1.6 Application
Computer Algebra Solutions

Computer algebra systems typically include commands for the "automatic" solution of differential equations. However, two different systems often give different results whose equivalence is not clear, and a single system may give the solution in an overly complicated form. Consequently, computer algebra solutions of differential equations often require considerable "processing" or simplification by a human user in order to be yield concrete and applicable information. Here we illustrate these issues using the interesting differential equation

$$\frac{dy}{dx} = \sin(x-y) \tag{1}$$

that appeared in the Section 1.3 Application. The *Maple* command

```
dsolve( D(y)(x) = sin(x - y(x)), y(x));
```

yields the nice-looking result

$$y(x) = x - 2\tan^{-1}\left(\frac{x-2-C1}{x-C1}\right) \tag{2}$$

that was cited there. However, the supposedly equivalent *Mathematica* command

```
DSolve[ y'[x] == Sin[x - y[x]], y[x], x]
```

returns the decidedly more complicated-looking result

$$y(x) = x \pm 2\cos^{-1}\left(\pm\frac{\sqrt{4+4x+x^2+4C1+2x\,C1+C1^2}}{\sqrt{4+4x+2x^2+4C1+4x\,C1+2C1^2}}\right) \tag{3}$$

This apparent disparity is not unusual, and with another differential equation it might be the other way around. As an alternative to attempting to reconcile the results in (2) and (3), a common tactic is to simplify the differential equation in advance, before submitting it to a computer algebra system. Exercises 1 through 5 below outline a preliminary paper-and-pencil investigation of Eq. (1).

Exercise 1 Show that the plausible substitution $v = x - y$ in (1) yields the separable equation

$$\frac{dv}{dx} = 1 - \sin v\,. \tag{4}$$

Now the *Mathematica* command

```
Integrate[1/(1 - Sin[v]), v] //  TrigReduce
```

yields

$$\int \frac{dv}{1-\sin v} = \frac{\cos v}{1-\sin v} \qquad \text{(plus a constant)} \tag{5}$$

Exercise 2 Apply the trig identities $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$ to deduce from (5) that

$$\int \frac{dv}{1-\sin v} = \frac{1+\tan(v/2)}{1-\tan(v/2)} \qquad \text{(plus a constant)} \tag{6}$$

Exercise 3 Deduce from (6) that Eq. (4) has the general solution

$$v(x) = 2\tan^{-1}\left(\frac{x-1+C}{x+1+C}\right)$$

and hence that Eq. (1) has the general solution

$$y(x) = x - 2\tan^{-1}\left(\frac{x-1+C}{x+1+C}\right). \tag{7}$$

Exercise 4 Finally, reconcile the forms in (2) and (7). What is the relation between the arbitrary constants C and $C1$?

Exercise 5 Show that the integral in (5) yields immediately the graphing calculator implicit solution shown in Fig. 1.6.10 of the text.

Investigation
For your own personal differential equation, let p and q be two distinct nonzero integers in your student ID number, and consider the differential equation

$$\frac{dy}{dx} = \frac{1}{p}\cos(x - q\,y). \tag{8}$$

(a) Find a symbolic general solution using a computer algebra system and/or some combination of the techniques illustrated above.

(b) Determine the symbolic particular solution corresponding to several typical initial conditions of the form $y(x_0) = y_0$.

(c) Determine the possible values of a and b such that the straight line $y = ax + b$ is a solution curve of (8).

(d) Plot a direction field and some typical solution curves. Can you make a connection between the symbolic solution and your (linear and nonlinear) solution curves?

The sections that follow illustrate the use of computer algebra systems to implement the substitution techniques of Section 1.6. You try out the technique of computer-algebra substitution with the examples and with Problems 1–30 in Section 1.6.

Using *Maple*

As an example we consider the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2 \tag{9}$$

of Example 1 in the text, which calls for the substitution

$$v = x + y + 3\,. \tag{10}$$

We first enter our differential equation (9) in terms of y,

```
yDE :=   diff(y(x),x) = (x+y(x)+3)^2;
```

$$yDE := \frac{\partial}{\partial x} y(x) = (x + y(x) + 3)^2$$

and then the substitution

```
vSubst :=   v(x) = x + y(x) + 3;
```

$$vSubst := v(x) = x + y(x) + 3$$

in (10) for v in terms of y. The inverse substitution for y in terms of v will be

```
ySubst := y(x) = solve(vSubst, y(x));
```

$$ySubst := y(x) = v(x) - x - 3$$

After calculating the derivative dy/dx in terms of dv/dx,

```
DySubst := diff(ySubst,x);
```

$$DySubst := \frac{\partial}{\partial x} y(x) = \left(\frac{\partial}{\partial x} v(x)\right) - 1$$

we substitute for $y(x)$ and $y'(x)$ in the original y-equation to get our transformed differential equation

```
vDE := eval(subs(ySubst, DySubst, yDE));
```

$$vDE := \left(\frac{\partial}{\partial x} v(x)\right) - 1 = v(x)^2$$

to be solved for $v(x)$. Having gotten this far with the computer, we might as well call on *Maple*'s **`dsolve`** function to solve this differential equation.

```
vSolution := dsolve( vDE, v(x) );
```

$$vSolution := v(x) = \tan(x + \_C1)$$

It remains only to re-substitute for v in terms and y and solve explicitly for the solution $y(x)$ of our original differential equation.

```
ySolution := subs(vSubst, vSolution );
```

$$ySolution := x + y(x) + 3 = \tan(x + \_C1)$$

```
y(x) = solve( y_solution, y(x) );
```

$$y(x) = -x - 3 + \tan(x + \_C1)$$

The *Maple* commands shown above provide a "template" that can be used with more complicated differential equations. You need only re-enter your own differential equation **`yDE`** and your desired substitution **`vSubst`**, and then proceed to re-execute the remaining commands.

Using *Mathematica*

Here we illustrate the use of *Mathematica* to solve a homogeneous differential equation of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{11}$$

using the standard substitutions

$$v = \frac{y}{x}, \quad y = v\,x, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}. \tag{12}$$

To solve the differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2 \tag{13}$$

of Example 2 in the text, for instance, we first enter the equation as

```
yDE  =  2 x y[x] y'[x] == 4 x^2 + 3 y[x]^2;
```

The substitution

```
y[x_] := x v[x]
```

in (12) then yields the transformed differential equation

```
vDE = yDE
```

$$2\,x^2 v(x)\big(v(x) + x\,v'(x)\big) == 3\,v(x)^2\,x^2 + 4\,x^2$$

when we re-evaluate **yDE**. Since we're prodeeding entirely by computer, let's just use *Mathematica*'s **DSolve** function to solve for $v(x)$.

```
vSolution = DSolve[vDE, v[x] ,x ]
```

$$\left\{\left\{v(x) \to -\sqrt{x}\,\sqrt{c_1 - \frac{4}{x}}\right\}, \left\{v(x) \to \sqrt{x}\,\sqrt{c_1 - \frac{4}{x}}\right\}\right\}$$

```
v[x_] = v[x] /. vSolution
```

$$\left\{-\sqrt{x}\,\sqrt{c_1 - \frac{4}{x}},\ \sqrt{x}\,\sqrt{c_1 - \frac{4}{x}}\right\}$$

We get two distinct v-solutions differing by sign, and the corresponding y-solutions result when we re-evaluate $y(x)$, thereby automatically substituting each formula for $v(x)$ via $y(x) = x\,v(x)$.

```
ySolution = y[x]
```

$$\left\{-x^{3/2}\,\sqrt{c_1 - \frac{4}{x}},\ x^{3/2}\,\sqrt{c_1 - \frac{4}{x}}\right\}$$

The implicit solution $y^2 + 4x^2 = Cx^3$ found in the text results when we square both sides.

```
implicitSolution =
y^2 == Expand[Simplify[ First[y[x]]^2 ]] /. C[1] -> C
```

$y^2 = Cx^3 - 4x^2$

The *Mathematica* commands shown above provide a "template" that can be used with more complicated differential equations. You need only re-enter your own differential equation **yDE** and your desired substitution — defining **y[x]** in terms of **x** and **v[x]** — and then proceed to re-execute the remaining commands.

Using MATLAB

As an example we consider the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2 \tag{9}$$

of Example 1 in the text, which calls for the substitution

$$v = x + y + 3. \tag{10}$$

We first enter our differential equation (9) in terms of y,

```
syms x y v
yDE = 'Dy = (x+y+3)^2'

yDE =
Dy = (x+y+3)^2
```

and the substitution

```
vSubst = 'x+y+3'

vSubst =
x+y+3
```

for v in terms of y. To invert this substitution, we set up the equation **v = vSubst** and solve for y in terms of v:

```
ySubst = solve(v-vSubst,y)

ySubst =
v-x-3
```

We calculate the derivative

```
DySubst = 'Dv-1'

DySubst =
Dv-1
```

of y in terms of the derivative of v, and then transform the original differential equation **`yDE`** by succesively substituting **`ySubst`** for **`y`** and **`DySubst`** for **`Dy`**.

```
tempDE = subs(yDE, ySubst, y)

tempDE =
Dy = v^2

vDE = subs(tempDE, DySubst, 'Dy')

vDE =
Dv-1 = v^2
```

Now we would like to do **`dsolve(vDE,'x')`**, but for some reason this gives an error message. Hence we write in the differential equation ourselves to solve for v in terms of x:

```
vSolution = dsolve('Dv-1=v^2','x')

vSolution =
tan(x-C1)
```

Finally, we resubstitute for v in terms of y and solve for $y(x)$:

```
vEquation = subs(v-vSolution,vSubst,v)

vEquation =
x+y+3-tan(x-C1)

y = solve(vEquation, y)

y =
-x-3+tan(x-C1)
```

The MATLAB commands shown above provide a "template" that can be used with more complicated differential equations. You need only re-enter your own differential equation **`yDE`** and your desired substitution **`vSubst`**, and then proceed to re-execute the remaining commands.