The Vibrating String

Applications of Differential Equations

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Introduction

- The purpose of our Differential Equations project is to derive the wave equation to model the behavior of a vibrating string.
- After deriving the wave equation we will then be able to produce an image that represents the motion of a vibrating string. This will allow us to study the behavior of the vibrating string.

Applications of the Vibrating String

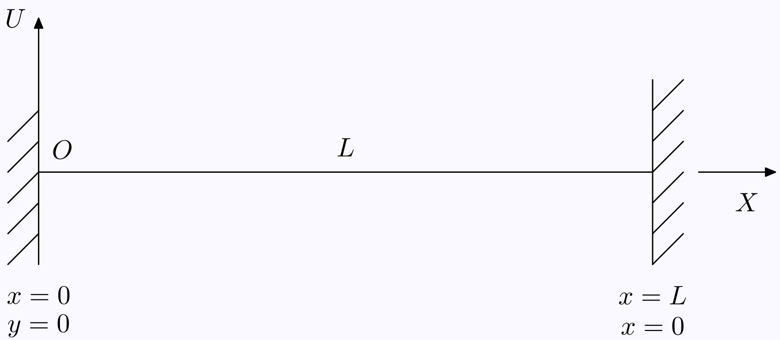
- The vibrating string has many applications:
- The most obvious application of the vibrating string is in musical instruments that have strings, such as guitars and violins.
- The equation for the vibrating string also provides a basis for the mathematics behind electromagnetic theory and quantum mechanics.

Assumptions

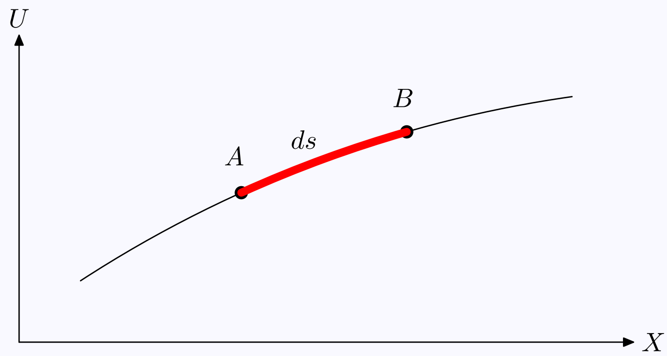
- We must assume that the string has a uniform density (μ). This means that the only force acting on the string is a uniform tension (T).
- We assume that the force due to gravity is very small when compared to the tension (T) therefore, the force due to gravity will be neglected.
- We also assume that the displacement is significantly small, so that the horizontal component of the tension may be ignored. Therefore, we will only be considering the vertical component of the tension.

Derivation of the Wave Equation

- First we will start with a string that is fixed at both ends, and has a set length (L).
- The string is in equilibrium along the x-axis.



Next



- In order to obtain the differential equation that best describes the motion of the string, we will first look at a small portion of the string bounded by the points A and B.

- The length between points A and B is represented by ds.

Tension

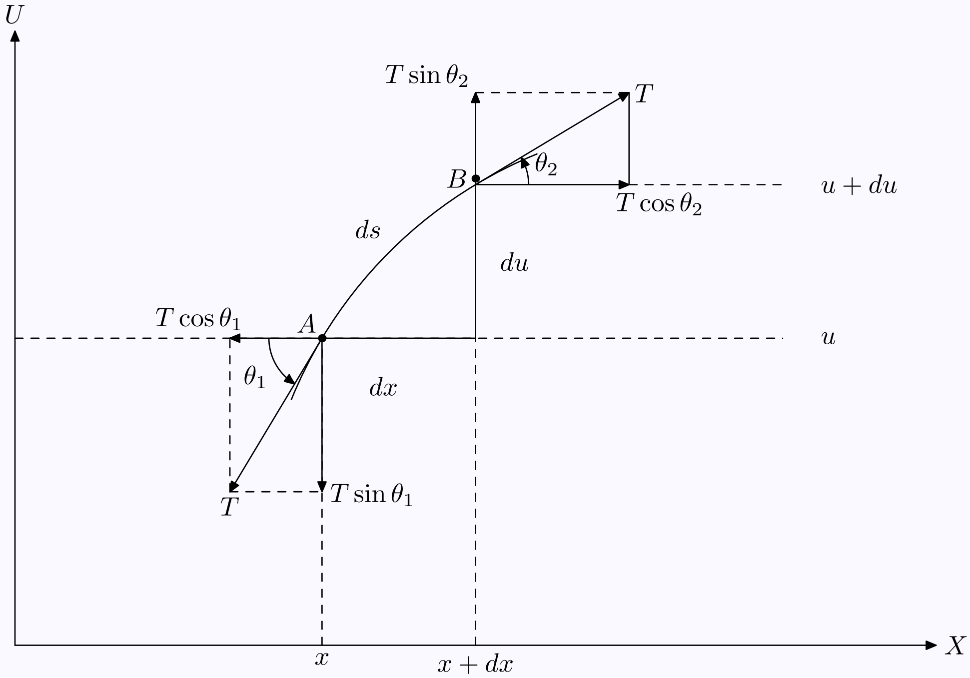
- The horizontal length will be called dx between x and $x + dx$, as shown in the figure on the next slide.
- We will assume that the magnitude of the tension at each end of ds is identical.
- The sum of the forces in the x-direction is

$$\sum F_x = T\cos\Theta_2 - T\cos\Theta_1.$$

- The sum of the forces in the u-direction is

$$\sum F_u = T\sin\Theta_2 - T\sin\Theta_1.$$

Figure 3



U
$T\sin\theta_2$
T
θ_2
B
$u + du$
$T\cos\theta_2$
ds
du
$T\cos\theta_1$
A
u
θ_1
dx
$T\sin\theta_1$
T
X
x
$x + dx$

Displacements

- As we mentioned earlier, we can assume there is no net horizontal force. This is because when Θ_1 and Θ_2 are very small, $\cos\Theta_1 \approx \cos\Theta_2$, so
$$\sum F_x = T\cos\Theta_2 - T\cos\Theta_1 = 0.$$
- Because of these assumptions we will only be considering the vertical motion of the string in the xu-plane.
- Also for small angles or displacements we can replace sine with tangent.

$$\sin\Theta_1 \approx \tan\Theta_1 \qquad \text{and} \qquad \sin\Theta_2 \approx \tan\Theta_2$$

Resultant Force

- Now we can rewrite the resultant force in the u direction as

$$\sum F_u = T\sin\theta_2 - T\sin\theta_1 \approx T\tan\Theta_2 - T\tan\Theta_1.$$

- The vertical displacement of the string is a function of position of time.
- We will let $u(x,t)$ represent the vertical displacement of the string at a point x and a time t.

Newton's Second Law

- By using Newton's second law (F = ma) we are able to rewrite the sum of the forces in the u direction. That is,

$$\sum F_u = ma_u$$

$$T\tan\theta_2 - T\tan\theta_1 = m\frac{\partial^2 u}{\partial t^2}$$

where m is the mass of the string between points A and B.

- The function $u(x,t)$ describes the displacement in x at an instant in time t, and mass is density times length, so $m = \mu dx$.

$$\mu dx\,\frac{\partial^2 u}{\partial t^2} = T\tan\Theta_2 - T\tan\theta_1 \qquad (1)$$

Calculating

- The tangent of the angle of inclination is given by the first derivative. Thus, at points A and B ,

$$\tan\theta = \frac{\partial u}{\partial x}.$$

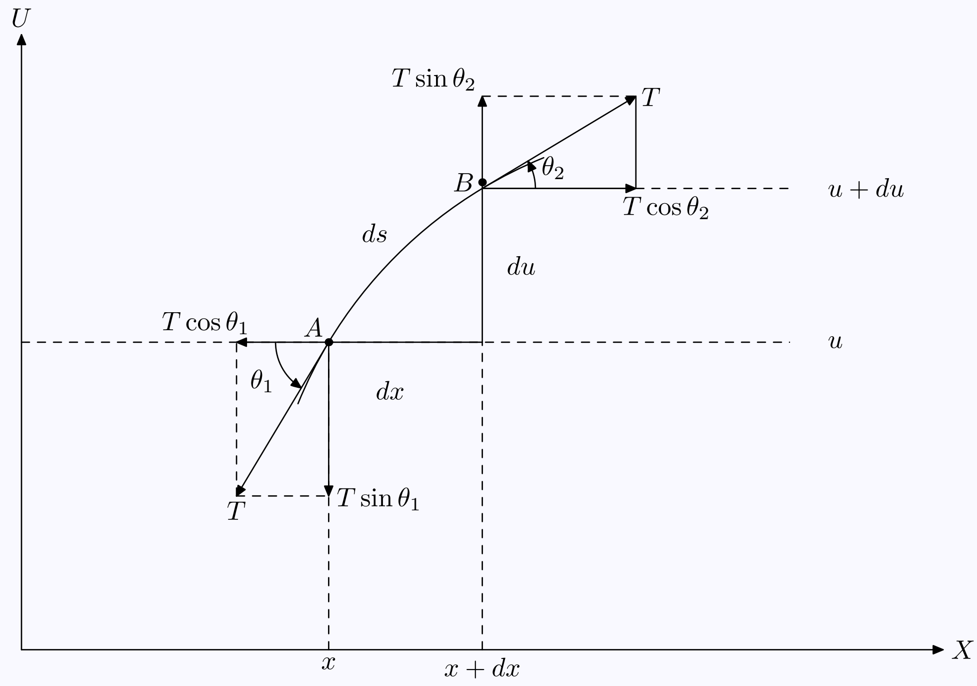
- By multiplying both sides by the tension T we obtain

$$T\tan\theta = T\frac{\partial u}{\partial x}.$$

- Thus we can write the net vertical force as

$$\sum F_u = T\tan\theta_2 - T\tan\theta_1 = T\left(\frac{\partial u}{\partial x}\right)_B - T\left(\frac{\partial u}{\partial x}\right)_A. \qquad (2)$$

Figure 3



U
$T\sin\theta_2$
T
θ_2
B
$u + du$
$T\cos\theta_2$
ds
du
$T\cos\theta_1$
A
u
θ_1
dx
T
$T\sin\theta_1$
X
x
$x + dx$

Taylor Series

- At this point, recall the Taylor Series expansion.

$$f(x+dx) = f(x) + f'(x)dx + \frac{f''(x)}{2!}(dx)^2 + \cdots$$

- The slope of the string at point B is expanded using the Taylor series.

$$\left(\frac{\partial u}{\partial x}\right)_B = \left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx + \frac{1}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_A (dx)^2 + \cdots$$

- Using equations (1) and (2),

$$\mu dx \frac{\partial^2 u}{\partial t^2} = T \tan\theta_2 - T \tan\theta_1$$

$$\mu dx \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial u}{\partial x}\right)_B - T\left(\frac{\partial u}{\partial x}\right)_A$$

- Replacing $\left(\frac{\partial u}{\partial x}\right)_B$ with Taylor's series expansion and deleting terms of higher order we have,

$$\mu dx \frac{\partial^2 u}{\partial t^2} = T\left[\left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx\right] - T\left(\frac{\partial u}{\partial x}\right)_A.$$

$$\mu dx \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2}\right)_A dx$$

- Dividing by μdx we discover that,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu}\left(\frac{\partial^2 u}{\partial x^2}\right)_A.$$

- Now we will let $\frac{T}{\mu} = a^2$ to obtain the wave equation.

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Initial Boundary Problem

- The string has length 2 and is held fixed at both ends. Therefore, the boundary conditions are

$$u(0,t) = u(2,t) = 0. \tag{4}$$

- We are assuming that the string is given an initial displacement and then released from rest; therefore, the initial conditions of the string will be

$$u(x,0) = \phi(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0 \tag{5}$$

Separation of Variables

- When solving the initial boundary problem problem (3) with homogenous boundary conditions, we will use a method called *separation of variables*. First, set

$$u(x,t) = f(x)g(t),$$

where f is a function of x and g is a function of t.

- Substituting $u(x,t) = f(x)g(t)$ into the wave equation (3), we obtain

$$f(x)g''(t) = a^2 g(t) f''(x).$$

- After some simple algebra we obtain

$$\frac{f''(x)}{f(x)} = \frac{1}{a^2}\frac{g''(t)}{g(t)}$$

.

- The only way a function of x can equal a function of t for all x and t is if they both are equal to a constant, therefore,

$$\frac{f''(x)}{f(x)} = \frac{1}{a^2}\frac{g''(t)}{g(t)} = -\lambda. \tag{6}$$

  Thus,

$$\frac{f''(x)}{f(x)} = -\lambda \qquad \text{and} \qquad \frac{1}{a^2}\frac{g''(t)}{g(t)} = -\lambda \tag{7}$$

  After a little algebra,

$$f''(x) + \lambda f(x) = 0 \qquad \text{and} \qquad g''(t) + \lambda a^2 g(t) = 0.$$

- It can be shown that $\lambda = 0$ and $\lambda < 0$ lead to the trivial solutions. We are not interested in trivial solutions, so, we will assume that $\lambda > 0$.

- To find the particular solution to the equation $f''(x) + \lambda f(x) = 0$, we will let

$$f(x) = e^{wx}.$$

- This yields the characteristic equation

$$w^2 + \lambda = 0,$$

where the roots are

$$w = \pm i\sqrt{\lambda}.$$

- Using Euler's identity the solution will take the form

$$f(x) = e^{wx} = e^{i\sqrt{\lambda}x} = \cos\sqrt{\lambda}x + i\sin\sqrt{\lambda}x,$$

which leads to the general solution

$$f(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x.$$

- The boundary condition $u(0,t) = 0$ forces

$$0 = u(0,t) = f(0)g(t)$$

for all t, which in turn implies that $f(0) = 0$.

- Substituting $f(0) = 0$,

$$0 = C_1 \cos \sqrt{\lambda}(0) + C_2 \sin \sqrt{\lambda}(0),$$

thus determining $C_1 = 0$ and $f(x) = C_2 \sin \sqrt{\lambda} x$.

- The boundary condition $u(2, t) = 0$ forces

$$0 = u(2, t) = f(2) g(t)$$

for all t, which in turn implies that $f(2) = 0$.

- After substituting $f(2) = 0$ in $f(x) = C_2 \sin \sqrt{\lambda} x$.

$$0 = f(2) = C_2 \sin \sqrt{\lambda}(2).$$

- We don't want C_2 to equal zero because this will again lead to a trivial solution. Therefore,

$$\sin 2\sqrt{\lambda} = 0.$$

This leads us to

$$\begin{aligned}\sin 2\sqrt{\lambda} &= 0\\ 2\sqrt{\lambda} &= n\pi\\ \sqrt{\lambda} &= \frac{n\pi}{2},\end{aligned}$$

for $n = 1, 2, 3 \ldots.$

- Substituting back into $f(x) = C_2 \sin \sqrt{\lambda} x$ gives

$$f(x) = C_2 \sin \frac{n\pi x}{2}. \tag{8}$$

- Next we will find the particular solution to

$$g''(t) + \lambda a^2 g(t) = 0.$$

- Subbing $g(t) = e^{wt}$ gives the characteristic equation

$$w^2 + \lambda a^2 = 0.$$

- Because $\sqrt{\lambda} = \frac{n\pi}{2}$,

$$w^2 + \frac{n^2\pi^2a^2}{4} = 0$$

and

$$w = \pm\frac{n\pi a}{2}i.$$

Therefore,

$$g(t) = e^{\frac{n\pi ait}{2}}.$$

- After using Euler's formula we have the general solution

$$g(t) = D_1 \cos\frac{n\pi at}{2} + D_2 \sin\frac{n\pi at}{2}.$$

- Given $u(x,t) = f(x)g(t)$,

$$\frac{\partial}{\partial t}u(x,t) = f(x)g'(t).$$

But the initial condition $\frac{\partial}{\partial t}u(x,0) = 0$ gives

$$f(x)g'(0) = 0.$$

for all x, we deduce that $g'(0) = 0$. Differentiating $g(t)$ above,

$$g'(t) = -\frac{D_1 n\pi a}{2} \sin \frac{n\pi a t}{2} + \frac{D_2 n\pi a}{2} \cos \frac{n\pi a t}{2}.$$

- Using the fact that $g'(0) = 0$ we have

$$0 = g'(0) = \frac{D_2 n\pi a}{2},$$

which implies that $D_2 = 0$. Therefore, we have a particular solution

$$g(t) = D_1 \cos \frac{n\pi a t}{2}. \tag{9}$$

- After multiplying Equation (8) and (9), together we have

$$u(x,t) = f(x)g(t) = b_n \sin \frac{n\pi x}{2} \cos \frac{n\pi a t}{2}$$

where b_n is an arbitrary constant.

- In addition, any linear combination must also be a solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cos\frac{n\pi at}{2}. \tag{10}$$

- Subbing in the initial condition $u(x,0) = \phi(x)$ yields the equation

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}. \tag{11}$$

- In determining the "Fourier" coefficients (b_n), we will make use of an important fact.

$$\int_0^2 \sin\left(\frac{m\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

- Now by multiplying both sides of the equation (11) by $\sin\frac{m\pi x}{2}$ and then integrating from 0 to 2 we get

$$\begin{aligned}
\int_0^2 \phi(x)\sin\frac{m\pi x}{2}dx &= \int_0^2 \sin\frac{m\pi x}{2}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}dx \\
&= \int_0^2 \sum_{n=1}^{\infty} b_n \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}dx \\
&= \sum_{n=1}^{\infty} b_n \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}dx \\
&= b_m \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{m\pi x}{2}dx \\
&= b_m
\end{aligned}$$

- Now replacing m with n,

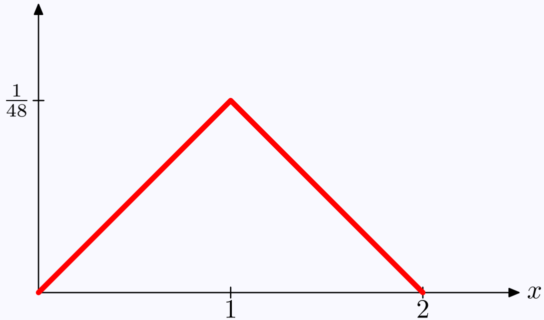
$$b_n = \int_0^2 \phi(x)\sin\frac{n\pi x}{2}dx \tag{12}$$

Example One

- In example 1 we will consider an initial position when

$$\phi(x) = \begin{cases} \frac{x}{48}, & 0 \le x \le 1 \\ \frac{2-x}{48}, & 1 < x \le 2 \end{cases}$$

- The graph of this initial condition.



- Next, calculate the Fourier coefficients.

$$b_n = \int_0^2 \phi(x) \sin \frac{n\pi x}{2}\, dx$$

- Integrating on $[0, 1]$.

$$\int_0^1 \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_0^1 \frac{x}{48} \sin \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{24n^2\pi^2}\left[-n\pi \cos \frac{n\pi}{2} + 2 \sin \frac{n\pi}{2}\right] \qquad (13)$$

- Integrating on $[1, 2]$.

$$\int_1^2 \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_1^2 \frac{2-x}{48} \sin \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{24n^2\pi^2}\left[n\pi \cos \frac{n\pi}{2} + 2 \sin \frac{n\pi}{2}\right] \qquad (14)$$

- Adding equations (13) and (14) to obtain our Fourier coefficient.

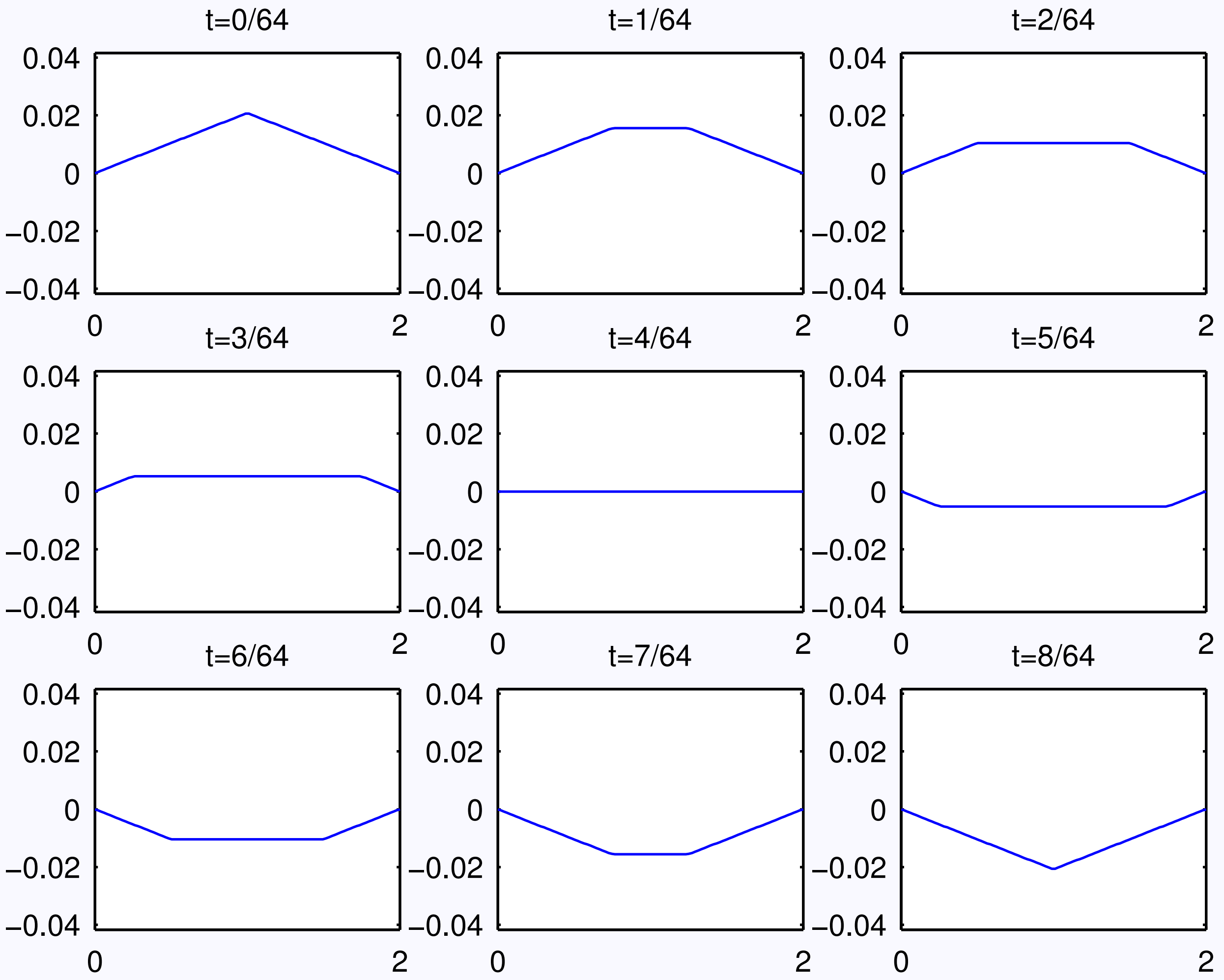
$$b_n = \frac{1}{6n^2\pi^2} \sin \frac{n\pi}{2}$$

- Substituting this result for b_n back into equation (10) and using

$a = 16$,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{6n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right) \cos(8n\pi t).$$

Snapshots of this solutions at times $t = 0$, $t = 1/64$, ..., $t = 8/64$ are shown in the graph on the next page.

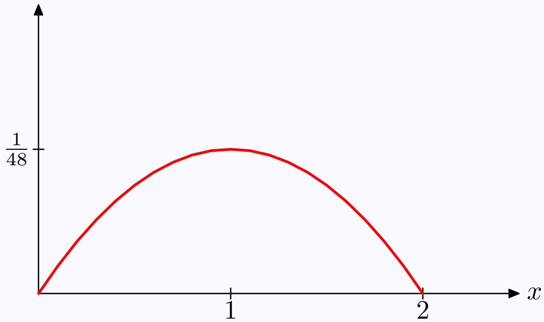

t=0/64
t=1/64
t=2/64
t=3/64
t=4/64
t=5/64
t=6/64
t=7/64
t=8/64
0.04
0.02
0
−0.02
−0.04
0
2

Example Two

- Suppose the initial position of the string is given by the function.

$$\phi(x) = (2x - x^2)/48, 0 \le x \le 2$$

- The graph of this initial condition.



- Next, calculate the Fourier coefficients.

$$b_n = \int_0^2 \phi(x) \sin \frac{n\pi x}{2}\, dx$$

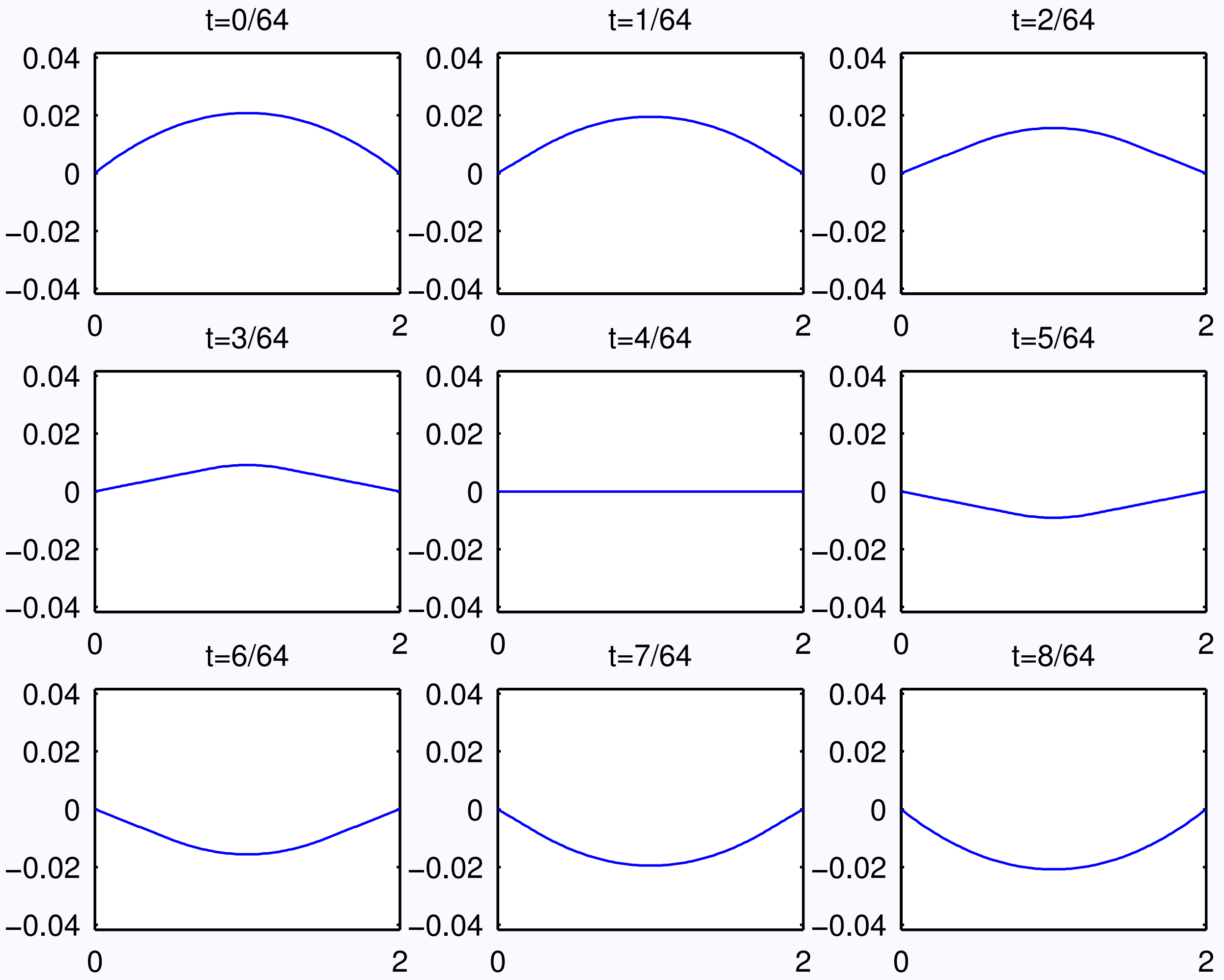
- Integrating on $[0, 2]$.

$$
\begin{aligned}
b_n &= \int_0^2 \phi(x) \sin \frac{n\pi x}{2} \, dx \\
&= \int_0^2 \frac{2x - x^2}{48} \sin \frac{n\pi x}{2} \, dx \\
&= \frac{1}{3n^3\pi^3} [1 - \cos n\pi]
\end{aligned}
$$

- Thus, subbing this result in equation (10) and adding $a = 16$ again,

$$
u(x,t) = \sum_{n=1}^{\infty} \frac{1}{3n^3\pi^3} [1 - \cos n\pi] \sin\left(\frac{n\pi x}{2}\right) \cos(8n\pi t).
$$

Snapshots of this solutions at times $t = 0$, $t = 1/64$, $\ldots$, $t = 8/64$ are shown in the graph on the next page.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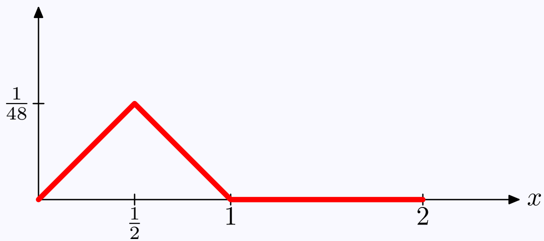
t=7/64
t=8/64
0.04
0.02
0
−0.02
−0.04
0
2

Example 3

- Suppose the initial position of the string is given by the function

$$\phi(x) = \begin{cases} x/24, & 0 \le x \le 1/2 \\ (x-1)/24, & 1/2 < x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$$

- The graph of this initial condition.



- Next, calculate the Fourier coefficients.

$$b_n = \int_0^2 \phi(x) \sin \frac{n\pi x}{2}\, dx$$

- Integrating on $[0, 1/2]$.

$$\int_0^{1/2} \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_0^{1/2} \frac{x}{24} \sin \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{24n^2\pi^2}\left[-n\pi \cos \frac{n\pi}{4} + 4 \sin \frac{n\pi}{4}\right] \quad (15)$$

- Integrating on $[1/2, 1]$.

$$\int_{1/2}^{1} \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_{1/2}^{1} \frac{1-x}{24} \sin \frac{n\pi x}{2}\, dx$$
$$= \frac{1}{24n^2\pi^2}\left[n\pi \cos \frac{n\pi}{4} + 4 \sin \frac{n\pi}{4} - 4 \sin \frac{n\pi}{2}\right] \quad (16)$$

- Adding equations (15) and (16),

$$b_n = \frac{1}{24n^2\pi^2}\left[8 \sin \frac{n\pi}{4} - 4 \sin \frac{n\pi}{2}\right] = \frac{1}{6n^2\pi^2}\left[2 \sin \frac{n\pi}{4} - \sin \frac{n\pi}{2}\right]$$

- Thus, subbing back into equation (10) with $a = 16$,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{6n^2\pi^2} \left[2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}\right] \sin\left(\frac{n\pi x}{2}\right) \cos(8n\pi t).$$

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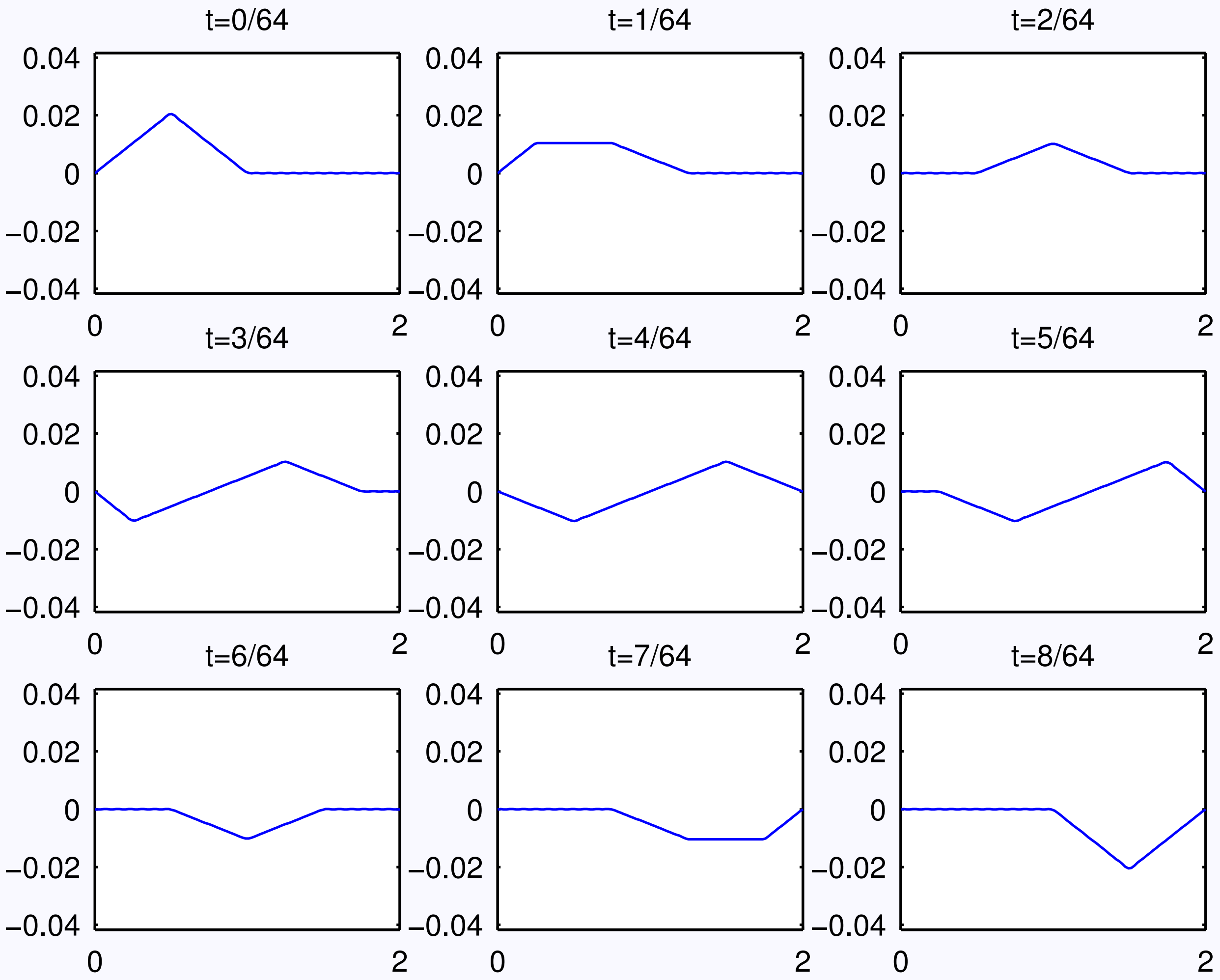
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2