Analysis of The Motion of Pumping on A Swing

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Abstract

The motion of a person pumping a swing is extremely complex. However, the modelling can be accomplished in the two-dimensional plane by assuming the swing to be a massless rigid rod pivoting at the origin. Another simplification to the system is by modelling the person that is pumping the swing as a set of three masses (the torso, the hips, and the legs) on another massless rigid rod, where the center of mass is connected to the first massless rod. Even with these simplifications, the problem of analyzing the motion is still somewhat difficult. With this model, we will be able get a close feel for the overall motion and the forces that are involved.

1. Background

The analysis of the swing system is possible because of the calculus started by Sir Isaac Newton and Gottfried von Leibniz in 18th century. Isaac Newton derived three Laws of Motion based on the relationship between force and acceleration.

In 1746, Pierre Louis Moreau de Maupertuis formulated the Principle of Least Action. Then Leonhard Euler developed calculus of variations, useful for finding curves that are a minima for the length with respect to a given set of conditions.

Next, Joseph-Louis Lagrange took Euler's calculus of variations and applied them to Newton's physics. The main result of Lagrange's work uses this Principle of Least Action. In the Principal of Least Action, the differential equation of motion of the system is derived by taking the minimum action of the system. This minimum action is often referred to as geodesy (the tendency of physical changes and processes to take the easiest or minimum path). The action is the integral with respect to time of the Lagrangian, which depends on the positions and velocities in the system.

$$Action = \int_{t_i}^{t_f}$$

The differential equation that describe the motion of the system are determined by the action being at a minimum (or maximum) value, where the functional differential of the action is zero. This yields the Euler-Lagrange equation

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right)$$

which gives the equations of motion for the system.

Finally, in 1834, Dublin mathematician William Rowan Hamilton applied his work in optics to Newtonian mechanics, resulting in Hamilton's principle of least action.[7][6]

2. Introduction

In modelling the motion, we will first use the Euler-Lagrange equations to define the periodic motion. The Lagrangian for the system will be determined by first finding the position of the masses and their respective velocities. Then, using the position and velocities of the respective masses, we will determine the kinetic and potential energy, which will define the Lagrangian equation. Next, we will use Euler's equation with respect to the angle of displacement from the vertical to find the second order differential equation which will reveal the terms that drive the system. We will look at these terms and analyze their effect on the motion of the system. Although much approximation is involved, the results of the analysis of driving forces in the model will be close to that of an actual system. This simplification will help us to understand the overall motion of the system.

3. The Lagrangian Equation

The Lagrangian (L) for a system has the form

$$L = K - U \tag{1}$$

where K is the total kinetic energy U is the total potential energy of the system. By looking at the diagram of the system, in Figure 1, we write the coordinates of the masses in generalized form. For simplicity, the following substitution will be made–$\theta_1 = \phi$ and $\theta_2 = \phi + \theta$.

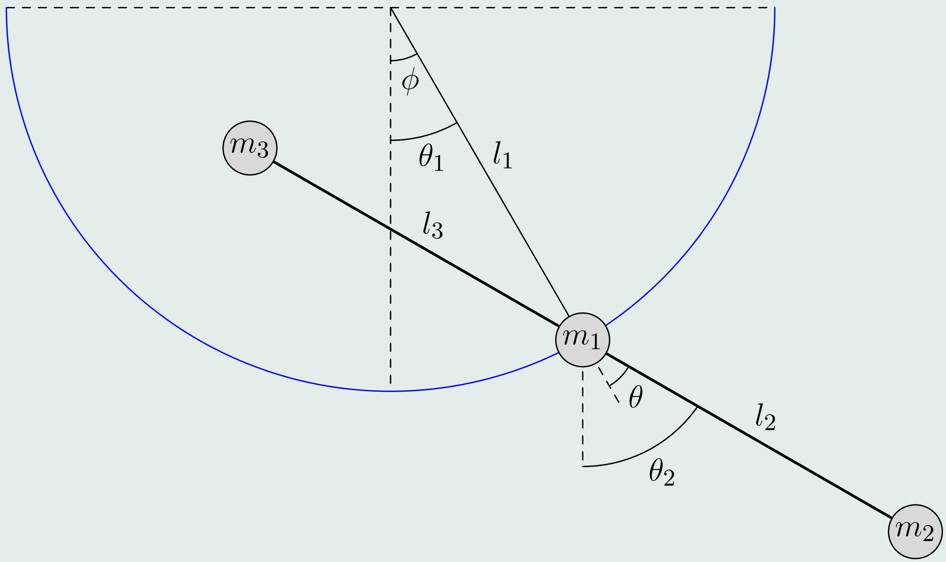

Figure 1: Graph of the system.

3.1. Coordinates of the System

In order to compute the Lagrangian, the position of the masses must be written in the form of generalized coordinates. The coordinates of m_1 are

$$x_1 = l_1 \sin(\theta_1) \qquad \text{and} \qquad y_1 = l_1 \cos(\theta_1). \tag{2}$$

The coordinates of m_2 are

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \qquad \text{and} \qquad y_2 = l_1 \cos(\theta_1) + l_2 \cos(\theta_2). \tag{3}$$

The coordinates of m_3 are

$$x_3 = l_1 \sin(\theta_1) - l_3 \sin(\theta_2) \qquad \text{and} \qquad y_3 = l_1 \cos(\theta_1) - l_3 \cos(\theta_2). \tag{4}$$

The components of the velocity some mass m at the point (x, y) with respect to θ is

$$v_x = \dot{x} = \frac{dx}{dt} \qquad \text{and} \qquad v_y = \dot{y} = \frac{dy}{dt}.$$

Thus, the components of the velocity for m_1 are

$$\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 \qquad \text{and} \qquad \dot{y}_1 = -l_1 \sin(\theta_1)\dot{\theta}_1. \tag{5}$$

The components of the velocity for m_2 are

$$\dot{x}_2 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2 \qquad \text{and} \qquad \dot{y}_2 = -l_1 \sin(\theta_1)\dot{\theta}_1 - l_2 \sin(\theta_2)\dot{\theta}_2. \tag{6}$$

The components of the velocity for m_3 are

$$\dot{x}_3 = l_1 \cos(\theta_1)\dot{\theta}_1 - l_3 \cos(\theta_2)\dot{\theta}_2 \qquad \text{and} \qquad \dot{y}_3 = -l_1 \sin(\theta_1)\dot{\theta}_1 + l_3 \sin(\theta_2)\dot{\theta}_2. \tag{7}$$

3.2. The Kinetic Energy of the System

The kinetic energy, $k = \frac{1}{2}mv^2$, of some mass m, where v^2 at a point (x, y) is equal to $(v_x^2 + v_y^2)$, is

$$k = \frac{1}{2}m(v_x^2 + v_y^2).$$

The coordinates of the masses are used to compute the individual kinetic energies of the individual masses. The kinetic energy for m_1 is

$$k_1 = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2).$$

Substituting equations (5) in for $\dot{x}_1$ and $\dot{y}_1$, respectively, we have

$$k_1 = \frac{1}{2}m_1[(l_1\cos(\theta_1)\dot{\theta}_1)^2 + (-l_1\sin(\theta_1)\dot{\theta}_1)^2].$$

This can be rewritten as

$$k_1 = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2. \tag{8}$$

Next, the kinetic energy for m_2 is

$$k_2 = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2).$$

Substituting equations (6) in for $\dot{x}_2$ and $\dot{y}_2$,

$$k_2 = \frac{1}{2}m_2[(l_1\cos(\theta_1)\dot{\theta}_1 + l_2\cos(\theta_2)\dot{\theta}_2)^2 + (-l_1\sin(\theta_1)\dot{\theta}_1 - l_2\sin(\theta_2)\dot{\theta}_2)^2].$$

Distributing the square and grouping like terms yields

$$\begin{aligned} k_2 =& \frac{1}{2}m_2[l_1^2\cos(\theta_1)^2\dot{\theta}_1^2 + l_1^2\sin(\theta_1)^2\dot{\theta}_1^2 + l_2^2\cos(\theta_2)^2\dot{\theta}_2^2 + l_2^2\sin(\theta_2)^2\dot{\theta}_2^2 \\ &+ 2l_1l_2\cos(\theta_1)\cos(\theta_2)\dot{\theta}_1\dot{\theta}_2 + 2l_1l_2\sin(\theta_1)\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2]. \end{aligned}$$

Factoring out common terms, using the pythagorean identity, and distributing the common factor $\frac{1}{2}m_2$ we get

$$k_2 = \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)). \tag{9}$$

Thirdly, the kinetic energy for m_3 is

$$k_3 = \frac{1}{2}m_3(\dot{x}_3^2 + \dot{y}_3^2).$$

Substituting equation (7) in for $\dot{x}_3$ and $\dot{y}_3$,

$$k_3 = \frac{1}{2}m_3[(l_1\cos(\theta_1)\dot{\theta}_1 - l_3\cos(\theta_2)\dot{\theta}_2)^2 + (-l_1\sin(\theta_1)\dot{\theta}_1 + l_3\sin(\theta_2)\dot{\theta}_2)^2].$$

Distributing the square and grouping like terms yields

$$
\begin{aligned}
k_3 = &\frac{1}{2}m_3[l_1^2\cos(\theta_1)^2\dot{\theta}_1^2 + l_1^2\sin(\theta_1)^2\dot{\theta}_1^2 + l_3^2\cos(\theta_2)^2\dot{\theta}_2^2 + l_3^2\sin(\theta_2)^2\dot{\theta}_2^2 \\
&- 2l_1l_3\cos(\theta_1)\cos(\theta_2)\dot{\theta}_1\dot{\theta}_2 - 2l_1l_3\sin(\theta_1)\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2].
\end{aligned}
$$

Factoring out common terms, using the pythagorean identity, and distributing the common factor $\frac{1}{2}m_3$ we get

$$
k_3 = \frac{1}{2}m_3l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_3l_3^2\dot{\theta}_2^2 - m_3l_1l_3\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)). \tag{10}
$$

Therefore, the total kinetic energy of the system $K_T = k_1 + k_2 + k_3$. Substituting the values of k_1, k_2, and k_3, computed in equations (8), (9),and (10), we get

$$
\begin{aligned}
K_T = &\frac{1}{2}m_1l_1^2\dot{\theta}_1^2 \\
&+ \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) \\
&+ \frac{1}{2}m_3l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_3l_3^2\dot{\theta}_2^2 - m_3l_1l_3\dot{\theta}_1\dot{\theta}_2(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)).
\end{aligned}
$$

Grouping like terms and factoring out common factors we get

$$
\begin{aligned}
K_T = &\frac{1}{2}l_1^2\dot{\theta}_1^2[m_1 + m_2 + m_3] + \frac{1}{2}\dot{\theta}_2^2[m_2l_2^2 + m_3l_3^2] \\
&+ l_1\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)][m_2l_2 - m_3l_3].
\end{aligned}
$$

If we let

$$
\begin{aligned}
M &= m_1 + m_2 + m_3 \\
I_1 &= Ml_1^2 \\
I_2 &= m_2l_2^2 + m_3l_3^2 \\
N &= m_3l_3 - m_2l_2,
\end{aligned}
$$

then substituting M, N, and I_2 into the equation yields

$$
K_T = \frac{1}{2}l_1^2\dot{\theta}_1^2(M) + \frac{1}{2}\dot{\theta}_2^2(I_2) + l_1\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)](-N).
$$

Using the cosine expansion trig-identity and substituting I_1 into the equation gives

$$K_T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - l_1 N\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1 - \theta_2)].$$

Finally, substituting back ϕ and θ, where $\theta_1 = \phi$ and $\theta_2 = \phi + \theta$, the equation becomes

$$K_T = \frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2(\dot{\phi} + \dot{\theta})^2 - l_1 N\dot{\phi}(\dot{\phi} + \dot{\theta})\cos(\theta). \tag{11}$$

3.3. The Potential Energy of the System

The potential energy for some mass m at a point (x, y) is $u = -mgy$. In this instance, gravity is always pulling against the direction of positive motion. If the swing is moving to a higher potential height, the masses always experience a negative acceleration through the highest point in the arc of the motion until they return to the lowest potential energy value. The potential energy for m_1 is

$$u_1 = -m_1 g y_1.$$

The coordinates of the masses are used to compute the individual potential energies of the individual masses. Substituting the value of y_1, computed in equation (2), into the previous equation yields

$$u_1 = -m_1 g(l_1\cos(\theta_1)). \tag{12}$$

Next, the potential energy for m_2 is

$$u_2 = -m_2 g y_2.$$

Substituting the value of y_2, computed in equation (3), and distributing the factor $-m_2 g$ yields

$$u_2 = -m_2 g l_1\cos(\theta_1) - m_2 g l_2\cos(\theta_2)). \tag{13}$$

Thirdly, the potential energy for m_3 is

$$u_3 = -m_3 g y_3.$$

Substituting the value of y_3, computed in equation (4), and distributing the factor $-m_3 g$ yields

$$u_3 = -m_3 g l_1\cos(\theta_1) + m_3 g l_3\cos(\theta_2)). \tag{14}$$

Therefore, the total potential energy of the system, as computed in equations (12), (13), and (14), is $U_T = u_1 + u_2 + u_3$. Substituting the values of u_1, u_2, u_3 and factoring common terms yields

$$U_T = -gl_1 \cos(\theta_1)(m_1 + m_2 + m_3) - g\cos(\theta_2)(m_2 l_2 - m_3 l_3).$$

Again,

$$\begin{aligned} M &= m_1 + m_2 + m_3 \\ N &= m_3 l_3 - m_2 l_2, \end{aligned}$$

then substituting M and N into the equation yields

$$U_T = -gl_1 \cos(\theta_1)(M) - g\cos(\theta_2)(-N).$$

Substituting back ϕ and θ, where $\theta_1 = \phi$ and $\theta_2 = \phi + \theta$, the equation becomes

$$U_T = -Mgl_1 \cos(\phi) + Ng\cos(\phi + \theta). \tag{15}$$

4. The Lagrangian Equation of the System

Now that the kinetic and potential energies have been attained, we can now compute the Lagrangian for the system. As is well known, the equation for the Lagrangian is is given by the equation

$$L = K_T - U_T. \tag{16}$$

After substituting the kinetic and potential energies, that are computed in equations (11) and (15), into equation (16) we get

$$\begin{aligned} L = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_2 (\dot{\phi} + \dot{\theta})^2 - l_1 N \dot{\phi}(\dot{\phi} + \dot{\theta}) \cos(\theta) \\ + Mgl_1 \cos(\phi) - Ng\cos(\phi + \theta). \end{aligned} \tag{17}$$

5. Euler-Lagrange Equation

Next, we use Euler's formula. The differential equation for the motion of the system is derived by taking the motion of the masses that requires a minimum of action for the system. The Euler-Lagrange equation is given by

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j},$$

where we set the right hand side of the equation equal to zero. Thus, setting j equal to the angler time-dependant displacement of the masses from the vertical y-axis (ϕ), we get

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi}. \tag{18}$$

Then, substituting the value of the Lagrangian, computed in equation (17), into equation (18), we solve for $\partial L/\partial \dot{\phi}$

$$\begin{aligned}\frac{\partial L}{\partial \dot{\phi}} =& \frac{\partial}{\partial \dot{\phi}}\Big(\frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2(\dot{\phi}+\dot{\theta})^2 - l_1 N\dot{\phi}(\dot{\phi}+\dot{\theta})\cos(\theta) \\ &+ Mgl_1\cos(\phi) - Ng\cos(\phi+\theta)\Big) \\ =& I_1\dot{\phi} + I_2(\dot{\phi}+\dot{\theta}) - l_1 N\cos(\theta)(2\dot{\phi}+\dot{\theta}).\end{aligned} \tag{19}$$

Next,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) =& \frac{d}{dt}[I_1\dot{\phi} + I_2(\dot{\phi}+\dot{\theta}) - l_1 N\cos(\theta)(2\dot{\phi}+\dot{\theta})] \\ =& I_1\ddot{\phi} + I_2(\ddot{\phi}+\ddot{\theta}) - l_1 N[\cos(\theta)(2\ddot{\phi}+\ddot{\theta}) - \sin(\theta)\dot{\theta}(2\dot{\phi}+\dot{\theta})].\end{aligned} \tag{20}$$

Next,

$$\begin{aligned}\frac{\partial L}{\partial \phi} =& \frac{\partial}{\partial \phi}\Big(\frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_2(\dot{\phi}+\dot{\theta})^2 - l_1 N\dot{\phi}(\dot{\phi}+\dot{\theta})\cos(\theta) \\ &+ Mgl_1\cos(\phi) - Ng\cos(\phi+\theta)\Big) \\ =& -Mgl_1\sin(\phi) + Ng\sin(\phi+\theta).\end{aligned} \tag{21}$$

Substituting the computed values in equations (19), (20), and (21) into equation (18) we get

$$\begin{aligned}0 =& \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} \\ =& I_1\ddot{\phi} + I_2(\ddot{\phi}+\ddot{\theta}) - l_1 N[\cos(\theta)(2\ddot{\phi}+\ddot{\theta}) - \sin(\theta)\dot{\theta}(2\dot{\phi}+\dot{\theta})] \\ &- [-Mgl_1\sin(\phi) + Ng\sin(\phi+\theta)] \\ =& I_1\ddot{\phi} + I_2\ddot{\phi} + I_2\ddot{\theta} - 2l_1 N\cos(\theta)\ddot{\phi} - l_1 N\cos(\theta)\ddot{\theta} + 2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} + l_1 N\sin(\theta)\dot{\theta}^2 \\ &+ Mgl_1\sin(\phi) - Ng\sin(\phi+\theta).\end{aligned}$$

This results in the path that requires a minimum of action for the swing system which is described by the following equation

$$\begin{aligned} 0 =& I_1\ddot{\phi} + I_2\ddot{\phi} + I_2\ddot{\theta} - 2l_1 N\cos(\theta)\ddot{\phi} - l_1 N\cos(\theta)\ddot{\theta} + 2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} + l_1 N\sin(\theta)\dot{\theta}^2 \\ &+ Mgl_1\sin(\phi) - Ng(\sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta)). \end{aligned} \tag{22}$$

We can get a close approximation for the last term in equation (22). For small values of $\phi, \sin(\phi)$ is approximately equal to ϕ and, likewise, for small values of $\phi, \cos(\phi)$ is approximately equal to one. However, for this approximation to yield an accurate solution, ϕ must be less than .7 radians or less than approximately $40°$. Thus, the last term can be approximated as

$$\begin{aligned} -Ng[\sin(\phi)\cos(\theta) + \cos(\phi)\sin(\theta)] &\approx -Ng[(\phi)\cos(\theta) + (1)\sin(\theta)] \\ &\approx -Ng\phi\cos(\theta) - Ng\sin(\theta). \end{aligned}$$

Substituting this approximated value into equation(22) we get

$$\begin{aligned} 0 =& I_1\ddot{\phi} + I_2\ddot{\phi} + I_2\ddot{\theta} - 2l_1 N\cos(\theta)\ddot{\phi} - l_1 N\cos(\theta)\ddot{\theta} + 2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} + l_1 N\sin(\theta)\dot{\theta}^2 \\ &+ Mgl_1\phi - Ng(\phi)\cos(\theta) - Ng\sin(\theta). \end{aligned} \tag{23}$$

Moving the first, second, and eighth terms to the right hand side of equation(23) we obtain

$$\begin{aligned} I_1\ddot{\phi} + I_2\ddot{\phi} + Mgl_1\phi =& - I_2\ddot{\theta} + 2l_1 N\cos(\theta)\ddot{\phi} + l_1 N\cos(\theta)\ddot{\theta} - 2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} - l_1 N\sin(\theta)\dot{\theta}^2 \\ &+ Ng(\phi)\cos(\theta) + Ng\sin(\theta). \end{aligned} \tag{24}$$

6. Approximation of the Terms

The two terms on the left hand side of the equation are the standard harmonic oscillatory terms of a harmonic system. We now make the assumption that the pumper pumps the swing by rotating their mass as θ varies as $\theta_0\cos(\omega t)$. The θ_0 factor is a time independent amplitude that represents a fixed parameter.[1]

6.1. The Driving Terms of the System

The first four terms on the right hand side of the equation(24) are the driving terms of the system.[1] The driving terms are characterized by the pumper rotating their body about the center mass (m_1).

With the assumption that $\theta = \theta_0 \cos(\omega t)$, the first derivative becomes

$$-I_2\ddot{\theta} = \omega^2 I_2 \theta_0 \cos(\omega t). \tag{25}$$

We wish to write the equation of motion in terms of fixed parameters and one variable (ϕ). To do this, the Taylor series is used to expand the functions of θ. The expansions are carried out only to the fifth order of θ. The next equation on the right is expanded to

$$-l_1 N \sin(\theta)\dot{\theta}^2 \approxeq -l_1 N \left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)[-\omega\theta_0 \sin(\omega t)]^2.$$

Substituting in $\theta_0 \cos(\omega t)$ in for θ yields

$$-l_1 N \sin(\theta)(\theta_0 \cos(\omega t)')^2 \approxeq -l_1 N\omega^2 \left(\frac{\theta_0^3 \cos(\omega t)}{1} - \frac{\theta_0^5 \cos^3(\omega t)}{6} + \frac{\theta_0^7 \cos^5(\omega t)}{120}\right)\sin^2(\omega t).$$

Using a trig-identity and distributing the $1 - \cos^2(\omega t)$ we have

$$-l_1 N \sin(\theta)(\theta_0 \cos(\omega t)')^2 \approxeq -l_1 N\omega^2 \left(\frac{\theta_0^3[\cos(\omega t) - \cos^3(\omega t)]}{1} - \frac{\theta_0^5[\cos^3(\omega t) - \cos^5(\omega t)]}{6}\right). \tag{26}$$

Next, we expand $\cos^2(\omega t)$, $\cos^3(\omega t)$, $\cos^4(\omega t)$, and $\cos^5(\omega t)$ out as a combination of first order functions of ωt and $2\omega t$ to rewrite all functions of θ of order greater than one. Solving for $\cos^2(\omega t)$ yields

$$\begin{aligned} \cos(2\omega t) &= 2\cos^2(\omega t) - 1 \\ \cos^2(\omega t) &= \frac{1}{2}\cos(2\omega t) + \frac{1}{2}. \end{aligned} \tag{27}$$

Next, solving for $\cos^3(\omega t)$ yields

$$\begin{aligned} \cos(3\omega t) &= 4\cos^3(\omega t) - 3\cos(\omega t) \\ \cos^3(\omega t) &= \frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t). \end{aligned} \tag{28}$$

Then, solving for $\cos^4(\omega t)$ yields

$$\begin{aligned} \cos(4\omega t) &= 8\cos^4(\omega t) - 8\cos^2(\omega t) + 1 \\ \cos^4(\omega t) &= \frac{1}{8}\cos(4\omega t) + \cos^2(\omega t) - \frac{1}{8} \\ &= \frac{1}{8}\cos(4\omega t) + \frac{1}{2}\cos(2\omega t) + \frac{3}{8}. \end{aligned} \tag{29}$$

Lastly, solving for $\cos^5(\omega t)$ yields

$$\begin{aligned}\cos(5\omega t) =& 16\cos^5(\omega t) - 20\cos^3(\omega t) + 5\cos(\omega t)\\ =& 16\cos^5(\omega t) - 20\left[\frac{1}{4}\cos(3\omega t) + \frac{3}{4}\cos(\omega t)\right] + 5\cos(\omega t)\\ \cos^5(\omega t) =& \frac{1}{16}\cos(5\omega t) + \frac{5}{16}\cos(3\omega t) + \frac{10}{16}\cos(\omega t).\end{aligned} \tag{30}$$

However, we only keep the terms with frequencies of ω for the driving force terms because terms at multiples of the ω frequency have very little effect, thus, we approximate the terms as

$$\cos^3(\omega t) \approxeq \frac{3}{4}\cos(\omega t) \qquad \text{and} \qquad \cos^5(\omega t) \approxeq \frac{10}{16}\cos(\omega t). \tag{31}$$

We only keep the terms with frequencies of 2ω for the parametric terms because terms at multiples of the 2ω frequency have very little effect, thus, we approximate the terms as

$$\cos^2(\omega t) = \frac{1}{2}\cos(2\omega t) + \frac{1}{2} \qquad \text{and} \qquad \cos^4(\omega t) \approxeq \frac{1}{2}\cos(2\omega t) + \frac{3}{8}. \tag{32}$$

An example of how frequencies at multiples of the natural frequency hardly affect the overall motion of the system is given in Figure 2. The following Matlab M-file plots graphs: one of a set frequency and a second with the addition of multiple frequencies. Note that the amplitudes of the graphs are virtually indistinguishable.

```
function multiple

close all tspan=[0,100]; init=[0,0]; [t1,y1]=ode45(@f,tspan,init);
subplot(1,2,1) plot(t1,y1(:,1)) title('Natural frequency')
subplot(1,2,2) [t2,y2]=ode45(@g,tspan,init);
plot(t1,y1(:,1),'b',t2,y2(:,1),'r') title('Natural frequency with
multiples of the natural frequency')

function yprime=f(t,y) yprime=zeros(2,1); yprime(1)=y(2);
yprime(2)=-16*y(1)+cos(4*t);

function yprime=g(t,y) yprime=zeros(2,1); yprime(1)=y(2);
yprime(2)=-16*y(1)+cos(4*t)+cos(8*t)+cos(16*t);
```

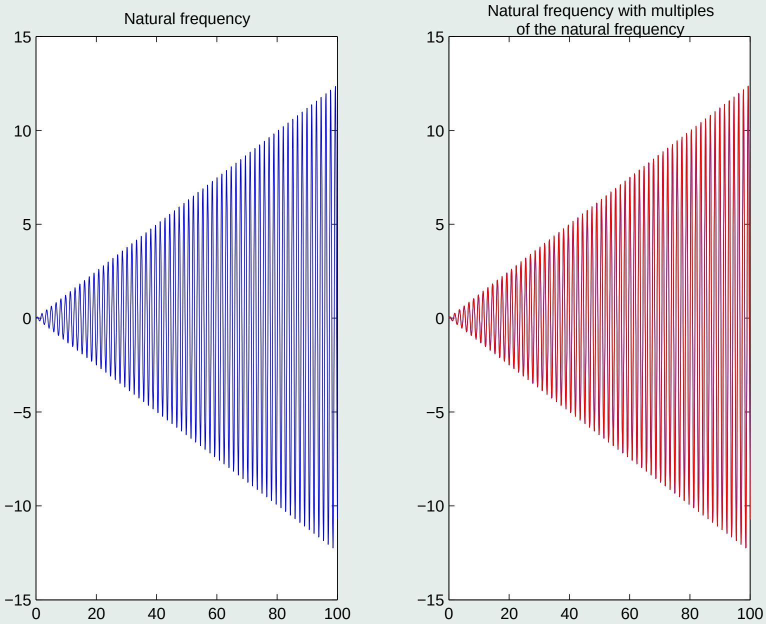
Natural frequency
Natural frequency with multiples
of the natural frequency
15
10
5
0
−5
−10
−15
0
20
40
60
80
100

Figure 2:

Then, substituting these approximations, calculated in equation (31), into equation (26) we get

$$-l_1 N \sin(\theta)\dot{\theta}^2 \approxeq -l_1 N\omega^2 \left[\theta_0^3 \left(\cos(\omega t) - \frac{3}{4}\cos(\omega t)\right) - \frac{1}{6}\theta_0^5 \left(\frac{3}{4}\cos(\omega t) - \frac{10}{16}\cos(\omega t)\right)\right].$$

Finally, subtracting terms and factoring out the $\cos(\omega t)/4$ gives

$$-l_1 N \sin(\theta)\dot{\theta}^2 \approxeq -\frac{1}{4} l_1 N\omega^2 \left(\theta_0^3 - \frac{1}{12}\theta_0^5\right)\cos(\omega t). \tag{33}$$

Expanding the next term on the right hand side of equation (24) we get

$$l_1 N \cos(\theta)\ddot{\theta} \approxeq l_1 N \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right)\left[-\omega^2\theta_0 \cos(\omega t)\right].$$

Substituting in $\theta_0 \cos(\omega t)$ in for θ and distributing yields

$$l_1 N \cos(\theta)\ddot{\theta} \approxeq -l_1 N\omega^2 \left(\theta_0 \cos(\omega t) - \frac{\theta_0^3 \cos^3(\omega t)}{2} + \frac{\theta_0^5 \cos^5(\omega t)}{24}\right). \tag{34}$$

Next, substituting the approximations from equation (31) into equation (34) yields

$$\begin{aligned} l_1 N \cos(\theta)\ddot{\theta} &\approxeq -l_1 N\omega^2 \left[\theta_0 \cos(\omega t) - \frac{1}{2}\theta_0^3 \left(\frac{3}{4}\cos(\omega t)\right) + \frac{1}{24}\theta_0^5 \left(\frac{10}{16}\cos(\omega t)\right)\right] \\ &\approxeq -l_1 N\omega^2 \left[\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right]\cos(\omega t). \end{aligned} \tag{35}$$

Expanding the next term on the right hand side of equation (24) we get

$$Ng\sin(\theta) \approxeq Ng\left(\frac{\theta^1}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right).$$

Substituting in $\theta_0 \cos(\omega t)$ in for θ yields

$$Ng\sin(\theta) \approxeq Ng\left(\frac{\theta_0 \cos(\omega t)}{1} - \frac{\theta_0^3 \cos^3(\omega t)}{6} + \frac{\theta_0^5 \cos^5(\omega t)}{120}\right). \tag{36}$$

Then, substituting these approximations, calculated in equation (31), into equation (36) and factoring out the $\cos(\omega t)$ we get

$$Ng\sin(\theta) \approxeq Ng\left[\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right]\cos(\omega t). \tag{37}$$

6.2. The Parametric Terms of the System

The next four terms on the right hand side of the equation are the parametric terms of the system.[1] They are called parametric terms because they have a certain time-dependance in that they are ϕ-dependant.[2] Expanding the next term on the right hand side of equation (24) we get

$$N g\phi\cos(\theta) \approxeq N g\phi\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}\right).$$

Substituting in $\theta_0\cos(\omega t)$ in for θ yields

$$N g\phi\cos(\theta) \approxeq N g\phi\left(1-\frac{\theta_0^2\cos^2(\omega t)}{2}+\frac{\theta_0^4\cos^4(\omega t)}{24}\right). \tag{38}$$

Next, substituting the approximations from equation (32) into equation (38), distributing the fractions, and grouping like terms yields

$$N g\phi\cos(\theta) \approxeq N g\phi\left[\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right)+\left(-\frac{1}{4}\theta_0^2\cos(2\omega t)+\frac{1}{48}\theta_0^4\cos(2\omega t)\right)\right].$$

Finally, factoring out common factors and distributing the $N g\phi$ gives

$$N g\phi\cos(\theta) \approxeq N g\phi\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right)-\frac{1}{4}N g\phi\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\cos(2\omega t) \tag{39}$$

Expanding the next term on the right hand side of equation (24) we get

$$-2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq -2l_1 N\left(\frac{\theta^1}{1!}-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right)\dot{\phi}\dot{\theta}.$$

Substituting in $\theta_0\cos(\omega t)$ in for θ and distributing the θ_0 yields

$$-2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq 2l_1 N\omega\dot{\phi}\left(\frac{\theta_0^2\cos(\omega t)}{1}-\frac{\theta_0^4\cos^3(\omega t)}{6}+\frac{\theta_0^6\cos^5(\omega t)}{120}\right)\sin(\omega t). \tag{40}$$

Remember that the terms are being expanded out to the fifth order functions of ωt, thus, equation (40) becomes

$$-2l_1 N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq 2l_1 N\omega\dot{\phi}\left(\frac{\theta_0^2\cos(\omega t)}{1}-\frac{\theta_0^4\cos^3(\omega t)}{6}\right)\sin(\omega t).$$

Factoring out the $\cos(\omega t)$ and using a trig-identity we get

$$-2l_1N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq l_1N\omega\dot{\phi}\left(\frac{\theta_0^2}{1} - \frac{\theta_0^4\cos^2(\omega t)}{6}\right)\sin(2\omega t) \tag{41}$$

Next, substituting the approximations from equation (32) into equation (41) yields

$$-2l_1N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq l_1N\omega\dot{\phi}\left[\theta_0^2 - \frac{1}{6}\theta_0^4\left(\frac{1}{2}\cos(2\omega t) + \frac{1}{2}\right)\right]\sin(2\omega t).$$

Distributing the factors $-(1/6)\theta_0^4$, $\sin(2\omega t)$, and using a trig-identity we have

$$-2l_1N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq l_1N\omega\dot{\phi}\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\sin(2\omega t) - l_1N\omega\dot{\phi}\frac{1}{12}\theta_0^4\left(\frac{1}{2}\sin(4\omega t)\right). \tag{42}$$

Remember that we only keep the terms with frequencies of 2ω for the parametric terms because terms at multiples of the 2ω frequency have very little effect, thus, we approximate equation (42) as

$$-2l_1N\sin(\theta)\dot{\phi}\dot{\theta} \approxeq l_1N\omega\dot{\phi}\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\sin(2\omega t). \tag{43}$$

Expanding the next term on the right hand side of equation (24) we get

$$2l_1N\cos(\theta)\ddot{\phi} \approxeq 2l_1N\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\right)\ddot{\phi}.$$

Substituting in $\theta_0\cos(\omega t)$ in for θ yields

$$2l_1N\cos(\theta)\ddot{\phi} \approxeq 2l_1N\ddot{\phi}\left(1 - \frac{\theta_0^2\cos^2(\omega t)}{2} + \frac{\theta_0^4\cos^4(\omega t)}{24}\right). \tag{44}$$

Next, substituting the approximations from equation (32) into equation (44) yields

$$2l_1N\cos(\theta)\ddot{\phi} \approxeq 2l_1N\ddot{\phi}\left[1 - \frac{1}{4}\theta_0^2\cos(2\omega t) - \frac{1}{4}\theta_0^2 + \frac{1}{48}\theta_0^4\cos(2\omega t) + \frac{1}{64}\theta_0^4\right].$$

Finally, grouping like terms, factoring out the $\cos(\omega t)$ and distributing the factor $N\ddot{\phi}$ yields

$$2l_1N\cos(\theta)\ddot{\phi} \approxeq 2l_1N\ddot{\phi}\left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) - \frac{1}{2}l_1N\ddot{\phi}\left[\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\cos(2\omega t)\right]. \tag{45}$$

By substituting the terms, computed in equations (26), (33), (35), (37), (39) (41), (43), and (45) into equation (24) gives

$$
\begin{aligned}
&(I_1 + I_2)\ddot{\phi} + Mgl_1\phi \approxeq \\
&\omega^2 I_2 \theta_0 \cos(\omega t) - \frac{1}{4} l_1 N \omega^2 \left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) \cos(\omega t) \\
&- l_1 N \omega^2 \left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right) \cos(\omega t) + Ng \left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right) \cos(\omega t) \\
&+ Ng\phi \left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) - \frac{1}{4} Ng\phi \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) \cos(2\omega t) + l_1 N \omega \dot{\phi} \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) \sin(2\omega t) \\
&+ 2 l_1 N \ddot{\phi} \left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) - \frac{1}{2} l_1 N \ddot{\phi} \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) \cos(2\omega t).
\end{aligned}
$$

Grouping like terms and factoring out periodic factors of ωt and $2\omega t$ we get

$$
\begin{aligned}
&(I_1 + I_2)\ddot{\phi} + Mgl_1\phi \approxeq \\
&\left[\omega^2 I_2 \theta_0 - \frac{1}{4} l_1 N \omega^2 \left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) - l_1 N \omega^2 \left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right) + Ng \left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right)\right] \cos(\omega t) \\
&+ \left[-\frac{1}{4} Ng\phi \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) - \frac{1}{2} l_1 N \ddot{\phi} \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\right] \cos(2\omega t) \\
&+ l_1 N \omega \dot{\phi} \left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) \sin(2\omega t) \\
&+ Ng\phi \left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) + 2 l_1 N \ddot{\phi} \left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right).
\end{aligned}
$$

Grouping the parametric terms on the right hand side of the equation in terms of $\phi, \dot{\phi}$, and $\ddot{\phi}$ we get

$$\begin{aligned}
&(I_1 + I_2)\ddot{\phi} + Mgl_1\phi \approxeq \\
&\quad Ng\phi\left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) + 2l_1N\ddot{\phi}\left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) \\
&\quad + \left[\omega^2 I_2\theta_0 - \frac{1}{4}l_1N\omega^2\left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) - l_1N\omega^2\left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right)\right. \\
&\quad \left. + Ng\left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right)\right]\cos(\omega t) \\
&\quad - \frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\cos(2\omega t)\phi \\
&\quad + l_1N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\sin(2\omega t)\dot{\phi} \\
&\quad - \frac{1}{2}l_1N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\cos(2\omega t)\ddot{\phi}
\end{aligned}$$

To have a differential equation that is easy to work with, the equation should have the form

$$\ddot{\phi} + \omega\phi \approxeq F\cos(\omega t) + A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}, \tag{46}$$

where the terms are grouped according to their dependance on ϕ. To do this, we begin by moving the terms without a periodic factor varying with the natural frequency to the left hand side of the equation we get

$$\begin{aligned}
&(I_1 + I_2)\ddot{\phi} + Mgl_1\phi + Ng\phi\left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) + 2l_1N\ddot{\phi}\left(1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4\right) \approxeq \\
&\left[\omega^2 I_2\theta_0 - \frac{1}{4}l_1N\omega^2\left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) - l_1N\omega^2\left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right) + Ng\left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right)\right]\cos(\omega t) \\
&- \frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\cos(2\omega t)\phi \\
&+ l_1N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\sin(2\omega t)\dot{\phi} \\
&- \frac{1}{2}l_1N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)\cos(2\omega t)\ddot{\phi}.
\end{aligned}$$

We then combine the like terms on the right hand side of the equation and factor out the ϕ and $\ddot{\phi}$ factors

$$
\begin{aligned}
&\left[(I_1+I_2)+2l_1N\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right)\right]\ddot{\phi}+\left[Mgl_1+Ng\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right)\right]\phi \approxeq \\
&\left[\omega^2I_2\theta_0-\frac{1}{4}l_1N\omega^2(\theta_0^3-\frac{1}{12}\theta_0^5)-l_1N\omega^2\left(\theta_0-\frac{3}{8}\theta_0^3+\frac{5}{192}\theta_0^5\right)+Ng\left(\theta_0-\frac{1}{8}\theta_0^3+\frac{1}{192}\theta_0^5\right)\right]\cos(\omega t) \\
&-\frac{1}{4}Ng\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\cos(2\omega t)\phi \\
&+l_1N\omega\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\sin(2\omega t)\dot{\phi} \\
&-\frac{1}{2}l_1N\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\cos(2\omega t)\ddot{\phi}.
\end{aligned}
\tag{47}
$$

Next, we make the following substitutions:

$$
\begin{aligned}
I_0 &= (I_1+I_2)+2l_1N\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right) \\
K_0 &= Mgl_1+Ng\left(1-\frac{1}{4}\theta_0^2+\frac{1}{64}\theta_0^4\right).
\end{aligned}
\tag{48}
$$

Making the substitutions of equation (48) into equation (47) yields

$$
\begin{aligned}
&I_0\ddot{\phi}+K_0\phi \approxeq \\
&\left[\omega^2I_2\theta_0-\frac{1}{4}l_1N\omega^2\left(\theta_0^3-\frac{1}{12}\theta_0^5\right)-l_1N\omega^2\left(\theta_0-\frac{3}{8}\theta_0^3+\frac{5}{192}\theta_0^5\right)+Ng\left(\theta_0-\frac{1}{8}\theta_0^3+\frac{1}{192}\theta_0^5\right)\right]\cos(\omega t) \\
&-\frac{1}{4}Ng\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\cos(2\omega t)\phi \\
&+l_1N\omega\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\sin(2\omega t)\dot{\phi} \\
&-\frac{1}{2}l_1N\left(\theta_0^2-\frac{1}{12}\theta_0^4\right)\cos(2\omega t)\ddot{\phi}.
\end{aligned}
$$

Dividing both sides of the equation by I_0 we have

$$
\begin{aligned}
&\ddot{\phi} + [K_0/I_0]\phi \approxeq \\
&\Bigg(\bigg[\omega^2 I_2\theta_0 - \frac{1}{4}l_1 N\omega^2\left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) - l_1 N\omega^2\left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right) \\
&+ Ng\left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right)\bigg]/I_0\Bigg)\cos(\omega t) \\
&- \left[\frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right]\cos(2\omega t)\phi \\
&+ \left[l_1 N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right]\sin(2\omega t)\dot{\phi} \\
&- \left[\frac{1}{2}l_1 N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right]\cos(2\omega t)\ddot{\phi}.
\end{aligned}
$$

Simplifying the coefficient on the $\cos(\omega t)$ term yields

$$
\begin{aligned}
&\left(\omega^2 I_2\theta_0 - \frac{1}{4}l_1 N\omega^2\left(\theta_0^3 - \frac{1}{12}\theta_0^5\right) - l_1 N\omega^2\left(\theta_0 - \frac{3}{8}\theta_0^3 + \frac{5}{192}\theta_0^5\right) + Ng\left(\theta_0 - \frac{1}{8}\theta_0^3 + \frac{1}{192}\theta_0^5\right)\right)/I_0 \\
&= \theta_0\left[\left(\omega^2 I_2 - \frac{1}{4}l_1 N\omega^2\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) - l_1 N\omega^2\left(1 - \frac{3}{8}\theta_0^2 + \frac{5}{192}\theta_0^4\right) + Ng\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right)/I_0\right] \\
&= \theta_0\left[\left(\omega^2 I_2 + N\left[-\frac{1}{4}l_1\omega^2\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) - l_1\omega^2\left(1 - \frac{3}{8}\theta_0^2 + \frac{5}{192}\theta_0^4\right) + g\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right] \\
&= \theta_0\left[\left(\omega^2 I_2 + N\left[l_1\omega^2\left[-\frac{1}{4}\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right) - \left(1 - \frac{3}{8}\theta_0^2 + \frac{5}{192}\theta_0^4\right)\right] + g\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right] \\
&= \theta_0\left[\left(\omega^2 I_2 + N\left[l_1\omega^2\left(-\frac{2}{8}\theta_0^2 - \frac{4}{192}\theta_0^4 - 1 + \frac{3}{8}\theta_0^2 - \frac{5}{192}\theta_0^4\right) + g\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right] \\
&= \theta_0\left[\left(\omega^2 I_2 + N\left[l_1\omega^2\left(-1 + \frac{1}{8}\theta_0^2 - \frac{1}{192}\theta_0^4\right) + g\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right] \\
&= \theta_0\left[\left(\omega^2 I_2 + N\left[(g - l_1\omega^2)\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right].
\end{aligned}
$$

The equation then becomes

$$\ddot{\phi} + (K_0/I_0)\,\phi \approxeq$$
$$\theta_0\left[\left(\omega^2 I_2 + N\left[(g - l_1\omega^2)\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right]\cos(\omega t)$$
$$-\left(\frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right)\cos(2\omega t)\phi$$
$$+\left(l_1 N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right)\sin(2\omega t)\dot{\phi}$$
$$-\left(\frac{1}{2}l_1 N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right)\cos(2\omega t)\ddot{\phi}.$$

Finally, letting

$$\begin{aligned}
\omega_0 &= K_0/I_0 \\
F &= \theta_0\left[\left(\omega^2 I_2 + N\left[(g - l_1\omega^2)\left(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4\right)\right]\right)/I_0\right] \\
A &= -\frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0 \\
B &= l_1 N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0 \\
C &= -\frac{1}{2}l_1 N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0,
\end{aligned} \tag{49}$$

we get the following equation by substituting equation (49) into equation (48)

$$\ddot{\phi} + \omega_0^2\phi \approxeq F\cos(\omega t) + A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}. \tag{50}$$

7. Analysis of the Driving and Parametric Terms

To better understand the system, we will look at the system as two separate systems: one with a harmonic oscillator with a driving term and the other as a parametric oscillator. The forced harmonic system displays an increase in amplitude from rest or from a small initial value of ϕ . The parametric system displays an increase in amplitude only from a non-zero initial value of ϕ. It is for this reason that it is expected that the driving force will have a dominant effect over the motion of the system for small values of ϕ whereas the parametric terms will have a minor effect over the system for these small values of ϕ. Because ϕ is small and the driving term is dominant, the system will be approximated by the equation

$$\ddot{\phi} + \omega_0^2\phi = F\cos(\omega t), \tag{51}$$

in which the last three terms of equation (49) are dropped. However, as ϕ increases in magnitude, and because of the previous assumption for the forced-harmonic system, the parametric terms should have a dominant effect on the motion of the system, thus the system will be approximated as

$$\ddot{\phi} + \omega_0^2\phi = A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}. \tag{52}$$

where the first terms of equation (49) is dropped.

7.1. The Effects of Dropping Terms

Before we approximate the solutions of the differential equation by approximating the system as two different systems, we have to look at if this approach is even reasonable. An answer to these considerations is the energy of the system. This will be the subject of focus. First, both sides of the equation are multiplied by the integrating factor $\dot{\phi}$.

$$\begin{aligned}\ddot{\phi}\dot{\phi} + \omega_0^2\phi\dot{\phi} &\approxeq F\cos(\omega t)\dot{\phi} + A\cos(2\omega t)\phi\dot{\phi} + B\sin(2\omega t)\dot{\phi}^2 + C\cos(2\omega t)\ddot{\phi}\dot{\phi}\\ \frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega_0^2\phi^2\right) &\approxeq F\cos(\omega t)\dot{\phi} + A\cos(2\omega t)\phi\dot{\phi} + B\sin(2\omega t)\dot{\phi}^2 + C\cos(2\omega t)\ddot{\phi}\dot{\phi}.\end{aligned} \tag{53}$$

It is assumed that will have an approximate solution $\phi \approxeq \phi_0\cos(\omega t + \delta)$, where ωt is the frequency and δ is the phase shift. Both the angle ϕ_0 and δ change slowly with respect to t. Substituting this value of ϕ into the differential equation we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega_0^2\phi^2\right) \approxeq\ & F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]'\\ &+ A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]'\\ &+ B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\ &+ C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]'.\end{aligned}$$

The terms on the left hand side of the equation are the rate of the change of the energy in the oscillatory system with respect to time.[1] The four terms on the right hand side of the equation are the means by which system is driven. We want to write these driving and parametric terms as an expression of periodic functions where the terms will be at a minimum when ϕ is zero. This is done by rewriting the them in terms of the sin function.

Starting with the driving term, we factor out the ϕ_0 and take the derivative which yields

$$F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]' = F\phi_0\cos(\omega t)[-\omega\sin(\omega t+\delta)].$$

Then, factoring out the $-\omega$ we have

$$F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]' = -\omega F\phi_0\cos(\omega t)\sin(\omega t+\delta).$$

Next, using the product to sum trig-identity we get

$$F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]' = -\omega F\phi_0\left(\frac{1}{2}\sin[(\omega t+\delta)-(\omega t)]+\frac{1}{2}\sin[(\omega t+\delta)+(\omega t)]\right),$$

and factoring out the one-half yields

$$F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]' = -\frac{1}{2}\omega F\phi_0[\sin([\omega t+\delta]-[\omega t])+\sin([\omega t+\delta]+[\omega t])].$$

Lastly, combing like terms inside the periodic function we have

$$F\cos(\omega t)[\phi_0\cos(\omega t+\delta)]' = -\frac{1}{2}\omega F\phi_0[\sin(\delta)+\sin(2\omega t+\delta)]. \tag{54}$$

Next, we rewrite the parametric term with an amplitude of A starting by factoring out the ϕ_0 and taking the derivative, which yields

$$A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]' = A\phi_0^2\cos(2\omega t)\cos(\omega t+\delta)[-\omega\sin(\omega t+\delta)].$$

Then, factoring out the $-\omega$ we have

$$A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]' = -\omega A\phi_0^2\cos(2\omega t)\cos(\omega t+\delta)\sin(\omega t+\delta).$$

Next, using the product to sum trig-identity we get

$$A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]' = -\omega A\phi_0^2\cos(2\omega t)\left(\frac{1}{2}\sin(2\omega t+2\delta)\right),$$

and factoring out the one-half yields

$$A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]' = -\frac{1}{2}\omega A\phi_0^2\cos(2\omega t)\sin(2\omega t+2\delta).$$

Again, using the product to sum trig-identity we have

$$\begin{aligned}&A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]'\\&\quad= -\frac{1}{2}\omega A\phi_0^2\left(\frac{1}{2}\sin([2\omega t+2\delta]-[2\omega t])+\frac{1}{2}\sin([2\omega t+2\delta]+[2\omega t])\right).\end{aligned}$$

Finally, combing like terms inside the periodic function and factoring out the one-half we get

$$A\cos(2\omega t)[\phi_0\cos(\omega t+\delta)][\phi_0\cos(\omega t+\delta)]' = -\frac{1}{4}\omega A\phi_0^2[\sin(2\delta)+\sin(4\omega t+2\delta)]. \tag{55}$$

Thirdly, rewriting the parametric term with an amplitude of B starting by distributing the square and factoring out the ϕ_0^2 and taking the derivative, which yields

$$B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2 = B\phi_0^2\sin(2\omega t)[-\omega\sin(\omega t+\delta)]^2.$$

Next, distributing the square and factoring out the ω_0^2 we get

$$B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2 = \omega B\phi_0^2\sin(2\omega t)\sin(\omega t+\delta)\sin(\omega t+\delta).$$

Then, using the product to sum trig-identity we have

$$\begin{aligned}&B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\&\quad= \omega B\phi_0^2\left(\frac{1}{2}\cos([\omega t+\delta]-[2\omega t])-\frac{1}{2}\cos([\omega t+\delta]+[2\omega t])\right)\sin(\omega t+\delta).\end{aligned}$$

Combing like terms inside the periodic function and factoring out the one-half yields

$$B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2 = \frac{1}{2}\omega B\phi_0^2[\cos(-\omega t+\delta)-\cos(3\omega t+\delta)]\sin(\omega t+\delta).$$

Next, distributing the sin factor we get

$$\begin{aligned}&B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\&\quad= \frac{1}{2}\omega B\phi_0^2[\cos(-\omega t+\delta)\sin(\omega t+\delta)-\cos(3\omega t+\delta)\sin(\omega t+\delta)].\end{aligned}$$

Again, using the product to sum trig-identity and distributing the $\frac{1}{2}\omega B\phi_0^2$ factor we have

$$\begin{aligned}
&B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\
&\quad=\frac{1}{2}\omega B\phi_0^2\left(\frac{1}{2}\sin([\omega t+\delta]-[-\omega t+\delta])+\frac{1}{2}\sin([\omega t+\delta]+[-\omega t+\delta])\right)\\
&\qquad-\frac{1}{2}\omega B\phi_0^2\left(\frac{1}{2}\sin([\omega t+\delta]-[3\omega t+\delta])+\frac{1}{2}\sin([\omega t+\delta]+[3\omega t+\delta])\right).
\end{aligned}$$

Combing like terms and factoring out the one-half yields

$$\begin{aligned}
&B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\
&\quad=\frac{1}{4}\omega B\phi_0^2[\sin(2\omega t)+\sin(2\delta)]\\
&\quad-\frac{1}{4}\omega B\phi_0^2[\sin(-2\omega t)+\sin(4\omega t+2\delta)].
\end{aligned}$$

Next, factoring out the $\frac{1}{4}\omega B\phi_0^2$ we get

$$\begin{aligned}
&B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2\\
&\quad=\frac{1}{4}\omega B\phi_0^2[\sin(2\omega t)+\sin(2\delta)-\sin(-2\omega t)-\sin(4\omega t+2\delta)],
\end{aligned}$$

and finally, addition of like terms results in

$$B\sin(2\omega t)([\phi_0\cos(\omega t+\delta)]')^2=\frac{1}{4}\omega B\phi_0^2[\sin(2\delta)+2\sin(2\omega t)-\sin(4\omega t+2\delta)], \tag{56}$$

Lastly, rewriting the parametric term with an amplitude of C starting by distributing the square and factoring out the ϕ_0^2 and taking the derivative of the periodic factors, which yields

$$\begin{aligned}
&C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]'\\
&\quad=C\phi_0^2\cos(2\omega t)[-\omega\sin(\omega t+\delta)]'[-\omega\sin(\omega t+\delta)].
\end{aligned}$$

Then, factoring out the ω^2 and taking the second derivative of the middle periodic function we get

$$\begin{aligned}
&C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]'\\
&\quad=\omega^2C\phi_0^2\cos(2\omega t)[\omega\cos(\omega t+\delta)]\sin(\omega t+\delta).
\end{aligned}$$

Next, factoring out the ω and using the product to sum trig-identity we have

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \omega^3 C\phi_0^2\left(\frac{1}{2}\sin([\omega t+\delta]-[2\omega t]) + \frac{1}{2}\sin([\omega t+\delta]+[2\omega t])\right)\cos(\omega t+\delta). \end{aligned}$$

Combing like terms and factoring out the one-half yields

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \frac{1}{2}\omega^3 C\phi_0^2[\sin(-\omega t+\delta)+\sin(3\omega t+\delta)]\cos(\omega t+\delta). \end{aligned}$$

Distributing the $\cos(\omega t+\delta)$ term we get

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \frac{1}{2}\omega^3 C\phi_0^2[\sin(-\omega t+\delta)\cos(\omega t+\delta)+\sin(3\omega t+\delta)\cos(\omega t+\delta)]. \end{aligned}$$

Again, using the product to sum trig-identity an distributing the factor $\frac{1}{2}\omega^3 C\phi_0^2$ we have

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \frac{1}{2}\omega^3 C\phi_0^2\left(\frac{1}{2}\sin([-\omega t+\delta]-[\omega t+\delta]) + \frac{1}{2}\sin([-\omega t+\delta]+[\omega t+\delta])\right) \\ &\qquad + \frac{1}{2}\omega^3 C\phi_0^2\left(\frac{1}{2}\sin([3\omega t+\delta]-[\omega t+\delta]) + \frac{1}{2}\sin([3\omega t+\delta]+[\omega t+\delta])\right). \end{aligned}$$

Combing like terms and factoring out the one-half yields

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \frac{1}{4}\omega^3 C\phi_0^2[\sin(-2\omega t)+\sin(2\delta)] + \frac{1}{4}\omega^3 C\phi_0^2[\sin(2\omega t)+\sin(4\omega t+2\delta)]. \end{aligned}$$

Factoring out the $\frac{1}{4}\omega^3 C\phi_0^2$ term we get

$$\begin{aligned} &C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' \\ &\quad = \frac{1}{4}\omega^3 C\phi_0^2[-\sin(2\omega t)+\sin(2\delta)+\sin(2\omega t)+\sin(4\omega t+2\delta)], \end{aligned}$$

and combining like terms results in

$$C\cos(2\omega t)[\phi_0\cos(\omega t+\delta)]''[\phi_0\cos(\omega t+\delta)]' = \frac{1}{4}\omega^3 C\phi_0^2[\sin(2\delta)+\sin(4\omega t+2\delta)]. \tag{57}$$

Thus, substituting the equations, computed in equations (54), (55), (56), and (57) into equation (53) we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2+\frac{1}{2}\omega_0^2\phi^2\right) \approx& -\frac{1}{2}\omega F\phi_0[\sin(\delta)+\sin(2\omega t+\delta)]\\ &-\frac{1}{4}\omega A\phi_0^2[\sin(2\delta)+\sin(4\omega t+2\delta)]\\ &+\frac{1}{4}\omega B\phi_0^2[\sin(2\delta)+2\sin(2\omega t)-\sin(4\omega t+2\delta)]\\ &+\frac{1}{4}\omega^3 C\phi_0^2[\sin(2\delta)+\sin(4\omega t+2\delta)].\end{aligned} \tag{58}$$

It is assumed that ϕ_0 and δ vary slowly with time. Also, because ϕ_0 and δ vary slowly with respect to time, it is apparent that for small values of ϕ the main variation in the energy of the system comes almost solely from the the driving term.[1] This then means that we are able to overlook the terms $\dot{\phi}_0$, $\ddot{\phi}_0$, $\dot{\delta}$, and $\ddot{\delta}$.

8. Looking At The Change In Regimes

Before ϕ passes some critical angle $\phi_{critical}$, the driving term contributes more to the growth of the amplitude in the system than the parametric terms contribute. This is representative of when the growth contributed by the driving term in greater than the growth contributed by the parametric terms ($\Gamma_D > \Gamma_P$). When ϕ is greater than the critical angle the driving term contributes less to the growth of the amplitude in the system and the parametric terms contribute more ($\Gamma_D < \Gamma_P$). At the critical angle ($\phi_{critical}$), both the driving term and the parametric terms contribute equally to the growth of the amplitude ($\Gamma_D = \Gamma_P$).

The value of this critical angle is dependant on the parameters chosen such as the distribution of the mass, the lengths of the massless rigid rods, and the amplitude of θ defined to be θ_0. The energy contained in the system is proportional to the amplitude of ϕ. When the ϕ is small, there is little kinetic and potential energy in the system as a result of the parametric terms. As the value of ϕ increases, more and more energy is introduced into the system, and this energy is introduced by the driving term.

8.1. Looking At the Forced Harmonic System

Because the parametric terms are dependant on ϕ, for small values of ϕ we can consider the system to have values that of a harmonic oscillator system with the one driving term. Given that the initial conditions are $\phi(0) = 0$ and that $\dot{\phi}(0) = 0$, we solve the differential equation for this system by first finding the general solution to the homogeneous equation (50)

$$\ddot{\phi} + \omega_0^2 \phi = F\cos(\omega t).$$

Setting the driving term, $F\cos(\omega t)$, equal to zero yields

$$\ddot{\phi} + \omega_0^2 \phi = 0.$$

We find the characteristic polynomial for the system by substituting $e^{\lambda t}$ in for ϕ

$$(e^{\lambda t})'' + \omega_0^2 (e^{\lambda t}) = 0.$$

Next, taking the derivatives of the second equation we get

$$(\lambda^2 e^{\lambda t}) + \omega_0^2 (e^{\lambda t}) = 0.$$

Then, dividing by $e^{\lambda t}$ we obtain

$$\lambda^2 + \omega_0^2 = 0.$$

Solving for λ we use the quadratic formula

$$\lambda = \pm \frac{(-4\omega_0^2)^{1/2}}{2}.$$

Factoring yields

$$\lambda = \pm \omega_0 i.$$

Therefore, the homogeneous solution is

$$\phi_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t). \tag{59}$$

Next, we find a particular solution to equation (50) by looking for a solution of the form $Ze^{i\omega t}$.

$$\ddot{\phi} + \omega_0^2 \phi = Fe^{i\omega t} \tag{60}$$

By setting

$$\phi = Ze^{i\omega t}, \tag{61}$$

we get

$$\dot{\phi} = Zi\omega e^{i\omega t},$$

and

$$\ddot{\phi} = -Z\omega^2 e^{i\omega t},$$

Then, substituting these equations into the equation we get

$$-Z\omega^2 e^{i\omega t} + \omega_0^2 Ze^{i\omega t} = Fe^{i\omega t}. \tag{62}$$

Next, factoring out common terms we get

$$Ze^{i\omega t}(\omega_0^2 - \omega^2) = Fe^{i\omega t}.$$

Dividing both sides by the factor $e^{i\omega t}$ the equation is simplified to

$$Z(\omega_0^2 - \omega^2) = F,$$

and finally, solving for Z we have

$$Z = \frac{F}{\omega_0^2 - \omega^2}.$$

Next, substituting the value of Z back into the equation (61) we have

$$\begin{aligned}\phi &= \left(\frac{F}{\omega_0^2 - \omega^2}\right) e^{i\omega t} \\ &= \left(\frac{F}{\omega_0^2 - \omega^2}\right) [\cos(\omega t) + i\sin(\omega t)].\end{aligned}$$

Taking the real part of the right hand side we get

$$\phi_p = \left(\frac{F}{\omega_0^2 - \omega^2}\right) \cos(\omega t). \tag{63}$$

Thus, the general solution, from equations(59) and (63) for the harmonic system is[5]

$$\begin{aligned}\phi =&\phi_h + \phi_p \\ =&C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \left(\frac{F}{\omega_0^2 - \omega^2}\right) \cos(\omega t).\end{aligned} \tag{64}$$

Next, solving for the constants C_1 and C_2 by using the initial conditions, $\phi(0) = 0$, we have

$$\begin{aligned}\phi(0) =&C_1 \cos(0) + C_2 \sin(0) + \left(\frac{F}{\omega_0^2 - \omega^2}\right) \cos(0) \\ 0 =&C_1(1) + C_2(0) + \left(\frac{F}{\omega_0^2 - \omega^2}\right) \\ C_1 =& -\left(\frac{F}{\omega_0^2 - \omega^2}\right)\end{aligned}$$

Then, we solve for the remaining constant by using $\dot{\phi}_0$. Taking the derivative of ϕ with respect to time we get

$$\begin{aligned}\dot{\phi} =&\dot{\phi}_h + \dot{\phi}_p \\ =& -C_1 \omega_0 \sin(\omega_0 t) + C_2 \omega_0 \cos(\omega_0 t) - \left(\frac{F\omega_0}{\omega_0^2 - \omega^2}\right) \sin(\omega t).\end{aligned}$$

Again, using the initial condition that $\dot{\phi}(0) = 0$, we have

$$\begin{aligned}\dot{\phi}(0) =& -C_1 \omega_0 \sin(0) + C_2 \omega_0 \cos(0) - \left(\frac{F\omega_0}{\omega_0^2 - \omega^2}\right) \sin(0) \\ 0 =& -C_1(0) + C_2 \omega_0(1) - (0) \\ C_2 =&0.\end{aligned}$$

Substituting these values for C_1 and C_2 into equation (64) we obtain

$$\begin{aligned}\phi =& -\left(\frac{F}{\omega_0^2 - \omega^2}\right) \cos(\omega_0 t) + (0) \sin(\omega_0 t) + \left(\frac{F}{\omega_0^2 - \omega^2}\right) \cos(\omega t) \\ =& \left(\frac{F}{\omega_0^2 - \omega^2}\right) [\cos(\omega t) - \cos(\omega_0 t)]\end{aligned}$$

We also need to consider that ω could be very close to ω_0. If this assumption is made, we have a solution of the form $Zte^{i\omega t}$. Setting $\omega = \omega_0$ we have

$$\ddot{\phi} + \omega_0^2 \phi = Fe^{i\omega_0 t}$$

By setting

$$\phi = Zte^{i\omega_0 t},$$

we get

$$\begin{aligned}\dot{\phi} =& Z[ti\omega_0 e^{i\omega_0 t} + e^{i\omega_0 t}] \\ =& Ze^{i\omega_0 t}[ti\omega_0 + 1],\end{aligned}$$

and

$$\begin{aligned}\ddot{\phi} =& Z[i\omega_0(ti\omega_0 e^{i\omega_0 t} + e^{i\omega_0 t}) + i\omega_0 e^{i\omega_0 t}] \\ =& Z[i\omega_0 e^{i\omega_0 t}(ti\omega_0 + 1) + i\omega_0 e^{i\omega_0 t}] \\ =& Zi\omega_0 e^{i\omega_0 t}(ti\omega_0 + 2).\end{aligned}$$

Then, substituting these values for ϕ and $\ddot{\phi}$ into equation (60) yields

$$Zi\omega_0 e^{i\omega_0 t}(ti\omega_0 + 2) + \omega_0^2[Zte^{i\omega_0 t}] = Fe^{i\omega_0 t}.$$

Next, factoring out a $Ze^{i\omega t}$ we get

$$Ze^{i\omega_0 t}[(ti^2\omega_0^2 + 2i\omega_0) + (t\omega_0^2)] = Fe^{i\omega_0 t}.$$

Dividing both sides by the factor $e^{i\omega t}$ the equation is simplified to

$$Z[-t\omega_0^2 + 2i\omega_0 + t\omega_0^2] = F.$$

Cancelling terms we have

$$Z2i\omega_0 = F,$$

and finally, solving for Z we have

$$Z = \frac{F}{2i\omega_0}$$

Next, substituting the value of Z back into equation (61) we have

$$\begin{aligned}\phi &= \left(\frac{F}{2i\omega_0}\right) te^{i\omega_0 t} \\ &= \left(\frac{F}{2i\omega_0}\right) t[\cos(\omega_0 t) + i\sin(\omega_0 t)].\end{aligned}$$

Taking the real part of the right hand side we get

$$\phi_p = \left(\frac{Ft}{2i\omega_0}\right) i\sin(\omega_0 t),$$

and cancelling terms we have

$$\phi_p = \left(\frac{Ft}{2\omega_0}\right) \sin(\omega_0 t). \tag{65}$$

From looking at the equations, we can see that the growth per period is linear in the beginning stages of the system. The period is

$$T = \frac{2\pi}{\omega_0}. \tag{66}$$

Substituting this value of the period into the amplitude for the time value yields the growth per cycle approximated by just the harmonic driving term

$$\begin{aligned}\Gamma_D =& \frac{F(2\pi/\omega_0)}{2\omega_0} \\ =& \frac{F\pi}{\omega_0^2}.\end{aligned} \tag{67}$$

Because the driving term is independent of ϕ, for small values of ϕ we can consider the system to have values that of a driven oscillator system with only the driving term. To see this visually, compare Figures 3 and 4.

The following Matlab M-file plots the models for the forced harmonic system and the actual system with the driving and parametric terms. The graphs are Figures 3, 4, 5, and 6. Note the the y-scales are arbitrary.

```
function motion(M,m1,m2,m3,L1,L2,L3,th0,Ti,Tf1,Tf2,Tf3)
```

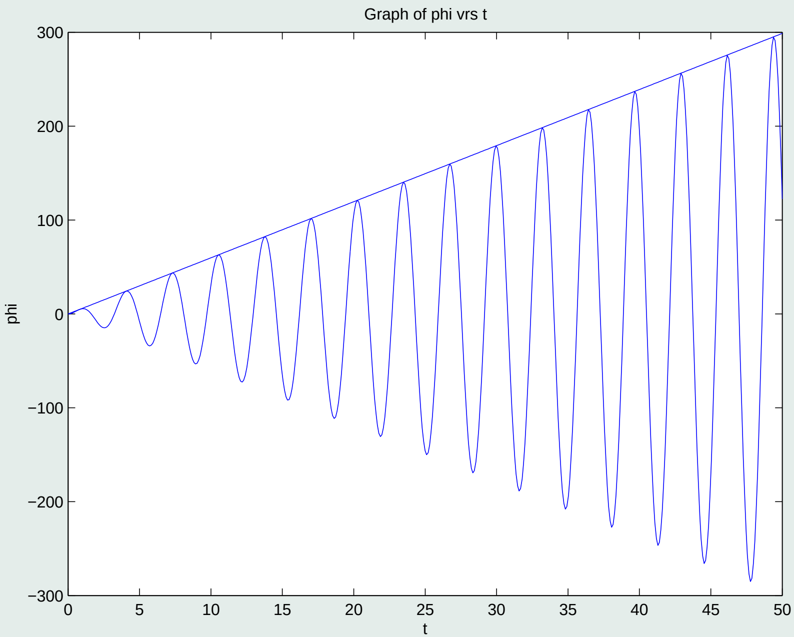

Figure 3: The forced harmonic system.

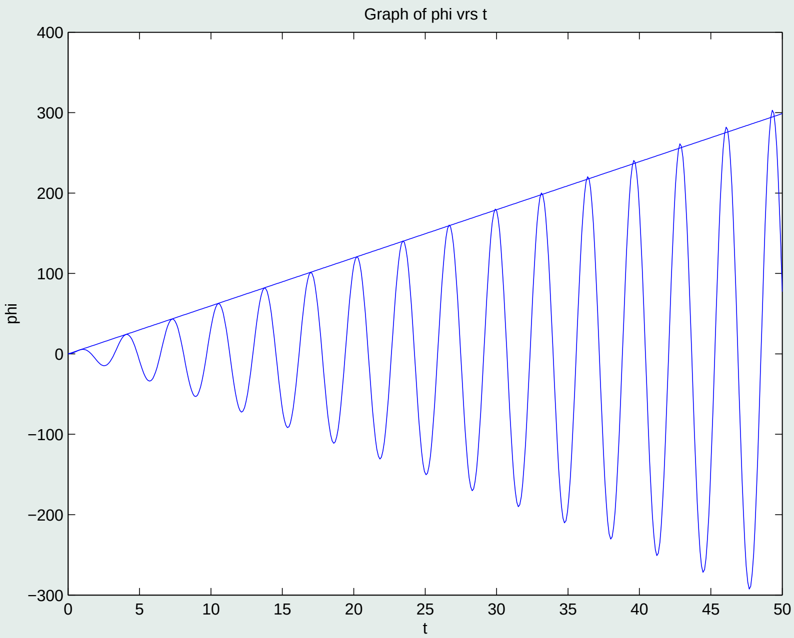

Figure 4: The actual approximated system.

```
% Closes all figures
close all
% Sets the default parameters:
    % M........is the total mass of the system.
    % m1.......is the center mass.
    % m2.......is the right mass.
    % m3.......is the left mass.
    % L1.......is the length between the center of the system and m1.
    % L2.......is the length between m1 and m2.
    % L3.......is the length between m1 and m3.
    % th0......is the initial theta angle.
    % Ti.......is the initial time.
    % Tf.......is the final time.
if nargin < 1
    M = 100;
    m1 = 0.4*M;
    m2 = 0.4*M;
    m3 = 0.2*M;
    L1 = 2.5;
    L2 = 0.4;
    L3 = L2;
    th0 = 0.7;
    Ti = 0;
    Tf1 = 50;
    Tf2 = 200;
    Tf3 = 400;
end
% Condenses combinations of like terms.
g=9.8; I1=M*L1^2; I2=m2*L2^2+m3*L3^2; N=m3*L3-m2*L2;
I0=(I1+I2)-2*L1*N*(1-(1/4)*th0^2+(1/64)*th0^4);
K0=(M*L1*g)-N*g*(1-(1/4)*th0^2+(1/64)*th0^4);
w0=(abs(K0/I0))^(1/2);
% Sets the frequency of the driving and parametric terms equal to the
% natural frequency.
```

```
w = w0
% Sets the oscillatory terms for the system:
    % F........is the coefficient on the driving term cos(wt).
    % A........is the coefficient on the parametric term cos(2wt)*phi.
    % B........is the coefficient on the parametric term sin(2wt)*phi'.
    % C........is the coefficient on the parametric term cos(2wt)*phi''.
F=th0*(w^2*I2+N*((g-w0^2*L1)*(1-(1/8)*th0^2+(1/192)*th0^4)));
A=-((1/4)*N*g*(th0^2-(1/12)*th0^4))/I0;
B=(L1*N*w*(th0^2-(1/12)*th0^4))/I0;
C=-((1/4)*L1*N*(th0^2-(1/12)*th0^4))/I0; Az=0; Bz=0; Cz=0;
L=((F*pi)/((w0)^2))/((2*pi)/(w0));
% Solving for phi'':
    % phi'' + (w0)^2(phi) = Fcos(wt) + Acos(2wt)phi + Bsin(2wt)phi' + Ccos(2wt)phi''
    % phi'' - Ccos(2wt)phi'' = Fcos(wt) + Acos(2wt)phi - (w0)^2(phi) + Bsin(2wt)phi'
    % phi''(1 - Ccos(2wt)) = Fcos(wt) + phi(Acos(2wt) - (w0)^2) + Bsin(2wt)phi'
    % phi'' = (Fcos(wt) + phi(Acos(2wt) - (w0)^2) + Bsin(2wt)phi')/(1 - Ccos(2wt))
% Sets the first time interval.
tspan1 = [Ti,Tf1];
% Sets the first time interval.
tspan2 = [Ti,Tf2];
% Sets the first time interval.
tspan3 = [Ti,Tf3];
% Sets the initial conditions for phi(0) and phi'(0).
x0 = [0;0];
% Sets options equal to an empty set.
options = [];
% Calls ode45 to solve the system of first order odes on tspan1.
[t,x] = ode45(@pz,tspan1,x0,options,Az,Bz,Cz,F,w0,w);
% Plots the data.
figure plot(t,x(:,1)) xlabel('t') ylabel('phi') title('Graph of
phi vrs t')
hold on plot(t,L*t)
% Calls ode45 to solve the system of first order odes on tspan1.
[t,x] = ode45(@ph,tspan1,x0,options,A,B,C,F,w0,w);
```

```
% Plots the data.
figure plot(t,x(:,1)) xlabel('t') ylabel('phi') title('Graph of
phi vrs t')
hold on plot(t,L*t)
% Calls ode45 to solve the system of first order odes on tspan2.
[t,x] = ode45(@ph,tspan2,x0,options,A,B,C,F,w0,w);
% Plots the data.
figure plot(t,x(:,1)) xlabel('t') ylabel('phi') title('Graph of
phi vrs t')
hold on plot(t,L*t)
% Calls ode45 to solve the system of first order odes on tspan3.
[t,x] = ode45(@ph,tspan3,x0,options,A,B,C,F,w0,w);
% Plots the data.
figure plot(t,x(:,1)) xlabel('t') ylabel('phi') title('Graph of
phi vrs t') hold on plot(t,L*t)

function xprime=ph(t,x,A,B,C,F,w0,w) xprime=zeros(2,1);
xprime(1)=x(2);
xprime(2)=(F*cos(w*t)+(A*cos(2*w*t)-(w0)^2)*x(1)+B*sin(2*w*t)*x(2))/(1-C*cos(2*w*t));

function xzprime=pz(t,x,Az,Bz,Cz,F,w0,w) xzprime=zeros(2,1);
xzprime(1)=x(2);
xzprime(2)=(F*cos(w*t)+(Az*cos(2*w*t)-(w0)^2)*x(1)+Bz*sin(2*w*t)*x(2))/(1-Cz*cos(2*w*t));
```

Next, substituting the value of F into equation (67) results in

$$\begin{aligned} \Gamma_D &= \frac{[\theta_0[(\omega^2 I_2 + N[(g - l_1\omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)])/I_0]]\pi}{\omega_0^2} \\ &= \pi\theta_0 \frac{[\omega^2 I_2 + N(g - l_1\omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)]}{I_0 \pi \omega_0^2} \end{aligned} \tag{68}$$

8.2. Looking At the Parametric System

Because the parametric terms are ϕ-dependant, we can consider the system to have motion like that of the parametric oscillator system with the three ϕ-dependant terms. However, one thing to note is that the parametric system will have no growth in amplitude if $\phi_0 = 0$.[4]

$$\ddot{\phi} + \omega_0^2\phi = A\cos(2\omega t)\phi + B\sin(2\omega t)\dot{\phi} + C\cos(2\omega t)\ddot{\phi}. \tag{69}$$

Because the parametric terms are dependent on ϕ, for large values of ϕ we can consider the system to have values that of only a parametric system. The solution to the equation has the form of

$$\phi = e^{\lambda t}(ae^{i\omega t} + \overline{a}e^{i\omega t}),$$

where a is complex and $\overline{a}$ is the complex conjugate of a. Solving for $\dot{\phi}$ we get

$$\dot{\phi} = e^{\lambda t}[i\omega ae^{i\omega t} - i\omega\overline{a}e^{-i\omega t}] + \lambda e^{\lambda t}[ae^{i\omega t} + \overline{a}e^{i\omega t}].$$

Factoring and grouping like terms we get

$$\dot{\phi} = ae^{i\omega t}e^{\lambda t}(\lambda + i\omega) + \overline{a}e^{-i\omega t}e^{\lambda t}(\lambda - i\omega).$$

Solving for $\ddot{\phi}$ we get

$$\begin{aligned}\ddot{\phi} &= e^{\lambda t}[(\lambda^2 - \omega^2)(ae^{i\omega t} + \overline{a}e^{-i\omega t}) + 2\lambda i\omega(ae^{i\omega t} - \overline{a}e^{-i\omega t})] \\ &= (ae^{i\omega t}e^{\lambda t}(\lambda + i\omega)^2) + (\overline{a}e^{i\omega t}e^{\lambda t}(\lambda - i\omega)^2).\end{aligned}$$

Substituting these equations for ϕ, $\dot{\phi}$, and $\ddot{\phi}$ into equation (69) we obtain

$$a[(\lambda + i\omega)^2 + \omega_0^2] = \frac{\overline{a}}{2}[A - iB(\lambda - i\omega) + C(\lambda - i\omega)^2]$$

If the real and imaginary parts of the complex numbers a and $\overline{a}$ then they are orthogonal and their magnitudes are equal. The equated magnitudes, computed in Maple, is

$$|a|^2(\lambda - \omega^2 + \omega_0)^2 + 4|a|^2\lambda^2\omega^2 = \frac{1}{4}|a|^2(A - b\omega + C(\lambda^2 - \omega^2))^2 + \frac{1}{4}|a|^2(-B\lambda - 2C\lambda\omega)^2$$

Then, dividing both sides by $|a|^2$, assuming that $|a|$ is non-zero, and setting $\omega = \omega_0$, yields

$$(\lambda)^4 + 4\lambda^2\omega_0^2 = \frac{1}{4}(A - B\omega_0 + C(\lambda^2 - \omega_0^2))^2 + \frac{1}{4}(-B\lambda - 2C\lambda\omega_0)^2.$$

Next, approximating the equation to only a second order of λ we get

$$4\lambda^2\omega_0^2 = \frac{1}{4}(A - B\omega_0 + C(\lambda^2 - \omega_0^2))^2 + \frac{1}{4}(-B\lambda - 2C\lambda\omega_0)^2.$$

Now, with λ much less ω_0 we set lambda on the right-hand side of the equation equal to zero, yielding

$$4\lambda^2\omega_0^2 = \frac{1}{4}(A - B\omega_0 + C\omega_0^2))^2.$$

Finally, solving for λ we have

$$\lambda = \pm\frac{1}{4\omega_0}(A - B\omega_0 + C\omega_0^2)).$$

Next, solving for ϕ

$$\phi = e^{\lambda t}(ae^{i\omega t} + \overline{a}e^{i\omega t}).$$

Writing the complex forms of a and $\overline{a}$ where x is the real part of the conjugates and iy is the complex part of the conjugates we get

$$\phi = e^{\lambda t}[(x + iy)e^{i\omega t} + (x - iy)e^{i\omega t}].$$

Distributing the exponentials gives

$$\phi = e^{\lambda t}[xe^{i\omega t} + iye^{i\omega t} + xe^{i\omega t} - iye^{i\omega t}],$$

and regrouping and factoring out the real and complex parts we have

$$\phi = e^{\lambda t}[x(e^{i\omega t} + e^{-i\omega t}) + iy(e^{i\omega t} - e^{-i\omega t})].$$

Then, expanding the exponential functions out yields

$$\begin{aligned}\phi = e^{\lambda t}[x([\cos(\omega t) + i\sin(\omega t)] + [\cos(\omega t) - i\sin(\omega t)])\\ + iy([\cos(\omega t) + i\sin(\omega t)] - [\cos(\omega t) - i\sin(\omega t)])].\end{aligned}$$

By cancelling terms we get

$$\phi = e^{\lambda t}[x(2\cos(\omega t)) + iy(2i\sin(\omega t))].$$

Redistributing we obtain

$$\phi = e^{\lambda t}[2x\cos(\omega t) - 2y\sin(\omega t)].$$

If the assumption that $x = y$ is made, we have

$$\phi = e^{\lambda t}[2x\cos(\omega t) - 2x\sin(\omega t)].$$

Factoring out the $2x$ yields

$$\phi = 2xe^{\lambda t}[\cos(\omega t) - \sin(\omega t)]. \tag{70}$$

Solving for x, we begin by solving for the magnitude of a

$$a^2 = (x^2 + y^2).$$

Taking the root of both sides we have

$$|a| = \pm\sqrt{x^2 + y^2}.$$

Now, because the assumption that x and y are equal in magnitude $|x|^2 = |y|^2$, we get

$$|a| = \pm\sqrt{x^2 + x^2} \quad = \pm\sqrt{2x^2} = \pm\sqrt{2}x$$

Thus,

$$x = \pm\frac{|a|}{\sqrt{2}}$$

Substituting this value of x into equation (70) we have

$$\phi = \pm 2\left(\frac{|a|}{\sqrt{2}}\right) e^{\lambda t}[\cos(\omega t) - \sin(\omega t)],$$

which is equal to

$$\phi = \pm\sqrt{2}|a|e^{\lambda t}[\cos(\omega t) - sin(\omega t)]. \tag{71}$$

Rewriting equation (71) we get[3]

$$\phi = \pm\phi_0\sqrt{2}e^{\lambda t}\cos(\omega t + \pi/4).$$

During the period where the parametric terms are the dominant contributors to the growth of the amplitude and this growth is exponential. See figures 5 and 6. The period is

$$T = \frac{2\pi}{\omega_0}.$$

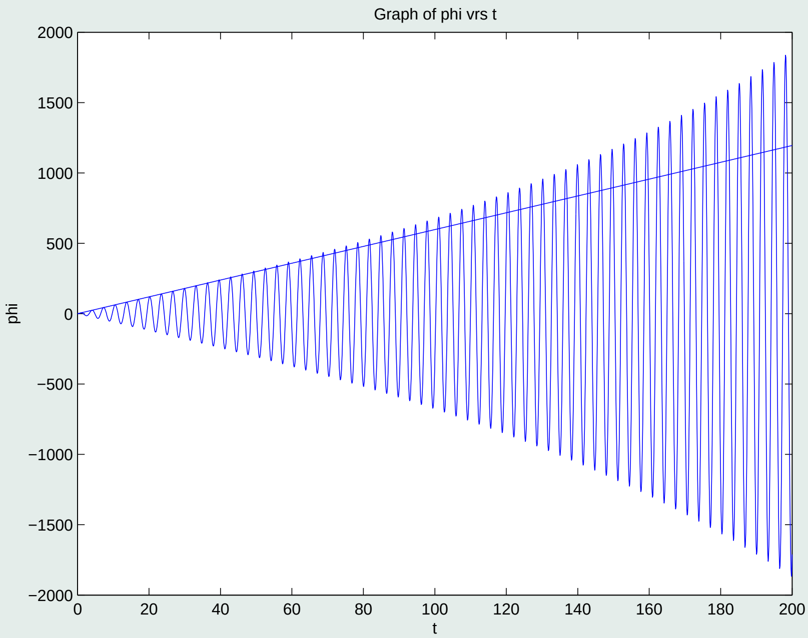

Figure 5: Graph of the system.

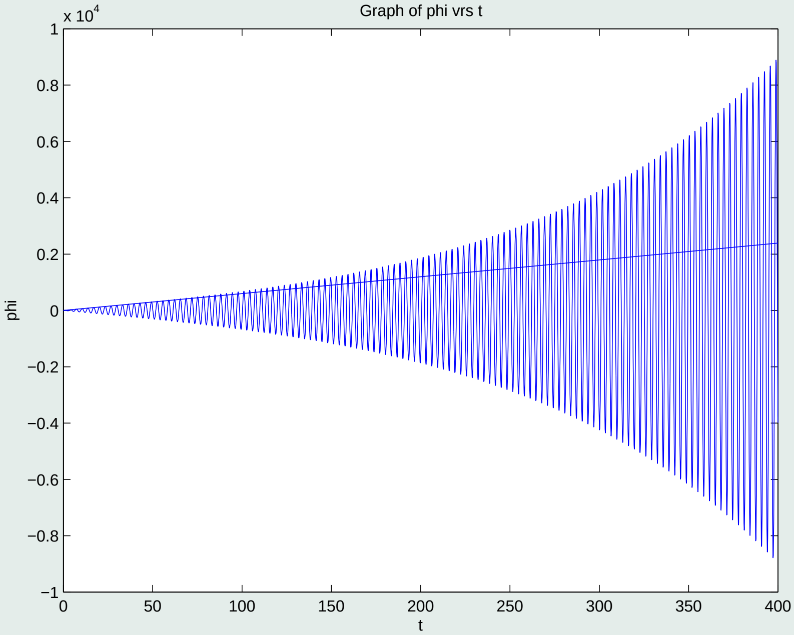

Figure 6: Graph of the system.

Substituting this value of the period into the time value in the exponent and multiplying by the time dependant amplitude of ϕ ($\phi_(0)$) yields the growth per cycle approximated by just the parametric terms

$$\Gamma_P = \lambda\left(\frac{2\pi}{\omega_0}\right)\phi_0 \quad = \frac{2\pi\lambda}{\omega_0}\phi_0.$$

Next, substituting in the value of λ using the negative sign, we have

$$\Gamma_P = \frac{2\pi}{\omega_0}\left[-\frac{1}{4\omega_0}(A - B\omega_0 - C\omega_0^2)\right]\phi_0.$$

Then, substituting in the values of A, B, and C we have

$$\Gamma_P = \frac{2\pi}{\omega_0}\Bigg(-\frac{1}{4\omega_0}\Bigg(\left[-\frac{1}{4}Ng\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right] - \left[l_1 N\omega\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right]\omega_0 \\ - \left[-\frac{1}{2}l_1 N\left(\theta_0^2 - \frac{1}{12}\theta_0^4\right)/I_0\right]\omega_0^2\Bigg)\Bigg)\phi_0.$$

Dropping the higher powers of λ, cancelling, factors and finding a common denominator we get

$$\Gamma_P = -\frac{\pi}{2\omega_0^2}\left(\frac{-Ng\theta_0^2}{4I_0} - \omega_0\frac{4l_1 N\omega\theta_0^2}{4I_0} + \omega_0^2\frac{2l_1 N\theta_0^2}{4I_0}\right)\phi_0.$$

Factoring out the $N\theta_0^2$, distributing the minus sign, and setting $\omega = \omega_0$ results in

$$\Gamma_P = \frac{\pi N\theta_0^2}{2\omega_0^2}\left(\frac{g + 4l_1\omega_0^2 - 2l_1\omega_0^2}{4I_0}\right)\phi_0.$$

Combining like terms gives

$$\Gamma_P = \frac{\pi N\theta_0^2}{2\omega_0^2}\left(\frac{g + 2l_1\omega_0^2}{4I_0}\right)\phi_0,$$

and distributing the $2\omega_0^2$ we have

$$\Gamma_P = \pi N\theta_0^2\left(\frac{g + 2l_1\omega_0^2}{8I_0\omega_0^2}\right)\phi_0.$$

This can be simplified to

$$\Gamma_P = \pi N \theta_0^2 \left(\frac{g/2 + l_1 \omega_0^2}{4 I_0 \omega_0^2} \right) \phi_0.$$

To determine $\phi_{critical}$ we set $\Gamma_D = \Gamma_P$ resulting with the equation

$$\pi \theta_0 \left(\frac{\omega^2 I_2 + N(g - l_1 \omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)}{I_0 \pi \omega_0^2} \right) = \pi N \theta_0^2 \left(\frac{g/2 + l_1 \omega_0^2}{4 I_0 \omega_0^2} \right) \phi_0.$$

Thus $\phi_{critical}$ is

$$\phi_{critical} = \pi \theta_0 \left(\frac{\omega^2 I_2 + N(g - l_1 \omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)}{I_0 \pi \omega_0^2} \right) \left(\frac{4 I_0 \omega_0^2}{(g/2 + l_1 \omega_0^2) \pi N \theta_0^2} \right)$$

Cancelling terms gives

$$\phi_{critical} = \left(\frac{\omega^2 I_2 + N(g - l_1 \omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)}{\pi} \right) \left(\frac{4}{(g/2 + l_1 \omega_0^2) N \theta_0} \right)$$

which equals

$$\phi_{critical} = \left(\frac{4\omega^2 I_2 + 4N(g - l_1 \omega^2)(1 - \frac{1}{8}\theta_0^2 + \frac{1}{192}\theta_0^4)}{\pi (g/2 + l_1 \omega_0^2) N \theta_0} \right)$$

9. Results

The next following Matlab M-file returns the values for the various substitutions, period, growths, and critical angle.

```
function critical(M,m1,m2,m3,L1,L2,L3,th0) format long
% Closes all figures
close all

% Sets the default parameters:
    % M........is the total mass of the system.
    % m1.......is the center mass.
    % m2.......is the right mass.
```

```
    % m3.......is the left mass.
    % L1.......is the length between the center of the system and m1.
    % L2.......is the length between m1 and m2.
    % L3.......is the length between m1 and m3.
    % th0......is the initial theta angle.

if nargin < 1
    M = 1;
    m1 = 0.4*M;
    m2 = 0.2*M;
    m3 = 0.4*M;
    L1 = 2.5;
    L2 = 0.4;
    L3 = L2;
    th0 = .7;
end

% Condenses combinations of like terms.
g=9.8; I1=M*L1^2 I2=m2*L2^2+m3*L3^2 N= m3*L3-m2*L2
I0=(I1+I2)-2*L1*N*(1-(1/4)*th0^2+(1/64)*th0^4)
K0=(M*L1*g)-N*g*(1-(1/4)*th0^2+(1/64)*th0^4) w0=(abs(K0/I0))^(1/2)
T=(2*pi)/w0

% Sets the frequency of the driving and parametric terms equal to the
% natural frequency.
w = w0

% Sets the oscillatory terms for the system:
    % F........is the coefficient on the driving term cos(wt).
    % A........is the coefficient on the parametric term cos(2wt)*phi.
    % B........is the coefficient on the parametric term sin(2wt)*phi'.
    % C........is the coefficient on the parametric term cos(2wt)*phi''.

F=th0*(w^2*I2+N*((g-w0^2*L1)*(1-(1/8)*th0^2+(1/192)*th0^4)))/I0
```

```
A=-((1/4)*N*g*(th0^2-(1/12)*th0^4))/I0
B=(L1*N*w*(th0^2-(1/12)*th0^4))/I0
C=-((1/2)*L1*N*(th0^2-(1/12)*th0^4))/I0
Lambda=-(1/(4*w0))*(A-w0*B-w0^2*C)
GammaD=((pi*F)/(w0)^2)
GammaP=(2*pi*Lambda)/(w0)
PhiCritical=((pi*F)/(w0)^2)/((2*pi*Lambda)/(w0))}
```

These are the following results when given an initial $m1 = 0.4M, m2 = 0.4M, m3 = 0.2M, L1 = 2.5, L2 = 0.4, L3 = L2$, and $th0 = 0.7$.

- $I_1 = 6.25M$
- $I_2 = 0.096M$
- $N = 0.08$
- $I_0 = 5.993499375$
- $K_0 = 23.809098775$
- $\omega_0 = 1.99310989928485$
- $T = 3.15245301296937$
- $\lambda = 0.00583519341812$
- $\Gamma_D = 0.03431258738378$
- $\Gamma_P = 0.01839517307222$
- $\phi_{critical} = 1.86530386254419$

10. Conclusions

Both types of systems display a type of growth whether enveloped by a linear function or enveloped by an exponential function. It appears from the graphs that after some point in time that the parametric terms dominate in the contribution of growth to the system. However, in the approximation of the terms for the differential equation, we approximated terms in equation (22) that hold fairly accurate to a value of $\phi < 0.7$ radians. However, Using realistic values for the lengths and distribution of masses, the computed critical angle far exceeds the values needed for the earlier approximation. Thus, this model cannot predict the contribution of growth to the amplitude of the system after the critical angle.

The model of the parametric oscillator is unstable in two possible phases. This is seen in the parametric solution for ϕ. The parametric solution for ϕ shows two possible phases of motion that have an equal effect. One, in which $\phi = -\sqrt{2}|a|e^{\lambda t}[\cos(\omega t) - \sin(\omega t)]$ is the closest to observation, in that the swinger raises their center of mass by rocking backward when the swing is at its lowest point. However, the second parametric solution $\phi = \sqrt{2}|a|e^{\lambda t}[\cos(\omega t) - \sin(\omega t)]$ never comes into play as the swinger never raises their center of mass by rocking forward while the swing is at the lowest point.[1][2]

The best model that best describes the motion of the swing is the forced harmonic system. The parametric model is too unstable and doesn't have any real, practical solution. The motion of the swing is approximated nicely during the driven oscillator regime, whereas, we cannot conclude any clear notions about the contributions of the parametric terms to the growth of the amplitude of the system due to how the system was approximated as well as the instability of the parametric terms.[1]

11. References

References

[1] William B. Case, and Mark A. Swanson *The Pumping Of A Swing From The Seated Position*, Departmant of Physics, Ginnell College, Ginnell, Iowa 50112, American Journal of Physics (Published May 1990), 463-467.

[2] William B. Case, *Parametric Instability: An Elementary Demonstration and Discussion*, Hobart and William Smith Colleges, Geneva, New York 14456, American Journal of Physics (Published March 1980), 218-221.

[3] William B. Case, *The Pumping of a Swing From The Standing Position*, Departmant of Physics, Ginnell College, Ginnell, Iowa 50112, American Journal of Physics (Published May 1996), 214-220.

[4] John T. McMullan, *On Initiating Motion in A Swing*, Physical Sciences The New York University of Ulster, Coleraine, N.Ireland, American Journal of Physics (Published May 1972), 764-766.

[5] John Polking, Albert Boggess, and David Arnold *Differential Equations With Boundary Value Problems*, Perntice Hall (2002), 91-99.

[6] *What Is Theoretical Physics?*, http://superstringtheory.com/basics/basic1a.html.

[7] Chris Davis *Idle Theory: Least Action Principles*. Last Edited: 28 feb 1998. http://ourworld.compuserve.com/homepages/cuius/idle/evolution/ref/leastact.html

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