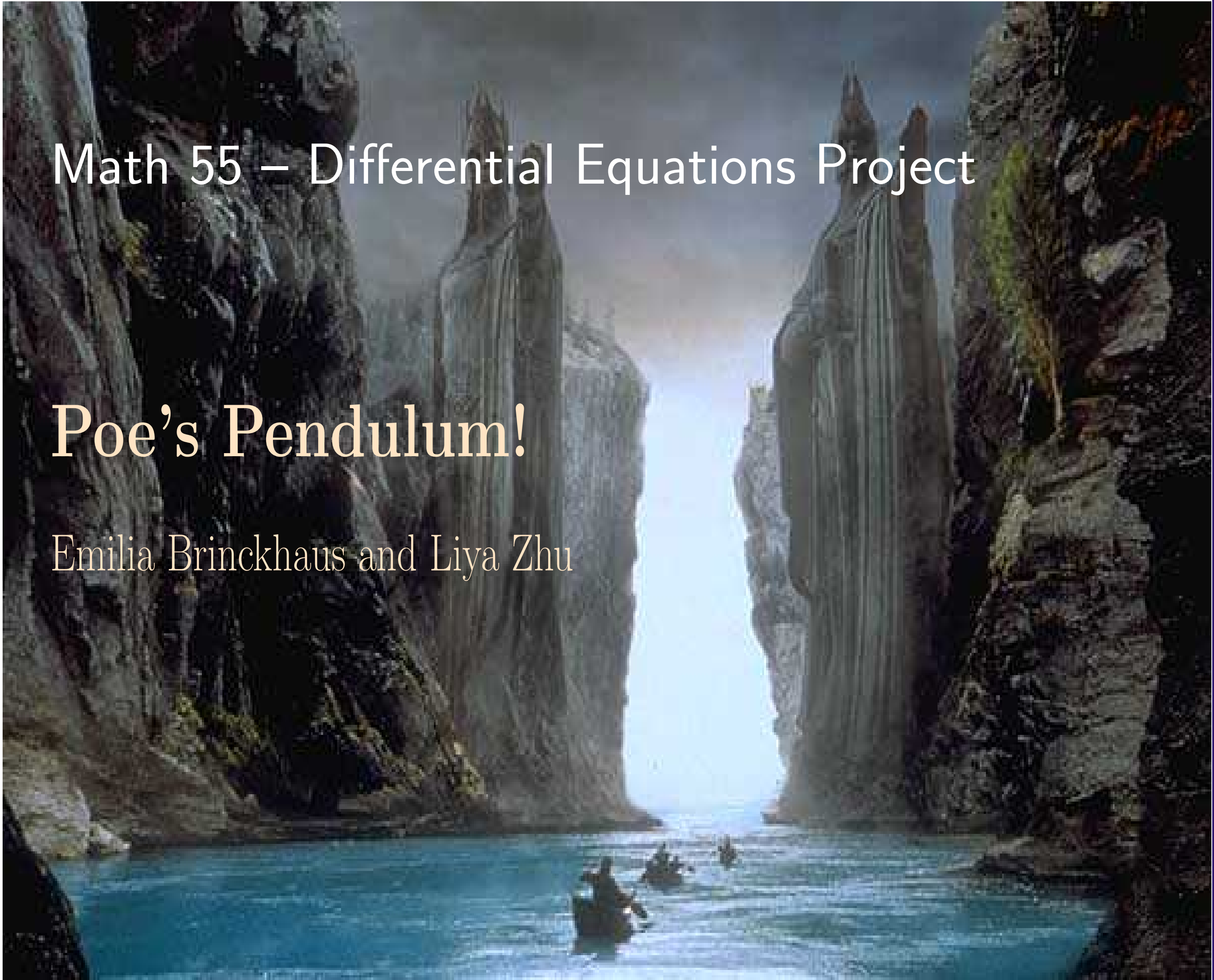


Math 55 – Differential Equations Project

Poe's Pendulum!

Emilia Brinckhaus and Liya Zhu

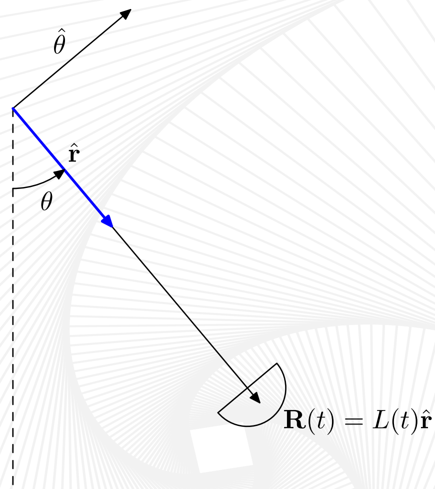
Introduction

This project is based on a story in Edgar Allan Poe's story, *The Pit and the Pendulum*, which tells of a prisoner tied on the floor,facing a sharp-edged pendulum descending toward him. Poe describes that the sweep of the descending pendulum increases as the swinging velocity goes faster. We'll try to model the pendulum's motion mathematically, and discover whether Poe's description is accurate.

Poe's Description of the Pendulum's Motion

...Looking upward, I surveyed the ceiling of my prison...I fancied that I saw [the pendulum] in motion. In a instant afterward the fancy was confirmed. *Its sweep was brief, and of course slow...*It might has been half an hour, perhaps even an hour...before I again cast my eyes upward. What I then saw confounded and amazed me. *The sweep of the pendulum had increased in extent by nearly a yard. As a natural consequence its velocity was also much greater...*I now observed–with what horror it is needless to say–that its nether extremity was formed of a crescent of glittering steel, about a foot in length from horn to horn; the horns upward, and the under edge as keen as that of a razor...and the whole hissed as it swung through the air..long, long hours of horror more than mortal during which I counted the rushing oscillations of the steel! Inch by inch–line by line–which a descent only appreciable at intervals that seemed ages–down and still down it came!...The vibration of the pendulum was at right angles to my length. I saw that the crescent was designed to cross the region of my heart...

The Model of a descending Pendulum



Length of the wire with respect to time: $L(t)$.

The angle that the wire makes with the downward vertical: $\theta(t)$.

Unit vector in direction of the wire: $\hat{\mathbf{r}}$

Unit vector perpendicular to $\hat{\mathbf{r}}$: $\hat{\theta}$.

The position vector for the pendulum bob: $\mathbf{R}(t) = L(t)\hat{\mathbf{r}}$

Position vector: $\mathbf{R} = L\hat{\mathbf{r}}$

Doing some work with the derivatives of $\hat{\mathbf{r}}$ and $\hat{\theta}$ and using the chain rule, we find and write the velocity and acceleration vectors with the inclusion of the terms $\hat{\theta}$ and θ .

$$\mathbf{R}' = L'\hat{r} + L\theta'\hat{\theta}$$

$$\mathbf{R}'' = (L'' - L\theta'\theta')\hat{\mathbf{r}} + (2L'\theta' + L\theta'')\hat{\theta}$$

From the last equation it is seen that the component of the acceleration vector in the $\hat{\theta}$ direction is:

$$2L'\theta' + L\theta''$$

Taking the mass of the pendulum bob to be m, and g being the gravitational constant, the component of the gravitational force in the $\hat{\theta}$ direction is $-mg\sin\theta$.

Equation of angular motion

By using Newton's second law, we have the equation of the pendulum's angular motion: $m(2L'\theta' + L\theta'')$ Setting our 2 equations equal to each other:

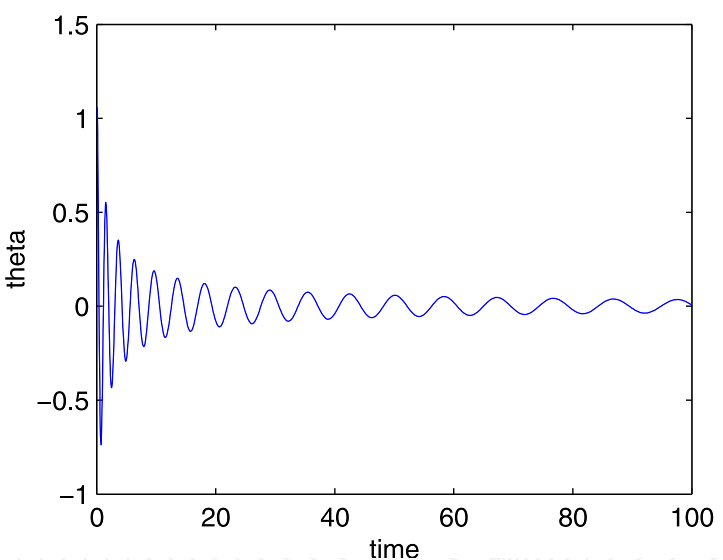
$$-mg\sin\theta = m(2L'\theta' + L\theta'') \tag{1}$$

Since the angle θ is relatively small, we use θ to replace $\sin\theta$ in our equation. After cancelling the mass factor, m, we have the equation of the pendulum's linear motion:

$$2L'\theta' + L\theta'' + g\theta = 0. \tag{2}$$

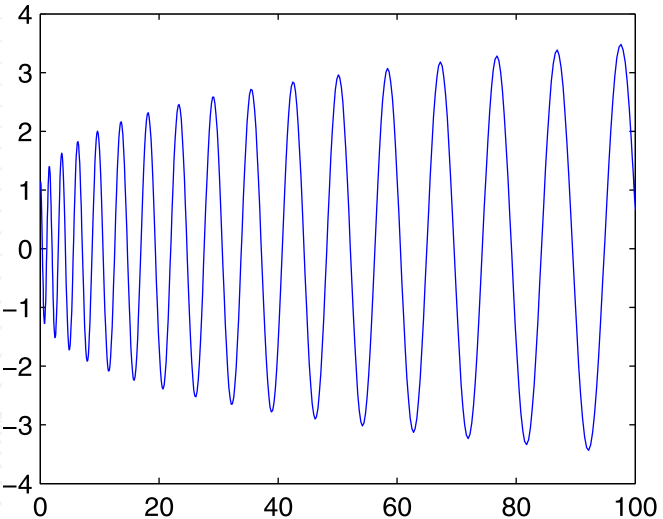
$L\theta$ and $(L\theta)'$ represent the pendulum's sweep and curvilinear velocity, respectively.

The decaying angle of a steadily descending pendulum



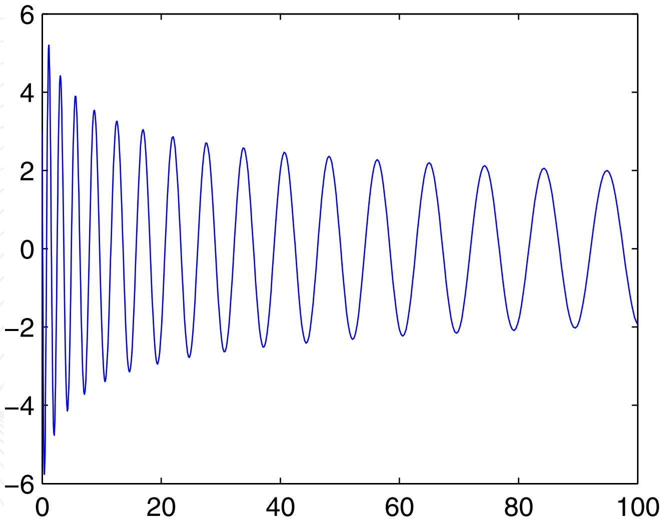
- Here is a graph drawn with the help of Matlab's ode45 solver of θ versus time.

The Sweep of the Pendulum



- This is the graph of $L\theta$, the pendulum's sweep.

The Curvilinear Velocity



- Lastly, this graph shows $(L\theta)'$ over time which is the velocity of the pendulum.

The Differential Equation of the Descending Pendulum

Assuming that the pendulum is descending at a constant rate, the length of the wire is expressed as

$$L(t) = a + bt. \tag{3}$$

a and b are positive constants. Substituting that equation for L in $2L'\theta' + L\theta'' + g\theta = 0$, we get the equation of the pendulum's angular motion:

$$(a + bt)\theta'' + 2b\theta' + g\theta = 0. \tag{4}$$

The Differential Equation Transformed

We would like to be able to change our equation into the form of an equation with known properties. Some intense work is required to figure out what new variables will transform our original equation. But we will simply introduce the two new variables x and y:

$$x = \frac{2}{b}\sqrt{(a+bt)g}, \qquad y = \theta\sqrt{a+bt}.$$

With the use of these new variables and the chain rule, we get the following equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0. \tag{5}$$

Our New Equation–Bessel's Equation!

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0. \tag{6}$$

The new equation is a Bessel's equation of order one! Bessel equations are special differential equations with solutions known as Bessel functions. Any solution $y(x)$ of this Bessel equation determines a solution of our original equation. For all solutions $y(x)$, the solution to our original equation is:

$$\theta(t) = \frac{1}{\sqrt{a+bt}}\,y\left(\frac{2}{b}\sqrt{(a+bt)g}\right)$$

Bessel Functions

One solution of the first order Bessel's equation is $J_1(x)$,which is the Bessel Function of order one of the First Kind. For a second independent solution, because we have an integer order Bessel's equation, we must turn to the first order Bessel Function of the Second Kind, depicted as $Y_1(x)$. Therefore our general solution is given by:

$$y(x) = AJ_1(x) + BY_1(x)$$

where A and B are arbitrary constants. No other solution can have any form but this.

The Solutions

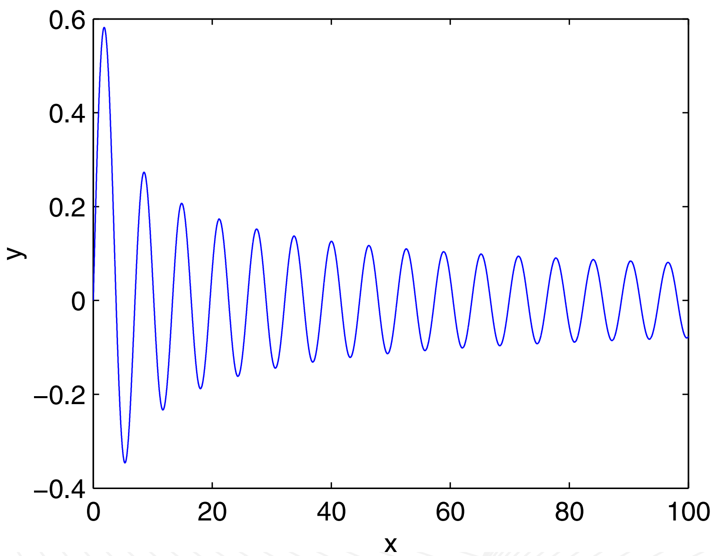
The only problem with the solutions to the Bessel equations is that they generally have the form of an infinite series, a series of ascending powers of x. We could solve these and get numerical approximations, but we are going to take a whole different approach. All Bessel's equations have several known properties. Properties of solutions of our first order Bessel equation become properties of the angular motion of the steadily descending pendulum.

Properties of Bessel Functions

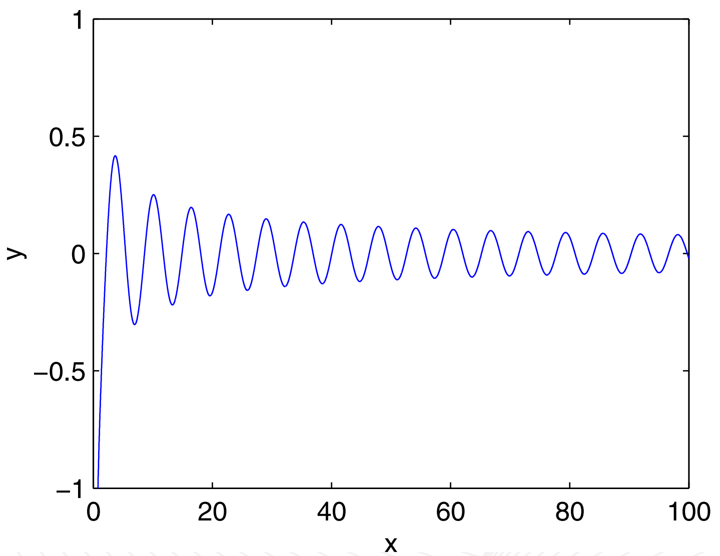
- $y(x)$ have an infinity of zeros.
- As $x \to \infty$, $y(x)$ decays to 0.
- Therefore, $y(x)$ is oscillating, with decreasing amplitude.

All these properties translate into properties of $\theta(t)$, the solution to our original differential equation.

$J_1(x)$: The Bessel Equation of the First kind



$Y_1(x)$: The Bessel Equation of the Second kind



Conclusion

Has our mathematical model fit Poe's description? Well, the sweep of our steadily descending pendulum was seen to increase over time, but this seemed to happen much more slowly than Poe's story suggested. As for the velocity, our model showed the pendulum decayed in velocity as it descended, which is contrary to Poe's comments about *rushing oscillations of steel* and the change from *its sweep was brief and of course slow* to *its velocity also was much greater*. Obviously, our model is not the pendulum of Poe's story.

This project shows that it is possible and sometimes very useful to find and analyze properties of solutions without even knowing the actual solutions. Without even solving for the Bessel functions, we could know for sure that the angle θ decays over time in an oscillatory manner.

Acknowledgements

1. David Arnold, Knowledge of LaTeXand Mathematics.

2. F.E. Relton *Applied Bessel functions* 1946.

3. Robert Borrelli, Courtney Coleman, and Dana Hobson. Poe's Pendulum,*Mathematics Magazine* Vol.58, No.2,(March 1985).

4. John Polking, Albert Boggess, and David Arnold, *Differential Equations.* 2001.

5. Greg Brown, Help with figures, computer issues, and LaTeX