpose of this paper is to show how differential equation solvers may give you "weird" answers.

Introduction

Computers, lies, and the fishing season takes a look at computer software programs. As mathematicians, we depend on computers for numerical answers. In this paper, various problems in our computer computations will be shown and explained. Possible ways to fix the false computer computations will be explained as well. What you may think is the correct answer, is in fact not. We will consider two examples of the pendulum that illustrate this point. Then we will consider two models of "harvested" populations and how the limitations of differential equation solver software may give "weird" or "incorrect" answers.

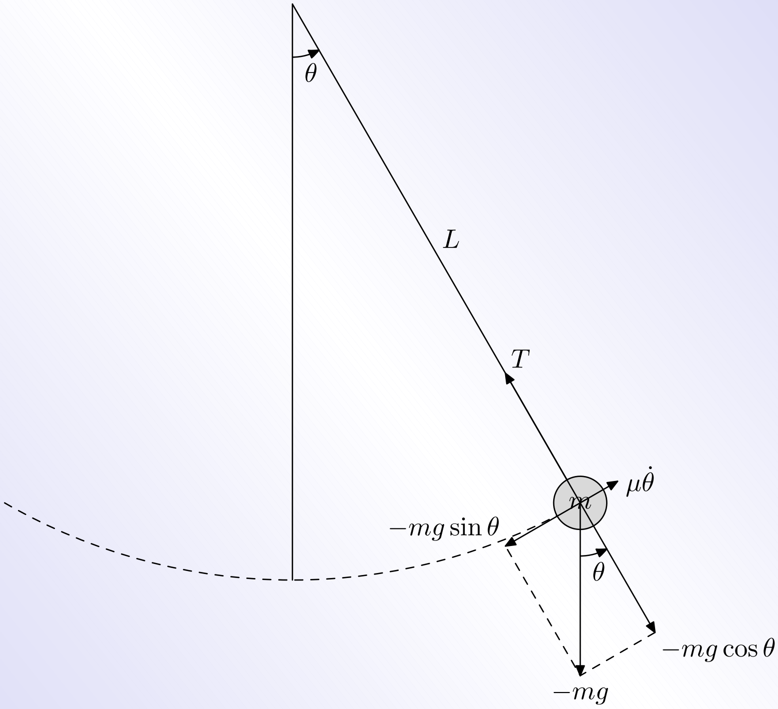

Figure 1: The Pendulum.

The Pendulum

The pendulum as shown in Figure 1 consists of a bob with a mass m, that is suspended by a rod of length L that is fixed at the upper end. The rod is considered to be massless. The angle θ is the angle between the rod and the vertical. The displacement of the bob from equilibrium is measured along the circle so it is denoted as $L\theta$. The bob is affected by the force of gravity, which points downward, and its magnitude is denoted by mg. The portion of the force that is parallel to the rod is balanced by the rod itself. The unbalanced portion of this force is tangential to the circle and perpendicular to the rod. The motion of the pendulum is driven by the force of gravity. The tangential force is denoted by $-mg\sin\theta$. The negative sign in the tangential force indicates that the force is always acting to decrease the magnitude of the angular displacement. If there is a damping force that is proportional to the velocity present, then it has the form $\mu L\dot{\theta}$, where μ is constant.

Finding the equation for the pendulum

To further complete our investigation of computer solvers, we need to come up with the equation of the pendulum. We know that

$$\theta = \frac{s}{r}.$$

However, in this case $r = L$, so $\theta = s/L$ and cross multiplying,

$$s = L\theta.$$

Now that we have the position, if we differentiate with respect to time (where L is constant) we get

$$\frac{ds}{dt} = L\frac{d\theta}{dt}$$
$$v = L\dot{\theta},$$

where v is the velocity. To find the acceleration, differentiate again with respect to time and we get

$$\frac{dv}{dt} = L\frac{d\ddot{\theta}}{dt}$$
$$a = L\ddot{\theta},$$

where a is the acceleration.

According to Sir Isaac Newton's second law, the sum of the forces acting on an object equals its mass times acceleration. Thus,

$$ma = \sum F,$$
$$ma = -mg\sin\theta - \ \mu L\dot{\theta} + f(t),$$

where $f(t)$ is the driving term. Substituting $a = L\ddot{\theta}$ we get

$$mL\ddot{\theta} = -mg\sin\theta - \mu L\dot{\theta} + f(t).$$

Dividing through by mL,

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\mu}{m}\dot{\theta} + g(t). \tag{1}$$

Linearization

Now that we have the equation of a pendulum, let's look at the case where there is no driving term. Our equations becomes

$$\ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{\mu}{m}\dot{\theta}. \tag{2}$$

Notice that the term $-g\sin\theta/L$ makes equation (2) non-linear. In fact, the $\sin\theta$ is the key term that is making equation (2) non-linear. If we assume that θ is small, we can use the approximation $\sin\theta \approx \theta$. This is true because as θ continues to go to zero, $\sin\theta$ will continue to go to zero. Now our equation becomes,

$$\ddot{\theta} = -\frac{g}{L}\theta - \frac{\mu}{m}\dot{\theta}. \tag{3}$$

Notice that equation (3) is now linear. So, for very small amplitudes of oscillations, the motion can be described by equation (3).

Example 1

Sampling Rates

To begin with, we are going to look at the linearized equation of a pendulum with no damping term involved. We are going to examine the simple harmonic oscillator equation,

$$\ddot{\theta} + 4\theta = 0, \theta(0) = 0, \dot{\theta}(0) = 1. \tag{4}$$

We are going to show eight plots of the solution that satisfies the initial conditions of $\theta(0) = 0$ and $\dot{\theta}(0) = 1$. The interval $0 \leq t \leq 100$ was divided by 1000 equally spaced points and a fourth-order Runge-Kutta method was used to approximate the solutions at these points. To graph the solutions, we are going to chose every point in Figure 2, every fifth point in Figure 4, every tenth point in Figure 6 and every twentieth point in Figure 8, and plot each of these solutions in the $t\theta$-plane. Then we are going to chose every point in Figure 3, every fifth point in Figure 5, every tenth point in Figure 7, and every twentieth point in Figure 9, and plot each of these solutions in the $\theta\dot{\theta}$-plane.

Plotting Solution Curves and orbits of the harmonic oscillator

As fewer and fewer points are plotted, notice how the graphs are modulated. Figure 2, and Figure 3 appear to be correct, but Figure 4, Figure 5, Figure 6, Figure 7, Figure 8, and Figure 9 appear to be modulated sinusoids. Also, notice that the period of the graphs plotting every point in Figure 2, every fifth point in Figure 4, and every tenth point in Figure 6, seem to have the same period of π, but the graph of plotting every twentieth point in Figure 8 seems to have a period that is doubled to 2π as seen in Figure 10.

What is going on here? Is the accuracy of the solver to blame? No, that is not it. The accuracy of all the points taken are just as good as the other ones. All of the points were sampled from the same list of 1000 approximate values. What we observed from looking at the graphs always occurs when an oscillatory curve is approximated by a finite number of equally spaced points connected by a straight line segments. As the number of sampled points per unit time decrease, the graphs appear to worsen.

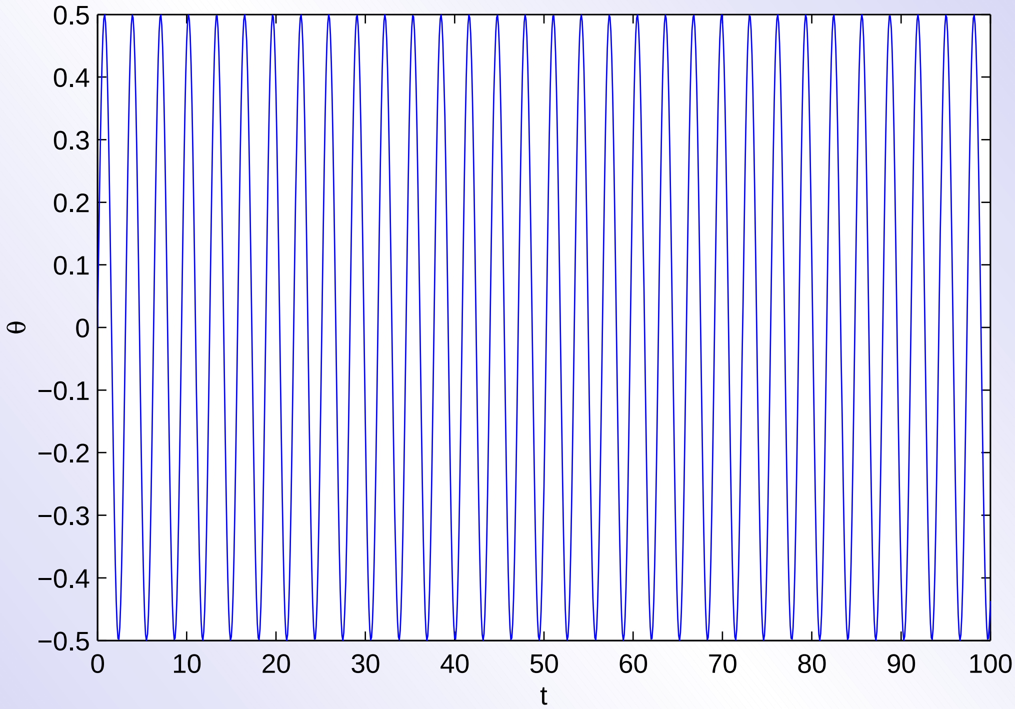

Figure 2: $t\theta$-plane using every point.

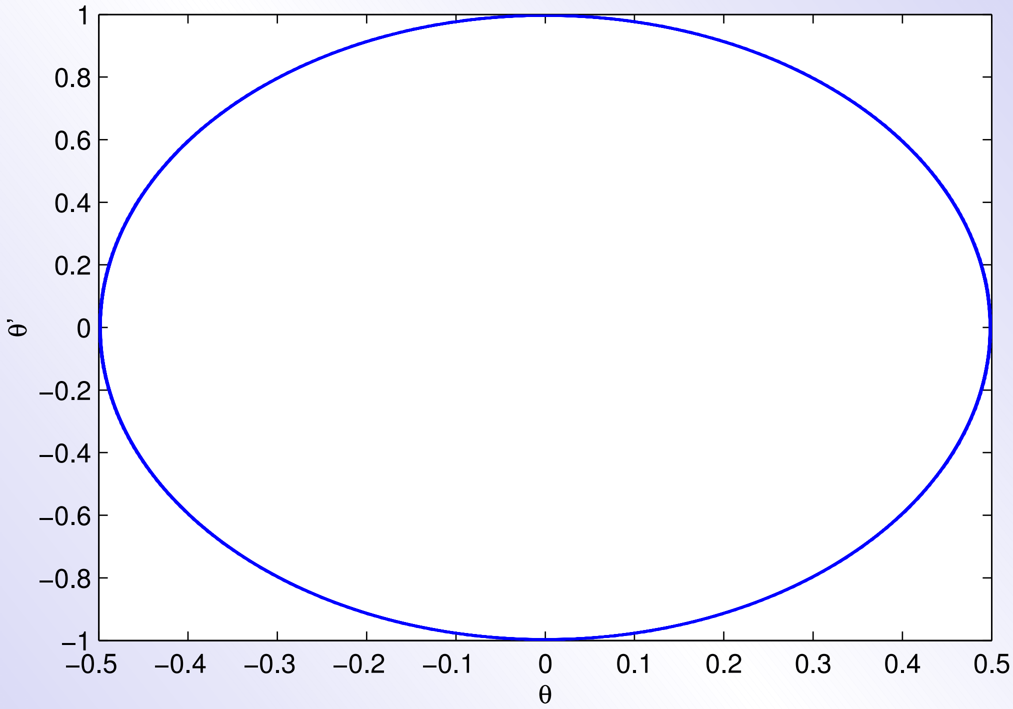

Figure 3: $\theta\dot{\theta}$-plane using every point.

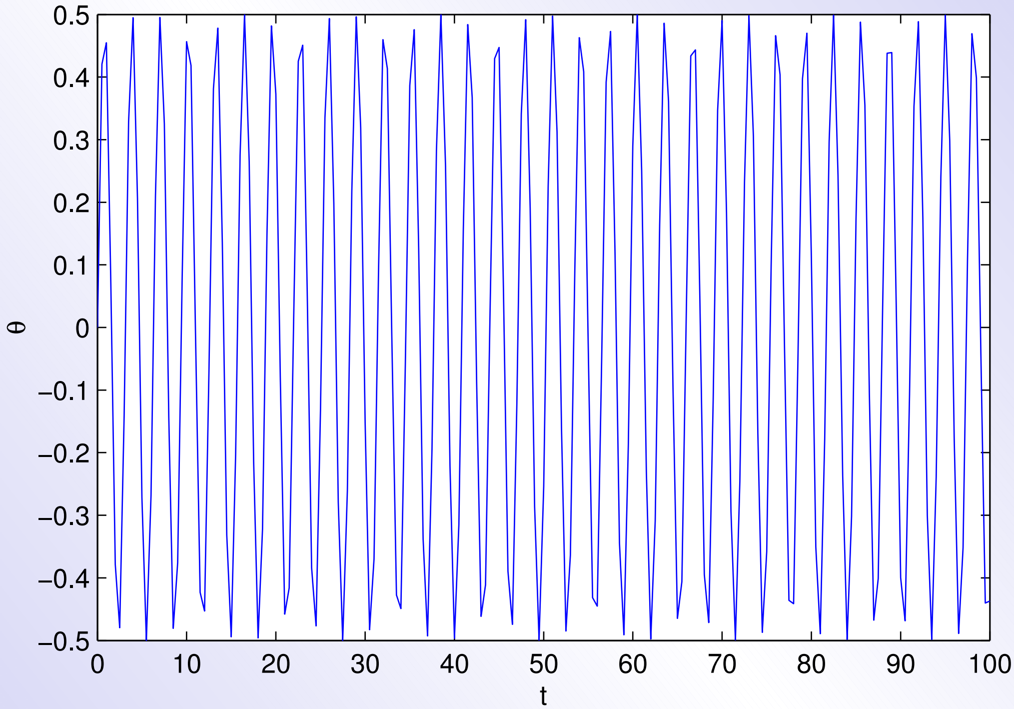

Figure 4: $t\theta$-plane using every fifth point.

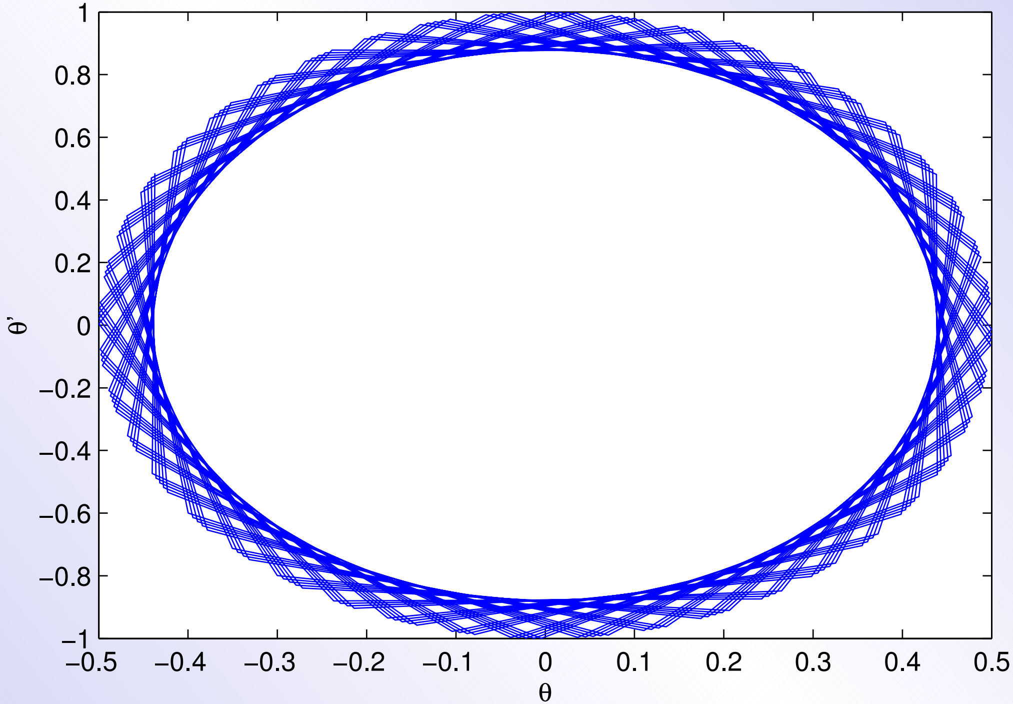

Figure 5: $\theta\dot{\theta}$-plane using every fifth point.

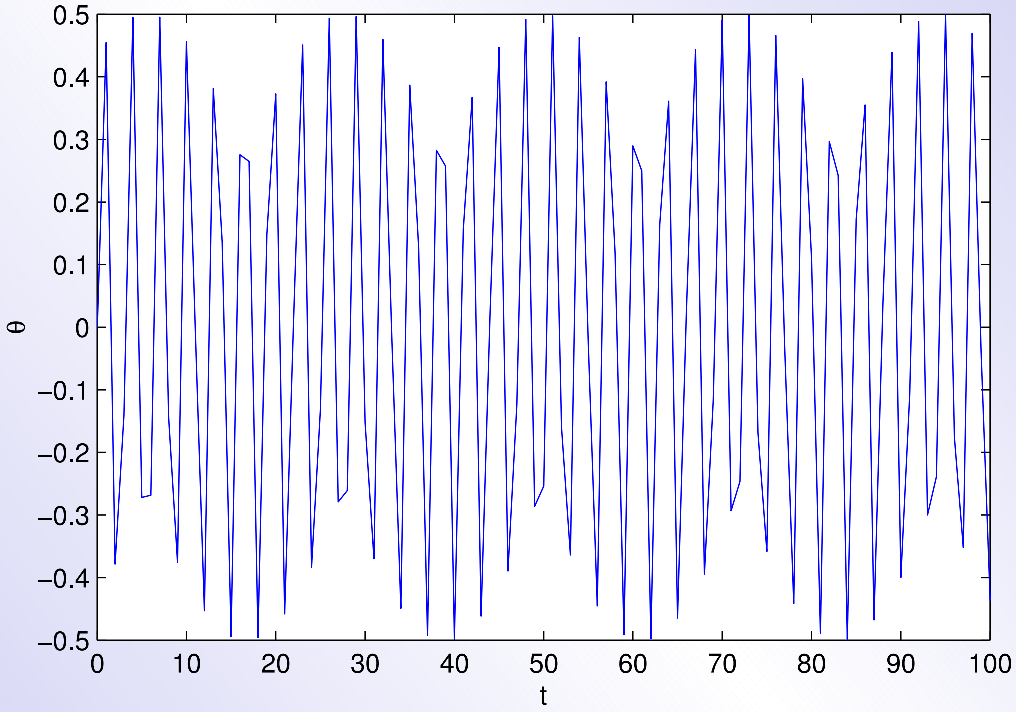

Figure 6: $t\theta$-plane using every tenth point.

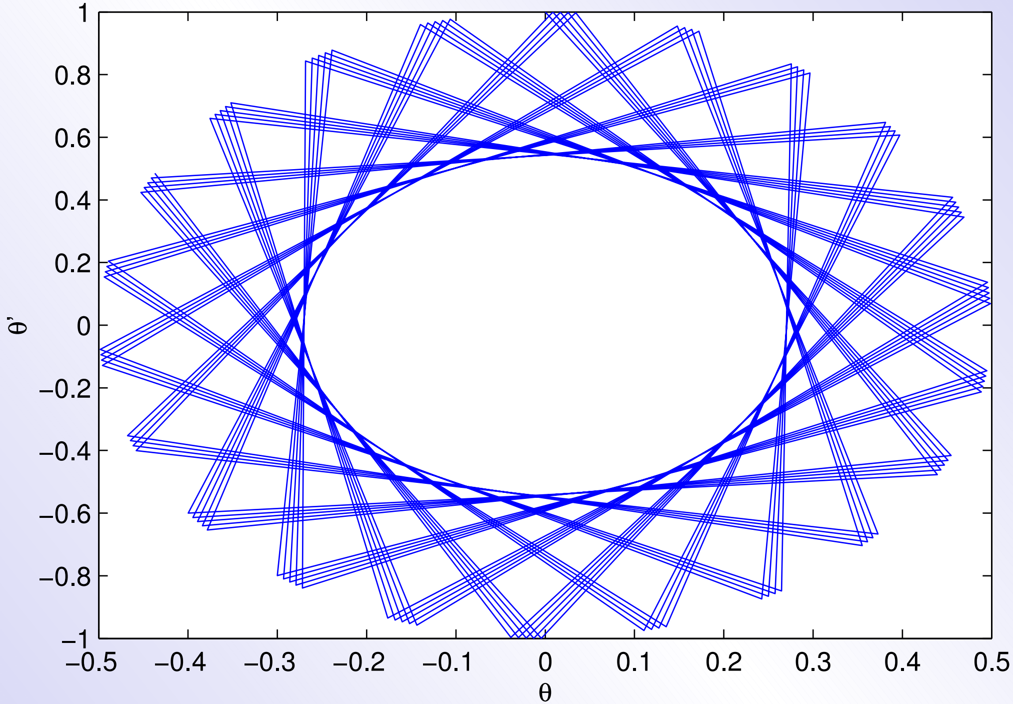

Figure 7: $\theta\dot{\theta}$-plane using every tenth point.

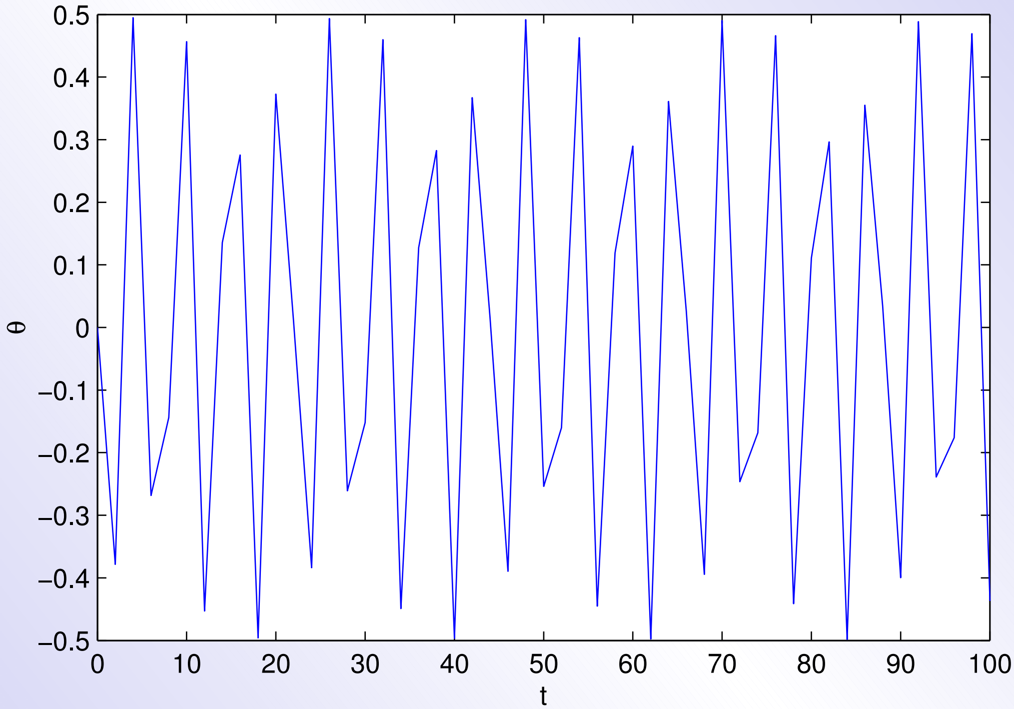

Figure 8: $t\theta$-plane using every twentieth point.

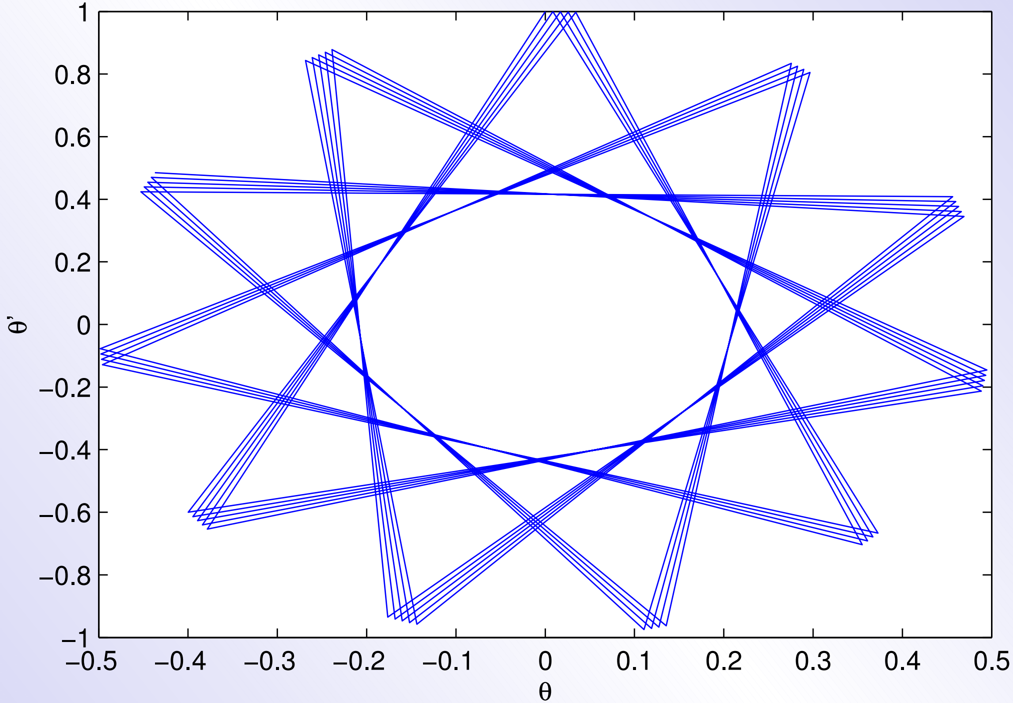

Figure 9: $\theta\dot{\theta}$-plane using every twentieth point.

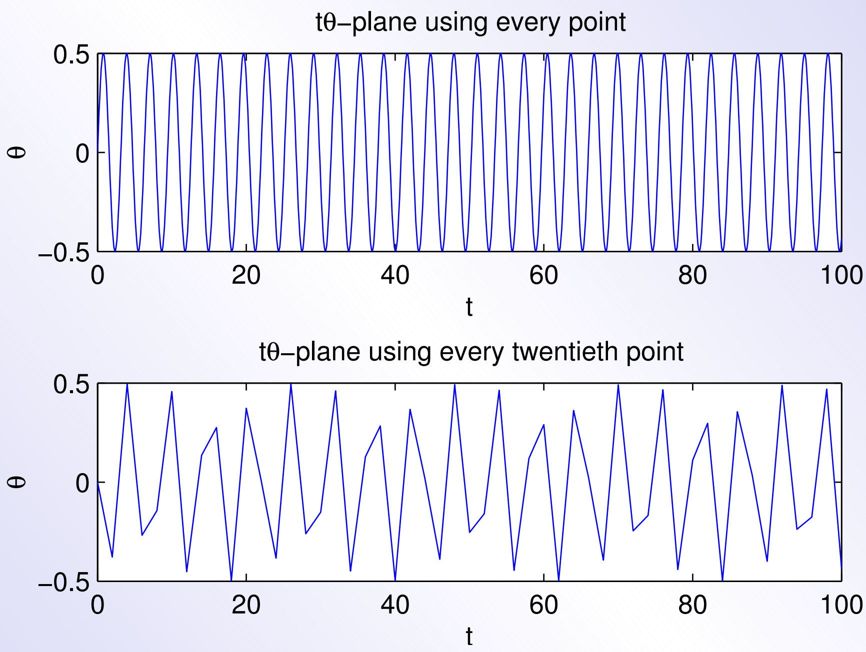

Figure 10: The period of second graph seems to have doubled the period of the first graph.

Aliasing

Aliasing occurs when there is not enough discrete points to reconstruct the shape of the original graph. Notice how Figure 2 looks when plotting every point. But when you sample fewer than two sample points per period as in Figure 8, which is plotting every twentieth point, the graphical representation appears to have a larger period as seen in Figure 10.

Conclusion

What can we do to correct the misrepresentation in representing a continuous periodic function by a discrete point set? If the experimenter knew before hand the period of oscillation, then the "sampling rate" could be set high enough to decrease the problem.

Example 2

The Driven Pendulum

We have just examined equation (4) of a pendulum that contains no damping term. Now, we are going to look at the equation of a pendulum with a damping term involved. Recall that the equation for a pendulum with a damping term is equation. The equation that we are going to examine is equation (5) as shown,

$$\ddot{\theta} = -\sin\theta - 0.1\dot{\theta} + \cos t. \tag{5}$$

Equation (5) describes the angular position θ of a damped, sinusoidally-driven pendulum moving in a plane.

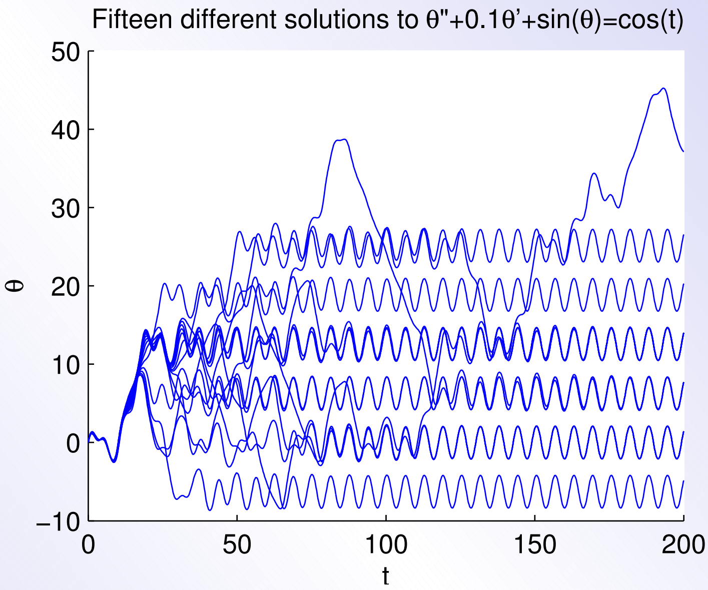

Figure 11: 15 different solutions.

What Does The Motion Look Like?

Figure 11 shows the motion of our pendulum resulting from 15 different sets of initial angular velocities. Each solution starts with the position $\theta(0) = 0$, while the initial angular velocities are evenly spaced between 1.85 and 2.1. The graphs are plotted on the intervals $0 < t < 200$ and $-10 < \theta < 50$.

Notice how the 15 solutions are different from one another even though they have the same starting position. The initial angular velocities affect each solution differently. The difference in the initial angular velocities is only 0.25 and the solutions behave in this way. Imagine how the solutions would look like if the difference between the initial angular velocities was greater than 0.25.

We saw what happens to the damped pendulum when you change the initial angular velocities slightly. Now, let's look at the case where we reduce the relative and absolute local error tolerance. We are going to continue to use equation (5) with the initial position of $\theta(0) = 0$ and the initial angular velocity of $\dot{\theta}(0) = 2$.

Graph a) as seen in Figure 12 is the graph of the pendulum with an absolute local error tolerance of 4×10^{-4} and a relative local error tolerance of 4×10^{-4}.

Graph b) as seen in Figure 12 is the graph of the pendulum with an absolute local error tolerance of 4×10^{-6} and a relative local error tolerance of 4×10^{-6}.

Notice how graph a) as seen in Figure 12 is settling down at about 25 while graph b) as seen in Figure 12 is settling down at about 12.

Which Graph Is The Correct Solution?

Just by reducing the relative and absolute local error tolerance, you see that we get two completely different answers for the same problem. Which one is the correct answer? The answer is the graph with the smaller relative and absolute local tolerances, which is graph b) as seen in Figure 12.

When graphing your solutions, you should experiment with different values of the relative and local error tolerances until you get two graphs that look similar and then you know that the graph you have is the correct solution.

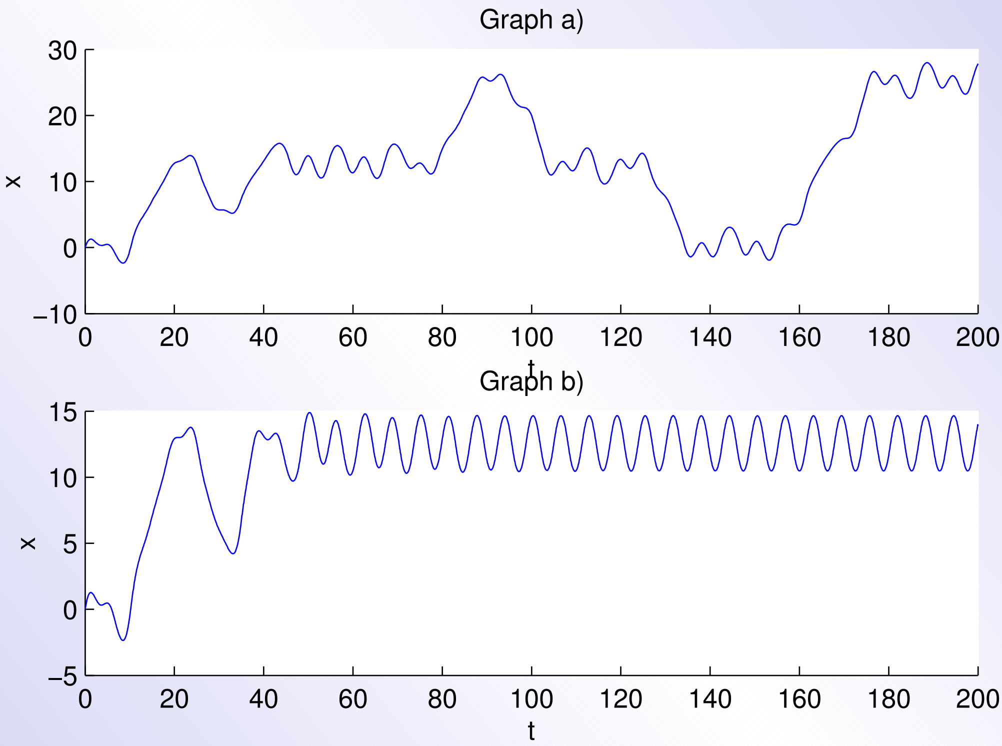

Figure 12: Graph a) and Graph b).

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Conclusion

As we saw, when graphing a damped harmonic oscillator, you can have different solutions just by increasing or decreasing the relative and absolute local tolerances. You must experiment with different values for the and relative and local error tolerances until you are satisfied that you have a correct representation of the graph.

Example 3

The Logistic Model With A Fishing Season

We are now going to look at model for a bounded population with a harvesting rate. The best known equation for this model is

$$x' = rx\left(1 - \frac{x}{K}\right) - F, \tag{6}$$

where $x(t)$ is the population at time t, r and K are positive constants, and F is the harvesting rate. The value $x(t)$ is measured in tons, and t is measured in years. The equation that we are going to examine is

$$x' = x(1 - x/12) - F. \tag{7}$$

If there is no fishing $(F = 0)$, then the fish population tends to the equilibrium level K as shown in Figure 13.

Continuous fishing at a constant but low rate $(F = 2)$, as shown in Figure 14, leads to a stable equilibrium below K. Notice that if the initial population level is too low, extinction for the fish population is inevitable.

Continuous fishing at a high constant rate $(F = 6)$, as shown in Figure 15, leads to extinction eventually, regardless of the initial condition.

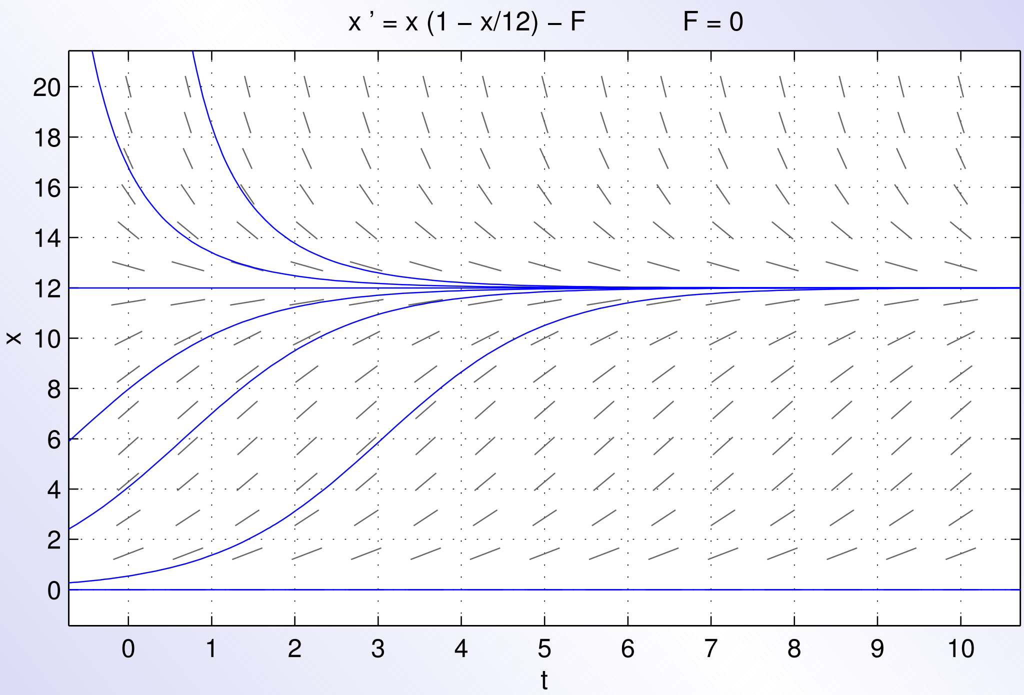

Figure 13: No fishing.

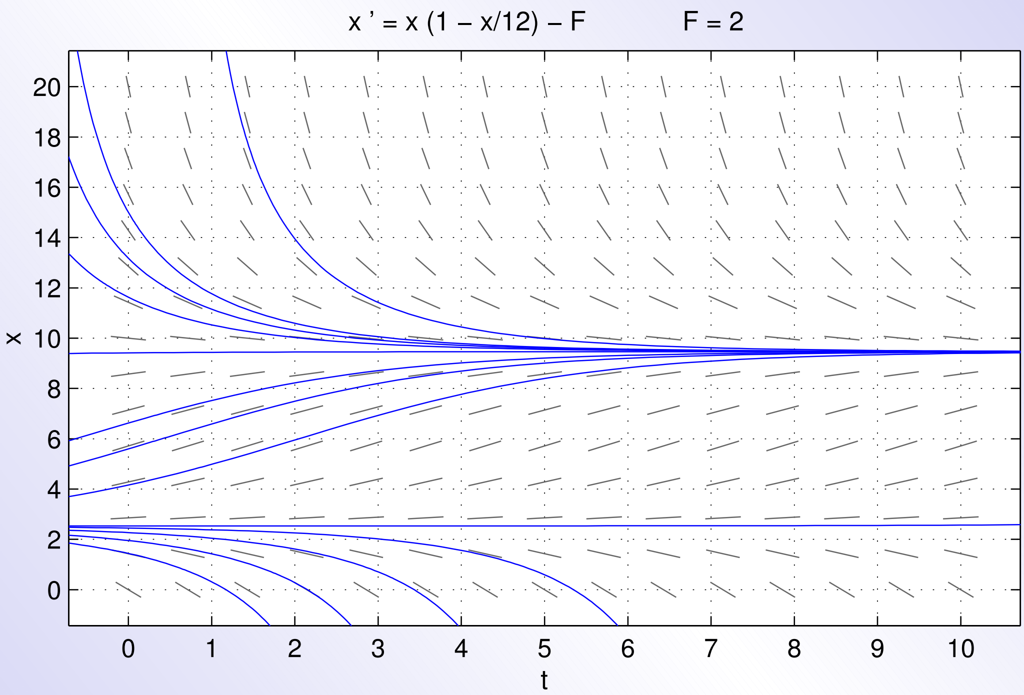

Figure 14: Continuous light fishing.

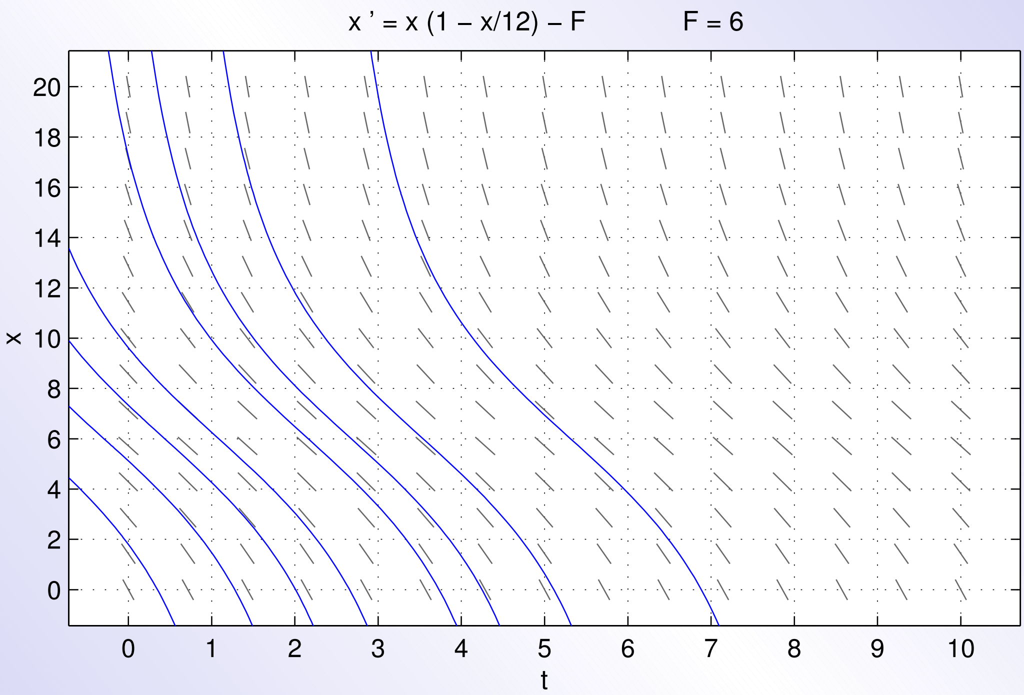

Figure 15: Continuous heavy fishing.

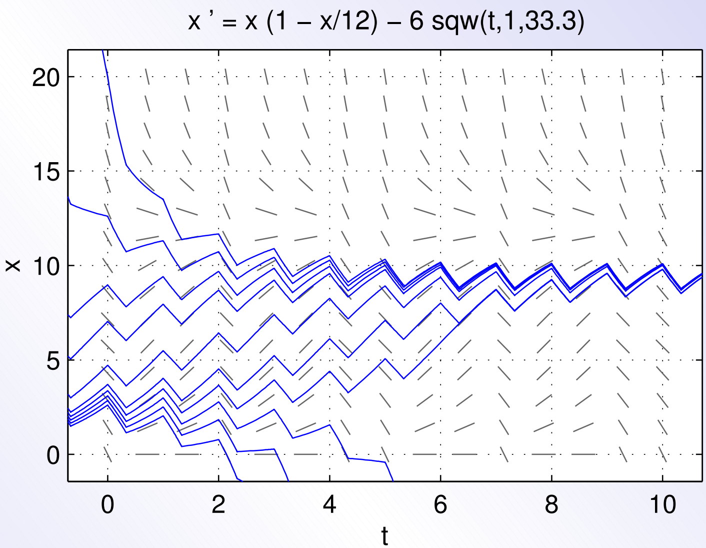

Figure 16: Four-month fishing season.

A high fishing rate will not lead to extinction if the fishing season is restricted to a few months each year. Using the high fishing rate equation (7), look what happens to the fish population maintained over a four-month fishing season as seen in Figure 16 as well as an eight-month fishing season as seen in Figure 17.

As you see, when having a high rate of fishing and a four-month fishing season, the fish population will continue to remain if there are more than approximately 3.2 tons of fish as seen in Figure 16. For the case of an eight-month fishing season you see in Figure 17, the fish population is headed for extinction regardless of the initial population.

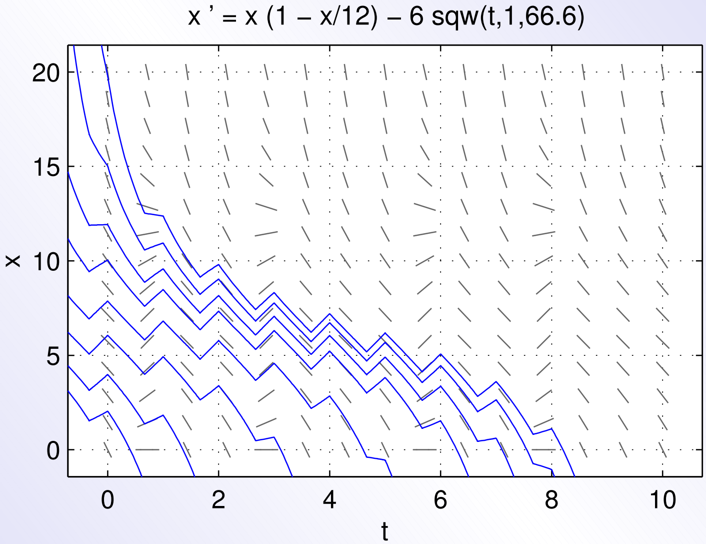

Figure 17: Eight-month fishing season.

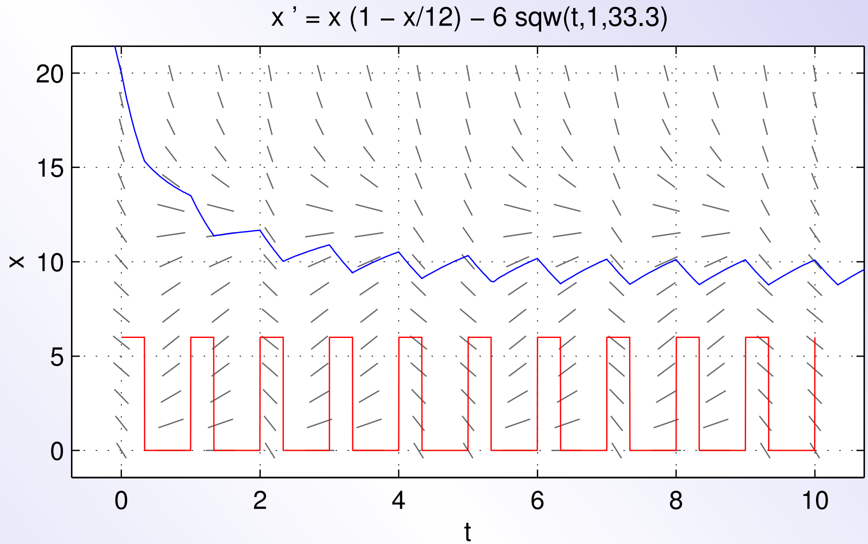

Figure 18: Small step size.

Computer Lies

How good are the computed zigzag solution curves in Figure 16, and Figure 17? The zigzag solutions are are pretty accurate if the maximal step size of the solver is kept small enough so that the solver recognizes the points in time when the harvesting function F switches on or off as in Figure 18. On the other hand, if the maximal step size is too big, the solver can't recognize all the points in time when the harvesting function F switches on or off as in Figure 19.

Alternatively, a large maximal step size is allowed if the local error bounds of the

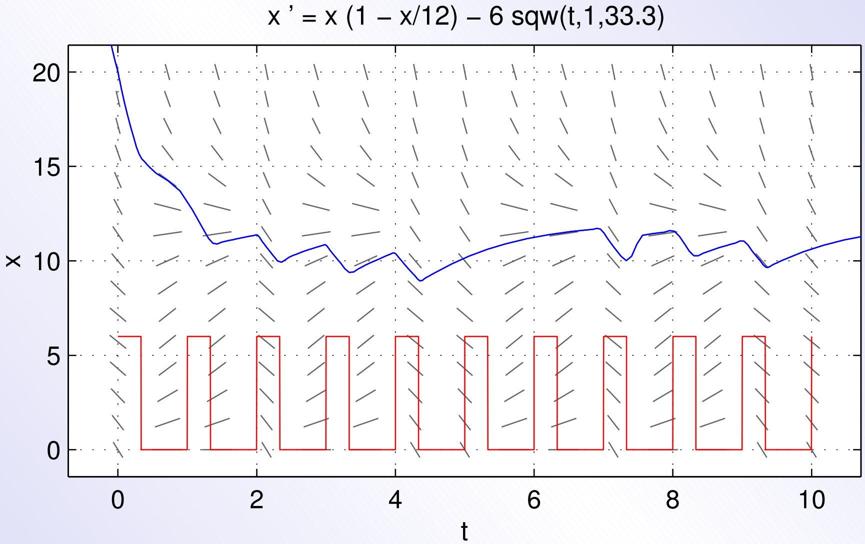

Figure 19: Large step size.

solver are set to very small levels. If neither condition is met, then the solver may lie about the effects of the fishing season on the overall population levels.

Figure 20, reading from top to bottom, shows with accuracy the consequences of no fishing, a four month fishing season, an eight month fishing season and continuous fishing on an initial population of $x(0) = 20$ tons of fish. The relative local error tolerance is 1×10^{-7} and the absolute local error tolerance is 1×10^{-10}.

Now, when we increase the local error bounds of the solver, notice what happens to our solutions in Figure 21. The relative local error tolerance is increased to 1×10^{-2} and the absolute local error tolerance is increased to 1×10^{-2}.

You can see that the graph in Figure 21 is telling "lies" about the effect of a four- or eight- month fishing season.

What happens when changing the step size of the solver set? Using the method of the fourth order Runge-Kutta, the graph in Figure 22 shows the effects of increasing the step size. You can see that the computer tells more "lies." When the step size is changed, you get different solutions.

Conclusion

When working with computers and graphs, keeping the step size of your solver small enough will help the solver recognize the points in time that are important and that will lead to accurate solutions. Also, if you have a large step size, then setting the local error bounds low enough will lead to accurate solutions as well. If neither of these conditions are met, then your computer solver may tell "lies" about your solution.

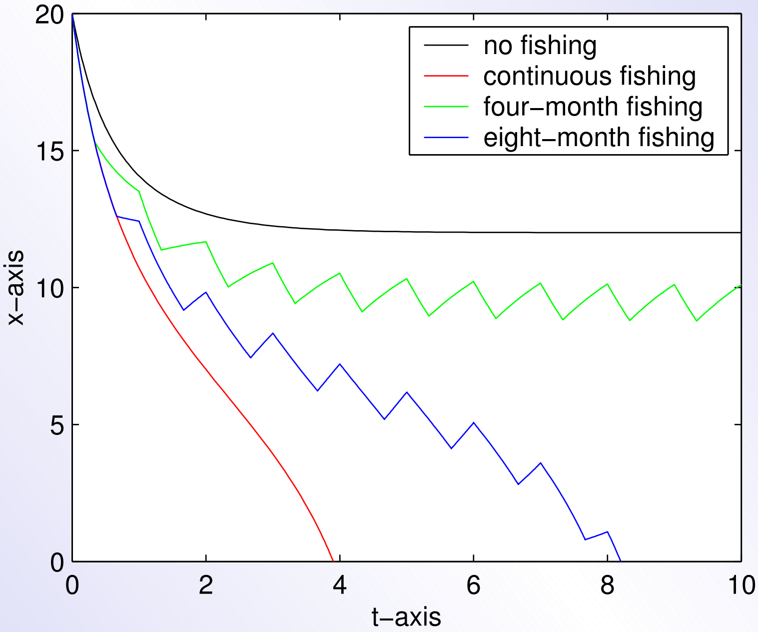

Figure 20: Rel.Tol (1×10^{-7}), Abs.Tol (1×10^{-10}).

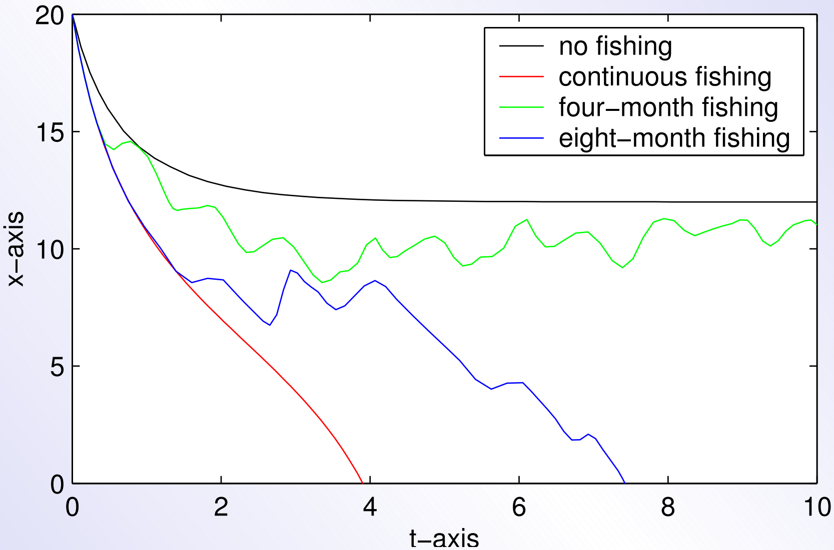

Figure 21: Rel.Tol (1×10^{-2}), Abs.Tol (1×10^{-2}).

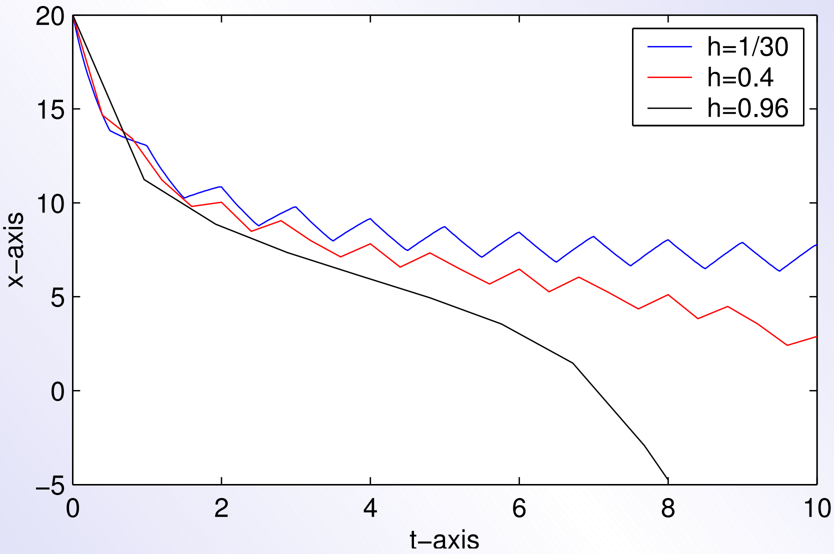

Figure 22: Six-month fishing season: RK4 with different step sizes.

Example 4

Predator-Prey Dynamics With Fishing

Finally, we are going to look at a predator-prey system of equations. The best known system of equations for the predator species $x(t)$ and the prey species $y(t)$ are

$$\begin{aligned} x' &= ax(-1+by) - H_1 x \\ y' &= cy(1-dx) - H_2 y, \end{aligned} \tag{8}$$

where a, b, c, and d are positive constants and H_1 and H_2 are nonnegative "harvesting" coefficients. The harvesting terms $H_1 x$ and $H_2 y$ model constant effort harvesting rather than the constant rate harvesting that we examined previously. Depending on the magnitudes of the harvesting coefficients, increased harvesting efficiency will shift the population cycles in the direction of larger average prey populations and smaller average predator populations.

We are going to plot the solutions of

$$\begin{aligned} x' &= x(-1+y) \\ y' &= y(1-x), \end{aligned} \tag{9}$$

on the xy-plane,tx-plane, and the ty-plane. Equation (9) examines no harvesting. Now, we are going to plot the solutions of

$$\begin{aligned} x' &= x(-1+y) - 0.7x \\ y' &= y(1-x) - 0.7y \end{aligned} \tag{10}$$

on the xy-plane, tx-plane, and the ty-plane. Equation (10) examines continuous light fishing.

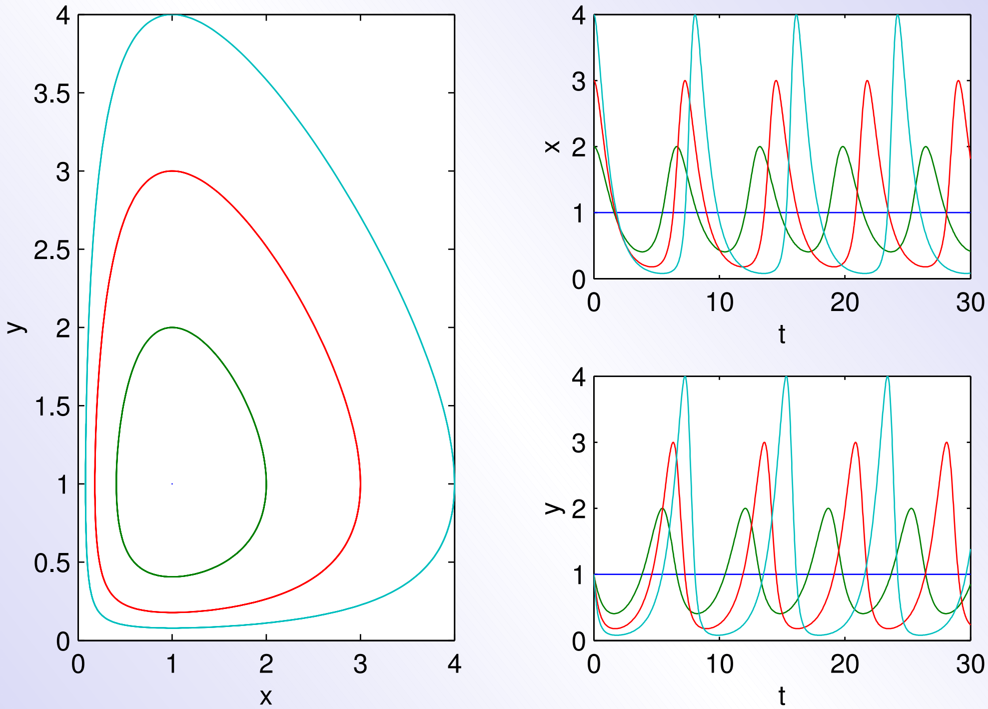

Figure 23: No harvesting.

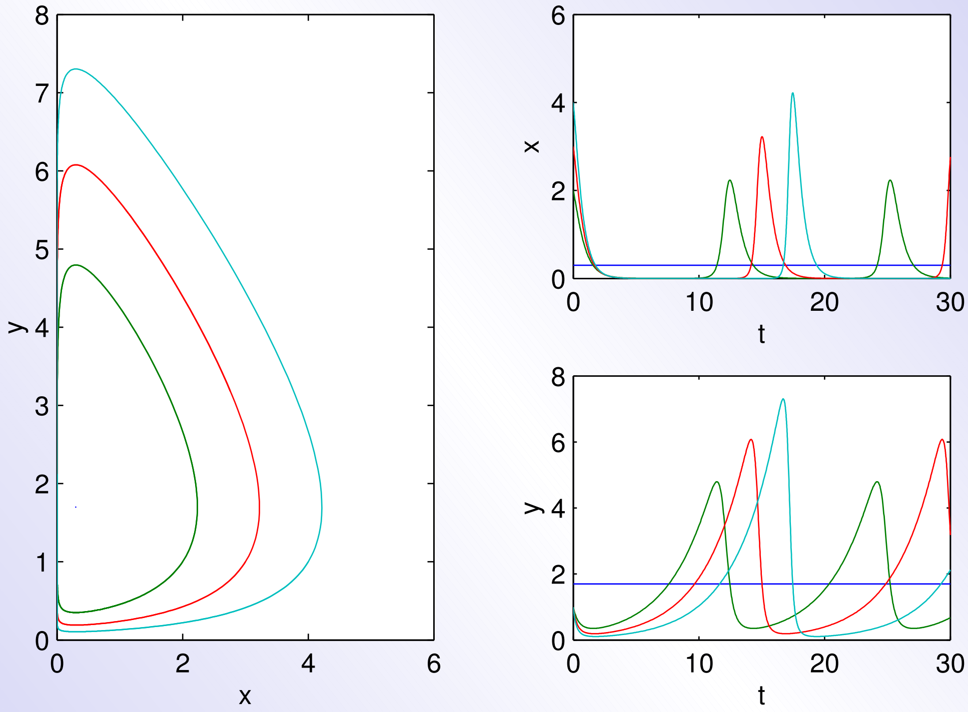

Figure 24: Light fishing.

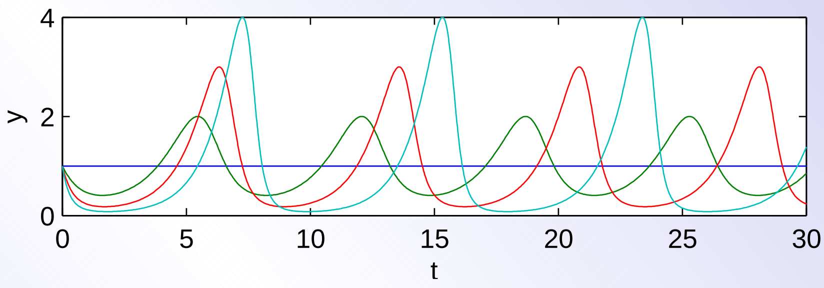

Figure 25: Nonlinear phenomenon.

Nonlinear Phenomenon

A distinctly nonlinear phenomenon is occurring with the graph containing no harvesting shown in Figure 25. When the periods of the solutions increase with the amplitude of the cycle, a nonlinear phenomenon has occurred.

Seasonal Harvesting

Now, we are going to show what happens to a particular population cycle if both species are harvested with the high harvesting coefficients, $H_1 = H_2 = 4$, but with a short, one month harvest season per year. The graph in Figure 26 shows an oval orbit of the no fishing case and an extinction orbit of the constant, heavy fishing case, and an orbit of fishing one month per year. The maximal step size is 0.01 and the relative local error tolerance is 1×10^{-7} and the absolute local error tolerance is 1×10^{-10}. The time span is 100 years and the number of points plotted is $10,000$.

Each arc of the "bracelet" is an arc of one of the population ovals in the no-harvest case or an arc of an extinction curve in the continuous high-rate harvest case.

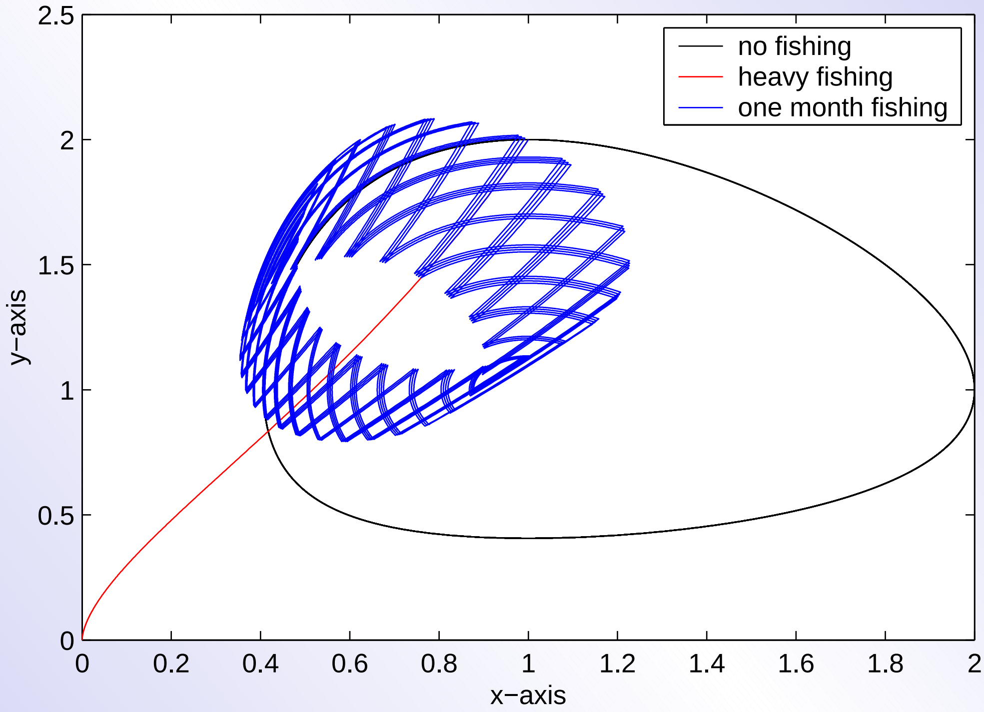

Figure 26: Three predator-prey orbits.

For clarity, if we reduce the time span to 20 years and the number of plotted points to 2000, we get the graphs in as seen in Figure 27.

If we increase the step size to 0.20, we get the graphs as seen in Figure 28. The computer is now telling lies about the population. The maximal step size is too large so the solver can't "see" the sudden breaks in the harvesting coefficient when they occur so the solver produces the graph in Figure 28.

If we now leave the step size at 0.20, and decrease the relative local tolerance to 1×10^{-9}, and decrease the absolute local tolerance to 1×10^{-12}, we get the graphs in Figure 29.

Conclusion

As we have seen in the case of the harmonic oscillator, the driven pendulum, the logistic model with a fishing season and the predator-prey dynamics with fishing, what you may think is a correct answer is in fact not. Every differential solver has is "kinks," so the user must experiment with their equations. You should not rely on the first graph that you see because the graph could be a lie.

References

[1] David Arnold, *Profesor at College of the Redwoods*

[2] *Computers, Lies, and the Fishing Season* by Robert L. Borrelli.

[3] *Differential Equations with Boundary Value Problems* by Arnold, Boggess and Polking

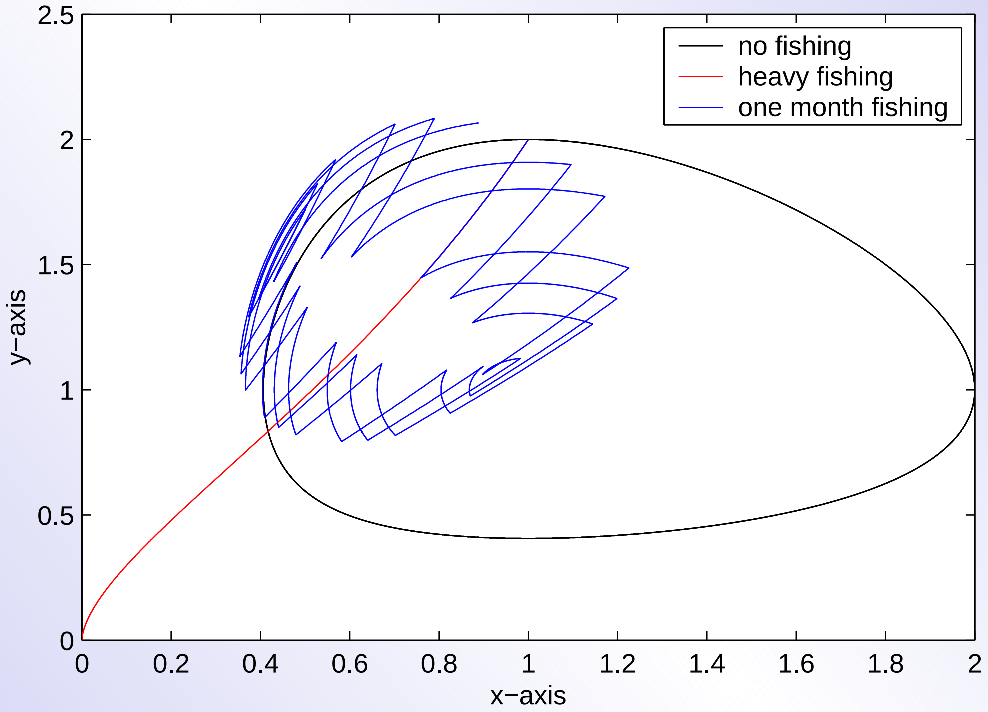

Figure 27: Step size 0.01; AbsTol. 1×10^{-10}, RelTol. 1×10^{-7}.

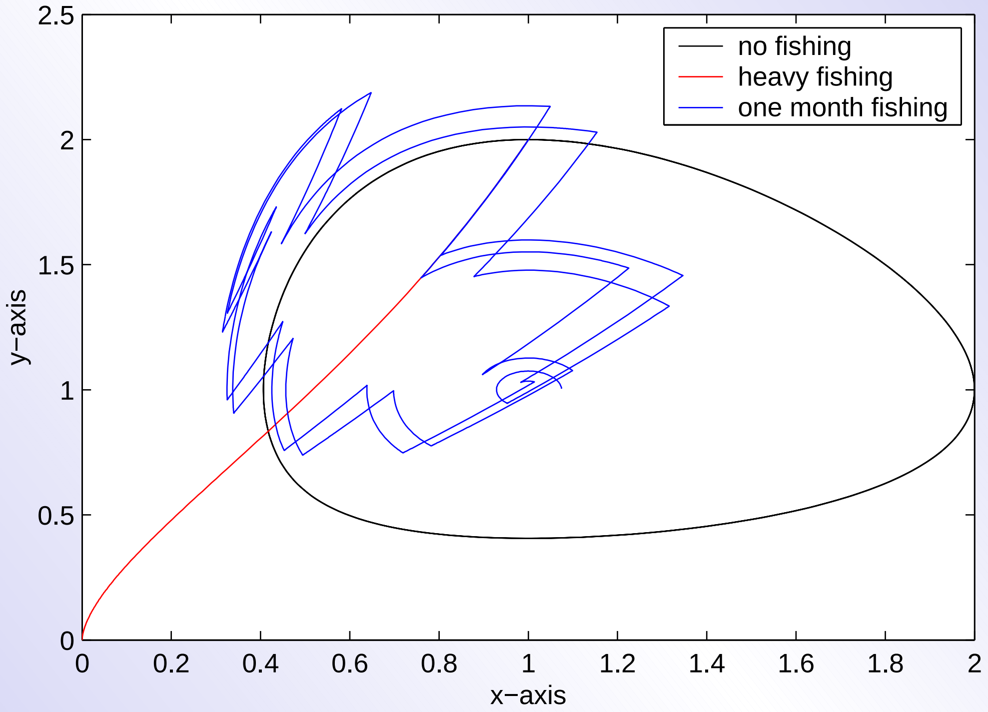

Figure 28: Step size 0.10; AbsTol. 1×10^{-10}, RelTol. 1×10^{-7}.

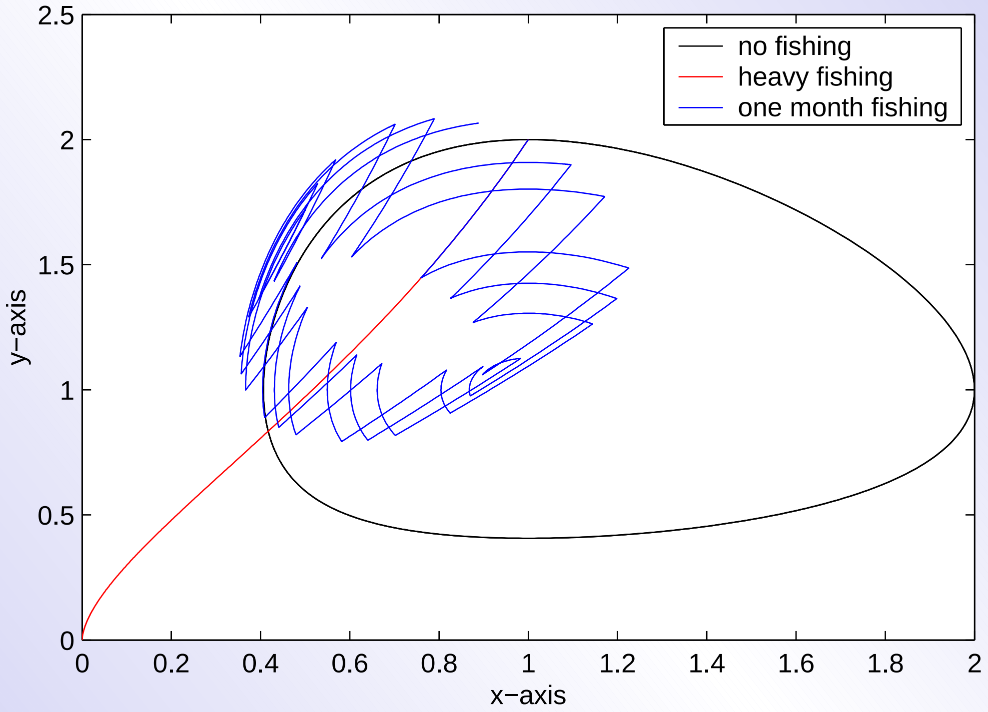

Figure 29: Step size 0.10; AbsTol. 1×10^{-12}, RelTol. 1×10^{-9}