A Low Pass Filter

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Introduction

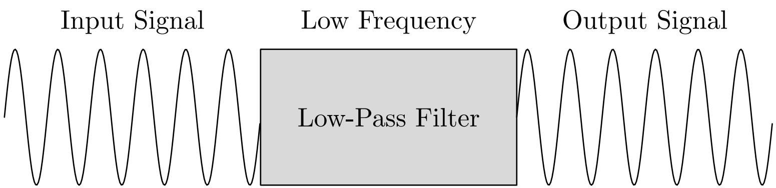
- Filters separate one unwanted quantity from another.
- Some circuit require that an unwanted signal be eliminated.
- Electrical filters eliminate one or more unwanted quantities while preserving the integrity of a desired signal.
- An ideal filter will separate and pass sinusoidal input signals based upon their frequency.

Low Pass Filters

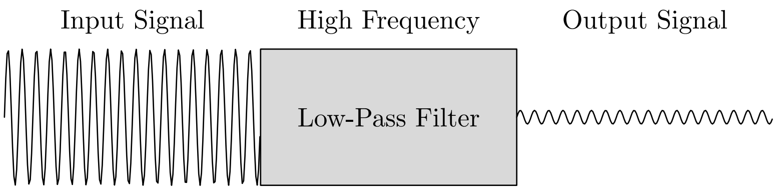
Gain is a dimensionless quantity that represents the ratio of the amplitude of output signal to that of the input signal:

$$Gain = \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|$$

A low pass filter will effectively pass signals of low frequency,



while signals of high frequency are severely attenuated (eliminated).

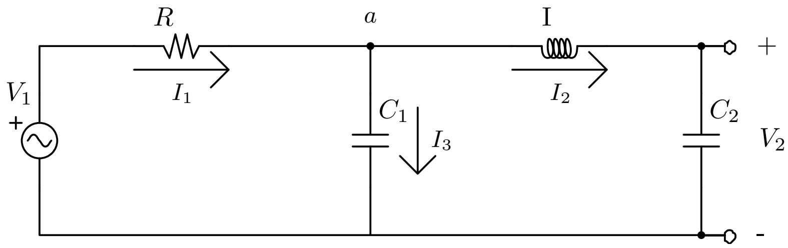


Periodic Steady State Theorem

Theorem 1 Periodic Steady State: *Suppose all eigenvalues of the constant matrix A have negative real parts and that $\mathbf{F}(t)$ is periodic with period T. Then the system $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$ has a unique steady-state, which is periodic of period T.*

Our Example Filter

Use Kirchoff's Laws on the circuit



to find:

$$V_2' = \frac{1}{C_2} I_2,$$

$$I_2' = \frac{V_1}{L} - \frac{RI_1}{L} - \frac{V_2}{L},$$

$$I_1' = \frac{V_1'}{R} - \frac{I_1}{RC_1} + \frac{I_1}{RC_2},$$

Putting Together a Linear System

- Linear system of first order differential equations,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t).$$

- The answer vector $\mathbf{x}$, the Matrix A and the driving force $\mathbf{F}(\mathbf{t})$ are as follows:

$$\mathbf{x} = \begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \frac{1}{C_2} \\ 0 & -\frac{1}{RC_1} & \frac{1}{RC_1} \\ -\frac{1}{L} & -\frac{R}{L} & 0 \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} 0 \\ \frac{V_1{}'}{R} \\ \frac{V_1}{L} \end{bmatrix}$$

Applying the Periodic Steady State Theorem

- Let us look to see if the eigenvalues of our system will have negative, real parts.
- Find the characteristic polynomial, $p(\lambda) = \mid A - \lambda I \mid$, where I represents the identity matrix and λ represents the eigenvalues of matrix A.
- Which becomes

$$p(\lambda) = -\left[\lambda^3 + \frac{1}{RC_1}\lambda^2 + \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\lambda + \frac{1}{LRC_1C_2}\right].$$

The Routh-Hurwitz Table and Stability Criterion

Let

$$p(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

be a polynomial such that all the coefficients are real and a_0 is non-zero.

- To create a Routh-Hurwitz Table, put all the coefficients with even subscripts into the first row and all the coefficients with odd subscripts into the second row.

$$\begin{array}{c|cccc} x^n & a_0 & a_2 & a_4 & \ldots \\ x^{n-1} & a_1 & a_3 & a_5 & \ldots \end{array}$$

More On Routh Tables

- Find the remaining rows with the following procedure:

$$
\begin{array}{c|cccc}
x^n & a_0 & a_2 & a_4 & \dots \\
x^{n-1} & a_1 & a_3 & a_5 & \dots \\
x^{n-2} & b_1 & b_2 & b_3 & \dots \\
x^{n-2} & c_1 & c_2 & c_3 & \dots
\end{array}
$$

- where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \dots,$$

This can also be written as

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \dots,$$

Example 1

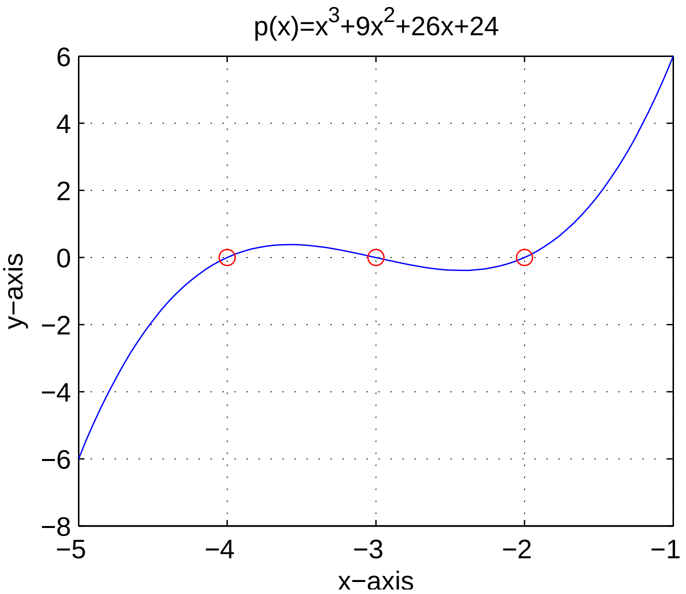
- Create a Routh Table for the polynomial $p(x) = x^3 + 9x^2 + 26x + 24$. Starting as before,

$$\begin{array}{c|ccc} x^3 & 1 & 26 & 0 \\ x^2 & 9 & 24 & 0 \\ x^1 & \frac{70}{3} & 0 & 0 \\ x^0 & 24 & 0 & 0 \end{array}$$

- Note that all the coefficients in the first column all have the same sign. Therefore we know that all the roots of $p(x)$ have negative real parts.

Graph of Example 1

- When we factor $p(x)$, it becomes $p(x) = (x+2)(x+3)(x+4)$.
- It can then easily be seen that the roots are $x = -2, -3$, and -4, all of which have negative real parts. This is illustrated below.



Our Circuit

Now let us analyze our characteristic cubic polynomial $p(\lambda) = \mid A - \lambda I \mid$.

$$
\begin{array}{c|ccc}
\lambda^3 & -1 & -\dfrac{1}{L}\left(\dfrac{1}{C_1}+\dfrac{1}{C_2}\right) & 0 \\
\lambda^2 & -\dfrac{1}{RC_1} & -\dfrac{1}{LRC_1C_2} & 0 \\
\lambda^1 & -\dfrac{1}{LC_1} & 0 & 0 \\
\lambda^0 & -\dfrac{1}{LRC_1C_2} & 0 & 0
\end{array}
$$

- Looking at the Routh Table above, all of the pivot coefficients are negative.

Analyzing Steady-State Output

- Therefore by the Routh-Hurwitz test, there are no unstable roots and all the eigenvalues have real, negative parts.
- We may then conclude by that there is a unique steady-state solution for our specific circuit.
- Lets examine the output in greater detail. We are most interested in the gain of the circuit.
- We begin by choosing an arbitrary sinusoidal input function of the form $V_1 = a_0 e^{\imath\omega t}$, then

$$\mathbf{F} = \begin{bmatrix} 0 \\ V_1'/R \\ V_1/L \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \imath\omega/R \\ a_0/L \end{bmatrix} e^{\imath\omega t} = \alpha e^{\imath\omega t}, \quad \text{where } \alpha = \begin{bmatrix} 0 \\ a_0\imath\omega/R \\ a_0/L \end{bmatrix}.$$

Finding the Steady-State Output

- We know that there will be a unique steady-state output, whose period, and therefore frequency, is identical to that of the input.
- We may assume a steady-state output of the form $x^s = \beta e^{\imath\omega t}$.
- Which we substitute into $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$, to produce

$$\imath\omega e^{\imath\omega t}\beta = e^{\imath\omega t}A\beta + e^{\imath\omega t}\alpha.$$

- Now we solve for β,

$$\beta = [\imath\omega I - A]^{-1}\alpha.$$

Continuing Our Work On β

- We use the fact that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

  to find β.

- We are only interested in V_2, the output of the circuit.

$$\mathbf{x}^{\mathrm{s}} = \begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix},$$

- We only need to expand the adjoint across the first column, for use in our calculation of $x^s = \beta e^{\imath\omega t}$.

β Found

Now,

$$\beta = \frac{1}{p(\imath\omega)} \begin{bmatrix} \begin{vmatrix} (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} & * & * \end{bmatrix}^T \begin{bmatrix} 0 \\ a_0\imath\omega/R \\ a_0/L \end{bmatrix}.$$

Now Look For V_2

Next,

$$\mathbf{x}^{\mathrm{s}} = \beta e^{\imath\omega t}$$

$$\begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{p(\imath\omega)} \begin{bmatrix} \begin{vmatrix} (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} & * & * \end{bmatrix}^T \begin{bmatrix} 0 \\ a_0\imath\omega/R \\ a_0/L \end{bmatrix} e^{\imath\omega t}$$

V_2 Found

Because we are only interested in the output voltage V_2,

$$V_2 = \frac{1}{p(\imath\omega)} \left\{ 0 - \frac{a_o \imath\omega}{R} \begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} + \frac{a_o}{L} \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} \right\} e^{\imath\omega t}.$$

After calculating the determinants and reducing

$$\frac{1}{p(\imath\omega)} \left\{ -\frac{a_o \imath\omega}{LC_2} + \frac{a_o \imath\omega}{LC_2} + \frac{a_o}{LRC_1C_2} \right\} e^{\imath\omega t}.$$

Finally, when the first two terms are cancelled, we get V_2, the output voltage,

$$V_2(t) = \frac{a_o}{p(\imath\omega) LRC_1C_2} e^{\imath\omega t}.$$

Gain In Relation to Frequency

Now that we have $V_2(t)$ as a function of ω, L, R, C_1 and C_2, we wish to find the gain of the circuit. We must recall that we originally chose $V_1 = a_0 e^{\imath\omega t}$. So

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{\frac{a_o e^{\imath\omega t}}{p(\imath\omega)LRC_1C_2}}{a_0 e^{\imath\omega t}}\right|,$$

and with some simplification

$$\left|\frac{V_2}{V_1}\right| = \frac{1}{|p(\imath\omega)LRC_1C_2|}.$$

Gain

Now, to examine the gain's response to frequency in more depth, let us substitute the characteristic polynomial, $p(\imath\omega)$ into the above equation and reduce,

$$\text{Gain} = \left|1 - LC_2\omega^2 + \imath R\omega(C_1 + C_2 - \omega^2 LC_1C_2)\right|^{-1}$$

Because we are looking at the magnitude of a vector in the imaginary plane, we can rewrite the above equation as

$$Gain = \frac{1}{\sqrt{(1 - LC_2\omega^2)^2 + R^2\omega^2\left(C_1 + C_2 - \omega^2 LC_1C_2\right)^2}}$$

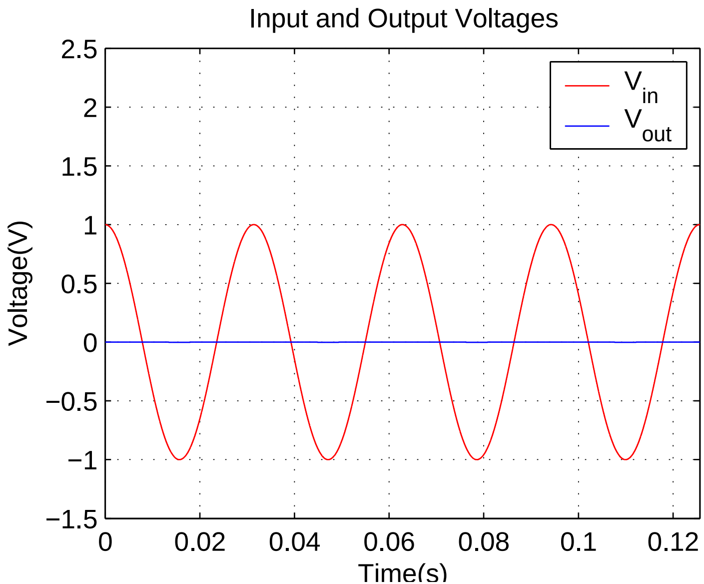
Effect of ω on Gain

- We can now see that if the denominator is big, the value of the gain will be small.
- If the denominator is small then the value of the gain will be large.
- Looking closely at what changes the value of the gain, we can see that if ω is large then the denominator will be large and therefore the gain will be small and if ω is small, then the denominator will be small and the gain will be large.
- This is why we call this circuit a Low-Pass filter. The low frequencies are passed relatively unchanged while the high frequencies are almost completely attenuated.
- Since our analysis will take place with component values declared constant, our gain is merely a function of ω, the angular frequency of the input voltage.

Analysis of Specific Component Sets

- We may now analyze the output of our circuit in response to a sinusoidal input voltage of varying frequency.
- For example, when we choose component values of $R = 20$, $L = 1$, $C_1 = 0.002$, $C_2 = 0.004$, and an input source with $\omega = 200$, (let us assume from this point forward that we will use input signals of amplitude one) we are able to produce a plot of V_1 and V_2 versus time. The result is shown below.

Graph of gain with a frequency of 200



It is obvious that the circuit is totally attenuating the input source of $\omega = 200$.

Analyzing the Gain in Response to ω

- We can also observe how this same circuit behaves with different choices of ω. When we plot gain versus ω over the interval $0 \leq \omega \leq 150$ we arrive at the next figure.
- The result is of a form that intuitively makes sense from what we understand of low pass filters. The gain is ≈ 1 when the frequency is extremely low, however, as the frequency increases, the gain begins to drop, with the rate of change becoming extremely large when $\omega \approx 20$.

Graph of gain versus ω