Student Projects in Differential Equations

http://online.redwoods.edu/instruct/darnold/deproj/index.htm

Modeling a Simple Toy

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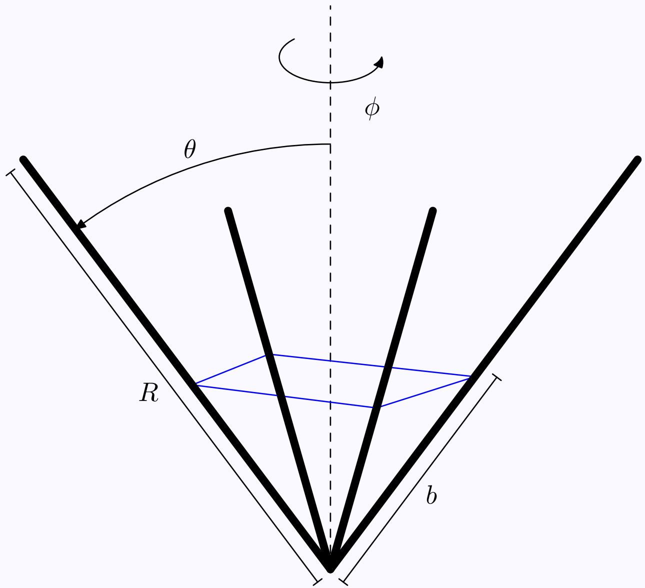
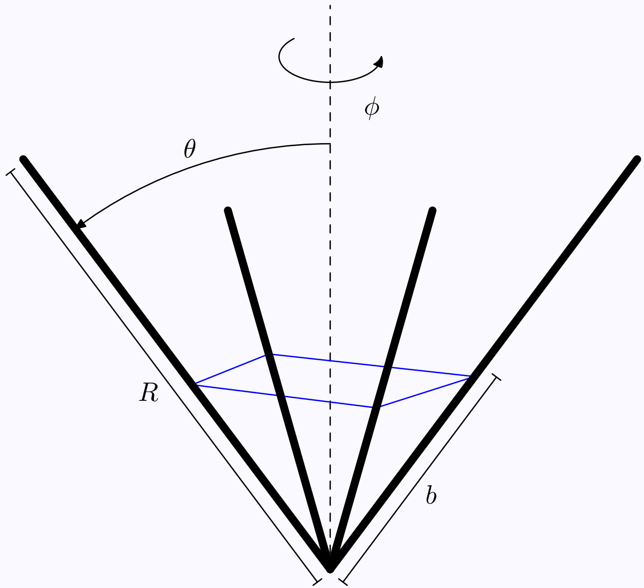
The Model



- θ is the angle off the vertical axis
- ϕ is the angle off the positive x axis

The Model (cont.)

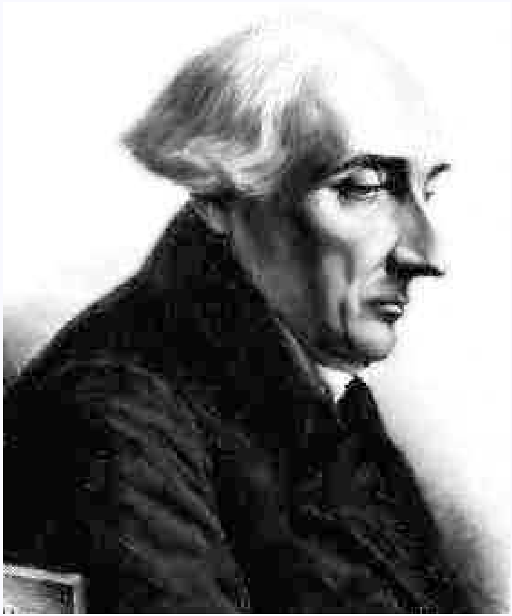


- The length of a single rod is R
- A spring is attached some distance b along the rod

The Lagrangian

- $L = K - V$
  - $\star$ K is the kinetic energy of the system
  - $\star$ V is the potential energy of the system
- To solve:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$
$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$



The Kinetic Energy

- There are two possible rotations, in θ and ϕ
- Kinetic energy for rotation:
  - ⋆ $K = \frac{1}{2}I\omega^2$
  - ⋆ I is the moment of inertia in the plane of rotation
  - ⋆ ω is the rotational velocity
- $K_\theta = \frac{1}{2}I\dot{\theta}^2$
- $K_\phi = \frac{1}{2}I\sin^2(\theta)\dot{\phi}^2$
- So the total kinetic energy of the system is:

$$\begin{aligned} K &= K_\theta + K_\phi \\ &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I\sin^2(\theta)\dot{\phi}^2 \end{aligned}$$

The Potential Energy

- Two components: gravitational and spring
- Gravitational:

$$
\begin{aligned}
V_G &= mgh \\
&= mg\frac{1}{2}R\cos(\theta)
\end{aligned}
$$

- Spring:

$$
\begin{aligned}
V_S &= \frac{1}{2}kx^2 \\
&= \frac{1}{2}k(\sqrt{2}b\sin(\theta))^2 \\
&= kb^2\sin^2(\theta)
\end{aligned}
$$

The Potential Energy (cont.)

- The total potential energy is:

$$
\begin{aligned}
V &= V_S + V_G \\
&= kb^2 \sin^2(\theta) + \frac{1}{2} mgR \cos(\theta)
\end{aligned}
$$

Solving The Lagrangian

- The Lagrangian for all 4 rods:

$$
\begin{aligned}
L &= K - V \\
&= 2I(\dot{\theta}^2 + \dot{\phi}^2 \sin^2(\theta)) - 4kb^2 \sin^2(\theta) - 2mgR\cos(\theta)
\end{aligned}
$$

Solving The Lagrangian (cont.)

- Finding $\ddot{\theta}$:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$4I\dot{\phi}^2 \sin(\theta)\cos(\theta) - 8kb^2 \sin(\theta)\cos(\theta) + 2mgR\sin(\theta) - 4I\ddot{\theta} = 0$$

- Finding $\ddot{\phi}$:

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 0$$

$$-\frac{d}{dt}(\dot{\phi}\sin^2(\theta)) = 0$$

$$-4\ddot{\phi}\sin^2(\theta) - 8\dot{\phi}\dot{\theta}\sin(\theta)\cos(\theta) = 0$$

Setting Up Our DEs

- Solve the previous equations for $\ddot{\theta}$ and $\ddot{\phi}$:

$$\ddot{\theta} = \frac{mgR}{2I}\sin(\theta) - \frac{2kb^2}{I}\sin(\theta)\cos(\theta) + \dot{\phi}^2\sin(\theta)\cos(\theta)$$
$$\ddot{\phi} = \frac{-2\dot{\phi}\dot{\theta}\sin(\theta)\cos(\theta)}{\sin^2(\theta)}$$

- Variable substitution to get a system of first order equations:

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = \phi$$
$$x_4 = \dot{\phi}$$

Setting Up Our DEs (cont.)

- The previous substitutions lead to the following system in the usual manner:

$$
\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= \frac{mgR}{2I}\sin(x_3) - \frac{2kb^2}{I}\sin(x_3)\cos(x_3) + (x_4)^2\sin(x_3)\cos(x_3) \\
\dot{x_3} &= x_4 \\
\dot{x_4} &= \frac{-2x_4x_2\sin(x_1)\cos(x_1)}{\sin^2(x_1)}
\end{aligned}
$$

Analyzing Behavior of the Model

- System is non-linear and we are interested in behavior over a large reigon, so linearization is not an option.
- A numeric solver will be user to model the motion of the system
- Behavior of the system will be analyzed via the effective potential

Effective Potential

- Take the total energy of the system and throw out any terms that vary with velocity.
- This is, in effect, adopting the refrence frame of the object we are studying.

Effective Potential (cont.)

- Energy of our system:

$$\begin{aligned} E &= K + V \\ &= 2I\dot{\theta}^2 + 2I\dot{\phi}^2 \sin^2(\theta) + 2mgR\cos(\theta) + 4kb^2 \sin^2(\theta) \end{aligned}$$

- It would seem that the $\dot{\theta}$ and $\dot{\phi}$ terms have to go. However, if we recall from earlier:

$$\frac{d}{dt}(\dot{\phi}\sin^2(\theta)) = 0$$

Integrating both sides with respect to t yields:

$$\dot{\phi}\sin^2(\theta) = h$$

Where h is some constant.

Effective Potential (cont.)

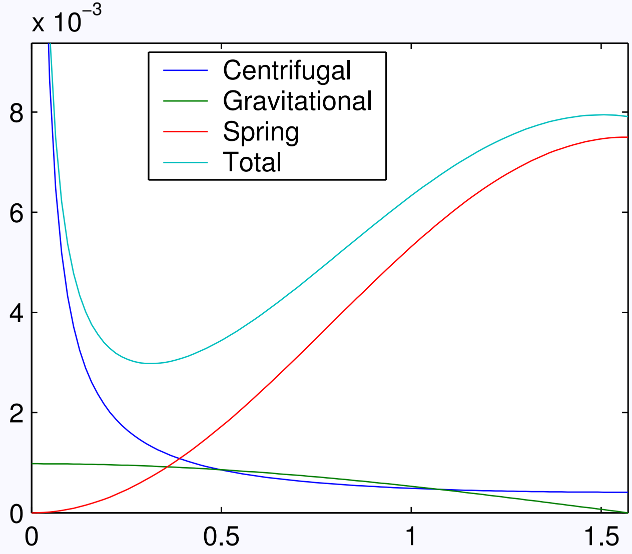
- Thus the energy equation becomes:

$$E(\theta, \dot{\theta}) = 2I\dot{\theta}^2 + \frac{2Ih^2}{\sin^2(\theta)} + 2mgR\cos(\theta) + 4kb^2\sin^2(\theta)$$

- And the effective potential is:

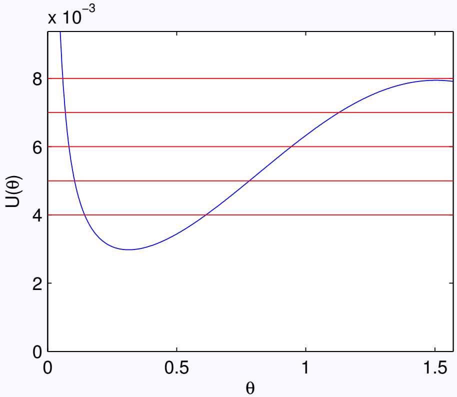
$$U(\theta) = \frac{2Ih^2}{\sin^2(\theta)} + 2mgR\cos(\theta) + 4kb^2\sin^2(\theta)$$

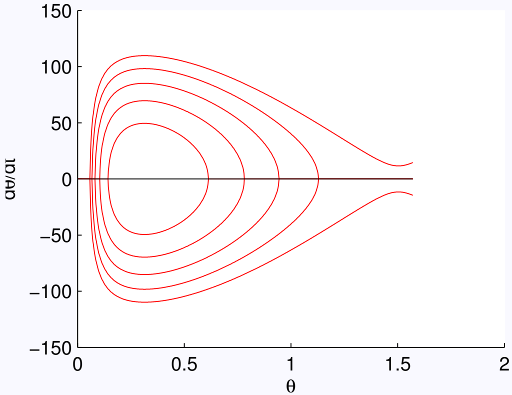
Effective Potential (cont.)



- Note that if the spring term is too small the "bowl" in the graph will be very shallow, if it is too large the "bowl" will be very steep. We want it in between these two extremes.

Effective Potential (cont.)





- The relative minimum of $U(\theta)$ yields the value of θ around which the system will oscillate.
- The relative maximum of $U(\theta)$ yields the maximum energy for which the system will oscillate.

OK, So Let's See It!

- Seeing is believing :)