Fun With The Brachistochrone!

Rhyan Pink

May 11, 2002

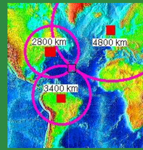
Abstract

In 1646, Johan Bernoulli proposed the general problem: given a wire bent into arbitrary curve, which curve of the infinitely many possibilities yields fastest descent? This path of fastest descent is called the brachistochrone. The Brachistochrone problem is to find the curve joining two points along which a frictionless bead will descend in minimal time. Earthquake energy propagates through the earth according to Fermat's Principle of Least Time, and thus earthquake energy will follow the general shape of this Brachistochrone curve.

1. Background

Earthquakes are a common occurrence around the world, which provides an excellent opportunity for geologists to study them. It is well known that there is a higher frequency of earthquakes in some regions versus others, but the precise reasons for why this discrepancy exists is still being studied. Earthquakes are a result of a pressure release as the earth shifts its crust, according to plate tectonics. When two plates interact with one another, a certain amount of shear stress can be tolerated by the geologic material that composes the subsurface. Once this range of tolerance is surpassed, an earthquake occurs, releasing a tremendous amount of energy that propagates (travels) in the form of elastic waves: body waves and surface waves. See Figure 1.

Due to the nature of how energy is released in this setting, some distinct aspects of the energy can be observed. Love and Rayleigh waves are surface waves. Rayleigh waves have a particle motion that is in the direction of energy propagation. Love waves occur when there is an increase of S-wave velocity with depth, and have particle motion that is transverse to the direction of energy propagation. P (pulse) waves and S (shear) waves are body waves. Body waves propagate (travel) through the



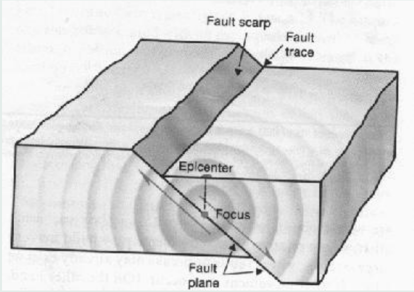


Figure 1: Focus

Earth similar to that of the propagation of light waves. See Figure 2. Through shearing of subsurface blocks during an earthquake, a wave of energy is emitted (a shear, or S-wave) that has a fluctuation of energy perpendicular to its axis of propagation. Through compression of subsurface blocks during an earthquake, a resultant compressional wave is emitted (a pulse, or P-wave) which has a fluctuation of energy parallel to its axis of propagation. The intrinsic nature of these two waves, as well as body wave dependence upon subsurface structure and density results in P-waves (velocity, v_p)

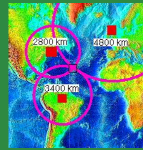
$$v_p = \sqrt{\frac{\kappa + \frac{4}{3}\mu}{\rho}}$$

travelling faster than S-waves (velocity, v_s)

$$v_s = \sqrt{\frac{\mu}{\rho}}$$

(where κ is the incompressibility, μ is rigidity, and ρ is density) which is key to my eventually being able to interpret the subsurface structure of the earth. That P and S wave velocities are dependent upon density (ρ) is not immediately obvious. But, this linear relationship between density and seismic velocity is Birch's Law

$$v = a\rho + b, \qquad \text{where } a \text{ and } b \text{ are constants.} \tag{1}$$



Background
Defining the problem
Establish Some Equations
Solving for I Don't . . .
Final Remarks
Acknowledgements
Home Page
Title Page
Page 2 of 18
Go Back
Full Screen
Close
Quit

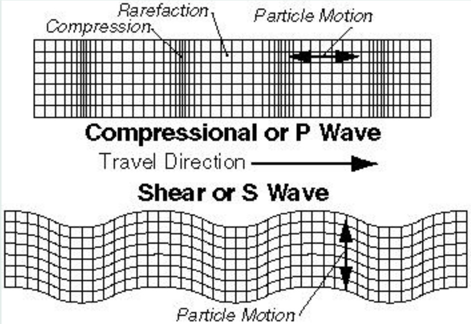
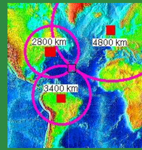


Figure 2: P and S propagation.

Also included in the emission of energy are surface waves, due to a projection of the energy released to the surface of the earth. Because this energy can physically vibrate the earthen material, it can be observed by a seismograph. For reference, we are most sensitive to surface waves (intense shaking is felt), somewhat sensitive to S-waves (‘rolling’ motion is felt), and weakly sensitive to P-waves (pulsating or sporadic motion is rarely or weakly felt.) The neat thing that can be perceived here is when one can feel all three types of waves, it is obvious which waves travel the fastest (in descending order): P, S, then surface.

Observations made concerning earthquakes are simply interpretations of the shaking that occurs when energy passes through the earth. To make such observations, the seismograph See Figure 3. is used. This instrument is fixed to the rigid bedrock of the region (compared to loose sediment) where the seismograph will shake along with the earth. Also involved is a weighted pen which resists the shaking of the seismograph, and is set up to write continuously on a piece of paper. This piece of paper is known as a seismogram. See Figure 4.

This was especially important in last semester’s Linear Algebra semester project, *Iterative Relocation of Earthquake Hypocenters.* By considering the relative speeds of each of the energy waves, it can be determined which wave is represented on the seismogram, and more importantly, the wave can be assigned a specific arrival time. Once identifying the arrival times for both the P and S waves and comparing the travelling velocities of each of the waves, we inferred the distance from the source of the waves (hypocenter) to the receiving seismograph. To accurately consider observations of a release of seismic energy, a sufficient number of seismographs must be utilized. By having multiple observation stations, the error associated with interpreting the hypocentral location is minimized. The focus



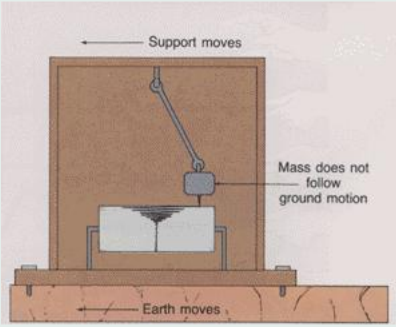


Figure 3: Seismograph.

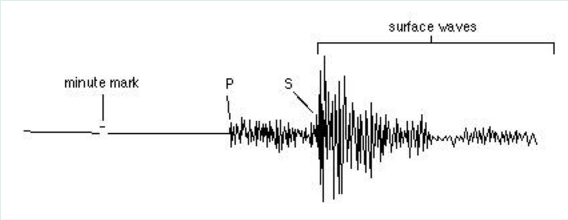
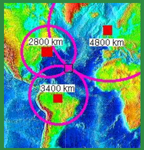


Figure 4: Seismogram.

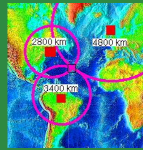


(hypocenter) is the three-dimensional location of the energy release. This stress release occurs along the fault planes, not exactly at a point. The epicenter is the two-dimensional location of the energy release (the surface projection of the hypocenter.) By considering these values, we can begin to study and understand earthquakes.

Once we begin to understand earthquakes, we can piece together vital information about the regions in which the earthquake occurs. By comparing similar information from adjacent areas, regions can be interpreted (which is vital in the study of geology.) The items being understood here are the three dimensional structure of the earth, something that cannot be directly observed at this point in time (the deepest drilling holes only scratch the surface.) If the arrival times of the earthquake waves can be compared with the location of the hypocenter and the corresponding velocities of energy waves, we have the possibility of understanding what the earthquake has to reveal.

This understanding is due in large to seismology. Seismology is the study of earthquakes and the passage of seismic (elastic) waves through the earth; it is the most powerful method we have for obtaining the internal structure of the earth. Other (less powerful) geophysical techniques include the study of gravity, magnetism and the electrical properties of Earth. Seismology reveals the velocity and density (ρ) structure of the earth. In this project, I assume a homogeneous make-up in each velocity layer. Birch's Law shows an empirical relationship between density and seismic velocity. See Equation (1). The velocity of the igneous and metamorphic layers are higher than that of the sedimentary layer. This follows the laws of energy; energy propagates more rapidly through more tightly structured material than it does through more loosely constructed material. The time for a seismic wave to travel from earthquake source to receiver can be expressed as a function of the path the energy traced and a model of the subsurface structure (allowing the assumption of the energy propagation velocity of Birch's Law (1).)

The study of seismology led (in part) to our understanding of plate tectonics. Plate Tectonics is the 'carpentry' or 'architecture' of the Earth's surface, the system of large lithospheric plates which move across the Earth's surf as spherical caps; most igneous and tectonic activity occurs along the boundaries between (rather than within) the plates. Earthquake focii give tectonic plate boundaries, which will help me in (eventually) mapping the subsurface material of the internal structure of the Earth. The three-dimensional structure of the earth can be determined by examining the travel time rays from all directions. This technique can recognize regions that are relatively fast or slow (when compared to the model.) This powerful research method is tomography. Earthquake distribution shows where Earth is active (near the surface) and seismic wave passage through the earth yields interior composition.



2. Defining the Brachistochrone problem

In 1696, Johann Bernoulli posed the general problem.

> I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The general problem is given a wire bent into arbitrary curve; which curve of the infinitely many possibilities yields fastest descent? This path of fastest descent is called the brachistochrone. The Brachistochrone problem is to find the curve joining two points along which a frictionless bead will descend in minimal time. The word *brachistochrone* comes from the Greek words *brachistos*, shortest + *chronos*, time.

3. Necessary Insight

Neglecting curvature, the magnitude of the friction force is less at steep points on a curve. The magnitude has a range from 0 at a vertical tangent to the *wholeweight* at a horizontal tangent.

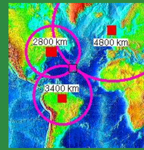
The classical Brachistochrone problem says that steepness is most important initially. This suggests that steepness is more weighted than the path length. The optimal curve (with friction) still needs an initial vertical tangent, but will be slightly steeper or below the cycloid.

The normal component of acceleration is proportional to the square of the speed, but expect the opposite to occur in the frictional model.

Starting off steeper forces more curvature for the latter portion of path, when there is a greater velocity.

4. Prior To The Derivation

Fermat's Principle of Least Time provides a basis for Snell's Law of Refraction. Fermat's Principle is based on that an energy wave will travel from one point to another along the path of least time. This



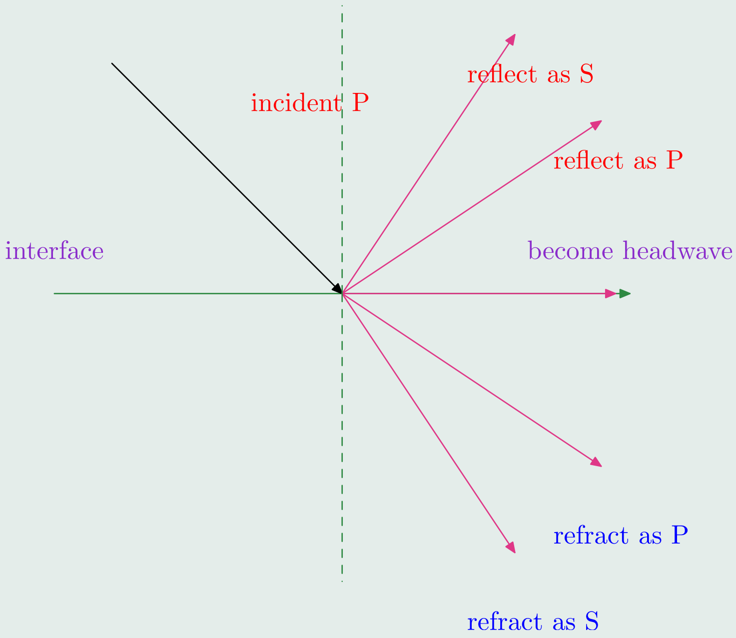
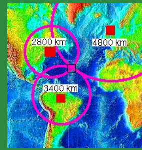


Figure 5: WaveAtInterface.

can be applied to Snell's Law, and that the path the energy follows through variable density, in which the paths found are curved! When a wave encounters a velocity or density difference (an interface) one of five things can happen: the wave will either be reflected as a P-wave, reflected as an S-wave, refracted as a P-wave, refracted as an S-wave, or become a headwave. See Figure 5. There are two approaches in determining the seismic velocity variations in Earth; one is the inverse method where velocities are obtained directly from the travel times (see last semester's Linear Algebra semester project, *Iterative Relocation of Earthquake Hypocenters*.)

Seismic energy finds the path of shortest travel time in accordance with Fermat's Principle of Least Time. Earthquake energy attenuates through the earth on curved ray paths. The principle of the conservation of elastic energy says that the velocity attained is solely determined by the loss in potential energy, not the path that brought it there.



Velocity differences occur at Earth structure boundaries, and increases with depth (in the mantle and the core). This causes ray paths to bend away from the normal. The decrease in velocity at the mantle-core boundary is caused by refraction of the ray into the core to bend towards the normal (which goes back to Fermat/Snell and the five things that can happen to a wave at an interface.)

4.1. Important Note

Let's not forget that the Earth is not perfectly elastic. This application is valid if and only if this is done inside a vacuum and we ignore friction.

5. The Derivation

Goal: to find the fastest curve, $y(x)$ from initial point $(0,0)$ to an arbitrary point, (a,b).

The acting force: gravity ($F_{gravity}$).

The positive y-axis is oriented downward, and g is positive because it is in the direction of motion.

$$F_{gravity} = mg\hat{j}$$

According to the way the coordinate system has been defined (with the positive y-axis downward), the value of the velocity will be positive (the curve is positive, and increasing.)

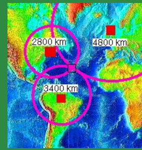
5.1. Developing the model (so integrand is independent of x)

In a frictionless environment, a decrease in potential energy (PE) is accompanied by an equal increase in kinetic energy (KE).

$$PE = mgh \qquad \text{and} \qquad KE = \frac{1}{2}mv^2$$

So the change in potential energy will be equal (in magnitude) to the change in kinetic energy.

$$\Delta PE = mg\Delta h \qquad \text{and} \qquad \Delta KE = \frac{1}{2}m\Delta v^2$$



Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 8 of 18
Go Back
Full Screen
Close
Quit

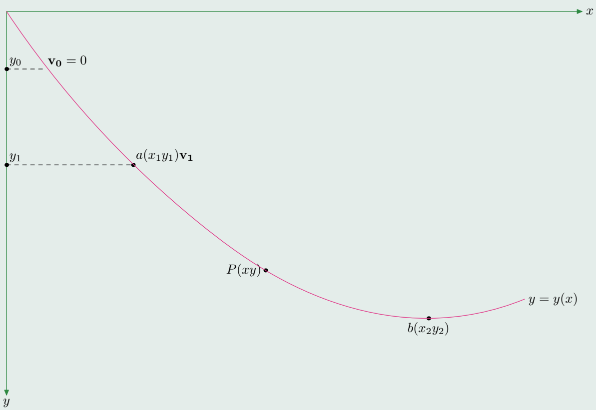


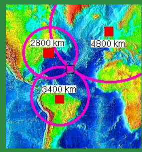
Figure 6: Frictionless.

$$\Delta PE = mg(y - y_1) \qquad \text{and} \qquad \Delta KE = \frac{1}{2}m(v - v_1)^2 \tag{2}$$

Combining the equations for potential and kinetic energies, See Equation (2) with the idea of their equality in a frictionless environment,

$$\begin{aligned}
\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 &= mgy - mgy_1 \\
\frac{1}{2}mv^2 - \frac{1}{2}mv_1^2 &= mg(y - y_1) \\
mv^2 - mv_1^2 &= 2mg(y - y_1) \\
mv^2 &= mv_1^2 + 2mg(y - y_1)
\end{aligned}$$

Solving for the velocity:



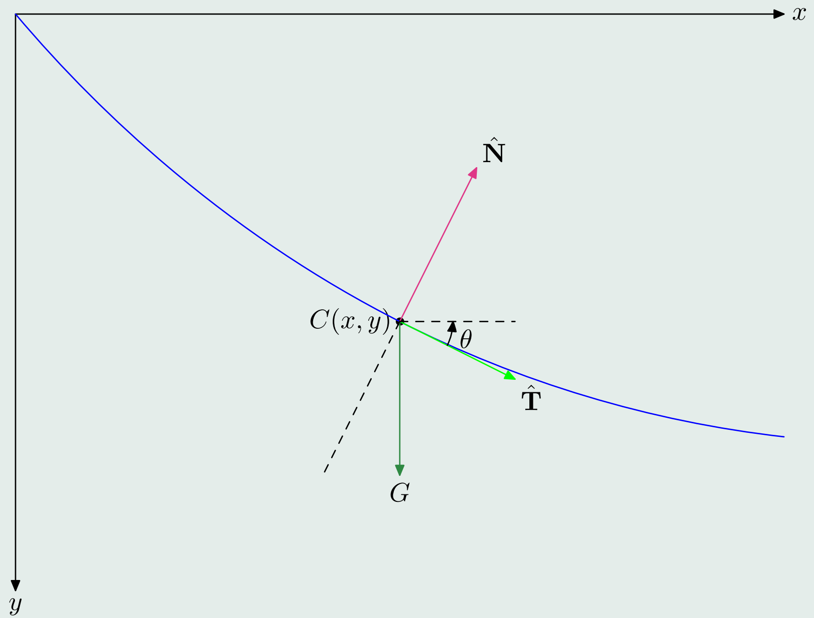
Background
Defining the problem
Establish Some Equations
Solving for I Don't . . .
Final Remarks
Acknowledgements
Home Page
Title Page
Page 9 of 18
Go Back
Full Screen
Close
Quit

$$v^2 = v_1^2 + 2g(y - y_1)$$
$$v^2 = 2g(\frac{v_1^2}{2g} + y - y_1)$$
$$v^2 = 2g\left(y - \left(y_1 \frac{v_1^2}{2g}\right)\right)$$
$$v^2 = 2g(y - y_0)$$

where $y_0 = y_1 - v_1^2/2g$

$$v = \sqrt{2g}\sqrt{y - y_0}$$

Which is the general velocity equation.



The point on the curve, (x, y) with unit Normal $(\hat{N})$ and unit Tangent $(\hat{T})$ vectors, can be written in terms of arc length, s:

$$\hat{N} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \qquad \hat{T} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$$

My curve is parametrized as $x = x(t)$ and $y = y(t)$, where x' and y' are continuous functions.

From the Fundamental Theorem of Calculus:

$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du\right] = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Now, I have the following relationships between s and t:

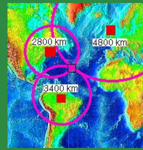
$$s = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du \qquad \text{and} \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Using the chain-rule on ds/dt,

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt}$$

Solving for dx/dt

$$\frac{dx}{dt} = \frac{\frac{ds}{dt}}{\frac{ds}{dx}}$$



Inverting:

$$\frac{1}{\frac{dx}{dt}} = \frac{\frac{ds}{dx}}{\frac{ds}{dt}}$$

which is an equation that yields a time value. Remember, dx/ds is arclength, ds/dt is velocity, and distance divided by velocity is equal to time.

The arclength, s of the planar curve, $y = f(x)$ is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

where $f'(x)$ is a continuous function. Equivalently,

$$s = \int_a^b \sqrt{1 + (y')^2} dx$$

Arc-length over velocity is equal to the time of descent time = dist/vel.

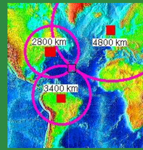
$$T = \int \frac{ds}{v} = \int_a^b \frac{\sqrt{1 + (y')^2}}{\sqrt{2g}\sqrt{y - y_0}}$$

The model equation, where the integrand, f is independent of the independent variable x:

$$T = \frac{1}{\sqrt{2g}} \int_a^b \frac{\sqrt{1 + (y')^2}}{\sqrt{y - y_0}} \tag{3}$$

5.2. Euler-Lagrange Equation (the degenerate case)

$$\frac{d}{dx}(y^{'} \frac{\partial f}{\partial y^{'}} - f) = y^{'} \frac{d}{dx}\left(\frac{\partial f}{\partial y^{'}}\right) + \frac{\partial f}{\partial y^{'}} \frac{d}{dx} y^{'} - \frac{d}{dx} f$$



$$
\begin{aligned}
\frac{d}{dx}(y^{'}\frac{\partial f}{\partial y'} - f) &= y^{'}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}\frac{d}{dx}y^{'} - \frac{d}{dx}f \\
&= y^{'}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y'}\frac{d}{dx}y^{'} - \left[\frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{d}{dx}y^{'}\right] \\
&= y^{'}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}y^{'} \\
&= y^{'}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right] - \frac{\partial f}{\partial x}
\end{aligned}
$$

So, I have

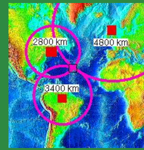
$$
\frac{d}{dx}(y^{'}\frac{\partial f}{\partial y'} - f) = y^{'}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right] - \frac{\partial f}{\partial x}. \tag{4}
$$

The Euler-Lagrange Equation says that if this differentiable function exists, is continuous and minimizes the integral, then the integrand satisfies the following:

$$
\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0
$$

and the following is true because my integrand, f is independent of the independent variable x.

$$
\frac{\partial f}{\partial x} = 0
$$



Therefore, the left-hand side of Equation (4) is equal to zero

$$\frac{d}{dx}\left(y^{'}\frac{\partial f}{\partial y^{'}} - f\right) = 0$$

The following is a conserved quantity because it does not change with respect to x.

$$\left(y^{'}\frac{\partial f}{\partial y^{'}} - f\right)$$

which leads to

$$y^{'}\frac{\partial f}{\partial y^{'}} - f = c_1$$

where c_1 is an arbitrary constant to be determined later, and f is the integrand of the time equation, Equation (3)

Applying the first integral of the Euler-Lagrange Equation to the integrand, f (which is explicitly independent of the independent variable, x):

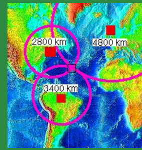
$$y^{'}\left[\frac{y^{'}}{\sqrt{y-y_0}\sqrt{1+(y^{'})^2}}\right] - \frac{\sqrt{1+(y^{'})^2}}{\sqrt{y-y_0}} = c_1$$

Solving for $y^{'} = \frac{dy}{dx}$,

$$y^{'} = \sqrt{\frac{1-c_1^2(y-y_0)}{c_1^2(y-y_0)}}$$

Letting $c_1 = (2a)^{-1/2}$

$$y^{'} = \sqrt{\frac{2a-(y-y_0)}{(y-y_0)}}$$



Background
Defining the problem
Establish Some Equations
Solving for I Don't . . .
Final Remarks
Acknowledgements
Home Page
Title Page
Page 14 of 18
Go Back
Full Screen
Close
Quit

And, equivalently

$$\frac{dy}{dx} = \sqrt{\frac{2a - (y - y_0)}{(y - y_0)}}$$

Inverting, and separating the variables

$$dx = \frac{\sqrt{(y - y_0)}}{\sqrt{2a - (y - y_0)}} dy$$

Letting

$$\tan\phi = \frac{\sqrt{(y - y_0)}}{\sqrt{2a - (y - y_0)}}$$

Such that $(y - y_0) = 2a\sin^2\phi$ and $dy = 4a\sin\phi\cos\phi d\phi$ the following occurs:

$$\begin{aligned} dx &= \tan\phi dy \\ &= 4a\sin^2\phi d\phi \end{aligned}$$

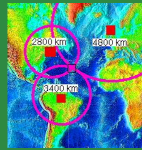
Using the double angle theorem and integrating,

$$dx = 2a(1 - \cos 2\phi)d\phi$$

$$\begin{aligned} (x - x_0) &= 2a\int(1 - cos2\phi)d\phi \\ x &= 2a\int(1 - cos2\phi)d\phi + x_0 \end{aligned}$$

Applying the double angle theorem to y,

$$\begin{aligned} (y - y_0) &= 2a\sin^2\phi \\ y &= a(1 - \cos 2\phi) + y_0 \end{aligned}$$



Home Page

Title Page

Page 15 of 18

Go Back

Full Screen

Close

Quit

Letting $2\phi = \theta$, and substituting into x and y,

$$x = a(\theta - \sin\theta) + x_0 \qquad \text{and} \qquad y = a(1 - \cos\theta) + y_0 \tag{5}$$

Which are the parametric equations of the cycloid!

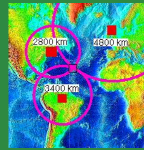
6. Final Remarks

I have derived this equation of the path of fastest descent, the cycloid. This path is almost exactly the path followed by the Brachistochrone. The only difference is that the cycloid takes into account only the gravitational force, neglecting the frictional force brought in by the Brachistochrone. The full problem is one yet to be tackled semester project. The math is a bit too complex for me to handle, but this problem will be addressed on a deeper (frictional) level at a later date. (So be watching for it!)

This idea of the path of fastest descent is in agreement with the laws of energy propagation: body waves propagate (travel) through the Earth similar to that of the propagation of light waves. This energy propagates more rapidly through more tightly structured material than it does through more loosely constructed material. The time for a seismic wave to travel from earthquake source to receiver can be expressed as a function of the path the energy traced and a model of the subsurface structure (allowing the assumption of the energy propagation velocity of Birch's Law.) The intrinsic nature of body waves, as well as their dependence upon subsurface structure and density results in P-waves travelling faster than S-waves. This is key to my eventually being able to interpret the subsurface structure of the earth.

This eventual understanding will be due in large to the findings of seismology (studying earthquakes and the results found from the passage of seismic (elastic) waves through the earth); it is the most powerful method we have for obtaining the internal structure of the earth. Seismology reveals the velocity and density (ρ) structure of the earth. In this project, I assume a homogeneous make-up in each velocity layer.

The three-dimensional structure of the earth can be determined by examining the travel time rays from all directions. This technique can recognize regions that are relatively fast or slow (when compared to the model.) This powerful research method is tomography. Earthquake distribution shows where Earth is active (near the surface) and seismic wave passage through the earth yields interior composition. That seismic energy finds the path of shortest travel time is Fermat's Principle of Least Time.



Earthquake energy attenuates through the earth on curved ray paths. The principle of the conservation of elastic energy says that the velocity attained is solely determined by the loss in potential energy, not the path that brought it there.

7. Acknowledgements

7.1. Special Thanks To:

David Arnold for his Latex and Meta-Post expertise, his patience, and his assistance with the derivations.
Dave Bazard for his Global Geophysics textbook.
Bob McPherson for his explanation of tomography in Exam 2 from my Geology 15 course.
Adam Abrahamsen for his knowledge of .log files.

References

[Anton] Anton, Howard. *Calculus, with Analytic Geometry, Second Edition* John Wiley and Sons, New York, 1984. pp.374-376, 754-761.

[Fowler] Fowler,C.M.R. *An Introduction to Global Geophysics* Cambridge University Press, Cambridge, 1990. Chapter 4, pp.76-115.

[Haws/Kiser] Haws, LaDawn and Kiser, Terry. *Exploring the Brachistochrone Problem* American Mathematical Monthly , Vol. 102, Issue 4 (April 1995), pp.328-336.

[Simmons] Simmons, George F. *Differential Equations, with Applications and Historical Notes* McGraw-Hill Publishing Co., New York, 1972. pp.25-30.

[Weinstock] Weinstock, Robert. *Calculus of Variations, with Applications to Physics and Engineering* Dover Publications, Inc., New York, 1974. pp.17-19, 22-26, 28-29.