The Ovals of Cassini

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May 15, 2007

Background

This curve was studied by Giovanni Domenico Cassini in 1680. It was his belief that stellar bodies followed paths traced out by one of these curves. If we have two fixed points F_1 and F_2 and a constant c, then the locus of points where the $|PF_1| \cdot |PF_2| = c$ defines a Cassinian Oval. This is similar to the ellipse but rather than adding the distances, their product is what is used.

General Equation

We will start with the general Cartesian equation for the curve:

$$[(x-a)^2+y^2][(x+a)^2+y^2]=b^4$$

To convert this to polar coordinates we start with the general conversion. Let $x = r\cos\theta$ and $y = r\sin\theta$.

Polar Form

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = b^4.$$

We can expand the left-hand side of this equation as follows.

$$\begin{aligned}
&[(x-a)^2 + y^2][(x+a)^2 + y^2] \\
&= [(r\cos\theta - a)^2 + (r\sin\theta)^2][(r\cos\theta + a)^2 + (r\sin\theta)^2] \\
&= [r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta]\cdot \\
&\qquad [r^2\cos^2\theta + 2ar\cos\theta + a^2 + r^2\sin^2\theta] \\
&= r^4\sin^4\theta + r^4\cos^4\theta - 2a^2r^2\cos^2\theta + 2a^2r^2\sin^2\theta \\
&\qquad + 2r^4\sin^2\theta\cos^2\theta + a^4 \\
&= r^4\sin^4\theta + r^4\cos^4\theta - 2a^2r^2(\cos^2\theta - \sin^2\theta) + 2r^4\sin^2\theta\cos^2\theta + a^4 \\
&= r^4\sin^4\theta + r^4\cos^4\theta - 2a^2r^2\cos 2\theta + 2r^4\sin^2\theta\cos^2\theta + a^4
\end{aligned}$$

Polar Form

Continuing in this manner,

$$\begin{aligned}
&[(x-a)^2+y^2][(x+a)^2+y^2] \\
&= r^4(\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta) - 2a^2r^2\cos 2\theta + a^4 \\
&= r^4(\sin^2\theta + \cos^2\theta)^2 - 2a^2r^2\cos 2\theta + a^4 \\
&= r^4 - 2a^2r^2\cos 2\theta + a^4
\end{aligned}$$

Which results in our polar equation. We need to push this further and try to find the parametric equations for this curve. We'll need to solve for r.

Parametric Form

$$
\begin{aligned}
r^2 &= \frac{2a^2\cos 2\theta \pm \sqrt{4a^4\cos^2 2\theta - 4(a^4 - b^4)}}{2} \\
&= a^2\cos 2\theta \pm \sqrt{a^4\cos^2 2\theta - a^4 + b^4} \\
&= a^2\cos 2\theta \pm \sqrt{a^4(\cos^2 2\theta - 1) + b^4} \\
&= a^2\cos 2\theta \pm \sqrt{-a^4(\sin^2 2\theta) + b^4} \\
&= a^2\left[\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}\right] \\
r &= \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}}
\end{aligned}
$$

Now using this r we'll need to substitute into our original

Parametric Form

$$x = \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}\,[\cos\theta]}$$

$$y = \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}\,[\sin\theta]}$$

Definitely some ugly equations, but better yet, if you have noticed there can be values of the radicand $\left(\left(\frac{b}{a}\right)^4 - \sin^2 2\theta\right)$ where complex numbers will result. So we will need to restrict our θ values.

Problem Solved

$$\left(\left(\frac{b}{a}\right)^4 - \sin^2 2\theta\right) = 0$$

$$\left(\frac{b}{a}\right)^2 = \sin 2\theta$$

$$\frac{1}{2}\arcsin\left(\frac{b}{a}\right)^2 = \theta$$

$\therefore \theta$ must be in the range of $\pm\frac{1}{2}\arcsin\left(\frac{b}{a}\right)^2$.

Phew!

Now that we finally have all of that out of the way it's time for some refreshing. Remember the analogy to the ellipse? Using the Cartesian equation

$$[(x-a)^2+y^2][(x+a)^2+y^2]=b^4$$

our value for a dictates the distance from the origin our foci will be located and our earlier c will be b^2.

Not Just Any Oval

The biggest eyebrow-raiser this equation holds is the ratio of $\frac{b}{a}$. Depending on what values are chosen we will receive an oval shape, 2 separate egg shaped curves, or the Lemniscate of Bernoulli, which is merely a special case of Cassinian Ovals. These ratios are listed below for easy reference.

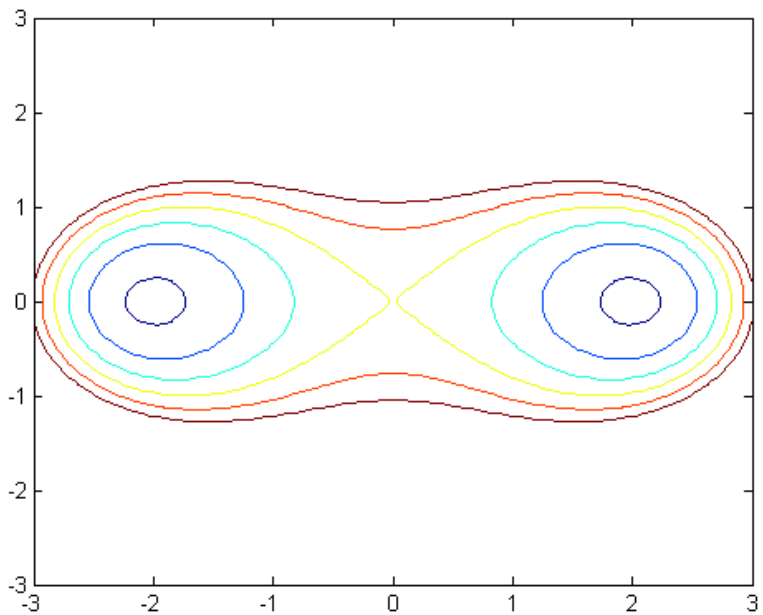
$$\frac{b}{a} > 1 \text{ Results in 1 loop}$$

$$\frac{b}{a} < 1 \text{ Results in 2 separate loops}$$

$$\frac{b}{a} = 1 \text{ Results in the Lemniscate}$$

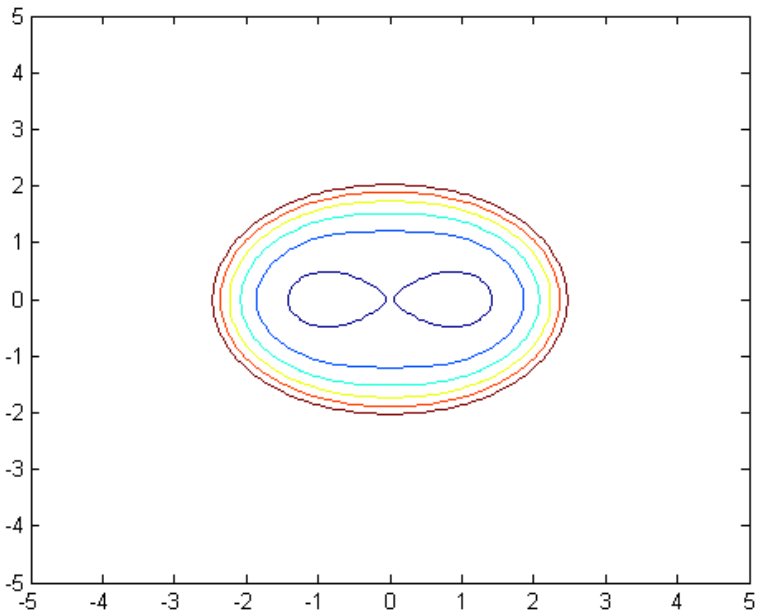
Pictures!

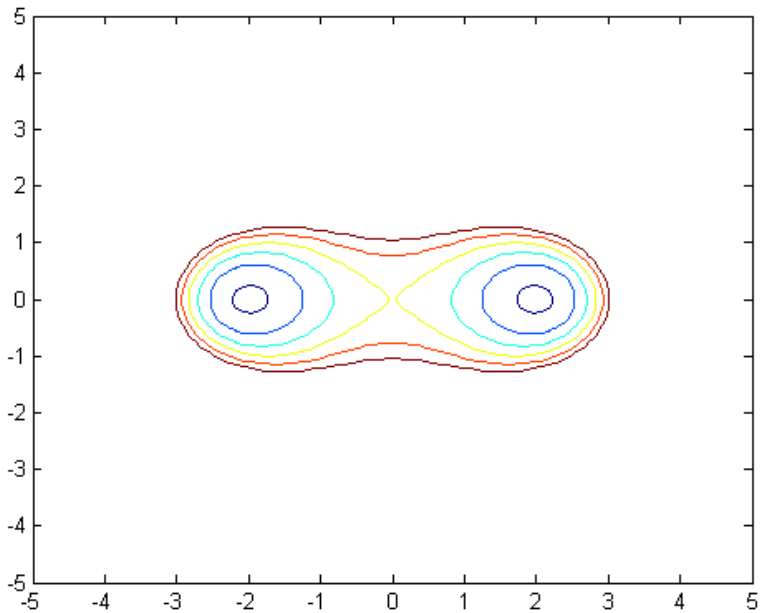
Now that we have an understanding of this curve we can now create some representations with it using *Matlab*®.

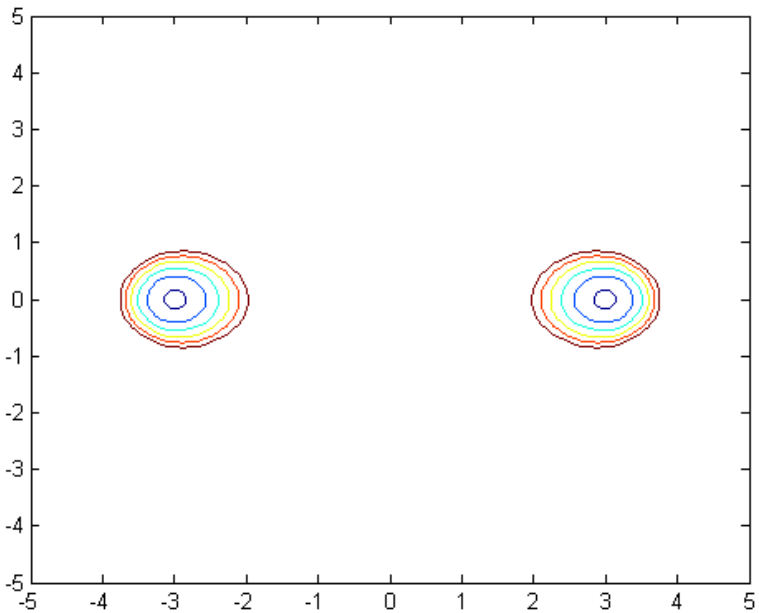


(Here the yellow curve is the Lemniscate.)

Here is some examples of changing the focus a.



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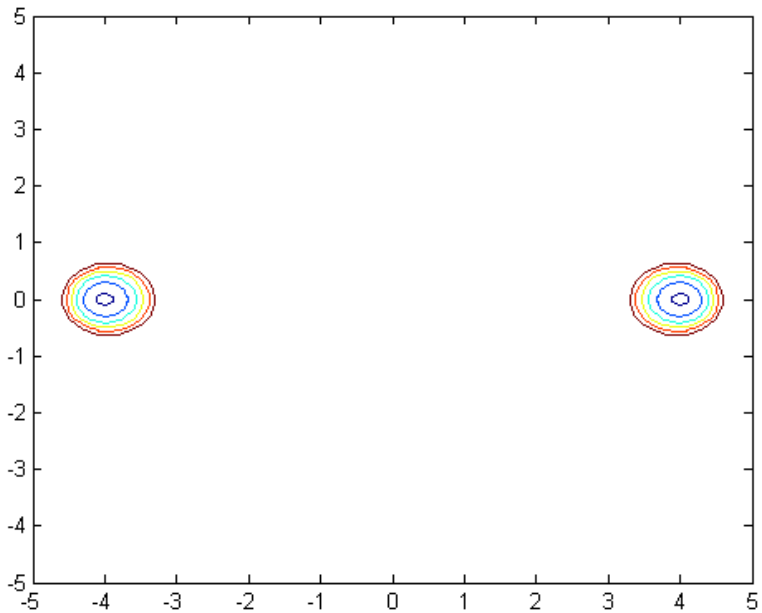
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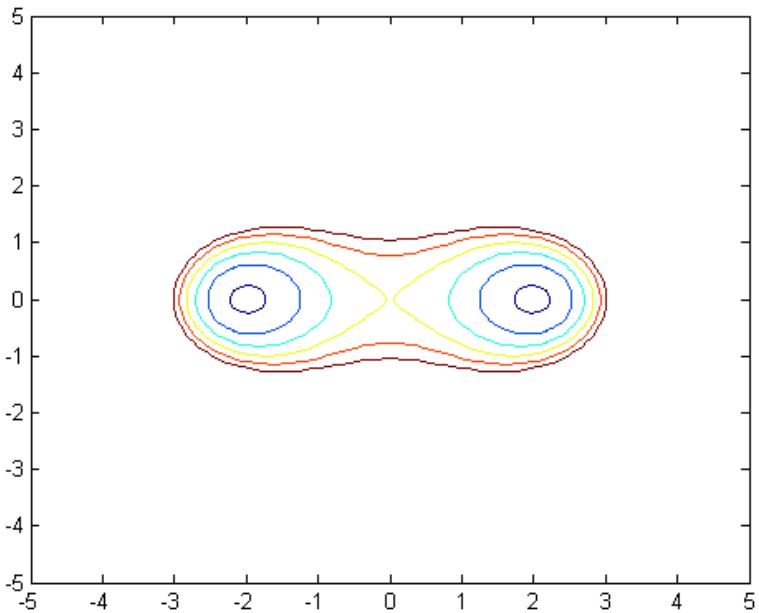
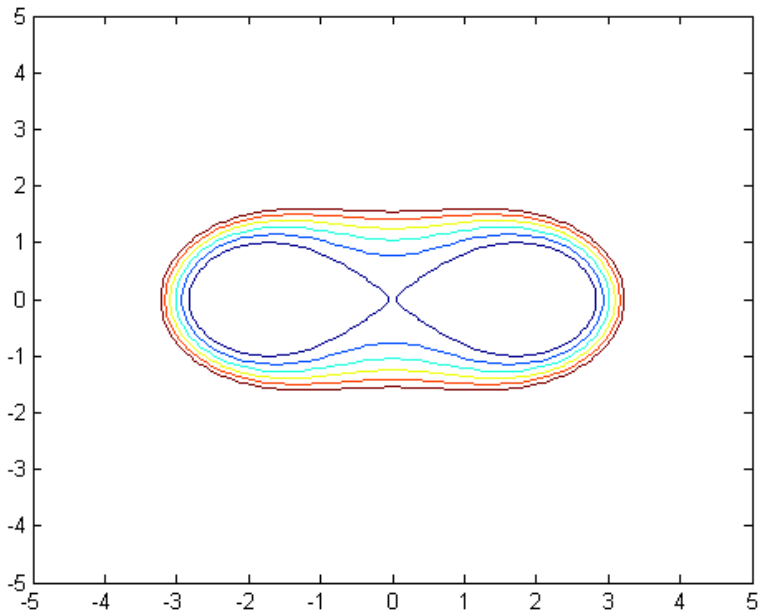
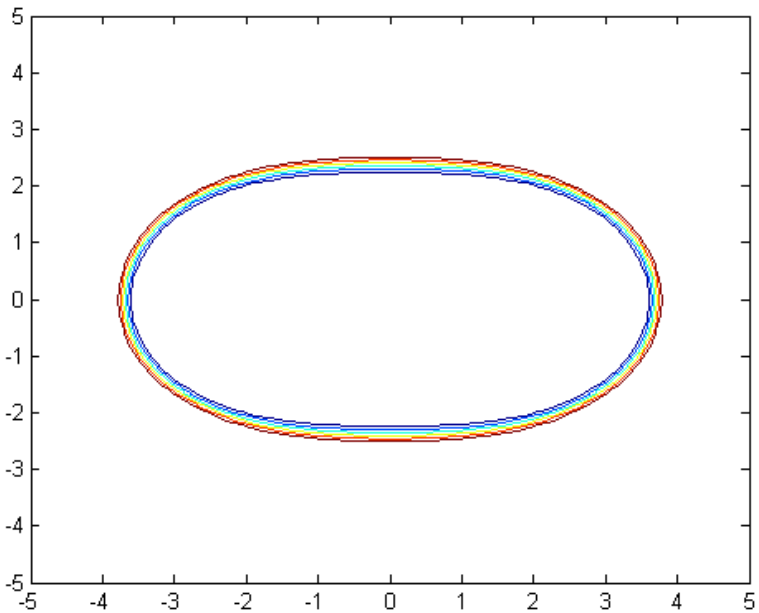
And changing b.



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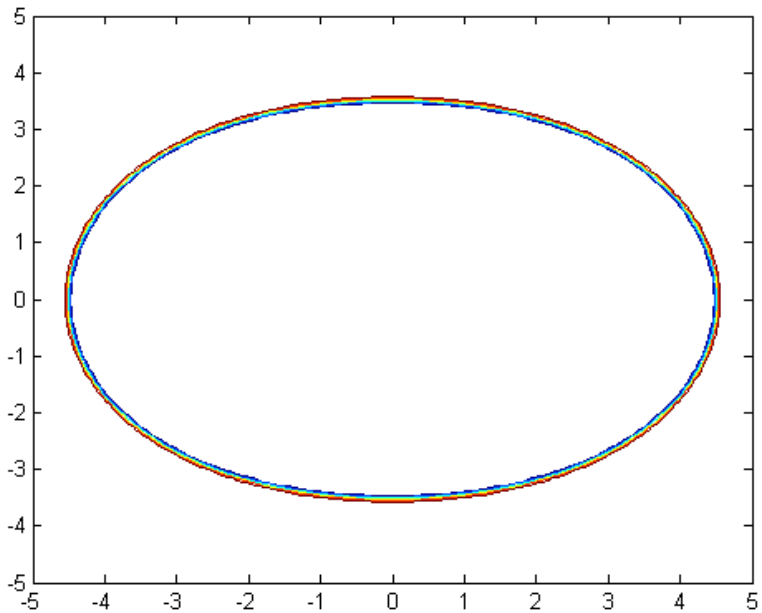
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-5 -4 -3 -2 -1 0 1 2 3 4 5

There's More to Explore

While this is a very in depth analysis of the curve there is more that can be done. Most notably exploring three dimensional representations of the curve. While conic sections reproduce ellipses how can one reproduce this curve in a similiar manner and how is the Möbius Strip related to Cassinian Ovals?