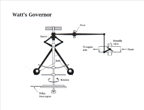

A Differential Look at the Watt's Governor

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May 15, 2003

Abstract

In an effort to harness the revolutionary potential of the steam engine, James Watt invented his governor in the late 1700's. It wasn't until almost one hundred years later that the mathematical models to describe the principals behind the Watt's governor were derived by James Maxwell. Herein we will describe the characteristics of Mr. Watt's device and how bifurcations and damping played critical rolls in its successful implementation.

Introduction

A governor is a device that is found in many machines where some degree of automation of the engine speed is needed. There are many methods to achieve this control. In particular we will investigate the Watt's governor invented by James Watt in the late 1700s.

As depicted in Figure 1, Watt's governor is comprised of a rotating vertical shaft which has two hinged arms connected near the top. At the end of each arm is a ball of a given mass which is free to swing in a vertical plane. The speed of the governor's rotating shaft is directly linked to the speed of the engine it is controlling. An increase or decrease in rotational speed causes the arms to swing up or down respectively, thus changing the angle θ they make with the vertical axis. Connecting the motion of these arms with a throttle or brake control will allow the governor to affect the engine speed as a function of θ.

Since this is an exceptionally complicated situation to model, we will begin by looking at a simpler model in an effort to get a handle on the forces involved and discover the relationship

Watt's Governor

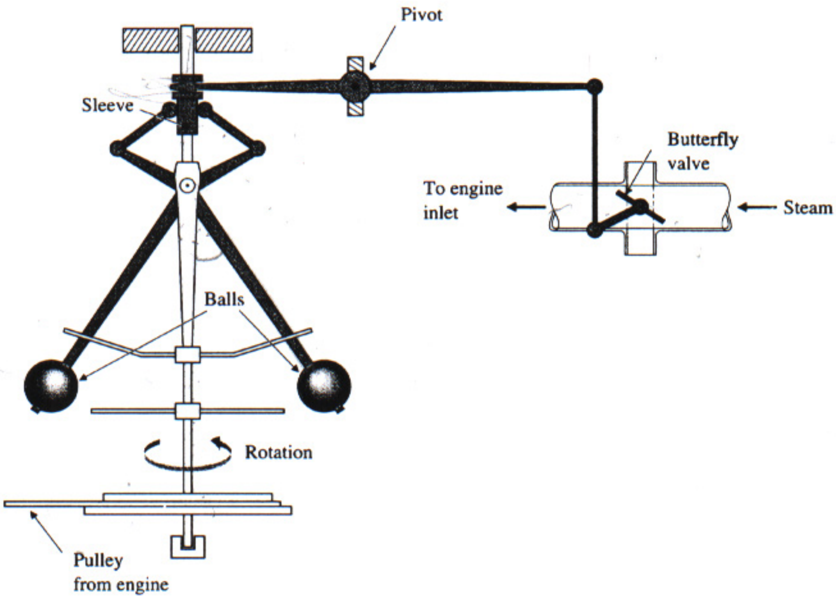
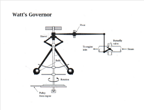


Figure 1: Watt's governor controlling a steam engine.



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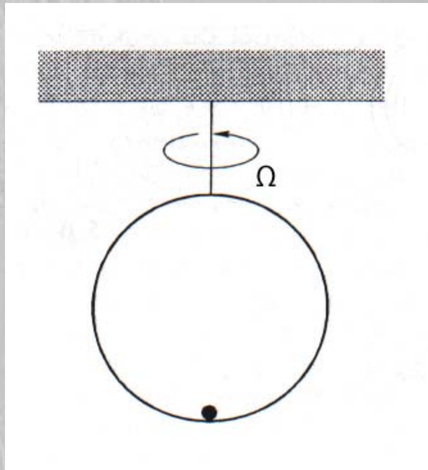


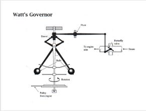
Figure 2: Ball-bearing in a rotating hoop.

between the rotational speed Ω and the angle θ of the mass with the vertical.

A Simplified Version

In Figure 2, we have a hoop that rotates about the central axis with a ball-bearing free to move within a track in the hoop.

If the ball is perfectly centered at the bottom of the hoop then it will remain there for all time regardless of how fast the hoop rotates. However, given the slightest deviation of angle, and a low rotational speed of the hoop, the ball will oscillate about a single equilibrium point at the bottom. See Figure 3 for a graphical representation of this in Matlab's pplane. Here we used equations which we will derive shortly. We gave the ball an initial angle of $.5$ rad and a slow rotational speed of 1 rad/sec. In Figure 4 it is easier to see this behavior as perpetual oscillations about the single stable point. The behavior depicted in these graphs will be the



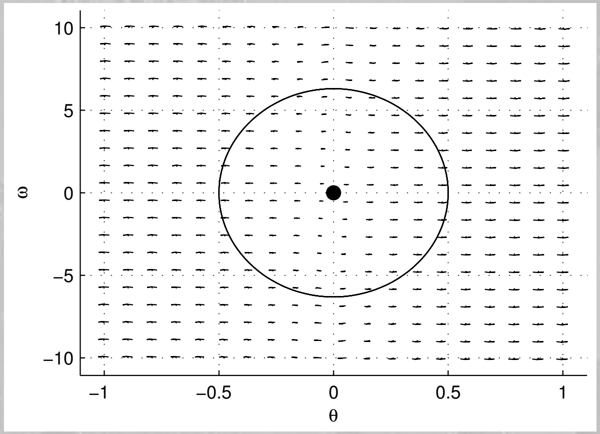
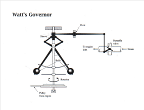


Figure 3: Phase plane for $\Omega = 1$ rad/sec.

case for a whole range of speeds until we reach a critical speed Ω_0.

As we increase the rotational speed past the critical Ω_0 the behavior changes dramatically. Suddenly the position at the bottom of the hoop is no longer a stable point and the ball begins to move up the side of the hoop. The behavior of this system endured a fundamental change because one of the parameters had surpassed a critical value. Because we now have more equilibrium points than we did before, this phenomenon is called a bifurcation. In our governor, this bifurcation means that below the critical rotational speed Ω_0 the device is useless since the equilibrium point is at $\theta = 0$ and our device's control is a function of θ.

In Figure 5 the hoop is rotating at somewhere beyond Ω_0 and the ball has left its position at the bottom of the hoop to oscillate about some other point θ. The equilibrium point at the bottom of the hoop has morphed into three possible equilibrium points: One at the bottom of the hoop and two that are equidistant from the bottom, θ radians either left or right from the bottom. Again, a Matlab graph is helpful here. Figure 6 shows the phase plane after bifurcation. The point at the bottom of the hoop is now an unstable equilibrium while the other two are stable equilibrium points and show perpetual oscillatory behavior.



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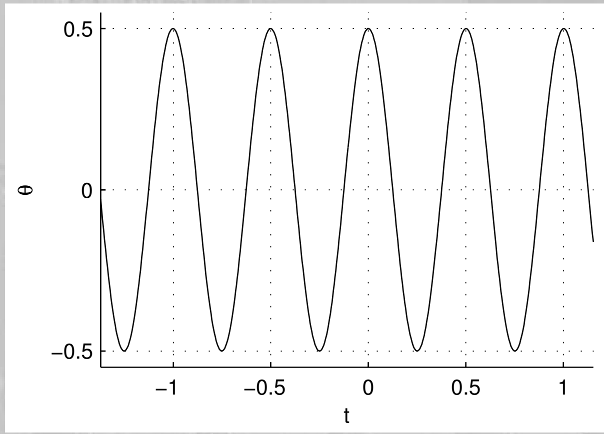


Figure 4: θ vs. t for $\Omega = 1$ rad/sec.

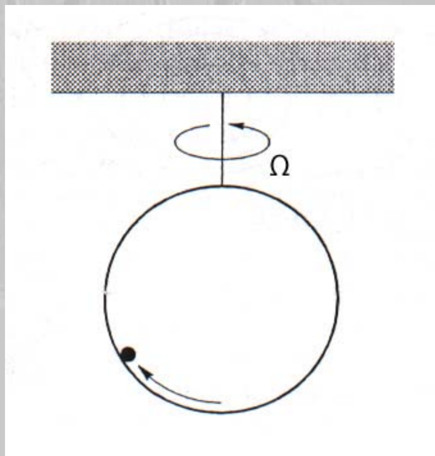
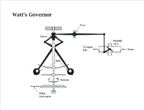


Figure 5: For $\Omega > 12.8$ rad/sec the ball moves toward a new equilibrium point.



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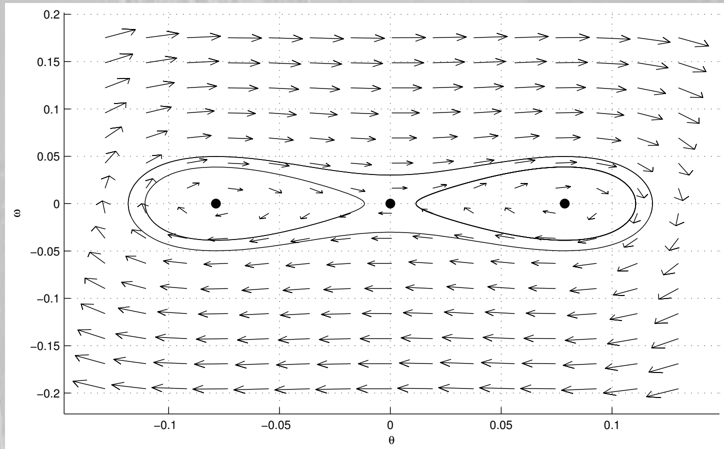


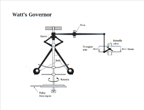
Figure 6: Phase plane for $\Omega = 13$ rad/sec.

To obtain the equation that describes this motion we need to sum the forces acting on the ball.

Identifying the Forces

Figure 7 shows that there are three forces acting on the ball at all times. They are gravity $-mg$, the centrifugal force F_{cent} and the normal force from the inside of the hoop. The normal force is actually a function of the rotational velocity Ω about the vertical axis and is analogous to the tension in the arms of the governor, so we label it T

In Figure 8 we have broken the forces into their perpendicular components which are aligned radially and tangentially to the hoop. At any point along the curved path of the ball's travel in the vertical plane we see that the normal force is always balanced by some combination of the components of the gravitational force and the centrifugal force. The components of these two forces that act directly opposed to T have been highlighted in blue and are labeled



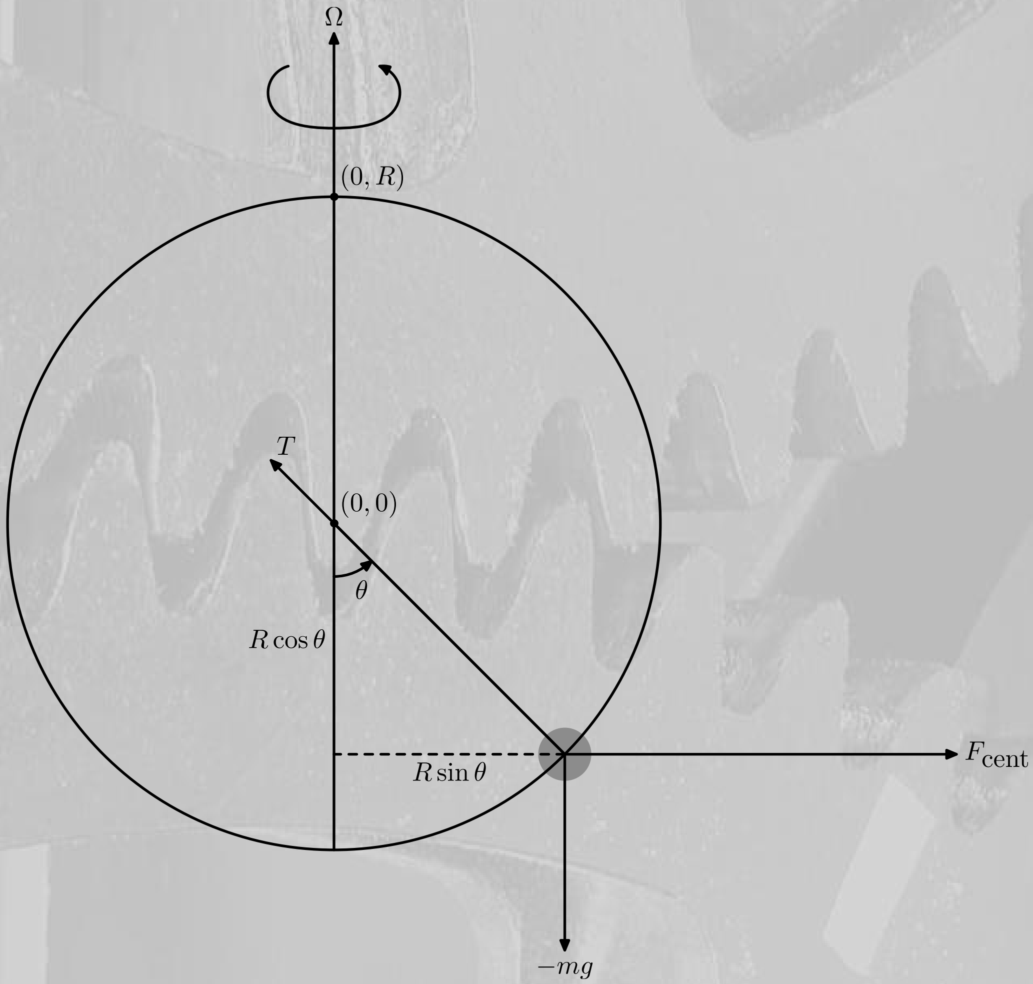


Figure 7: Forces acting on the mass.

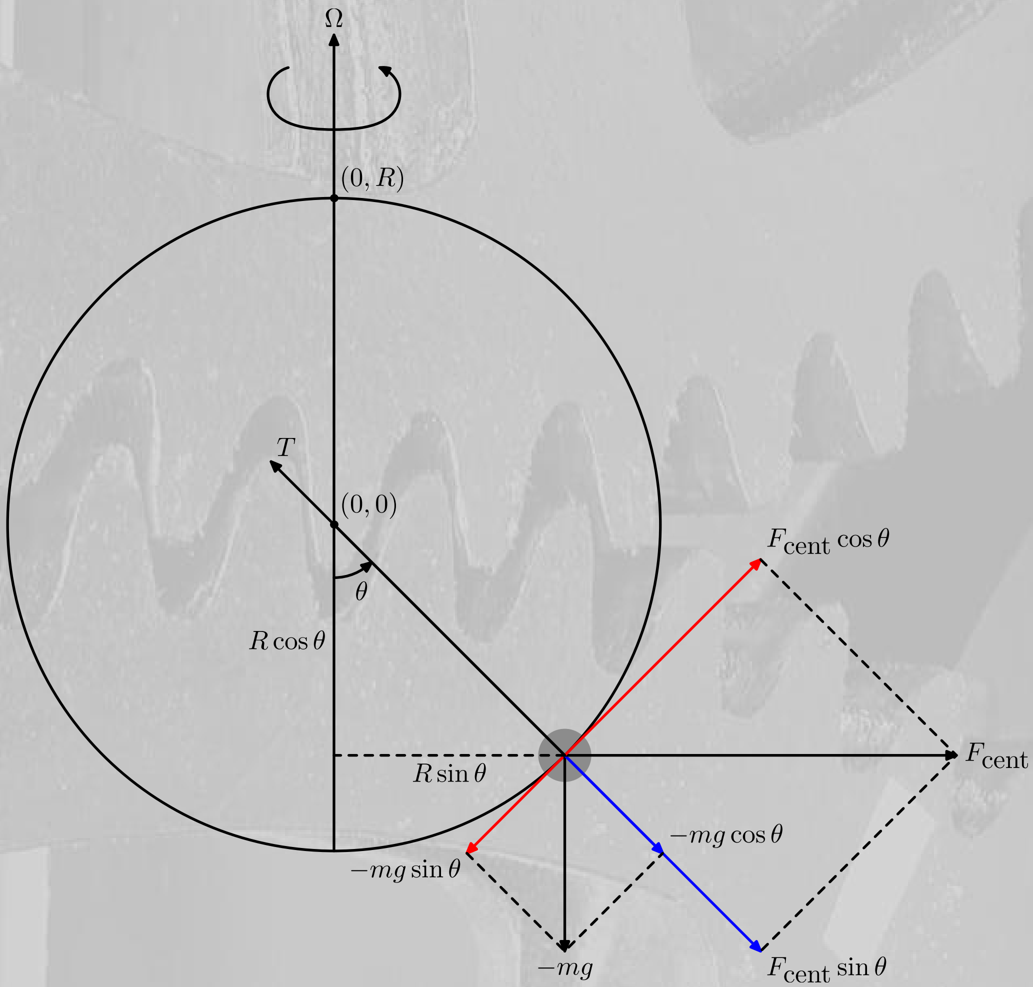


Figure 8: Tangential forces (in the vertical plane) in red and radial forces in blue.

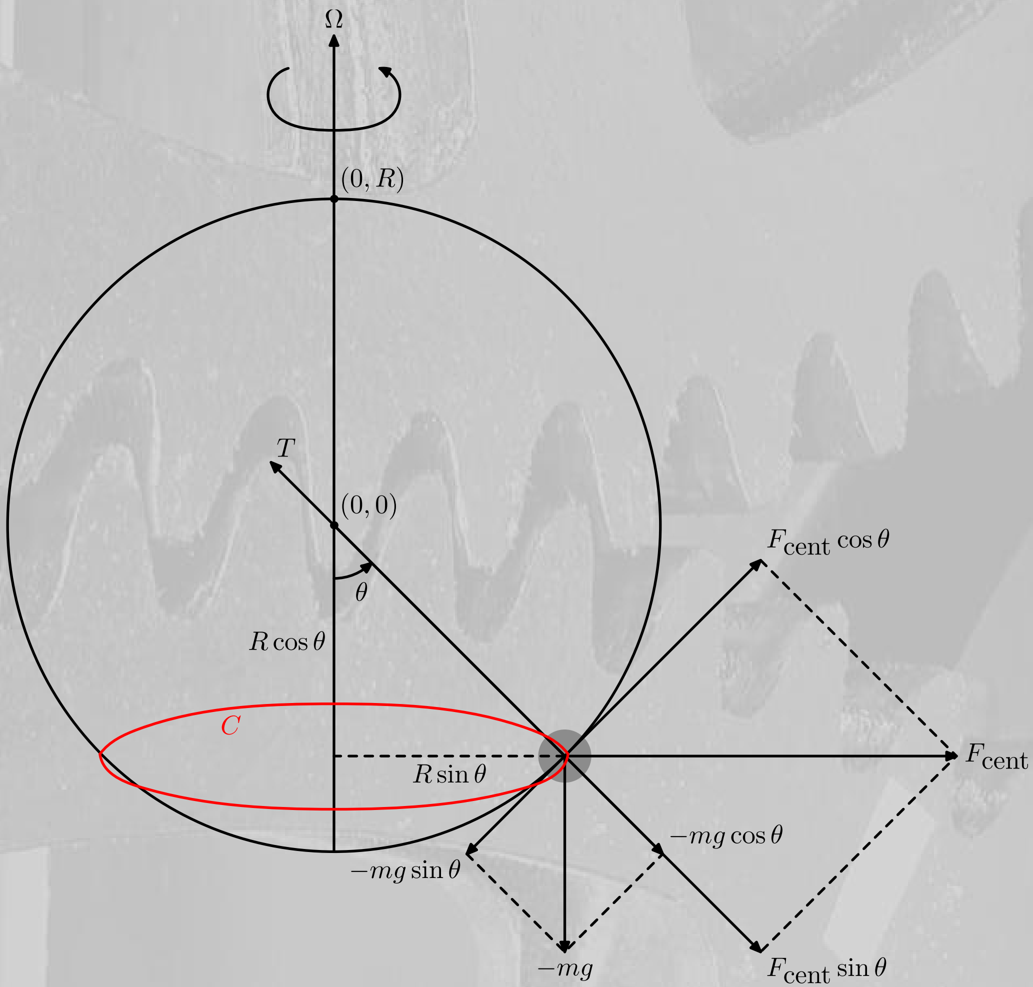
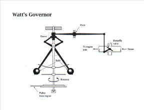


Figure 9: The horizontal path of the ball.



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$-mg\cos\theta$, and $F_{\text{cent}}\sin\theta$.

However, there are forces tangent to the circular path in the vertical plane that sometimes do not balance. When this happens there will be a tangential acceleration a_T that is proportional to the ball's angular acceleration $\ddot{\theta}$ in the vertical plane. These tangential forces, highlighted in red, are $-mg\sin\theta$, and $F_{\text{cent}}\cos\theta$, which is the tangential component of the centrifugal force. We must find a mathematical expression for the centrifugal force in order to proceed.

Consider the horizontal circle C swept out by the ball in Figure 9. The radius of C will change in proportion to Ω, but it is important to note that the centrifugal force on any object moving in a circle is equal to the mass times the radial acceleration. From rotational kinematics, the radial acceleration is equal to the square of the linear velocity divided by the radius. As well, the linear velocity is equal to the radius times the angular velocity.

$$\begin{cases} v_{\text{lin}} = rv_{\text{ang}} \\ a_r = \dfrac{v_{\text{lin}}^2}{r} \\ F_{\text{cent}} = ma_r \end{cases}$$

In our case, Ω is the angular velocity about the center of C and the radius is $R\sin\theta$. Therefore, our linear velocity is,

$$v_{\text{lin}} = (R\sin\theta)\Omega.$$

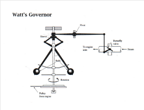
Squaring and dividing by the radius gives our radial acceleration toward the center of C,

$$a_r = \frac{[(R\sin\theta)\Omega]^2}{R\sin\theta} = (R\sin\theta)\Omega^2.$$

Finally, the centrifugal force acting on the ball is the mass times a_r,

$$F_{\text{cent}} = m(R\sin\theta)\Omega^2 = m\Omega^2 R\sin\theta.$$

Now that we have determined the tangential forces that induce the ball's acceleration in the vertical plane, we sum these forces to arrive at the second order equation that describes



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the motion of the ball. Again from rotational kinematics we recall that the net tangential acceleration a_T is the radius R times the angular acceleration which is the second derivative of the position θ.

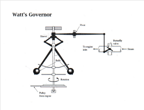
$$\begin{aligned} ma_T &= F_{\text{cent}}\cos\theta - mg\sin\theta \\ mR\ddot{\theta} &= m\Omega^2 R\sin\theta\cos\theta - mg\sin\theta \\ \ddot{\theta} &= \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \end{aligned} \tag{1}$$

Equation 1 is a second order, non linear equation, so in order to use this equation we must first transpose it into two first order equations. The following conversion is what allowed us to use Matlab's phase plane tool in the previous graphs.

$$\begin{cases} \theta = \theta \\ \omega = \dot{\theta} \end{cases} \quad \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \ddot{\theta} = \Omega^2\sin\theta\cos\theta - \dfrac{g}{R}\sin\theta \end{cases}$$

The previous graphs and all that follow are based on functionally determined dimensions of the governor. We set our radius R to be $6\,\text{cm}$ and the masses to be $10\,\text{g}$.

Intuitively we would expect that there will be a unique, stable angle θ that corresponds to each Ω above the critical speed Ω_0. We want to find a relationship so that given an Ω we can calculate the corresponding angle θ and vice versa. First we must realize that an equilibrium angle for a certain Ω means that the tangential forces on the ball (see Figure 8) are balanced and therefore the acceleration is zero. If we set the right side of equation 1 equal to zero we will have θ as a function of Ω.



$$\begin{aligned}
\ddot{\theta} &= 0 \\
\Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta &= 0 \\
\sin\theta(\Omega^2\cos\theta - \frac{g}{R}) &= 0
\end{aligned}$$

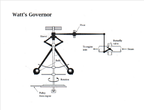
Therefore,

$$\sin\theta = 0 \qquad \text{or} \qquad \Omega^2\cos\theta - \frac{g}{R} = 0.$$

When $\sin\theta = 0$ it is implied that $\theta = 0$ or π, so these equilibrium points are at the bottom and top of the hoop. Since the top of the hoop is not a realistic possibility it can be disregarded. It is a reality of the mathematical model, but not of the physical model. As we have seen earlier, the bottom of the hoop remains a realistically possible equilibrium point for all Ωs when the mass is perfectly centered and the system is perfectly balanced. Solving for θ in the other term gives us the θs that correspond to any Ω greater than Ω_0.

$$\begin{aligned}
\Omega^2\cos\theta - \frac{g}{R} &= 0 \\
\cos\theta = \frac{g/R}{\Omega^2} &= \frac{g}{R\Omega^2} \qquad (2)
\end{aligned}$$

Because the cosine of an angle is never greater than 1, we only want to consider Ω values that make the right side of equation 2 less than or equal to 1. In the physical model this corresponds to restricting the rotational speed of the governor to a particular range of Ωs in order for it to remain functional.

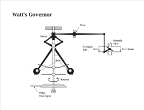


$$
\begin{aligned}
\frac{g}{R\Omega_0^2} &\le 1. \\
\frac{g}{R} &\le \Omega_0^2 \\
\sqrt{\frac{g}{R}} &\le \Omega_0 \qquad (3)
\end{aligned}
$$

Using equation 3 we can determine the Ω_0 for any radius. In our case the radius $R = 6\,\text{cm}$ and the acceleration due to gravity $g = 9.8$.

$$
\begin{aligned}
\sqrt{\frac{9.8}{.06}} &\le \Omega_0 \\
12.78 &\le \Omega_0 \qquad (4)
\end{aligned}
$$

A rotational speed below 12.78 rad/sec will result in the single equilibrium case seen in Figure 3, whereas Ωs above 12.78 will result in the bifurcated equilibrium points of some θ radians from the bottom as in Figure 6. We want our governor to maintain an equilibrium angle of $\pi/4$ radians while the engine that it is controlling is at it's optimal operational speed. If the drive system from the engine to the governor is geared such that this angle is maintained, the governor will have the maximum range of motion on either side of the equilibrium angle. This equates to the maximum range of control. Solving equation 2 for Ω we can find the ideal rotational speed for our governor.



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$$\begin{aligned}\cos\theta &= \frac{g/R}{\Omega^2}\\ \cos\frac{\pi}{4} &= \frac{9.8/.06}{\Omega^2}\\ \Omega &= \sqrt{\frac{9.8}{.06\cos\frac{\pi}{4}}}\\ \Omega &= 15.2. \end{aligned}\tag{5}$$

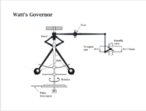
According to equation 5 a rotational speed of 15.2 rad/sec will result in an equilibrium angle of $\pi/4$. While the mathematical model represents two realistic equilibrium possibilities we want to look at just one of these since our two-mass governor will always have positive θs. In Figure 10 we have the phase plane solution of the right side equilibrium point where the perpetual oscillations about the θ value of $.785 = \pi/4$ should be noticed. In Figure 11 these oscillations are easier to see.

Putting It All Together

So, we have a governor design that will maintain the desired equilibrium angle at our engine's optimal speed. This is great except for the fact that the masses are **oscillating** about that angle. Remembering that this is a real world device, it should be apparent that transmitting these oscillations through the linkage to the throttle control (Figure 1) will not provide the steady control of engine speed that Mr. Watt was seeking. The final element of success for Watt's governor came from proper damping.

In Figure 12 we have introduce a damping term into equation 1. This term is proportional to the angular velocity (in the vertical plane) and is divided by the mass.

$$\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{R} mg\sin\theta - k\frac{\dot{\theta}}{m}\tag{6}$$



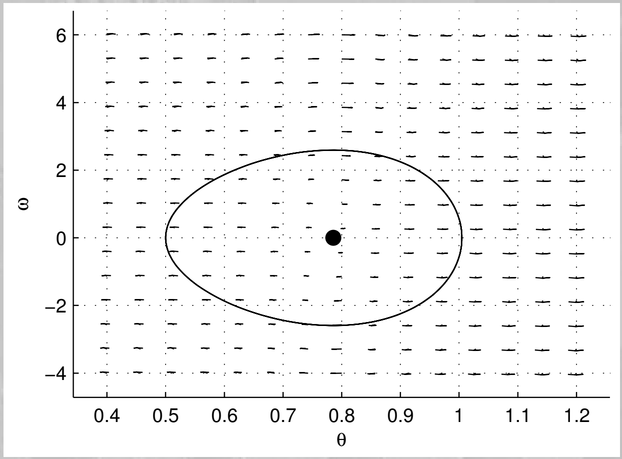


Figure 10: For $\Omega = 15.2$ rad/sec.

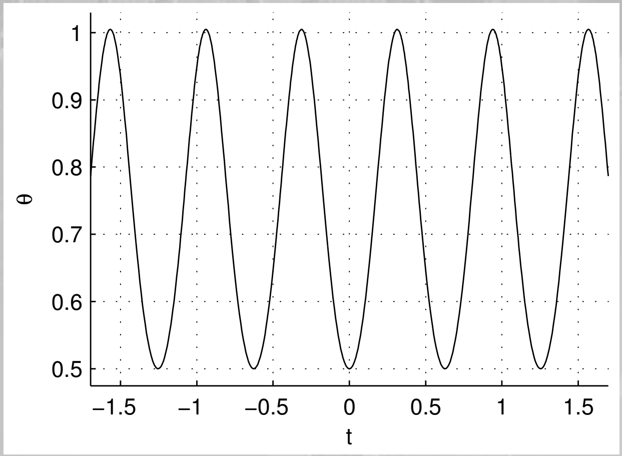
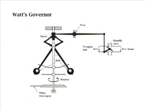


Figure 11: θ vs. t for $\Omega = 15.2$ rad/sec.



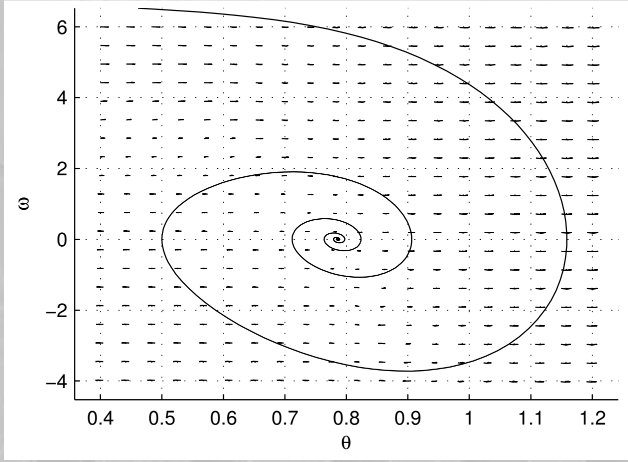


Figure 12: $\Omega = 15.2$ rad/sec with damping term.

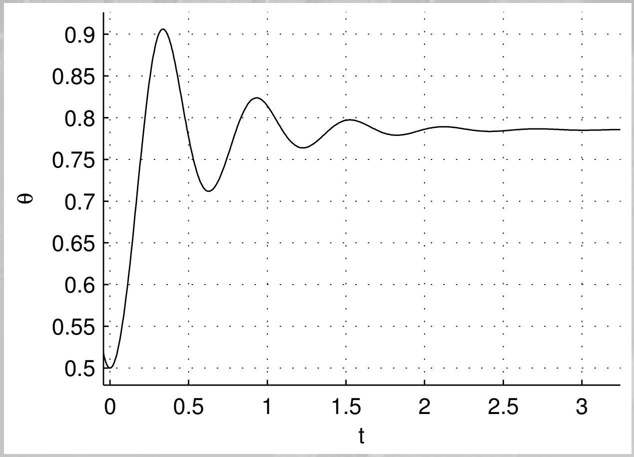
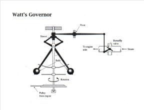


Figure 13: θ vs. t for $\Omega = 15.2$ and damping term.



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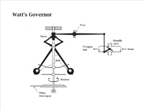
Again we see a dramatic change in the behavior of the system. In Figure 13 we see the damped oscillations that we desire. At $t = 0$ seconds a load has been applied to our engine and it's speed is suddenly reduced. Through the drive mechanism, Ω is also reduced which causes a change in θ. Secondary to this change in θ the linkage seen in Figure 1 opens the valve, which in turn increases the engine speed. In typical rebound behavior, this compensation crosses the equilibrium point which again affects the valve position and engine speed. Because of the introduced damping term each subsequent oscillation is reduced in magnitude until the equilibrium point is once again achieved.

But is this the tightest response that we can get? Ideally we want as few oscillations as possible. In our first model, the change in Ω perturbed the ball just $-.29$ radians (about $-\pi/11$ rad.) which causes it to oscillate past the equilibrium point about seven times before coming to rest. Recall that in equation 6 the damping term is inversely related to the mass. This means that increasing the mass of the ball will lessen the effects of damping and decreasing the mass will exaggerate the effects. In our first example the ball had a mass of $10\,\mathrm{g}$ so let's try some different masses and observe the effects.

In Figure 14 the increase in mass causes many more oscillations than our first example. Constructing a governor of these dimensions would be a poor regulating device. In Figure 15 we see that decreasing the mass results in fewer oscillation than our first example.

We found that varying the arm length R did not effect the oscillatory behavior whatsoever. Different Rs only effected the critical rotational speeds, in other words, at which speeds Ω_0 and $\theta = \pi/4$ occurs. This would be a useful parameter to adjust when trying to match the engine speed to the Ω that produces the preferred angle $\theta = \pi/4$.

We seek a governor that returns the engine to it's optimal speed as quickly as possible. The most responsive governor for any given circumstance will require some experimentation with arm length, mass weight, damping, and proper gearing between the engine and the governor. With the help of a numeric solver like MatLab these experiments can occur in the digital realm and the optimum design arrived at before any prototypes are attempted.



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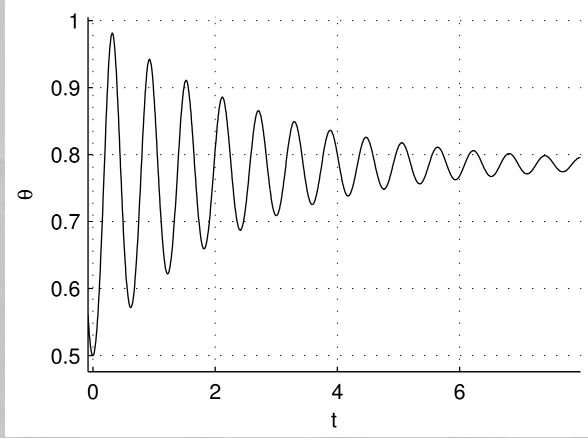


Figure 14: θ vs. t for $\Omega = 15.2$, damping term, and $m = 50\,\mathrm{g}$.

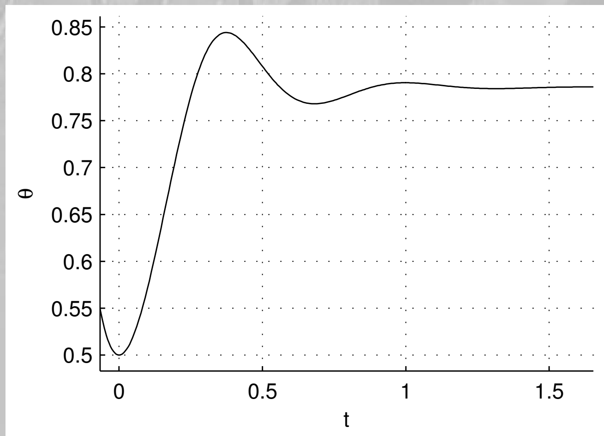
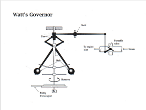


Figure 15: θ vs. t for $\Omega = 15.2$, damping term, and $m = 5\,\mathrm{g}$.



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