The One Dimensional Heat Equation

Adam Abrahamsen and David Richards

May 22, 2002

Abstract

In this document we will study the flow of heat in one dimension through a small thin rod. We will use the derivation of the heat equation, and Matlab's `pdepe` solver to model the motion and show graphical solutions of our examples.

1. Introduction

Before we begin our discussion of the mathematics of the heat equation, we must first determine what is meant by the term **heat**? A common example of the misunderstanding of the term heat is the classic physics question of what contains more heat, a bathtub of warm water or a boiling cup of water? We all know the boiling cup of water has a higher temperature but contains less heat than the bathtub.

Therefore, in calculating problems concerned with heat we must distinguish between the two types of measurements, the measurement of **temperature** and the measurement of the **quantity of heat contained in an object**. In this discussion when we mention the term **heat** we will be talking about the **quantity of heat in an object**.

2. Derivation of the heat equation

There are two methods used to solve for the rate of heat flow through an object. The first method is derived from the properties of the object. The second method is derived by measuring the rate of heat flow through the boundaries of the object.

2.1. Method One

Experimental calculations show that the heat Q in ΔV at time t can be defined by

$$\Delta Q = c\rho u \Delta V. \tag{1}$$

where c is the specific heat, ρ is the density, u is the temperature, and ΔV is a small volume. Consider this thin rod, made of a homogenous material and perfectly insulated along its length so that heat can only flow through its ends. Any position along the rod is denoted as x, and the length of the rod is denoted as L such that $0 \le x \le L$.
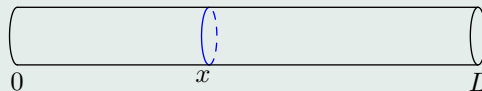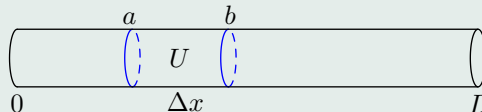

Therefore we find the temperature u is the only condition that depends on position x and time t. Thus,

$$\Delta Q = c\rho u(x,t)\Delta V. \tag{2}$$

Now consider a small section of the rod U defined as the interval from $x = a$ to $x = b$.
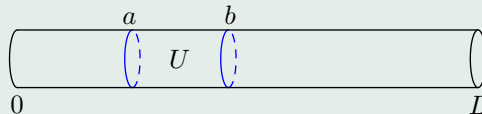
The cross sectional area is defined as S, and the width of this section is Δx. This gives $\Delta V = S\Delta x$.

We can now express the amount of heat in the cross-sectional area as

$$\Delta Q = c\rho u(x,t)S\Delta x. \tag{3}$$

To find the amount of heat in the section U at time t, we take the integral the integral.

$$Q(t) = \int_a^b c\rho u(x,t)Sdx. \tag{4}$$

Since the rod has uniform thickness, S doesn't change with respect to time, and because we are dealing with homogenous materials c and ρ do not change with respect to time. Thus, by differentiating we take the partial of u to find the change in heat with respect to time.

$$\frac{dQ}{dt} = \int_a^b c\rho \frac{\partial u}{\partial t} dxS. \tag{5}$$

2.2. Method Two

Our second method of finding the change in heat with respect to time is also determined experimentally, with a rod similar to that in method one. The rate of heat flow through U is inversely proportional to the width of U, and directly proportional to the cross-sectional area. This is logical because the longer the length of the section, the longer it would take for the heat to flow through it. This is similar for the cross sectional area, if you have two rods, one with a large diameter and one with a small diameter it would take less time for the heat to flow through the larger diameter rod than the smaller diameter rod. Using the property that when two objects of different temperature are placed together (touching) heat will flow from the hotter object to the cooler one. If the temperature at $b > a$ then heat will flow from $b \rightarrow a$.
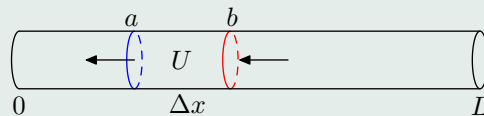
Combining these properties we find

$$\Delta Q = -C \frac{u(a+\Delta x,t) - u(a,t)}{\Delta x} S. \tag{6}$$
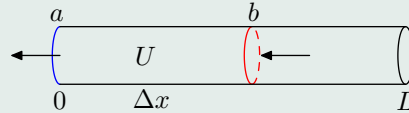
The proportionality constant C is known as the thermal conductivity. This varies depending on the type of material being evaluated. To show this is true, we need to show the flow of heat through the section U where the boundaries of the section are defined as

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Go Back

Full Screen

Close

Quit

$$a = x = 0,$$

$$b = a + \Delta x.$$

If the temperature at $u(a + \Delta x, t) > u(a, t)$, then ΔQ would be negative which is logical because the temperature at b is greater than the temperature at a, so the heat would be flowing from b to a, thus heat would be flowing out of the rod. Letting $\Delta x \to 0$ in equation (6), the difference quotient approaches $\partial u / \partial x$ then the rate of heat flow through U at $x = a$ is given by

$$-C\frac{\partial u}{\partial x}(a, t)S. \tag{7}$$

Following the same argument we can show that the rate of heat flow through U at b is defined as

$$C\frac{\partial u}{\partial x}(b, t)S. \tag{8}$$

Therefore, the amount of heat that U obtains at time t can be given by

$$\frac{dQ}{dt} = C\left[\frac{\partial u}{\partial x}(b, t) - \frac{\partial u}{\partial x}(a, t)\right]S. \tag{9}$$

If we apply the fundamental theorem of calculus to equation (9) we have

$$\frac{dQ}{dt} = \int_a^b \frac{d}{dx}\left(C\frac{\partial u}{\partial x}S\right)dx.$$

Since we are dealing with homogeneous materials with constant cross-sectional area we define C and S to be constant. Therefore the expression becomes

$$\frac{dQ}{dt} = C\int_a^b \frac{\partial^2 u}{\partial x^2}dxS. \tag{10}$$

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Page 4 of 23

Go Back

Full Screen

Close

Quit

3. Combining the two Methods

Now we have two equations for the rate of heat flow into and out of the section U.

Method 1 :

$$\frac{dQ}{dt} = \int_a^b c\rho \frac{\partial u}{\partial t} dx S$$

Method 2 :

$$\frac{dQ}{dt} = C \int_a^b \frac{\partial^2 u}{\partial x^2} dx S$$

Since both equations model the flow of heat through a rod, we set these equations equal to each other.

$$c\rho \int_a^b \frac{\partial u}{\partial t} dx S = C \int_a^b \frac{\partial^2 u}{\partial x^2} dx S \qquad (11)$$

Moving the second equation to the other side and losing the common terms,

$$c\rho \int_a^b \frac{\partial u}{\partial t} dx - C \int_a^b \frac{\partial^2 u}{\partial x^2} dx = 0,$$

which becomes

$$\int_a^b \left(c\rho \frac{\partial u}{\partial t} - C \frac{\partial^2 u}{\partial x^2} \right) dx = 0.$$

But, the integral can only be zero if the integrand equals zero, this gives

$$c\rho \frac{\partial u}{\partial t} - C \frac{\partial^2 u}{\partial x^2} = 0.$$

Then dividing through by $c\rho$ we obtain

$$\frac{\partial u}{\partial t} - \frac{C}{c\rho} \frac{\partial^2 u}{\partial x^2} = 0.$$

Combining the constants into one expression k, where $k = C/c\rho$ the equation becomes

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0. \qquad (12)$$

This equation is more popularly written with subscript notation as

$$u_t = ku_{xx}. \tag{13}$$

Where k is defined to be the thermal diffusivity. We have now developed the heat equation, also known as the diffusion equation. This equation models the flow of heat through a rod by modelling the temperature along the rod at time t.

4.   Initial Boundary Value Problems (IBV)

In order to solve the heat equation for many solutions or even just a single solution we must give the problem some initial conditions. So we must define the temperature of every point along the rod at time $t = 0$, which we do with the function

$$u(x,0) = u_0(x) \qquad \text{for} \qquad 0 \le x \le L. \tag{14}$$

This function is known as the initial temperature distribution. Since heat can only enter or exit the rod at its boundaries we must define some "boundary conditions," for the rod. Therefore we need to define the conditions of the rod at the boundaries of the rod $(0, L)$,

$$u(0,t) = T_0, \qquad u(L,t) = T_L \qquad \text{for all} \qquad t > 0. \tag{15}$$

For example if one end of the rod was submerged in a liquid that is a constant 0°, and the other end in a liquid at 100°, then

$$u(0,t) = 0, \qquad \text{and} \qquad u(L,t) = 100 \qquad \text{for all} \qquad t > 0. \tag{16}$$

These are known as Dirichlet conditions.

Now we can set up an initial boundary value (IBV) problem. Combining the heat equation with the initial conditions and boundary conditions. The problem is to find $u(x,t)$ such that

$$u_t(x,t) = ku_{xx}(x,t), \qquad \text{for} \qquad 0 \le x \le L \qquad \text{and} \qquad t > 0,$$
$$u(x,0) = u_0(x), \qquad \text{for} \qquad 0 \le x \le L$$
$$u(0,t) = T_0, \qquad \text{and} \qquad u(L,t) = T_L, \qquad \text{for} \qquad t > 0.$$

This is the general form for the heat equation as an IBV problem.

5. Examples

For examples of the heat equation we will set some initial conditions and boundary conditions.

5.1. Example 1

If we have a rod of length $L = 1$, and $k = 0.02$. Setting the initial temperature distribution, $u(x, 0) = 0$, and the boundary conditions defined with system (16). We arrive with the problem

$$u_t(x, t) = 0.02 u_{xx}(x, t), \qquad \text{for} \qquad 0 \leq x \leq 1 \qquad \text{and} \qquad t > 0,$$
$$u(x, 0) = 0, \qquad \text{for} \qquad 0 \leq x \leq 1$$
$$u(0, t) = 0, \qquad \text{and} \qquad u(1, t) = 100, \qquad \text{for} \qquad t > 0.$$

We start by saying that the operators in the heat equation are linear operators, such that

$$\frac{\partial}{\partial x} \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2} \qquad \text{are linear.}$$

Since the heat equation is linear we can find a linear combination of two solutions to equal another solution. The first of these solutions is the steady state solution $u_s(x)$ such that

$$u_s(x) = \left(\frac{T_L - T_0}{L} \right) x + T_0.$$

Therefore in our example

$$u_s(x) = \frac{(100 - 0)}{1} x + 0 = 100x.$$

The remaining part of $u = v + u_s$ is v. Where $v = u - u_s$, so that it satisfies the IBV problem,

$$v_t(x, t) = k u_{xx}(x, t), \qquad \text{for} \qquad 0 \leq x \leq L \qquad \text{and} \qquad t > 0,$$
$$u(x, 0) = u_0(x) - u_s(x), \qquad \text{for} \qquad 0 \leq x \leq L$$
$$u(0, t) = 0 = u(1, t), \qquad \text{for} \qquad t > 0.$$

The purpose of doing this is, v has homogenous boundary conditions which makes finding a solution much easier. This allows us to use separation of variables to find a soluton to v because separation of

variables requires that the problem be homogeneous. Therefore we want to find product solutions of the form,

$$v(x, t) = X(x)T(t).$$

Using separation of variables will lead us to two ODE's for T and X. If we plug $v = X(x)T(t)$ into the heat equation we get

$$X(x)T'(t) = kX''(x)T(t).$$

Dividing both sides by $kT(t)$ and $X(x)$ gives

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

Since x and t are independent variables the only way for these two equations to be equal is if they equal a constant. Conventionally this constant is $-\lambda$. Thus,

$$\frac{T'(t)}{kT(t)} = -\lambda \qquad \text{and} \qquad \frac{X''(x)}{X(x)} = -\lambda$$

or,

$$T' + \lambda kT = 0 \qquad \text{and} \qquad X'' + \lambda X = 0.$$

We find the first equation has the general solution of the form

$$T(t) = Ce^{-\lambda kt}.$$

Now we need to find the solution for

$$X'' + \lambda X = 0 \qquad \text{with} \qquad X(0) = X(L) = 0.$$

This is called a two point boundary value problem also known as a Sturm-Liouville problem. We need to find the values of λ that return non-zero solutions. We will first show where $\lambda < 0$. By setting $\lambda = -r^2$,

$$X'' - r^2 X = 0, \qquad X = e^{\lambda t}.$$

This becomes

$$\lambda^2 e^{\lambda t} - r^2 e^{\lambda t} = 0,$$

which intern implies that

$$\lambda^2 - r^2 = 0.$$

Therefore,

$$\lambda = \pm r.$$

We find the general solution has the form

$$X(t) = C_1 e^{rt} + C_2 e^{-rt},$$

and the boundary conditions are as follows.

$$\begin{aligned} 0 = X(0) = C_1 + C_2 \\ 0 = X(L) = C_1 e^{rt} + C_2 e^{-rt} \end{aligned} \tag{17}$$

From equation (17) we find that $C_2 = -C_1$, if we plug this equation into the second equation we find

$$0 = C_1(e^{rL} - e^{-rL}).$$

Since $r \neq 0$, the factor in parenthesis is not zero. This implies that $C_1 = 0$ and $C_2 = 0$, which in turn makes $X(x) = 0$. This is logical because if $\lambda < 0$ then the product solution would be $v(x, t) = e^{-\lambda k t} X(x)$. So with $\lambda < 0$ and the absence of heat source this equation would grow exponentially, which we know is not what happens to objects when there is no heat source.

For $\lambda = 0$ the differential equation is $X'' = 0$ and has solutions of the general form

$$X(x) = ax + b,$$

with boundary conditions

$$0 = X(0) = b \qquad \text{and} \qquad 0 = X(L) = aL + b.$$

Now we find that $a = b = 0$, again we arrive at a zero solution.

This leaves $\lambda > 0$. If we set $\lambda = \omega^2$, then

$$X'' + \omega^2 X = 0.$$

This has the general solution of the form

$$X(x) = a \cos \omega x + b \sin \omega x.$$

With the boundary conditions

$$X(0) = 0, \qquad \text{which means} \qquad a = 0, \qquad \text{and}$$
$$X(L) = 0 \qquad \text{becomes} \qquad b \sin \omega L = 0.$$

Because $a = 0$ the cosine term disappears which means we need to solve

$$b \sin \omega L = 0.$$

We do this by finding where $\sin \omega L = 0$. Since sine is equal to zero for positive integer values of π, $\sin \omega L = 0$ will only happen when $\omega L = n\pi$. This leads us to

$$\lambda = \omega^2 = \frac{n^2 \pi^2}{L^2}.$$

Therefore,

$$X(x) = b \sin \left(\frac{n \pi x}{L} \right).$$

Since we are looking for non-zero solutions we set $b = 1$, which also helps to simplify calculations. Therefore we need to find

$$\lambda_n = \frac{n^2 \pi^2}{L^2},$$

and

$$X_n(x) = \sin \left(\frac{n \pi x}{L} \right).$$

Here λ_n is an eigenvalue of the Sturm-Liouville problem and $X_n(x)$ is an eigenfunction. The complete solution to the Sturm-Liouville problem is the group of eigenvalues and eigenfunctions for $n = 1$ to $n = \infty$. For the solutions to the heat equation we have the product solution

$$v_n(x, t) = e^{-n^2 \pi^2 k t / L} \sin \left(\frac{n \pi x}{L} \right),$$

which satisfies the boundary conditions. Though we still need it to satisfy the initial conditions. Again, any linear combination of two solutions is a solution. Therefore we have

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 \pi^2 kt/L} \sin\left(\frac{n\pi x}{L}\right), \tag{18}$$

which is also a solution to the heat equation. This solution satisfies all the conditions except the initial conditions. So we need to find the coefficients b_n that satisfy the initial conditions. We use the initial condition to get,

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L} x.$$

For our example,

$$-100x = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

This is known as the half-range sine series expansion, and the b_n are calculated with

$$b_n = \frac{2}{L} \int_0^L v_0(x) \sin(n\pi x/L) dx.$$

Expanding this gives

$$b_n = 2 \int_0^1 (-100x) \sin n\pi x\, dx$$

$$= -200 \int_0^1 x \sin n\pi x\, dx$$

$$= (-1)^n \frac{200}{n\pi}.$$

Plugging this function of b_n into equation (18) gives a complete solution to the IBV problem $v(x,t)$. Therefore,

$$v(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-0.02n^2\pi^2 t} \sin n\pi x.$$

We conclude that the temperature in the rod is

$$u(x, t) = u_s(x) + v(x, t),$$

which is

$$u(x, t) = 100x + \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-0.02n^2\pi^2 t} \sin n\pi x.$$

Of course we don't want to plot this equation by hand so we look to Matlab to do the work for us. Conveniently enough, Matlab has a partial differential equation solver `pdepe` built in which can solve IBV problems.
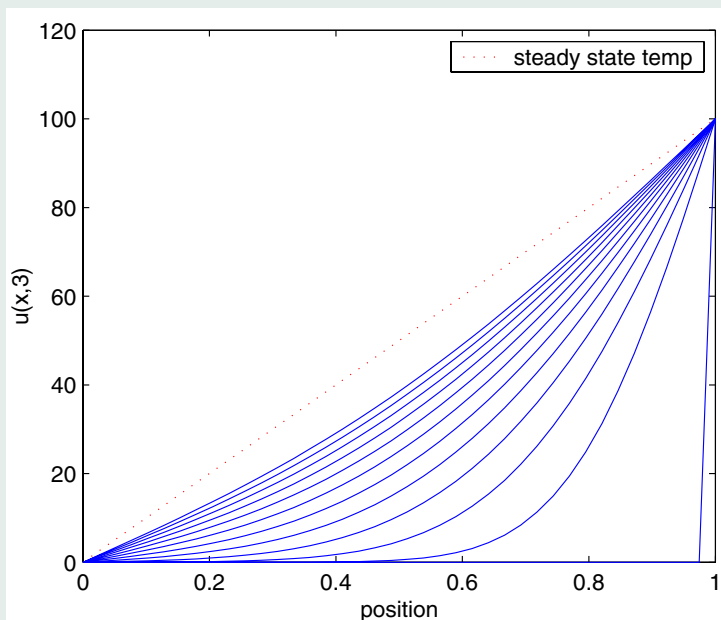
Introduction

Derivation of the heat . . .

Combining the two . . .

Initial Boundary Value . . .

Examples

Conclusion

How to use the pdepe . . .
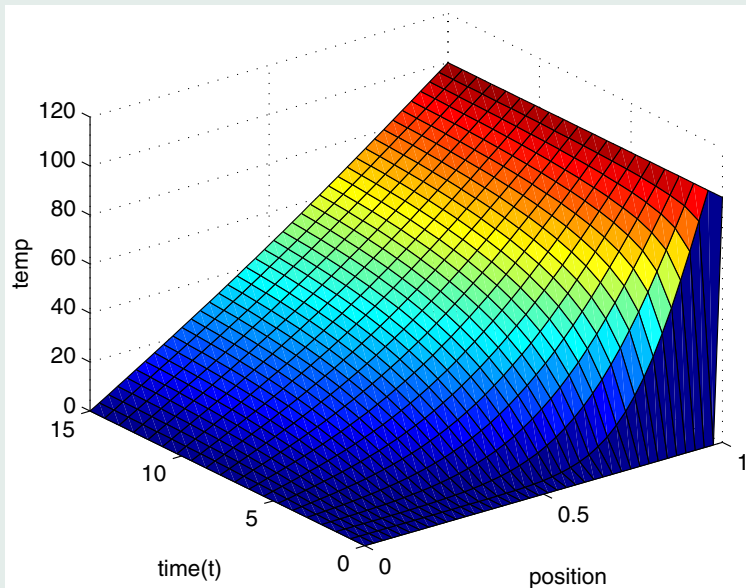
Figure 1: Collection of solutions to Example 1.

Figure 2: Mesh of solutions to Example 1.

In the image in Figure 1 we can see a plot of a collection of solutions converging towards the steady state solution of $u_s(x) = 100x$, as time increases. The image in Figure 2 is a mesh of the solutions to the heat equation. You can see the temperature stays the same at the endpoints for the Dirichlet conditions.

5.2. Example 2

Next, an example with different initial conditions and boundary conditions. Where $k = 1$, $L = 1$, the initial temperature distribution equal to $u_0(x)$, and insulated boundaries $u_x(0, t)$. These conditions are known as Neumann conditions.

$$u_t(x, t) = u_{xx}(x, t), \qquad \text{for} \qquad 0 \le x \le 1 \qquad \text{and} \qquad t > 0,$$
$$u(x, 0) = u_0(x), \qquad \text{for} \qquad 0 \le x \le 1$$
$$u_x(0, t) = 0, \qquad \text{and} \qquad u_x(1, t) = 0, \qquad \text{for} \qquad t > 0.$$

Where $u_0(x)$ is a piecewise function,

$$u_0(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ (1 - x), & 1/2 \le x \le 1. \end{cases}$$
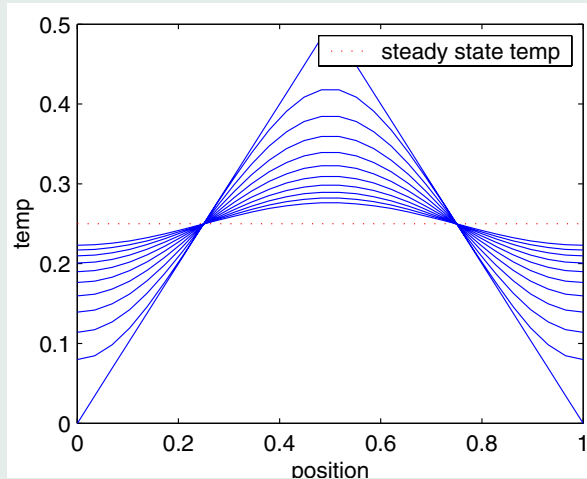

Figure 3: Collection of solutions for Example 2.

Here we can see the example of Neumann conditions which means the ends of the rod are insulated. Again in the image in Figure 3 we can see a collection of solutions converging towards the steady state solution. Since no heat is exchanged through the ends of the rod the heat in the rod averages out to the steady state solution.
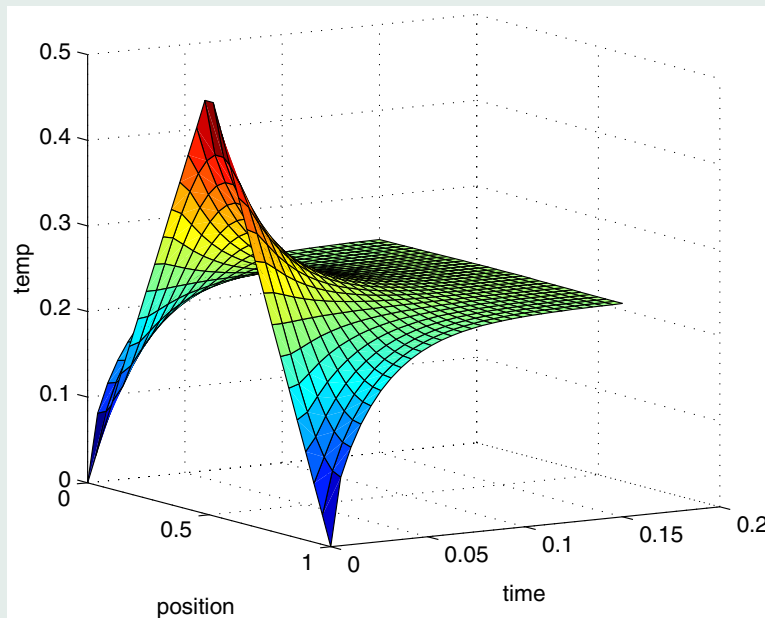
Figure 4: Mesh of the solutions for Example 2.

The image in Figure 4 is a mesh of the solutions to Example 2. We can see how as time increases the initial temperature distribution approaches the steady state solution.

5.3. Example 3

Here we have example with Dirichelet conditions,

$$u_t(x,t) = u_{xx}(x,t) + e^{-x}, \qquad \text{for} \qquad 0 \le x \le 1 \qquad \text{and} \qquad t > 0,$$
$$u(x,0) = \sin(2\pi x), \qquad \text{for} \qquad 0 \le x \le 1$$
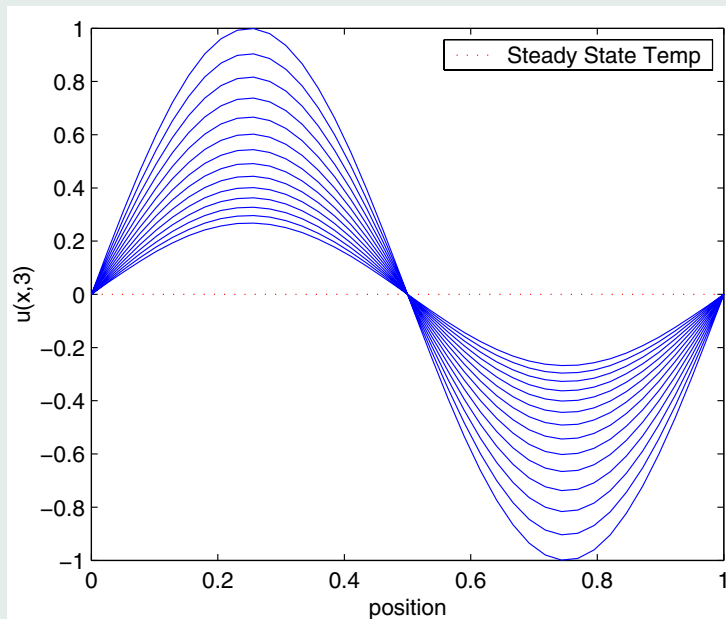$$u(0,t) = 0, \qquad \text{and} \qquad u(1,t) = 0, \qquad \text{for} \qquad t > 0.$$

Figure 5: Collection of solutions for Example 3.

The image in Figure 5 you can see that the solutions converge toward the boundary temperatures of 0° which is the steady state solution.

Home Page

Title Page

◀◀   ▶▶

◀   ▶
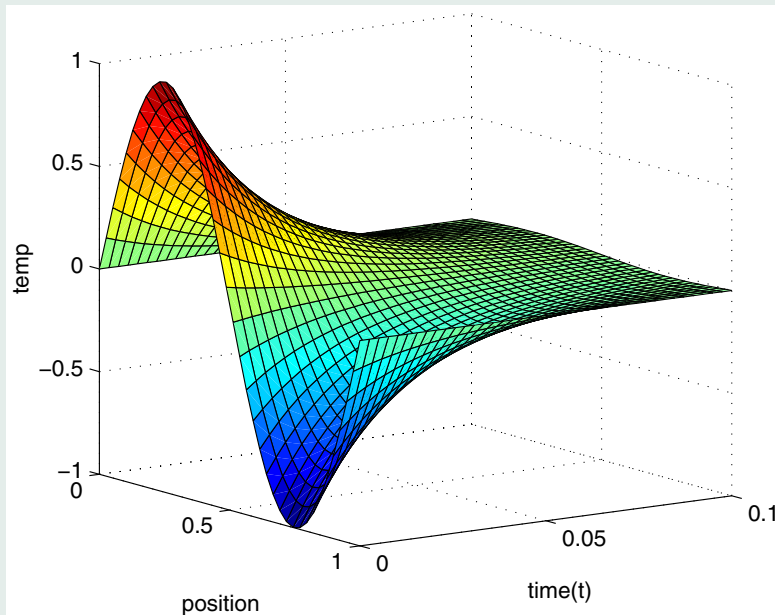
Go Back

Full Screen

Close

Quit

Figure 6: Mesh of the solutions for Example 3.

The image in Figure 6 is the mesh of the solutions to Example 3. Again see how the initial temperature distribution approaches the steady state solution.

6.   How to use the pdepe Function

Understanding the use of `pdepe` can be a bit confusing at first but we will try to explain it in simple terms. We will demonstrate the use of `pdepe` with the example used in Example 1. To better understand how this is written we recommend having the M-file that is posted along with this paper open so you can refer to it while the functions are being developed. To begin we start by explaining the syntax of the `pdepe` function.

```
sol = pdepe(m,@pdex,@pdexic,@pdexbc,x,t)
```

Home Page

Title Page

◀◀   ▶▶

◀   ▶

Go Back

Full Screen

Close

Quit

6.1. The Symmetry

The first variable m is called the "symmetry" of the problem. This is the m that appears in equation (19). There are 3 values for m, 0-slab which basically means the problem is one dimensional. Next two are, 1-cylindrical, 2-spherical. All our examples deal with the one dimensional heat equation therefore $m = 0$ for our examples.

6.2. The Function Handle

Next we define the use of the function handle @. Function handles are data types that contain information used in referencing a function. When a function handle is first created, Matlab stores into the function handle all the information about the function that is needed to execute it later on . The benefits of using a function handle are many. First, handles improve performance because function handles only search the path when they are created. Second, there is more control and maintainability. You can create a handle for a subfunction (as we are going to do) to pass context that would not ordinarily be visible. This is nice because we no longer have to create separate M-files for each function we want to pass to `feval`. This is only the beginning for the uses of function handles. If you'd like to know more about function handles there are a few web sites that go into more detail about them. One we found that we recommend is posted here.

www.icam.vt.edu/cliff/2074/notes/function_handles.pdf

6.3. Subfunctions

Now we need to explain the functions @pdex, @pdexic, @pdexbc. The first, @pdex, is a function that calculates the components of the pde. For `pdepe` to be able to interpret the pde the equation must be written in the form of,

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^m f\left(x, t, u, \frac{\partial u}{\partial x}\right)\right) + s\left(x, t, u, \frac{\partial u}{\partial x}\right). \tag{19}$$

Where c is a diagonal matrix, f is called the flux term, and s is called the source term. Placing our pde in the form as shown in (19), we find

$$u_t(x, t) = 0.02 u_{xx}(x, t),$$

which becomes

$$1\left(\frac{\partial u}{\partial t}\right) = \left(0.02\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right) + 0.$$

Therefore,

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right) = 1,$$

$$f\left(x, t, u, \frac{\partial u}{\partial x}\right) = 0.02 * \frac{\partial u}{\partial x},$$

$$s\left(x, t, u, \frac{\partial u}{\partial x}\right) = 0.$$

We create the following function M-file to represent this problem.

```
function [c,f,s] = pdex(x,t,u,DuDx)
c = 1;
f = 0.02*DuDx;
s = 0;
```

The function `pdex` computes the terms `c`, `f`, `s` as column vectors, but note that these are different than the `c`, `f`, `s` in the general form equation for `pdepe`.

Next is `pdexic`. This is the function that evaluates the initial conditions of the pde. Since the initial temperature distribution for this example is $u_0(x) = 0$, the function has the following form.

```
function u0 = pdexic(x)
u0 = 0;
```

This returns the solution components of `x` in the column vector `u0`.

If we were to write an initial condition function for a piecewise function, as in example 2, the function would be written as follows.

```
function u0 = pdexic(x)
if   x < 1
    u0 = x;
else
    u0 = 1 - x;
end
```

The function `pdexbc` is the function that evaluates the boundary conditions. Again for `pdepe` to be able to evaluate these conditions they must be put into a form that it can interpret. The general form for the boundary conditions is

$$p(x,t,u) + q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right).$$

With the boundary conditions of example 1 being

$$u(0,t) = 0, \qquad \text{and} \qquad u(1,t) = 100,$$

we need to find the values for `pl` and `ql`. Where `pl` and `ql` are the functions in the general form for `pdepe` of the left boundary conditions.

$$u(0,t) + 0 * \frac{\partial u}{\partial x}(0,t) = 0$$

This gives `pl=ul` where `ul` is the temperature function at the left boundary, and `ql=0`. Next we find the `pdepe` form for `pr` and `qr`, where these are the values for the right boundary condition.

$$u(1,t) + 0 * \frac{\partial u}{\partial x}(1,t) = 100.$$

Therefore, `pr=ur-100`, and `qr=0`. This function is written as follows.

```
function [pl,ql,pr,qr] = pdexbc(xl,ul,xr,ur,t)
pl = ul;
ql = 0;
pr= ur-100;
qr = 0;
```

For an example of the Newmann conditions as in example 2, where $u_x(0,t) = 0$ and $u_x(1,t) = 0$, are the boundary conditions. We find the `pdepe` form for `pl` and `ql`.

$$0 + 1 * \frac{\partial u}{\partial x}(0,t) = 0.$$

Therefore, `pl=0`, and `ql=1`. Evaluating the right boundary,

$$0 + 1 * \frac{\partial u}{\partial x}(1,t) = 0$$

gives `pr=0`, and `qr=0`. Putting these into the function `pdexbc` we get the results for example 2.

6.4.   Variables x and t

Now we have the variables x, and t. Where x is a vector corresponding to the length of the rod such that the values of x are $0 \leq x \leq L$. The vector x contains the points along the length of the rod that are to be evaluated. There must be ≥ 3 elements in this vector for pdepe to be able to evaluate it .

The time vector t, is a vector containing the points in time that the temperature of the rod is to be evaluated at. This too must have ≥ 3 elements in it.

6.5.   Driver function

Now that we have described all of the components to the pdepe function we need to write the main function "driver" program that combines all these subfunctions together. This "driver" executes pdepe and plots the solutions to the IBV problem. We first begin with the main function name. We are going to call it pdexfunc.

```
function  pdexfunc
close all
```

We also like to follow it with a "close all" statement to close out any other figure windows. Next declare the symmetry of the pde, m=0 for this example.

```
m=0;
```

Next declare the x, and t vectors,

```
x = linspace(0,1,30); %this creates 30 equal points along the length of the rod
t = linspace(0,15,20);  %this creates 20 equal time pionts
```

Now we execute pdepe. We do this with the code shown below.

```
sol = pdepe(m,@pdex,@pdexic,@pdexbc,x,t);
```

The output argument sol is a three-dimensional array, such that:

- sol(:,:,k) approximates component k of the solution u.

- sol(i,:,k) approximates component k of the solution at time tspan(i) and mesh points xmesh(:).

- sol(i,j,k) approximates component k of the solution at time tspan(i) and the mesh point xmesh(j).

Now we need to extract the first solution component.
```
u=sol(:,:,1);
```

To get a mesh plot of the solution we used the `surf` command.
```
surf(x,t,u)
```

This will produce a mesh of your selected `x`, and `t` values vs temp `u`. We then proceeded to open a new figure window and plotted the 2-dimensional plot, so that we could demonstrate the solutions convergence toward the steady state temperature. This was done using consecutive plot commands.

```
plot(x,u(t,:))
```

Replacing `t` with values that made the graph presentable, with `t` being $0 \leq t \leq \#$ of elements in the vector t. Following this code with the functions written earlier we were able to obtain graphical solutions to our IBV problems. To see the exact code used to plot the graphs please result to the M-files that are posted along with this paper.

7. Conclusion

In concluding we can imagine how the heat equation has many applications from engines to structural mechanics. The equation also has immense amounts of use in Biology where it is know as the **diffusion equation** and models the diffusion of a substance through a system. We have only examined the case where heat flows in one direction through a thin rod. You can imagine the complexity of heat moving in two dimensions or even three.

References

[1] Polking, John. Albert, Bogges. Dave, Arnold . **Differential Equations.** Prentice Hall, 2002.

[2] Arnold, David. His matlab and LaTeX expertise.

[3] Cooper, Jeffery **Introduction to Partial Differential Equations with MATLAB.** Birkhauser, 1998.

[4] The Sauceman. His LaTeX expertise.

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit