

## Application 11.4

### Riccati Equations and Modified Bessel Functions

A *Riccati equation* is a first-order differential equation of the form

$$y' = A(x) y^2 + B(x) y + C(x)$$

(with a single nonlinear  $A y^2$  term). Many Riccati equations like the ones listed below can be solved explicitly in terms of Bessel functions.

$$y' = x^2 + y^2 \quad (1)$$

$$y' = x^2 - y^2 \quad (2)$$

$$y' = y^2 - x^2 \quad (3)$$

$$y' = x + y^2 \quad (4)$$

$$y' = x - y^2 \quad (5)$$

$$y' = y^2 - x \quad (6)$$

For instance, Problem 33 in Section 11.4 of the text says that the general solution of (1) is given by

$$y(x) = x \cdot \frac{J_{3/4}\left(\frac{1}{2}x^2\right) - c J_{-3/4}\left(\frac{1}{2}x^2\right)}{c J_{1/4}\left(\frac{1}{2}x^2\right) + J_{-1/4}\left(\frac{1}{2}x^2\right)}. \quad (7)$$

See whether the symbolic DE solver command in your computer algebra system, such as

`dsolve(diff(y(x),x) = x^2 + y(x)^2, y(x))` (Maple)

or

`DSolve[y'[x] == x^2 + y[x]^2, y[x], x]` (Mathematica)

or

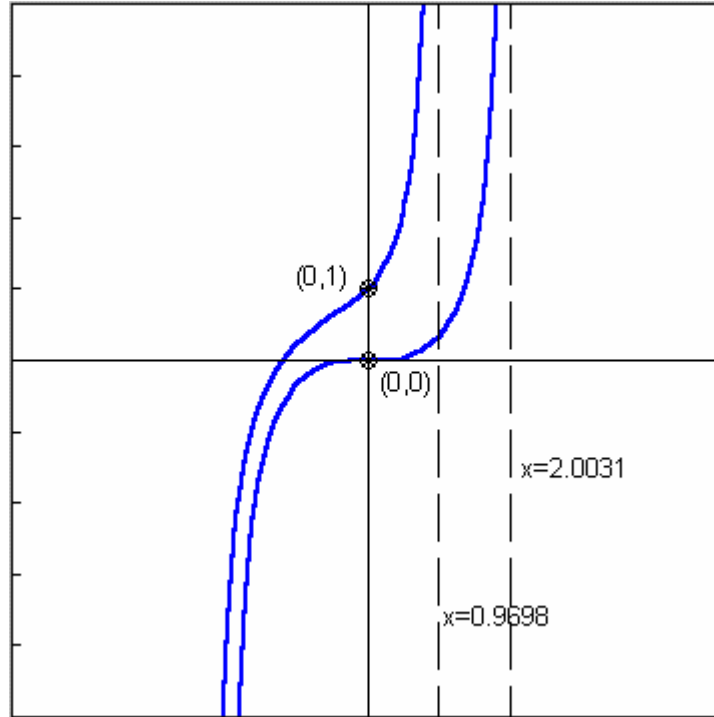
`dsolve('Dy = x^2 + y^2')` (MATLAB)

agrees with (7).

If Bessel functions other than those appearing in (7) are involved, you may need to apply Identities (26) and (27) in Section 11.4 of the text to transform the computer's "answer" to (7). Then see whether your system can take the limit as  $x \rightarrow 0$  in (7) to show that the arbitrary constant  $c$  is given in terms of the initial value  $y(0)$  by

$$c = -\frac{y(0) \Gamma(\frac{1}{4})}{2 \Gamma(\frac{3}{4})} \quad (8)$$

Now you should be able to use built-in Bessel functions to plot typical solution curves like those illustrated below.



This figure shows the trajectories of (1) satisfying the initial conditions  $y(0)=0$  and  $y(0)=1$ , together with apparent vertical asymptotes. Can you use (7) and (8) to verify that these asymptotes are given approximately by  $x = 2.00315$  and  $x = 0.96981$ , respectively? (*Suggestion:* You are looking for zeros of the denominator in (7).)

Next, investigate similarly one of the other equations in (2)–(6). Each has a general solution of the same general form as in (7) — a quotient of linear combinations of Bessel functions. In addition to  $J_p(x)$  and  $Y_p(x)$ , these solutions may involve the *modified Bessel functions*

$$I_p(x) = i^{-p} J_p(ix)$$

and

$$K_p(x) = \frac{\pi}{2} i^{-p} [J_p(ix) + Y_p(ix)]$$

that satisfy the *modified Bessel equation*

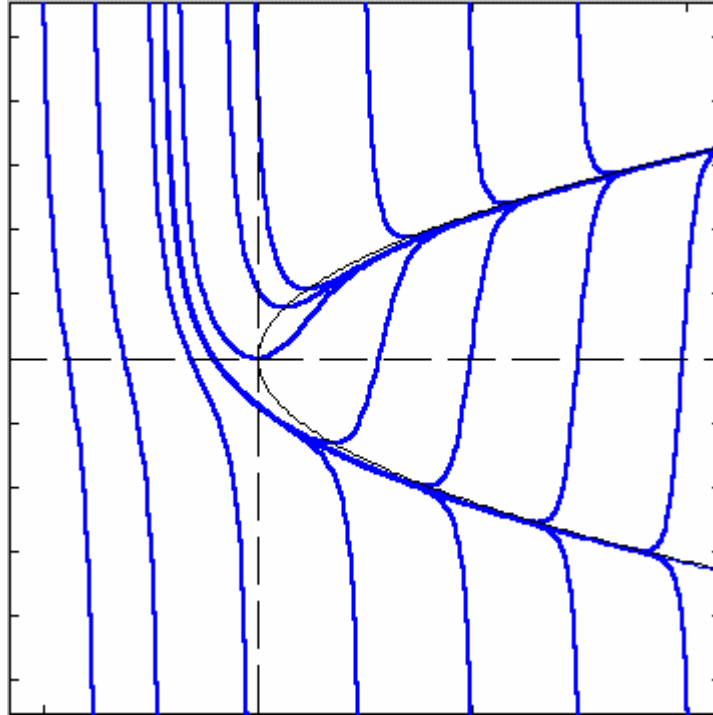
$$x^2 y'' + x y' - (x^2 + p^2) y = 0$$

of order  $p$ . For instance, the general solution of Equation (5) is given for  $x > 0$  by

$$y(x) = \sqrt{x} \cdot \frac{I_{2/3}\left(\frac{2}{3}x^{3/2}\right) - c I_{-2/3}\left(\frac{2}{3}x^{3/2}\right)}{I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) - c I_{1/3}\left(\frac{2}{3}x^{3/2}\right)} \quad (9)$$

where

$$c = -\frac{y(0) \Gamma\left(\frac{1}{3}\right)}{\sqrt[3]{3} \Gamma\left(\frac{2}{3}\right)}. \quad (10)$$



The figure above shows some typical solution curves for Equation (5), together with the parabola  $y^2 = x$  that appears to bear an interesting relation to these curves — we see a funnel near  $y = +\sqrt{x}$  and a spout near  $y = -\sqrt{x}$ .

The Bessel functions with imaginary argument that appear in the definitions of  $I_p(x)$  and  $K_p(x)$  may look exotic, but the power series of the modified function

$I_n(x)$  is simply that of the unmodified function  $J_n(x)$ , except without the alternating minus signs. For instance,

$$I_0(x) = 1 + \frac{x^2}{4} + \frac{x^4}{64} + \frac{x^6}{2304} + \dots$$

and

$$I_1(x) = \frac{x}{2} + \frac{x^3}{16} + \frac{x^5}{384} + \frac{x^7}{18432} + \dots$$

Check these power series expansions using your computer algebra system — look at **BesselI** in either *Maple* or *Mathematica* — and compare them with Eqs. (17)–(18) in Section 11.4 of the text.

In the paragraphs that follow we illustrate the use of *Maple*, *Mathematica*, and MATLAB to investigate solutions of the Riccati equation  $y' = x^2 + y^2$  in Eq. (1).

## Using *Maple*

The general solution of the Riccati equation

$$\text{de} := \text{diff}(y(x), x) = x^2 + y(x)^2:$$

is given by

$$\begin{aligned} \text{soln} &:= \text{dsolve}(\text{de}, y(x)): \\ y &:= \text{subs}(\_C1=c, \text{rhs}(\text{soln})); \end{aligned}$$

$$y0 := -\frac{x \left( c \text{BesselJ}\left(\frac{-3}{4}, \frac{1}{2}x^2\right) + \text{BesselY}\left(\frac{-3}{4}, \frac{1}{2}x^2\right) \right)}{c \text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) + \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)}$$

Note that this form of the solution differs from (7) in that it involves the Bessel functions  $Y_{-3/4}$  and  $Y_{1/4}$  of the second kind rather than the Bessel functions  $J_{-3/4}$  and  $J_{-1/4}$  of the first kind.

In order to impose an initial condition, we must therefore evaluate the limit as  $x \rightarrow 0$  instead of using (8). The left-hand and right-hand limits of  $y(x)$  at  $x = 0$  are given by

$$\text{limit}(y, x=0, \text{right});$$

$$\frac{\Gamma\left(\frac{3}{4}\right)^2 (c+1)}{\pi}$$

```
limit(y, x=0, left);
```

$$-\frac{\Gamma\left(\frac{3}{4}\right)^2 (c+1)}{\pi}$$

We therefore see that the particular solution satisfying the initial condition  $y(0) = 0$  is obtained with the arbitrary constant value  $c = -1$ :

```
y0 := subs(c=-1, y);
```

$$y_0 := -\frac{x\left(-\text{BesselJ}\left(\frac{-3}{4}, \frac{1}{2}x^2\right) + \text{BesselY}\left(\frac{-3}{4}, \frac{1}{2}x^2\right)\right)}{-\text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) + \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)}$$

When we graph this particular solution  $y_0(x)$ ,

```
plot(y0, x=-2..2, -5..5);
```

we see an apparent vertical asymptote near  $x = 2$ . This vertical asymptote corresponds to a zero of the denominator of  $y_0(x)$ , which we proceed to locate:

```
denom(y0);
```

$$\text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}x^2\right) - \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}x^2\right)$$

```
fsolve(denom(y0)=0, x, 1.9..2.1);
```

```
2.003147359
```

Thus this vertical asymptote is given by  $x \approx 2.0031$ .

When you investigate similarly the particular solution  $y_1(x)$  satisfying the initial condition  $y(0) = 1$ , note (from the left-hand and right-hand limits above) that different values of the arbitrary constant  $c$  must be chosen for  $x < 0$  and for  $x > 0$ .

## Using *Mathematica*

The general solution of the Riccati equation

$$\text{de} = y'[x] == x^2 + y[x]^2;$$

is given by

```
soln = DSolve[de, y[x], x];
soln =
y[x] /. (soln /. {(x^4)^(1/2) -> x^2}) // First
```

$$\frac{J_{-\frac{3}{4}}\left(\frac{x^2}{2}\right)x^2 - J_{\frac{5}{4}}\left(\frac{x^2}{2}\right)x^2 + J_{-\frac{5}{4}}\left(\frac{x^2}{2}\right)c_1x^2 - J_{\frac{3}{4}}\left(\frac{x^2}{2}\right)c_1x^2 + J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) + J_{\frac{1}{4}}\left(\frac{x^2}{2}\right)c_1}{-2xJ_{\frac{1}{4}}\left(\frac{x^2}{2}\right) - 2xJ_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)c_1}$$

To "get rid of" the Bessel functions of order  $\pm 5/4$  we need to apply the reduction rules given in Eqs. (26) and (27) of Section 11.4 in the text:

```
rule =
BesselJ[p+s,x] -> (2p/x)BesselJ[p,x] - BesselJ[p-s,x];
rule1 = rule /. {p -> 1/4, s -> 1, x -> x^2/2}
```

$$J_{\frac{5}{4}}\left(\frac{x^2}{2}\right) \rightarrow \frac{J_{\frac{1}{4}}\left(\frac{x^2}{2}\right)}{x^2} - J_{-\frac{3}{4}}\left(\frac{x^2}{2}\right)$$

```
rule2 = rule /. {p -> -(1/4), s -> -1, x -> x^2/2}
```

$$J_{-\frac{5}{4}}\left(\frac{x^2}{2}\right) \rightarrow \frac{J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)}{x^2} - J_{\frac{3}{4}}\left(\frac{x^2}{2}\right)$$

When we make these substitutions and "adjust" the arbitrary constant notation, we get the following simple form

```
soln =
soln /. {rule1, rule2, C[1] -> 1/c} // Simplify
```

$$\frac{x \left( J_{\frac{3}{4}}\left(\frac{x^2}{2}\right) - c J_{-\frac{3}{4}}\left(\frac{x^2}{2}\right) \right)}{c J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) + J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)}$$

that agrees with the general solution given in Eq. (7) above.

In order to impose an initial condition, we evaluate the limit as  $x \rightarrow 0$  explicitly (instead of merely using (8)). The left-hand and right-hand limits of  $y(x)$  at  $x = 0$  are given by

$$\begin{aligned} y_0 &= \text{Limit}[\text{soln}, x \rightarrow 0, \text{Direction} \rightarrow 1] \quad (* \text{ with } x < 0 *) \\ &= \frac{2c \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \\ y_0 &= \text{Limit}[\text{soln}, x \rightarrow 0, \text{Direction} \rightarrow -1] \quad (* \text{ with } x > 0 *) \\ &= -\frac{2c \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \end{aligned}$$

Thus we must use different values of  $c$  for  $x < 0$  and for  $x > 0$ . Using the second value of  $y_0$  above, the value of  $c$  for  $x > 0$  is given by

$$\begin{aligned} c &= c /. \text{Solve}[y_0 == 1, c] // \text{First} \\ &= -\frac{\Gamma\left(\frac{1}{4}\right)}{2 \Gamma\left(\frac{3}{4}\right)} \end{aligned}$$

Finally, then, the particular solution (for  $x > 0$ ) of Eq. (1) such that  $y(0) = 1$  is

$$\begin{aligned} y_1 &= \text{soln} // \text{Simplify} \\ &= \frac{x \left( \Gamma\left(\frac{1}{4}\right) J_{-\frac{3}{4}}\left(\frac{x^2}{2}\right) + 2 \Gamma\left(\frac{3}{4}\right) J_{\frac{3}{4}}\left(\frac{x^2}{2}\right) \right)}{\Gamma\left(\frac{1}{4}\right) J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) - 2 \Gamma\left(\frac{3}{4}\right) J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right)} \end{aligned}$$

When we graph this particular solution  $y_1(x)$ ,

$$\text{Plot}[y_1, \{x, 0, 1\}, \text{PlotRange} \rightarrow \{0, 10\}];$$

we see an apparent vertical asymptote near  $x = 1$ . This vertical asymptote corresponds to a zero of the denominator of  $y_1(x)$ , which we proceed to locate:

$$\begin{aligned} &\text{Denominator}[y_1] \\ &= \Gamma\left(\frac{1}{4}\right) J_{\frac{1}{4}}\left(\frac{x^2}{2}\right) - 2 \Gamma\left(\frac{3}{4}\right) J_{-\frac{1}{4}}\left(\frac{x^2}{2}\right) \end{aligned}$$

```
FindRoot[Denominator[y1]==0, {x,1}]
{x → 0.969811}
```

Thus this vertical asymptote is given by  $x \approx 0.9698$ .

## Using MATLAB

The solution  $y_0(x)$  of the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 0$$

is given by

```
y0 = dsolve('Dy = x^2 + y^2','y(0)=0','x')
pretty(y0)
```

$$-\frac{x \left( -\text{besselj}\left(-\frac{3}{4}, \frac{1}{2} x^2\right) + \text{bessely}\left(-\frac{3}{4}, \frac{1}{2} x^2\right) \right)}{-\text{besselj}\left(\frac{1}{4}, \frac{1}{2} x^2\right) + \text{bessely}\left(\frac{1}{4}, \frac{1}{2} x^2\right)}$$

When we plot this particular solution,

```
ezplot(y0, [0 3])
```

we see an apparent vertical asymptote located near  $x = 2$ . This vertical asymptote corresponds to a zero of the denominator of  $y_0(x)$ , which we proceed to locate:

```
[num, den] = numden(y0);
den
den =
besselj(1/4,1/2*x^2)-bessely(1/4,1/2*x^2)

fzero('besselj(1/4,1/2*x^2)-bessely(1/4,1/2*x^2)', 2)
ans =
2.0031
```

Thus this vertical asymptote is given by  $x \approx 2.0031$ .

The solution  $y_1(x)$  of the initial value problem

$$y' = x^2 + y^2, \quad y(0) = 1$$

is given by



```
y1 = dsolve('Dy = x^2 + y^2','y(0)=1','x');
```

A fairly simple form for this particular solution  $y_1(x)$  is given by

```
y1 = simplify(y1);
pretty(y1)
```

$$-x \left( -\text{besselj}\left(-\frac{3}{4}, \frac{1}{2} x\right) \text{gamma}\left(\frac{3}{4}\right) + \text{bessely}\left(-\frac{3}{4}, \frac{1}{2} x\right) \text{gamma}\left(\frac{3}{4}\right) \right. \\ \left. + \pi \text{besselj}\left(-\frac{3}{4}, \frac{1}{2} x\right) \right) \bigg/ \left( -\text{besselj}\left(\frac{1}{4}, \frac{1}{2} x\right) \text{gamma}\left(\frac{3}{4}\right) \right. \\ \left. + \text{bessely}\left(\frac{1}{4}, \frac{1}{2} x\right) \text{gamma}\left(\frac{3}{4}\right) + \pi \text{besselj}\left(\frac{1}{4}, \frac{1}{2} x\right) \right)$$

When we graph this solution,

```
ezplot(y1, [0 1])
```

we see an apparent vertical asymptote located near  $x = 1$ .

The observed vertical asymptote to the graph  $y = y_1(x)$  corresponds to a zero of the denominator of  $y_1(x)$ , which we proceed to locate:

```
[num, den] = numden(y1);
den
den =
-besselj(1/4,1/2*x^2)*gamma(3/4)^2+
bessely(1/4,1/2*x^2)*gamma(3/4)^2+
pi*besselj(1/4,1/2*x^2)

fzero('-besselj(1/4,1/2*x^2)*gamma(3/4)^2+
      bessely(1/4,1/2*x^2)*gamma(3/4)^2+
      pi*besselj(1/4,1/2*x^2)', 1)
ans =
    0.9698
```

Thus this vertical asymptote is given by  $x \approx 0.9698$ .