Tacoma Narrows Bridge

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Abstract

The exact reasons behind the failure of the Tacoma Narrows Bridge are still being debated. This paper explores the error in the mathematic model that was used to estimate the bridges real life oscillations, which is believed to be a large part of its failure, and compares it to the model that should have been used which more accurately estimates those oscillations.

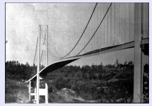
1. Introduction

In 1940 the wind started blowing, much like it had on many other days. Yet on this day something went wrong. The Tacoma Narrows bridge experienced normal vertical oscillations, no worries. The problem was the oscillations did not stay vertical, they changed to torsional. This made the bridge undergo a showcase of large oscillations and an eventual collapse. To understand this failure in more detail a model was developed to study the mathematics involved. This model takes in account both the vertical and torsional movement that was involved. We will only talk about the torsional movement in this paper. To do this a cross section of the bridge will be isolated for the model (Figure 2). The cables of the bridge will be thought of as springs and the cross section of the bridge as a rod.

2. Energy

First we will show the potential energy of the cables using the physics equation for springs:

$$PE = \frac{1}{2}ky^2. \tag{1}$$



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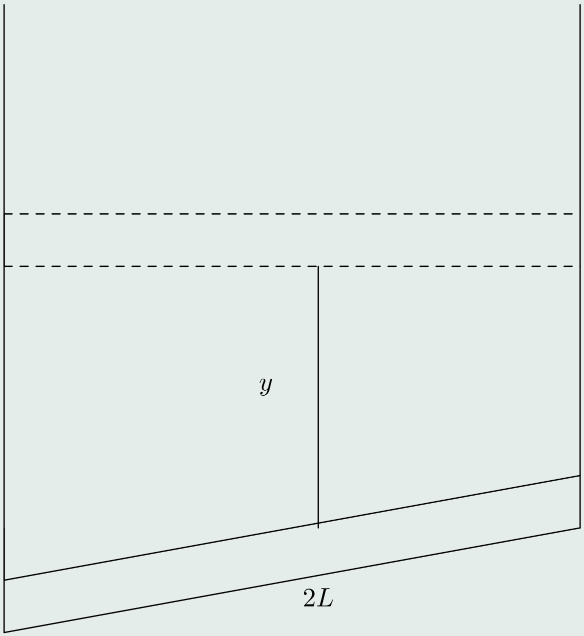


Figure 1: Cross section of the bridge.

The kinetic energy for the rod's vertical motion can be found from the physics equation:

$$KE = \frac{1}{2}mv^2. \tag{2}$$

There is also a KE from the rod rotating around it's center (equation 8). This is derived from the kinetic energy equation, equation 2.

$$KE = \frac{1}{2}v^2\,dm \tag{3}$$

Using (Figure 3) we see that the length of the arc carved out by the toggle of the rod is:

$$y = r\theta. \tag{4}$$

Differentiating y with respect to t, we get $v = \omega r$, and plugging into equation 3

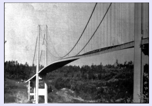
$$KE = \frac{1}{2}dm(\omega r)^2. \tag{5}$$

We set dm equal to $\lambda\,dr$ (where λ is mass per unit length):

$$KE = \frac{1}{2}\lambda dr(\omega r)^2. \tag{6}$$

Now integrate with respect to r over the length of the rod:

$$\begin{aligned}
KE &= \frac{1}{2}\lambda\omega^2\int_{-L}^{L} r^2 dr\\
KE &= \frac{1}{6}\lambda\omega^2 r^3\Big|_{-L}^{L}\\
KE &= \frac{1}{6}\lambda\omega^2[L^3 + L^3]\\
KE &= \frac{1}{3}\lambda\omega^2 L^3. \qquad (7)
\end{aligned}$$



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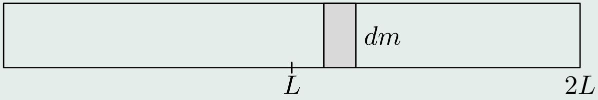


Figure 2: Treating cross section as a rod.

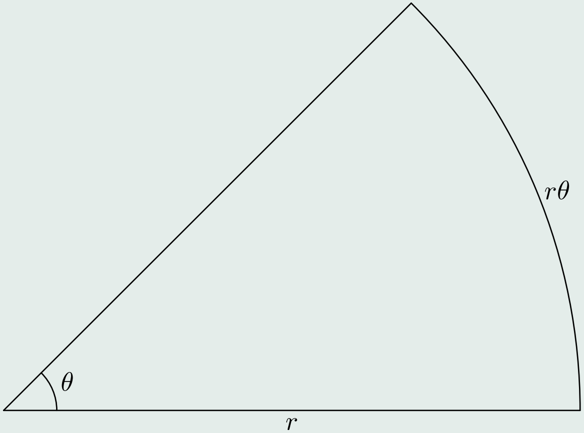


Figure 3: Arc.

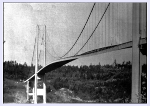
Since λ is mass per unit length (i.e. $\lambda = m/2L$), the equation for the kinetic energy becomes

$$KE = \frac{1}{3}\frac{m}{2L}\omega^2 L^3 = \frac{1}{6}m\omega^2 L^2. \tag{8}$$

Now we sub $\dot{\theta}$ back into equation 8. The total kinetic energy should be in the form kinetic energy of the vertical motion of the bridge cross section + the kinetic energy from rotation equation 8.

$$T = \frac{1}{2}mv^2 + \frac{1}{6}mL^2\dot{\theta}^2 \tag{9}$$

Subbing in we get:



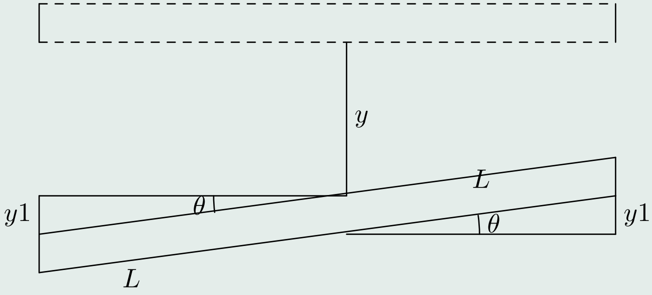


Figure 4: Total displacement.

$$\frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$$
$$T = \frac{1}{2}m\dot{y}^2 + (\frac{1}{6}mL^2\dot{\theta}^2). \tag{10}$$

To find the potential energy from the rod we must take into account 2 parts. First the potential energy due to gravity given by -mgy. The second part comes from the rotation of the rod about it's center creating a toggle displacement aspect (Figure 4). On the left side the vertical deflection is:

$$Y_t = (y + y_1)^+, \tag{11}$$

and on the right the vertical deflection is:

$$Y_t = (y - y_1)^+. \tag{12}$$

These come from the fact that the total displacement (on each side) is the displacement from the center of gravity y1 + toggle displacement y which will equal Y(t) or y(total). The + power represents the fact that the cables only act as springs when pulled.

Looking at the picture (Figure 5) we see that y(1) is:

$$L \sin\theta, \tag{13}$$

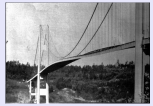
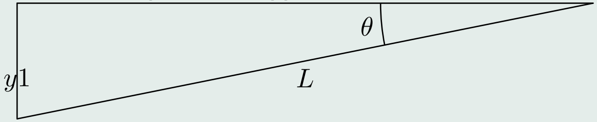


Figure 5: Toggle displacement.



subbing this into the y(total) equations above we get:

$$
\begin{aligned}
y_t &= (y - L\sin\theta)^+ \\
y_t &= (y + L\sin\theta)^+.
\end{aligned}
\tag{14}
$$

The potential energy due to gravity is:

$$PE_g = mgy, \tag{15}$$

adding equation 1 and mgy from gravity gives the total potential energy:

$$PE = \frac{1}{2}ky^2 - mgy. \tag{16}$$

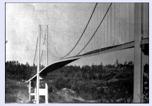
Plugging our y(total) equations into this we get the total potential energy equation 17. This leaves us with two equations that sum the potential and kinetic energy of the model, the total potential energy (V) and the total kinetic energy (T):

$$V = \frac{k}{2}[((y - L\sin\theta)^2)^+ + ((y + L\sin\theta)^2)^+] - mgy \tag{17}$$

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{6}mL^2\dot{\theta}^2. \tag{18}$$

We will form the Lagrangian and let $\mathcal{L} = T - V$ given that:

$$\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{\theta}} = \frac{\delta\mathcal{L}}{\delta\theta} \tag{19}$$



$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{y}} = \frac{\delta \mathcal{L}}{\delta y}, \tag{20}$$

$$\begin{aligned}
\mathcal{L} &= \frac{m\dot{y}^2}{2} + \frac{1}{6}mL^2\dot{\theta}^2 - \frac{1}{2}k[((y - L\sin\theta)^2)^+ + ((y + L\sin\theta)^2)^+] + mgy \\
\frac{\delta L}{\delta \theta} &= \frac{1}{2}k[2(y - L\sin\theta)^+(-L\cos\theta) + 2(y + L\sin\theta)^+(L\cos\theta)] \\
&= kL\cos\theta[(y - L\sin\theta)^+ - (y + L\sin\theta)^+]
\end{aligned} \tag{21}$$

Now taking the time derivative of partial $\mathcal{L}$ partial $\dot{\theta}$:

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\theta}} = \frac{1}{3}mL^2\ddot{\theta}. \tag{22}$$

From equation 19 we see that:

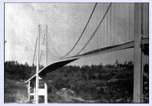
$$\frac{1}{3}mL^2\ddot{\theta} = kL\cos\theta[(y - L\sin\theta)^+ - (y + L\sin\theta)^+]. \tag{23}$$

Now taking the time derivative of partial $\mathcal{L}$ partial y:

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta y} &= -\frac{1}{2}k[2(y - L\sin\theta)^+ + 2(y + L\sin\theta)^+] + mg \\
&= -k[(y - L\sin\theta)^+ + (y + L\sin\theta)^+] + mg
\end{aligned} \tag{24}$$

Now taking the time derivative of partial $\mathcal{L}$ partial $\dot{y}$:

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta \dot{y}} &= (m\dot{y}) \\
\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{y}} &= \frac{d}{dt}(m\dot{y}) = m\ddot{y}.
\end{aligned} \tag{25}$$



From equation 20 we see that:

$$m\ddot{y} = -k[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + mg. \tag{26}$$

We now simplify equation 23 and equation 26 to get:

$$\ddot{\theta} = \frac{3k}{mL}\cos\theta[(y - L\sin\theta)^{+} - (y + L\sin\theta)^{+}] \tag{27}$$

$$\ddot{y} = -\frac{k}{m}[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + g. \tag{28}$$

We then add a damping term,

$$\delta\dot{\theta} \tag{29}$$

to equation 27 and

$$\delta\dot{y} \tag{30}$$

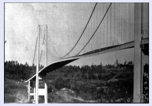
to equation 28. We also add a external forcing term f(t) to equation 27. Doing this we get:

$$\ddot{\theta} = -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[(y - L\sin\theta)^{+} - (y + L\sin\theta)^{+}] + f(t) \tag{31}$$

$$\ddot{y} = -\delta\dot{y} - \frac{k}{m}[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + g. \tag{32}$$

Assume that the cables never loose tension, this lets us remove the + exponent that was introduced in equation 11. We will now simplify equations 31 and 32:

$$\begin{aligned}
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[(y - L\sin\theta) - (y + L\sin\theta)] + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[y - L\sin\theta - y - L\sin\theta] + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta(-2L\sin\theta) + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} - \frac{6k}{m}\cos\theta(\sin\theta) + f(t). \qquad (33)
\end{aligned}$$



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Simplifying equation 32:

$$
\begin{aligned}
\ddot{y} &= -\delta\dot{y} - \frac{k}{m}[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + g \\
y &= -\delta\dot{y} - \frac{k}{m}(y - Lsin\theta + y + L\sin\theta) + g \\
y &= -\delta\dot{y} - \frac{k}{m}(2y) + g \\
y &= -\delta\dot{y} - \frac{2k}{m}(y) + g. \qquad (34)
\end{aligned}
$$

Now that we have equations 33 and 34 we can add in the error that was the key to the Tacoma Narrows bridge. We will linearize the equations, this is only an error if the oscillations are large. If they are small then this would have simply been an approximation.
We will set:

$$
\begin{aligned}
\sin\theta &= \theta \\
\cos\theta &= 1.
\end{aligned} \qquad (35)
$$

This will result in the equation:

$$\ddot{\theta} = -\delta\dot{\theta} - \frac{6k}{m}\theta + f(t), \qquad (36)$$

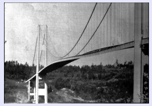
and leaving the other equation just as it was:

$$y = -\delta\dot{y} - \frac{2k}{m}(y) + g. \qquad (37)$$

3. The Constants

The mass of the bridge was about 5,000lbs so we will pick the m in our equations to be 2,500Kg. The width of the bridge was 12 meters, and since we called the width 2L we will say that L is 6 meters.
The bridge would naturally deflect about .5m for every 100Kg for every .3m of bridge. Since there are 2 springs we have the equation:

$$2ky = mg, \qquad (38)$$



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plugging in the numbers from above we have:

$$2k(.5) = 100(9.8). \tag{39}$$

Solving this gives us a approximate value for k of about 1000.
It is clear from the movie clip that the bridge oscillates, because of this we will pick a external driving term that will cause oscillation and yet be able to (with the right constants) match the frequency and amplitude that was seen in the Tacoma Narrows bridge.
Our driving term will be:

$$f(t) = \lambda \sin \mu t. \tag{40}$$

The movie clip was analyzed and right before the collapse the frequency of the motion was about 12 to 14 cycles a minute, because of this we pick μ to between 1.2 and 1.6.

After years a consensus for the damping term was reached. It was decided that it should be about .01.

We will pick lambda so that the amplitude is not more than 3 degrees or .052 Radians

We will now compare the non-linearized equation 33 with the linearized equation 36

4. Behavior linear/torsional oscillator

We have 2 models for the Tacoma Narrows bridge (equations 36 and 33), a linear one and a non-linear accurate one. First we will explore the long-term behavior of the linear model, with different initial conditions and driving terms of different periods (μ controls the period and is 1.2 for all linear graphs). The initial push is controlled by θ we will increase θ and hold $\dot{\theta}$ at 0. The vertical axis is θ for all graphs the horizontal axis is time. Figure 6 is the linear model with initial conditions θ =1.2 $\dot{\theta} = 0$ $\lambda = .05$ (large push). Figure 7 is the linear model with initial conditions $\theta = 2$ $\dot{\theta} = 0$ $\lambda = .05$ (very large push). Figure 8 is the linear model, $\theta = 2$ $\dot{\theta} = 0$ $\lambda = .08$ (very large push and large driving term). We can easily see that the linear model always settles down when given enough time.

Now let us explore the long-term behavior of the non-linear model. Here we see an interesting behavior. If the period of the driving term is allowed to change over a range consistent with what was seen before the collapse, the bridge can go crazy, all it needs is a large initial push. With sufficient push the torsional oscillations will never die down.

Figure 9 is the non-linear model with a large initial push $\lambda = .05$ $\mu = 1.26$. Figure 10 is the non-linear model with a small initial push $\lambda = .05$ $\mu = 1.26$. We will now pick a different frequency while holding λ constant and see if it makes any difference in the long term behavior. Figure 11 is the non-linear model

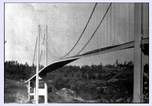


Figure 6: Linear model $\lambda = .05$ (large push).

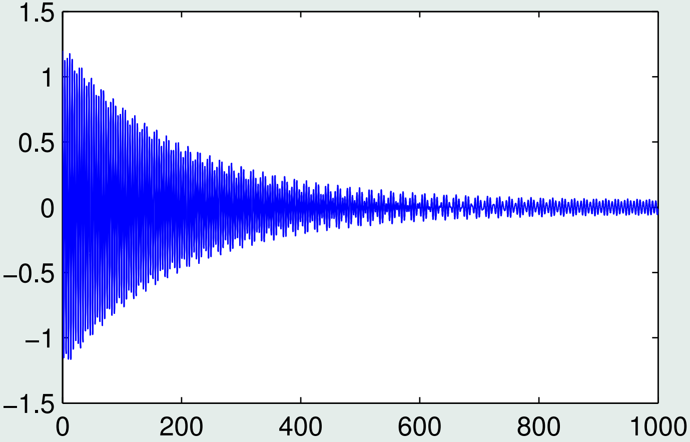
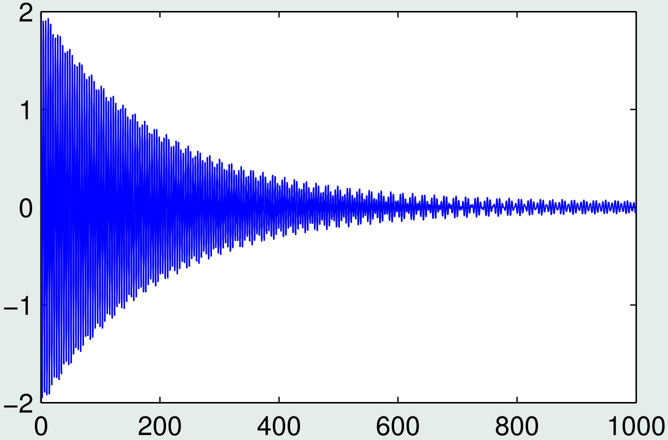


Figure 7: Linear model $\lambda = .05$ (very large push).



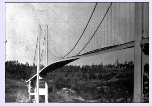


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Figure 8: Linear model $\theta = 2$ $\dot{\theta} = 0$ $\lambda = .08$ (very large push and large driving term).

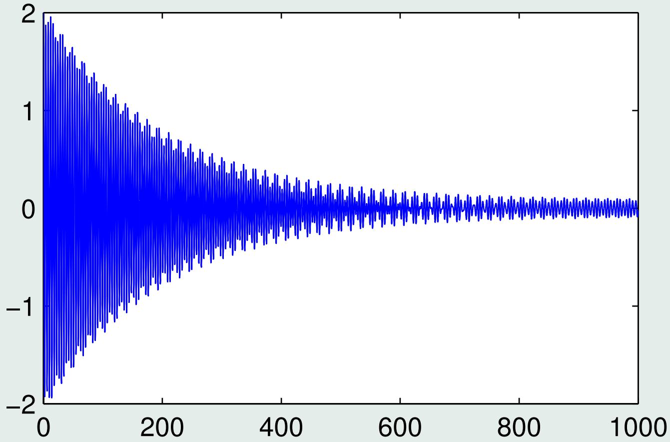
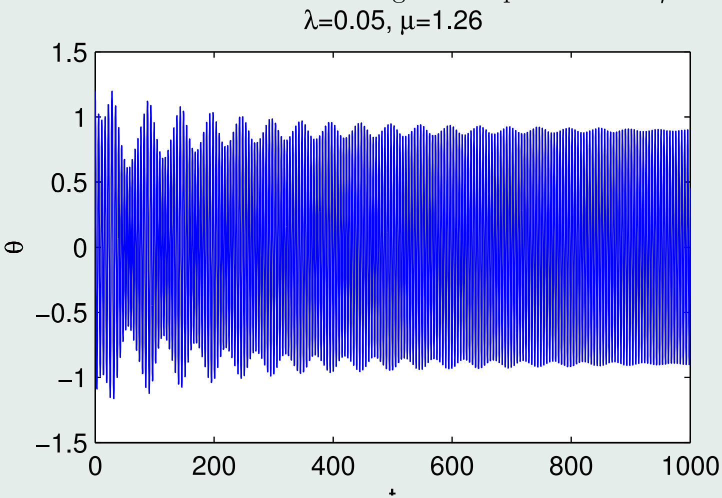


Figure 9: non-linear model with a large initial push $\lambda = .05$ $\mu = 1.26$.



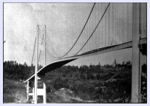


Figure 10: non-linear model with a small initial push $\lambda = .05$ $\mu = 1.26$.

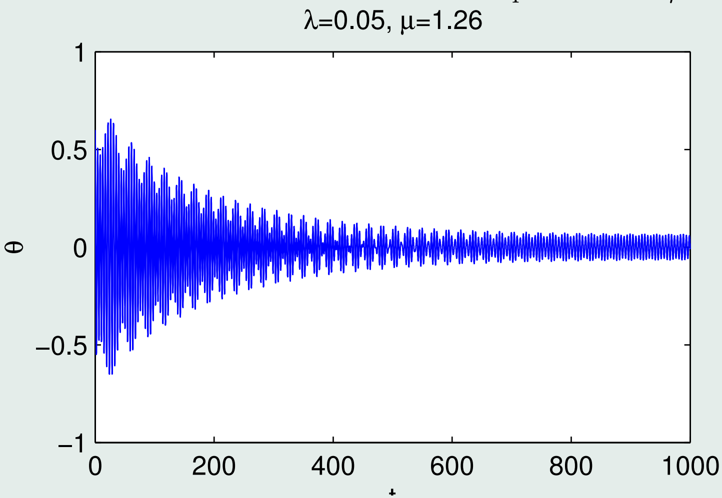
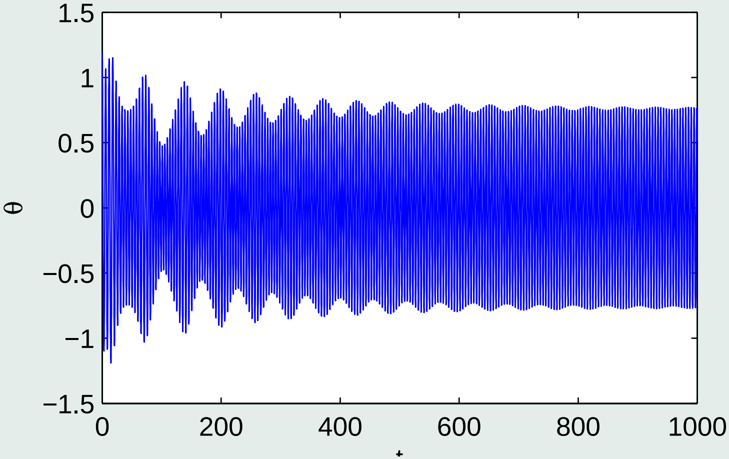


Figure 11: non-linear model with a large initial push $\mu = 1.35$ $\lambda = .05$.



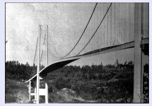


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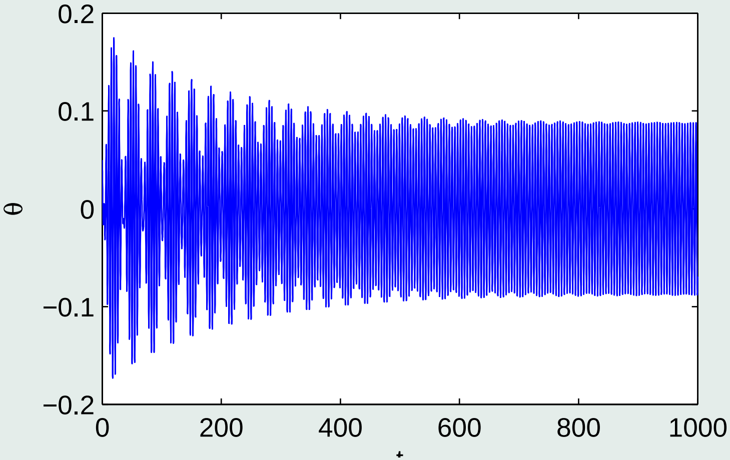
Figure 12: non-linear model with a small initial push $\mu = 1.35$ $\lambda = .05$.



with a large initial push and a different frequency $\mu = 1.35$ $\lambda = .05$. Figure 12 is the non-linear model with a small initial push and a different frequency $\mu = 1.35$ $\lambda = .05$.

This is an important fact to remember. Over a range of frequencies the non-linear model can exhibit long term behavior similar to the linear model or very different. It all depends on the initial push (initial conditions). This means that it is not important that we get a driving term with exactly the frequency that was seen at Tacoma Narrows.

5. So What Happened

At the time of the bridge construction, it was not possible to solve the non-linear model so the engineers chose to use the linear one. After seeing the behavior of the linear and non-linear model you can probably guess where we are going. We will start by doing a direct comparison of the linear and non-linear models. When given a large initial push and no driving terms (figure 13) both the models are very similar, so far so good.

Now we start both models at equilibrium but add a small driving term (lambda =.05) (figure 15). Once again both models appear very similar, and this is good.

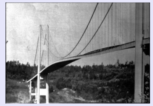


Figure 13: Linear model.

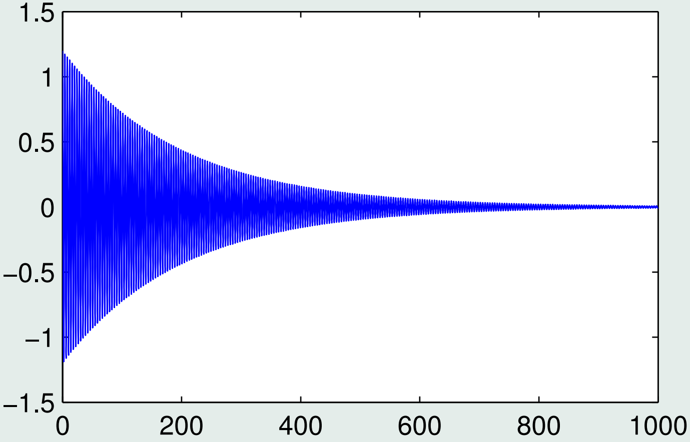
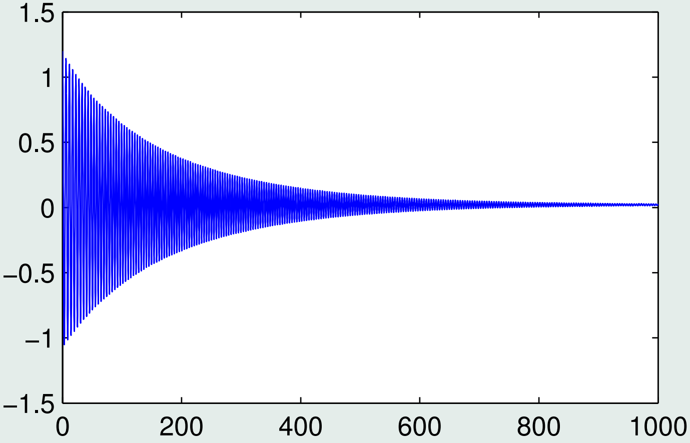


Figure 14: Non-linear model.



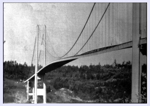


Figure 15: Linear model.

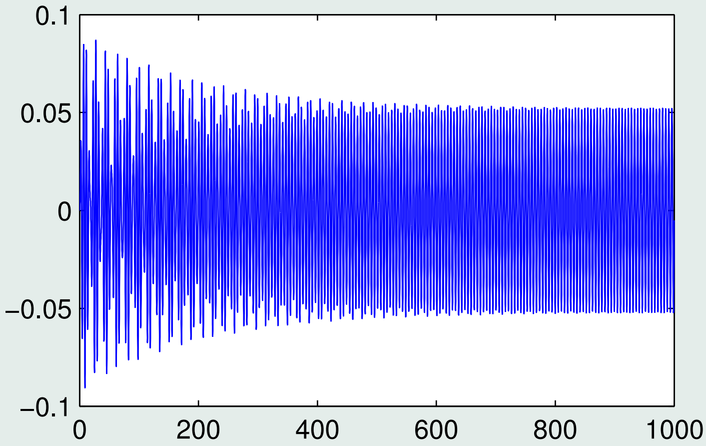
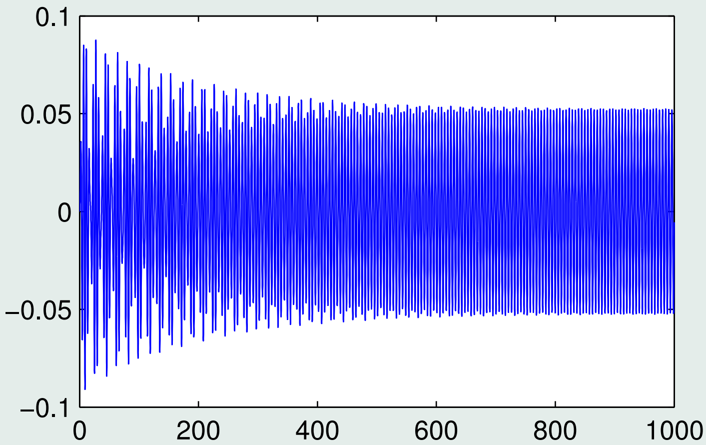
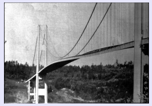


Figure 16: Non-linear model.





Finally we will look at both models starting with a large initial push and a driving term (figure 17). Here we see what you would probably expect, the linear model dies down, but the non-linear model does not. Large oscillations continue on for all time or until the eventual collapse of the bridge.

This brings us to the point of our discussion. The linear model used to design the bridge was flawed. It failed to take into account the nature of the real bridge, when given the right amount of initial push, oscillations will begin that will never stop on their own. It is these oscillations that lead to the collapse of the bridge. This explains why the bridge failed after it started to oscillate, but it does not address why it started to oscillate. To this date there is no consensus on this point. Some people believe that it was a small structural failure that made the bridge jolt and start to oscillate. Others think that maybe the position of one of the cables may have shifted, but this is beyond the scope of this paper. It is a problem that can not be modelled with math.

6. References

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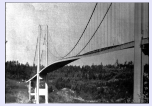
(2) Professor Dave Arnold, College of the Redwoods.

(3) Professor Scott Pilzer, College of the Redwoods.



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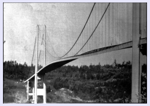

Figure 17: Linear model.

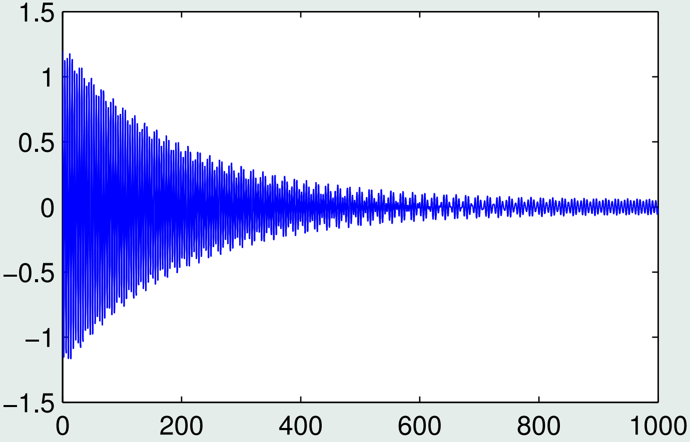

Figure 18: Non-linear model.