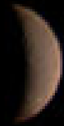
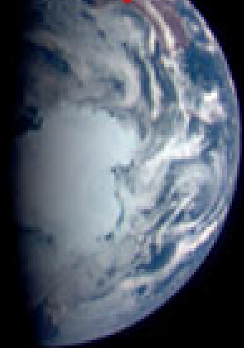
Applications of Differential Equations



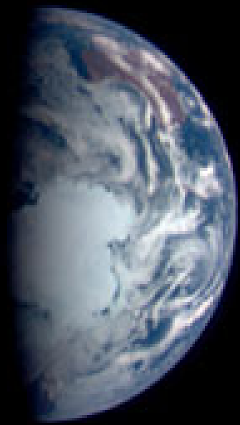
A Presentation of the Two-Body Problem

Noah Brisby and Robin R. Rumple



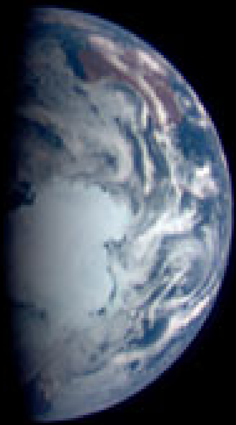
Introduction

In this presentation we will examine the motion of two bodies in space according to their mutual gravitational attraction-as proposed by Isaac Newton.



Conceptual Development

The first step with developing the solution to the Two-Body Problem is to present an illustration of the geometry. Figure 1 shows the basic concept of the two bodies of interest in three space.



As seen from Three Space

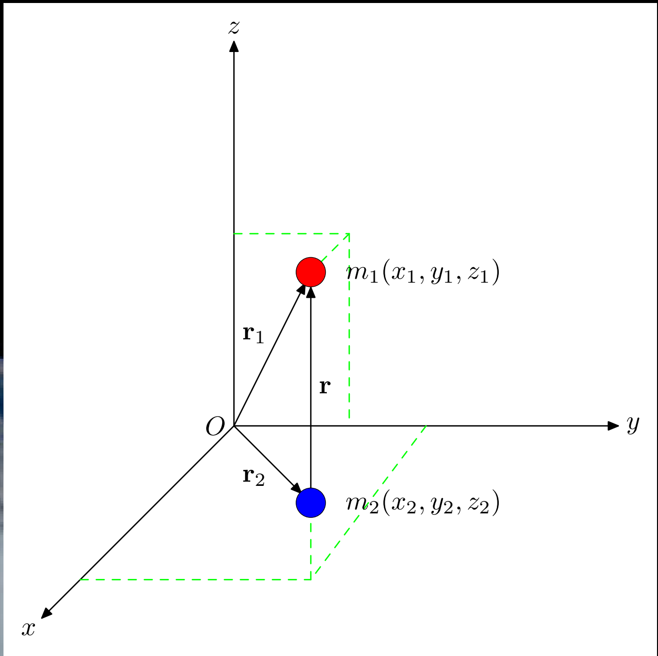


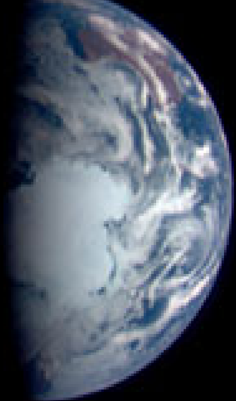
Figure 1

Mathematical Development

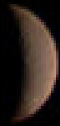
To begin to quantify the motion of the two bodies in respect to origin 0 we begin with Newton's second law of motion

$$\mathbf{F} = m\mathbf{a}.$$

In the process of deriving the equations of motion, the dimension of the model has been reduced to two-space through geometry magic. Figure 2 shows our simplified model.



As seen from Two Space



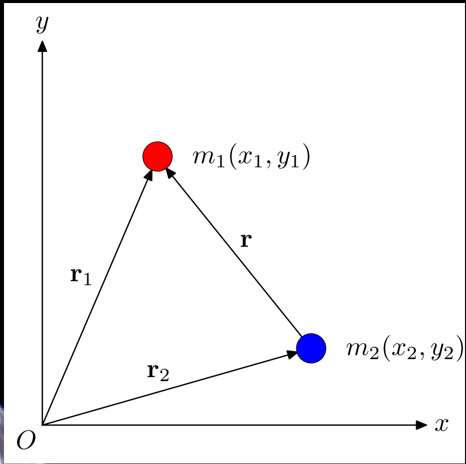


Figure 2

With the simplified model and waves of the magical mathematical wand,

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{bmatrix} = \frac{Gm_2}{(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})^3} \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)$$

and

$$\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = \frac{Gm_1}{(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})^3} \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

are derived.

Plotting these four second order differential equations in Matlab provides figure 3.

Motion of the Two Bodies with specific Initial Conditions

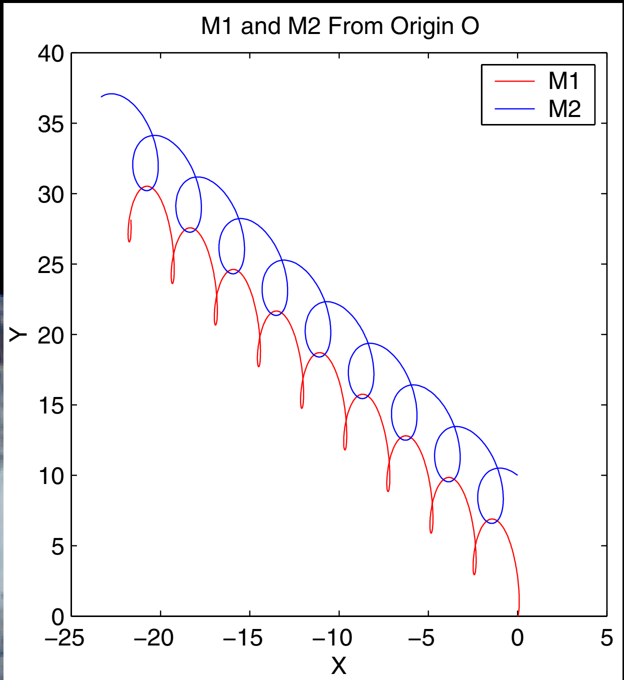


Figure 3

The One-Body Problem

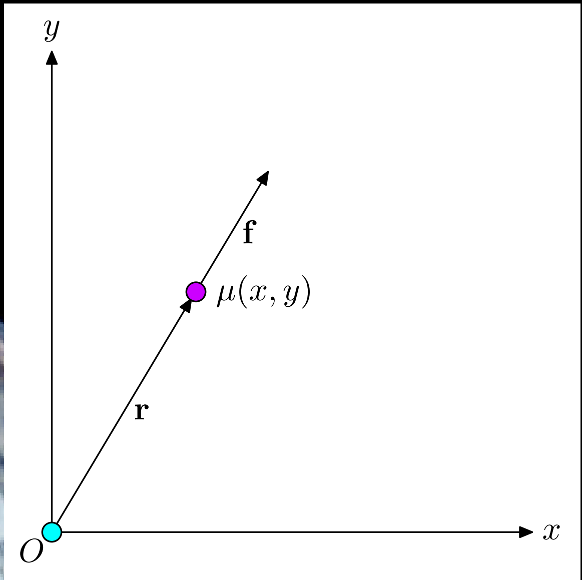


Figure 4

To derive of the motion of one body as seen from another we begin with

$$\mathbf{F} = m\mathbf{a}$$

once again. Further wand magic produces

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \frac{G(m_1 + m_2)}{(\sqrt{x^2 + y^2})^3} \begin{bmatrix} x \\ y \end{bmatrix}.$$

These two second order differential equations provide pictures of the orbits of one body in relation to another. Depending on the initial conditions there are three different orbits that one body can make in relation to another. The three orbit options are: ellipse, hyperbola, and parabola. The three orbit types depend on the eccentricity of the system. We will visualize two of them.

The Elliptical Orbit

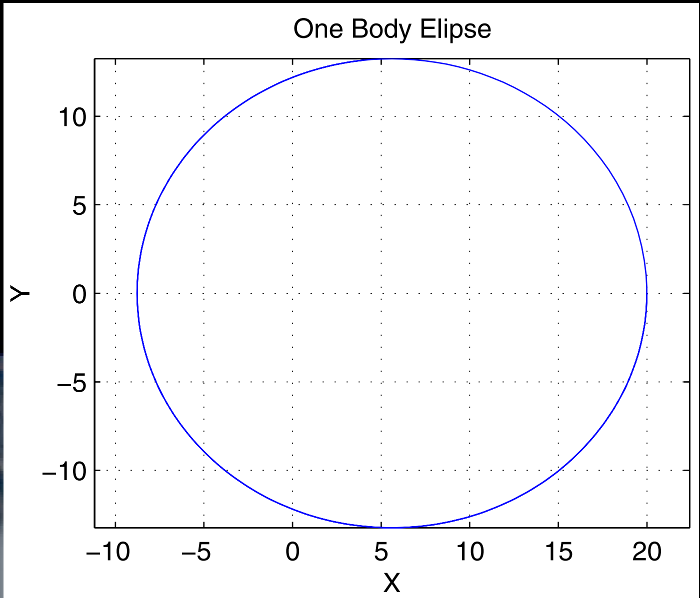


Figure 5

The Hyperbolic Orbit

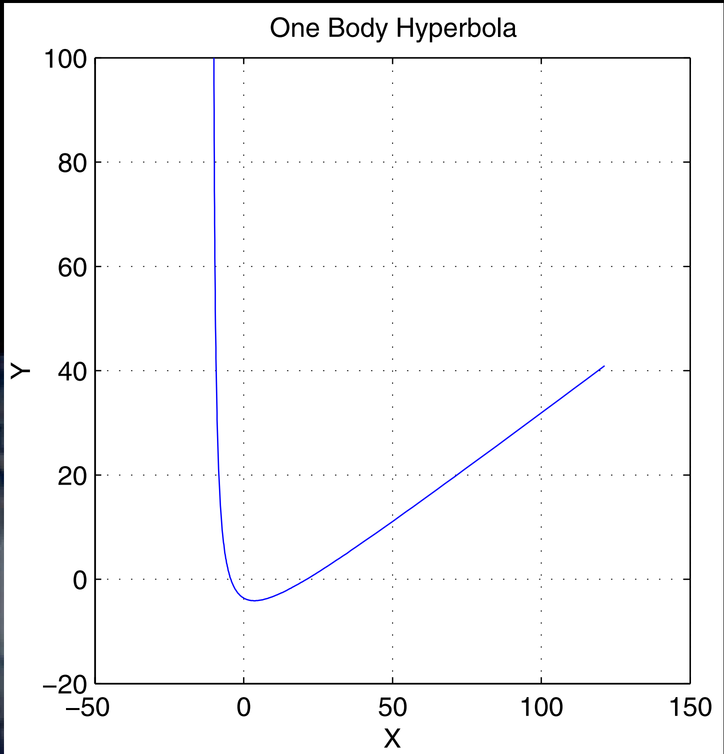
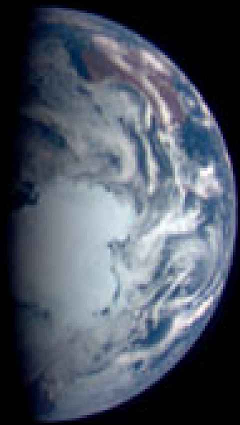


Figure 6

Figures 5 and 6 demonstrate two of the three possible orbits of one body in relation to another via the One-Body Problem. The elliptical orbit in particular provides much interest.

Our One-Body Model is easily manipulated to mathematically derive Kepler's first and second laws of planetary motion - which he coined through observational data.



Derivations for two of Kepler's Laws

First we will derive Kepler's first law which shows that planetary orbits are ellipses. We begin with

$$\mathbf{F} = m\mathbf{a}.$$

With mathematical derivation based on the geometry of the One-Body Problem we obtain

$$\mu * \mathbf{a} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}.$$

From the above equation we then derive

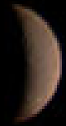
$$r = \frac{ed}{1 + e\cos\theta}.$$

This is the polar equation of a conic section, specifically the ellipse, if e is less than one.

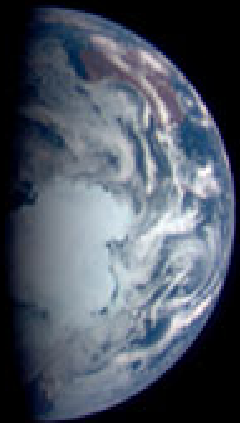
It is also easy to show Kepler's second law of planetary motion. That law states that equal areas are swept out by the vector $\mathbf{r}$ in an equal

time span. The second law is quantified by

$$\frac{dA}{dt} = \frac{L}{2\mu}$$

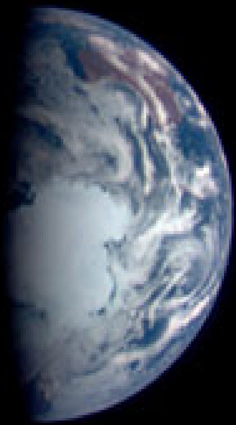


where L is the angular momentum, which is constant, and μ is the "reduced mass" of the system, which is also constant.



Conclusion

The solution to the Two-Body Problem has had a large impact on the mathematical community because it demonstrates a quantifiable proof observational laws.



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