Lines and Planes in MATLAB

Math 50C — Multivariable Calculus

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Abstract

In this activity you will learn how to plot lines and and planes. *Prequisites: Some knowledge of how to enter vectors and matrices in MATLAB. Some familiarity with MATLAB's array operations and* `plot` *command is useful.*

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Introduction

This is an interactive document designed for online viewing. We've constructed this onscreen documents because we want to make a conscientious effort to cut down on the amount of paper wasted at the College. Consequently, printing of the onscreen document has been purposefully disabled. However, if you are extremely uncomfortable viewing documents onscreen, we have provided a print version. If you click on the Print Doc button, you will be transferred to the print version of the document, which you can print from your browser or the Acrobat Reader. We respectfully request that you only use this feature when you are at home. Help us to cut down on paper use at the College.

Much effort has been put into the design of the onscreen version so that you can comfortably navigate through the document. Most of the navigation tools are evident, but one particular feature warrants a bit of explanation. The section and subsection headings in the onscreen and print documents are interactive. If you click on any section or subsection header in the onscreen document, you will be transferred to an identical location in the print version of the document. If you are in the print version, you can make a return journey to the onscreen document by clicking on any section or subsection header in the print document.

Finally, the table of contents is also interactive. Clicking on an entry in the table of contents takes you directly to that section or subsection in the document.

Working with Matlab

This document is a working document. It is expected that you are sitting in front of a computer terminal where the Matlab software is installed. You are not supposed to read this document as if it were a short story. Rather, each time your are presented with a Matlab command, it is expected that you will enter the command, then hit the Enter key to execute the command and view the result. Furthermore, it is expected that you will ponder the result. Make sure that you completely understand why you got the result you did before you continue with the reading.

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Lines

Most of us are familiar with the concept of slope that we learned in our college algebra classes.

Definition 1

The slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Along with this concept come the familiar equations of a line.

Definition 2

The equation of a line having slope m and y-intercept b is

$$y = mx + b.$$

This called the *slope-intercept* form. The equation of the line having slope m that passes through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

This is called the *point-slope* form of the line

When we add an extra dimension, the concept of slope is lost in 3-space and the equations in **Definition 2** become useless. However, lines in 3-space still have a definite "direction," which can be completely described by a vector pointing in the direction of the line. Thus, if we are given a point on the line, say (x_0, y_0, z_0), and a vector pointing in the direction of the line, say $\mathbf{v}$, then our line is completely determined.

For example, consider a line in the plane, shown **Figure 1**, passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

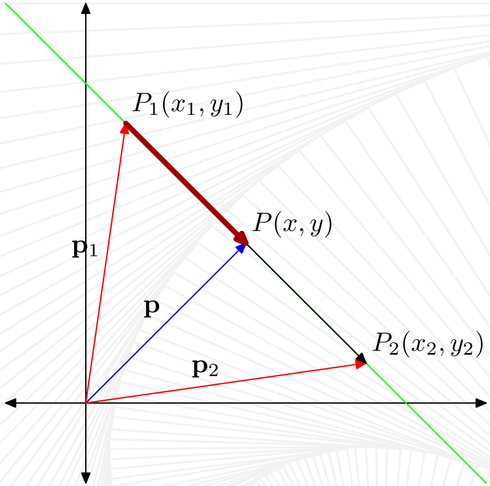

Figure 1 A line through two points.

Clearly, the vector $\overrightarrow{P_1P_2}$ is a vector pointing in the direction of the line. This vector is easily calculated, because it is the difference of the position vectors $\mathbf{p}_1$ and $\mathbf{p}_2$.

$$\begin{aligned}\overrightarrow{P_1P_2} &= \mathbf{p}_2 - \mathbf{p}_1\\ &=< x_2, y_2 > - < x_1, y_1 >\\ &=< x_2 - x_1, y_2 - y_1 >\end{aligned}$$

Now, let $P(x, y)$ be an arbitrary point on the line. Let $\mathbf{p}$ be its position vector, as shown in **Figure 1**. Note that vector $\overrightarrow{P_1P}$ is parallel to vector $\overrightarrow{P_1P_2}$. Consequently, $\overrightarrow{P_1P}$ must be a multiple of vector $\overrightarrow{P_1P_2}$. In symbols,

$$\overrightarrow{P_1P} = t\overrightarrow{P_1P_2}$$

where t is a scalar (real number).

Next, note that the position vector $\mathbf{p}$ shown in **Figure 1** is the sum of the vector $\mathbf{p}_1$ and the vector $\overrightarrow{P_1P}$. That is,

$$\mathbf{p} = \mathbf{p}_1 + \overrightarrow{P_1P}$$

Thus,

$$\mathbf{p} = \mathbf{p}_1 + t\overrightarrow{P_1P_2}, \tag{1}$$

We need to pause and make a very important point before continuing with the development of the equation of our line. Note that if $t = 1$, **equation 1** becomes

$$\mathbf{p} = \mathbf{p}_1 + \overrightarrow{P_1P_2},$$

In this case, point P would coincide with point P_2. If we were to let $t = 1/2$, then **equation 1** becomes

$$\mathbf{p} = \mathbf{p}_1 + (1/2)\overrightarrow{P_1P_2},$$

and point P would be located on the line at the midpoint of the segment connecting points P_1 and P_2. If we let $t = 2$, then point P would be located beyond point P_2, at a distance that is twice as far away from point P_1 as is point P_2. If we let $t = -1$, then point P moves in the reverse direction, beyond point P_1, so that P is located opposite P_2, but at the same distance as P_2 is from P_1. If we let $0 \le t \le 1$, then P ranges from P_1 to P_2, defining the line *segment* joining point P_1 to point P_2.

Continuing with the development of the equation of our line, **equation 1** can be written in component form.

$$<x, y> = < x_1, y_1 > + t < x_2 - x_1, y_2 - y_1 > \tag{2}$$

If we equate components in this last equation, we have the so-called *parametric equations* of the line.

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \end{aligned} \tag{3}$$

An interesting comparison can be made if we eliminate t from these equations. Solve each equation for t, then equate to get

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

This is easily place in the equivalent form

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$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Note that this is exactly the same as the point-slope form of the line given in **Definition 2**.

By simply adding a third dimension to the image in **Figure 1**, the parametric equations given in **3** become

$$\begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \\ z &= z_1 + t(z_2 - z_1) \end{aligned} \tag{4}$$

Let's pause to use MATLAB to draw lines in the plane and 3-space.

Example 1

Use MATLAB to draw the line passing through the points $P_1(1,1)$ and $P_2(3,-3)$.

Solution. The line through P_1 in the direction of $\overrightarrow{P_1P_2}$ is

$$\mathbf{p} = \mathbf{p}_1 + t\overrightarrow{P_1P_2}. \tag{5}$$

Calculate $\overrightarrow{P_1P_2}$.

$$\overrightarrow{P_1P_2} =< 3-1, -3-1 >=< 2, -4 >$$

Hence, in component form, equation ?? becomes

$$< x, y >=< 1, 1 > +t < 2, -4 >,$$

or, in parametric form,

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - 4t \end{aligned}.$$

We will plot the segment that runs from P_1 to P_2 (see comments above) by restricting t to the interval $[0, 1]$.

```
>> t=linspace(0,1);
>> x=1+2*t;y=1-4*t;
>> plot(x,y)
```

These commands produce the image shown in **Figure 2**.

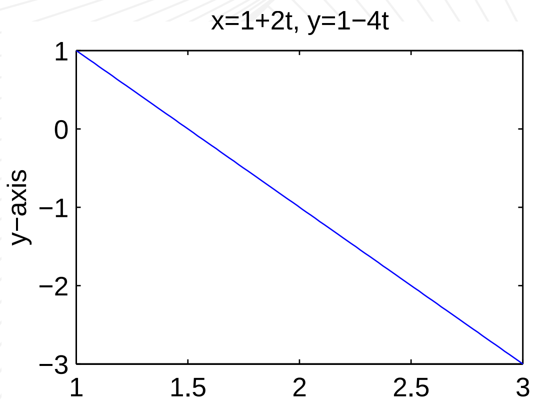

Figure 2 A line segment connecting $P_1(1,1)$ and $P_2(3,-3)$.

It is a simple matter to extend this last example to 3-space.

Example 2

Use MATLAB to plot the line passing through the points $P_1(1,-1,2)$ and $P_2(2,5,-2)$.

Solution. Calculate $\overrightarrow{P_1P_2}$.

$$\overrightarrow{P_1P_2} = < 2-1, 5-(-1), -2-2 > = < 1, 6, -4 >$$

Thus, in component form, equation ?? becomes

$$< x, y, z > = < 1, -1, 2 > + t < 1, 6, -4 > .$$

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Comparing components reveals the parametric form of the line.

$$x = 1 + t$$
$$y = -1 + 6t$$
$$z = 2 - 4t$$

We again plot the segment running from P_1 to P_2 by restricting t to $[0, 1]$.

```
>> t=linspace(0,1);
>> x=1+t; y=-1+6*t; z=2-4*t;
>> plot3(x,y,z)
>> xlabel('x-axis')
>> ylabel('y-axis')
>> zlabel('z-axis')
>> title('x=1+t, y=-1+6t, z=2-4t')
>> axis tight
>> grid
>> box on
>> view([150,30])
```

There are some new commands here that warrant explanation. Clearly, the `plot3` command is the counterpart of the `plot` command in 3-space and we've used the `zlabel` command to label the z-axis. The `axis tight` command tightens the plot to the axis constraints and the `grid` command toggles the grid on. The `box on` command adds a three dimensional box which improves interpretation. Finally the `view` command rotates the axis (`[azimuth,elevation]`) so that the axis are aligned according to the right-hand rule used on the classroom white boards during lecture. The effect of each of these commands is evident in **Figure 3**.

Planes

Like the line in space, a plane is completely determined by a minimal set of information. Imagine a point in space. There are an infinite number of planes that pass through this given point. However, there is only one plane that passes through this point that is orthogonal to a given vector.

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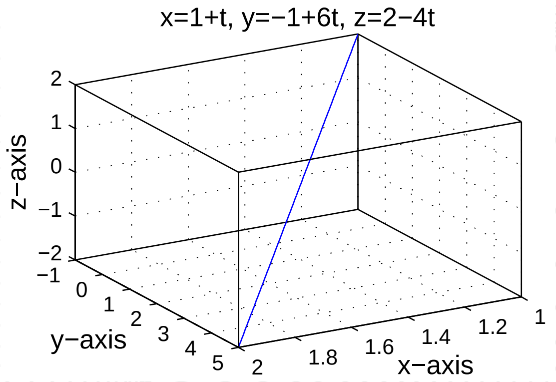

Figure 3 A line segment connecting $P_1(1,-1,2)$ and $P_2(2,5,-2)$.

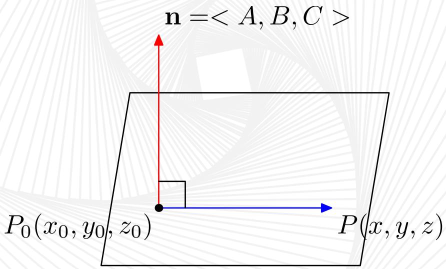

Figure 4 Vector **n** is orthogonal to the plane.

Consider the plane shown in **Figure 4**. Note that the plane contains a given point $P_0(x_0, y_0, z_0)$ and is orthogonal to a given vector $\mathbf{n} =< A, B, C >$.

Let $P(x, y, z)$ be an arbitrary point in the plane. The key to finding the equation of the plane is to note that the vectors $\overrightarrow{P_0P}$ and $\mathbf{n}$ are orthogonal. Therefore,

$$\overrightarrow{P_0P} \cdot \mathbf{n} = 0 \tag{6}$$

is the equation of the plane. This is the key idea, the rest is simple arithmetic.

$$
\begin{aligned}
< x - x_0, y - y_0, z - z_0 > \cdot < A, B, C > &= 0 \\
A(x - x_0) + B(y - y_0) + C(z - z_0) &= 0 \\
Ax + By + Cz - (Ax_0 + By_0 + Cz_0) &= 0
\end{aligned}
$$

Letting $D = -Ax_0 - By_0 - Cz_0$, the equation of the plane is

$$Ax + By + Cz + D = 0. \tag{7}$$

Many students of multivariable calculus find it useful to memorize this equation of the plane. Indeed, it is handy for quick solutions. For example, suppose that we wish to find the equation of the plane passing through $(1, 1, 1)$ and orthogonal to the vector $\mathbf{n} =< 1, 2, 3 >$. A typical approach is to substitute 1, 2, and 3, for A, B, and C, getting $x + 2y + 3z + D = 0$. Now, use the point $(1, 1, 1)$ to find the value of D with this calculation:

$$
\begin{aligned}
x + 2y + 3z + D &= 0 \\
1 + 2(1) + 3(1) + D &= 0 \\
D &= -6.
\end{aligned}
$$

Therefore, the equation of the plane is $x + 2y + 3z - 6 = 0$. However, this computation does nothing to strengthen our knowledge of vector geometry, so I'll avoid this approach like the plague. I prefer to work with the image shown in **Figure 4** and the resulting **equation 6**.

However, I do find it useful to remember that the vector normal to the plane $Ax + By + Cy + D = 0$ is $\mathbf{n} =< A, B, C >$. For example, a normal vector to the plane $2x - 3y + 5z - 11 = 0$ is $< 2, -3, 5 >$. I will use this often.

Surfaces in Matlab

In single variable calculus, we studied functions that mapped the real numbers into the real numbers. In symbols, $f : \mathbf{R} \to \mathbf{R}$. In the plane ($\mathbf{R}^2$), the graph of this function f is defined as follows.

Definition 3

Suppose that $f : \mathbf{R} \to \mathbf{R}$. Then the graph of f is

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$$\{(x, f(x)) : \ x \in \text{Domain of } f\}$$

That is, the graph of f is the set of all ordered pairs that satisfy the equation of f. For example, suppose that $f : \mathbf{R} \to \mathbf{R}$ is defined by the equation $f(x) = x^2$. An easy calculation shows that $f(2) = 4$. Therefore, the ordered pair $(2, 4)$ lies on the graph of f.

Now, suppose that we have a function that maps elements in the plane onto the real line. That is, $f : \mathbf{R}^2 \to \mathbf{R}$. Then the graph of f lies in 3-space ($\mathbf{R}^3$) and is defined as follows.

Definition 4

Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$. Then the graph of f is

$$\{(x, y, f(x, y)) : \ (x, y) \in \text{Domain of } f\}$$

Thus, the graph of f is the set of all ordered triples that satisfy the equation of f. For example, suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by the equation $f(x, y) = x^2 + y^2$. Then an easy calculation shows that $f(2, 1) = 5$. Therefore, $(2, 1, 5)$ is an ordered triple that lies on the graph of f. This point is shown in Figure ??.

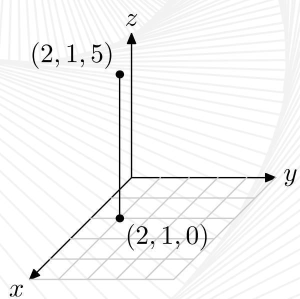

Figure 5 Point $(2, 1, 5)$ is on the graph of f.

To plot the graph of $f(x) = x^2$ in the plane, we begin by making a table of points that satisfy the equation, as shown in **Table 1**

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x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9

Table 1 A table of points satisfying $f(x) = x^2$.

However, if $f(x, y) = x^2 + y^2$, the domain of f lies in $\mathbf{R}^2$. Thus, for each ordered pair (x, y), our function computes an output $z = f(x, y)$. It is the ordered triple (x, y, z) that must be plotted. We must come up with a new strategy for creating a table of points that satisfy the equation $f(x, y) = x^2 + y^2$. MATLAB accomplishes this with the `meshgrid` command.

```
>> [X,Y]=meshgrid([1,2,3,4,5])
X =
     1     2     3     4     5
     1     2     3     4     5
     1     2     3     4     5
     1     2     3     4     5
     1     2     3     4     5
Y =
     1     1     1     1     1
     2     2     2     2     2
     3     3     3     3     3
     4     4     4     4     4
     5     5     5     5     5
```

This rather cryptic output warrants extensive explanation. Actually, the output is easily understood if one superimposes the matrix Y onto the matrix X to obtain a grid of ordered pairs.

Therefore, **Table 2** contains a set of points in the plane that we will substitute into the function $f(x, y) = x^2 + y^2$. MATLAB's array smart operators make this an easy proposition.

```
>> Z=X.^2+Y.^2
Z =
     2     5    10    17    26
     5     8    13    20    29
```


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$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$
$(1,1)$	$(2,1)$	$(3,1)$	$(4,1)$	$(5,1)$

Table 2 Interpreting the output of `meshgrid`.

```
10    13    18    25    34
17    20    25    32    41
26    29    34    41    50
```

The alert reader will want use their calculator (mental calculations are also good) to check that these points actually satisfy the equation $f(x,y) = x^2 + y^2$. It is now an easy task to plot the surface to which these points belong. The following command was used to produce the image in **Figure 6**.

```
>> mesh(X,Y,Z)
```

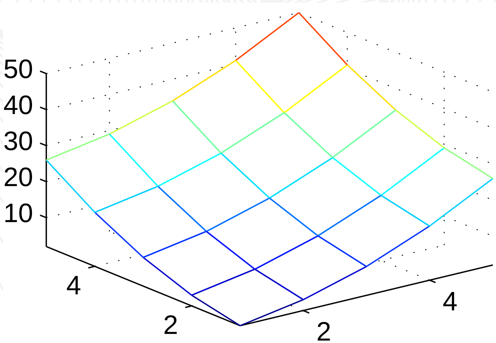

Figure 6 Plotting the surface $f(x,y) = x^2 + y^2$.

Recall how MATLAB's `plot` command was used to draw the graphs of functions.

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```
>> x=linspace(0,2*pi);
>> y=sin(2*x);
>> plot(x,y)
```

This set of commands would plot the points in the vectors x and y, then line segments were used to connect consecutive points. If you plotted enough points, then the plot took on the shape of a smooth curve. Too few points and your plot had a "jagged" look.

When MATLAB plots a surface, a similar thing occurs. MATLAB plots the points, then neighboring points are connected with segments. The surface takes on the appearance of a mesh, where each set of four neighboring points seemed to be joined with small quadrilaterals. Again, if you plot too few points, the surface takes on a "jagged" look and feel. To draw a smoother surface, plot more points. The following commands were used to draw the image in **Figure 7**.

```
>> [X,Y]=meshgrid(1:.2:5);
>> Z=X.^2+Y.^2;
>> mesh(X,Y,Z)
```

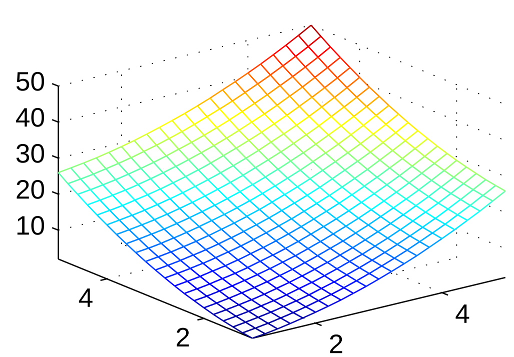

Figure 7 A smoother plot of the surface $f(x, y) = x^2 + y^2$.

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A Plane is a Surface

But this activity is about plotting lines and planes, so let's use our newly found knowledge of surface plotting in MATLAB. After all, a plane is an example of a surface.

Example 3

Plot the plane that passes through the point $(-3, 4, -2)$ that is orthogonal to the vector $\mathbf{n} =< 1, 2, 3 >$ ..

Solution. Sketch the plane, the given point, and the normal vector.

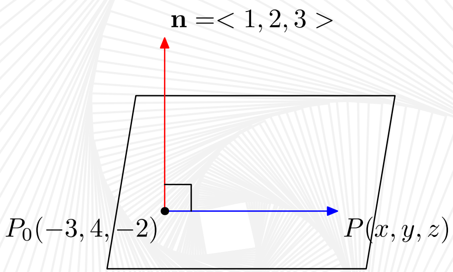

Figure 8 A plane through $(-3, 4, -3)$ orthogonal to $< 1, 2, 3 >$.

The equation of the plane is

$$\begin{aligned} \overrightarrow{P_0P} \cdot \mathbf{n} &= 0 \\ < x + 3, y - 4, z + 2 > \cdot < 1, 2, 3 > &= 0 \\ 1(x + 3) + 2(y - 4) + 3(z + 2) &= 0 \\ x + 2y + 3z + 1 &= 0. \end{aligned}$$

Solve this last equation for z.

$$z = \frac{-1 - x - 2y}{3}$$

Set up a grid of points. We will let $-5 \le x \le 5$ and $-3 \le y \le 3$.

```
>> x=-5:5; y=-3:3;
>> [X,Y]=meshgrid(x,y);
```

Calculate the z-value for each ordered pair (x, y).

```
>> Z=(-1-X-2*Y)/3;
```

Finally, plot the plane.

```
>> mesh(X,Y,Z)
>> axis tight
>> grid
>> view([150,30])
```

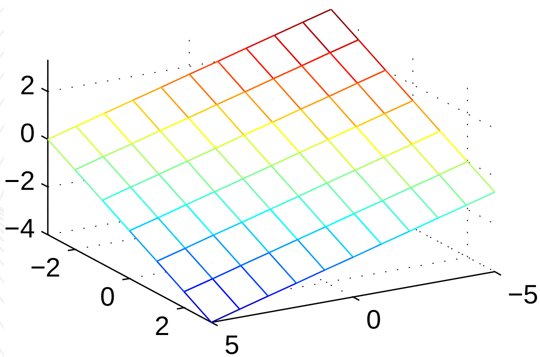

Figure 9 The graph of $x + 2y + 3z + 1 = 0$ for $-5 \le x \le 5$ and $-3 \le y \le 3$.

Exercises

1. Show that the equation of a line passing through the point (x_0, y_0, z_0) in the direction of vector $\mathbf{v} = < a, b, c >$ has parametric equations

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$$x = x_0 + at$$
$$y = y_0 + bt\,.$$
$$z = z_0 + ct$$

Further, assuming that a, b, and c are nonzero, eliminate the parameter t to show that

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

2. Use MATLAB to sketch the graph of the line passing through the points $(2, -3, 5)$ and $(-1, -1, -6)$.

3. Use MATLAB to sketch the graph of the line passing through the point $(1, 1, 1)$ that is parallel to the line given by $x = 2 - 3t$, $y = 5t$, and $z = 2 + 2t$.

4. Sketch the plane passing through the point $(2, 3, -1)$ that is orthogonal to the vector $< 1, -1, 3 >$.

5. Sketch the plane passing through the point $(1, 2, 3)$ that is perpendicular to the line defined by

$$\frac{x - 1}{2} = \frac{y + 1}{-3} = \frac{z - 2}{4}.$$

6. Sketch the plane passing through the points $(0, 0, 0)$, $(1, 2, -1)$, and $(3, 0, -4)$.

7. Find the equation of the line of intersection of the planes $x + 2y + z + 4 = 0$ and $2x - y - z - 4 = 0$. On one plot, plot the two planes, then superimpose the plot of the line of intersection. *Note: There are a number of commands that you might find useful:* `hold`, `line`, `hidden`. *Help for each of these is available in MATLAB's help files. For example, click the ? on MATLAB's toolbar and from the View menu select Help View Options, then check Show Help Navigator. Use the index tab to search for the line command. Read the help file. There are some real gems in this file. It will also take some experimentation to find a viewing window that nicely show both planes and the line of intersection.*

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