Predator-Prey Modeling

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Introduction

Predator-prey modelling is population modelling with two distinct populations, one of which is a food source for the other.

The General System

H is the prey population. (Herbivores)

P is the predator population.

$$
\begin{aligned}
H' &= h(H, P) \\
P' &= p(H, P)
\end{aligned}
$$

or

$$
\begin{bmatrix} H \\ P \end{bmatrix}' = F\left(\begin{bmatrix} H \\ P \end{bmatrix}\right)
$$

Relevant Concepts

- The Malthusian and logistic models for a single population.
- Autonomous systems ($\vec{x}' = F(\vec{x})$).
- Linearization and the Jacobian.

Developing Models

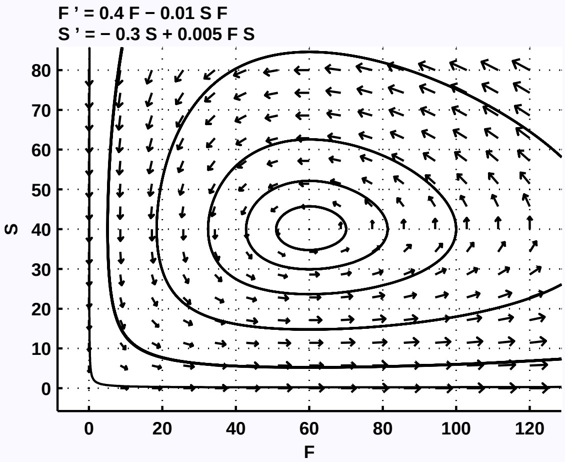
The Malthusian model:

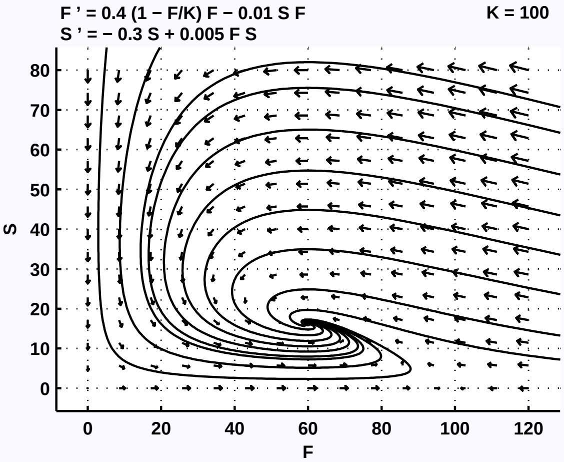
$$
\begin{aligned}
H' &= rH \\
\frac{H'}{H} &= r
\end{aligned}
$$

The logistic model:

$$
\begin{aligned}
H' &= r\left(1 - \frac{H}{K}\right)H \\
\frac{H'}{H} &= r\left(1 - \frac{H}{K}\right)
\end{aligned}
$$

Why is predator-prey modelling important?





$$
\begin{aligned}
F' &= 0.4F - 0.01FS & \quad F' &= 0.4(1 - F/K)F - 0.01FS \\
S' &= -0.3S + 0.005FS & \quad S' &= -0.3S + 0.005FS
\end{aligned}
$$

As $K \to \infty$, the logistic system behaves more like the malthusian system.

This suggests that a practical method for stabilizing wildly oscillating populations with large K might be to lower K.

Advanced Predator-Prey Modelling

- Develop more realistic models.
- Linearize the systems around interesting equilibrium points and examine their behavior.
- Make suggestions for manipulating populations.

Model 1

$$\begin{aligned} H' &= rH - dHP \\ 0 &= H(r - dP) \\ H = 0 \quad \text{or} \quad 0 &= r - dP \\ P &= r/d \end{aligned}$$

$$\begin{aligned} P' &= -sP + fHP \\ 0 &= P(-s + fHP) \\ P = 0 \quad \text{or} \quad 0 &= -s + fH \\ H &= s/f \end{aligned}$$

Equilibrium points: $(0, 0), (s/f, r/d)$

$$\frac{\partial H'}{\partial H} = r - dP \qquad \frac{\partial H'}{\partial P} = -dH$$

$$\frac{\partial P'}{\partial H} = fP \qquad \frac{\partial P'}{\partial P} = -s + fH$$

$$J = \begin{bmatrix} r - dP & -dH \\ fP & -s + fH \end{bmatrix}$$

Model 1 Continued

$$J = \begin{bmatrix} r - dP & -dH \\ fP & -s + fH \end{bmatrix}$$

$$J\left(\begin{bmatrix} s/f \\ r/d \end{bmatrix}\right) = \begin{bmatrix} 0 & -ds/f \\ fr/d & 0 \end{bmatrix}$$

$$\begin{aligned} \text{Trace } &= 0 \\ \text{Determinant } &= sr \end{aligned}$$

There is center-like behavior near the interesting equilibrium point.

Model 2

$$\begin{aligned} H' &= r(1 - H/K)H - dHP \\ 0 &= H(r(1 - H/K) - dP) \\ H = 0 \quad &or \quad dP = r(1 - H/K) \\ &P = r/d(1 - H/K) \end{aligned}$$

$$\begin{aligned} P' &= -sP + fHP \\ 0 &= P(-s + fHP) \\ P = 0 \quad &or \quad H = s/f \end{aligned}$$

Equilibrium points: $(0,0), (K,0), (s/f, r/d - sr/Kfd)$

$$\frac{\partial H'}{\partial H} = r - (2r/K)H - dP \qquad \frac{\partial H'}{\partial P} = -dH$$

$$\frac{\partial P'}{\partial H} = fP \qquad \frac{\partial P'}{\partial P} = -s + fH$$

$$J = \begin{bmatrix} r - (2r/K)H - dP & -dH \\ fP & -s + fH \end{bmatrix}$$

Model 2 Continued

$$J = \begin{bmatrix} r - (2r/K)H - dP & -dH \\ fP & -s + fH \end{bmatrix}$$

$$J\left(\begin{bmatrix} s/f \\ r/d - sr/Kfd \end{bmatrix}\right) = \begin{bmatrix} -rs/Kf & -ds/f \\ (rKf - rs)/dK & 0 \end{bmatrix}$$

$$\text{Trace} = -rs/Kf$$
$$\text{Determinant} = sr(Kf - s)/Kf$$

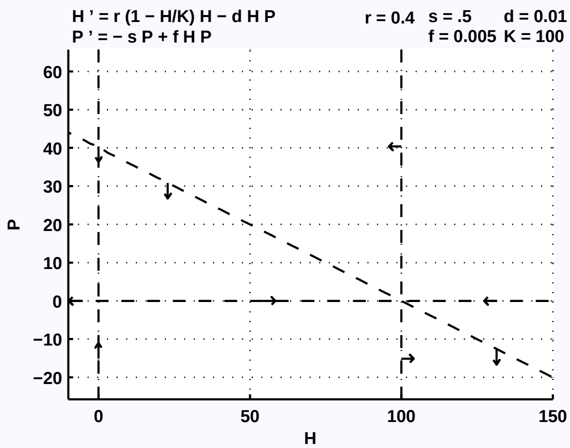
There is sink-like behavior near the interesting equilibrium point as long as $Kf - s > 0$.

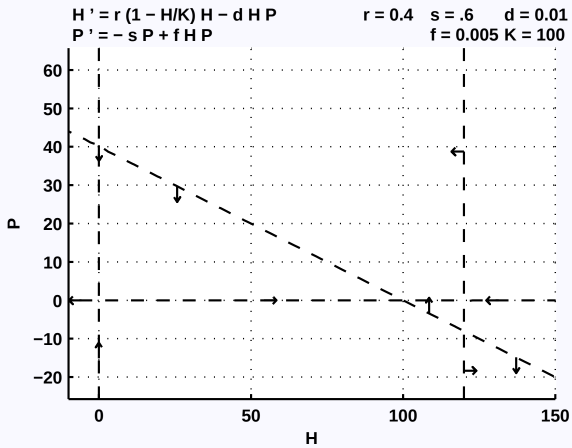
Model 2 Continued

$Kf > s$ means the determinant is greater than 0.

What if $Kf \leq s$?

Equilibrium point: $(s/f, r/d(1 - s/Kf))$





A New Model

From model 2: $H' = r\left(1 - \frac{H}{K}\right) H - dHP$

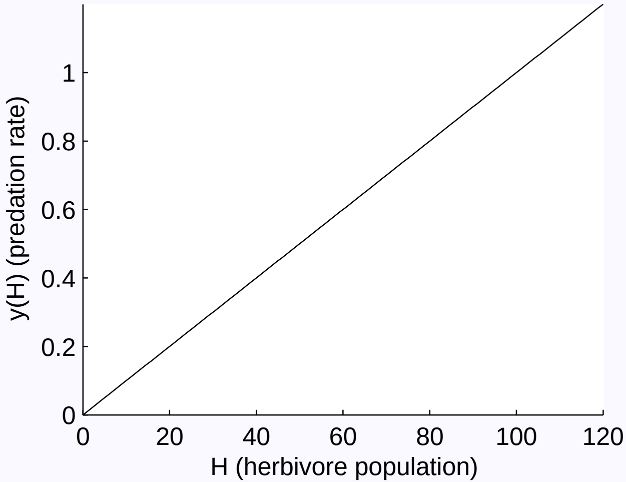
Harm done to prey population growth rate from predator interaction: dHP

Harm done to prey population growth rate per predator (predation rate): dH

Is that realistic?

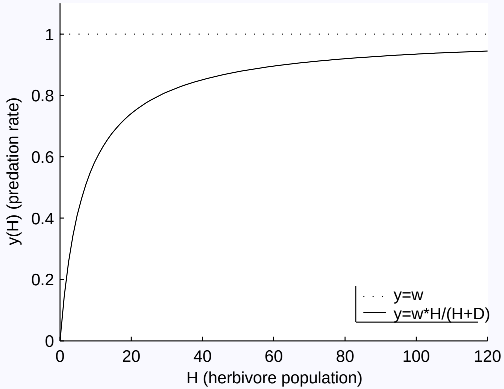
Linear Predation

That is not realistic.



Individual predators are limited in how much they can and will kill.

A Better Predation Model



$$y = w\frac{H}{D + H}$$

- $y(0) = 0$
- $\lim_{H\to\infty} y(H) = w$
- The parameters have meaning. w is the maximum predation rate, and D is proportional to predator search time or alternately the level of cover offered to the prey by the environment.

The Entire Prey Equation

$$H' = r\left(1 - \frac{H}{K}\right)H - Py$$

$$y = w\frac{H}{H+D}$$

$$H' = r\left(1 - \frac{H}{K}\right)H - Pw\frac{H}{H+D}$$

A New Predator Model

$$P' = s\left(1 - \frac{P}{HJ^{-1}}\right)P$$

J is the number of prey required to support one predator at equilibrium. This is very similar to the logistic model, except a constant carrying capacity K is replaced by HJ^{-1}, the number of predators that a population of H prey could support at equilibrium.

The Complete System (Model 3)

$$
\begin{aligned}
H' &= r\left(1-\frac{H}{K}\right)H - Pw\frac{H}{H+D}\\
P' &= s\left(1-\frac{P}{HJ^{-1}}\right)P
\end{aligned}
$$

The parameters:

- r and s are the natural growth rates of the prey and predators, respectively.
- K is the carrying capacity of the environment for the prey.
- J is the number of prey required to support one predator at equilibrium.
- w is the maximum predation rate.
- D is the predator search time, or alternately the quality of the cover afforded the prey by the environment.

Model 3

The process of finding the nullclines proceeds in the usual manner, but involves many more steps. The equation of the non-trivial prey nullcline is

$$P = \frac{r}{w}\left(1 - \frac{H}{K}\right)(H + D)$$

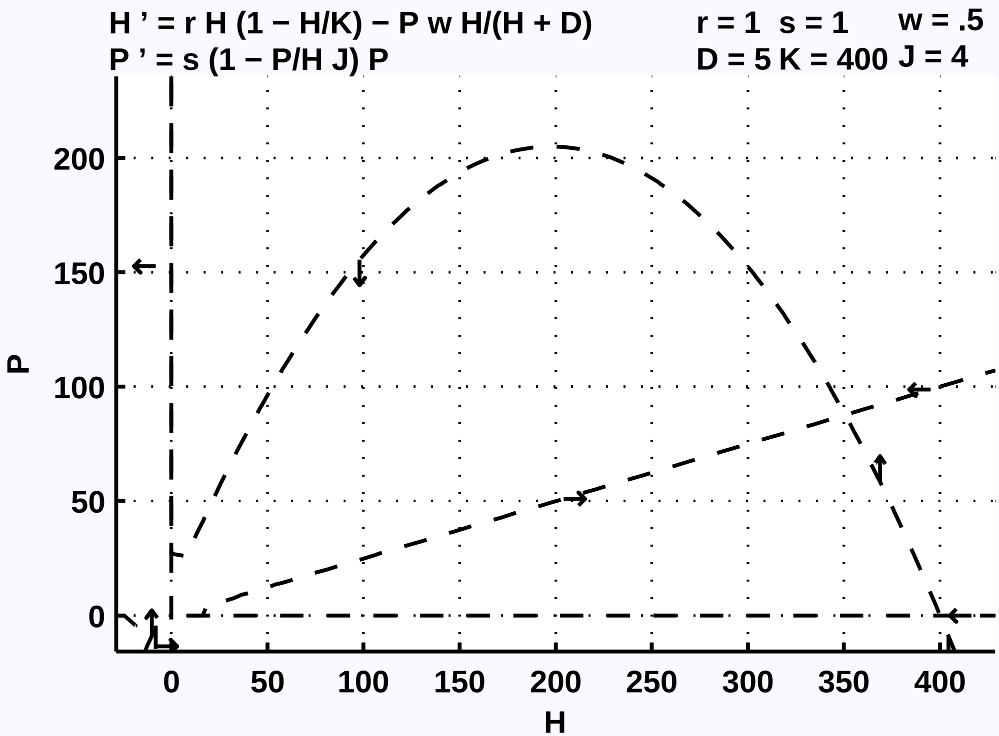
or

$$P = -\frac{r}{wK}H^2 + \frac{rK - rD}{wK}H + \frac{rD}{w}.$$

It is a parabola with roots at $H = K$ and $H = -D$, and a vertex at $H = (K - D)/2$. It opens down. The equation of the non-trivial predator nullcline is

$$P = \frac{H}{J}.$$

Model 3 Continued



Model 3 Continued

It is possible to follow the same procedure as before to find the Jacobian at the equilibrium point. However, another method yields more useful information. First, declare the equilibrium point to be (H^*, P^*). Then, scale the variables H and P and the parameters K and D by H^*. This gives

$$\begin{aligned} H^* &= 1 \\ P^* &= \frac{H^*}{J} = J^{-1} = \frac{r}{w}\left(1 - \frac{1}{K}\right)(1 + D). \end{aligned}$$

The Jacobian, evaluated at (H^*, P^*) is

$$J\left(\begin{bmatrix} H^* \\ P^* \end{bmatrix}\right) = \begin{bmatrix} r(-K^{-1} + w(rJ)^{-1}(1+D)^{-2}) & -\frac{w}{1+D} \\ \frac{s}{J} & -s \end{bmatrix}$$

Model 3 Continued

$$\begin{vmatrix} r(-K^{-1} + w(rJ)^{-1}(1+D)^{-2}) & -\frac{w}{1+D} \\ \frac{s}{J} & -s \end{vmatrix} = sr\left(\frac{1}{K} + \frac{wD}{rJ(1+D)^2}\right)$$

The determinant is always positive.

In order to be a stable equilibrium point, the trace must be less than 0. This can be written as

$$r(-K^{-1} + w(rJ)^{-1}(1+D)^{-2}) - s < 0$$

or

$$s/r > \frac{2\left(\frac{K-D}{2} - 1\right)}{K(1+D)}.$$

Recall that the vertex of the prey nullcline is located at $H = \frac{K-D}{2}$, and that in the above equation involving the trace, the parameters K and D have been scaled by H^*.

Model 3 Continued

$$s/r > \frac{2\left(\frac{K-D}{2} - 1\right)}{K(1+D)}$$

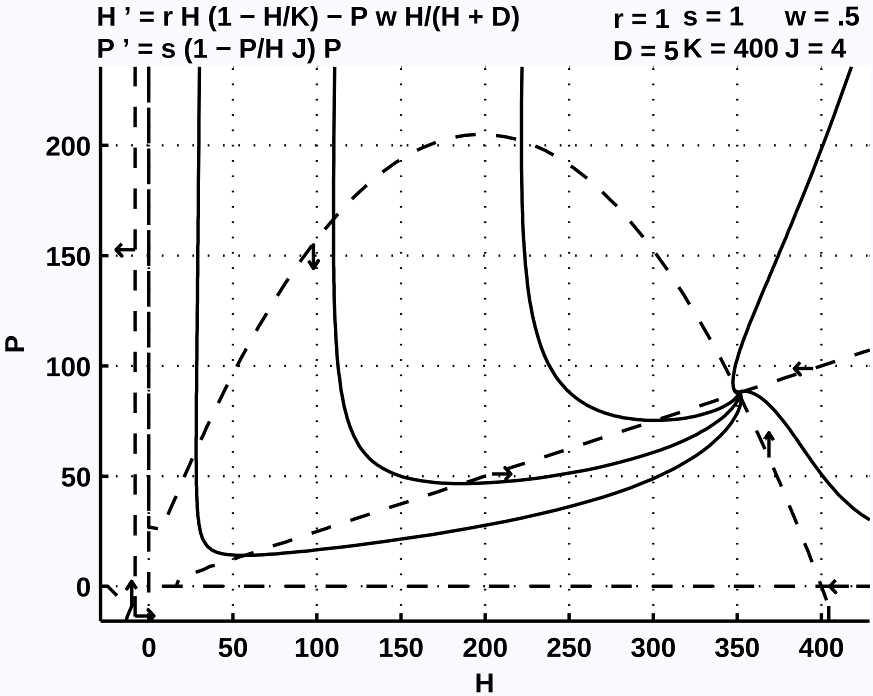
$$(H^* = 1) = \frac{K-D}{2}$$

If $(K-D)/2 < (H^* = 1)$ (i.e. the vertex is to the left of the equilibrium point), then the trace must be less than 0. If $(K-D)/2 > (H^* = 1)$ (i.e. the vertex is to the right of the equilibrium point), then the trace is only 0 if the ratio of s to r is greater than $(K-D-2)/(K(1+D))$.

Model 3 Continued

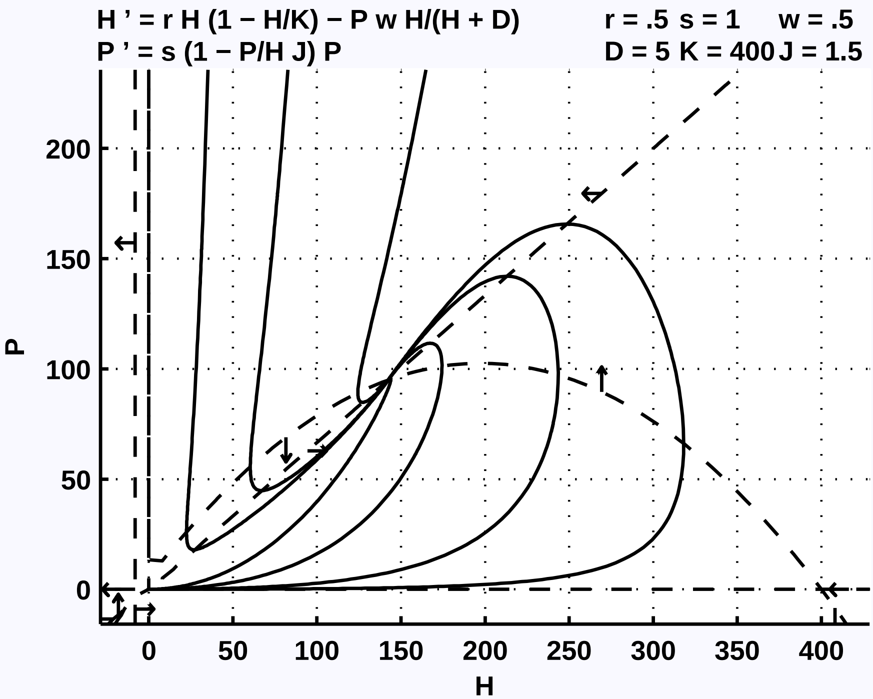
In other words, the equilibrium point is stable if a sufficiently large number of prey are required to support a single predator (i.e. J is large enough), or failing that, if the predator population in ideal predator conditions responds sufficiently more quickly than the prey population in ideal prey conditions (i.e. s/r is large enough).

Model 3 Continued



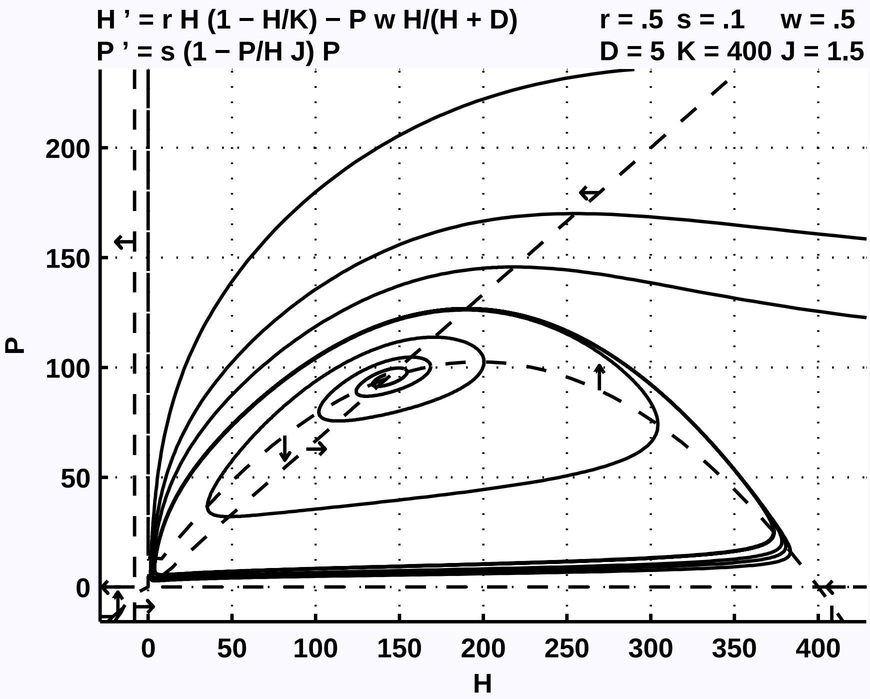
The equilibrium point is to the right of the vertex.

Model 3 Continued



The equilibrium point is to the left of the vertex and is stable.

Model 3 Continued



The equilibrium point is to the left of the vertex and is unstable.

Final Thoughts

Algebraically, it is easy to manipulate the parameters to turn an unstable equilibrium point into a stable equilibrium point. One might

- raise J,
- raise D,
- lower K,
- raise s,
- lower r,
- or lower w.

Final Thoughts

However, it manipulating these parameters in the real world can be more difficult. Raising s or lowering r would mean interfering with the natural reproductive rate of one species while leaving its other behavior unchanged. One might raise J by lowering the nutritional value of the prey to the predator (e.g. by reducing the nutritional value of the prey's own food), but how would this affect the prey's health and longevity?

Final Thoughts

Also, it is important to keep in mind that these systems are definitely not linear. While solutions that start near equilibrium points will behave similarly to their counterparts in the linearization, there is no guarantee of this further away. Do solutions that start near unstable equilibrium points spiral off into increasingly unstable loops, or approach a limit cycle. Further analysis is necessary to fully determine the behavior of the systems.

The End