A Simple Microeconomics Model

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Abstract

Two common examples of everyday business.

1 Introduction

For Dave Arnold's spring semester differential equations(DE) class we all presented DE models. Here's a rundown of the parameters, assumptions, and qualitative solutions of the model I presented in class. The model I chose came from our textbook, the preliminary edition Differential Equations written by the authors Blanchard, Devaney, and Hall. The simple micro-economics model is referred to numerous times throughout chapter three, Linear Systems. The model first shows up in section 3.1 page 224. The following is a quote from the text.

"After retiring from writing differential equations text books full of silly examples, Paul and Bob both decide to open small stores selling compact discs. *Paul's Rock and Roll CDs* and *Bob's Opera Only Discs* are located on the same block, and Paul and Bob soon become concerned about the effect the stores have on each other. On the one hand, having two CD stores on the same block might help both stores by attracting customers to the neighborhood. On the other hand, the stores may compete with each other for limited customers."

After much arguing a simple system is finally proposed.

$$\frac{dx}{dt} = ax + by \tag{1}$$

$$\frac{dy}{dt} = cx + dy \tag{2}$$

Let $x(t)$ = profit per day of Paul's store at time t. If $x(t)$ is greater than zero Paul's store is making money, but if $x(t)$ is less than zero his store is losing money. Let $y(t)$ = profit per day of Bob's store at time t. If $y(t)$ is greater than zero Bob's store is making money, but if $y(t)$ is less than zero his store is losing money.

For this model a, b, c, and d are parameters. Now this autonomous linear system allows each store to affect the other in other words the rate of change of

Paul's profits depends linearly on both Paul's profits and Bob's profits. These same assumptions apply for Bob's profits as well.

1.1 Example #1

Adjust the parameters a, b, c, and d to account for the two stores competing for limited customers. For this example we assume that the typical customer doesn't have a preference to either rock and roll music or opera music and will only shop in one of the two stores.

Consider the following system of differential equations

$$\frac{dx}{dt} = 2x - 3y \tag{3}$$
$$\frac{dy}{dt} = -3x + 2y \tag{4}$$

Recall that $\frac{dx}{dt}$ is the rate of change in profit for Paul's store. The parameter $a = 2$ allows Paul to profit more quickly if his store is making a profit. In other words the ax term will be greater than zero. However if Bob's store is making money, y greater than zero, and the parameter $b = -3$ then Paul's profit will increase more slowly. Hence the by term will be less than zero. In this sense the parameter b can be thought of as Bob stealing customers away from Paul. These same assumptions apply to the parameters c and d of Bob's store.

In order to find the general solution to this system, we need to find the eigenvalues and the eigenvectors. The characteristic equation is

$$\lambda^2 - T\lambda + \Delta = 0 \tag{characteristic equation}$$

where $T = a + b$ and $\Delta = ab - cd$. Substituting the parameter values, from equations 3 and 4 into this equation yields

$$\lambda^2 - 4\lambda - 5 = 0$$

This equation can be rewritten and factored solving for lambda

$$\begin{aligned}
\lambda^2 + \lambda - 5\lambda - 5 &= 0 \\
\lambda(\lambda + 1) - 5(\lambda + 1) &= 0 \\
(\lambda - 5)(\lambda + 1) &= 0 \\
\lambda &= -1, 5
\end{aligned}$$

Having one positive and one negative eigen value indicates that the origin is a saddle. Now we're ready to find the eigen vectors.

Let A be our two by two matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and Y be a two by one column vector

$$Y = \begin{bmatrix} x \\ y \end{bmatrix}$$

and I be the identity matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now consider the following moves.

$$\begin{aligned} AY &= \lambda Y \\ AY - \lambda Y &= 0 \\ AY - \lambda IY &= 0 \\ (A - \lambda I)\, Y &= 0 \end{aligned} \qquad \text{(formula \#1)}$$

Rewriting the system of D.E.s in this last form

$$\begin{aligned} (2-\lambda)\, x - 3y &= 0 \\ -3x + (2-\lambda)\, y &= 0 \end{aligned}$$

This last move gives two redundant equations, by substituting in our eigen values in to either of these equations will allow us to pick smart eigenvectors. By subbing in lambda equals minus one into the top equation gives

$$3x - 3y = 0$$

Choose a smart eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Substitute lambda equals five into the top equation

$$-3x - 3y = 0$$

and pick a smart eigenvector

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Now we have enough information to write out the general solution of our system

$$Y(t) = k_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

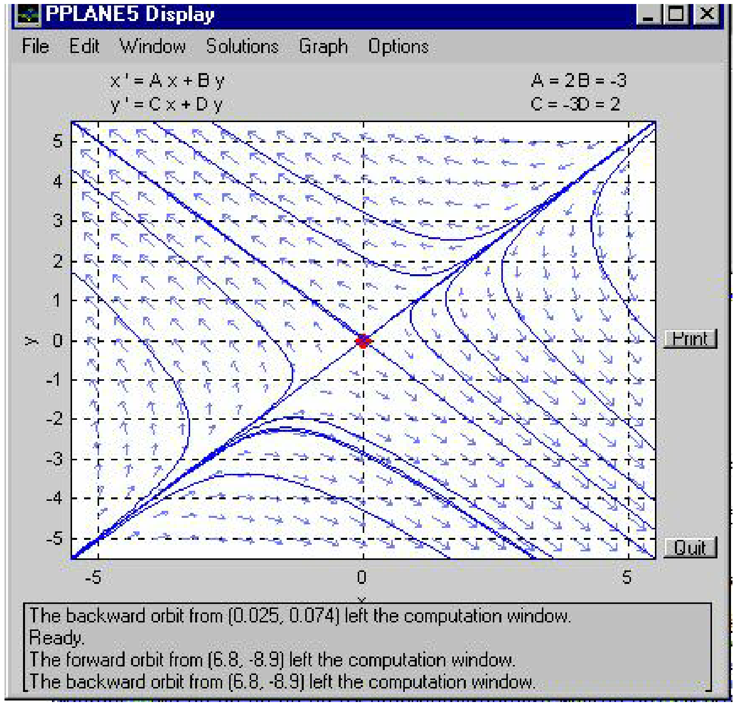

Figure 1:

1.1.1 Conclusions of example #1

The direction field (figure #1) shows that the long term solutions to this model are different for either side of the straight line solution ($y = x$). If at time $t = 0$ both stores are making the same amount of profit $x(0) = y(0)$ then their profits will gradually decrease to zero and stay there as time moves forward, which means eventually both stores will just break even, they won't make any money or loose any money.

If at time $t = 0$ both stores are losing the same amount of profit $x(0) = y(0)$ then the rate at which their losing profit will gradually decrease until zero and again both stores will just break even.

However If one of the stores is making slightly more money than the other at time $t = 0$ the long term solution is quite different from the first two cases. For example if Paul's store is making just slightly more than Bob's store at time $t = 0$ Paul gets richer and richer as time moves forward, but Bob loses money and his store falls deeper and deeper in to debt as time moves forward.

How ever if at time $t = 0$ Bob's store is making slightly more money than Paul's store, Bob will eventually become rich and Paul will become broke.

1.2 Example #2

Changing the parameters a, b, c, and d to reflect new market conditions is easy. Consider the following system of differential equations.

$$\frac{dx}{dt} = -2x - 5y \tag{5}$$

$$\frac{dy}{dt} = 3x + y \tag{6}$$

Like before $\frac{dx}{dt}$ is the rate of change in profit for Paul's store and $\frac{dy}{dt}$ represents the profits of Bob's store. Again in order to find the general solution to the system we need to find the eigen values and the eigen vectors. Starting out with the characteristic polynomial and substituting in the new values of the trace T and the determinant Δ.

$$\lambda^2 + \lambda + 13 = 0$$

Solving this equation for lambda using the quadratic formula yields complex numbers for lambda.

$$\lambda = \frac{-1 + i\sqrt{51}}{2} \tag{lambda \#1}$$

and

$$\lambda = \frac{-1 - i\sqrt{51}}{2} \tag{lambda \#2}$$

Because the real part of the eigenvalues is negative, we know that the origin is a spiral sink. Now substituting our equations, 5 and 6, in to formula #1 we get a similar situation to example #1, two redundant equations.

$$\begin{aligned} (-2 - \lambda)\, x - 5y &= 0 \\ 3x + (1 - \lambda)\, y &= 0 \end{aligned}$$

Next rewrite the top equation

$$y = \frac{(-2 - \lambda)\, x}{5}$$

and choose a smart eigenvector

$$v = \begin{bmatrix} 5 \\ \frac{-3 - i\sqrt{51}}{2} \end{bmatrix}$$

Now we have enough information to write out the complex solution of our system

$$Y_1(t) = e^{\left(\frac{-1 + i\sqrt{51}}{2}\right)t} \begin{bmatrix} 5 \\ \frac{-3 - i\sqrt{51}}{2} \end{bmatrix}$$

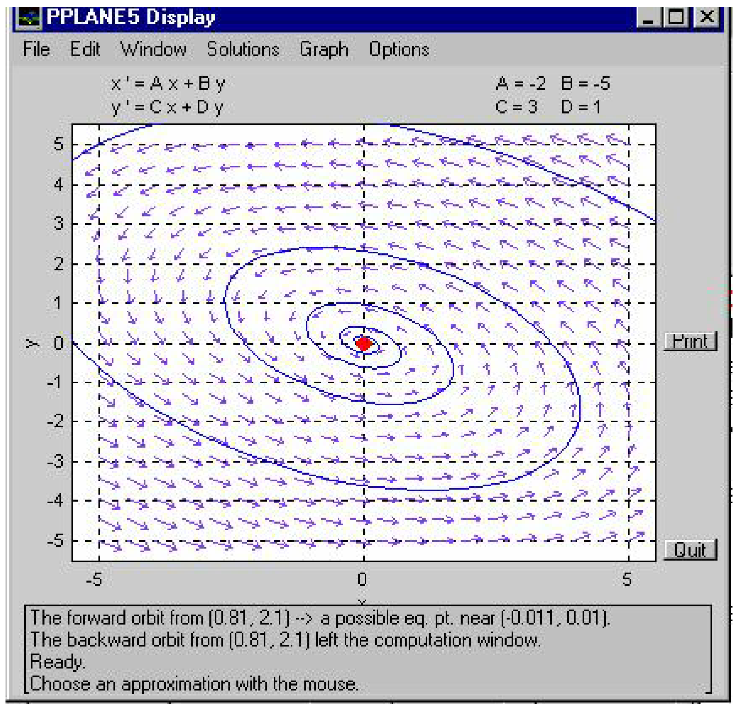

Figure 2:

Rewriting this using Euler's formula

$$Y_1(t) = e^{-\frac{1}{2}t}\begin{pmatrix} 5\cos\frac{\sqrt{51}}{2}t \\ \frac{\sqrt{51}}{2}\sin\frac{\sqrt{51}}{2}t - \frac{3}{2}\cos\frac{\sqrt{51}}{2}t \end{pmatrix} + ie^{-\frac{1}{2}t}\begin{pmatrix} 5\sin\frac{\sqrt{51}}{2}t \\ -\frac{\sqrt{51}}{2}\cos\frac{\sqrt{51}}{2}t - 3\sin\frac{\sqrt{51}}{2}t \end{pmatrix}$$

the general solution is

$$Y(t) = k_1 e^{-\frac{1}{2}t}\begin{pmatrix} 5\cos\frac{\sqrt{51}}{2}t \\ \frac{\sqrt{51}}{2}\sin\frac{\sqrt{51}}{2}t - \frac{3}{2}\cos\frac{\sqrt{51}}{2}t \end{pmatrix} + k_2 e^{-\frac{1}{2}t}\begin{pmatrix} 5\sin\frac{\sqrt{51}}{2}t \\ -\frac{\sqrt{51}}{2}\cos\frac{\sqrt{51}}{2}t - 3\sin\frac{\sqrt{51}}{2}t \end{pmatrix}$$

where k_1 and k_2 are arbitrary constants depending on the initial conditions.

1.2.1 Conclusions of example #2

As you can see from the direction field (figure #2) all solutions spirals in to the origin in smaller and smaller loops as time t moves forward. This means that the profits of Paul's store and Bob's store will oscillate decreasingly less and less as time moves forward, in a boom-to-bust-to-boom business cycle, and eventually all solutions will dive in to the origin meaning that both stores eventually break even.

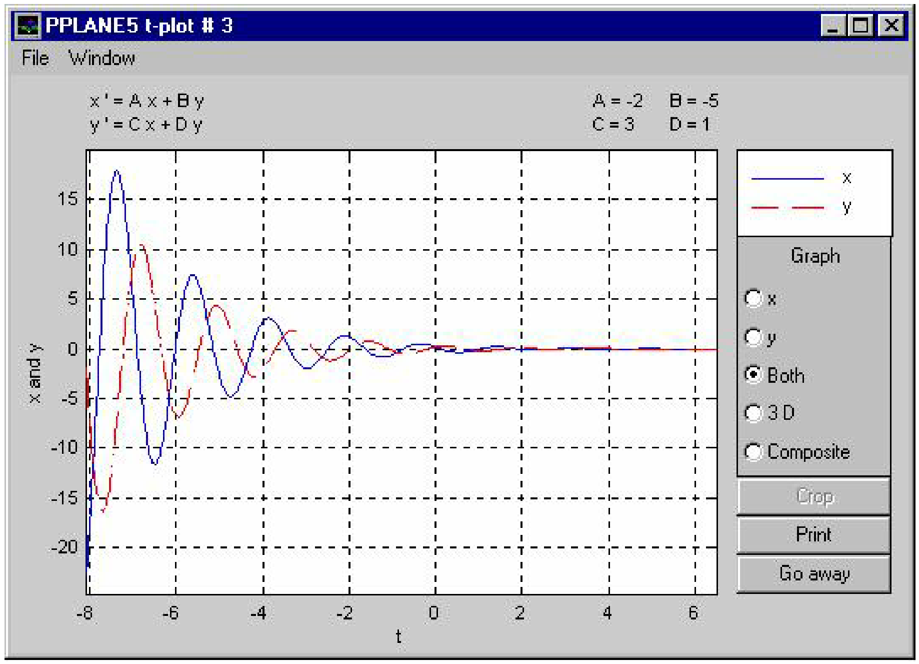

Figure 3:

Figure #3 is a graph of $x(t)$ and $y(t)$, from this graph you can see how the profits of the two CD stores occillate up and down in periods of $\frac{\sqrt{51}}{2}$ and with a decaying amplitude of $e^{-\frac{1}{2}t}$ eventually breaking even.