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# Lanchester's Square Law Modeling the Battle of the Atlantic

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## 1. Introduction

F.W. Lanchester was a British engineer who attempted to translate a wartime battle into a series of mathematical problems. Lanchester systems can be used to model specific combat situations. More precisely they can be used to predict the outcome and number of soldiers surviving a given battle. With the proper amount of variables, these equations can be used to analyze extremely complex battles. The most basic form of this system is the same as the population model:

$$\frac{dx}{dt} = \alpha x$$

This first order differential equation is based on the assumption that the rate of growth of the population is proportional to the size of the population. The rate of change of a population depends only on the size of the population and nothing else.

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## 1.1. Expanding the Model

From the population model we can then create a system of two equations which will simulate a modified predator-prey model. However, in the Lanchester system there are two predators, each representing the number of troops in an army.

$$\begin{aligned}\frac{dx}{dt} &= -\alpha y, x > 0 \\ \frac{dy}{dt} &= -\beta x, y > 0\end{aligned}$$

Note that  $x$  and  $y$  each represents the number of troops in each army. We are also not interested in the numbers that are less than 0. The  $\alpha$  and  $\beta$  in each equation each represent the efficiency which the respective army kills with. An army with superior firepower or better trained troops would have a higher coefficient of destruction. Note also that the larger number troops in the opposing army also leads to a higher rate of death for the army.

## 1.2. Solving Explicitly

The above differential equations can now be solved explicitly. We could do this by solving each equation separately and introducing time as the independent variable, however, in this instance it is more informative to eliminate time as a variable and solve both equations simultaneously.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\alpha x}{\beta y} \\ \beta(y^2 - y_o^2) &= \alpha(x^2 - x_o^2)\end{aligned}$$

With this explicit solution we can now compare the effects of destruction on each troop

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population. First we solve the equation when

$$y_f = 0$$
$$x_f = \sqrt{x_o^2 - \frac{\beta}{\alpha} y_o^2}.$$

We can then solve when

$$x_f = 0$$
$$y_f = \sqrt{y_o^2 - \frac{\alpha}{\beta} x_o^2}$$

This explicit derivation shows the power of prediction that the modified population model holds for models of combat.

### 1.3. Analyzing with Matlab

Next we can examine the numerical solution of this equation. We can use Matlab to generate a graph of this numerical solution. The solution shown in Figure 1 shows the decay of one side of the army down to zero. This is the point where one army is victorious in combat and the other has been completely annihilated.

## 2. Battle of the Atlantic

The Battle of the Atlantic was a conflict during World War II. During the conflict, German submarines attempted to cut off the allied supply route to the British Isles. They attempted to do this by destroying all of the merchant ships that were using the supply route across the Atlantic Ocean.

Since we have a basic understanding of the modeling of warfare, we can now attempt to model a large scale and complex battle. Using earlier techniques, we can now try to model the Battle of the Atlantic and simulate the numbers of merchant ships that were



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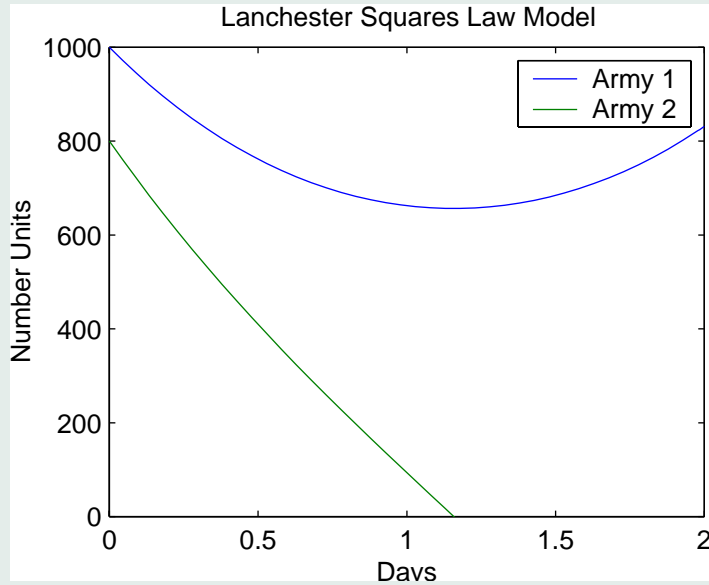


Figure 1: A solution of the Lanchester model.



Figure 2: A dramatic moment in the battle of the Atlantic.



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destroyed during the conflict. In doing this we will also be able to analyze the numbers of submarines destroyed as well as escort ships for the merchant marine.

## 2.1. Destruction Efficiency

First we must find some estimations for the efficiency of the submarine destruction of the merchant marine. We must also find the reciprocal efficiency with which escort ships managed to destroy the German submarines.

$$\alpha = \frac{5n}{c}$$

This will be the rate at which the merchant ships are destroyed.

$$\beta = \frac{nc}{100}$$

This will be the destruction of submarines by escort ships.

$$\lambda = \frac{nc}{500}$$

This will be the destruction of escort ships by submarines, where  $n$  is the size of the attacking convoy of submarines while  $c$  is the number of escorts assigned to the merchant convoy.

## 2.2. Applying Lanchester Equations

Now we use the original Lanchester equations, slightly modified for our purposes.

$$\frac{dS}{dt} = -e \left( \frac{nc}{100} \right) - f_s + R_s$$

This is the actual rate of submarine destruction



$$\frac{dE}{dt} = -e \left( \frac{nc}{500} \right) + R_e$$

The actual rate of caravan escort destruction.

$$\frac{dM}{dt} = e \left( \frac{5n}{c} \right)$$

The rate of the merchant ships being destroyed.

Note that  $e$  is the rate at which submarines and convoys will find each other and engage on the high seas. Also note that  $dM/dt$  is positive, which signifies an increase in the value of  $M$ . This is because  $M$  is the total number of destroyed merchant ships over the course of the war rather than the actual population of ships.

## 2.3. Matlab Code for Numerical Analysis

We use the following Matlab code to craft a plot of each population over time:

```
p_c=.01;    %Probability that a convoy is attacked
t_c=30;     %Number of days a convoy
            %must be escorted (both ways)
T_c=50;     %Number of days for an escort to return to service
f_c=30/50;  %Number of escorts escorting convoys
t_s=20;     %Number of days a submarine can patrol
T_s=50;     %Number of days for a submarine to return to service
f_s=20/50;  %Number of submarines on patrol
r=1;        %Number of convoys that leave each day
m=40;       %Number of merchants in each convoy
n=4;        %Number of submarines in each patrol
p_s=.04;    %Probability that a submarine will
            %be lost to random circumstances
r_s=.7;     %Number of submarines that will be replaced
```

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```
%every day
r_e=.5;      %Number of escorts replaced each day

[t,x]=ode45('lanchester',[0,2000],[0;500;100],[...
    p_c,t_c,T_c,f_c,t_s,T_s,f_s,r,m,n,p_s,r_s,r_e);

plot(t,x)
axis([0, 3000, 0, 3000])
title('Battle of the Atlantic Model')
xlabel('Days')
ylabel('Number Units')
legend('Dead Merchants','Subs','Escorts')
```

We also will need the following function file as well.

```
function xprime=lanchester(t,x,flag,p_c,t_c,T_c,...
    f_c,t_s,T_s,f_s,r,m,n,p_s,r_s,r_e)
xprime=zeros(3,1); xprime(1)=10*r^2*T_c*p_c*f_s*(x(2)/x(3));
xprime(2)=-(((2*p_c*f_s)/(100*T_c))*x(3)+p_s/T_s)*x(2)+r_s;
xprime(3)=-((2*p_c*f_s)/(500*T_c))*x(3)*x(2)+r_e
```

Note that the ellipses are Matlab's line continuation character.

## 2.4. Creation of Matlab Plot

Using the initial conditions  $M(0) = 0$ ,  $S(0) = 50$ ,  $E(0) = 100$ , for the values of merchant ships, submarines and escort ships respectively we can craft the following plot. The Matlab plot in Figure 3 leads to many interesting conclusions. One is that the total number of submarines eventually decays to zero over time. Another is that the merchant marine loses a large number of ships over the course of the war and their escorts manage to retain a fairly normal amount of units.





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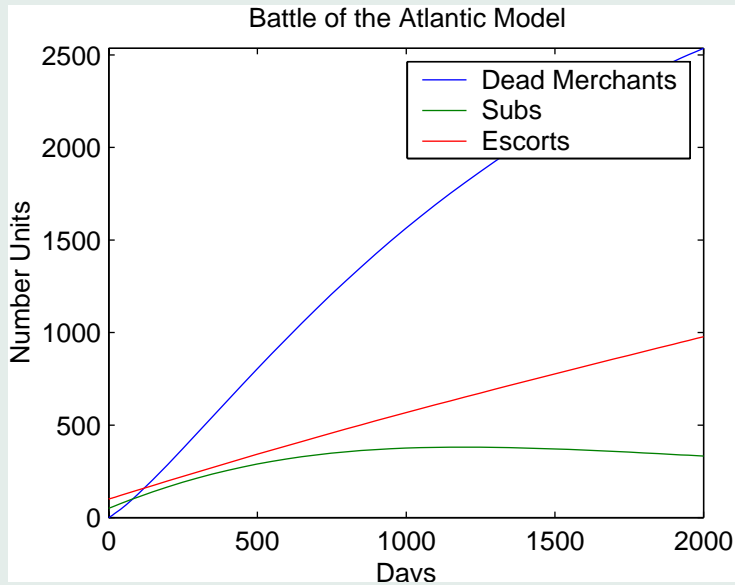


Figure 3: Simulation of the Battle of the Atlantic.



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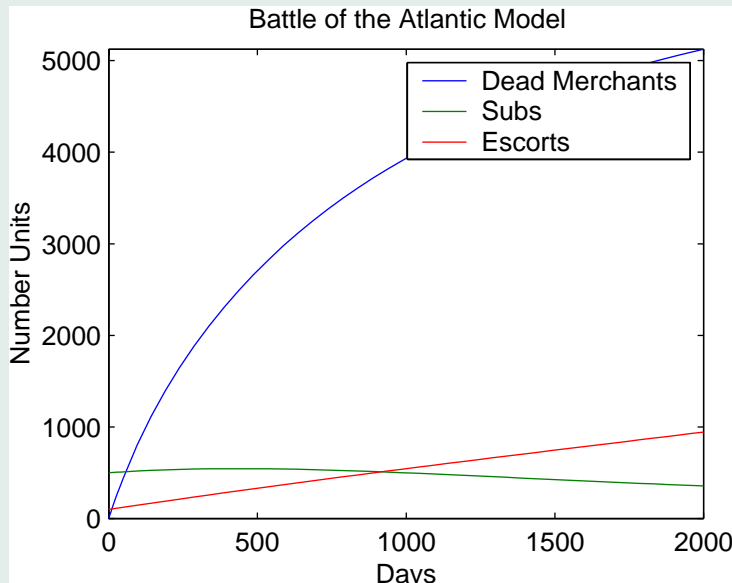


Figure 4: Changing the initial conditions.

The initial conditions  $M(0) = 0$ ,  $S(0) = 500$ ,  $E(0) = 100$ , for the values of the merchant ships, submarines and escort ships, respectively, lead to the solution shown in Figure 4.

### 3. Conclusions

The conclusion reached from the examination of these two graphs shows that over time, regardless of the initial amount of submarines, the allies will prevail in the Atlantic theater.



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The superiority of the cruiser compared to the submarine as a weapon of warfare makes this battle one that the allies are destined to win. Unfortunately the higher the initial amount of submarines, the higher number of merchant marine casualties. In the end, according to this model, allied troops would most likely have won the Battle of the Atlantic, regardless of the amount of German submarines initially involved.

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