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The Spring Pendulum

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May 16, 2000

Abstract

The spring pendulum is analyzed in three dimensions using differential equations and the Ode45 solver in Matlab. Newton's second law is used to write second order differential equations that describe the path of the spring pendulum. The spring pendulum has aspects of both the spring-mass oscillator and the simple pendulum.

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1. Introduction

A spring pendulum is simply a spring with a mass on the end. If it has no initial horizontal displacement it acts like a simple spring mass system, oscillating up and down. If it has an initial displacement to the side it will oscillate from side to side as well as up and down. In this case its motion will have elements of both a simple pendulum and a simple harmonic oscillator. If the horizontal component of its initial displacement is parallel to the horizontal component of its initial velocity it will move only in two dimensions. If these components are not parallel the mass will move in all three dimensions.

2. Derivation of the Mathematical Model of the System

The mathematical model of the system can be determined by analyzing the forces acting on the mass at the end of the spring. Several assumptions will be made to simplify the model of the system. They are: the spring is ideal, there is no air resistance acting on the spring, the magnitude of the force due to air resistance acting on the mass at the end of the spring is directly proportional to its speed, and the mass's only loss of energy is due to this air resistance.

There are three forces acting on the mass: the force due to gravity, the spring force, and the force due to air resistance. The z-axis will be the vertical axis with positive z being upwards. The spring will be hung from the origin.

$\mathbf{F}_g$	force due to gravity
$\mathbf{F}_s$	spring force
$\mathbf{F}_r$	force due to air resistance
L	relaxed spring length
K	spring constant
M	mass of object at end of spring
G	acceleration due to gravity
R	air resistance constant

$$\mathbf{F}_{net} = \mathbf{F}_g + \mathbf{F}_s + \mathbf{F}_r$$

The force due to gravity is equal to mass times the acceleration of gravity and it is in the downward direction:

$$\mathbf{F}_g = -MG\mathbf{k}$$

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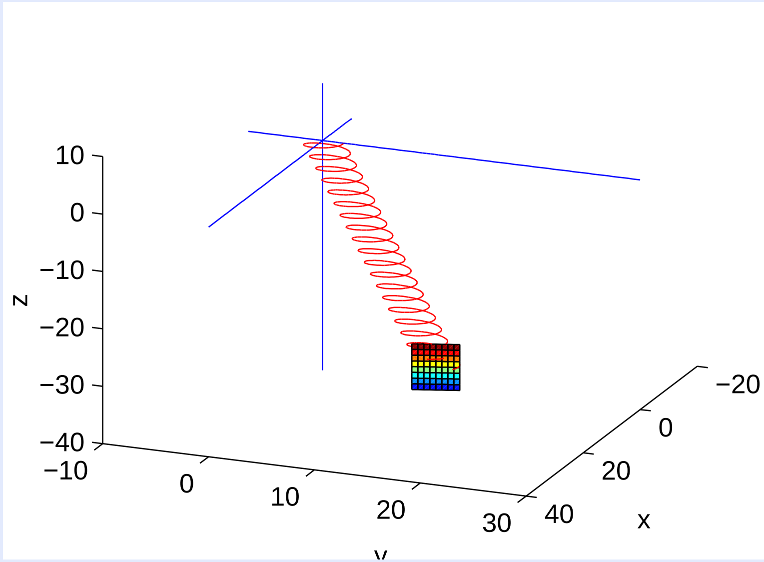

Figure 1: A model of the spring pendulum.

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The spring force is in the direction of the displacement of the mass and its magnitude is the negative of the spring constant times the magnitude of its displacement from relaxed spring length. The displacement vector is:

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

The magnitude is of the displacement is:

$$\sqrt{x^2 + y^2 + z^2}$$

Dividing the displacement vector by its magnitude you get a unit vector in the direction of the spring force:

$$\mathbf{u}_s = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Now we need to find the magnitude of the spring force. This is the negative of the spring constant times the displacement from the natural length of the spring. If the spring is stretched past its natural length the spring force will be toward the origin, so to get the displacement form the natural length of the spring we take the distance from the origin of the mass:

$$\sqrt{x^2 + y^2 + z^2}$$

And subtract from this the natural length of the spring to get the displacement from the natural length of the spring:

$$\sqrt{x^2 + y^2 + z^2} - L$$

Multiplying this by the negative of the spring constant we get the magnitude of the spring force:

$$F_s = -K\left(\sqrt{x^2+y^2+z^2}-L\right)$$

By multiplying the magnitude of the spring force by the unit vector in its direction we get a vector that is the spring force:

$$\mathbf{F}_s = -K\left(\sqrt{x^2+y^2+z^2}-L\right)\frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$$

This can then be simplified a little:

$$\mathbf{F}_s = K\left(\frac{L}{\sqrt{x^2+y^2+z^2}}-1\right)\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$$

We now have one more force to find, the force due to air resistance. This force is in the opposite direction of the velocity and its magnitude is directly proportional to the velocity. The velocity is:

$$x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}$$

Multiplying this by the negative of the air resistance constant we have the air resistance in vector form:

$$\mathbf{F}_r = -R\left(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}\right)$$

We now have an expression for all three forces and can add the to get the net force:

$$\begin{aligned}\mathbf{F}_{net} &= \mathbf{F}_g+\mathbf{F}_s+\mathbf{F}_r\\ &= -MG\mathbf{k}+K\left(\frac{L}{\sqrt{x^2+y^2+z^2}}-1\right)\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)-R\left(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}\right)\end{aligned}$$

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Adding the components along the different axis together and separating them into a force along each axis we have:

$$
\begin{aligned}
\mathbf{F}_x &= \left[Kx\left(\frac{L}{\sqrt{x^2+y^2+z^2}} - 1\right) - Rx' \right]\mathbf{i} \\
\mathbf{F}_y &= \left[Ky\left(\frac{L}{\sqrt{x^2+y^2+z^2}} - 1\right) - Ry' \right]\mathbf{j} \\
\mathbf{F}_z &= \left[Kz\left(\frac{L}{\sqrt{x^2+y^2+z^2}} - 1\right) - Rz' - MG \right]\mathbf{k}
\end{aligned}
$$

If we divide both sides of each of these equations by the mass we get expressions for the acceleration along each axis due to Newton's second law:

$$
\begin{aligned}
\mathbf{F}_{net} &= M\mathbf{a} \\
\mathbf{a} &= \frac{\mathbf{F}_{net}}{M}
\end{aligned}
$$

At the same time we can drop the vector notation and express each acceleration as a scalar:

$$
\begin{aligned}
x'' &= \frac{KLx}{M\sqrt{x^2+y^2+z^2}} - \frac{Kx}{M} - \frac{Rx'}{M} \\
y'' &= \frac{KLy}{M\sqrt{x^2+y^2+z^2}} - \frac{Ky}{M} - \frac{Ry'}{M} \\
z'' &= \frac{KLz}{M\sqrt{x^2+y^2+z^2}} - \frac{Kz}{M} - \frac{Rz'}{M} - G
\end{aligned}
$$

These equations completely describe the motion of the spring pendulum.

3. Using Ode45 to Model the Spring Pendulum

These equations have no solutions, so x, y, and z can not be written as functions of t. Another method is needed to analyze the system. Matlab's ode45 routine can be used find solutions to the system that can then be graphed in three dimensions to show the path of the spring pendulum. To do this I will first change the above system of three second order differential equations into a system of six first order differential equations. Let:

$$\begin{aligned} x' &= v_x \\ y' &= v_y \\ z' &= v_z \end{aligned}$$

We can then write:

$$\begin{aligned} x' &= v_x \\ v_x' &= \frac{KLx}{M\sqrt{x^2+y^2+z^2}} - \frac{Kx}{M} - \frac{Rv_x}{M} \\ y' &= v_y \\ v_y' &= \frac{KLy}{M\sqrt{x^2+y^2+z^2}} - \frac{Ky}{M} - \frac{Rv_y}{M} \\ z' &= v_z \\ v_z' &= \frac{KLz}{M\sqrt{x^2+y^2+z^2}} - \frac{Kz}{M} - \frac{Rv_z}{M} - G \end{aligned}$$

The following Matlab commands when put into an m-file will make “proj” a function that describes the spring pendulum.

```
function sprime=proj(t,s,flag,L,K,M,G,R)
sprime=zeros(6,1);
sprime(1)=s(2);
sprime(2)=-(K*s(1)/M)+(K*L*s(1)/...
   (M*sqrt((s(1))^2+(s(3))^2+(s(5))^2)))-R*s(2)/M;
sprime(3)=s(4);
sprime(4)=-(K*s(3)/M)+(K*L*s(3)/...
   (M*sqrt((s(1))^2+(s(3))^2+(s(5))^2)))-R*s(4)/M;
sprime(5)=s(6);
sprime(6)=-(K*s(5)/M)+(K*L*s(5)/...
   (M*sqrt((s(1))^2+(s(3))^2+(s(5))^2)))-R*s(6)/M-G;
```

These commands will use ode45 to solve the set of equations. The refine option changes the accuracy of ode45. Increasing it will make it more accurate. The values of L, K, M, G, and R are put in after the options command. The initial conditions are put in before the options command.

```
options = odeset('refine',10);
[t,s]=ode45('proj',[0,100],[1;1;1;2;1;3],options,...
   1,1,2,9.81,.015);
```

The path of the mass at the end of the pendulum can be plotted with the following commands. s(:,1) is a vector containing the x positions,s(:,3) contains the y positions, and s(:,5) contains the z positions.

```
plot3(s(:,1),s(:,3),s(:,5))
xlabel('x')
```

```
ylabel('y')
zlabel('z')
```

Altering the initial conditions and the values of the constants can make many different and interesting looking graphs. If the only initial displacement and velocity are in the z direction the mass will only oscillate in one dimension, along the z-axis. If the z-axis, the initial displacement, and the initial velocity all lie in a plane the mass will move in only two dimensions. If neither of these is the case the mass will move in three dimensions. The motion is not periodic in three dimensions. The mass will never have the same position and velocity at two different times. The one and only equilibrium position of the system is at:

$$\left(0, 0, -L - \frac{MG}{K}\right)$$

where it hangs motionless.

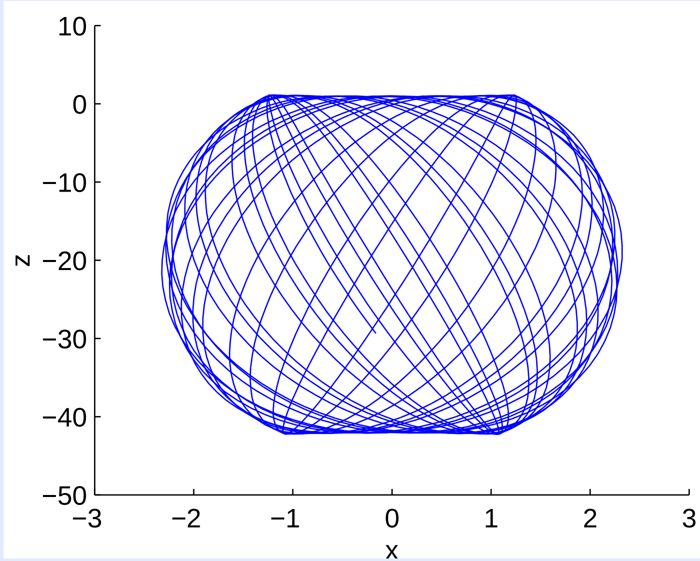

Figure 2: In this example there is no damping and the motion is in only two dimensions.

4. Examples

Figures 2-4 are some examples that have been generated in Matlab by using ode45 to solve the system. In those with no damping you can see that the motion is not periodic. The accuracy of ode45 is sufficiently high to make the graphs quit accurate.

You can look at my PowerPoint presentation for more examples or use the

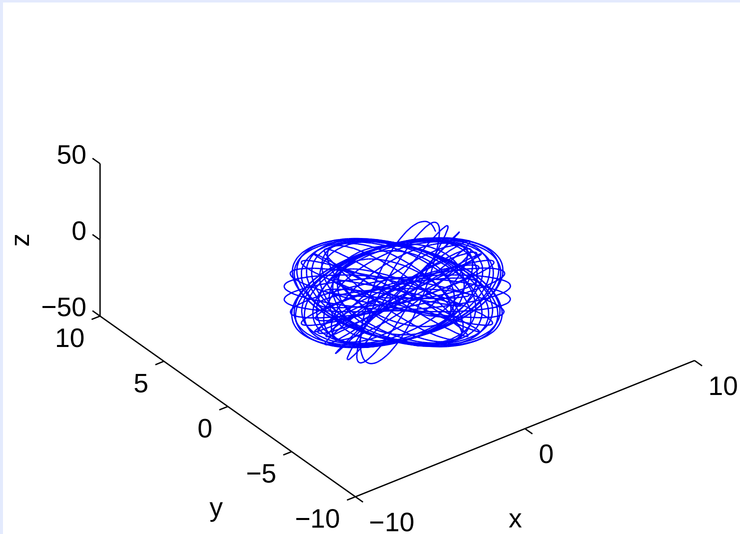

Figure 3: This is an example with no damping in three dimensions.

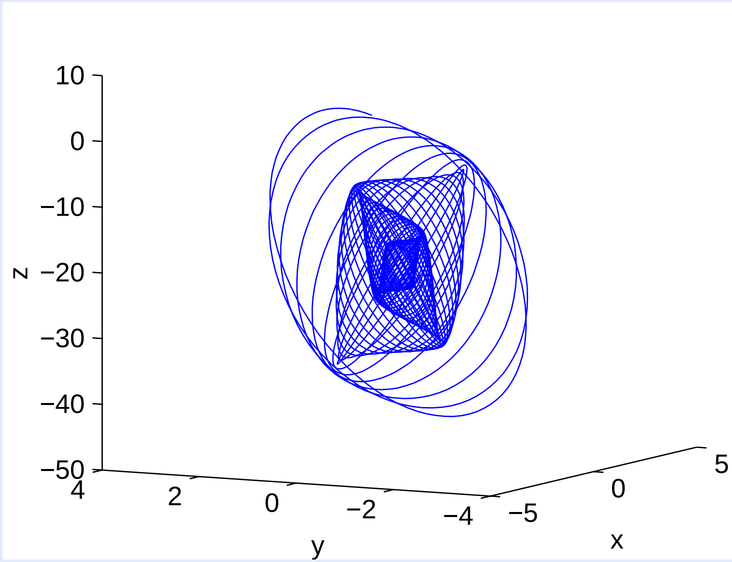

Figure 4: This is an example with damping.

Matlab commands to make your own. The PowerPoint presentation, located at http://online.redwoods.cc.ca.us/instruct/darnold/deproj/Sp00/NickWhitman/presentation/index.htm, also has a diagram of the mass and spring that shows the forces acting on the mass.