Tacoma Narrows Bridge

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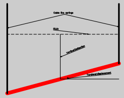
Abstract

The spectacular collapse of the Tacoma Narrows Bridge on November 7^{th}, 1940, soon after its completion on July 1^{st}, 1940, is possibly the most famous engineering failure of this century. The sunken remains of the bridge lie at the bottom of Puget Sound to this day.

Among the reams of articles, term papers, theses, and other scholarly writings about the collapse of this bridge one can find numerous mathematical models analyzing the failure. These range from simple trigonometric models to systems of non-linear second order differential equations.

The engineering drawings have been subject to intense scrutiny, and physical constants have been gleaned from extensive analysis of motion picture footage shot during the collapse. Recent advances in computer science have given us the necessary tools to evaluate the more sophisticated, non-linear models, resulting in renewed interest in this fascinating failure analysis.



In this project we will examine one of the simpler models, as well as one of the more complex models. Our purpose is not to validate or debunk either of these models. Rather we wish to gain some understanding of this fascinating civil engineering problem.

1. History

In the mid 1930's, the City of Tacoma and the Pierce County Board of commissioners asked the State of Washington to build a bridge linking the Washington mainland to the Olympic Peninsula. After spending $25,000 for feasibility studies, The State of Washington submitted an application to the Public Works Administration for funds for design and construction.

The design team was headed by Clark Eldridge, a bridge engineer with the Washington State Department of Highways. Eldridge's plan was for a 5,000 foot, two lane suspension bridge. The bridge was to be the third longest suspension bridge in the world.

After reviewing the plans, the Public Works Administration committed to finance 45% of the project with the proviso that the State of Washington retain a board of independent engineering consultants to examine the design. The State hired the firm of Moran and Proctor to study the substructure plans. A world renown suspension bridge engineer, Leon S. Mosseiff, was retained to review the superstructure plans.

Moran and Proctor designed an entirely new substructure. Mosseiff made significant changes to the superstructure as well, replacing Eldridge's 25 foot open stiffening truss with a shallow eight foot plate girder, which made the bridge significantly lighter.

Before the project went out to bid, a group of contractors told the State Department of Highways that the substructure could not be built as designed. Subsequently, Eldridge's substructure design was combined with Mosseiff's superstructure design.

The bridge was built by Pacific Bridge Company, with Bethlehem Steel Company supplying and erecting the steel and wire. Work began early in 1939, and the bridge was completed on July 1, 1940 for a price tag of $6,400,000.

During construction, large amplitude vertical oscillations occurred frequently, calling into question the design's integrity. Vertical oscillation was sometimes observed under conditions of gentle breezes as low as four miles per hour, while oscillation might be absent during winds of 35 miles per hour. Hydraulic dampers were installed on the towers as control measures, with little effect. Studies made by the University of Washington recommended tie down cables in the side spans, but this did not cure the problem.

When the bridge opened, it quickly earned the moniker 'Galloping Gertie' and many people drove hundreds of miles to experience the roller coaster like sensation of crossing the center span. Traffic on the bridge was near triple the anticipated volume, and the Washington Toll Bridge Authority was happy as a clam.

During the early morning hours of November 7^{th}, 1940, the center span was undulating at an amplitude of three to five feet in winds of 35-46 miles per hour. At 10:00 AM, concerned officials closed the bridge. Shortly after the bridge was closed, the character of the motion changed from vertical oscillation to two-wave torsional motion. The torsional motion caused the roadbed to tilt as much as 45° from horizontal. The center span, remarkably, endured the vertical and torsional oscillation for about a half hour, but then a center span floor panel broke off and dropped into the water below. Soon, chunks of concrete were breaking off the roadbed and raining into the sound. Shortly after 11:00 AM, a portion of the western end of the center span twisted free. A few minutes later, the rest of the center span ripped free and splashed into the sound. Without the tension from the center span, the 1,100 foot side spans dropped 60 feet, rebounded, then came to rest 30 feet below their original position.

2. A Generally Accepted Simple Model

First, let us examine a relatively simple model. A common model, which we believe to be much too simplistic, suggests that there was some natural frequency of the bridge that resonated with some external periodic force at or close to that frequency. This is easily modelled by the familiar equation of the single degree of freedom harmonic oscillator with sinusoidal forcing, $m\ddot{x} + \mu\dot{x} + kx = \alpha\cos\omega t$. [6] If the frequency of the forcing term, $\alpha\cos(\omega t)$ is very near to the natural frequency of the bridge, $\sqrt{k/m}$, resonance occurs and large amplitude oscillation is observed. [3] It is difficult to generate a graph, using reasonable parameters, that could account for the destructive motion. Using $m = 2268$, $k = 980$, $\mu = 0.01$, $\omega = \sqrt{980/2286}$, $\alpha = 5$, and initial conditions $y(0) = 0$, $\dot{y}(0) = 3$, we were able to generate the graph in Figure 1, but these initial conditions aren't very realistic. Another problem with this model is that it does not address the torsional oscillation that is clearly evident in the film.

3. Design of the Bridge and Physical Constants

We will convert all angle measure to radians and other quantities to SI and SI derived units in order to simplify the mathematical models.

The bridge was designed to use lighter, less expensive materials by distributing part of the dynamic loads to the main cables rather than depending on stiffening trusses below the bridge deck to support the loads. Light traffic was expected, and so the bridge design called for two lanes with a width of 39 feet (**11.9 M**) between cable centers.

Because of the nature of the channel bottom and the swift currents, a center span length of 2800 feet (**853.0 M**) was necessary. The stiffening girders of the center span superstructure had a depth of only 8 feet (**2.4 M**) and a depth to length ratio of 1:350 [3] (Eldridge's original design specified a depth to length ratio of 1:112), making the Tacoma Narrows bridge the most flexible of the five largest suspension bridges built during the period. The closest in torsional flexibility was the Golden Gate Bridge, and the Tacoma Narrows Bridge was over 3.5 times as flexible. The closest in vertical flexibility was the San Francisco Bay Bridge, and the Tacoma Narrows Bridge was more than twice as flexible.

Traditional suspension bridges had rigid towers with rollers that allowed the cables to move along the tower tops. The Tacoma Narrows Bridge was designed with flexible towers with the main cables fixed to the tower tops.

During the short life of the bridge, several attempts were made to stabilize the bridge

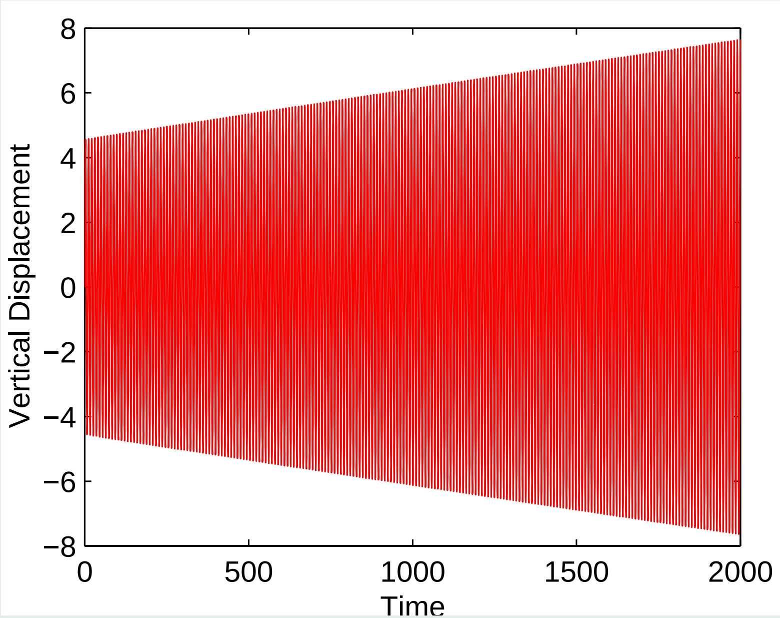
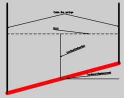


Figure 1: The damped harmonic oscillator.



deck. The hydraulic dampers and tie down cables in the side spans were mentioned previously. The hydraulic dampers were ineffective because the seals had been damaged by sandblasting operations prior to painting. [5] Additionally, cables were attached to the girders which were then anchored to fifty ton blocks on the shore. They snapped shortly after installation. Large cable stays from the main cable to the bridge deck at mid center span were added. They remained operational until the day of the collapse.

After the failure, the Federal Works Agency established a commission to determine the cause of the collapse. All experts agreed that the shift from relatively safe vertical motion to the destructive torsional motion was the result of the cable bands on the side spans to which the center cable stays were attached slipping. [3]

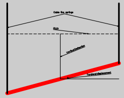
The main source for physical constants is [1]. The width of the deck was mentioned previously as 11.9 M (call 12 M). The bridge was closed when it collapsed so we can accurately deduce the mass at about 5,000 pounds per foot of length (call 2300 kg).

The spring constant of the cables requires some simple calculation.The deck would deflect about 0.5 M when loaded with 300 kg per meter of length. [4] Since there is a spring on each side of the deck, this yields the equation $2Ky = mg$. Substituting and solving for k gives $2(0.5)K = 100(9.8)$, $K = 980$.

Now we need to find some constants for the forcing term. The oscillations can be observed in the film to be in the range of about 0.2 hz (13 cycles per minute) which gives us a value of about 1.26 for ω. Amplitude of the forcing term is a little trickier, but [4] suggests a small forcing amplitude sufficient to cause a $\pm\pi/60$ torsional oscillation. We will leave this constant to be determined by investigation.

4. Mathematical Models - Predicting Failure?

A more realistic model is proposed by P. J. McKenna in [4]. Extend a spring with spring constant K a distance y. The potential energy is $Ky^2/2$. If a beam of mass m and length $2l$ rotates about its centroid with an angular velocity $\dot{\theta}$, its kinetic energy is $m(l\theta)^2/6$. [7] Consider a beam suspended at its ends by steel cables that act as springs with spring



constant K as in Figure 2.

Let y be the downward displacement of the centroid from the position of the unloaded springs, and θ be the angle of the beam from the horizontal. Since cables have no compression spring constant, we will have to let $y^+ = \max\{y, 0\}$. The potential energy from gravity is $-mgy$. The spring extension is $(y - l\sin\theta)^+$ for one cable and $(y + l\sin\theta)^+$ for the other. The total potential energy of the system is

$$V = \frac{K}{2}\left(\left[(y - l\sin\theta)^+\right]^2 + \left[(y + l\sin\theta)^+\right]^2\right) - mgy$$

and the total kinetic energy is [4]

$$T = \frac{m\dot{y}^2}{2} + \frac{ml^2\dot{\theta}^2}{6}.$$

First we form the Lagrangian.

$$L = T - V$$

$$L = \frac{m\dot{y}^2}{2} + \frac{ml^2\dot{\theta}^2}{6} - \frac{K}{2}\left(\left[(y - l\sin\theta)^+\right]^2 - \left[(y + l\sin\theta)^+\right]^2\right) + mgy$$

According to the principle of least action, the motion of the beam obeys the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0.$$

In preparation for the use of the first Euler-Lagrange equation, we calculate these

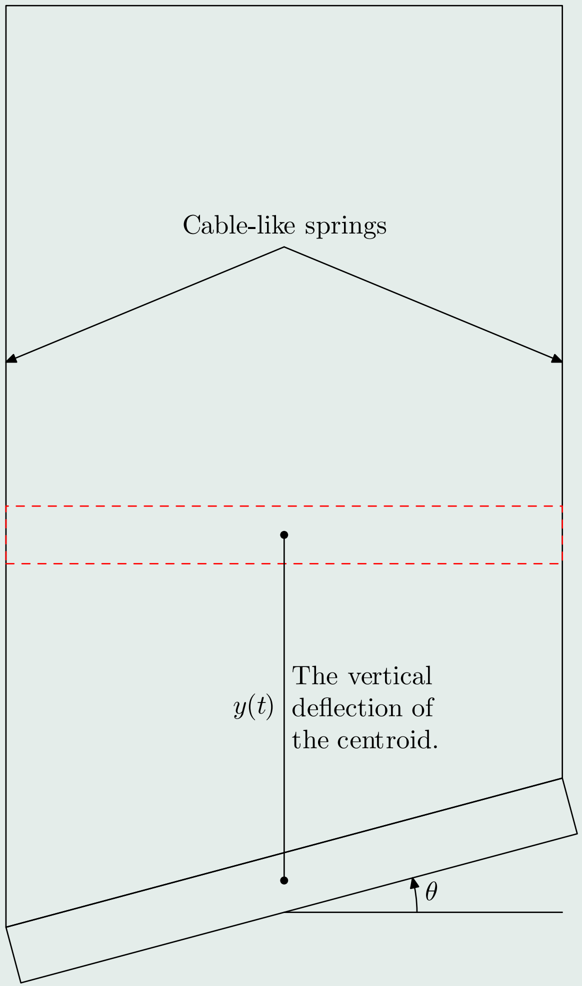


Figure 2: The McKenna Model

derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial \dot\theta} &= \frac{mL^2\dot\theta}{3} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) &= \frac{ml^2\ddot\theta}{3} \\
\frac{\partial L}{\partial \theta} &= Kl\cos\theta\left[(y - l\sin\theta)^+ - (y + l\sin\theta)^+\right].
\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

becomes

$$\frac{ml^2\ddot\theta}{3} = (Kl)\cos\theta\left[(y - l\sin\theta)^+ - (y + l\sin\theta)^+\right]. \tag{1}$$

Similarly, for the second Euler-Lagrange equation, we calculate these derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial \dot y} &= m\dot y \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) &= m\ddot y \\
\frac{\partial L}{\partial y} &= -K\left[(y - l\sin\theta)^+ + (y + l\sin\theta)^+\right] + mg
\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = 0$$

becomes

$$m\ddot y = -K\left[(y - l\sin\theta)^+ + (y + l\sin\theta)^+\right] + mg. \tag{2}$$

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If we simplify and add damping terms $\delta\dot{y}$ and $\delta\dot{\theta}$ respectively, along with a periodic forcing term of the form $f_t = \lambda \sin \omega t$, where λ is the amplitude and $\mu/2\pi$ is the frequency, we get the following system of second order differential equations

$$\ddot{\theta} = -\delta\dot{\theta} + \left(\frac{3K}{ml}\right)\cos\theta\left[(y - l\sin\theta)^{+} - (y + l\sin\theta)^{+}\right] + \lambda\sin\mu t$$

$$\ddot{y} = -\delta\dot{y} - \left(\frac{K}{m}\right)\left[(y - l\sin\theta)^{+} + (y + l\sin\theta)^{+}\right] + g$$

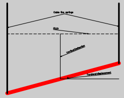
If we make the assumption, as does McKenna in [4], that the cables never lose tension, these simplify even further to

$$\ddot{\theta} = -\delta\dot{\theta} - \left(\frac{6K}{m}\right)\cos\theta\sin\theta + \lambda\sin\omega t \tag{3}$$

$$\ddot{y} = -\delta\dot{y} - \left(\frac{2K}{m}\right)y + g \tag{4}$$

None of this is controversial. These equations were derived long ago. [2] However, we see some problems with this model. First, our observation of the film leads us to believe that the cables do indeed lose tension. Secondly, we are not convinced that the vertical and torsional motion are uncoupled. Since our purpose is to understand the model, we will go along with Dr. McKenna for the time being.

What is new is the advances in computational science that make it possible to examine these models in all their exquisite complexity. It is no longer necessary to use a linear approximation to equations like equation 3 and be content with examining the system near some equilibrium point. We want to see how this model will react when values of θ and y are not anywhere near an equilibrium point, when the bridge approaches total failure.



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5. Examining the Model

We were unable to reproduce the graphs shown in [4]. Our results, while different, show that there is some validity to the model. The main thrust of [4] is that a relatively small sinusoidal forcing term for torsional motion can translate to different steady state solutions depending on the initial condition. If a large torsional push occurs, such as might be caused by one of the cable bands on the side spans slipping, the model exhibits behavior that accounts for the multi-nodal torsional oscillation that was observed as well as a large steady state torsional oscillation of amplitude close to the observed value of $\pi/4$. Using the parameters $\lambda = 0.06$, $\omega = 1.4$, $m = 2300$, $K = 980$, $0 < t < 1800$ (equivalent to 30 minutes), and initial conditions $y(0) = 5$, $\dot{y}(0) = 0$, $\theta(0) = 1.2$, $\dot{\theta}(0) = 0$ produced the graph shown in Figure 3. We have translated the y graph 23 units downward for clarity.

Figure 4 clearly shows multi-nodal transients in the torsional solution, while Figure 5 shows the steady state torsional amplitude of around $\pi/4$.

The phase portraits for this system are not surprising. The phase portrait for vertical motion is a spiral sink, while the phase portrait for the torsional motion is a limit cycle. The phase portraits are shown in Figure 6.

In [4], Dr. McKenna predicts that for values of ω not very near to 1.4, the torsional motion will go to zero as time goes to infinity. The model bears that out. Figure 7 shows what happens to the model when we change the value of ω to 0.5.

6. Our Conclusions

As shown in Figure 3 and Figure 7, the steady state solution is sensitive to a change in the forcing frequency. In order for the model to exhibit motion similar to the observed motion of the bridge, a large torsional push is required. We believe that such a push could have been the result of slipping of one of the side cable stays.

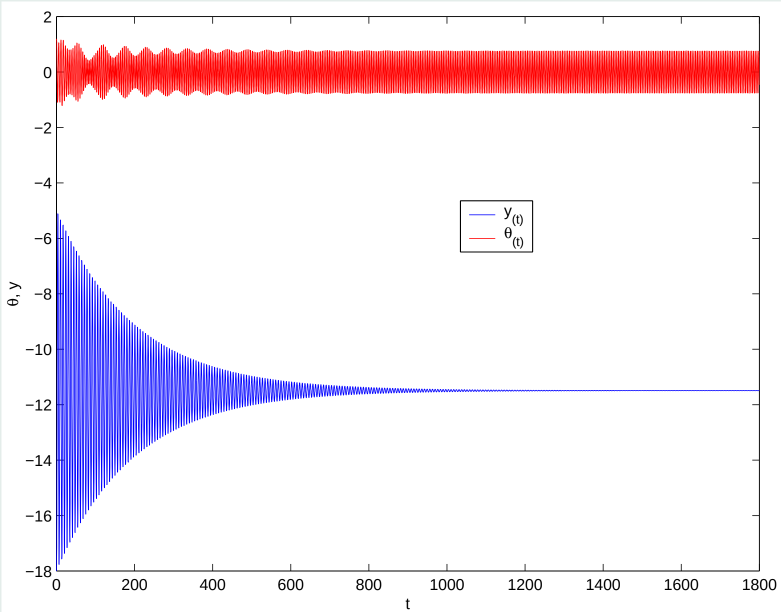

Figure 3: Solution to the system with a large initial torsional push.

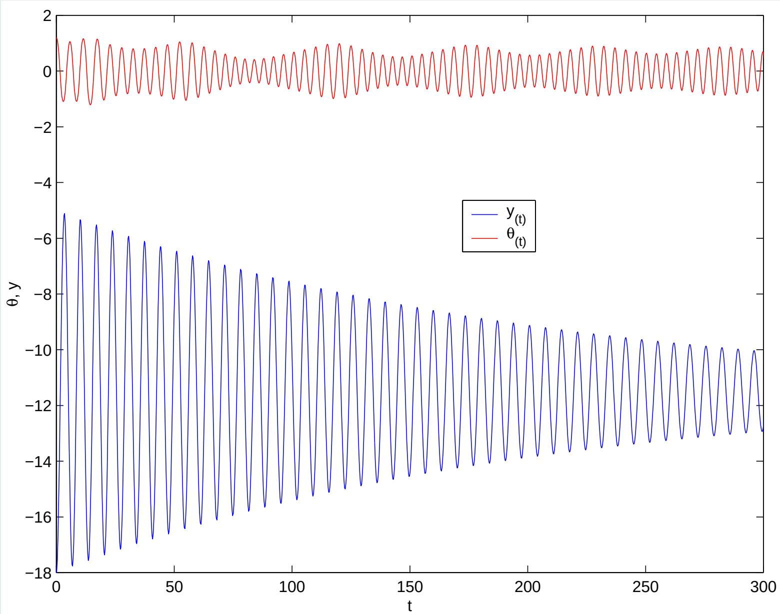

Figure 4: The first five minutes of the motion shown in Figure 3.

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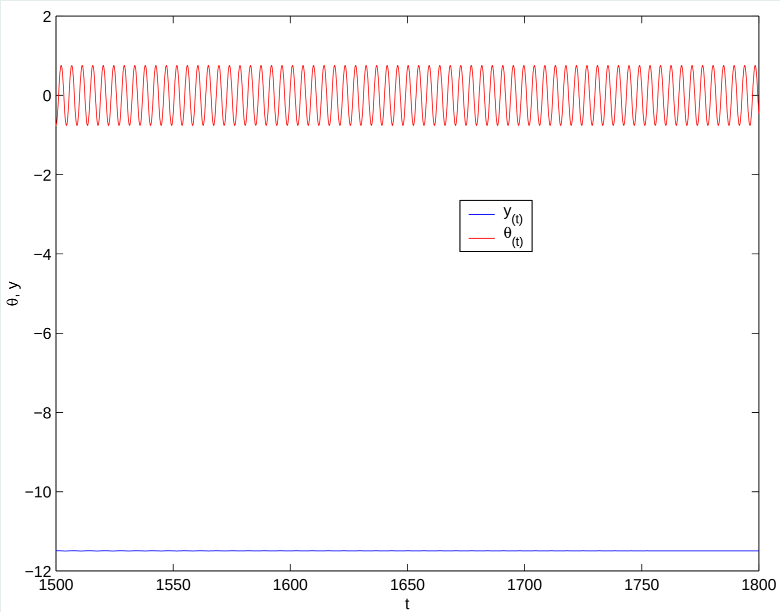

Figure 5: The last five minutes of the motion shown in Figure 3.

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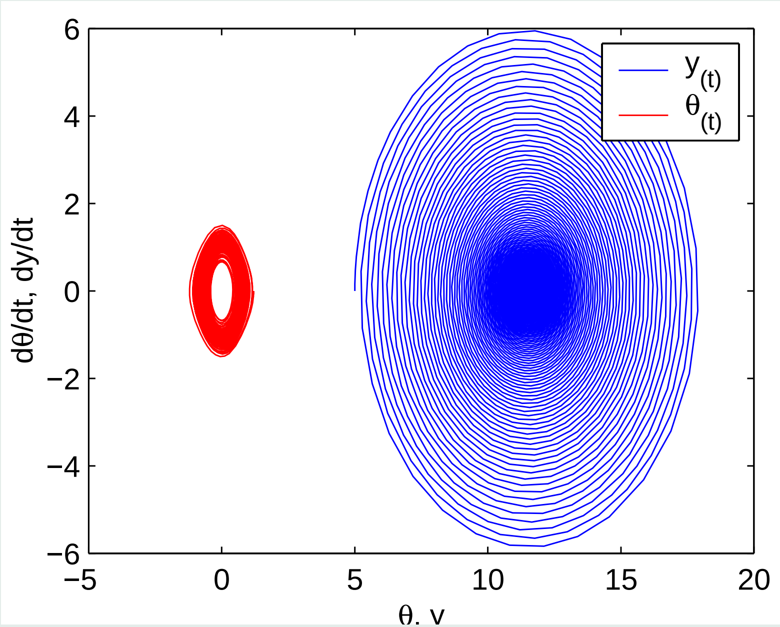


Figure 6: The Phase Portraits for θ and y.

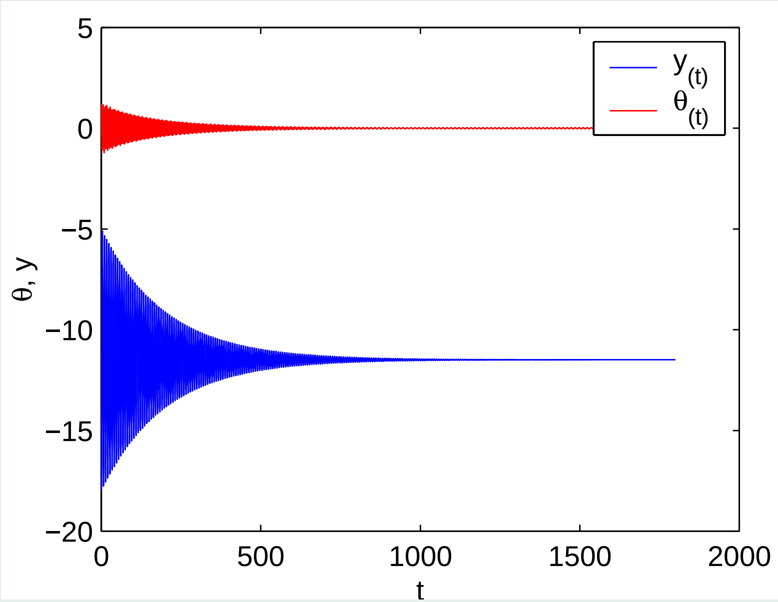


Figure 7: Results of changing the value of ω to 0.5.

Post failure examination revealed that this had indeed occurred. No model can incorporate all the variables involved in such a complex system however, we believe that this model is a good approximation.

It has historically been believed that the torsional forcing is the result of vortex formation behind the bridge deck. This forcing would necessarily be very small for so massive a system. We are not entirely convinced that this is the source of the forcing.

Our model does not allow for the cables to go slack. In order for this to happen, the vertical amplitude would have to have been on the order of 11.5 m. The film of the collapse does not show motion of this magnitude.

Our project has accomplished its stated goal. Our understand of this civil engineering disaster has increased as the result of this project.

7. Glossary of Bridge Terms

Abutment Land structure supporting bridge superstructure at either end of a bridge.

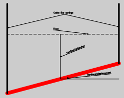
Bent or Column Vertical bridge structural support.

Channel Any navigable waterway in fact by vessels or artificially improved or created as to be navigable by vessels, including the structures and facilities created to facilitate navigation.

Cofferdam A watertight temporary structure that prevents water from entering an enclosed area. The enclosed area can be pumped dry to allow work below the normal waterline.

Footing The enlarged foundation under a column or bent to spread the weight of the bridge and prevent settling. Footings may be entirely above or below ground.

Girder Longitudinal element of the superstructure, typically a large steel beam the length of the span.



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Pier Vertical bridge support in open water. The pier is situated between the footing and the column or bent.

Pile A heavy steel, concrete, or timber vertical structural member driven or cast into the ground to anchor a bridge footing.

Span The distance between bents.

Substructure The portion of the bridge that supports the roadway, deck, railing and girders. In a suspension bridge, this would include the towers and support cables.

Superstructure The deck, roadway, railing, girders, lighting standards: the portion supported by the substructure.

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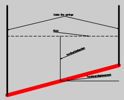

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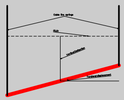

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