Application 7.3
Eigenvalue Calculations and Brine Tank Problems

Most computational systems offer the capability to find eigenvalues and eigenvectors readily. For instance, for the matrix

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \tag{1}$$

of Example 2 in Section 7.3 of the text, the TI-86 commands

```
[[-0.5,0,0][0.5,-0.25,0][0,0,25,-0.2]] → A
eigVl A
                              {-.2, -.25, -.5}
eigVc A
                              [[0.00   0.00   1.00]
                               [0.00   1.00  -2.00]
                               [1.00  -5.00   1.67]]
```

produce the three eigenvalues -0.2, -0.25, and -0.5 of the matrix $\mathbf{A}$ and display beneath each its (column) eigenvector. Note that with results presented in decimal form, it is up to us to guess (and verify by matrix multiplication) that the exact eigenvector associated with the eigenvalue $\lambda = -\frac{1}{2}$ is $\mathbf{v} = \begin{bmatrix} 1 & -2 & \frac{5}{3} \end{bmatrix}^T$. The *Maple* commands

```
with(linalg):
A := matrix(3,3,[-0.5,0,0, 0.5,-0.25,0, 0,0.25,-0.2]);
eigenvects(A);
```

the *Mathematica* commands

```
A = {{-0.5,0,0}, {0.5,-0.25,0}, {0,0.25,-0.2}}
Eigensystem[A]
```

and the MATLAB commands

```
[-0.5,0,0; 0.5,-0.25,0; 0,0.25,-0.2]
[V, E] = eig(A)
```

(where $\mathbf{E}$ will be a diagonal matrix displaying the eigenvalues of $\mathbf{A}$ and the column vectors of $\mathbf{V}$ are the corresponding eigenvectors) produce similar results. You can use these commands to find the eigenvalues and eigenvectors needed for any of the problems in Section 7.3 of the text.

Brine Tank Investigations

Consider a linear cascade of 5 full brine tanks whose volumes v_1, v_2, v_3, v_4, v_5 are given by

$$v_i = 10\, d_i \qquad \text{(gallons)}$$

where d_1, d_2, d_3, d_4, d_5 are the first five *distinct* non-zero digits of ***your*** student ID number. (Pick additional digits at random if your ID number has less than five distinct non-zero digits.)

Initially, Tank 1 contains one pound of salt per gallon of brine, whereas the remaining tanks contain pure water. The brine in each tank is kept thoroughly mixed, and the flow rate out of each tank is $r_i = 10$ gal/min. Your task is to investigate the subsequent amounts $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$, $x_5(t)$ of salt (in pounds) present in these brine tanks.

Open System

If fresh water flows into Tank 1 at the rate of 10 gal/min, then these functions satisfy the system

$$\begin{aligned} x_1' &= -k_1 x_1 \\ x_2' &= +k_1 x_1 - k_2 x_2 \\ x_3' &= +k_2 x_2 - k_3 x_3 \\ x_4' &= +k_3 x_3 - k_4 x_4 \\ x_5' &= +k_4 x_4 - k_5 x_5 \end{aligned} \tag{2}$$

where $k_i = r_i/v_i$ for i = 1, 2, ..., 5. Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and the corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ of the this system's coefficient matrix in order to write the general solution in the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} + c_4\mathbf{v}_4 e^{\lambda_4 t} + c_5\mathbf{v}_5 e^{\lambda_5 t}. \tag{3}$$

Use the given initial conditions to find the values of the constants c_1, c_2, c_3, c_4, c_5. Then observe that each $x_i(t) \to 0$ as $t \to \infty$, and explain why you would anticipate this result. Plot the graphs of the component functions $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)$ of $\mathbf{x}(t)$ on a single picture, and finally note (at least as close as the mouse will take you) the maximum amount of salt that is ever present in each tank.

Closed System
If Tank 1 receives as inflow (rather than fresh water) the outflow from Tank 5, then the first equation in (2) is replaced with the equation

$$x_1' = +k_5 x_5 - k_1 x_1 . \qquad (4)$$

Assuming the same initial conditions as before, find the explicit solution of the form in (3). Now show that — in this *closed* system of brine tanks — as $t \to \infty$ the salt originally in Tank 1 distributes itself with uniform concentration throughout the various tanks. A plot should make this point rather vividly.

Maple, *Mathematica*, and MATLAB techniques that will be useful for these brine tank investigations are illustrated in the sections that follow. We consider the "open system" of three brine tanks that is shown in Fig. 7.3.2 of the text (see Example 2 of Section 7.3). The vector $\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)]^T$ of salt amounts (in the three tanks) satisfies the linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \qquad (5)$$

where $\mathbf{A}$ is the 3×3 matrix in (1). If initially Tank 1 contains 15 pounds of salt and the other two tanks contain pure water, then the initial vector is $\mathbf{x}(0) = \mathbf{x}_0 = [15\ \ 0\ \ 0]^T$.

Using *Maple*

We begin by entering (as indicated previously) the coefficient matrix in (1),

```
with(linalg):
A := matrix(3,3,[-0.5,0,0, 0.5,-0.25,0, 0,0.25,-0.2]);
```

$$A := \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix}$$

and the initial vector

```
x0 := matrix(3,1, [15,0,0]);
```

$$x0 := \begin{bmatrix} 15 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvalues and eigenvectors of **A** are calculated with the command

```
eigs := eigenvects(A);
```

$$eigs := [-.5, 1, \{[1\ \ -2.000000000\ \ 1.666666667]\}],$$
$$[-.25, 1, \{[0\ \ 1\ \ -5.000000000]\}],$$
$$[-.2, 1, \{[0\ \ 0\ \ 1]\}]$$

Thus the first eigenvalue λ_1 and its associated eigenvector $\mathbf{v}_1$ are given by

```
eigs[1][1];
```

$$-.5$$

```
eigs[1][3][1];
```

$$[1\quad -2.000000000\quad 1.666666667]$$

We therefore record the three eigenvalues

```
L1 := eigs[1][1]:
L2 := eigs[2][1]:
L3 := eigs[3][1]:
```

and the corresponding three eigenvectors

```
v1 := matrix(1,3, eigs[1][3][1]):
v2 := matrix(1,3, eigs[2][3][1]):
v3 := matrix(1,3, eigs[3][3][1]):
```

The matrix **V** with these three column vectors is then defined by

```
V := transpose(stackmatrix(v1,v2,v3));
```

$$V := \begin{bmatrix} 1 & 0 & 0 \\ -2.000000000 & 1 & 0 \\ 1.666666667 & -5.000000000 & 1 \end{bmatrix}$$

To find the constants $c_1,\ c_2,\ c_3$ in the solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} \tag{6}$$

we need only solve the system $\mathbf{Vc} = \mathbf{x_0}$:

```
c := linsolve(V,x0);
```

$$c := \begin{bmatrix} 15. \\ 30. \\ 125.0000000 \end{bmatrix}$$

Recording the values of these three constants,

```
c1 := c[1,1]:
c2 := c[2,1]:
c3 := c[3,1]:
```

we can finally calculate the solution

```
x := evalm(c1*v1*exp(L1*t) + c2*v2*exp(L2*t) +
             c3*v3*exp(L3*t)):
```

in (6) with component functions

```
x1 := x[1,1];
x2 := x[1,2];
x3 := x[1,3];
```

$$x1 := 15.\, e^{-.5t}$$

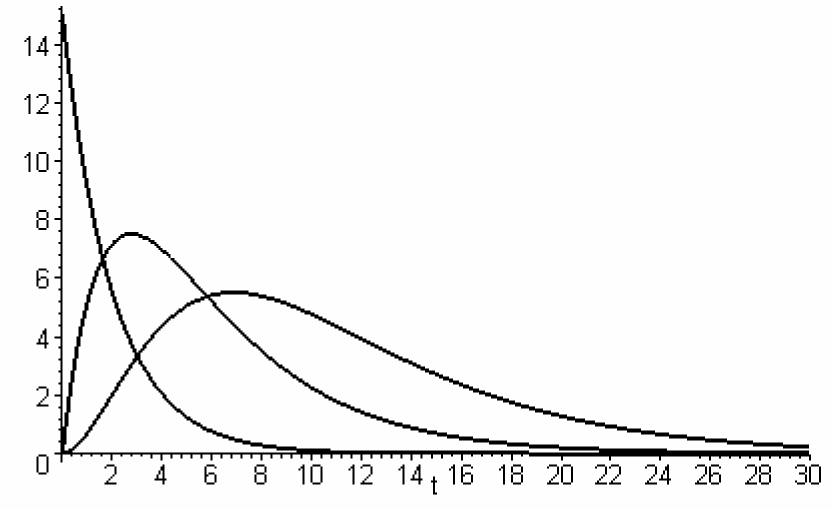
$$x2 := -30.00000000\, e^{-.5t} + 30.\, e^{-.25t}$$

$$x2 := 25.00000001\, e^{-.5t} - 150.00000000\, e^{-.25t} + 125.00000000\, e^{-.2t}$$

The command

```
plot( {x1,x2,x3}, t = 0..30 );
```

produces the figure on the next page showing the graphs of the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ giving the amounts of salt in the three tanks. We can approximate the maximum value of each $x_i(t)$ by mouse-clicking on the apex of the appropriate graph.



Using *Mathematica*

We begin by entering (as indicated previously) the coefficient matrix in (1),

```
A = {{-0.50 ,0,0}, {0.50,-0.25,0}, {0,0.25,-0.20}};
A // MatrixForm
```

```
-0.5    0       0

0.5     -0.25   0
0       0.25    -0.2
```

and the initial vector

```
x0 = {{15}, {0}, {0}};  x0 // MatrixForm
```

```
15
0
0
```

The eigenvalues and eigenvectors of **A** are calculated with the command

```
eigs = Eigensystem[A]
```

```
{{-0.5,-0.25,-0.2}, {{0.358569, -0.717137, 0.597614},
   {0., 0.196116, -0.980581}, {0., 0., 1.}}}
```

Thus the first eigenvalue λ_1 and its associated eigenvector $\mathbf{v}_1$ are given by

```
eigs[[1,1]]
```

```
-0.5
```

```
eigs[[2,1]]
```

```
{0.358569, -0.717137, 0.597614}
```

We therefore record the three eigenvalues

```
L1 = eigs[[1,1]];
L2 = eigs[[1,2]];
L3 = eigs[[1,3]];
```

and the corresponding three eigenvectors

```
v1 = eigs[[2,1]];
v2 = eigs[[2,2]];
v3 = eigs[[2,3]];
```

The matrix **V** having these three eigenvectors as its *column* vectors is then defined by

```
V = Transpose[ {v1, v2, v3} ];
V // MatrixForm
```

```
0.358569    0.          0.
-0.717137   0.196116    0.
0.597614    -0.980581   1.
```

To find the constants $c_1,\ c_2,\ c_3$ in the solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} \tag{6}$$

we need only solve the system $\mathbf{Vc} = \mathbf{x_0}$:

```
c = LinearSolve[V,x0]
```

```
{{41.833}, {152.971}, {125.}}
```

Recording the values of these three constants,

```
c1 = c[[1,1]];
c2 = c[[2,1]];
c3 = c[[3,1]];
```

we can finally calculate the solution

```
x = c1*v1*Exp[L1*t]+c2*v2*Exp[L2*t]+c3*v3*Exp[L3*t];
```

in (6) with component functions

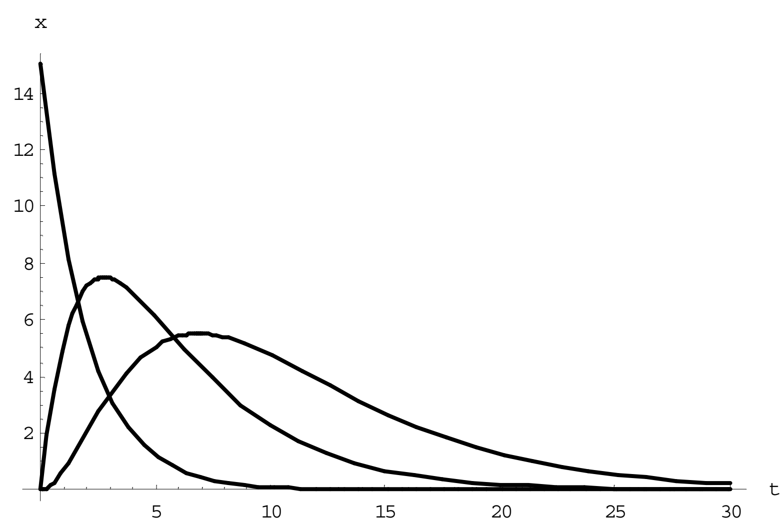
```
x1 = x[[1]]
x2 = x[[2]]
x3 = x[[3]]
```

$$\frac{0.}{e^{0.25t}}+\frac{15.}{e^{0.5t}}+\frac{0.}{e^{0.2t}}$$
$$\frac{30.}{e^{0.25t}}-\frac{30.}{e^{0.5t}}+\frac{0.}{e^{0.2t}}$$
$$-\frac{150.}{e^{0.25t}}+\frac{25.}{e^{0.5t}}+\frac{125.}{e^{0.2t}}$$

The command

```
Plot[ {x1,x2,x3}, {t,0,30} ]
```

then produces the figure below, showing the graphs of the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ that give the amounts of salt in the three tanks. We can approximate the maximum value of each $x_i(t)$ by mouse-clicking on the apex of the appropriate graph.



Using MATLAB

We begin by entering (as indicated previously) the coefficient matrix in (1),

```
A = [-0.5      0       0
      0.5   -0.25      0
       0     0.25    -0.2 ]
A =
   -0.5000         0         0
    0.5000   -0.2500         0
         0    0.2500   -0.2000
```

and the initial vector

```
x0 = [15; 0; 0];
```

The eigenvalues and eigenvectors of **A** are calculated with the command

```
[V, E] = eig(A)
V =
         0         0    0.3586
         0    0.1961   -0.7171
    1.0000   -0.9806    0.5976

E =
   -0.2000         0         0
         0   -0.2500         0
         0         0   -0.5000
```

The eigenvalues of **A** are the diagonal elements

```
L = diag(E)
L =
   -0.2000
   -0.2500
   -0.5000

L1 = L(1);     L2 = L(2);     L3 = L(3);
```

of the matrix **E**. The associated eigenvectors are the corresponding column vectors

```
v1 = V(:,1);  v2 = V(:,2);  v3 = V(:,3);
```

of the matrix **V**. To find the constants $c_1,\ c_2,\ c_3$ in the solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t} \qquad (6)$$

we need only solve the system $\mathbf{Vc} = \mathbf{x_0}$:

```
c = V\x0;    c'
ans =
  125.0000  152.9706   41.8330
```

Recording the values of these three constants,

```
c1 = c(1);    c2 = c(2);    c3 = c(3);
```

and defining an appropriate range

```
t = 0 : 0.1 : 30;
```

of values of t, we can finally calculate the solution

```
x = c1*v1*exp(L1*t)+c2*v2*exp(L2*t)+c3*v3*exp(L3*t);
```

in (6). We plot its three component functions

```
x1 = x(1,:);   x2 = x(2,:);   x3 = x(3,:);
```

using the command

```
plot( t,x1, t,x2, t,x3 )
```

The resulting figure (just like those exhibited in the preceding *Maple* and *Mathematica* discussions) shows the graphs of the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ giving the amounts of salt in the three tanks. We can approximate the maximum value of each $x_i(t)$ by mouse-clicking (after **`ginput`**) on the apex of the appropriate graph.

Complex Eigenvalues

Finally we consider the closed system of three brine tanks that is shown in Fig. 7.3.5 of the text (see Example 4 of Section 7.3). The vector $\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \ x_3(t)]^T$ of salt amounts (in the three tanks) satisfies the linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{5}$$

where $\mathbf{A}$ now is the 3×3 matrix defined by

```
A = [-0.2    0    0.2
      0.2  -0.4    0
       0    0.4  -0.2 ]
A =
   -0.2000         0    0.2000
    0.2000   -0.4000         0
         0    0.4000   -0.2000
```

If initially Tank 1 contains 10 pounds of salt and the other two tanks contain pure water, then the initial vector $\mathbf{x}(0) = \mathbf{x}_0$ is defined by

```
x0 = [10; 0; 0];
```

The eigenvalues and eigenvectors of $\mathbf{A}$ are calculated using the command

```
[V, E] = eig(A)
V =
  -0.6667      0.4976 + 0.0486i      0.4976 - 0.0486i
  -0.3333      0.0486 - 0.4976i      0.0486 + 0.4976i
  -0.6667     -0.5462 + 0.4491i     -0.5462 - 0.4491i

E =
  -0.0000      0                     0
        0     -0.4000 + 0.2000i      0
        0                 0         -0.4000 - 0.2000i
```

Now we see complex conjugate pairs of eigenvalues and eigenvectors, but let us nevertheless proceed without fear, hoping that "ordinary" real-valued solution functions will somehow result. First we pick off and record the eigenvalues that appear as the diagonal elements of the matrix **E**.

```
L = diag(E);
L'

ans =
  -0.0000     -0.4000 - 0.2000i     -0.4000 + 0.2000i

L1 = L(1);
L2 = L(2);
L3 = L(3);
```

The associated eigenvectors are the corresponding column vectors

```
v1 = V(:,1);
v2 = V(:,2);
v3 = V(:,3);
```

of the matrix **V**. To find the constants c_1, c_2, c_3 in the solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t} \tag{6}$$

we need only solve the system $\mathbf{Vc} = \mathbf{x_0}$:

```
c = V\x0;
c'
ans =
  -6.0000+0.0000i  5.7773+2.5735i  5.7773-2.5735i
```

Recording the values of these three constants,

```
c1 = c(1);
c2 = c(2);
c3 = c(3);
```

and defining an appropriate range

```
t = 0 : 0.1 : 30;
```

of values of t, we can finally calculate the solution

```
x = c1*v1*exp(L1*t)+c2*v2*exp(L2*t)+c3*v3*exp(L3*t);
```

We can plot the three component functions

```
x1 = x(1,:);
x2 = x(2,:);
x3 = x(3,:);
```

using the command

```
plot( t,x1, t,x2, t,x3 )
```

The result is the figure below. It shows the graphs of the functions $x_1(t)$, $x_2(t)$, and $x_3(t)$ that give the amounts of salt in the three tanks. Is it clear to you that the three solution curves "level off" as $t \to \infty$ in a way that exhibits a long-term uniform concentration of salt throughout the system?