9.2 Project B
Implementing Euler's Method

We illustrate below the implementation of Euler's method with graphing calculators and systems like *Maple*, *Mathematica*, and MATLAB. As a typical example, we will use the initial value problem

$$\frac{dy}{dx} = x + y, \qquad y(0) = 1 \tag{1}$$

For this project, you should implement Euler's method on your own calculator or in a programming language of your choice. First test your implementation by carrying through its application to the initial value problem in (1), and then apply it to solve some of the problems for Section 9.2 in the text. Then carry out the following investigation.

Famous Numbers Investigation

The problems below describe the numbers $e \approx 2.71828$, $\ln 2 \approx 0.69315$, and $\pi \approx 3.14159$ as specific values of certain initial value problem solutions. In each case, apply Euler's method with $n = 50, 100, 200, \cdots$ subintervals (doubling n each time). How many subintervals are needed to obtain — twice in succession — the correct value of the target number rounded off to 3 decimal places?

1. The number $e = y(1)$ where $y(x)$ is the solution of the initial value problem $y' = y, \quad y(0) = 1$.

2. The number $\ln 2 = y(2)$ where $y(x)$ is the solution of the initial value problem $y' = 1/x, \quad y(1) = 0$.

3. The number $\pi = y(1)$ where $y(x)$ is the solution of the initial value problem $y' = 4/(1+x^2), \quad y(0) = 0$.

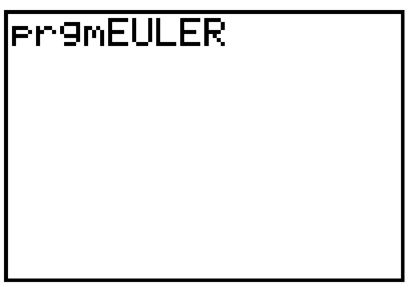
Also, explain in each problem what the point is — why the approximate value of the indicated famous number is, indeed, the expected numerical result.

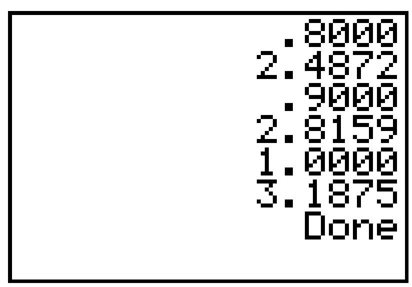
Using a Graphing Calculator

The I-83 program EULER listed in the first column below applies Euler's method to the initial value problem in (1). The comments in the final column should clarify the meanings of the successive program steps. The corresponding BASIC commands in the middle column are included for comparison.

TI-83	BASIC	COMMENT
PROGRAM:EULER	**Program EULER**	Program title
:10→N	**N = 10**	No. of steps
:0→X	**X = 0**	Initial x
:1→Y	**Y = 1**	Initial y
:1→B	**B = 1**	Final x
:(B-X)/N→H	**H = (B-X)/N**	Step size
:For(I,1,N)	**FOR I=1 TO N**	Begin loop
:X+Y→F	**F = X + Y**	Function value
:Y+H*F→Y	**Y = Y + H*F**	Euler iteration
:X+H→X	**X = X + H**	New x
:Disp X,Y	**PRINT X,Y**	Display results
:End	**NEXT I**	End loop

To increase the number of steps (and thereby decrease the step size) you need only change the value of **N** specified in the first line of the program. To apply Euler's method to a different equation $y' = f(x, y)$, you need only change the single line that calculates the function value **F**. Running Program EULER with $n = 10$ steps with $h = 0.1$ from $a = 0$ to $b = 1$, we see the following screens:





The output consists pairs of successively displayed x_i- and y_i-values. In particular, we see that Euler's method with n = 10 steps gives $y(1) \approx 3.1875$ for the initial value problem in (1). The exact solution is $y(x) = 2e^x - x - 1$, so the actual value at $x = 1$ is $y(1) = 2e - 2 \approx 3.4366$. Thus our Euler approximation underestimates the actual value by about 7.25%.

Using Maple

To apply Euler's method to the initial value problem in (1), we first define the right-hand side function $f(x, y) = x + y$ in the differential equation.

```
f := (x,y) -> x + y;
```

$$f := (x, y) \rightarrow x + y$$

To approximate the solution with initial value $y(x_0) = x_0$ on the interval $[x_0, x_f]$, we enter first the initial values of x and y and the final value of x.

```
x0 := 0:    y0 := 1:
xf := 1:
```

and then the desired number n of steps and the resulting step size h.

```
n := 10:
h := evalf((xf - x0)/n);
```

$$h := .10000$$

After we initialize the values of x and y,

```
x := x0:    y := y0:
```

Euler's method itself is implemented by the following **`for`**-loop, which carries out the iteration

$$y_{n+1} = y_n + h\, f(x_n, y_n), \qquad x_{n+1} = x_n + h$$

n times in successtion to take n steps across the interval from $x = x_0$ to $x = x_f$.

```
for i from 1 to n do
    k := f(x,y):          # the left-hand slope
    y := y + h*k:         # Euler step to update y
    x := x + h:           # update x
    print(x,y);           # display current values
    od:
```

.10000,	1.1000
.20000,	1.2200
.30000,	1.3620
.40000,	1.5282
.50000,	1.7210
.60000,	1.9431
.70000,	2.1974
.80000,	2.4871
.90000,	2.8158
1.0000,	3.1874

Note that x is updated after y in order that the computation $k = f(x, y)$ can use the left-hand values (with neither yet updated).

The output consists of x- and y-columns of resulting x_i- and y_i-values. In particular, we see that Euler's method with $n = 10$ steps gives $y(1) \approx 3.1874$ for the initial value problem in (1). The exact solution is $y(x) = 2e^x - x - 1$, so the actual value at $x = 1$ is $y(1) = 2e - 2 \approx 3.4366$. Thus our Euler approximation underestimates the actual value by about 7.25%.

If only the final endpoint result is wanted explicitly, then the print command can be removed from the loop and executed following it:

```
x := x0:    y := y0:     # re-initialize
for i from 1 to n do
    k := f(x,y):         # the left-hand slope
    y := y + h*k:        # Euler step to update y
    x := x + h:          # update x
    od:

print(x,y);
```

1.0000, 3.1874

For a different initial value problem, we need only enter the appropriate new function $f(x, y)$ and the desired initial and final values in the first two commands above, then re-execute the subsequent ones.

Using Mathematica

To apply Euler's method to the initial value problem in (1), we first define the right-hand side function $f(x, y) = x + y$ in the differential equation.

```
f[x_,y_] := x + y
```

To approximate the solution with initial value $y(x_0) = x_0$ on the interval $[x_0, x_f]$, we enter first the initial values of x and y and the final value of x.

```
x0 = 0;      y0 = 1;
xf = 1;
```

and then the desired number n of steps and the resulting step size h.

```
n = 10;
h = (xf - x0)/n  //  N
0.1
```

After we initialize the values of x and y,

```
x = x0;      y = y0;
```

Euler's method itself is implemented by the following **Do**-loop, which carries out the iteration

$$y_{n+1} \;=\; y_n + h\, f(x_n, y_n), \qquad x_{n+1} \;=\; x_n + h$$

n times in successtion to take n steps across the interval from $x = x_0$ to $x = x_f$.

```
Do[  k = f[x,y];           (* the left-hand slope     *)
     y = y + h*k;          (* Euler step to update y  *)
     x = x + h;            (* update x                *)
     Print[x,"    ",y],    (* display updated values  *)
     {i,1,n} ]

0.1     1.1
0.2     1.22
0.3     1.362
0.4     1.5282
0.5     1.72102
0.6     1.94312
0.7     2.19743
0.8     2.48718
0.9     2.8159
1.      3.18748
```

Note that x is updated after y in order that the computation $k = f(x, y)$ can use the left-hand values (with neither yet updated).

The output consists of x- and y-columns of resulting x_i- and y_i-values. In particular, we see that Euler's method with $n = 10$ steps gives $y(1) \approx 3.1875$ for the initial value problem in (1). The exact solution is $y(x) = 2e^x - x - 1$, so the actual value at $x = 1$ is $y(1) = 2e - 2 \approx 3.4366$. Thus our Euler approximation underestimates the actual value by about 7.25%.

If only the final endpoint result is wanted explicitly, then the print command can be removed from the loop and executed following it:

```
Do[  k = f[x,y];           (* the left-hand slope     *)
     y = y + h*k;          (* Euler step to update y  *)
     x = x + h,            (* update x                *)
     {i,1,n} ]

Print[x,"    ",y]
1.     3.18748
```

For a different initial value problem, we need only enter the appropriate new function $f(x, y)$ and the desired initial and final values in the first two commands above, then re-execute the subsequent ones.

Using MATLAB

To apply Euler's method to the initial value problem in (1), we first define the right-hand function $f(x, y)$ in the differential equation. User-defined functions in MATLAB are defined in (ASCII) text files. To define the function $f(x, y) = x + y$ we save the MATLAB function definition

```
function  yp = f(x,y)
yp = x + y;    % yp = y'
```

in the text file **`f.m`**.

To approximate the solution with initial value $y(x_0) = x_0$ on the interval $[x_0, x_f]$, we enter first the initial values

```
x0 = 0;    y0 = 1;
xf = 1;
```

and then the desired number n of steps and the resulting step size h.

```
n = 10;
h = (xf - x0)/n
h =
    0.1000
```

After we initialize the values of x and y,

```
x = x0;     y = y0;
```

and the column vectors $\mathbf{X}$ and $\mathbf{Y}$ of approximate values

```
X = x;     Y = y;
```

Euler's method itself is implemented by the following **`for`**-loop, which carries out the iteration

$$y_{n+1} = y_n + h\, f(x_n, y_n), \qquad x_{n+1} = x_n + h$$

n times in successtion to take n steps across the interval from $x = x_0$ to $x = x_f$.

```
for i = 1 : n                % for i = 1 to n do
    k = f(x,y);              % the left-hand slope
    y = y + h*k;             % Euler step to update y
    x = x + h;               % update x
    X = [X; x];              % adjoin new x-value
    Y = [Y; y];              % adjoin new y-value
    end
```

Note that x is updated after y in order that the computation $k = f(x, y)$ can use the left-hand values (with neither yet updated).

As output the loop above produces the resulting column vectors **X** and **Y** of x- and y-values that can be displayed simultaneously using the command

```
[X,Y]
ans =
         0    1.0000
    0.1000    1.1000
    0.2000    1.2200
    0.3000    1.3620
    0.4000    1.5282
    0.5000    1.7210
    0.6000    1.9431
    0.7000    2.1974
    0.8000    2.4872
    0.9000    2.8159
    1.0000    3.1875
```

In particular, we see that $y(1) \approx 3.1875$ for the initial value problem in (1). If only this final endpoint result is wanted explicitly, then we can simply enter

```
[X(n+1), Y(n+1)]
ans =
    1.0000    3.1875
```

The index **n+1** (instead of **n**) is required because the initial values x_0 and y_0 are the initial vector elements **X(1)** and **Y(1)**, respectively.

The exact solution of the initial value problem in (1) is $y(x) = 2e^x - x - 1$, so the actual value at $x = 1$ is $y(1) = 2e - 2 \approx 3.4366$. Thus our Euler approximation underestimates the actual value by about 7.25%.

For a different initial value problem, we need only define the appropriate function $f(x, y)$ in the file **f.m**, then enter the desired initial and final values in the first command above and re-execute the subsequent ones.

Automating Euler's Method

The **`for`**-loop above can be automated by saving the MATLAB function definition

```
function  [X,Y] = euler1(x,xf,y,n)
h = (xf - x)/n;                 % step size
X = x;                          % initial x
Y = y;                          % initial y
for i = 1 : n                   % begin loop
     y = y + h*f(x,y);          % Euler iteration
     x = x + h;                 % new x
     X = [X;x];                 % update x-column
     Y = [Y;y];                 % update y-column
     end                        % end loop
```

in the text file **`euler1.m`** (we use the name **`euler1`** to avoid conflict with MATLAB's built-in **`euler`** function). This function assumes that the function $f(x, y)$ has been defined and saved in the MATLAB file **`f.m`**.

The function **`euler1`** applies Euler's method to take n steps from x to x_f starting with the initial value y of the solution. For instance, with **`f`** as previously defined, the command

```
[X,Y] = euler1(0,1, 1, 10);  [X,Y]
```

is a one-liner that generates table **`[X,Y]`** displayed above to approximate the solution of the initial value problem $y' = x + y, \; y(0) = 1$ on the x-interval $[0, 1]$.