Using Differential Equations to Model a Vibrating String

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Abstract

The purpose of this paper is to develop a mathematical model to describe the motion of a vibrating string. The string will be fixed rigidly at either end. We will derive the wave equation from Newton's laws and solve for the general solution using separation of variables. Then we choose initial conditions and determine the specific solutions for several initial value problems.

1. Introduction

The science of waves and wave motion is essential to a wide range of applications. In it's simplest form, a wave is a disturbance traveling through some medium. From studying the behavior of waves in water, in the air as sound, in the Earth as

Figure 1: Pythagoras circa 550 B.C.

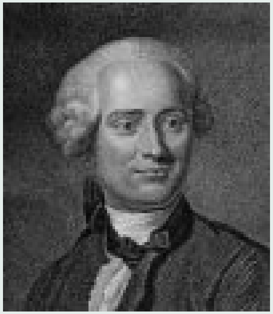

Figure 2: French mathematician and scientist Jean Le Rond d'Alembert

earthquakes, and in the aether as light, scientist and mathematicians have been able to bring important technologies to us as well as make powerful connections between disciplines. The study of acoustics was one of the first to use the idea of waves and led to much of our current understanding of wave motion. In 550 B.C. the Pythagorians observed that vibrating strings produced sound and studied the mathematical relationship between the frequency of the sound and the length of the string.

In the sevententh century, the science of wave propagation received attention from Galileo Galilei, Robert Boyle, and Isaac Newton. It was not until the Eighteenth Century that French mathematician and scientist Jean Le Rond d'Alembert derived the wave equation. This elegant description of waves has led to advancements in everything from geology to quantum physics and continues

to offer insights into the world around us. In this paper will will derive the wave equation, and use it to model a vibrating string.

2. Derivation of the Wave Equation

In order to derive the equation a few assumptions must be made. The string of length L will be of uniform mass density and fixed rigidly at both ends as in Figure 3. The vertical displacement $(u(x,t))$ will be small compared to L. We will also neglect stretching and bending of the string; meaning energy losses associated with dampening in the distortion of the string will be neglected. All other external forces such as gravity will also be ignored.

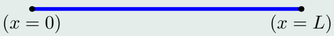

Figure 3: A string of length L fixed at the ends.

Consider a segment of string from x to $x+\Delta x$ as in Figure 4. The Tension (T) is always tangential to the string. Breaking this vector into its horizontal and vertical components, as shown in Figure 5, we have

$$T_u = -T\sin(\theta) \tag{1}$$

$$T_x = -T\cos(\theta)$$

where θ is the angular displacement from the horizontal. The horizontal component (T_x) can be neglected since every point on the string has another tension vector in the opposite direction. The slope of the string at any point x can be

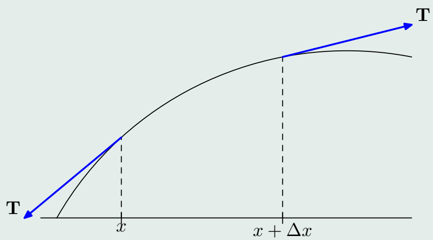

Figure 4: Section of string from x to Δx

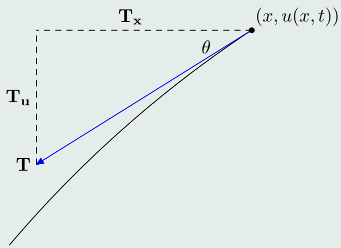

Figure 5: Vector representation of tension at x

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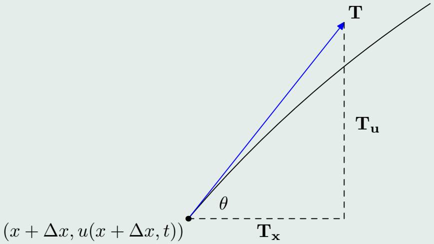

Figure 6: Vector representation of tension at $x + \Delta x$

described as $\partial u/\partial x = \tan(\theta)$ (Figure 5). Since $u(x,t)$ is small compared to L, $\cos(\theta) \approx 1$ so $\tan(\theta) \approx \sin(\theta)$. Using these facts with (1) gives us

$$T_u = -T\frac{\partial u}{\partial x}(x,t).$$

From Figure 6, the vertical component of the tension at $x+\Delta x$ is approximately

$$T\frac{\partial u}{\partial x}(x+\Delta x,t).$$

This means that the total force in the vertical direction is

$$F \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right).$$

We can define the vertical acceleration as $\partial^2 u/\partial t^2$ and the mass per unit length as ρ. Putting it together using $f = ma$ we have

$$T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right) = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

Now we can divide both sides by Δx and take the limit as Δx approaches zero.

$$\rho \frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right) \tag{2}$$

From the definition of a derivative

$$\lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t) \right)}{\Delta x} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

Combining (2) and (3) while letting $c^2 = T/\rho$ gives us the wave equation as defined by d'Alembert.

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2} \tag{4}$$

3. The General Solution

Now that we have the wave equation a general solution for $u(x, t)$ can be obtained easily by separation of variables. First we will define $u(x, t)$ as a product of two independent functions.

$$u(x, t) = X(x)T(t) \tag{5}$$

Substituting this into (4) gives the following.

$$[X(x)T(t)]_{xx} = \frac{1}{c^2}[X(x)T(t)]_{tt} \tag{6}$$

Taking the second derivative and grouping variables gives

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)}$$

Since both sides of this equation are independent and equal we will choose a constant, $-\lambda$ and equate both sides of the equation to this constant.

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\frac{T''(t)}{c^2T(t)} = -\lambda$$

This gives us a system of two second order ordinary differential equations.

$$X''(x) + \lambda X(x) = 0 \tag{7}$$

$$T''(t) + \lambda c^2 T(t) = 0 \tag{8}$$

3.1. The General Solution for $X(x)$

We will start with the first equation and solve for $X(x)$. There are three cases depending on the value of λ. These cases will be $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. By using our boundary conditions we will identify the appropriate case for the specific solutions.

3.1.1. Case 1: $\lambda < 0$

Let $\lambda = -\omega^2, \omega > 0$. Substituting into (7) we have

$$X'' - \omega^2 X = 0$$

Now we choose an integrating factor $X(x) = e^{rx}$ giving us

$$r^2 e^{rx} - \omega^2 e^{rx} = 0$$

Diving both sides by e^{rx} and solving for r gives us

$$r = \pm\omega.$$

Now that we have two independent solutions they can be written as a linear combination.

$$X(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} \tag{9}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_1 = -C_2$$

Substituting into (8) and using our second boundary condition $X(L) = 0$ gives us

$$e^{2\omega L} = 1.$$

This is a false statement since neither L nor ω can be zero. This means that λ is not less than zero.

3.1.2. Case 2: $\lambda = 0$

When $\lambda = 0$, $X'' = 0$ and

$$X(x) = C_1 x + C_2 \tag{10}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_2 = 0$$

Substituting into (10) and using our second boundary condition $X(L) = 0$ gives us

$$C_1 = 0$$

If both C_1 and C_2 equal zero then our string doesn't ever move. This is trivial and therefore λ cannot be equal to zero.

3.1.3. Case 3: $\lambda > 0$

Let $\lambda = \omega^2, \omega > 0$. Substituting into (7) we have

$$X'' + \omega^2 X = 0$$

Now we choose an integrating factor $X(x) = e^{rx}$ giving us

$$r^2 e^{rx} + \omega^2 e^{rx} = 0$$

Diving both sides by e^{rx} and solving for r gives us $r^2 = -\omega^2$ or

$$r = \pm \omega i.$$

This means that $X(x) = e^{\pm\omega i}$. Using Euler's identity

$$e^{\omega x i} = [\cos\omega x + i\sin\omega x]$$

The general solution is a linear combination of the $\mathrm{Re}X(x)$ and $\mathrm{Im}X(x)$.

$$X(x) = C_1\cos\omega x + C_2\sin\omega x \tag{11}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_1 = 0$$

Substituting into (11) and using our second boundary condition $X(L) = 0$ gives us

$$0 = C_2\sin\omega L$$

which is only satisfied when

$$\omega = \frac{n\pi}{L}, \qquad \text{for } n = 1, 2, 3, \ldots$$

Therefore our general solution for $X(x)$ is

$$X(x) = \sin\frac{n\pi}{L}x \tag{12}$$

3.2. The General Solution for $T(t)$

We take (8) and let $\omega^2 = \lambda c^2$.

$$T'' + \omega^2 T = 0$$

This equation is the same as case 3 replacing $X(x)$ with $T(t)$ and ω^2 replacing λ. Now we can see that the general solution of $T(t)$ will have the same form as the solution for (11) except now

$$\omega = \frac{n\pi c}{L}$$

Now we can see that our fundamental set of solutions is defined as

$$T_n(t) = \sin \frac{n\pi c}{L} t + \cos \frac{n\pi ct}{L} \qquad \text{for } n = 1,\ 2,\ 3,\ \ldots$$

3.3. Putting $T(t)$ and $X(x)$ Together

Now we can express $u_n(x,t)$ as the product of $X_n(x)$ and $T_n(t)$ (5)

$$u_n(x,t) = \sin \frac{n\pi x}{L} \left(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \right)$$

Combining these in a linear combination we have the final solution for $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t \right] \tag{13}$$

4. Specific Solutions with Initial Conditions

We will define the initial shape of the string as $f(x)$ and the initial vertical velocity of the string as $g(x)$. We will choose the length of the string to be

$L = \pi$. To reduce the complexity of our solution we will only choose initial conditions where $g(x) = 0$ so $b_n = 0$.

To satisfy our initial condition for $f(x)$

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin nx$$

To find the fourier coefficients for a_n to satisfy the initial condition we define

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx \qquad (14)$$

4.1. A Standing Wave

Choose the initial condition $u(x,0) = f(x) = \sin 2x$, for $0 < x < \pi$ as in Figure 7.

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{L} \sin 2x \sin nx dx \\ &= \frac{2 \sin n\pi}{n^2 - 4} \end{aligned}$$

From this we can see that $a_n = 0$ for all values of n except $n = 2$ where the denominator is zero. This case must be done separately.

$$a_2 = \frac{2}{\pi} \int_0^{\pi} \sin 2x \sin 2x dx = 1$$

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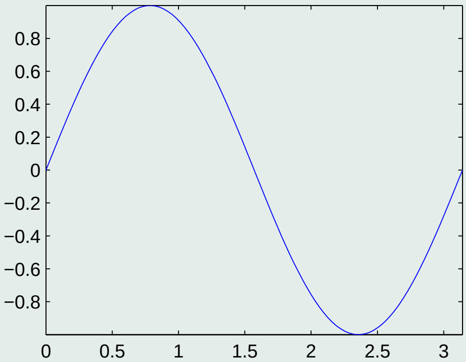

Figure 7: The initial condition $u(x, 0) = f(x) = \sin 2x$

Thus,

$$u(x, t) = \sin(nx)\cos(nct).$$

See Figure 8 for the graph of this solution.

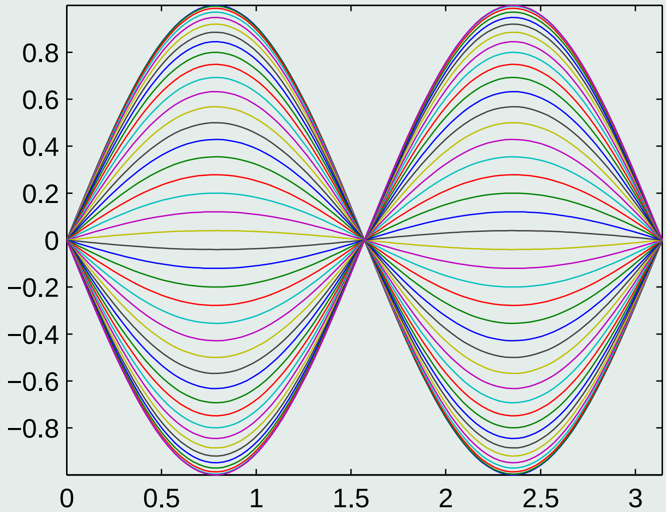

Figure 8: The solution at various times for one period.

4.2. Displacement of the string at L/2

The initial condition,

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \pi/2 \\ \pi - x, & \text{if } \pi/2 \leq x \leq \pi \end{cases}$$

defines a string displaced from the center of the string as in Figure 9.

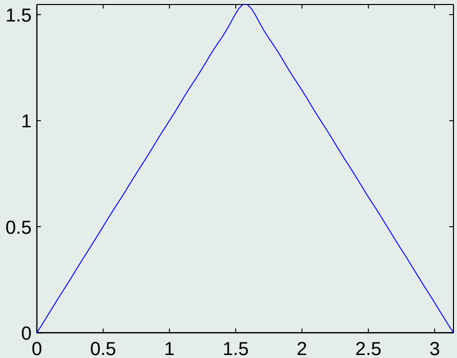

Figure 9: Displacing the string at $L/2$

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From (14) we can see that from $0 \leq x \leq \pi/2$,

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx dx$$
$$= \frac{2}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2}$$

Similarly, from $\pi/2 \leq x \leq \pi$

$$a_n = \frac{2}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2}$$

Combining these two terms gives us

$$a_n = \frac{4}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2}$$

Giving us the following solution which is graphed in Figure 10.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin(nx) \sin\left(\frac{n\pi}{2}\right) \cos(nct)$$

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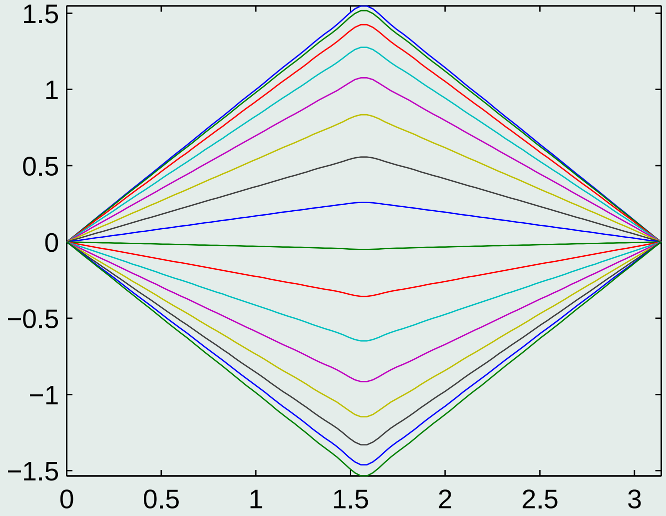

Figure 10: The solution at various times for one period.

4.3. A More Interesting Example

We will choose our initial conditions to produce the displacement shown in Figure 11.

$$f(x) = \begin{cases} x - 3/8, & \text{if } 3/8 \leq x \leq 1/2 \\ 5/8 - x, & \text{if } 1/2 \leq x \leq 5/8 \\ 0, & \text{otherwise} \end{cases}$$

provide a more interesting case because the result is two waves traveling in

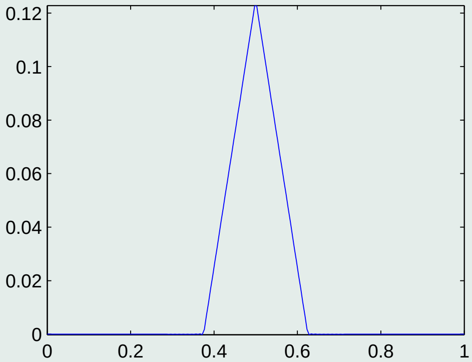

Figure 11: A more complicated displacement

opposite directions along the string. The length of the string will be changed

to $L = 1$ and

$$a_n = \frac{2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{8} - \sin\frac{5n\pi}{8}\right).$$

So the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\sin\left(n\pi x\right)\cos\left(n\pi ct\right)\left(2\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{8} - \sin\frac{5n\pi}{8}\right)$$

The graph of which can be found in Figure 12.

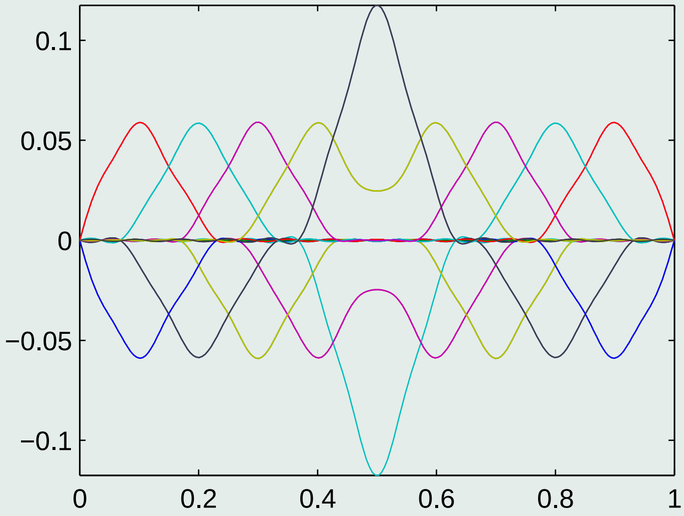

Figure 12: The solution at various times for one period.

5. Matlab M-files

The following matlab code will produce a live animation of the string for each initial condition previously described. When the code is executed in matlab a static figure of the initial displacement of the string will be produced. Just press the space bar to begin the animation.

In the first example, the animation will exhibit the solution for the initial condition presented in subsection 4.1, *A Standing Wave*

```
close all
figure x=linspace(0,pi); t=0;
 s=cos(2*.002.*t).*sin(2*x);

 hndl=line(x,s);
set(hndl,'EraseMode','xor') pause

for t=linspace(0,pi/0.002,1000);
    s=cos(2*.002.*t).*sin(2*x);
    set(hndl,'YData',s)
    drawnow
end;
```

In this example, the animation will exhibit the solution for the initial condition presented in subsection 4.2, *Displacement of the string at* $L/2$.

```
close all
```

```
figure
axis([0,pi,-1.6,1.6])
x=linspace(0,pi);
N=15;
t=0;
s=zeros(size(x));
for k=0:N;
    s=s+((4/pi).*(((-1).^k)./(2*k+1).^2).*...
        sin((2*k+1).*x).*cos((.002*2*k+1).*t));
end;
hndl=line(x,s);
set(hndl,'EraseMode','xor');
pause;

for t=linspace(0,2*pi,700);
    s=zeros(size(x));
    for k=0:N;
        s=s+((4/pi).*(((-1).^k)./(2*k+1).^2).*...
            sin((2*k+1).*x).*cos((.002*2*k+1).*t));
    end;
    set(hndl,'YData',s);
    drawnow
end;
```

In our final example, the animation will exhibit the solution for the initial condition presented in subsection 4.3, *A more Interesting Example.*

```
close all
figure
axis([0,1,-.12,.12])
x=linspace(0,1,200);
N=25;

t=0;
s=zeros(size(x));
for k=1:N;
    s=s+(2/(k^2*pi^2)*(2*sin(k*pi/2)-sin(3*k*pi/8)...
        -sin(5*k*pi/8))*sin(pi*k*x).*cos(pi*k*t));
end;
hndl=line(x,s);
set(hndl,'EraseMode','xor')
pause
for t=linspace(0,2,1000)
    s=zeros(size(x));
    for k=1:N;
        s=s+(2/(k^2*pi^2)*(2*sin(k*pi/2)-sin(3*k*pi/8)...
            -sin(5*k*pi/8))*sin(pi*k*x).*cos(pi*k*t));
    end;
    set(hndl,'YData',s)
    drawnow
end
```

References

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[3] Slater, Frank *1947 Mechanics*

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