Application 5.5
Automated Variation of Parameters

The method of variation of parameters — as described in the final part of Section 5.5 in the text — is readily implemented using a computer algebra system. In the paragraphs below we illustrate the use of *Maple*, *Mathematica*, and MATLAB in finding a particular solution of the differential equation

$$y'' + y = \tan x \tag{1}$$

of Example 11 in Section 5.5, to which the method of undetermined coefficients does not apply. In each case the initial commands serve to enter the two independent homogeneous solutions $y_1 = \cos x$ and $y_2 = \sin x$ and the nonhomogeneous term $f(x) = \tan x$ in Equation (1). The final commands implement the variation of parameters formula

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx \tag{2}$$

To solve similarly another second-order linear differential equation

$$y'' + P(x)\,y' + Q(x)\,y = f(x) \tag{3}$$

whose complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$ is known, we need only insert the corresponding definitions of $y_1(x)$, $y_2(x)$, and $f(x)$ in the initial commands. Find in this way the indicated particular solution $y_p(x)$ of each of the nonhomogeneous equations in Problems 1–6 below.

1. $y'' + y = 2\sin x$ $\qquad y_p(x) = -x\cos x$

2. $y'' + y = 4x\sin x$ $\qquad y_p(x) = x\sin x - x^2\cos x$

3. $y'' + y = 12x^2\sin x$ $\qquad y_p(x) = 3x^2\sin x + \left(3x - 2x^3\right)\cos x$

4. $y'' - 2y' + 2y = 2e^x\sin x$ $\qquad y_p(x) = -xe^x\cos x$

5. $y'' - 2y' + 2y = 4xe^x\sin x$ $\qquad y_p(x) = e^x\left(x\sin x - x^2\cos x\right)$

6. $y'' - 2y' + 2y = 12x^2 e^x \sin x$ $y_p(x) = e^x\left[3x^2 \sin x + \left(3x - 2x^3\right)\cos x\right]$

Using *Maple*

First we enter the independent complementary solutions

```
y1 := cos(x):
y2 := sin(x):
```

and the nonhomogeneous term

```
f := tan(x):
```

in Eq. (1). Then we calculate and simplify the Wronskian

```
W := y1*diff(y2,x) - y2*diff(y1,x):
W := simplify(W);
```

$$W := 1$$

of y_1 and y_2. It remains only to calculate the desired particular solution

```
yp := -y1*int(y2*f/W,x)+y2*int(y1*f/W,x):
yp := simplify(yp);
```

$$yp := -\cos(x)\ln\left(\frac{1+\sin(x)}{\cos(x)}\right)$$

using formula (2) above. Do you see that this result is equivalent to the particular solution

$$y_p = -(\cos x)\ \ln\left(\sec x + \tan x\right)$$

found in the text?

Using *Mathematica*

First we enter the independent complementary solutions

```
y1 = Cos[x];
y2 = Sin[x];
```

and the nonhomogeneous term

```
f = Tan[x];
```

in Eq. (1). Then we calculate and simplify the Wronskian

```
W  = y1*D[y2,x] - y2*D[y1,x];
W = Simplify[W]
1
```

of y_1 and y_2. It remains only to calculate the desired particular solution

```
yp = -y1*Integrate[y2*f/W,x] + y2*Integrate[y1*f/W,x];
yp = Simplify[yp]
```

$$-\cos(x)\left(\log\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)-\log\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)\right)$$

using formula (2) above. If you write this result in the form

$$y_p = -(\cos x)\ln\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}$$

and begin by multiplying numerator and denominator of the fraction by $\cos(x/2)-\sin(x/2)$, you should be able to use familiar trigonometric identities to show that the result is equivalent to the particular solution

$$y_p = -(\cos x)\ \ln\left(\sec x+\tan x\right)$$

found in the text.

Using MATLAB

First we enter the independent complementary solutions

```
syms x
y1 = cos(x);
y2 = sin(x);
```

and the nonhomogeneous term

```
f = tan(x)
```

in Eq. (1). Then we calculate and simplify the Wronskian

```
W = y1*diff(y2,x) - y2*diff(y1,x);
```

```
W = simplify(W)

W =
   1
```

of y_1 and y_2. It remains only to calculate the desired particular solution

```
yp = -y1*int(y2*f/W,x)+y2*int(y1*f/W,x);
yp = simplify(yp)
yp =
-cos(x)*log((1+sin(x))/cos(x))
```

using formula (2) above. Thus we find the particular solution

$$y_p(x) = -\cos(x)\ln\left(\frac{1+\sin(x)}{\cos(x)}\right)$$

Do you see that this result is equivalent to the particular solution

$$y_p = -(\cos x)\ \ln(\sec x + \tan x)$$

found in the text?