Quantum Junctions

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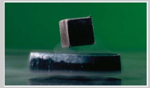
Abstract

Have you ever looked at the stars and thought how can someone even detect the faintest pinpricks of light after they have traveled 800 million lightyears. Or looked at certain scientific instruments and wondered how they worked. The devices that were used in the two examples are based, primarily, around a specific phenomenon. This phenomenon is called the Josephson effect, or more precisely a **Josephson Junction**.

1. Exordium

First one must explain what a junction consists of in order to delve to further understanding of it. A junction is two (or more) very closely spaced superconductors separated by a weak connection. One thing must be further explained within that sentence and that is the word *superconductor*. A superconductor is a material that when cooled to below a very critical temperature can have electricity pass through it without any electrical resistance, no voltage is lost by traveling through it. Recent findings have also shown that some material create a superconducting state at higher temperatures, in excess of 90 Kelvin even. Back to the explanation: The weak connection that is talked about is provided by some material that weakly couples the two superconductors, such as a normal metal, a semiconductor or a weakened superconductor. As the weak separator's thickness is reduced to 0 the phase of the 2 superconductors becomes connected, as in the two super conductors are acting as one superconductor. Each of the superconducting regions



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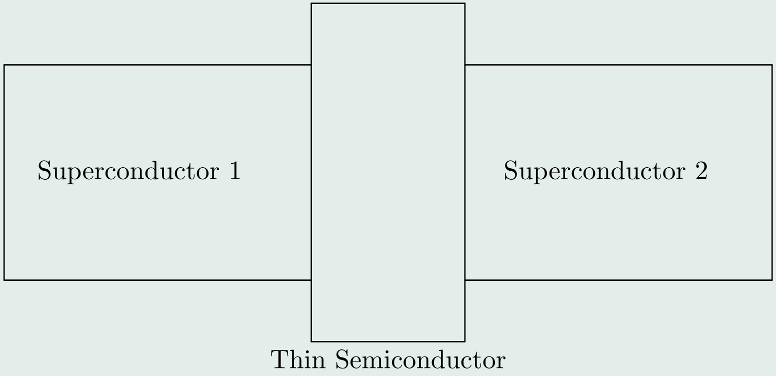
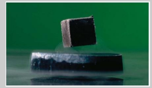


Figure 1: A very simple Josephson Junction

can be looked at in another way, as quantum mechanical wave functions. $\Psi_1 e^{i\phi_1}$ and $\Psi_2 e^{i\phi_2}$ will represent superconductor 1 and superconductor 2 respectively. These two superconductor equation were first described by Ginzburg and Landau, and they are complex order parameters that look like: $\psi(r,t) = \mid \psi(r,t) \mid \exp\left[i\psi(r,t)\right]$. A Josephson Junction is the idea that the $\sim 10^{23}$ electrons between the superconductors create "Cooper Pairs" which can then be described as a single macroscopic wave function. The density of the cooper pairs are given by the first equation: $\mid \psi(r,t) \mid^2$, which was also first described by Ginzburg and Landau. The total movement of the condensed cooper's pairs are described by a single wave function: $\psi(r,t)$, and the phase of the wave function is described by $\phi(r,t)$. And since the phase is a reflection of all of the pairs rather than just one it is considered a physical observable.

In 1962 Brian Josephson, a 22 year graduate student, suggested that electrons could flow between the two S.conductors, even though there was *no* voltage difference between them. This idea was thought to be impossible in classical physics but in quantum physics it could occur. Anderson and Rowell, two physicists, observed this Josephson effect in 1963. Josephson won the Nobel Prize in 1973 for physics, but shortly thereafter lost interest with physics and was rarely heard from again.



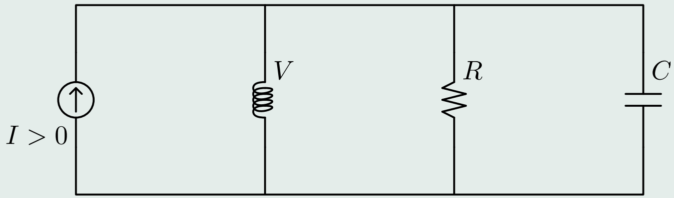


Figure 2: The DC Circuit

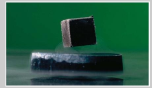
2. Core

2.1. DC Effect

Suppose a J.Junction is connected to a dc power source, such that a constant current $I > 0$ is driven through. In Quantum physics if that current is less then a certain critical current I_c of the J. Junction, no voltage will be developed across the junction. When $I < I_c$ though the phases of the two s.conductors will be driven apart to a new constant incoherent phase difference described as: $\Delta\phi = \phi_2 - \phi_1$ in which $\Delta\phi$ satisfies the Josephson current-phase relation:

$$I = I_c sin(\Delta\phi) \tag{1}$$

And (1) is the formula for the supercurrent of the cooper's pairs bypassing the interposing structure. I_c is said to be the *critical current* because if I ever equals I_c then the two s.conductors can no longer sustain the supercurrent through the barrier. There is an energy that is coupling the two s.conductors, which is given by: $E_c(\Delta\phi) = -(\hbar/2e)I_c cos(\delta\phi)$. Using (1) it can be said that as the bias current, I increases the phase difference increases as well. When $I_c = I$ then the coupling energy, E_c has been reduced to zero and the junction can no longer sustain the supercurrent. But when I *exceeds* I_c, then a constant phase difference can no longer be maintained and voltage develops across the junction. This is called the Josephson AC effect and will be described in the next section.



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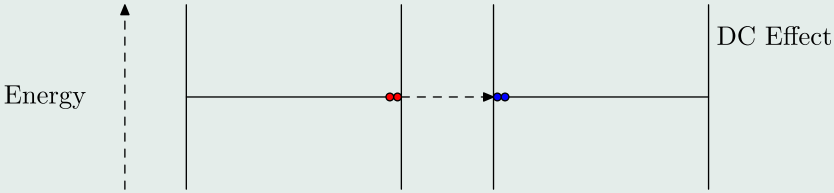


Figure 3: A representation of the DC effects energy level

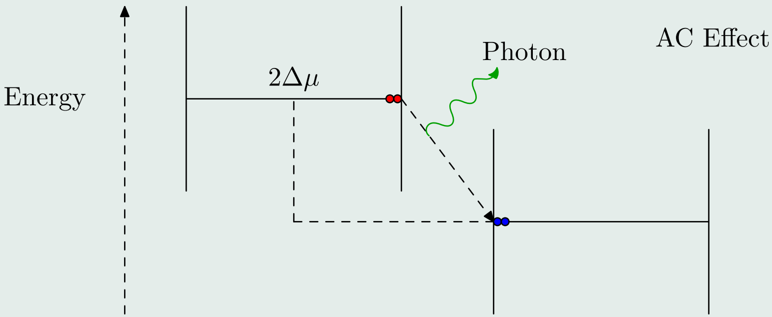


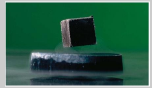
Figure 4: The energy in the AC effect

2.2. AC Effect

As I is increased beyond I_c a small voltage is created across the junction. The new super current is no longer a direct supercurrent like in the DC Effect. The new supercurrent now oscillates with time. (1) now changes to become:

$$I = I_c sin(\omega\phi) \tag{2}$$

This is the AC Josephson effect. The alternating supercurrents within the effect may be thought of as being tunneled through by the cooper's pairs, *along* with the emission, or absorption of a photon.If there is a finite electrochemical potential difference, defined as $\Delta\mu$, between the two



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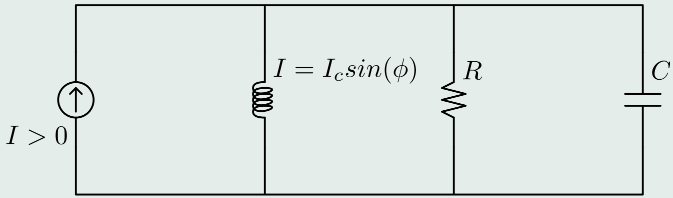


Figure 5: The Junction,Ordinary,and Displacement Currents

s. conductors then the Cooper pairs on either side of the barrier differ by an amount of energy governed by: $2\Delta\mu = 2eV$. Thus, the photon frequency (υ) will be given by the Josephson voltage-frequency relation:

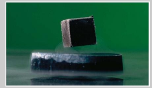
$$h\upsilon = 2\Delta\mu$$

3. Analogous Situation(s)

If we now turn our attention back to the phase difference, ϕ then we will be able to further analyze this ODE. If we look at the relationship between the Phase and the Voltage we would look at the *Josephson Phase-Voltage Relation*:

$$V = \frac{\hbar}{2e}\dot{\phi} \tag{3}$$

In this situation $V(t)$ is the instantaneous voltage across the junction, $\hbar$ is Planck's constant divided by 2π, or Dirac's constant, and e is the charge on the electron. The average charge on the electron is roughly $1.60217646 \times 10^{-19}$ coulombs. Equation (1) applies only to the supercurrent carried by the electron pairs. In general though the total current passing through the junction will also contain contributions from a *displacement current* and an *ordinary current*. If we represent the displacement by a capacitor and the ordinary by a resistor we get a circuit similar to 3:



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3.1. Finally Analogous

In order to gain a more insightful understanding of a J. Junction one must under stand the 1st and 2nd law of Gustav Kirchhoff. The 1^{st} law is his Current Law or KCL for short. This is the conservation of charge equation, which states that at any point in an electrical circuit where charge density is not changing in time, the sum of currents flowing towards a nodal point is equal to the sum of currents flowing away from that point. As in I is going into a node and diverging into three wires out of it. the current equal to the sum of the three outgoing wires or: $I_{in} = \sum outgoingcurrent$. Kirchhoff's 2^{nd} law is called Kirchhoff's Voltage Law or KVL for short. His 2nd law states that sum of the electrical voltage differences around any closed circuit must be zero. Another way to look at KVL would be to say that if a charge drops from a resistor, then from a capacitor, the total must be zero, or in symbols: $\sum VoltageDrops = 0$.

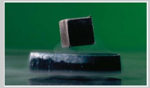
Now, using Kirchhoff's voltage and current laws, the voltage across each branch must be equal, so all voltages are equal to the voltage across the J. junction. The capacitor current is CV' and the resistor current is V/R. The sum of these currents and the *supercurrent*, $I_c sin(\phi)$, must equal the bias current, I therefore:

$$CV' + \frac{V}{R} + I_c sin(\phi) = I \tag{4}$$

we can now use the information garnered from (3) to rewrite (4) in terms of the phase difference ϕ by saying that $V = (\hbar/2e)\dot{\phi}$, and that $V' = (\hbar/2e)\ddot{\phi}$ to create:

$$\frac{C\hbar}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c sin(\phi) = I \tag{5}$$

which is analogous to the equation governing a damped pendulum driven by a constant torque! Typically when analyzing this ODE one must look at regular parameter values for J. Junctions. The critical current is regularly between $I_c \approx 1\mu A - 1mA$, while the typical voltage is $I_c R \approx 1mV$. $\frac{2e}{h} \approx 4.83x10^{14}$ Hz/V, therefore a typical frequency is on the order of 10^{11} Hz. And finally the regular length scale used for J.Junctions is $\approx 1\mu m$.



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3.2. Damped Pendulum to Nonuniform oscillator

But (5) isn't all that useful to someone of my level of mathematics as there are a lot of constants that need to be put in and the equation itself is nonlinear which makes it much more difficult if not impossible to solve. So let's try and clean this equation up a little bit. If we can get this into a more simplistic model life would be much easier. The first thing to do would be to create a dimensionless equation. In order to do this one would start by stating the new t which is τ:

$$\tau = \frac{2eI_cR}{\hbar}t \tag{6}$$

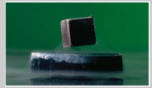
This will get us out of trouble don't worry about it right now. Next we will have to redefine ϕ in terms of τ instead of t. In order to do that we take the derivative of (6) and say that

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{d\phi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{d\phi}{d\tau}\frac{2eI_cR}{\hbar}$$

And we will have to do the same thing for the second derivative of ϕ which is $\ddot{\phi}$. And in order to change it we must do as follows:

$$\begin{aligned}
\ddot{\phi} = \frac{d^2\phi}{dt^2} = \frac{d}{dt}\left(\frac{d\phi}{dt}\right) &= \frac{d}{dt}\left(\frac{2eI_cR}{\hbar}\frac{d\phi}{d\tau}\right) \\
&= \left(\frac{2eI_cR}{\hbar}\right)\frac{d}{dt}\left(\frac{d\phi}{d\tau}\right) \\
&= \left(\frac{2eI_cR}{\hbar}\right)\left(\frac{2eI_cR}{\hbar}\right)\frac{d}{d\tau}\left(\frac{d\phi}{d\tau}\right) \\
&= \left(\frac{2eI_cR}{\hbar}\right)^2\frac{d^2\phi}{d\tau^2}
\end{aligned}$$

and instead of writing $d^2\phi/d\tau^2$ for the second derivative we will now call it ϕ'' and the same goes for $d\phi/d\tau$ which will become ϕ'. Now we can replace $\ddot{\phi}$ and $\dot{\phi}$ with the corresponding



values of ϕ'' and ϕ' respectively. Let's remind ourselves of what we found earlier:

$$\begin{aligned} \dot{\phi} &= \left(\frac{2eI_cR}{\hbar}\right)\phi' \\ \ddot{\phi} &= \left(\frac{2eI_cR}{\hbar}\right)^2\phi'' \end{aligned} \tag{7}$$

First we will divide (5) by I_c, then we will make the necessary substitutions of (7) into (5) and we come out with a new equation which looks like:

$$\left(\frac{\hbar C}{I_c 2e}\right)\left(\frac{2eI_cR}{\hbar}\right)^2\phi'' + \left(\frac{\hbar}{2eRI_c}\right)\left(\frac{2eI_cR}{\hbar}\right)\phi' + sin(\phi) = \frac{I}{I_c}$$

Though this may look even worse you can see that much of the equation cancels out until you end up with:

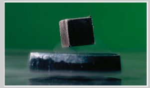
$$\left(\frac{2eI_cCR^2}{\hbar}\right)\phi'' + \phi' + sin(\phi) = \frac{I}{I_c}$$

Although better lastly we would like to replace the coefficient on ϕ'' with something that is easier to understand like β which will change the equation into:

$$\begin{aligned} \beta\phi'' + \phi' + sin(\phi) &= \frac{I}{I_c} \\ \beta &= \frac{2eI_cR^2C}{\hbar} \end{aligned} \tag{8}$$

3.3. Overdamped Limit

Now that we have found a much simpler equation in (8) we can now better analyze the equation itself. First and foremost though we must figure out a way to only look at a first order deferential equation. And as the title of the section should show that there is a way, we can look at the overdamped limit of this "pendulum" analogy. In the overdamped limit the $\beta \ll 1$ and is *so*



much smaller than one that ϕ'' and β can be ignored . . . for now. If you ignore the them both (8) turns into a much simpler DE:

$$\phi' = \frac{I}{I_c} - sin(\phi) \tag{9}$$

And (9) is just a nonuniform oscillator, crazy huh! We can use (9) to find the current-voltage curve, which is a very peculiar graph. Before we delve into anymore mathematics we must analyze (9). If we can visualize the graph of (9) then when $I \leq I_c$ the sine curve dips below the x-axis creating two equilibrium points instead of the standard none. One of these equilibrium points is an asymptotically stable one and the other is unstable which means that all of the possible solutions would go to a fixed point. This fixed point can be shown as $\phi^* = sin^{-1}(I/I_c)$, where $-\pi/2 \leq \phi^* \leq -\pi/2$. Since ϕ' has two equilibrium points, then ϕ' is equal to zero. And so $V = 0$ for all $I \leq I_c$. Well since we found what happens if $I \leq I_c$ what happens if $I > I_c$? As we remember from earlier as the current I is increased beyond I_c the current becomes an AC current that oscillates. When the solutions become periodic we can find a period for them by using a period solution for a nonuniform oscillator which in a general form looks like this:

$$T = \frac{2\pi}{\sqrt{(\omega)^2 - (\alpha)^2}} \tag{10}$$

In the J.Junction equation, (9), the $\omega = I/I_c$ and the $\alpha = 1$. More specifically the period is $T = 2\pi/\sqrt{(I/I_c)^2 - (1)^2}$. We can now compute ϕ' by taking the average of one cycle like so:

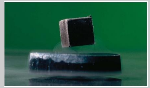
$$< \phi' >= \frac{1}{T}\int_0^T \frac{d\phi}{d\tau} dt = \frac{1}{T}\int_0^T \frac{d\phi}{d\tau}\frac{d\tau}{dt} dt = \frac{1}{T}\int_0^{2\pi} d\phi = \frac{2\pi}{T} \tag{11}$$

We must now change (3) into a form that is consistent with the rest of the τ universe by changing

$$< V >= \left(\frac{\hbar}{2e}\right) < \dot{\phi} >$$

by substituting (7) to create:

$$< V >= I_c R < \phi' > \tag{12}$$



Now we can combine (10), (11), and (12) to create:

$$< V >= I_c R \sqrt{(\frac{I}{I_c})^2 - 1}$$

and this only applies for all $I > I_c$. To summarize we have found that for $I \leq I_c$ the voltage, V, is zero, but for all $I > I_c$ everything changes. To summarize visually:

$$< V >= \begin{cases} 0 \text{ if } I \leq I_c \\ \\ I_c R \sqrt{(\frac{I}{I_c})^2 - 1} \text{ if } I > I_c \end{cases} \tag{13}$$

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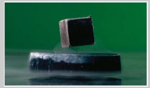

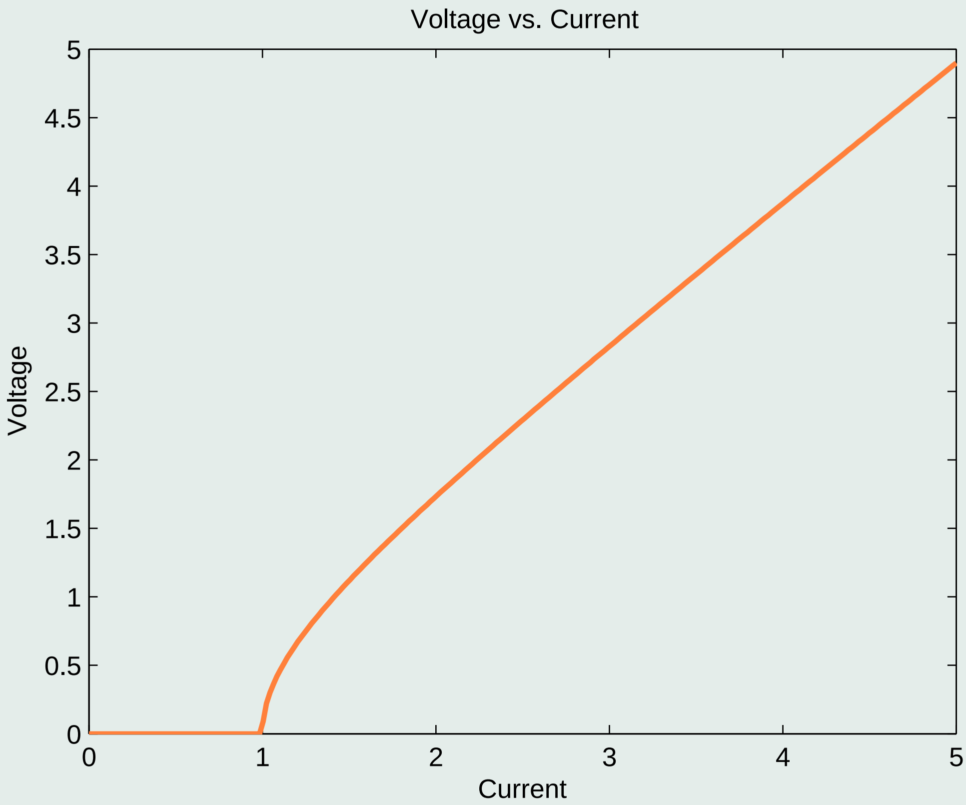

Figure 6: Relationship between voltage and current.