The Analysis of Communications Channels

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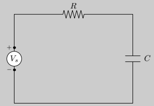
Abstract

In today's technological society communication is vital, and we would like to be able to send large amounts of data as fast as possible. However, there is a limit to how fast a channel will allow us to send data with out any loss. The purpose of this paper is to determine the maximum rate data can be transmitted through a communications channel and still be recognizable when it is received.

1. Modeling a Communications Channel

We would like to model the circuit shown in Figure 1 with a differential equation, to do this we will need to expand our knowledge of essential circuit theory laws.



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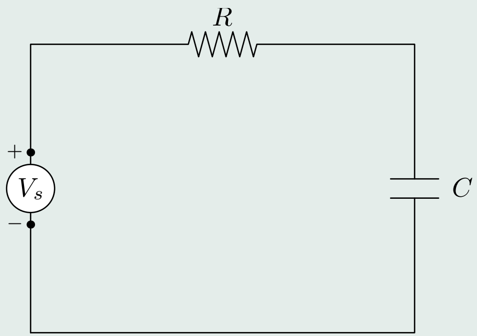


Figure 1: A simple RC circuit

1. **Kirchhoff's Voltage Law** Which states the sum of the voltage drops around a closed circuit is equal to zero, which is an extension of a conservation of energy law.

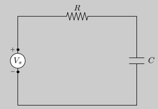
$$V_S - V_R - V_C = 0 \tag{1}$$

2. **Ohm's Law** States that the voltage drops between the end points of a resistor is proportional to the current flowing through the resistor (measured in Ohm's Ω)

$$V_R = IR \tag{2}$$

3. **Coulomb's Law** States that the voltage drop across a capacitor is proportional to the charge on the capacitor (measured in farads **F**)

$$V_C = \frac{q}{C} \tag{3}$$



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Now, if there is a initial charge on the capacitor, then

$$V_C = \frac{1}{C}\left(q_0 + \int_{t_0}^{t_f} I(s)ds\right), \tag{4}$$

where **C** represents the capacitance of the capacitor.

So, in our simple circuit,

- the **resistor** will simulate the effect of noise on the channel,
- the **capacitor** corresponds to the data interface, and
- the **driving force**, $\mathbf{V_S}$, symbolizes the original signal being sent through our circuit.

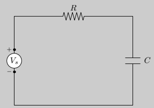
Keep in mind the objective in our analysis was to determine the maximum rate of data flow (known as the data rate) that can be sent through a communications channel and still be recognizable on the receiving end.

2. Derivation of the First Order Differential Equation:

$$V_C' + \frac{1}{RC}V_C = V_0 \sin 2\pi f t \tag{5}$$

$$V_C' + \frac{1}{RC}V_C = V_0\,\mathrm{sqw}(t, 1/f,\ \text{duty cycle}) \tag{6}$$

At this point we would like to make the connection between a sinusoidal function and a square wave function. Using techniques of the Fourier series one can represent a square wave as a infinite series of sine and cosine waves. It turns out that when we send either of the driving forces through our circuit the effect on the amplitude of these driving forces are the same. To keep our calculations manageable we will focus on sinusoidal input signals, which are easily solved using first semester Differential Equations techniques. We



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will, however, include pictorial models of the effect of our circuit on a square wave (easily obtained using a computer solver such as Matlab 5.3).

We begin our derivation of the first order differential equation.

$$V_S - V_R - V_C = 0 \tag{7}$$

$$V_S - RI - V_C = 0 \tag{8}$$

$$V_S - RCV_C' - V_C = 0 \tag{9}$$

$$V_C' + \frac{V_C}{RC} = \frac{1}{RC} V_0 \sin(2\pi f t) \tag{10}$$

Voila! A first order differential equation to model the behavior of the signal across the capacitor.

Remember that a solution to a driven differential equation is the sum of the homogeneous solution and the particular solution, $V_C = V_h + V_p$. At this time we will solve for the particular solution using the method of undermined coefficients, also known as the method of the "Lucky Guess:"

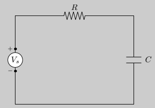
$$V_p = \frac{1}{\sqrt{1/RC + i2\pi f}} \frac{1}{RC} V_S e^{i2\pi f t} \tag{11}$$

With a little bit of trigonometry we can rearrange the particular solution to become:

$$V_p = \frac{1}{\sqrt{1 + (2\pi f RC)^2}} V_0 \sin(2\pi f t - \phi) \tag{12}$$

Therefore the general solution to our differential equation becomes:

$$V_C = C_1 e^{-t/RC} + \frac{1}{\sqrt{1 + (2\pi f RC)^2}} V_0 \sin(2\pi f t - \phi) \tag{13}$$



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3. Analyzing the Solution to the Forced Differential Equation

The contribution of the transient solution to the general solution is negligible. The astute reader will have noticed a phase shift of ϕ, this phase shift, however, does not effect the signal's amplitude which is the focus of this paper. We are more concerned with the gain, which represents the percent of the original data received.

To examine the effect of our simple circuit on the source signal we will graph the gain as a function of frequency, shown in Figure 2.

$$G(f) = \frac{1}{\sqrt{1+(2\pi f RC)^2}} \tag{14}$$

The reader may realize that if the frequency is very small, the value of the gain is nearly 1 corresponding to almost 100% data received. Conversely, as the frequency becomes larger, the value of the gain function will decrease resulting in great signal degradation. This is known as a low pass filter.

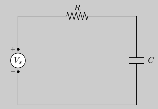
Using simple algebraic techniques we can solve the gain function for frequency as a function of percent data received, **P**:

$$f(\mathbf{P}) = \frac{1}{2RC\pi}\frac{\sqrt{1-\mathbf{P}^2}}{\mathbf{P}} \tag{15}$$

Once the acceptable percentage data received has been determine, this equation may be used to find the maximum data rate (the frequency) at which to transmit the information signal.

4. Driving it Home

If it has been determined that an acceptable data loss is 15% (or 85% data received) a frequency of 1000 Hz should be transmitted given a RC circuit using a $1\,K\Omega$ Resistor and



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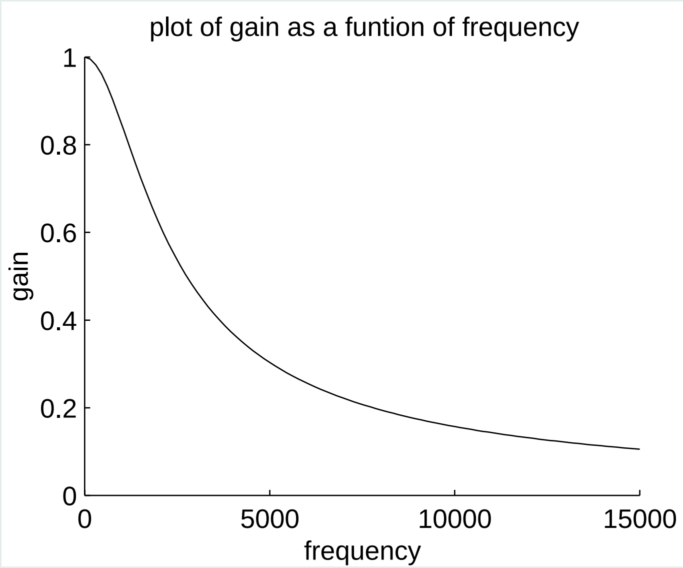
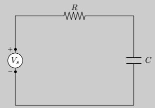


Figure 2: Gain vs. frequency



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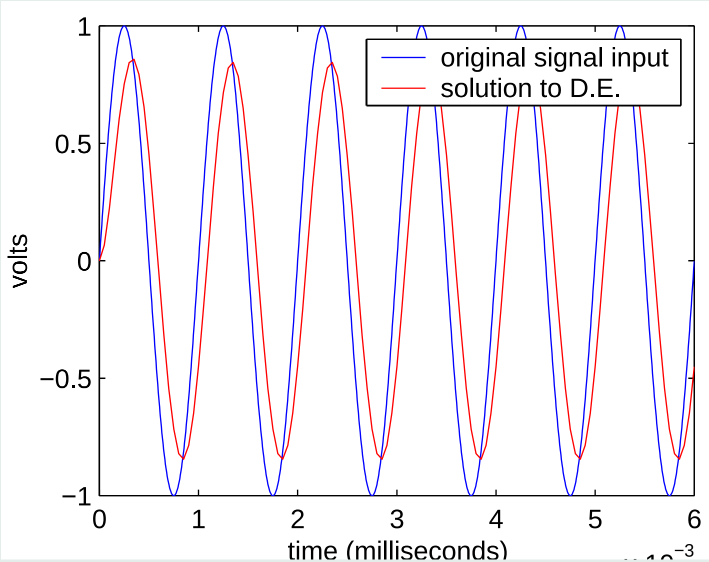


Figure 3: For a sinusiodal input signal

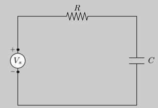
a $0.1\,\mu\mathrm{F}$ capacitor. Using Matlab 5.3 we have graphed the solution to the Initial Value Problem $V_0 = V_0' = 0$. In Figure 3, the driving force is sinusoidal; that is,

$$V_S = V_0 \sin 2\pi f t.$$

On the other hand, in Figure 4, the driving force is a square wave; that is,

$$V_S = V_0 \mathrm{sqw}(t, 1/f, 50),$$

a square that is "on" for the first 50% of its period.



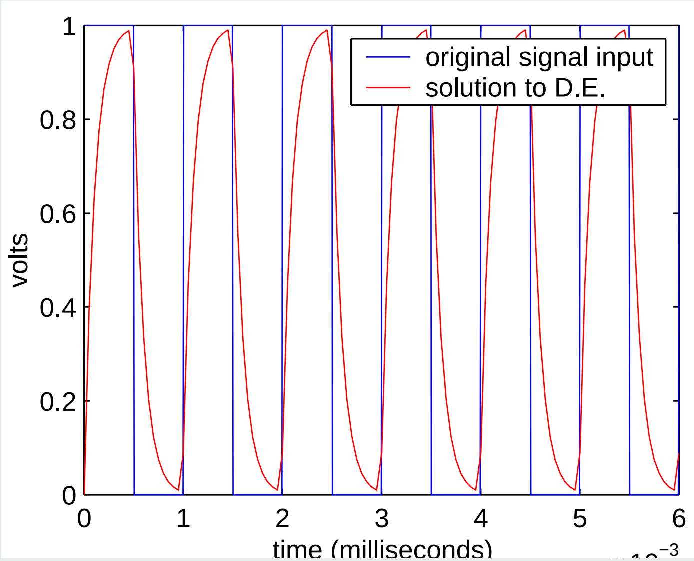
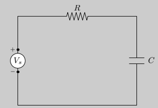


Figure 4: For a square wave input signal



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