College of the Redwoods

Math 55: Ordinary Differential Equations

The Motion of Pumping On A Swing

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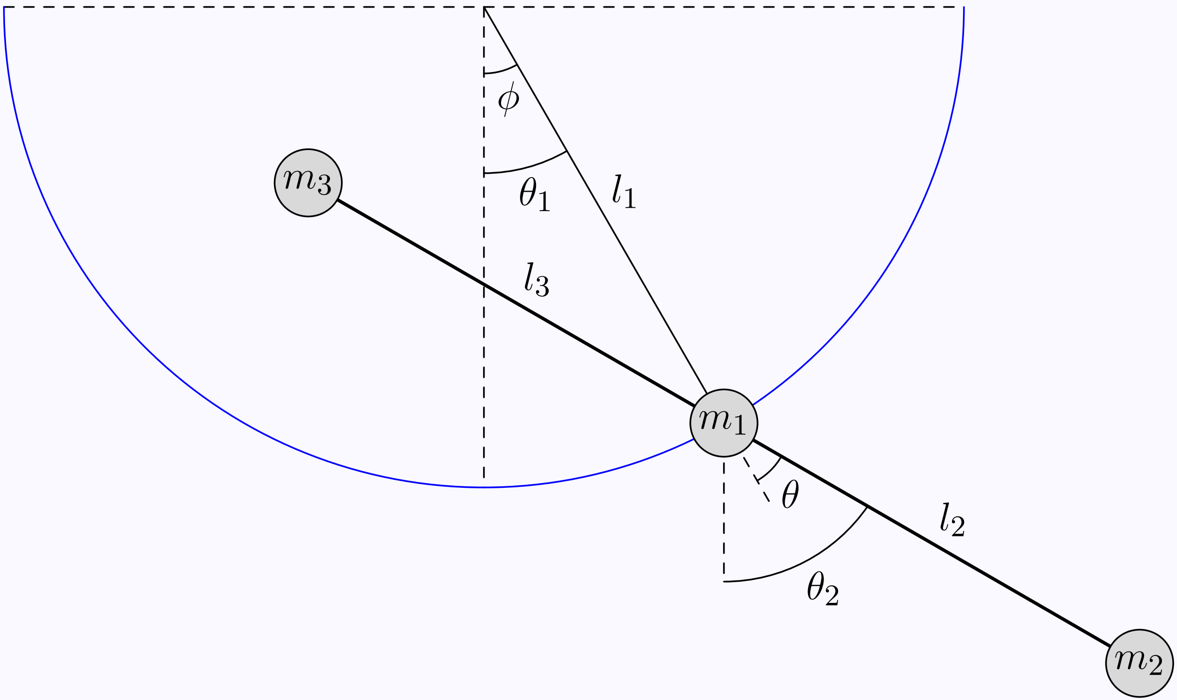
Pumping On A Swing

Layout

- Kinetic energy
- Potential energy
- Lagrangian
- Euler's formula
- Approximation of the terms
- The harmonic system
- The parametric system
- Linear and exponential resonance

The System



Variable Dictionary

- ϕ is the angle in which the system rotates about the origin.
- θ is the angle in which the masses m_2 and m_3 rotates about the center mass (m_1).
- θ_1 is equal to the angle ϕ.
- θ_2 is equal to the sum of the angles $\phi + \theta$.

Energy

- K = Total kinetic energy
- U = Total potential energy
- $L = K - U$

Position

Coordinates of mass m_1.

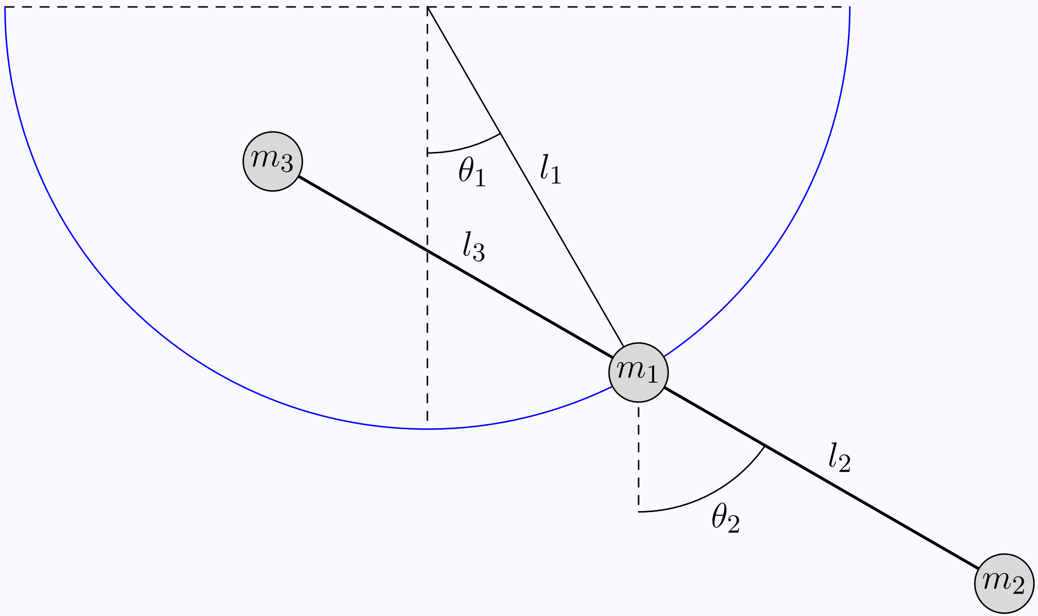
$$x_1 = l_1 \sin(\theta_1)$$
$$y_1 = l_1 \cos(\theta_1)$$

Coordinates of mass m_2.

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$$
$$y_2 = l_1 \cos(\theta_1) + l_2 \cos(\theta_2)$$

Coordinates of mass m_3.

$$x_3 = l_1 \sin(\theta_1) - l_3 \sin(\theta_2)$$
$$y_3 = l_1 \cos(\theta_1) - l_3 \cos(\theta_2)$$



$$coordinates:$$

$$(x_1, y_1) = (l_1 \sin(\theta_1), l_1 \cos(\theta_1))$$

$$(x_2, y_2) = (l_1 \sin(\theta_1) + l_2 \sin(\theta_2), l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$$

$$(x_3, y_3) = (l_1 \sin(\theta_1) - l_3 \sin(\theta_2), l_1 \cos(\theta_1) - l_3 \cos(\theta_2))$$

Velocity

Velocity of mass m_1.

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1$$
$$\dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1$$

Velocity of mass m_2.

$$\dot{x}_2 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2$$
$$\dot{y}_2 = -l_1 \sin(\theta_1)\dot{\theta}_1 - l_2 \sin(\theta_2)\dot{\theta}_2$$

Velocity of mass m_3.

$$\dot{x}_3 = l_1 \cos(\theta_1)\dot{\theta}_1 - l_3 \cos(\theta_2)\dot{\theta}_2$$
$$\dot{y}_3 = -l_1 \sin(\theta_1)\dot{\theta}_1 + l_3 \sin(\theta_2)\dot{\theta}_2.$$

Velocity Squared

$$\dot{v_1}^2 = (\dot{x_1}^2 + \dot{y_1}^2)$$
$$\dot{v_2}^2 = (\dot{x_2}^2 + \dot{y_2}^2)$$
$$\dot{v_3}^2 = (\dot{x_3}^2 + \dot{y_3}^2)$$

Substituting in the computed values of the $\dot{x_i}$ and $\dot{y_i}$ we have

$$\dot{v_1}^2 = l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 + l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2$$

$$\begin{aligned}\dot{v_2}^2 = {} & (l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 + 2 l_1 l_2 \cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \cos(\theta_2)^2 \dot{\theta}_2^2) \\ & + (l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2 + 2 l_1 l_2 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \sin(\theta_2)^2 \dot{\theta}_2^2)\end{aligned}$$

$$\begin{aligned}\dot{v_3}^2 = {} & (l_1^2 \cos(\theta_1)^2 \dot{\theta}_1^2 - 2 l_1 l_3 \cos(\theta_1) \cos(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_3^2 \cos(\theta_2)^2 \dot{\theta}_2^2) \\ & + (l_1^2 \sin(\theta_1)^2 \dot{\theta}_1^2 - 2 l_1 l_3 \sin(\theta_1) \sin(\theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_3^2 \sin(\theta_2)^2 \dot{\theta}_2^2)\end{aligned}$$

Kinetic Energy

The equation for the kinetic energy is given by the equation

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2$$

Substituting the equations from the previous page in for the squared velocities we get

$$\begin{aligned} K = {} & \frac{1}{2}l_1^2\dot{\theta}_1^2[m_1 + m_2 + m_3] + \frac{1}{2}\dot{\theta}_2^2[m_2l_2^2 + m_3l_3^2] \\ & + l_1\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)][m_2l_2 - m_3l_3]. \end{aligned}$$

Potential Energy

The equation for the potential energy is given by the equation

$$U = -m_1 g y_1 - m_2 g y_2 - m_1 g y_3$$

Substituting the equations for the y-coordinates in for y we get

$$U = -[g l_1 \cos(\theta_1)(m_1 + m_2 + m_3)] - [g \cos(\theta_2)(m_2 l_2 - m_3 l_3)]$$

Substitutions

We eliminate a number of parameters for simplicity. We let

$$
\begin{aligned}
M &= m_1 + m_2 + m_3 \\
I_1 &= M l_1^2 \\
I_2 &= m_2 l_2 + m_3 l_3 \\
N &= m_3 l_3 - m_2 l_2 \\
\theta_1 &= \phi \\
\theta_2 &= \phi + \theta
\end{aligned}
$$

Our Energy Equations Become

Kinetic energy:

$$K = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 (\dot{\phi} + \dot{\theta})^2 - l_1 N \dot{\phi} (\dot{\phi} + \dot{\theta}) \cos(\theta).$$

Potential energy:

$$U = -M g l_1 \cos(\phi) + N g \cos(\phi + \theta).$$

Lagrangian

The Lagrangian for the swing system is

$$L = K - U$$

Substituting the values for the kinetic and potential energies in for K and U we get

$$\begin{aligned} L =& \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 (\dot{\phi} + \dot{\theta})^2 - l_1 N \dot{\phi} (\dot{\phi} + \dot{\theta}) \cos(\theta) \\ &+ M g l_1 \cos(\phi)(M) - N g \cos(\phi + \theta) \end{aligned}$$

Euler-Lagrange Equation

The Euler-Lagrange Equation is given by the equation

$$0 = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\phi}}\right) - \left(\frac{\delta L}{\delta \phi}\right).$$

Finally, solving for $(d/dt)[(\delta L)/(\delta \dot{\phi})] - (\delta L)/(\delta \phi)$ we get

$$\begin{aligned} 0 = & I_1 \ddot{\phi} + I_2 \ddot{\phi} + I_2 \ddot{\theta} - 2 l_1 N \cos(\theta) \ddot{\phi} - l_1 N \cos(\theta) \ddot{\theta} \\ & + 2 l_1 N \sin(\theta) \dot{\phi} \dot{\theta} + l_1 N \sin(\theta) \dot{\theta}^2 + M g l_1 \sin(\phi) \\ & - N g (\sin(\phi) \cos(\theta) + \cos(\phi) \sin(\theta)) \end{aligned}$$

Approximations

Some of the approximation we have made include the following:

- approximation of sin and cosine factors using Taylor series.
- approximation of sin and cosine factors where $\sin(\phi) \approxeq 0$ and $\cos(\phi) \approxeq 1$ for small angles of ϕ.
- approximation of sin and cosine factors that are of some degree n to a sum of terms of degree 1 and eliminating terms with frequencies at some multiple of the natural frequency.

More Substitutions

We now make the following substitutions

$$I_0 = [(I_1 + I_2) + 2l_1N(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4)]$$

$$K_0 = [Mgl_1 + Ng(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4)]$$

$$\omega_0 = K_0/I_0$$

$$F = \theta_0[(\omega^2 I_2 + N[(g - l_1\omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)])/I_0]$$

$$A = -\frac{1}{4}Ng(\theta_0^2 - \frac{1}{12}\theta_0^4)/I_0$$

$$B = l_1N\omega[\theta_0^2 - \frac{1}{12}\theta_0^4/I_0$$

$$C = -\frac{1}{2}l_1N(\theta_0^2 - \frac{1}{12}\theta_0^4)/I_0.$$

The Approximated Swing System

The swing system becomes

$$\ddot{\phi} + \omega\phi \approxeq F\cos(\omega t) + A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}. \quad (1)$$

Notice that the F term isn't dependant on ϕ whereas the other three terms on the right hand side are ϕ-dependant. The F term is driving force and the other three terms are parametric terms, in that they are functions of the angle ϕ (they have a time-dependant piece).

The Forced Harmonic System

We can approximate the motion of the system when ϕ is small by looking at just F-term. Thus our new system becomes

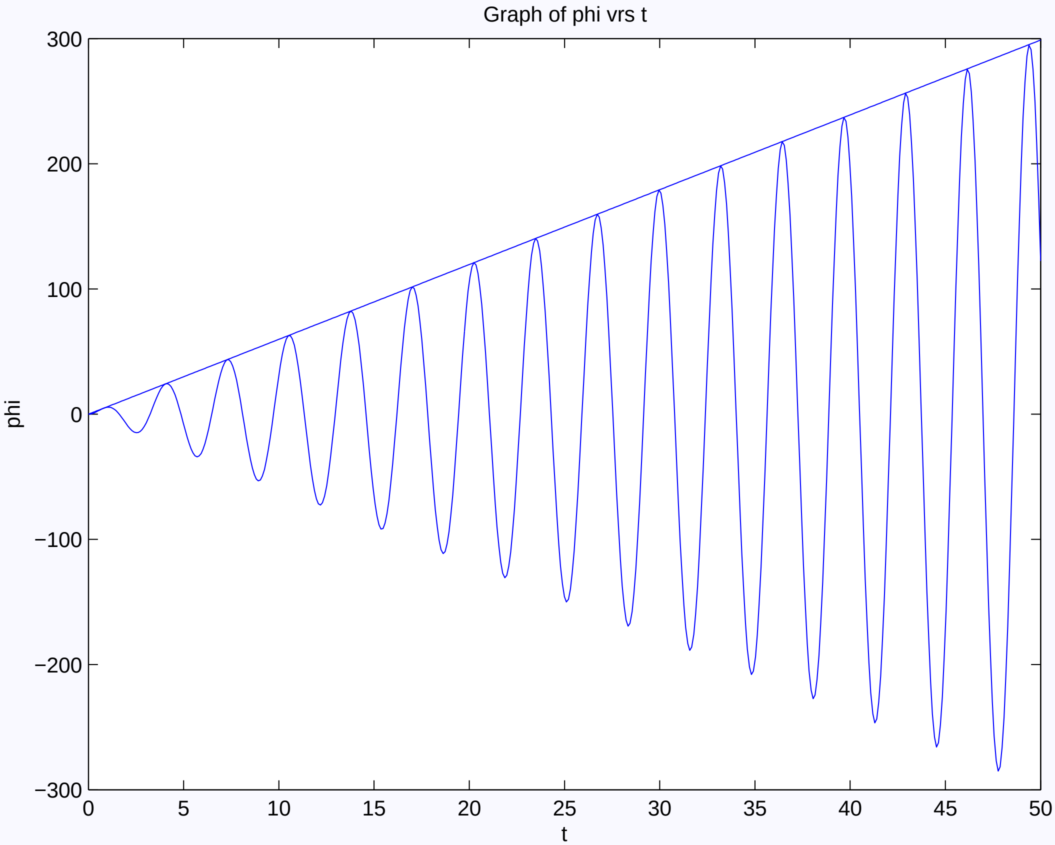
$$\ddot{\phi} + \omega\phi \approxeq F\cos(\omega t).$$

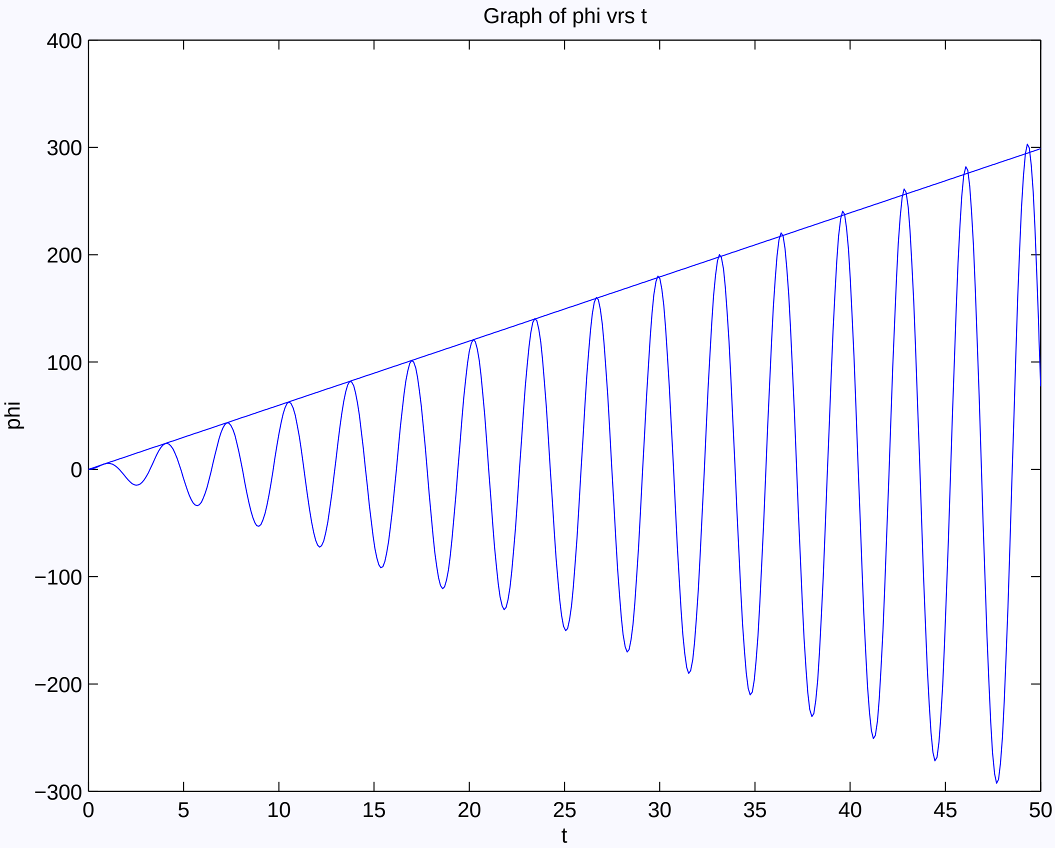
When solving the differential equation for ϕ, with initial conditions that the swing initially at rest at time $t = 0$ ($\phi(0) = 0$), we get

$$\phi \approxeq (Ft)/(2\omega_0)\sin(\omega_0 t),$$

which results in a linear growth per cycle

$$\Gamma_D = (F\pi)/(\omega_0^2).$$


Graph of phi vrs t
phi
t
300
200
100
0
−100
−200
−300
0
5
10
15
20
25
30
35
40
45
50


Graph of phi vrs t
phi
t
400
300
200
100
0
−100
−200
−300
0
5
10
15
20
25
30
35
40
45
50

Solving For The Parametric System

We can approximate the motion of the system when ϕ is large by looking at the ϕ-dependant terms. Thus our new system becomes

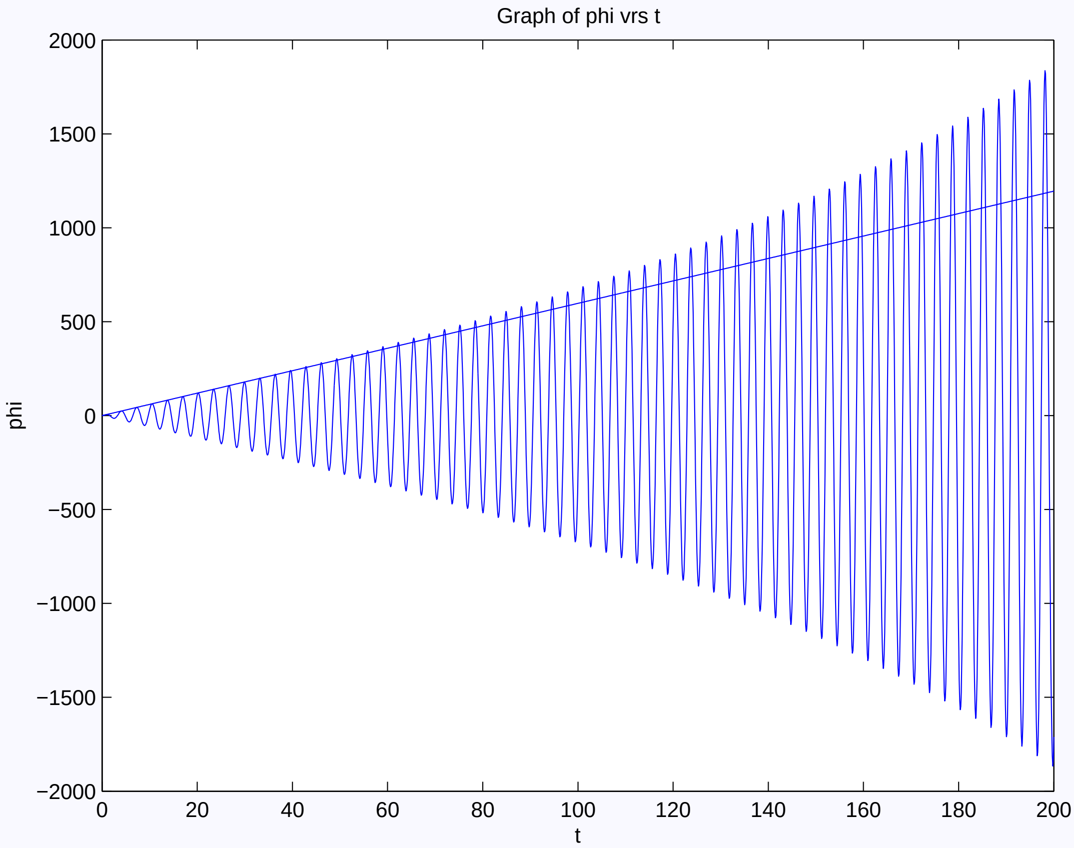
$$\ddot{\phi} + \omega\phi \approxeq A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}.$$

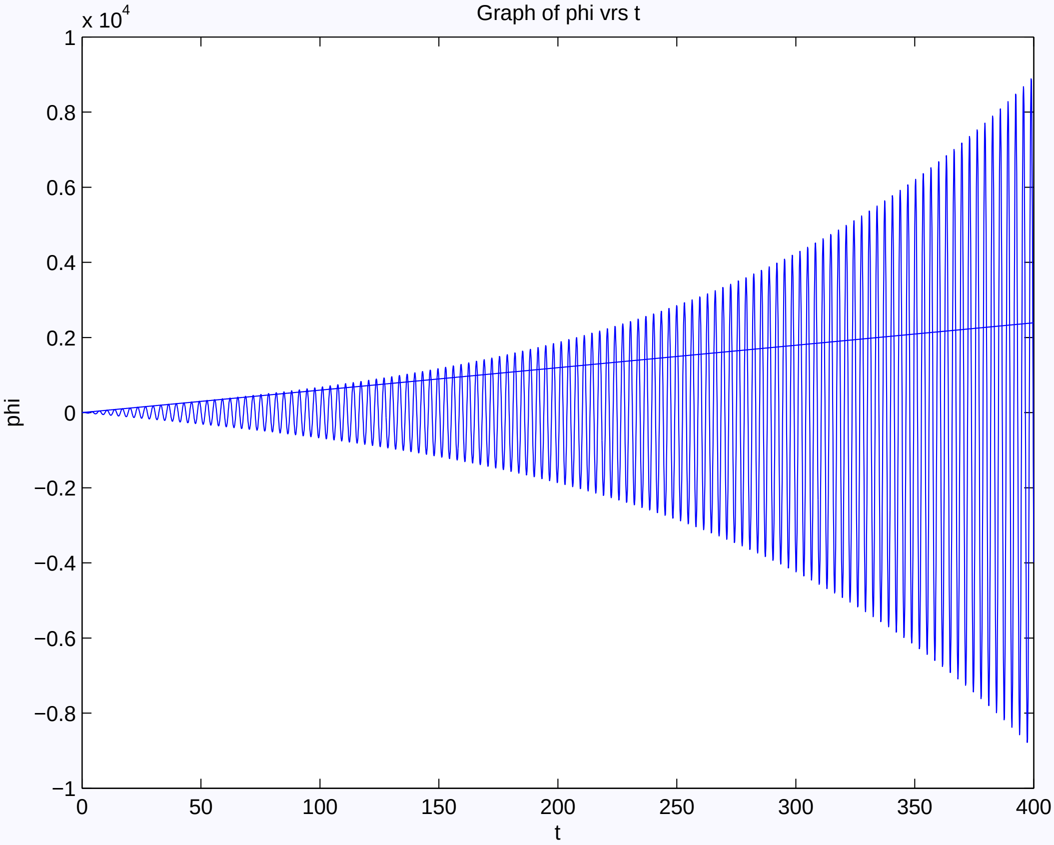
The difference between the forced harmonic oscillator system and the ϕ-dependant system is that an initial condition at time $t=0$ results in no amplitude growth in the system. Thus, we have to assume that the swing is pulled back to some initial angle of ϕ for there to be some positive contribution to the growth of the amplitude of the system. When solving the differential equation for ϕ, we get

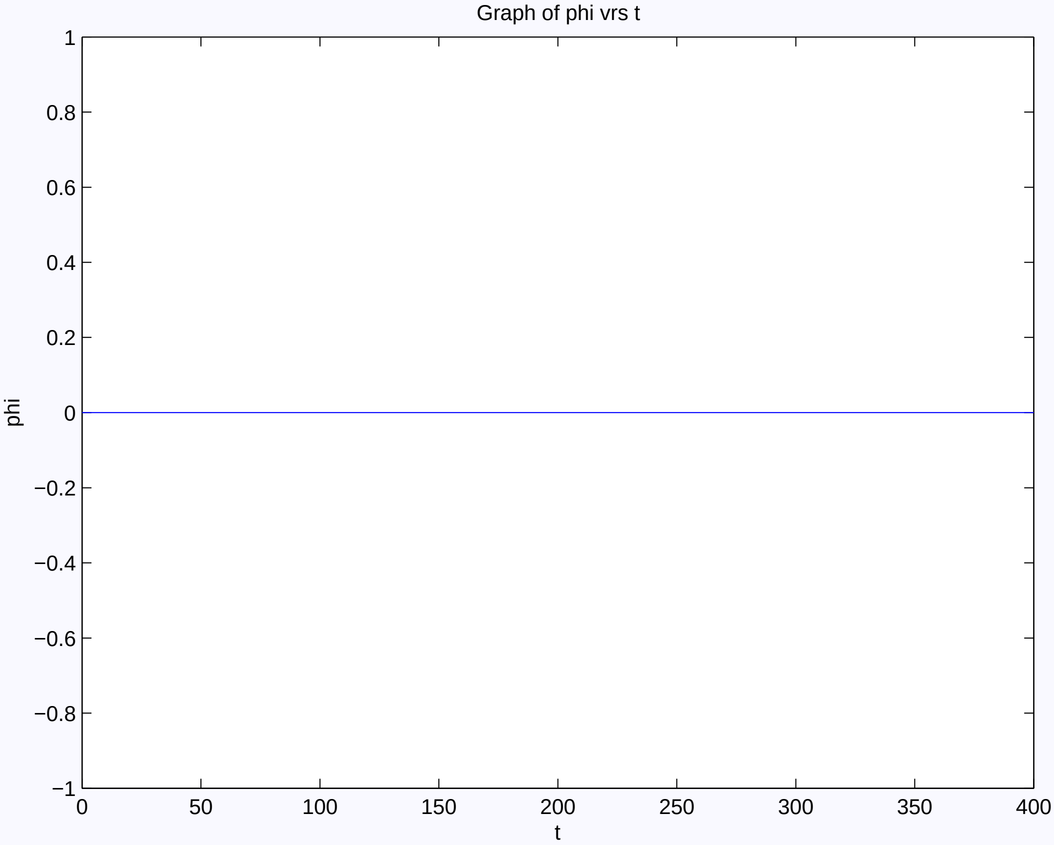
$$\phi \approxeq \pm\sqrt{2}|a|e^{-((A-\omega_0 B-\omega_0^2 C)t/(4\omega_0))}(\cos(\omega t)-\sin(\omega t))$$

which results in an exponential growth per cycle

$$\Gamma_P = (2\pi\lambda)/(\omega_0)\phi_0.$$


Graph of phi vrs t
phi
t
2000
1500
1000
500
0
−500
−1000
−1500
−2000
0
20
40
60
80
100
120
140
160
180
200


Graph of phi vrs t
x 10^4
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
phi
0
50
100
150
200
250
300
350
400
t


Graph of phi vrs t
phi
t
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
50
100
150
200
250
300
350
400

Conclusion

By looking at the two systems, we can see that the forced harmonic system

$$\ddot{\phi} + \omega\phi \approxeq F\cos(\omega t)$$

appears to be the dominating term early in the system when ϕ is small. However, as ϕ approaches some critical angle

$$\phi_{critical} \approxeq \frac{8I_2}{3\theta_0 l_1 N}$$

when the growth per cycle of the forced harmonic system equals the growth of the parametric system

$$\ddot{\phi} + \omega\phi \approxeq A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi},$$

the parametric terms start to dominate in the contribution of growth to the system.