Numerical Solutions For Geodesics on Two Dimensional Surfaces

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Abstract

This paper presents a brief introduction to geodesics on two dimensional surfaces in R^3 and includes brief instructions for solving the resulting geodesic equations numerically using Matlab. Included also are brief instructions for using Matlab's symbolic toolbox to evaluate complex derivations.

1. Introduction

A two dimensional surface is a space in which only two independent vectors are needed to span the entire space. For example, the surface of a sphere is two dimensional. Only two independent vectors such as angles of longitude and latitude are needed to locate every point on the surface.

Ever since we were children, we have been told that the shortest distance between two points is a straight line. This is true if the space you live in obeys the rules of Euclidean geometry. But what if the space you live in does not obey the rules of Euclidean geometry? Suppose that you are a two dimensional being living on the surface of a sphere. The shortest distance between two points on a sphere is not a straight line, it is a *great circle*.

Our goal in this paper is to find a curve on a two dimensional curved surface that is analogous to a straight line on a flat surface such as a plane. Such a curve is termed a *geodesic*. We shall consider some properties of straight lines in R^2 and relate them to properties of curves on two dimensional surfaces.

2. Analysis

A line $y = f(x)$ in R^2 minimizes the distance between two points p and q contained on $f(x)$. The first derivative of $f(x)$, $f'(x)$, will give the slope of the line. For $y = f(x)$ to be a line in R^2, $f'(x)$ must remain constant for all $x \in R$. The slope of the line must not change. Therefore, the second derivative of $f(x)$, $f''(x)$, must be 0 for all $x \in R$. It is this vanishing of the second derivative of $f(x)$ that we are interested in. We seek to describe a curve $\alpha(t)$ on a two dimensional curved surface M, that shares this property of the vanishing of the second derivative. Now $\alpha''(t) = 0$ for all $t \in R$ implies that $\alpha(t)$ is a straight line in R^3 and M is a plane, so $\alpha''(t) \neq 0$ in general on a curved surface.

To see the problem from a different perspective, lets look at the situation from the viewpoint of a two dimensional being living on M at some point p. Let's call this being Dave. Dave is completely unaware of everything that happens outside of M and the tangent plane to M at p. In other words, he is ignorant of everything that occurs in the third dimension. Suppose that Dave is initially at rest and somehow he receives a push that briefly accelerates him until he is moving at a constant speed on M. After his initial push, Dave feels no acceleration. As far a he can tell, he is moving with a constant velocity in a straight line. From our perspective looking down from R^3, we see our friend moving along on a curve contained on M, a geodesic. How do we determine what this geodesic curve is and where it will go? The answer lies in the fact that from the perspective of our two dimensional friend, there is no acceleration felt while moving along M at a constant speed. In R^3, Dave's acceleration vector at some point p on M can be expressed as a linear combination of any three independent vectors. If we chose two independent vectors x_1 and x_2 that lie in the tangent plane of M at p and a third vector $x_1 \times x_2$, then the acceleration can be expressed as

$$\alpha''(t) = c_1 x_1 + c_2 x_2 + c_3 x_1 \times x_2.$$

Or equivalently

$$\alpha''(t) = \alpha''_{\text{tan}} + \alpha''_{\perp},$$

where $\alpha''_{\text{tan}} = c_1 x_1 + c_2 x_2$ is the component of the acceleration vector that lies on the tangent plane of M at point p and $\alpha''_{\perp} = c_3 x_1 \times x_2$ is the component of the acceleration vector that is orthogonal to the tangent plane at p. This is illustrated in Figure 2.1.

Following the argument mentioned above, if Dave feels no acceleration, then $\alpha''_{\text{tan}} = 0$ and $\alpha''(t) = \alpha''_{\perp}$.

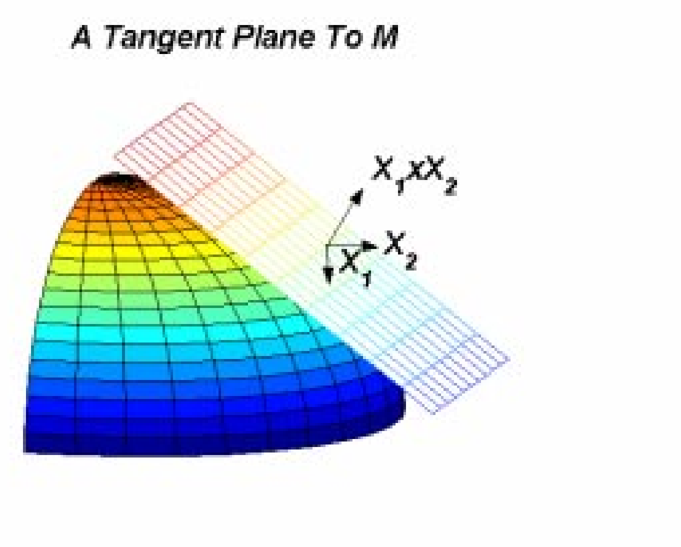

Figure 2.1: The tangent plane.

Definition 2.1. *A curve $\alpha(t)$ is a geodesic on M if $\alpha''_{\tan} = 0$*

Lemma 2.2. *A geodesic always has a constant speed.*

Proof. The speed of a geodesic $\alpha(t)$ is given by $v = |\alpha'(t)|$, so

$$v^2 = \alpha'(t) \cdot \alpha'(t).$$

Differentiating with respect to t yields

$$2v\frac{dv}{dt} = \alpha''(t) \cdot \alpha'(t) + \alpha'(t) \cdot \alpha''(t) = 2\alpha''(t) \cdot \alpha'(t).$$

Since $\alpha'(t)$ lies on the tangent plane of M, α'' is always orthogonal to $\alpha'(t)$ and their dot product is always 0. Thus,

$$2v\frac{dv}{dt} = 0$$

and

$$\frac{dv}{dt} = 0,$$

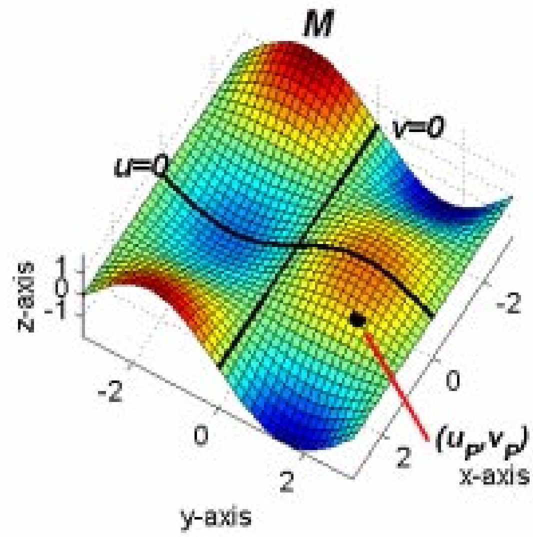

Figure 2.2: Dave's Coordinate System on M.

indicating that v remains constant.

This lemma reinforces the idea of a geodesic being analogous to a straight line on a plane. The *speed* of a line on a plane is given by it's first derivative and the first derivative of a line is always constant

Using the restriction of $\alpha''_{\text{tan}} = 0$, it is possible to find the geodesic curve $\alpha(t)$ taken by Dave on M. Suppose Dave has established a coordinate system on an area or patch of M such that he can location his position at any point p in the patch with the coordinates (u_p, v_p). This is illustrated in Figure 2.2. The same point p in R^3 has Cartesian coordinates (x_p, y_p, z_p). To locate all points on the patch of M in R^3, the Cartesian coordinates must be functions of u and v such that the vector

$$X(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} \tag{Eq1}$$

maps every point in the patch of M to a point in R^3. Now since Dave is moving on M, his coordinates u and v change with time such that $u = u(t)$ and $v = v(t)$.

Thus the vector

$$\alpha(t) = X(u(t), v(t)) = \begin{bmatrix} x(u(t), v(t)) \\ y(u(t), v(t)) \\ z(u(t), v(t)) \end{bmatrix} \qquad \text{(Eq2)}$$

maps the geodesic $\alpha(t)$ taken by Dave on M to points in R^3. Differentiating equation 2 will yield the velocity vector of the geodesic

$$\alpha'(t) = V = X'(u(t), v(t)) = X_u u' + X_v v', \qquad \text{(Eq3)}$$

where the subscript denotes the partial derivative of the vector X with respect to u or v. Differentiating again to get Dave's acceleration vector gives

$$\begin{aligned} \alpha'' &= X''(u(t), v(t)) \\ &= X_{uu}u'^2 + X_{uv}v'u' + X_u u'' + X_{vu}u'v' + X_{vv}v'^2 + X_v v'' \qquad \text{(Eq4)} \end{aligned}$$

Since $\alpha(t)$ is a geodesic, $\alpha''_{\tan} = 0$ for all $t \in R$. Now since $\alpha''_{\tan}$ can be expressed as a linear combination of any two independent vectors that lie on the tangent plane of M, we shall pick the vectors X_u and X_v. Then $\alpha''_{\tan} = c_1 X_u + c_2 X_v$ and $c_1 = c_2 = 0$. This can be expressed with vector notation as

$$\alpha''_{\tan} = Ax$$

where $A = \begin{bmatrix} X_u & X_v \end{bmatrix}$ and $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Let's look at the scene as shone in Figure 2.3. Dave's acceleration vector α'' is shown in green and it's tangent component $\alpha''_{\tan} = Ax$ shown in blue. The tangent component of Dave's acceleration vector is simply the vector projection of α'' onto the tangent plane of M. The vector $\alpha'' - Ax$ shown in red, is always orthogonal to the tangent plane of M. Thus $\alpha'' - Ax \in N(A^T)$ and

$$A^T(\alpha'' - Ax) = 0$$

Simplifying gives

$$\begin{aligned} A^T\alpha'' - A^TAx &= 0 \\ A^TAx &= A^T\alpha'' \end{aligned}$$

Multiplying both sides on the left by $(A^TA)^{-1}$ gives

$$\begin{aligned} (A^TA)^{-1}A^TAx &= (A^TA)^{-1}A^T\alpha'' \\ x &= (A^TA)^{-1}A^T\alpha'' \qquad \text{(Eq 5)} \end{aligned}$$

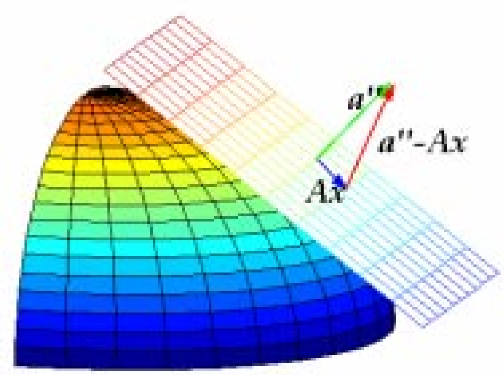

Figure 2.3: The Vector Projection onto the Tangent Plane.

The vector x is a function of u, v, u', v', u'', and v''. It contains a system of second order differential equations known as *geodesic equations*. Solving $x = 0$ for $u(t)$ and $v(t)$ will yield the functions $u(t)$ and $v(t)$, that when substituted back into equation 2, map the geodesic $\alpha(t)$ taken by Dave on M. All that is required is an initial point (u_0, v_0) and an initial velocity vector V_0. The velocity vector is determined by evaluating the vectors X_u and X_v in equation 3 at the point (u_0, v_0), and solving for u_0' and v_0'.

$$V_0 = X_u\rfloor_{(u_0,v_0)} u_0' + X_v\rfloor_{(u_0,v_0)} v_0' \tag{Eq6}$$

Thus the initial condition are u_0, v_0, u_0', and v_0'.

2.1. Orthogonal Coordinate Systems

Setting up equation 5 by hand is in general a very difficult task. In the special case in which $u(t)$ and $v(t)$ are orthogonal (i.e. $X_u \cdot X_v = 0$), a simpler approach may be taken. This involves expressing Dave's acceleration as seen in equation 4 as a linear combination of the vectors X_u, X_v, and a orthogonal vector $N = X_u \times X_v$. This gives

$$\alpha''(t) \quad = \quad c_1 X_u + c_2 X_v + c_3 N$$

$$= \quad X_{uu}u'^2 + X_{uv}v'u' + X_u u'' + X_{vu}u'v' + X_{vv}v'^2 + X_v v'' \qquad \text{(Eq7)}$$

The problem is that we need to know the vectors X_{uu}, X_{uv}, and X_{vv} in terms of X_u, X_v, and N such that

$$\begin{aligned} X_{uu} &= a_1 X_u + a_2 X_v + a_3 N && \text{(Eq8.1)} \\ X_{uv} &= a_4 X_u + a_5 X_v + a_6 N && \text{(Eq8.2)} \\ X_{vv} &= a_7 X_u + a_8 X_v + a_9 N && \text{(Eq8.3)} \end{aligned}$$

If the a_i's are known, then equations 8.1, 8.2, and 8.3 can be substituted into equation 7 giving Dave's acceleration in terms of X_u, X_v, and the orthogonal vector N. The coefficients a_i in equations 8.1, 8.2, and 8.3 can be determined by using some properties of the dot product of vectors. First, lets define three functions E, F,and G as

$$\begin{aligned} E &= X_u \cdot X_u \\ F &= X_u \cdot X_v \\ G &= X_v \cdot X_v. \end{aligned}$$

Note, that if u and v are orthogonal, $F = 0$. Next, let's dot equation 8.1 with X_u,

$$\begin{aligned} X_{uu} \cdot X_u &= a_1 X_u \cdot X_u + a_2 X_v \cdot X_u + a_3 N \cdot X_u, \\ X_{uu} \cdot X_u &= a_1 X_u \cdot X_u + 0 + 0, \\ X_{uu} \cdot X_u &= a_1 E. \end{aligned}$$

If we can compute $X_{uu} \cdot X_u$, then we can compute a_1. If we take the partial derivative of E with respect to u, then

$$\begin{aligned} E_u &= X_{uu} \cdot X_u + X_u \cdot X_{uu} = 2X_{uu} \cdot X_u, && \text{(Eq9.1)} \\ X_{uu} \cdot X_u &= \frac{E_u}{2} = a_1 E, && \text{(Eq9.2)} \\ a_1 &= \frac{E_u}{2E}. && \text{(Eq9.3)} \end{aligned}$$

Next, let's dot equation 8.1 with X_v

$$\begin{aligned} X_{uu} \cdot X_v &= a_1 X_u \cdot X_v + a_2 X_v \cdot X_v + a_3 N \cdot X_v, \\ X_{uu} \cdot X_v &= 0 + a_2 G + 0, \\ a_2 &= \frac{X_{uu} \cdot X_v}{G}. \end{aligned}$$

Now if we take the partial derivative of E with respect to v

$$E_v = X_{uv} \cdot X_u + X_u \cdot X_{uv} = 2X_u \cdot X_{uv},$$

or equivalently,

$$\frac{E_v}{2} = X_u \cdot X_{uv} = -X_{uu} \cdot X_v. \qquad \text{(Eq9.4)}$$

Then

$$a_2 = \frac{X_{uu} \cdot X_v}{G} = -\frac{E_v}{2G}.$$

Now, let's dot equation 8.2 with X_u,

$$\begin{aligned} X_{uv} \cdot X_u &= a_4 X_u \cdot X_u + a_5 X_v \cdot X_u + a_6 N \cdot X_u, \\ X_{uv} \cdot X_u &= a_4 E + 0 + 0, \\ a_4 &= \frac{X_{uv} \cdot X_u}{E}. \end{aligned}$$

By making a substitution from equation 9.4,

$$a_4 = \frac{E_v}{2E}.$$

Further, if we dot equation 8.2 with X_v,

$$\begin{aligned} X_{uv} \cdot X_v &= a_4 X_u \cdot X_v + a_5 X_v \cdot X_v + a_6 N \cdot X_v, \\ X_{uv} \cdot X_v &= 0 + a_5 G + 0, \\ a_5 &= \frac{X_{uv} \cdot X_v}{G}. \end{aligned} \qquad \text{(Eq9.5)}$$

Next, if we take the partial derivative of G with respect to u

$$G_u = X_{vu} \cdot X_v + X_v \cdot X_{vu} = 2X_{vu} \cdot X_v,$$

or equivalently,

$$\frac{G_u}{2} = X_{vu} \cdot X_v = -X_{uu} \cdot X_v. \qquad \text{(Eq9.6)}$$

By making a substitution from equation 9.6 into equation 9.5,

$$a_5 = \frac{G_u}{2G}$$

Now let's dot equation 8.3 with X_v,

$$\begin{aligned} X_{vv}\cdot X_v &= a_7X_u\cdot X_v + a_8X_v\cdot X_v + a_9N\cdot X_v, \\ X_{vv}\cdot X_v &= 0 + a_8G + 0 + 0, \\ a_8 &= \frac{X_{vv}\cdot X_v}{G}. \qquad \text{(Eq9.7)} \end{aligned}$$

Now let's take the partial derivative of G with respect to v

$$\begin{aligned} G_v &= X_{vv}\cdot X_v + X_v\cdot X_{vv} = 2X_{vv}\cdot X_v, \\ X_{vv}\cdot X_v &= \frac{G_v}{2}. \qquad \text{(Eq9.8)} \end{aligned}$$

Using substitutions from equation 9.8 into equation 9.7,

$$a_8 = \frac{G_v}{2G}$$

Finally, let's dot equation 8.3 with X_u

$$\begin{aligned} X_{vv}\cdot X_u &= a_7X_u\cdot X_u + a_8X_v\cdot X_u + a_9N\cdot X_u \\ X_{vv}\cdot X_u &= a_7E + 0 + 0 \\ a_7 &= \frac{X_{vv}\cdot X_u}{E} \qquad \text{(Eq9.9)} \end{aligned}$$

Next, let's take the partial derivative of F with respect to v

$$F_v = X_{uv}\cdot X_v + X_u\cdot X_{vv} = 0$$

or equivalently

$$-X_{vv}\cdot X_u = X_{uv}\cdot X_v \qquad \text{(Eq10)}$$

Using substitutions from equations 9.5 and 10 into equation 9.9 gives

$$\alpha_7 = -\frac{G_u}{2E}.$$

Substituting all of the a_i's back into equations 8.1, 8.2, and 8.3 gives

$$\begin{aligned} X_{uu} &= \frac{E_u}{2E}X_u - \frac{E_v}{2G} + a_3N \\ X_{uv} &= \frac{E_v}{2E}X_u + \frac{G_u}{2G}X_v + a_6N \\ X_{vv} &= -\frac{G_u}{2E}X_u + \frac{G_v}{2G}X_v + a_9N \end{aligned}$$

Substituting these equation into equation 7 and doing a bit of rearranging gives

$$\begin{aligned}\alpha''(t) &= (u'' + \frac{E_u}{2E}u'^2 + \frac{E_v}{E}u'v' - \frac{G_u}{2E}v'^2)X_u \\ &\quad +(v'' - \frac{E_v}{2G}u'^2 + \frac{G_u}{G}u'v' + \frac{G_v}{2G}v'^2)X_v \qquad \text{(Eq11)} \\ &\quad +(a_3u'^2 + 2a_6u'v' + a_9v'^2)N.\end{aligned}$$

Since $\alpha(t)$ is a geodesic, $\alpha''_{\text{tan}} = 0$ and we end up with the following system of geodesic equations for $u(t)$ and $v(t)$.

$$u'' + \frac{E_u}{2E}u'^2 + \frac{E_v}{E}u'v' - \frac{G_u}{2E}v'^2 = 0 \qquad \text{(Eq12)}$$

$$v'' - \frac{E_v}{2G}u'^2 + \frac{G_u}{G}u'v' + \frac{G_v}{2G}v'^2 = 0 \qquad \text{(Eq13)}$$

Equations 12 and 13 are much simpler to work with than equation 5. It is important to remember though that equations 12 and 13 only work when u and v are orthogonal. If u and v are not orthogonal, then equation 5 must be used.

2.2. Solving the Differential Equations For $u(t)$ and $v(t)$

The differential equations found in equations 5, 12, and 13 are usually very difficult to solve analytically, provided they even have closed form solutions. It is for this reason a numerical solver is used. Matlab's ODE solvers solve systems of first order differential equations. To solve a system of differential equations such as those seen in equations 12 and 13, the system must first be expressed as a first order system. This is done by introducing 4 new variables x_1, x_2, x_3, and x_4 such that

$$\begin{aligned} x_1 &= u \\ x_2 &= u' \\ x_3 &= v \\ x_4 &= v' \end{aligned}$$

Next, the new system is differentiated

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= u'' \\ x_3' &= x_4 \\ x_4' &= v'' \end{aligned}$$

Note that $x_2' = u''$ and $x_4' = v''$ both consist the entire functions u'' and v''. These functions will almost certainly contain terms other than u'' and v''. It is very important that these other terms are expressed in terms of the x_i's defined above. After this is done, a Matlab function file is written with the format

```
function xprime=myfunction(t,x)
xprime=zeros(4,1);
xprime(1)=x(2);
xprime(2)=function1;
xprime(3)=x(4);
xprime(4)=function2;
```

Both function1 and function2 mentioned above are generic for the real functions u'' and v''. All terms in u'' and v'' need to be expressed with the same code as used above. For example, if $u'' = x_3 + x_1$, then in the function file

```
xprime(2)=x(3)+x(1)
```

Once the function file is written, the system can be solved by executing the following command:

```
[t,X] =ode23s('myfunction',[t1,t2] ,[u_0;u'_0;v_0;v'_0]);
```

This command outputs a matrix X, that consist of 4 column vectors. Each vector corresponds to a numerical solution for each x_i. Since $u = x_1$ and $v = x_3$, we are only interested in the first and third columns of X. We can isolate these columns with the commands

```
u=X(:,1); v=X(:,3);
```

3. The Unit Sphere

The unit sphere can mapped by the vector

$$X(u,v) = \begin{bmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \sin(v) \end{bmatrix}$$

where u and v are angles of longitude and latitude. In this case, u and v are orthogonal so equations 12 and 13 can be used. Computing X_u gives

$$X_u = \begin{bmatrix} -\sin(u)\cos(v) \\ \cos(u)\cos(v) \\ 0 \end{bmatrix}$$

Computing X_v gives

$$X_v = \begin{bmatrix} -\cos(u)\sin(v) \\ -\sin(u)\sin(v) \\ \cos(v) \end{bmatrix}$$

Thus $E = \cos^2(v)$, $F = 0$, and $G = 1$ and the geodesic equations become

$$\begin{aligned} u'' - 2\tan(v)u'v' &= 0 \\ v'' + \sin(v)\cos(v)u'^2 &= 0. \end{aligned}$$

Writing this as a first order system with the substitution

$$\begin{aligned} x_1 &= u \\ x_2 &= u' \\ x_3 &= v \\ x_4 &= v' \end{aligned}$$

and differentiating gives

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= 2\tan(x_3)x_2x_4 \\ x_3' &= x_4 \\ x_4' &= -\sin(x_3)\cos(x_3)x_2^2. \end{aligned}$$

Next, we write a Matlab function file

```
function xp=cir(t,x)
xp=zeros(4,1);
xp(1)=x(2);
xp(2)=2*tan(x(3))*x(2)*x(4);
xp(3)=x(4);
xp(4)=-sin(x(3))*cos(x(3))*x(2)^2;
```

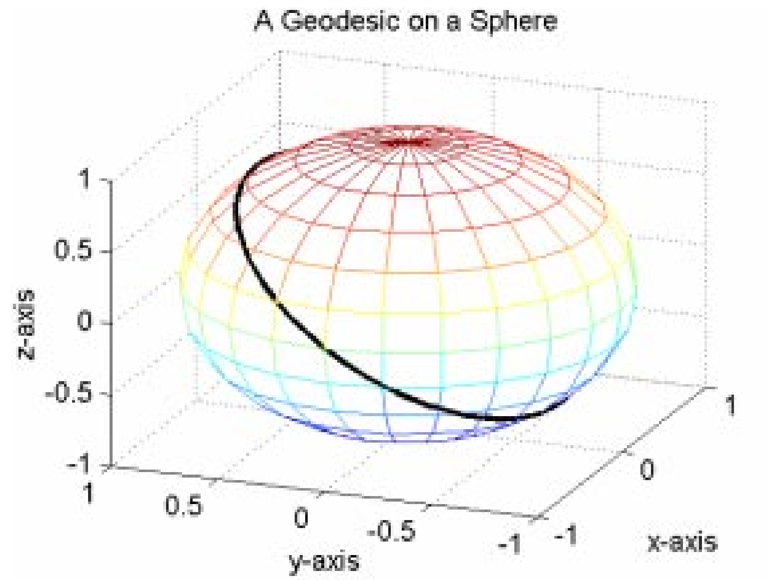

Figure 3.1: Geodesics on Spheres are great circles.

The following list of commands will produce the graph shown in Figure 3.1.

```
[u,v]=meshgrid(linspace(0,2*pi,25),linspace(0,2*pi,25));
x=cos(u).*cos(v);
y=sin(u).*cos(v);
z=sin(v);
mesh(x,y,z)
hold on
[t,X]=ode23s('cir',[0,18*pi],[0,.1,0,.1]);
u=X(:,1);
v=X(:,3);
x=cos(u).*cos(v);
y=sin(u).*cos(v);
z=sin(v);
plot3(x,y,z)
```

4. The Torus

The torus can be mapped by the vector

$$X(u,v) = \begin{bmatrix} (2+\cos(u))\cos(v) \\ (2+\cos(u))\sin(v) \\ \sin(u) \end{bmatrix}$$

Where u is an angle of rotation in the interior of the torus and v is an angle of rotation from the center of the torus. Again, in this case u and v are orthogonal so equations 12 and 13 can be used. Computing E, F, and G gives

$$\begin{aligned} E &= 1 \\ F &= 0 \\ G &= (2+\cos(u))^2. \end{aligned}$$

The geodesic equations for the torus become

$$\begin{aligned} u'' + (2+\cos(u))\sin(u)v'^2 &= 0 \\ v'' - 2\frac{\sin(u)}{2+\cos(u)}u'v' &= 0. \end{aligned}$$

Next, a Matlab function file simular to the sphere's is written.

```
function xp=tor(t,x)
xp=zeros(4,1);
xp(1)=x(2);
xp(2)=-(2+cos(x(1)))*sin(x(1))*x(4)^2;
xp(3)=x(4);
xp(4)=2*(sin(x(1))/(2+cos(x(1))))*x(2)*x(4);
```

The following list of commands will produce a graph similar to Figure 4.1.

```
[u,v]=meshgrid(linspace(0,2*pi,36),linspace(0,2*pi,36));
x=(2+cos(u)).*cos(v);
y=(2+cos(u)).*sin(v);
z=sin(u);
mesh(x,y,z)
hold on
[t,X]=ode23s('tor',[0,20*pi] ,[pi/2,.1,-pi/2,.2]);
```

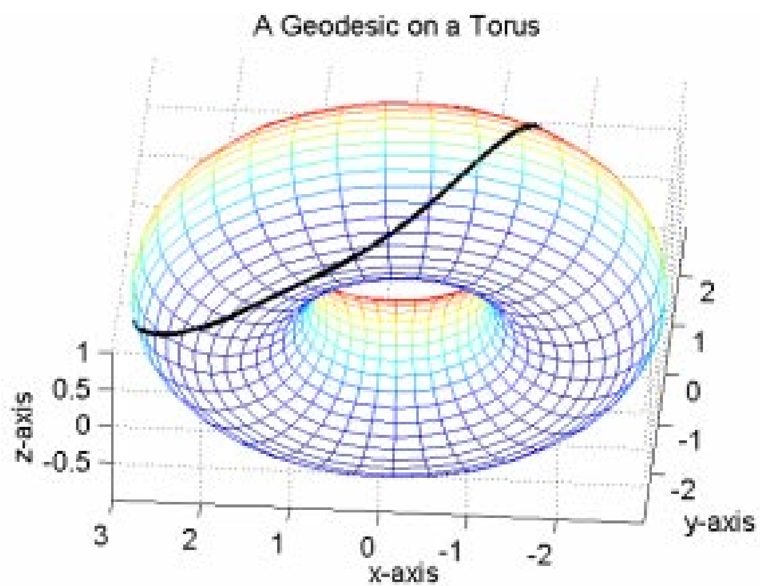

Figure 4.1: The Geodesic on the Torus will never reach the interior of the torus.

```
u=X(:,1);
v=X(:,3);
x=(2+cos(u)).*cos(v);
y=(2+cos(u)).*sin(v);
z=sin(u);
plot3(x,y,z)
```

5. The Egg Carton

The egg carton can be mapped by the vector

$$X(u,v) = \begin{bmatrix} v \\ u \\ \sin(u)\cos(v) \end{bmatrix}$$

Unfortunately, in this case u and v are not orthogonal. Therefore equation 5 must be used. The calculations done for the egg carton are very difficult to do by hand. In this case we shall use Matlab's symbolic toolbox to evaluate equation 5. The following commands will output the vector x in equation 5.

```
syms u v u1 v1 u2 v2 real
X=[v;u;sin(u)*cos(v)];
Xu=diff(X,u);
Xv=diff(X,v);
Xuu=diff(Xu,u);
Xvv=diff(Xv,v);
Xuv=diff(Xu,v);
Xvu=diff(Xv,u);
acel=Xuu*u1^2+Xuv*v1*u1+Xu*u2+Xvu*u1*v1+Xvv*v1^2+Xv*v2;
A=[Xu,Xv];
x=inv(A'*A)*A'*acel;
x1=simple(x(1,:))
x2=simple(x(2,:))
```

This routine will output the rows of the vector x, x_1 and x_2. The differential equations output by this routine are to long to be displayed. Matlab's solve command can be used to solve each equation, $x_1 = 0$ and $x_2 = 0$ for u'' and v''. After this is done, a Matlab function file simular to the sphere's and the torus' is written.

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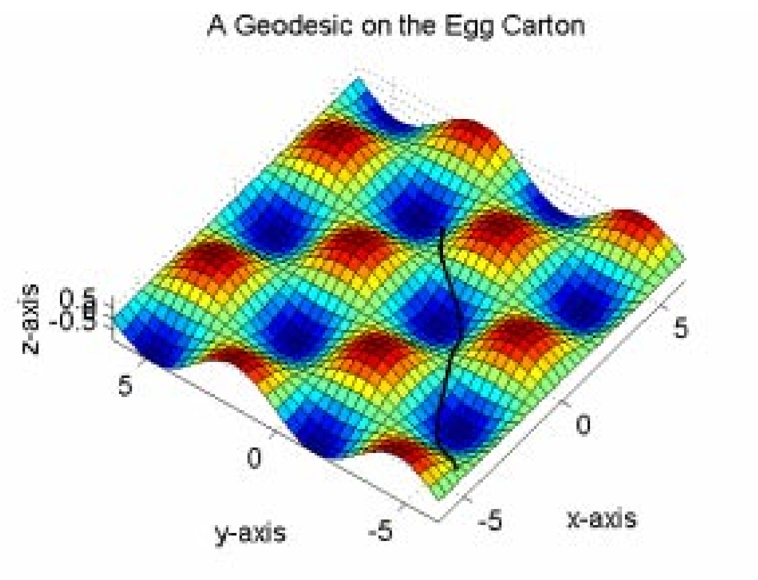

Figure 5.1: The acceleration vector is always orthogonal to the tangent plane along a geodesic.