Differential Equations
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A Differential Look at the Watt's Governor

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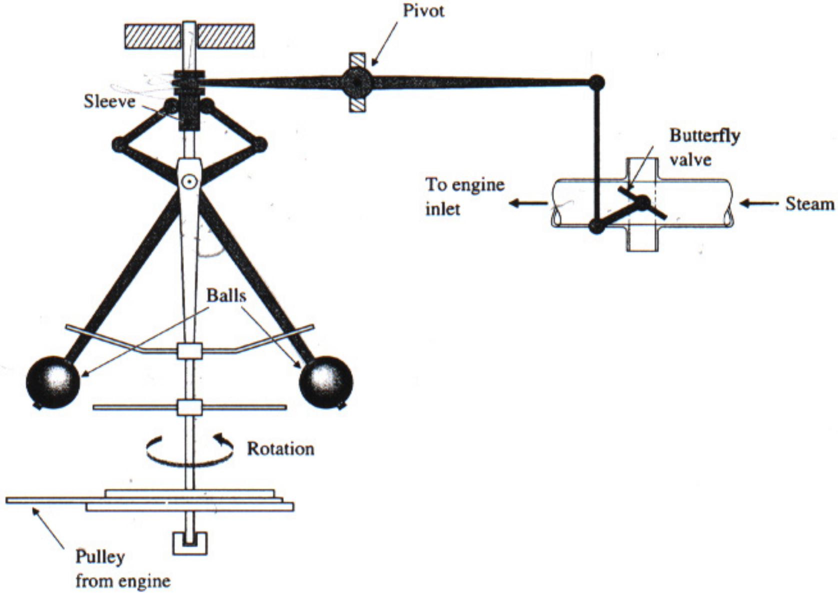
Introduction

Invented by James Watt in the late 1700s, a governor is an automated speed control that ushered in the industrial revolution.

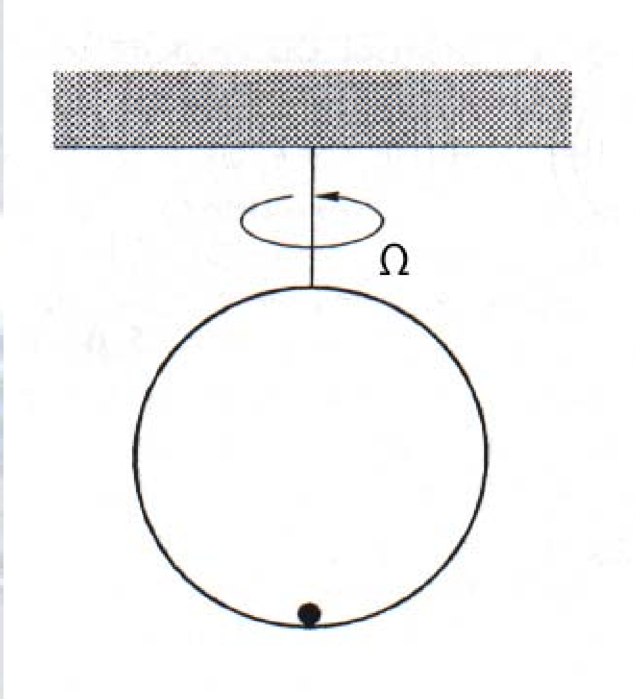
- Mathematical model.
- Bifurcation.
- Damping.

Watt's Governor

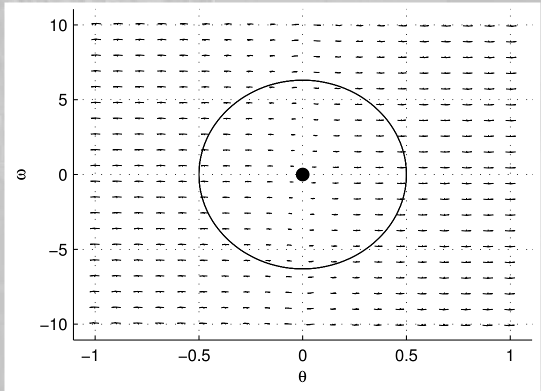


The Watt's governor controlling a steam engine.

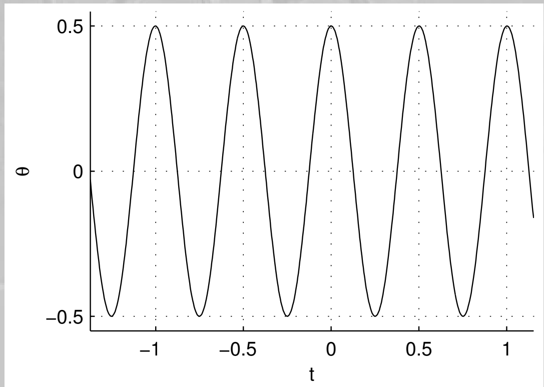
A Simplified Version



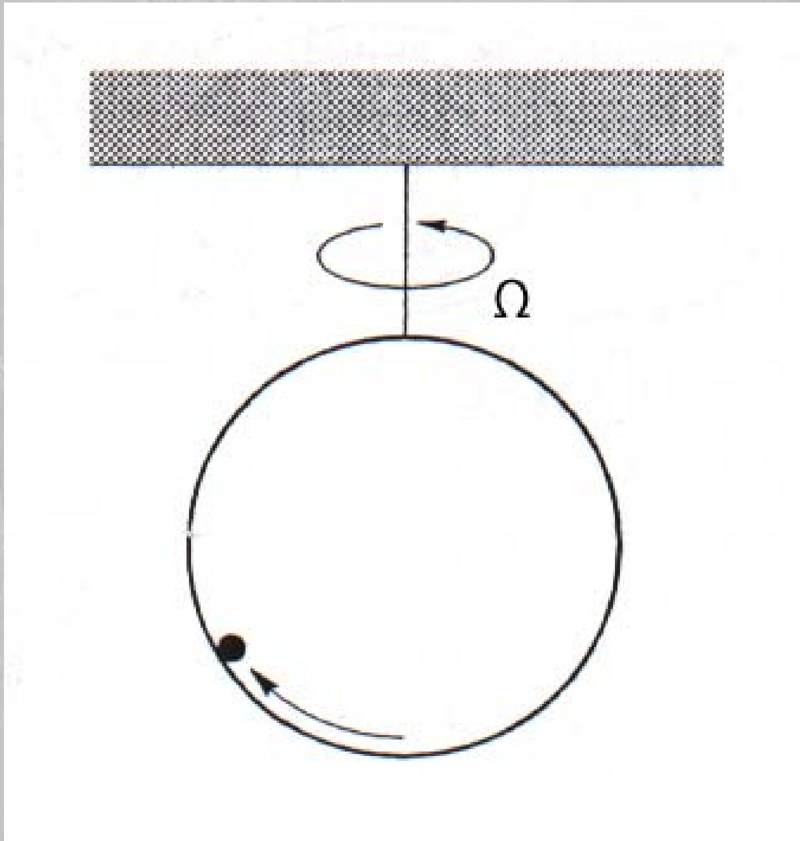
Ball-bearing in a rotating hoop.



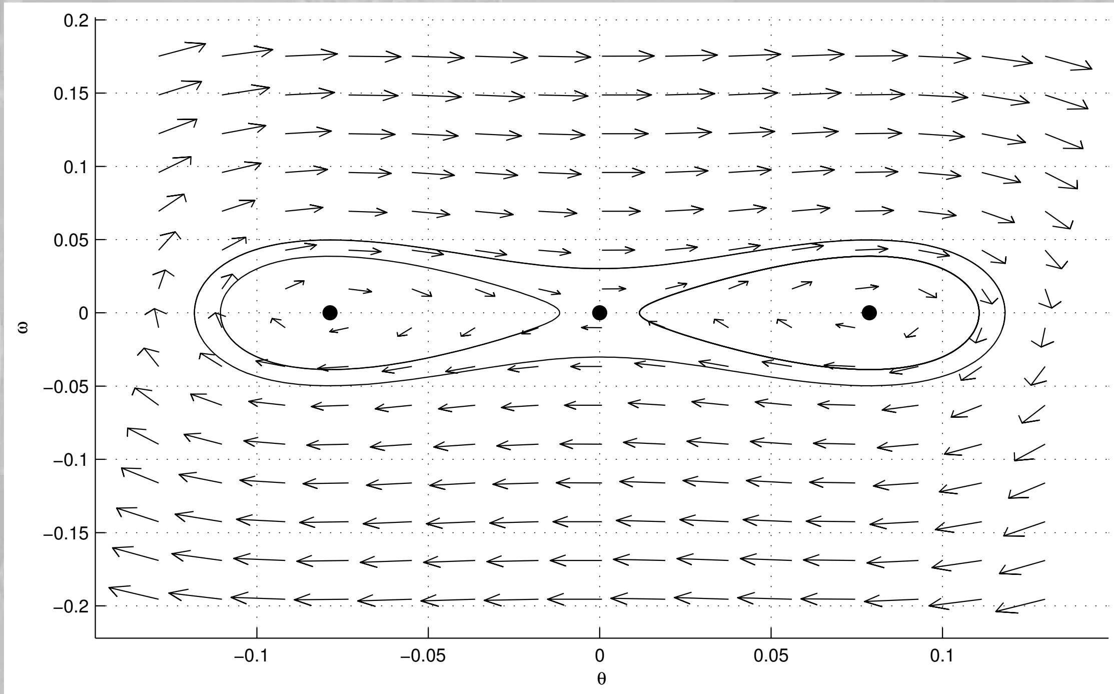
Phase plane for $\Omega = 1$ rad/sec.



θ vs. t for $\Omega = 1$ rad/sec.



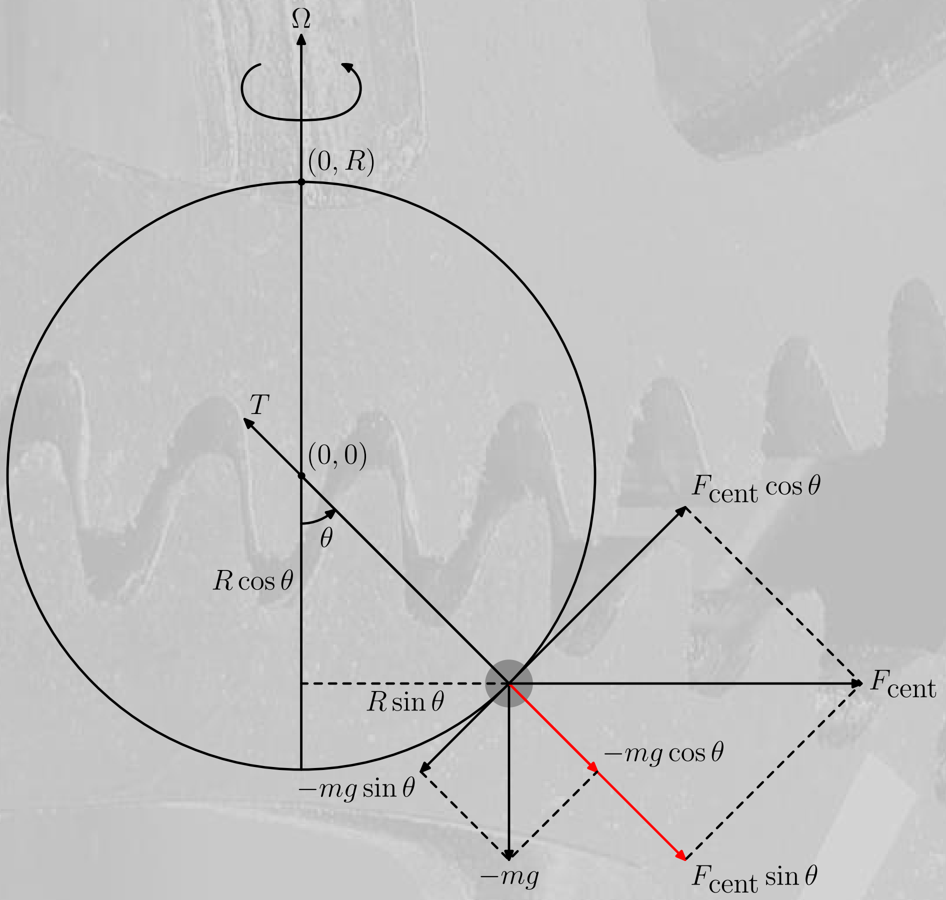
For $\Omega > 12$ rad/sec the ball moves towards a new equilibrium point.



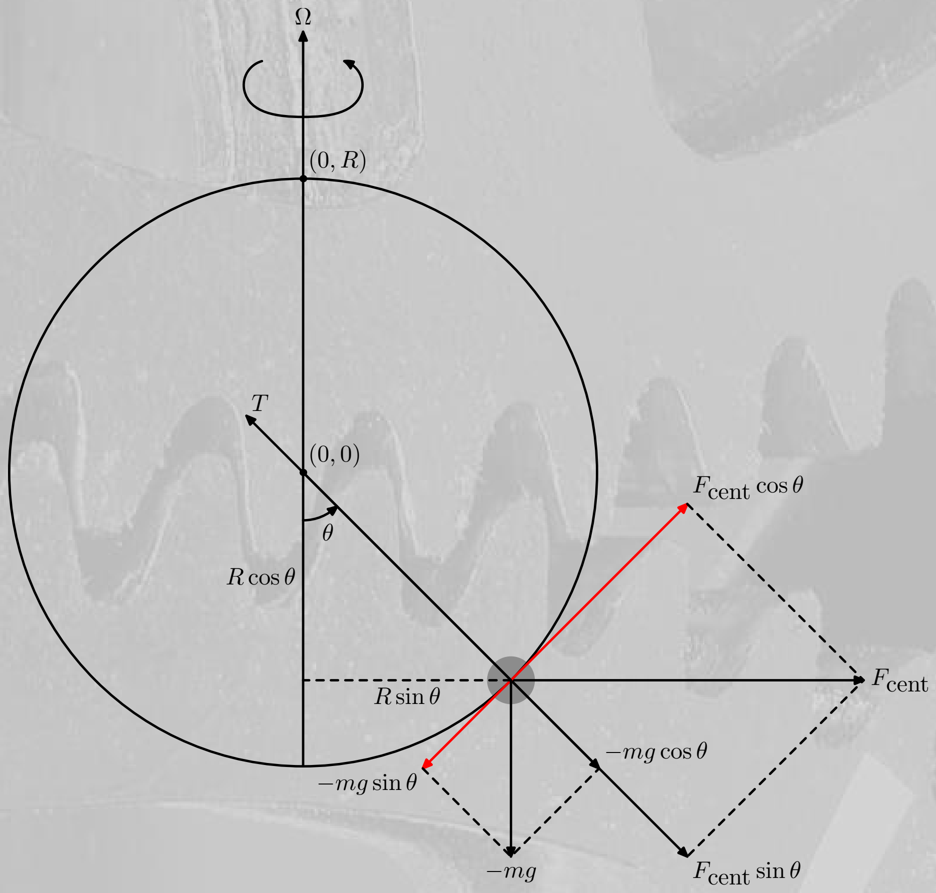
θ vs. t for $\Omega = 13$ rad/sec.

Identifying the Forces

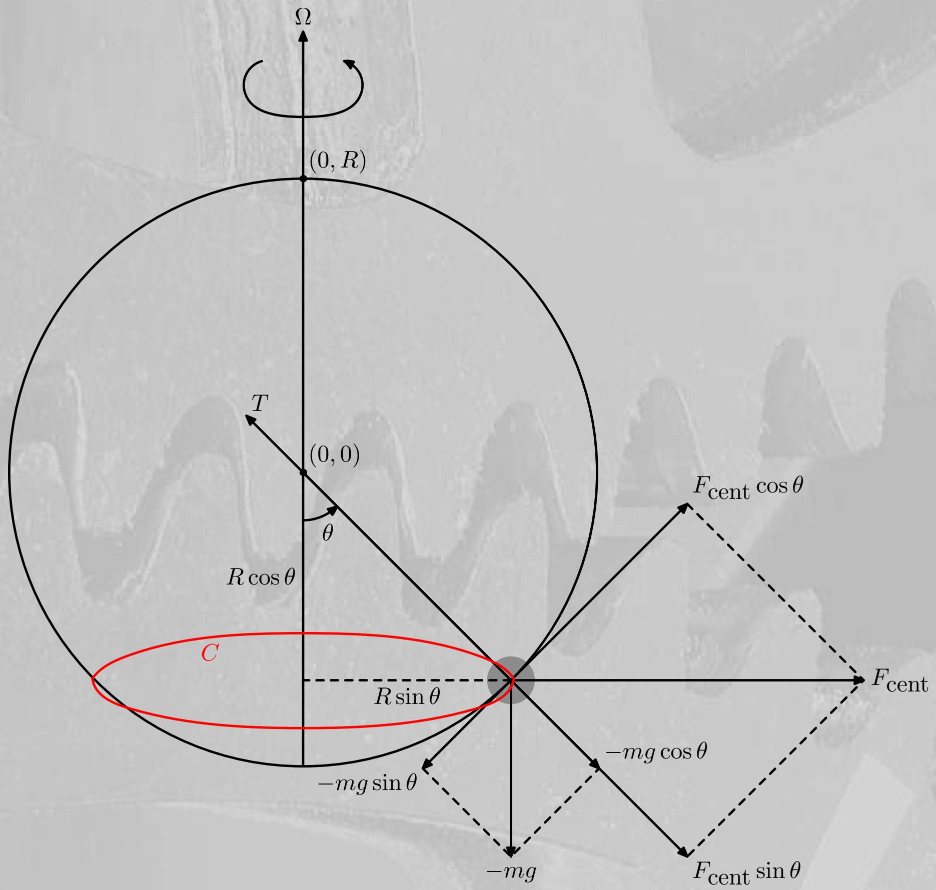
- Identify the forces that always balance.
- Identify the forces that do not always balance.
- Sum the forces to derive the equations.



Forces opposing the normal force.



Tangential forces in the vertical plane.



The horizontal path of the ball.

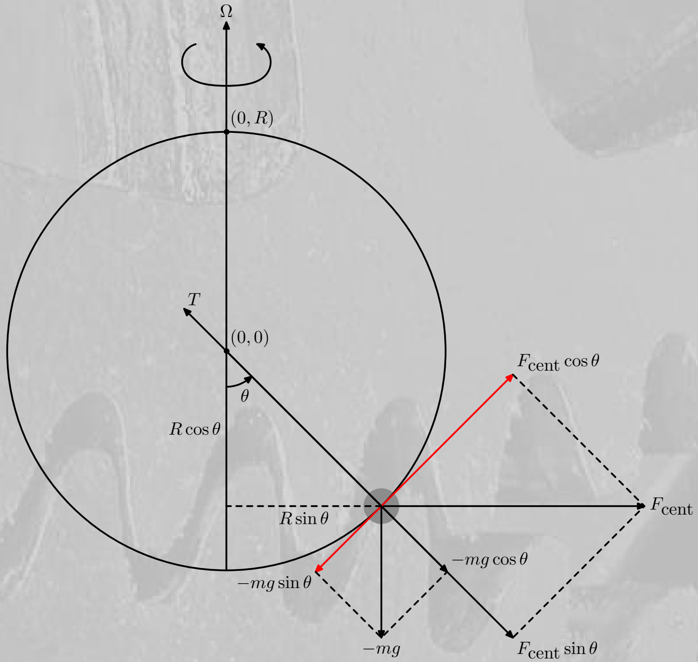
Finding F_{cent}

Recall the kinematic identities, and our values.

$$\begin{cases} v_{\text{lin}} = r v_{\text{ang}} \\ a_r = \frac{v_{\text{lin}}^2}{r} \\ F_{\text{cent}} = m a_r \end{cases} \qquad \begin{cases} v_{\text{lin}} = (R\sin\theta)\Omega \\ a_r = \frac{[(R\sin\theta)\Omega]^2}{R\sin\theta} = (R\sin\theta)\Omega^2 \\ F_{\text{cent}} = m(R\sin\theta)\Omega^2 \end{cases}$$

In our case, Ω is the angular velocity v_{ang}, about the center of C and the radius is $R\sin\theta$. The centrifugal force acting on the ball is the mass times a_r.

$$F_{\text{cent}} = m\Omega^2 R\sin\theta.$$



$$ma_T = F_{\text{cent}}\cos\theta - mg\sin\theta$$

$$mR\ddot{\theta} = m\Omega^2 R\sin\theta\cos\theta - mg\sin\theta$$

$$mR\ddot{\theta} = m\Omega^2 R\sin\theta\cos\theta - mg\sin\theta$$

$$\ddot{\theta} = \Omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \tag{1}$$

$$\ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

In order to use this equation we must first transpose it into two first order equations.

$$\begin{cases} \theta = \theta \\ \omega = \dot{\theta} \end{cases} \quad \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \ddot{\theta} = \Omega^2 \sin\theta\cos\theta - \dfrac{g}{R}\sin\theta \end{cases}$$

An equilibrium angle means that the forces are balanced and the acceleration is zero.

Set the right side equal to zero.

$$\begin{aligned} \ddot{\theta} &= 0 \\ \Omega^2 \sin\theta \cos\theta - \frac{g}{R}\sin\theta &= 0 \\ \sin\theta(\Omega^2 \cos\theta - \frac{g}{R}) &= 0 \end{aligned}$$

Therefore,

$$\sin\theta = 0 \qquad \text{or} \qquad \Omega^2\cos\theta - \frac{g}{R} = 0.$$

When $\sin\theta = 0$, $\theta = 0$ or π. To find other equilibrium angles we set the other factor equal to zero.

$$\begin{aligned} \Omega^2 \cos\theta - \frac{g}{R} &= 0 \\ \cos\theta &= \frac{g/R}{\Omega^2} \qquad (2) \end{aligned}$$

Cosine is never greater than 1 so we seek Ωs that make the right side less than or equal to 1.

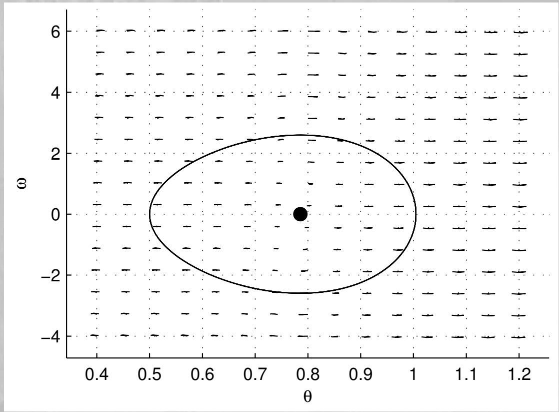
$$\begin{aligned} \frac{g}{R\Omega_0^2} &\leq 1. \\ \frac{g}{R} &\leq \Omega_0^2 \\ \sqrt{\frac{g}{R}} &\leq \Omega_0 \qquad (3) \end{aligned}$$

In our case the Ω where bifurcation occurs is,

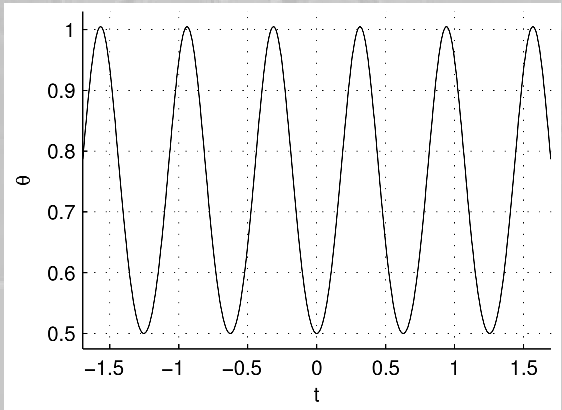
$$\begin{aligned} \sqrt{\frac{9.8}{.06}} &\le \Omega_0 \\ 12.78 &\le \Omega_0. \end{aligned} \tag{4}$$

Now we find the Ω that produces $\theta = \pi/4$.

$$\begin{aligned} \cos\theta &= \frac{g/R}{\Omega^2} \\ \cos\frac{\pi}{4} &= \frac{9.8/.06}{\Omega^2} \\ \sqrt{\frac{9.8}{.06\cos\frac{\pi}{4}}} &= \Omega \\ 15.2 &= \Omega. \end{aligned} \tag{5}$$

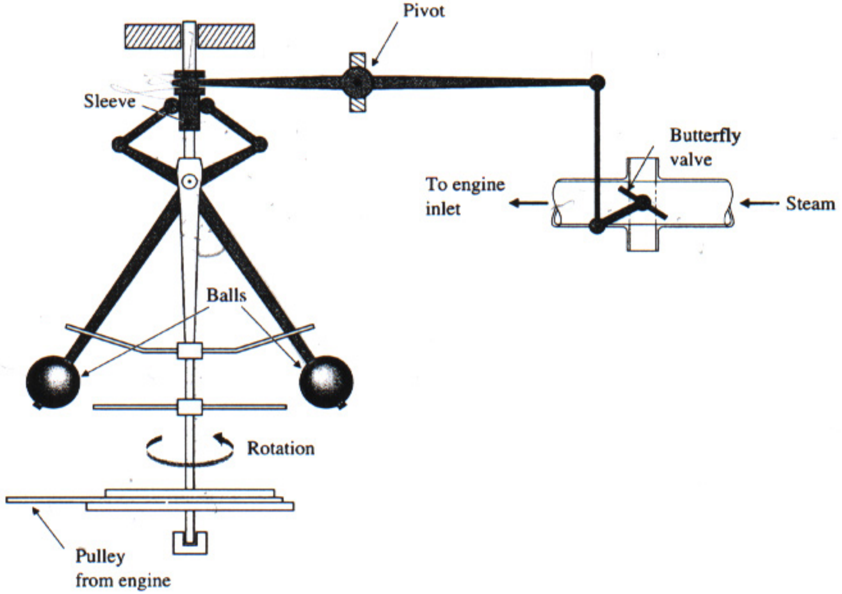


For $\Omega = 15.2$ rad/sec.



θ vs. t for $\Omega = 15.2$ rad/sec.

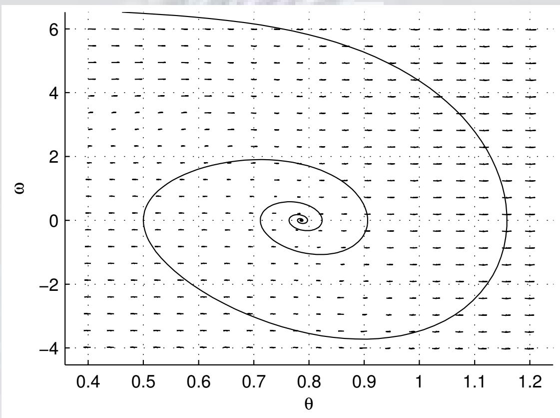
Watt's Governor



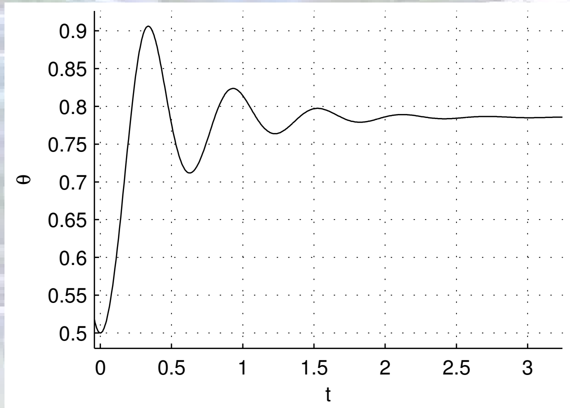
Now we have a governor that will maintain the desired angle but oscillates perpetually. How can we improve this performance?

$$\ddot{\theta} = \Omega^2 \sin\theta \cos\theta - \frac{g}{R} mg \sin\theta - k\frac{\dot{\theta}}{m} \qquad (6)$$

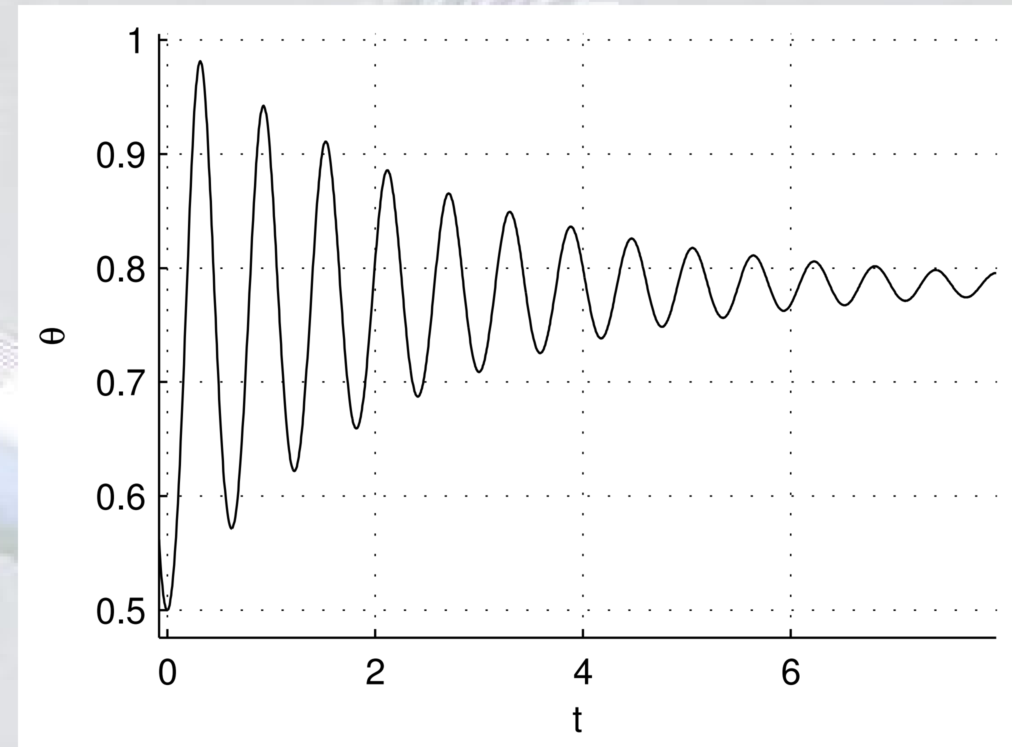
The damping term is proportional to the angular velocity (in the vertical plane) and is divided by the mass.



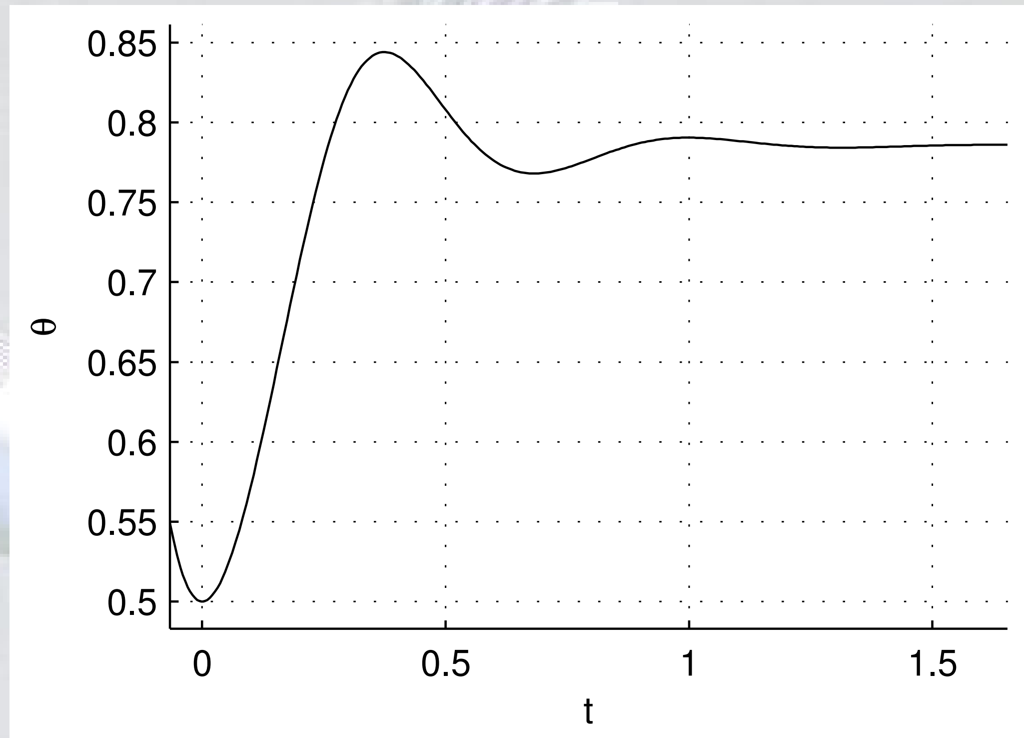
$\Omega = 15.2$ rad/sec with damping term.



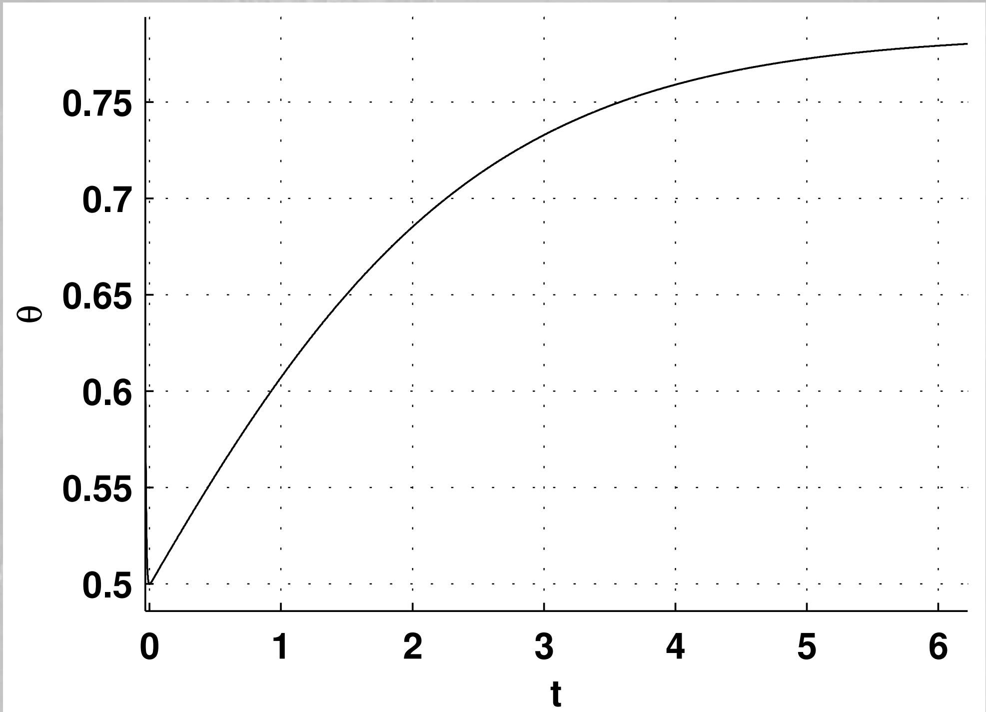
θ vs. t for $\Omega = 15.2$ and damping term.



θ vs. t for $\Omega = 15.2$, damping term, and $m = 50g$.



θ vs. t for $\Omega = 15.2$, damping term, and $m = 5g$.



θ vs. t for $\Omega = 15.2$, damping term, and $m = .25g$.

As you can see this also would not be a governor of optimum design. When designing a governor one would have to experiment with the parameters and would undoubtedly be somewhere between 5g and $1/4$g.

Putting It All Together

We have a governor design that will maintain the desired

- Changing R only effects the where the critical Ωs occur but not the oscillatory behavior.
- Increasing the mass reduces the effects of damping, reducing mass increases the effects of damping.
- Changing the damping term has an inverse effect as changing the mass.