Chapter 8

# Matrix Exponential Methods

## Application 8.1
## Automated Matrix Exponential Solutions

If $\mathbf{A}$ is an $n \times n$ matrix, then a computer algebra system can be used to calculate the fundamental matrix $e^{\mathbf{A}t}$ for the homogeneous linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Then Theorem 2 in Section 8.1 of the text says that the solution of the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

is given by the (matrix) product

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}_0. \tag{2}$$

In the paragraphs below we illustrate this approach by using *Maple*, *Mathematica*, and MATLAB to solve the initial value problem

$$\begin{aligned} x_1' &= 13x_1 + 4x_2, \qquad & x_1(0) &= 11 \\ x_2' &= 4x_1 + 7x_2, \qquad & x_2(0) &= 23. \end{aligned} \tag{3}$$

The success of this automated method and the usefulness of the results for a particular initial value problem may depend on whether sufficiently simple symbolic expressions for the eigenvalues and eigenvectors of the given matrix $\mathbf{A}$ can be found.

For a three-dimensional example to investigate, use this matrix exponential approach to solve the initial value problem

$$\begin{aligned} x_1' &= -149x_1 - 50x_2 - 154x_3, \qquad & x_1(0) &= 17 \\ x_2' &= 537x_1 + 180x_2 + 546x_3, \qquad & x_2(0) &= 43 \\ x_3' &= -27x_1 - 9x_2 - 25x_3, \qquad & x_3(0) &= 79. \end{aligned}$$

And here's a four-dimensional problem:

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 + 7x_4, \qquad & x_1(0) &= 15 \\ x_2' &= x_1 + 4x_2 + 10x_3 + x_4, \qquad & x_2(0) &= 35 \end{aligned}$$

$$x_3' = x_1 + 10x_2 + 4x_3 + x_4, \qquad x_3(0) = 55$$
$$x_4' = 7x_1 + x_2 + x_3 + 4x_4, \qquad x_4(0) = 75.$$

If at this point you're having too much fun with matrix exponentials to stop, make up some problems of your own. For instance, choose any homogeneous linear system appearing in this chapter and experiment with differential initial conditions. The exotic $5\times5$ matrix $\mathbf{A}$ of the 7.5 Application (Chapter 7) may suggest an interesting possibility.

## Using *Maple*

First we define the coefficient matrix

```
with(linalg):
A := matrix(2,2, [13, 4, 4, 7] );
```

$$A := \begin{bmatrix} 13 & 4 \\ 4 & 7 \end{bmatrix}$$

and the initial vector

```
x0 := matrix(2,1, [11, 23]);
```

$$x0 := \begin{bmatrix} 11 \\ 23 \end{bmatrix}$$

for the initial value problem in (3). Then a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

```
fundMatrix := exponential(A,t);
```

$$fundMatrix := \begin{bmatrix} \frac{1}{5}e^{(5t)} + \frac{4}{5}e^{(15t)} & -\frac{2}{5}e^{(5t)} + \frac{2}{5}e^{(15t)} \\ -\frac{2}{5}e^{(5t)} + \frac{2}{5}e^{(15t)} & \frac{4}{5}e^{(5t)} + \frac{1}{5}e^{(15t)} \end{bmatrix}$$

Hence Equation (2) gives the particular solution

```
solution := evalm(fundMatrix &* x0);
```

$$solution := \begin{bmatrix} -7e^{(5t)} + 18e^{(15t)} \\ 14e^{(5t)} + 9e^{(15t)} \end{bmatrix}$$

Thus the desired particular solution is given in scalar form by

$$x_1(t) := -7e^{5t} + 18e^{15t}, \qquad x_2(t) := 14e^{5t} + 9e^{15t}.$$

## Using *Mathematica*

First we define the coefficient matrix

```
A = {{13, 4}, {4, 7}};
A // MatrixForm
```

```
13   4
4    7
```

and the initial vector

```
x0 = {{11}, {23}};
x0 // MatrixForm
```

```
11
23
```

for the initial value problem in (3). Then a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

```
fundMatrix = MatrixExp[ A t ] // Expand
fundMatrix // MatrixForm
```

```
   5 t       15 t              5 t       15 t
  E       4 E              -2 E       2 E
  ---- + -------           ------- + -------
   5         5                5          5

     5 t       15 t          5 t     15 t
  -2 E       2 E           4 E       E
  ------- + -------        ------ + -----
     5          5             5        5
```

Hence Equation (2) gives the particular solution

```
solution = fundMatrix . x0 // Expand
solution // MatrixForm
```

```
    5 t        15 t
-7 E     + 18 E

    5 t       15 t
14 E     + 9 E
```

Thus the desired particular solution is given in scalar form by

$$x_1(t) := -7e^{5t} + 18e^{15t}, \qquad x_2(t) := 14e^{5t} + 9e^{15t}.$$

**Using MATLAB**

First we define the coefficient matrix

```
A = [13 4; 4 7]

A =
    13     4
     4     7
```

and the initial vector

```
x0 = [11; 23]
x0 =
    11
    23
```

for the initial value problem in (3). Then a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

```
syms t
fundMatrix = expm(A*t)

fundMatrix =

[ 4/5*exp(15*t)+1/5*exp(5*t),-2/5*exp(5*t)+2/5*exp(15*t)]
[-2/5*exp(5*t)+2/5*exp(15*t), 1/5*exp(15*t)+4/5*exp(5*t)]
```

Hence Equation (2) gives the particular solution

```
solution = fundMatrix*x0

solution =

[ 18*exp(15*t)-7*exp(5*t)]
[ 14*exp(5*t)+9*exp(15*t)]
```

Thus the desired particular solution is given in scalar form by

$$x_1(t) := -7e^{5t} + 18e^{15t},$$

$$x_2(t) := 14e^{5t} + 9e^{15t}.$$