Modeling An Economy's Growth

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Abstract

The intent of this paper is to explore a model of long term economic growth.

1. The Model

Suppose a community that produces a single product, whose output will be expressed as Y, a function of K, the community's stock of capital, and L, the supply of labor. Suppose further that each employable member of the community is employed in producing this product. The income of the community is then also expressed by Y. The fraction of the community's income that is saved will be expressed as the constant s, making the amount saved equivalent to sY. Investment is defined to be the rate of change of the community's stock of capital, and it is thus K'. Making a standard neoclassical economic assumption, that investment

equals savings, it can be said that

$$K' = sY(K, L) \tag{1}$$

The function Y is known as a production function, and even without defining its exact shape, it is possible for equation (1) to be transformed into a model that is easily studied qualitatively. To do this, it is helpful to introduce a new variable $r = K/L$ to represent the ratio of capital to labor in the economy. Alternatively, r can be understood as the amount of capital per worker in the economy. Multiplying each side by L and differentiating with respect to time yields

$$K' = r'L + rL'$$

Combining this with equation (1) and solving for r',

$$\begin{aligned} r'L + rL' &= sY(K, L) \\ r'L &= sY(K, L) - rL' \\ r' &= \frac{sY(K, L) - rL'}{L} \end{aligned} \tag{2}$$

It's necessary to define a supply of labor equation, so as to rid equation (2) of the L' term. Assuming that the labor supply grows proportional to itself, it can be said that $L' = nL$. This is a separable differential equation, and can be integrated

directly by moving L to the left side of the equation and dt to the right.

$$\begin{aligned} \frac{dL}{dt} &= nL \\ \int \frac{dL}{L} &= \int n dt \\ \ln L &= nt + c \\ L &= L_0 e^{nt} \qquad (3) \\ & \qquad (4) \end{aligned}$$

Replacing L' with nL, and further replacing L with $L_0 e^{nt}$, equation (2) becomes

$$r' = \frac{sY(K, L_0 e^{nt}) - nrL_0 e^{nt}}{L_0 e^{nt}} \tag{5}$$

It is here important to note that it is assumed that the production function, Y, exhibits constant returns to scale. Constant returns to scale implies that if the factors of production, in this case K and L, are increased by a factor of λ, then output, Y, also increases by a factor of λ. Equivalently, if Y exhibits constant returns to scale, then

$$Y(\lambda K, \lambda L) = \lambda Y(K, L). \tag{6}$$

Using this fact, equation (5) is further simplified.

$$r' = sY\left(\frac{K}{L_0 e^{nt}}, 1\right) - nr$$

Recalling that $r = K/L$, or now $r = K/L_0 e^{nt}$,

$$r' = sY(r, 1) - nr \tag{7}$$

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2. Economic Equilibrium

What is most interesting from this equation are the implications surrounding economic equilibrium. Economic equilibrium is defined to be a state of economic growth in which there are no induced changes in relative prices of the factors of production over time. For instance, the price of a unit of capital will grow at a rate equivalent to the price of a unit of labor. If $r' = 0$, r is a constant, and it is easy to show that this is necessary for economic equilibrium.

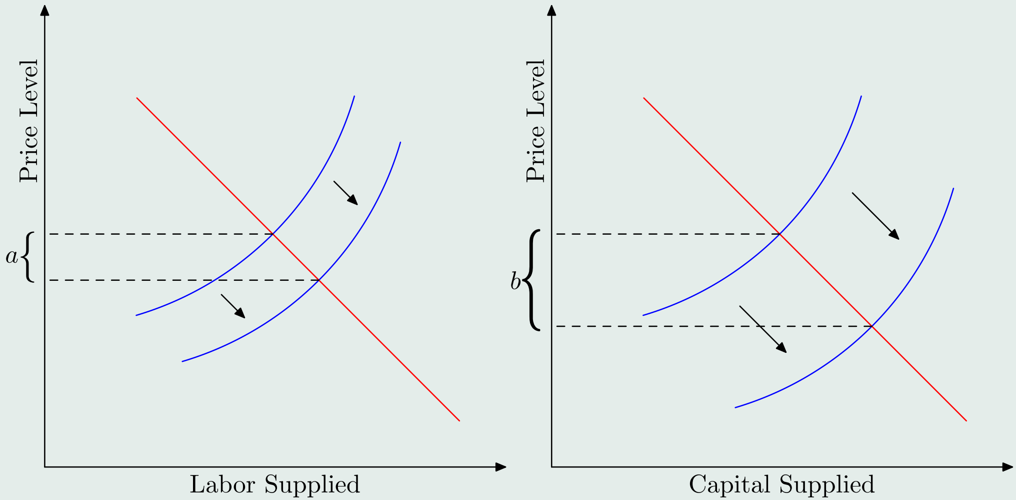

Figure 1: Changes in the Price Level by an Increasing Capital-Labor Ratio

Figure 1 illustrates a scenerio in which r is allowed to increase, or, equivalently, $r' > 0$. The supply of capital is thus increasing faster than the supply of labor.

The result is a decrease in the price of labor by a, accompanied by a visibly larger decrease in the price of capital, b. The price of capital decreasing at a faster rate than the price of capital is a direct contradiction of the definition of economic equilibrium. A similar analysis of when $r' < 0$ will result in a similar contradiction.

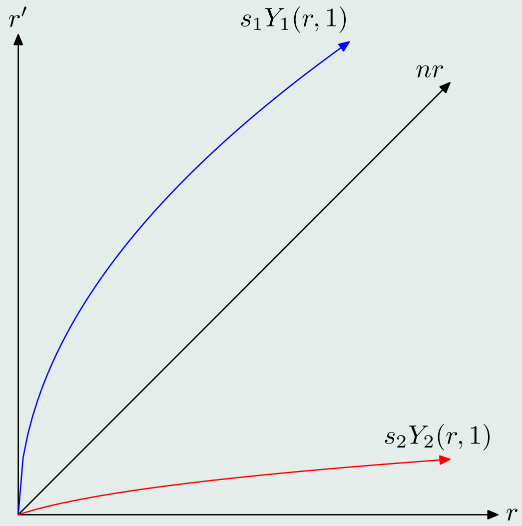

Figure 2: No Economic Equilibrium

So, economic equilibrium occurs when $r' = 0$, or, by equation (7), when $sY(r, 1) = nr$. Without a known shape for the production function Y, numerous scenarios surrounding possibilities for economic growth exist. For example, figure 2 shows two possible configurations that do not provide for any economic equilibrium. Y_1 represents such a high output per worker, and the community saves such a large portion of its income, s_1, that $s_1Y(r,1)$ is perpetually above the ray nr. By equation (7), $s_1Y_1(r,1) > nr$ implies $r' > 0$, and this scenario would result in a perpetually increasing capital-labor ratio. Alternatively, Y_2 represents such a low output per worker, and the community saves such a small portion of its income, s_2, that $s_2Y(r,1)$ is perpetually below the ray nr. Again by equation (7), $s_2Y_2(r,1) < nr$ implies $r' < 0$, and this scenario would result in a perpetually decreasing capital-labor ratio.

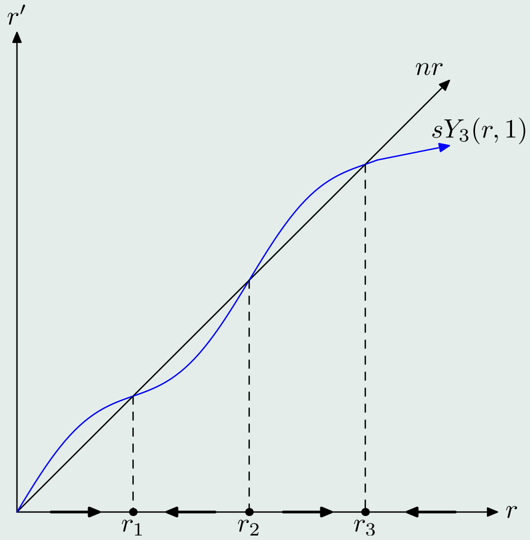

Figure 3: Multiple Economic Equilibriums

Figure 3 represents a completely different scenario. The multiple intersections of sY_3 with nr provide for multiple possible equilibriums in the economy's growth. With a lesser capital-to-labor ratio than r_1, $sY_3 > nr$ and so $r' > 0$. The capital-labor ratio will thus increase toward r_1. For a capital-labor ratio greater than r_1 but less than r_2, $sY_3 < nr$ and so $r' < 0$. The capital-labor ratio will thus decrease toward r_1 again. r_1 can now be classified as a stable equilibrium point. For a capital-labor ratio greater than r_2 but less than r_3, r' is again positive, and r will tend toward r_3. This enables r_2 to be classified as an unstable equilibrium point. Finally, a capital-labor ratio greater than r_3 implies that r' is negative, and r will decrease toward r_3, allowing r_3 to be classified as a second stable equilibrium point.

The implications are interesting. If Y_3 were a reasonable shape for the production curve, then it is possible for the economy's capital-labor ratio to be in a less productive equilibrium, such as r_1, and a large infusion of capital could "push" the capital-labor ratio above r_2, resulting in the growth of the capital-labor ratio to a more productive capital-labor equilibrium at r_3. Alternatively, the capital-labor ratio could begin at r_3, and a large detraction of capital could "push" the capital-labor ratio below r_2, resulting in the capital-labor ratio shrinking to a less productive capital-labor ratio, r_1.

3. The Cobb-Douglas Production Function

Instead of choosing arbitrary shapes for the production function, it would be best to choose a reasonable model. The Cobb-Douglas production function, P, is a natural choice. The Cobb-Douglas production function is based upon the following three assumptions. First, if either labor or capital falls to zero, then

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production will also fall to zero. Second, the marginal productivity of labor is proportional to the amount of production per unit of labor. Third, the marginal productivity of capital is proportional to the amount of production per capital.

Some definitions are in order. Production has already been defined to be P, labor as L, and capital as K. It follows that production per unit of labor is P/L, while production per unit of capital is P/K. Marginal productivity of labor is defined to be $\partial P/\partial L$, while marginal productivity of capital is defined to be $\partial P/\partial K$. The second assumption implies that, for some constant β,

$$\frac{\partial P}{\partial L} = \beta \frac{P}{L}$$

Keeping K constant at K_0, this partial differential equation becomes an ordinary separable differential equation easily solved.

$$\begin{aligned} \frac{dP}{dL} &= \beta \frac{P}{L} \\ \int \frac{dP}{P} &= \beta \int \frac{dL}{L} \\ \ln P &= \beta \ln L + C \\ e^{\ln P} &= e^{lnL^{\beta}+C} \\ P(K_0, L) &= C_1(K_0)L^{\beta} \qquad (8) \end{aligned}$$

Note that the constant C_1 in equation (8) has been written as a function of K_0, as its value could depend on the value of K_0. It's also important to acknowledge that $\beta > 0$ by the first assumption. If $\beta = 0$, capital tending toward zero would by no means imply production tending toward zero, contradicting the first assumption.

Similarly, if $\beta < 0$, capital tending toward zero would result in production tending toward infinity, again contradicting the first assumption. Therefore, $\beta > 0$. The same procedure used to produce equation (8) produces a similar result for $\partial P/\partial K$. For some constant α, and keeping L constant at some L_0,

$$P(K, L_0) = K^\alpha C_2(L_0) \tag{9}$$

C_2 is a written as a function of L_0, as its value could depend on the value of L_0. Also note that the first assumption leads to the conclusion that $\alpha > 0$. It would be reasonable that the form for $P(K, L)$ would be some combination of equations (8) and (9). Cobb and Douglas, careful to choose a convenient form, suggested that, for some constant b independent of both K and L,

$$P(K, L) = bK^\alpha L^\beta$$

This equation was further justified by showing its accuracy in describing the United States economy from 1899 to 1923. Recall that the production function was earlier defined in the derivation of equation (7) to have constant returns to scale, meaning equation (6) must hold for P. Increasing K and L by a factor of λ shows

$$P(\lambda K, \lambda L) = b(\lambda K)^\alpha (\lambda L)^\beta = \lambda^{\alpha+\beta} bK^\alpha L^\beta = \lambda^{\alpha+\beta} P(K, L)$$

Note that in order for P to satisfy equation (6), $\alpha + \beta = 1$, and P becomes the Cobb-Douglas production function.

$$P(K, L) = K^\alpha L^{1-\alpha} \tag{10}$$

Note that equation (10) does not contain the constant b. This is intentional; least squares regression calculation by Cobb and Douglas estimated $b = 1$, and this result is utilized for the remainder of this paper for simplicity.

4. Return to the Model

Now that a reasonable function production function has been found, what does it imply for economic equilibrium? As the capital-labor ratio decreases, the marginal productivity of capital increases indefinitely, leading to the conclusion that $sP(r,1)$ must initially rise above the ray nr immediately following their intersection at $(0,0)$. Further, as the capital-labor ratio increases, the marginal productivity of capital decreases, resulting in a curve that is continuously concave down. Since it's known that $\alpha > 0$, and that $1-\alpha > 0$, α can be more definitively bounded by acknowledging that $0 < \alpha < 1$. With an $\alpha < 1$, the curve of $sP(r,1)$ must eventually fall below the ray nr, resulting in one equilibrium capital-labor ratio. The result is figure 4, with the single equilibrium capital-labor ratio at r^*.

As evidenced by figure 4, if $r < r^*$, $sP(r,1) > nr$, which implies $r' > 0$, and r will increase toward r^*. Alternatively, if $r > r^*$, $sP(r,1) < nr$, which implies $r' < 0$, and r will decrease toward r^*. Thus, r^* is an asymptotically stable equilibrium point, which implies that as long as capital and labor exist within the economy, it will tend toward an economic equilibrium with a capital-labor ratio of r^*.

In order to find r^*, it's helpful to describe $P(K,L)$ as $P(r,1)$. As P exhibits constant returns to scale,

$$P(r,1) = P\left(\frac{K}{L},1\right) = \frac{P(K,L)}{L} = K^\alpha L^{-\alpha} = r^\alpha$$

Substituting this result into equation (7), $r' = sr^\alpha - nr$. By setting $r' = 0$,

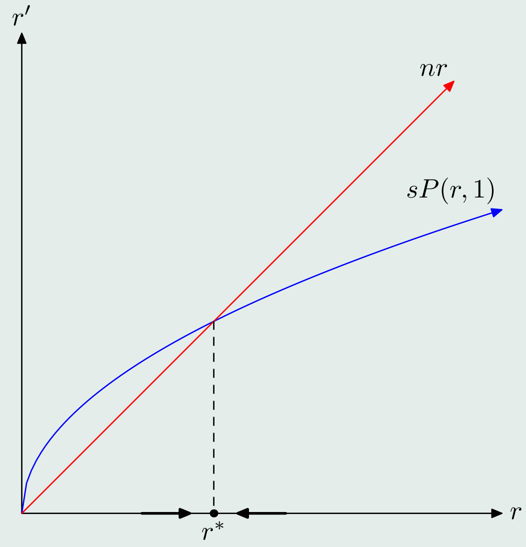

Figure 4: The Cobb-Douglas Production Function

the equilibrium point r^* can be found.

$$
\begin{aligned}
sr^\alpha - nr &= 0 \\
sr^\alpha &= nr \\
r^{1-\alpha} &= \frac{s}{n} \\
r &= \left(\frac{s}{n}\right)^{1/\beta}
\end{aligned}
$$

As β is a constant, the equilibrium capital-labor ratio is based only upon two variables: the fraction of income saved, s, and the natural rate of growth of the labor supply, n. Importantly, a larger s leads to a higher equilibrium capital-labor ratio. Additionally, a smaller n leads to an even higher equilibrium capital-labor ratio.

A higher capital-labor ratio can also now be shown to result in a higher real income per unit of labor. Recall that $P(K, L)$ represents the community's output and income as a function of K, the community's capital stock, and L, the community's supply of labor. $P(r, 1)$, then, represents the community's output and income as a function of r, the community's capital per unit of labor, and 1 unit of labor. Equivilently, $P(r, 1)$ represents the community's per capita income. Recall that $P(r, 1) = r^\alpha$. r will tend toward $(s/n)^{1/\beta}$, and so per capita income will tend toward $(s/n)^{\alpha/\beta}$. The benefits of a higher capital-labor ratio are now easy to relate to one's own wallet; a community with a higher fraction of income saved, s, and a lower labor supply growth rate, n, will result in a higher real per capita income.

It's now possible to obtain an equation for r, although the simplest method is to return to the untransformed differential equation for the model, equation (1).

Substituting equation (10) into equation (1), and recalling that $L = L_0 e^{nt}$,

$$K' = sK^{\alpha}(L_0 e^{nt})^{1-\alpha}$$

This is a separable differential equation, and can be integrated directly. Recall that $\beta = 1 - \alpha$.

$$\begin{aligned}
\int K^{-\alpha} dK &= sL_0^{\beta} \int e^{n\beta t} dt \\
\frac{1}{\beta} K^{\beta} &= \frac{sL_0^{\beta} e^{n\beta t}}{n\beta} + C \\
K &= \left(\frac{sL_0^{\beta} e^{n\beta t}}{n} \right)^{1/\beta} + C \\
&= \left(\frac{s}{n} \right)^{1/\beta} L_0 e^{nt} + C
\end{aligned}$$

Assuming that at time $t = 0$, $K = K_0$,

$$\begin{aligned}
K_0 &= \left(\frac{s}{n} \right)^{1/\beta} L_0 + C \\
C &= K_0 - \left(\frac{s}{n} \right)^{1/\beta} L_0 \\
\Rightarrow K &= K_0 + \left(\frac{s}{n} \right)^{1/\beta} L_0 \left(e^{nt} - 1 \right) \qquad (11) \\
& \qquad (12)
\end{aligned}$$

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Dividing through by L, and introducing the initial capital labor ratio $r_0 = K_0/L_0$,

$$
\begin{aligned}
\frac{K}{L} &= \frac{K_0 + \left(\frac{s}{n}\right)^{1/\beta} L_0 e^{nt} - \left(\frac{s}{n}\right)^{1/\beta} L_0}{L_0 e^{nt}} \\
r &= r_0 e^{-nt} + \left(\frac{s}{n}\right)^{1/\beta} - \left(\frac{s}{n}\right)^{1/\beta} e^{-nt} \\
r &= \left(\frac{s}{n}\right)^{1/\beta} + e^{-nt}\left(r_0 - \left(\frac{s}{n}\right)^{1/\beta}\right) \qquad (13)
\end{aligned}
$$

A close look at equation (13) will concur with previous results for r^*. As t grows infinitely large, e^{-nt} shrinks infinitely small, and it can be easily seen that r approaches $(s/n)^{1/\beta}$, the previously calculated r^*.

5. Numerical Results

Now with functions for K, L, and r, it is possible to analyze the model numerically. Time $t = 0$ will be the year 1899, the first year that capital and labor statistics are available from Cobb and Douglas' paper *A Theory of Production*. As no data for a reasonable value of n was able to be located, an n of .032 was estimated. This allows equation (3) to provide a decent approximation of Cobb and Douglas' data for the supply of labor, while accounting for variance in the labor supply due to World War I. Figure 5 shows a graphical representation of Cobb and Douglas' data for the labor supply and the calculated estimation from equation (3).

Similarly, no data for a reasonable value of s was able to be located. An s of .06 was estimated, as this allows equation (11) to provide a decent approximation of Cobb and Douglas' data for the total capital stock, while again accounting for

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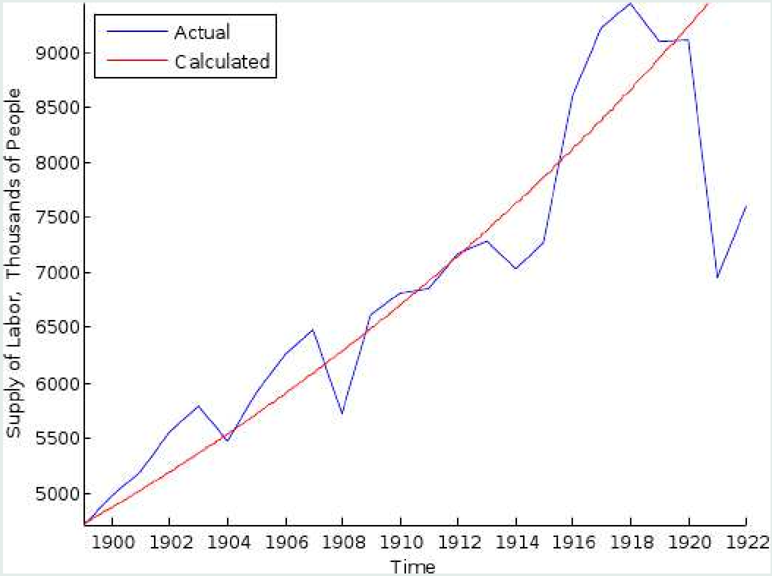

Figure 5: Supply of Labor 1899-1922

variance due to World War I. Figure 6 shows a graphical representation of Cobb and Douglas' data for the total capital stock and the calculated estimation from equation (11).

Finally, Figure 7 shows the capital-labor ratio from Cobb and Douglas' data, as well as the calculated estimation from equation (13). The model appears to be a fair approximation until the beginning of World War I. Table 1 provides the data represented Figures 5, 6, and 7.

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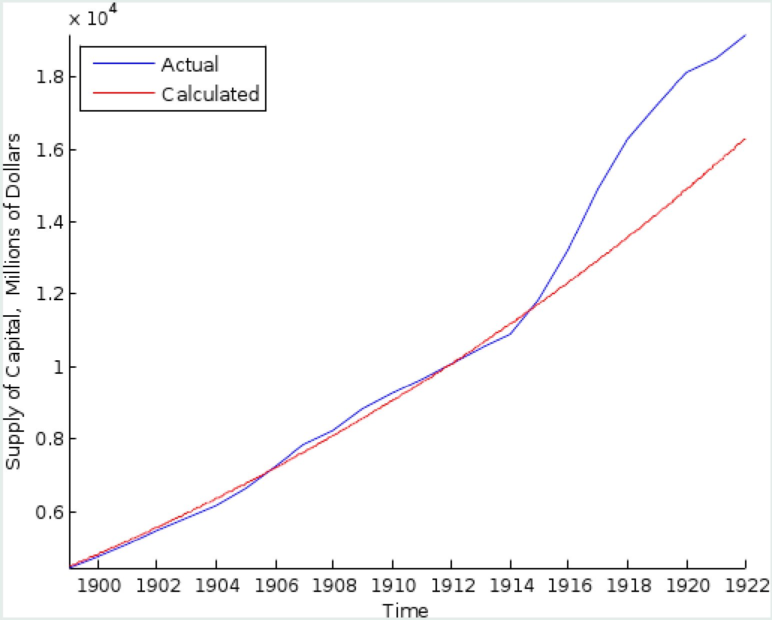

Figure 6: Total Capital Stock 1899-1922

The span of 23 years is hardly a long enough period of time to allow an economy to tend toward equilibrium. One readily available statistic that is easily relatable to the casual reader is per-capita income. Provided that the model for r is a good approximation of the true capital-labor ratio and the function P is a good approximation of the true output of the United States, then $P(r, 1)$ should be a good approximation of per-capita income. Recall that $P(r, 1)$ was earlier found to be r^α, and r^* was found to be $(s/n)^{1/\beta}$. Substituting r^* into $P(r, 1)$, and

Table 1: Cobb and Douglas' Data, Actual and Calculated Estimations

	Total Capital (Millions of 1880 Dollars)		Total Laborers (Thousands of Laborers)		Capital Labor Ratio ($1000 / Laborer)	
Year	Actual	Calculated	Actual	Calculated	Actual	Calculated
1899	4449	4449	4713	4713	.944	.944
1900	4746	4803	4968	4866	.955	.987
1901	5061	5169	5184	5025	.976	1.029
1902	5444	5547	5554	5188	.980	1.069
1903	5806	5937	5784	5357	1.004	1.108
1904	6132	6340	5468	5531	1.121	1.146
1905	6626	6756	5906	5711	1.122	1.183
1906	7237	7185	6251	5896	1.158	1.219
1907	7832	7628	6483	6088	1.208	1.253
1908	8229	8086	5714	6286	1.440	1.286
1909	8820	8558	6615	6490	1.333	1.319
1910	9240	9046	6807	6702	1.357	1.350
1911	9624	9550	6855	6919	1.404	1.380
1912	10067	10071	7167	7144	1.405	1.410
1913	10520	10608	7277	7377	1.446	1.438
1914	10873	11162	7026	7617	1.548	1.466
1915	11840	11735	7269	7864	1.629	1.492
1916	13242	12326	8601	8120	1.540	1.518
1917	14915	12937	9218	8384	1.618	1.543
1918	16265	13567	9446	8657	1.722	1.567
1919	17234	14218	9096	8938	1.895	1.591
1920	18118	14890	9110	9229	1.989	1.613
1921	18542	15584	6947	9529	2.669	1.635
1922	19192	16300	7602	9839	2.525	1.657

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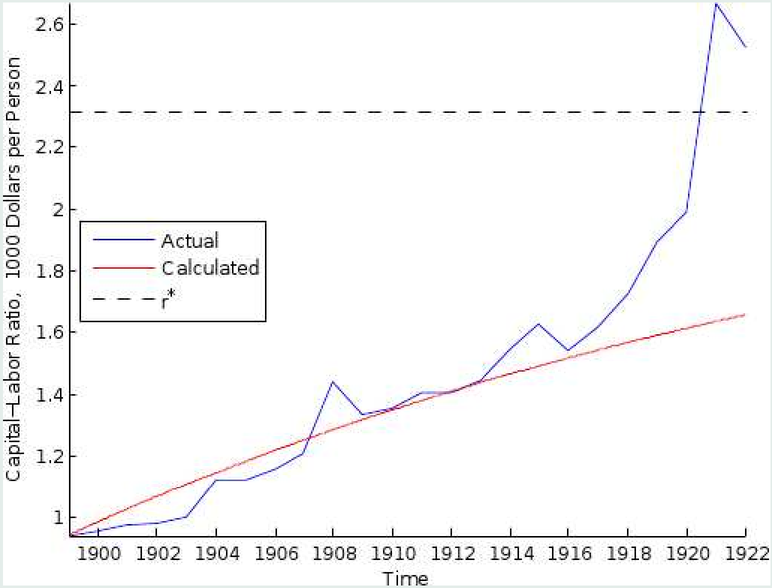

Figure 7: Capital-Labor Ratio 1899-1922

substituing earlier discussed values for s, n, and β,

$$\begin{aligned} P(r*,1) &= \left(\left(\frac{s}{n}\right)^{1/\beta}\right)^{\alpha} \\ &= \left(\frac{.06}{.032}\right)^{.25/.75} \\ &= 1.233 \end{aligned}$$

Recall that the units on this result are thousands of 1880 dollars per laborer. According to data provided by Robert Sahr of Oregon State University, \$1233 in 1880 is worth approximately \$24,176 in 2006. This amount only differs from the actual 2006 per-capita income of \$26,352 by 8%. While this analysis is not conclusive, it appears that the model provides a decent approximation, assuming the capital-labor ratio has been given sufficient time to reach an equilibrium.

6. Improvements on the Model

While the treatment of s, n, and β as constants simplifies the model, it is also highly inaccurate. The value of each of these is likely a function of numerous variables, not least of which is time. Furthermore, the supply of labor likely doesn't grow exogeneously, and a more realistic supply of labor function could be derived. Also, the use of the Cobb-Douglas production function is not necessarily the best. Intriguingly, the Cobb-Douglas production function has been modified to $A(t)K^{\alpha}L^{\beta}$, where the function A represents technological change. With these changes, it's likely that the model could be significantly improved.

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