Rigid Body Rotations

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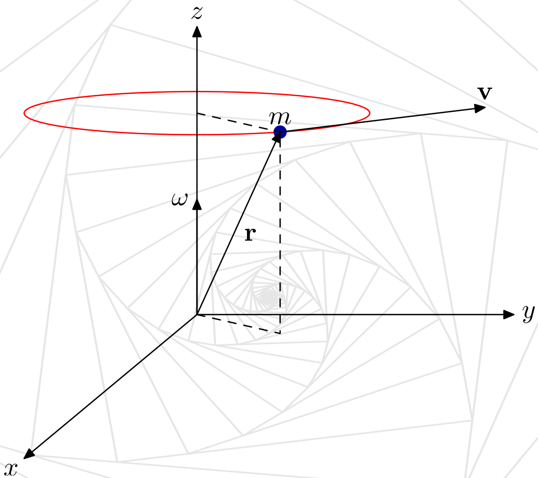
A Point Particle and Fundamental Quantities



Angular Velocity	$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$
Kinetic Energy	$K = \frac{1}{2} m v^2$
Momentum	$\mathbf{p} = m\mathbf{v}$

Rigid Bodies

- We treat a rigid body as a system of particles.
- Each particle on the rigid body rotates with the same angular velocity $\boldsymbol{\omega}$.
- The angular velocity vector $\boldsymbol{\omega}$ of a rigid body whose center of mass stays constant is directed along its axis of rotation.

Angular Momentum

The angular momentum of a point particle is defined to be

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$$

The angular momentum of a rigid body is the vector sum of the individual momenta of it's particles.

$$\mathbf{L} = \sum_{i=1} m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)$$

Computing these cross products yields the expression

$$\begin{aligned}\mathbf{L} =& [\omega_x \sum_{i=1} m_i(y_i^2 + z_i^2) - \omega_y \sum_{i=1} m_i x_i y_i - \omega_z \sum_{i=1} m_i x_i z_i]\hat{\boldsymbol{\imath}} + \\ & [-\omega_x \sum_{i=1} m_i x_i y_i + \omega_y \sum_{i=1} m_i(x_i^2 + z_i^2) - \omega_z \sum_{i=1} m_i y_i z_i]\hat{\boldsymbol{\jmath}} + \\ & [-\omega_x \sum_{i=1} m_i x_i z_i - \omega_y \sum_{i=1} m_i y_i z_i + \omega_z \sum_{i=1} m_i(x_i^2 + y_i^2)]\hat{\boldsymbol{\kappa}}\end{aligned}$$

Moments of Inertia

The sums in our previous expression for **L** have fundamental importance.

$$\sum_{i=1} m_i(y_i^2 + z_i^2) = \text{moment of inertia about the } x\text{-axis} = I_x.$$

$$\sum_{i=1} m_i(x_i^2 + z_i^2) = \text{moment of inertia about the } y\text{-axis} = I_y.$$

$$\sum_{i=1} m_i(x_i^2 + y_i^2) = \text{moment of inertia about the } z\text{-axis} = I_z.$$

$$\sum_{i=1} m_i x_i y_i = xy\text{-product of inertia} = I_{xy}.$$

$$\sum_{i=1} m_i x_i z_i = xz\text{-product of inertia} = I_{xz}.$$

$$\sum_{i=1} m_i y_i z_i = yz\text{-product of inertia} = I_{yz}.$$

We can now write our expression for angular momentum as

$$
\begin{aligned}
\mathbf{L} = (\omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz})\hat{\boldsymbol{\imath}} + \\
(-\omega I_{xy} + \omega_y I_y - \omega_z I_{yz})\hat{\boldsymbol{\jmath}} + \\
(-\omega I_{xz} - \omega_y I_{yz} + \omega_z I_z)\hat{\boldsymbol{\kappa}}
\end{aligned}
$$

Rotational Kinetic Energy

The rotational form of kinetic energy is obtained from it's linear form.

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\boldsymbol{\omega}\times\mathbf{r})\cdot\mathbf{v}$$

The total kinetic energy of a rigid body is the sum of the individual kinetic energies of its constituent particles.

$$K = \frac{1}{2}\sum_{i=1} m_i(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\mathbf{v}_i$$

Computing the cross product and then the dot product yields

$$K = \frac{1}{2}\boldsymbol{\omega}\cdot\sum_{i=1} m_i\mathbf{r}_i\times(\boldsymbol{\omega}\times\mathbf{r}_i) = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}$$

The Inertia Tensor

- Our expressions for $\mathbf{L}$ and K are complex and cumbersome.
- We need a more compact form of $\mathbf{L}$ and K.
- This is accomplished using what is known as the inertia tensor.

Recall the previous expression for $\mathbf{L}$.

$$\mathbf{L} = (\omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz})\hat{\boldsymbol{\imath}} + (-\omega I_{xy} + \omega_y I_y - \omega_z I_{yz})\hat{\boldsymbol{\jmath}} + (-\omega I_{xz} - \omega_y I_{yz} + \omega_z I_z)\hat{\boldsymbol{\kappa}}$$

In components

$$\begin{aligned} L_x &= \omega_x I_x - \omega_y I_{xy} - \omega_z I_{xz} \\ L_y &= -\omega_x I_{xy} + \omega_y I_y - \omega_z I_{yz} \\ L_z &= -\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_z \end{aligned}$$

This system of equations can be written in matrix form as

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or

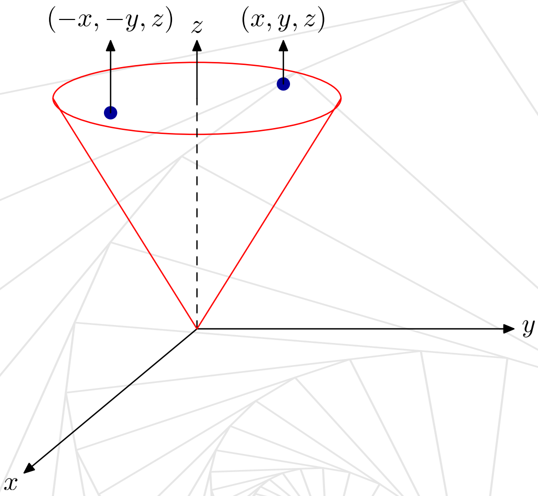
$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$ where $\mathbf{I}$ is the inertia tensor.

We can also express K in terms of $\mathbf{I}$

$$K = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} = \frac{1}{2}\boldsymbol{\omega}\cdot(\mathbf{I}\boldsymbol{\omega})$$

Principal Axes and Diagonalizing the Inertia Tensor

- Every rigid body has 3 principal axes.
- Rotations about these axes are stable. ($\boldsymbol{\omega}$ is constant in time.)
- Choosing the principal axes to be our coordinate axes makes the object's products of inertia zero.
- The inertia tensor then contains only the object's moments of inertia along it's main diagonal.
- This process is called diagonalizing the inertia tensor.
- We begin by letting the z-axis be the body symmetry axis.

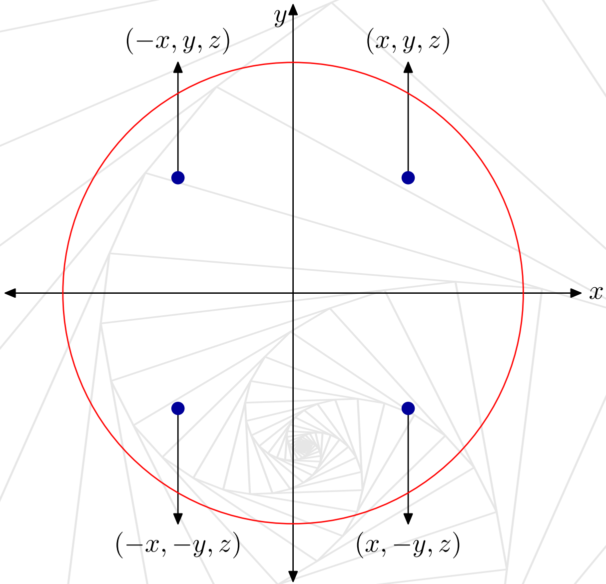


- Every point (x,y,z) has a corresponding point ($-x$,$-y$,z) which is a reflection across the z-axis.
- Therefore, in the sum of the xz and yz-products of inertia, we have

$$I_{xz} = \sum_{i=1} m_i x_i z_i = 0$$

$$I_{yz} = \sum_{i=1} m_i y_i z_i = 0$$

We now show that the xy-product of inertia is also zero.



Therefore, we can conclude that in the total sum:

$$I_{xy} = \sum_{i=1} m_i x_i y_i = 0$$

.

With this coordinate system orientation, the inertia tensor becomes

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Consider again, an object's rotational kinetic energy.

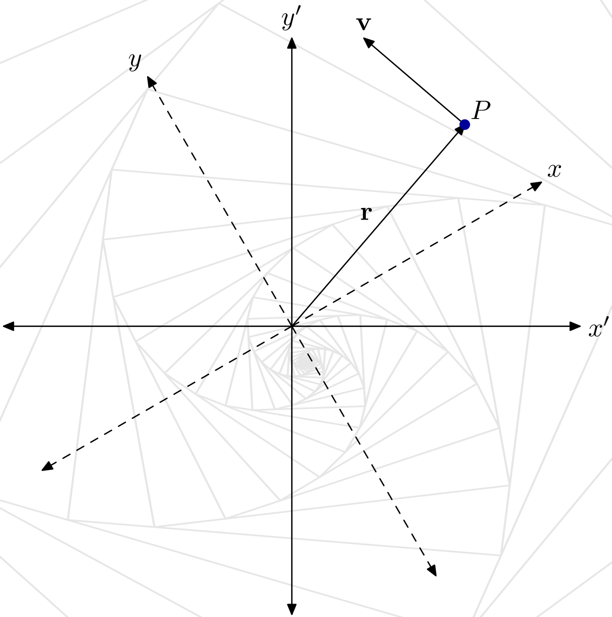
$$K = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbf{I}\boldsymbol{\omega})$$

$$K = \frac{1}{2}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \cdot \left[\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}\right]$$

As you can see, this greatly simplifies our computations, which give

$$K = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Rotating Coordinate Systems



$\mathbf{r}$ must be the same as measured from both coordinate systems.

$$\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'$$

If we take $\mathbf{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ and differentiate both sides from the *rotating* frame of reference we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} = \left(\frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}}\right)$$

Now we take the same derivative with from the *fixed* reference frame.

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \frac{dx}{dt}\hat{\boldsymbol{\imath}} + x\frac{d\hat{\boldsymbol{\imath}}}{dt} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}$$

Grouping:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{dx}{dt}\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\hat{\boldsymbol{\jmath}}\right) + \left(x\frac{d\hat{\boldsymbol{\imath}}}{dt} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}\right) \tag{1}$$

Substituting the previous result yields

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \left(x\frac{d\hat{\boldsymbol{\imath}}}{dt} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}\right) \tag{2}$$

Now, recall that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r}$$

Therefore, we can treat $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ as rotating position vectors and say that

$$\frac{d\hat{\boldsymbol{\imath}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{\imath}} \qquad \text{and} \qquad \frac{d\hat{\boldsymbol{\jmath}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{\jmath}}$$

Recalling equation (2) and making these substitutions

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \left(x\frac{d\hat{\boldsymbol{\imath}}}{dt} + y\frac{d\hat{\boldsymbol{\jmath}}}{dt}\right)$$

$$\begin{aligned}
\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + [x(\boldsymbol{\omega} \times \hat{\boldsymbol{\imath}}) + y(\boldsymbol{\omega} \times \hat{\boldsymbol{\jmath}})] \\
&= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) \\
&= \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r}
\end{aligned}$$

This gives us the general operator on any vector $\mathbf{A}$

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{A}$$

Euler's Equations

The rotational motion of a rigid body is completely described by Newton's second law in rotational form.

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}$$

To avoid a changing inertia tensor, we must adopt a coordinate system that rotates with the rigid body. Using our previous operator, we have

$$\sum \boldsymbol{\tau} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt} = \left(\frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

But since the inertia tensor is now constant in time

$$\boldsymbol{\tau} = \mathbf{I}\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

And again using our vector operator, we see that

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \boldsymbol{\omega}$$

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}}$$

And we now have

$$\boldsymbol{\tau} = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

Or, in matrix form

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} + \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Computing the cross product and matrix multiplication yields

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} \dot{\omega}_x I_x \\ \dot{\omega}_y I_y \\ \dot{\omega}_z I_z \end{pmatrix} + \begin{pmatrix} \omega_y \omega_z I_z - \omega_y \omega_z I_y \\ \omega_x \omega_z I_x - \omega_x \omega_z I_z \\ \omega_x \omega_y I_y - \omega_x \omega_y I_x \end{pmatrix}$$

Or, in components we have

$$\tau_x = \omega_y \omega_z I_z - \omega_y \omega_z I_y + \dot{\omega}_x I_x$$
$$\tau_y = \omega_x \omega_z I_x - \omega_x \omega_z I_z + \dot{\omega}_y I_y$$
$$\tau_z = \omega_x \omega_y I_y - \omega_x \omega_y I_x + \dot{\omega}_z I_z$$

These three equations are known as Euler's equations of motion for a rigid body.

Force Free Rotations and Precession

We wish to consider in detail the case when the net torque on a rotating object is zero. This is true for any object freely falling in a gravitational field. In this case, Euler's equations become

$$\dot{\omega}_x = \frac{I_y - I_z}{I_x}\omega_y\omega_z$$
$$\dot{\omega}_y = \frac{I_z - I_x}{I_y}\omega_x\omega_z$$
$$\dot{\omega}_z = \frac{I_x - I_y}{I_z}\omega_x\omega_y$$

This is a nonlinear system of three first order equations which is easily solved using ODE45. But let's first consider the nature of such rotations given some initial conditions.

Initial Conditions and Equilibrium Solutions

The equilibrium points are obtained by setting $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$.

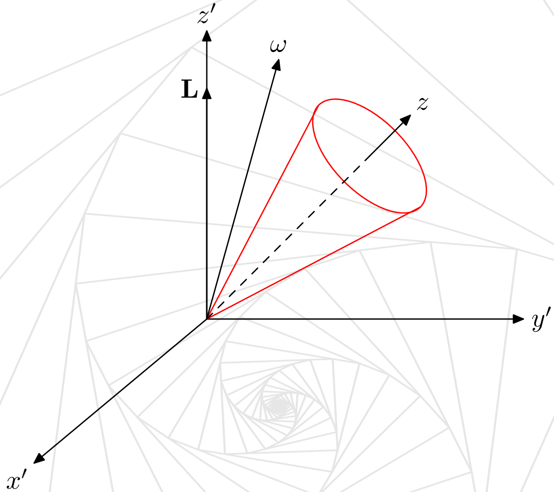
$$0 = \omega_y \omega_z$$
$$0 = \omega_x \omega_z$$
$$0 = \omega_x \omega_y$$

When solving this system simultaneously, we get any multiple of one of the following

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

In other words, any point on the coordinate axes is an equilibrium point. Therefore, any solution with an initial condition of one of these forms will remain constant for all time. That is, if $\boldsymbol{\omega}$ is directed along one of the three principal axes, it will remain constant.

Let us now consider an initial condition when $\boldsymbol{\omega}$ is in some arbitrary direction.



In the fixed coordinate system:

$$\sum \boldsymbol{\tau} = 0 = \frac{d\mathbf{L}}{dt} = \frac{d(\mathbf{I}\boldsymbol{\omega})}{dt}$$

In the rotating coordinate system:

$$\sum \boldsymbol{\tau} = 0 = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega})$$

Kinetic Energy and the Ellipsoid

Since we are considering force free motion, we expect, from the conservation of energy, that K will also remain constant in time. We can verify this by considering our previous expression for K.

$$K = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2)$$

Taking the time derivative of K

$$\frac{dK}{dt} = \frac{1}{2}(2I_x\omega_x\dot{\omega}_x + 2I_y\omega_y\dot{\omega}_y + 2I_z\omega_z\dot{\omega}_z)$$

Substituting in the differential equations for $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$

$$\frac{dK}{dt} = \frac{1}{2}\left(2I_x\omega_x\frac{I_y - I_z}{I_x}\omega_y\omega_z + 2I_y\omega_y\frac{I_z - I_x}{I_y}\omega_x\omega_z + 2I_z\omega_z\frac{I_x - I_y}{I_z}\omega_x\omega_y\right)$$

$$\begin{aligned}\frac{dK}{dt} &= \omega_x\omega_y\omega_z\left[(I_y - I_z) + (I_z - I_x) + (I_x - I_y)\right] \\ &= 0\end{aligned}$$

So, given an initial condition, $(\omega_x, \omega_y, \omega_z)$ the object's kinetic energy is given by

$$K = \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) = \text{constant}$$

- This gives the equation of an ellipsoid.
- This surface represents a surface of constant energy called the inertial ellipsoid.
- Solutions must stay on the ellipsoid for all time.
- The equilibrium points occur where the axes intersect the ellipsoid.

Example

Let $I_x = 2$, $I_y = 1$, and $I_z = 3$. Our system of equations now becomes:

$$\dot{\omega}_x = -\omega_y \omega_z$$
$$\dot{\omega}_y = \omega_x \omega_z$$
$$\dot{\omega}_z = \frac{1}{3}\omega_x \omega_y$$

And our equation for kinetic energy becomes

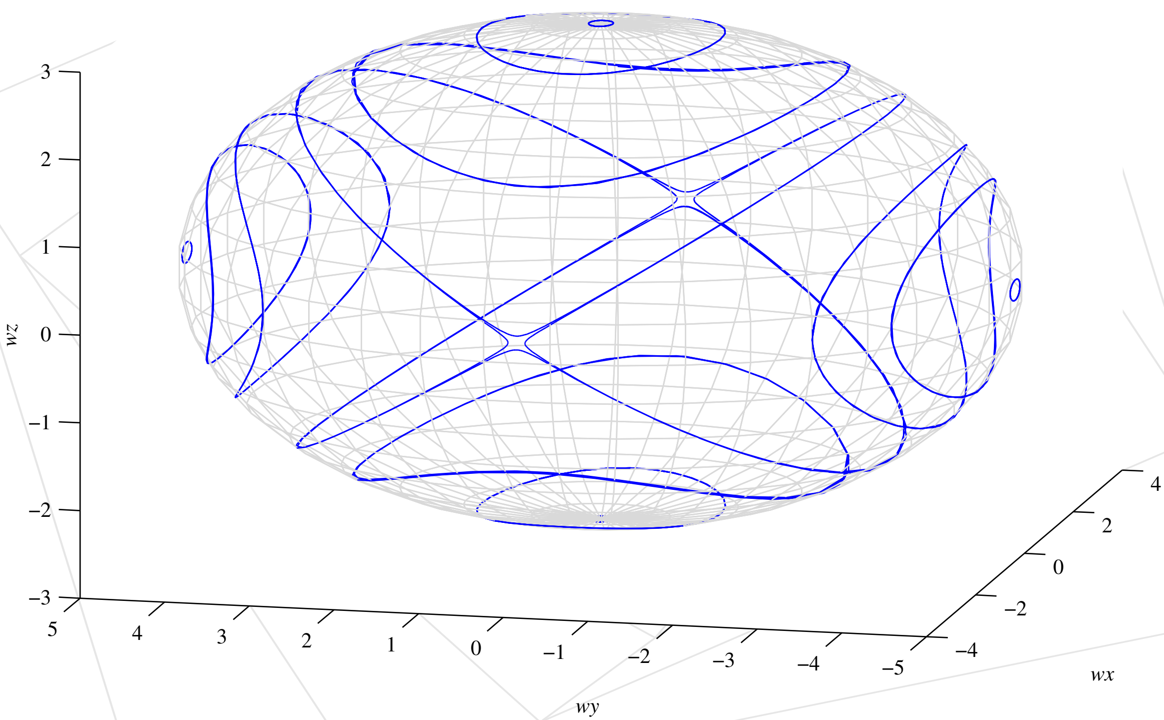
$$K = \frac{1}{2}(2\omega_x^2 + \omega_y^2 + 3\omega_z^2)$$

If we let $K = 12$, this becomes the equation of our inertial ellipsoid

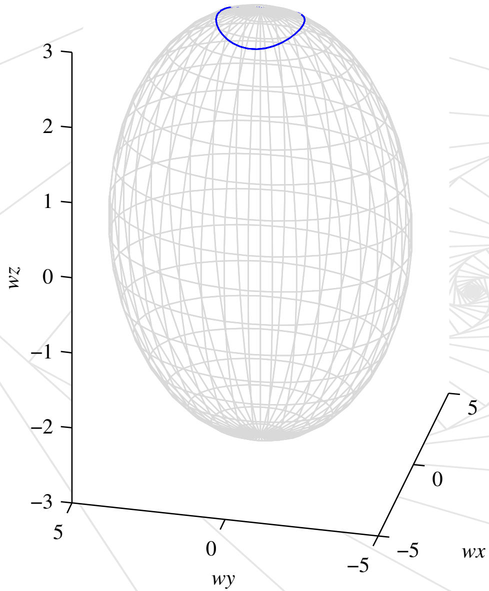
$$24 = 2\omega_x^2 + \omega_y^2 + 3\omega_z^2$$

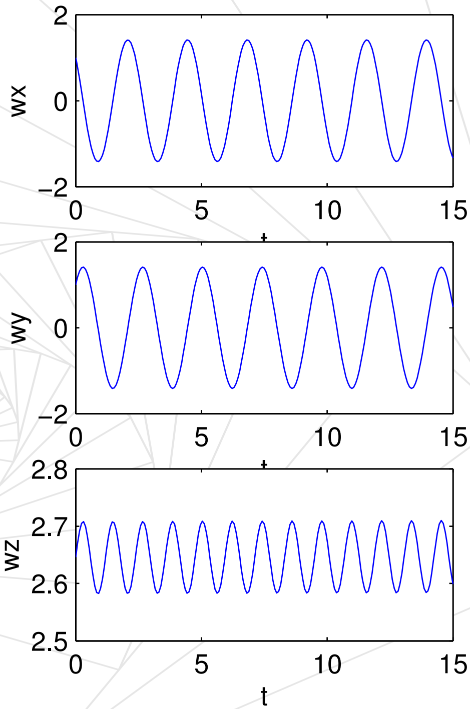
If we pick off any initial condition that satisfies this equation, the resulting solution trajectory must remain on the surface of the ellipsoid for all time.

This is observed using ODE45 to plot several solution trajectories



Examining a solution near a stable equilibrium point





Now consider a solution near an unstable equilibrium point