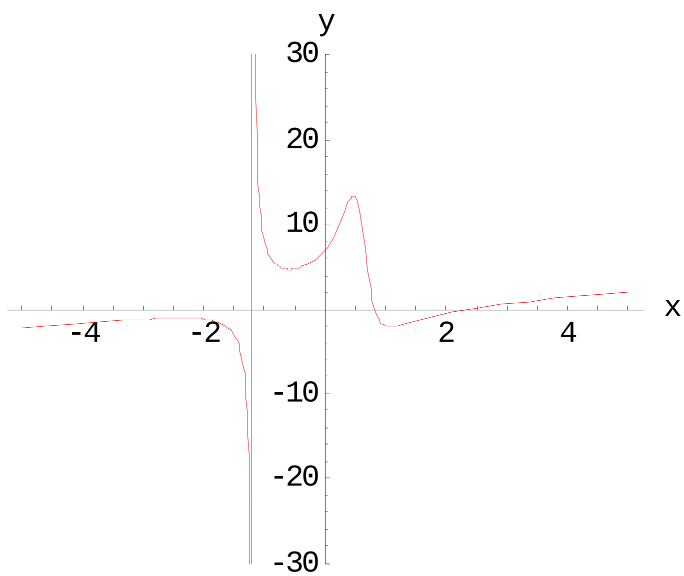
4.7 Project
Locating Critical Points and Inflection Points



This project is about locating the critical points and the inflection points on a given graph $y = f(x)$. Here we illustrate the use of various technologies for this purpose by analyzing the graph of the function

$$f(x) = \frac{x^4 - 5x^2 - 5x + 7}{2x^3 - 2x + 1} \tag{1}$$

that is shown in the figure above. (See Example 8 in Section 4.7 of the text for further discussion.) This graph appears to have several critical points and several inflection points as well as a vertical asymptote.

We will see that the denominator $2x^3 - 2x + 1$ in Eq. (1) has a single real zero near $x = -1.1915$, and this provides the location of the vertical asymptote. To find the critical points on the graph, we need first to calculate the first derivative

$$f'(x) = \frac{2x^6 + 4x^4 + 24x^3 - 32x^2 - 10x + 9}{\left(2x^3 - 2x + 1\right)^2}. \tag{2}$$

Because a fraction vanishes only if its numerator vanishes, the horizontal tangent lines to the graph $y = f(x)$ occur at the (real) zeros of the numerator, and thus where

$$g(x) = 2x^6 + 4x^4 + 24x^3 - 32x^2 - 10x + 9 = 0. \tag{3}$$

To find the inflection points on the graph, we need to calculate the second derivative

$$f''(x) = \frac{2\left(-16x^6 - 66x^5 + 120x^4 + 34x^3 - 18x^2 - 42x + 13\right)}{\left(2x^3 - 2x + 1\right)^3}. \qquad (4)$$

Then the possible inflection points on the graph $y = f(x)$ are those points where the numerator in (4) vanishes, so

$$h(x) = -16x^6 - 66x^5 + 120x^4 + 34x^3 - 18x^2 - 42x + 13 = 0. \qquad (5)$$

Thus our task of finding all the critical points and possible inflection points on the graph $y = f(x)$ boils down to finding the real solutions of Equations (3) and (5). Having done this, we can plot the graph near each such special point to determine visually whether it is a local maximum or minimum point or an actual inflection point.

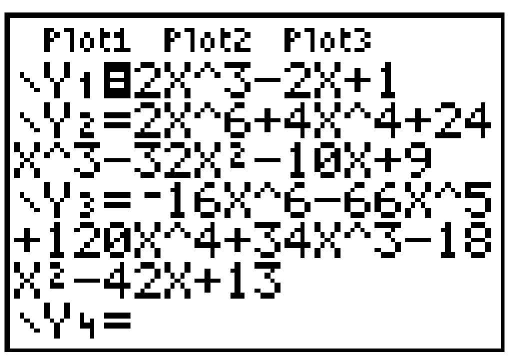
We will illustrate this procedure using typical calculating and computing systems. Having worked through these illustrative computations, you should be prepared to apply the same techniques to the investigations assigned in Project 4.7 in the text.

Using a Graphing Calculator

We have seen that the **vertical asymptote** of the function $f(x)$ in (1) above is given by the roots of its denominator polynomial $2x^3 - 2x + 1$, while

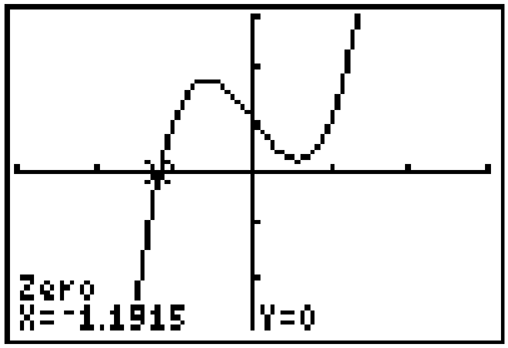
- its **critical points** are given by the roots of the polynomial $g(x)$ in (3) that is the numerator of the first derivative $f'(x)$, and
- the possible **inflection points** of $f(x)$ are the roots of the polynomial $h(x)$ in (5) that is the numerator of the second derivative $f''(x)$.

In order to use a TI-83 (for instance) to locate these special points on the graph $y = f(x)$, we first define these three polynomials. The following calculator screen shows them defined as **Y1**, **Y2**, and **Y3**, respectively.



The Asymptote

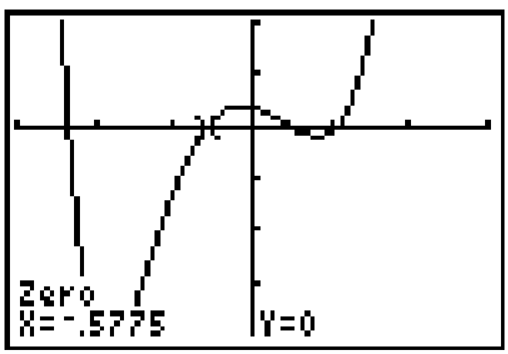
When we plot the graph of **Y1 =** $2x^3 - 2x + 1$ for $-3 \leq x, y \leq 3$, we see the following graph:



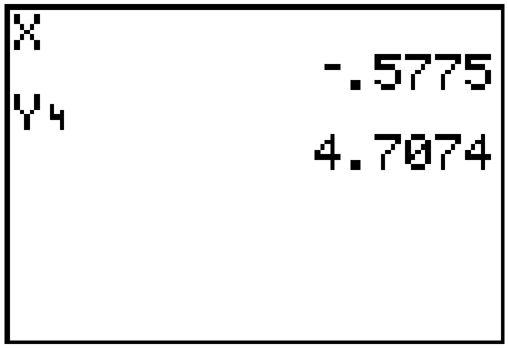
We have already used **2nd CALC zero** to locate the single *x*-intercept. Thus the curve has the single vertical asymptote with approximate location $x = -1.1915$.

The Critical Points

When we plot the graph of **Y2 =** $g(x)$ for $-3 \leq x \leq 3,\ -100 \leq y \leq 50$ we see the following graph:

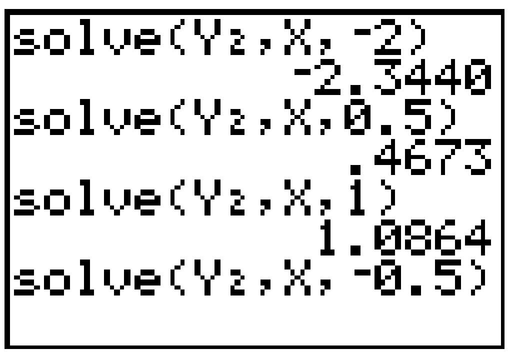


Four real solutions of $f'(x) = 0$ are visible, and we have already used **2nd CALC zero** to find that the second of these is approximately $x = -0.5775$. In order to calculate the ordinate of the corresponding critical point on the curve $y = f(x)$, we also define **Y4 =** $f(x)$. Then evaluation of **X** and **Y4** — as indicated in the calculator screen at the top of page 4 — yields the critical point $(-0.5775, 4.7074)$. Now the denominator $(2x^3 - 2x + 1)^2$ of $f'(x)$ is positive. Because the graph of the numerator $g(x)$ above therefore shows that $f'(x)$ is negative to the left and positive to the right, it follows from the first derivative test shows that this is a local minimum point.



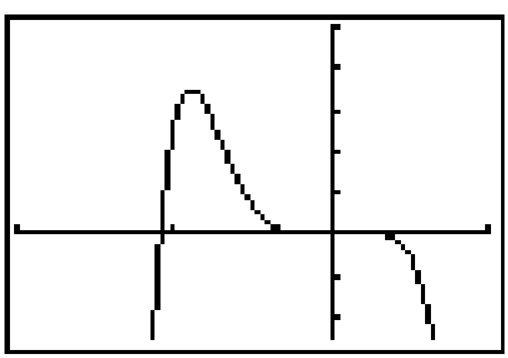
Exercise 1 Show similarly that the other three critical points on the graph $y = f(x)$ are the local maximum points (–2.3440, –1.0681) and (0.4673, 13.4304), and the local minimum point (1.0864, –2.1128).

Alternatively, you can use the **`solve`** function (entered from the **`CATALOG`** menu) if you want to find the abscissa of all four critical points immediately. In the following screen we have read the initial estimates x = –2, x = –0.5, x = 0.5, and x = 1 from the graph of $g(x)$ above.

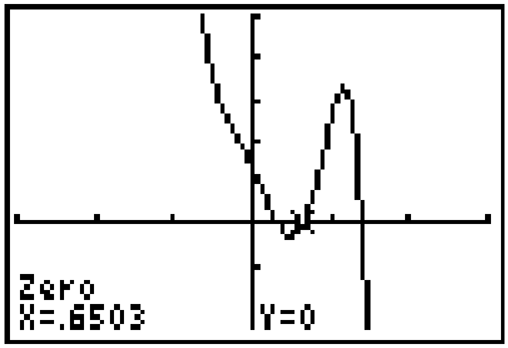


The Inflection Points

We must work some with the y-scale in order to see the "whole graph" of $y = h(x)$. When we plot the graph of **`Y3`** $= h(x)$ for $-10 \le x \le 5$, $-25000 \le y \le 50000$, we see the picture



which shows one root just to the left of $x = -5$ and some more "action" near the origin. With $-3 \le x \le 3,\ -25 \le y \le 50,$ we get the picture



which shows three positive roots. We have already used **`2nd CALC zero`** to find that the second of these is approximately $x = 0.6503$. Then evaluation of **`Y4`** yields the corresponding ordinate 7.2690. Thus the corresponding possible inflection point on the graph is (0.6503, 7.2690). Now the graph of **`Y1`** $= 2x^3 - 2x + 1$ above shows that the denominator $(2x^3 - 2x + 1)^3$ of $f''(x)$ is positive for $x > 0$. Because the graph of the numerator $h(x)$ above therefore shows that $f''(x)$ is negative to the left and positive to the right, it follows (why?) shows that this is an inflection point where the graph $y = f(x)$ is concave downward to the left and concave upward to the right.

Exercise 2 Show similarly (or using the **`solve`** function) that the other three roots of $h(x)$ yield the three inflection points (–5.4303, 2.4522), (0.3152, 11.4232), and (1.3937, –1.6289) on the graph $y = f(x)$, and check the concavity of the graph to the left and right of each.

Using Maple

First we define the function $f(x)$.

```
f := x -> (x^4-5*x^2-5*x+7)/(2*x^3-2*x+1):
```

Then we can let Maple proceed to calculate the first derivative

```
Dy := simplify(D(f)(x));
```

and the second derivative

```
D2y := simplify(D(D(f))(x));
```

The Asymptote

To find the asymptote, first let us pick out the denominator of $f(x)$.

```
denominator := denom(f(x));
```

denominator := 2*x^3-2*x+1

It zeros are given by

```
fsolve(denominator = 0, x, complex);
```

-1.19149, .595744-.254426*I, .595744+.254426*I

and we see the single real zero $x = -1.19149$ that gives the location of the vertical asymptote.

The Critical Points

We must find the real zeros of the numerator

```
g := x -> numer(Dy);
g(x);
```

2*x^6+4*x^4+24*x^3-32*x^2+9-10*x

of the derivative $f'(x)$. Its zeros are given by

```
zeros := [fsolve(g(x) = 0, x, complex)];
```

zeros := [-2.34399, -.577489, .467291, .683912-2.46595*I, .683912+2.46595*I, 1.08636]

We see that the first three and the last of these zeros are real.

```
realzeros := [op(zeros[1..3]),zeros[6]];
```

realzeros := [-2.34399, -.577489, .467291, 1.08636]

The respective ordinates of the corresponding critical points on the graph are given by

```
ordinates := map(f, realzeros);
```

ordinates := [-1.06809, 4.70742, 13.4304, -2.11277]

These four critical points themselves are defined by

```
critPoint := i -> [realzeros[i],ordinates[i]]:
```

For instance, the second of these four critical points has *xy*-coordinates given by

```
critPoint(2);
```

[-.577489, 4.70742]

and the graph

```
plot(f(x), x=-0.7..-0.4);
```

shows that it is a local minimum point.

The Inflection Points

Now we must find the real zeros of the numerator

```
h := x -> numer(D2y);
h(x);
        26-84*x+68*x^3+240*x^4-36*x^2-32*x^6-132*x^5
```

of the second derivative $f''(x)$. Its zeros are given by

```
zeros := [fsolve(h(x) = 0, x, complex)];

 zeros := [-5.43025, -.526977-.495982*I, -.526977+.495982*I,
                .315246, .650294, 1.39367]
```

We see that the first and the last three of these zeros are real.

```
realzeros := [zeros[1],op(zeros[4..6])];

     realzeros := [-5.43025, .315246, .650294, 1.39367]
```

The respective ordinates of the corresponding inflection points on the graph are given by

```
ordinates := map(f, realzeros);

     ordinates := [-2.45219, 11.4233, 7.26907, -1.62893]
```

These four inflection points themselves are defined by

```
inflectPoint := i -> [realzeros[i],ordinates[i]]:
```

For instance, the third of these four inflection points has *xy*-coordinates given by

```
inflectPoint(3);
                      [.650294, 7.26907]
```

and the graph

```
plot(f(x), x=0.5..0.8);
```

verifies that it *is* an inflection point — concave downward to the left and concave upward to the right.

Using Mathematica

First we define the function $f(x)$.

```
f[x_] := (x^4 - 5*x^2 - 5*x + 7)/(2*x^3 - 2*x + 1)
```

Then we can let *Mathematica* proceed to calculate the first derivative

```
Dy = Together[D[f[x], x]]
```

and the second derivative

```
D2y = Together[D[f[x], x,x]]
```

The Asymptote

To find the asymptote, first let us pick out the denominator of $f(x)$.

```
denom = Denominator[f[x]]
```

```
1 - 2 x + 2 x^3
```

Its zeros are given by

```
NSolve[denom == 0, x]
```

```
{{x -> -1.19149}, {x -> 0.595744 - 0.254426 I},
{x -> 0.595744 + 0.254426 I}}
```

and we see the single real zero $x = -1.19149$ that gives the location of the vertical asymptote.

The Critical Points

We must find the real zeros of the numerator

```
g[x_] = Numerator[Dy]
```

```
9 - 10 x - 32 x^2 + 24 x^3 + 4 x^4 + 2 x^6
```

of the derivative $f'(x)$. Its zeros are given by

```
zeros = NSolve[g[x] == 0, x]
```

```
{{x -> -2.34399}, {x -> -0.577489}, {x -> 0.467291},
{x -> 0.683912 - 2.46595 I},
{x -> 0.683912 + 2.46595 I}, {x -> 1.08636}}
```

We see that the first three and the last of these zeros are real.

```
realZeros = zeros[[{1, 2, 3, 6},1,2]]
```

```
{-2.34399, -0.577489, 0.467291, 1.08636}
```

The corresponding points on the graph $y = f(x)$ are tabulated by

```
critPoints =
Table[{realZeros[[i]], f[realZeros[[i]]]}, {i, 1, 4}]
```

```
{{-2.34399, -1.06809}, {-0.577489, 4.70742},
{0.467291, 13.4304}, {1.08636, -2.11277}}
```

Each (horizontal) row of this 4-by-2 array provides the coordinates $(x, f(x))$ of a critical point on the graph. For instance, the second of these points has *xy*-coordinates given by

```
critPoints[[2]]
```

```
{-0.577489, 4.70742}
```

and the graph

```
Plot[f[x], {x, -0.7, -0.4}];
```

shows that it is a local minimum point.

The Inflection Points

Now we must find the real zeros of the numerator

```
h[x_] = Numerator[D2y]
```

```
-2 (-13 + 42 x + 18 x^2 - 34 x^3 - 120 x^4 + 66 x^5 +
16 x^6 )
```

of the second derivative $f''(x)$. Its zeros are given by

```
zeros = NSolve[h[x] == 0, x]
```

```
{{x -> -5.43025}, {x -> -0.526977 - 0.495982 I},
{x -> -0.526977 + 0.495982 I}, {x -> 0.315246},
{x -> 0.650294}, {x -> 1.39367}}
```

We see that the first and the last three of these zeros are real.

```
realZeros = zeros[[{1, 4, 5, 6},1,2]]
```

```
{-5.43025, 0.315246, 0.650294, 1.39367}
```

The corresponding points on the graph $y = f(x)$ are tabulated by

```
inflectPoints =
Table[{realZeros[[i]], f[realZeros[[i]]]}, {i, 1, 4}]
```

```
{{-5.43025, -2.4522}, {0.315246, 11.4232},
{0.650294, 7.26901}, {1.39367, -1.62892}}
```

Each row of this 4-by-2 array provides the coordinates $(x, f(x))$ of an inflection point on the graph. For instance, the third of these points has *xy*-coordinates given by

```
inflectPoints[[3]]
```

```
{0.650294, 7.26901}
```

and the graph

```
Plot[f[x], {x, 0.5, 0.8}];
```

verifies that it *is* an inflection point — concave downward to the left and concave upward to the right.

Using MATLAB

First we define the function $f(x)$.

```
syms x
f = (x^4-5*x^2-5*x+7)/(2*x^3-2*x+1);
```

Then we can let MATLAB proceed to calculate the first derivative

```
Dy = simple(diff(f,x))
```

```
(2*x^6+4*x^4+24*x^3-32*x^2+9-10*x)/(2*x^3-2*x+1)^2
```

and the second derivative

```
D2y = simple(diff(f,x,2))
```

```
-2*(-13+16*x^6+42*x-34*x^3+66*x^5+18*x^2-120*x^4)/(2*x^3-
2*x+1)^3
```

You can use the commands **`pretty(Dy)`** and **`pretty(D2y)`** to see these fractions printed in nicely built-up form.

The Asymptote

To find the asymptote, first let us pick out the denominator of $f(x)$.

```
[numer,denom] = numden(f);
denom

denom =
2*x^3-2*x+1
```

It zeros are given by

```
zeros = solve(denom);
zeros = double(zeros)

zeros =
   -1.1915
   0.5957 - 0.2544i
   0.5957 + 0.2544i
```

and we see the single real zero $x = -1.1915$ that gives the location of the vertical asymptote.

The Critical Points

We must find the real zeros of the numerator

```
[numer,denom] = numden(Dy);
numer

numer =
2*x^6+4*x^4+24*x^3-32*x^2+9-10*x
```

of the derivative $f'(x)$. Its zeros are given by

```
zeros = solve(numer);
zeros = double(zeros)

zeros =
   -2.3440
   -0.5775
   0.4673
   0.6839 - 2.4659i
   0.6839 + 2.4659i
   1.0864
```

We see that the first three and the last of these zeros are real.

```
realzeros = zeros([1,2,3,6])

realzeros =
   -2.3440
   -0.5775
   0.4673
   1.0864
```

The respective ordinates of the corresponding critical points on the graph are given by

```
ordinates = double(subs(f,x,realzeros))

ordinates =
   -1.0681
   4.7074
   13.4304
   -2.1128
```

These four critical points themselves are defined by

```
critPoints = [realzeros,ordinates]

critPoints =
   -2.3440 -1.0681
   -0.5775 4.7074
   0.4673 13.4304
   1.0864 -2.1128
```

For instance, the second of these four critical points has *xy*-coordinates given by

```
critPoints(2,:)
ans =
   -0.5775 4.7074
```

and the graph

```
ezplot(f,[-0.7 -0.4])
```

shows that it is a local minimum point.

The Inflection Points

Now we must find the real zeros of the numerator

```
[numer,denom] = numden(D2y);
numer

numer =
26-32*x^6-84*x+68*x^3-132*x^5-36*x^2+240*x^4
```

of the second derivative $f''(x)$. Its zeros are given by

```
zeros = solve(numer);
zeros = double(zeros)

zeros =
```

```
   -5.4303
   -0.5270 - 0.4960i
   -0.5270 + 0.4960i
   0.3152
   0.6503
   1.3937
```

We see that the first and the last three of these zeros are real.

```
realzeros = zeros([1,4,5,6])

realzeros =
   -5.4303
   0.3152
   0.6503
   1.3937
```

The respective ordinates of the corresponding inflection points on the graph are given by

```
ordinates = double(subs(f,x,realzeros))

ordinates =
   -2.4522
   11.4232
   7.2690
   -1.6289
```

These four inflection points themselves are defined by

```
inflectPoint = [realzeros,ordinates]

inflectPoint =
   -5.4303   -2.4522
    0.3152   11.4232
    0.6503    7.2690
    1.3937   -1.6289
```

For instance, the third of these four inflection points has *xy*-coordinates given by

```
inflectPoint(3,:)
ans =
0.6503 7.2690
```

and the graph

```
ezplot(f,[0.5 0.8])
```

verifies that it *is* an inflection point — concave downward to the left and concave upward to the right.