The Study of the Vibrating String

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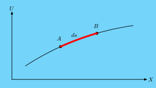
Abstract

In this document we will study and analyze the behavior of a vibrating string given different initial conditions. We will show the derivation of the wave equation, which we will use to mathematically model a vibrating string. We will also use Matlab and Metapost to show visual solutions to our examples.

Introduction

Before we begin to show the mathematics behind the model of a vibrating string, we must first explain the importance of studying the vibrating string. It is common for people to think that that the only application of the vibrating string is it's use in the musician's community, where strings are often stretched to make music and a living; however, studying the mathematics behind a vibrating string provides a starting point for understanding many results and concepts in physics. By learning about a vibrating string's dynamics, natural vibrations, and its response at different frequencies, we will



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receive the beginning of an introduction to the mathematics used in electromagnetic theory and quantum mechanics.

Derivation of the Wave Equation

The first step is to consider a string that is fixed at both ends and has a set length (L). The string must be in equilibrium along the x axis as seen in Figure 1. In order to accurately model the vibrating string without great complexity we will first have to make some assumptions about the string itself.

Assumptions

We will only be modelling the behavior of strings with a uniform density (mass/unit length) μ. We can assume that there will be no force due to gravity. This is because the force due to gravity is very small when compared to the tension in the string, and therefore the force of gravity can be neglected. Furthermore, we can assume that the displacement in the horizontal direction is significantly small. Therefore, we can ignore the horizontal component of tension and only consider the vertical component.

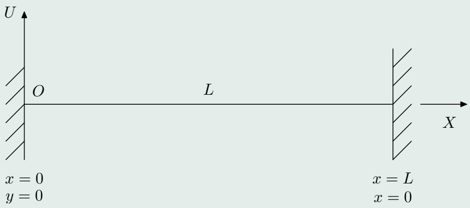
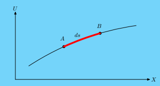


Figure 1: The String in Equilibrium



Deriving the Differential Equation

The next step is to obtain the differential equation that will best describe the motion of the vibrating string. In order to accomplish this we will look at a small portion of the string bounded by the points A and B as seen in Figure 2. We will let the length of the string between points A and B be called ds and the horizontal length between x and $x + dx$ be called dx as seen in Figure 3. By referring to Figure 3 you can also see that for small displacements the tension acting on the string remains the same.

This allows us to write the x and u components of the tension acting on the string as follows:

$$\Sigma F_x = T\cos\Theta_2 - T\cos\Theta_1 \tag{1}$$
$$\Sigma F_u = T\sin\Theta_2 - T\sin\Theta_1 \tag{2}$$

When Θ_1 and Θ_2 are very small,

$$\cos\Theta_1 \approx \cos\Theta_2.$$

This is the fact that lets us assume that there is no net horizontal force acting on the string, just as we mentioned earlier. Because there is no net horizontal force acting

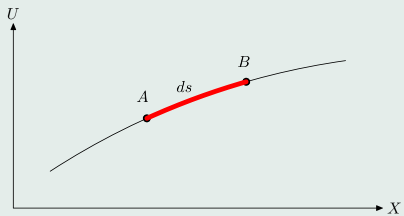
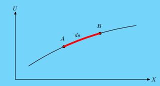


Figure 2: Determining ds



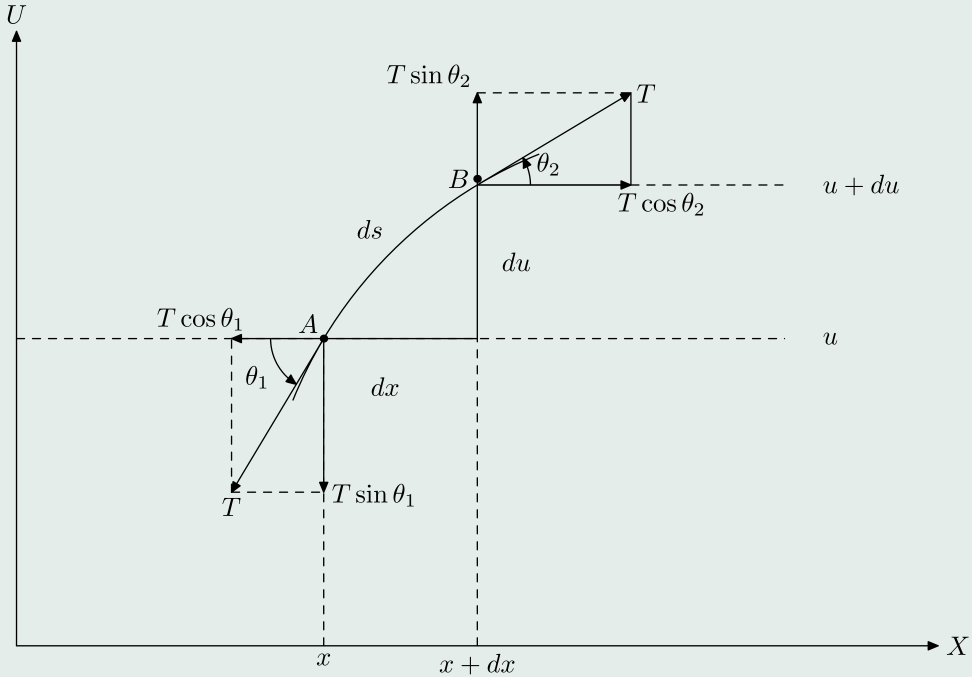
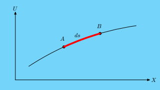


Figure 3: Transverse Vibrations



on the string, we are only concerned with the motion perpendicular to the length of the string. Furthermore, when Θ_1 and Θ_2 are very small, $\sin\Theta_1 \approx \tan\Theta_1$ and $\sin\Theta_2 \approx \tan\Theta_2$. We can now rewrite the sum of the forces in the u direction as:

$$\Sigma F_u = T\tan\Theta_2 - T\tan\Theta_1 \tag{3}$$

We will use a displacement function $u(x,t)$ to describe the motion of the string. This function will describe the string's position x at an instant in time t. Just as we let the horizontal displacement be called dx we will let the vertical displacement be called du, which is between u and $u+du$. (see Figure 3) By using Newton's second law we can express the sum of the forces in the u direction as a second order partial derivative times the mass of the string between points A and B. This second order partial derivative is the velocity of the string.

$$\Sigma F_u = ma_u = m\frac{\partial^2 u}{\partial t^2} \tag{4}$$

We use partial derivatives because $u = (x,t)$, where u is a function of both position and time. Now if we assume that $ds \approx dx$ and note that $m \approx \mu dx$, then we can combine the right sides of equations (3) and (4) and obtain equation (5):

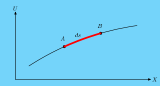
$$\mu dx\frac{\partial^2 u}{\partial t^2} = T\tan\Theta_2 - T\tan\Theta_1 \tag{5}$$

Next we will use the fact that $\tan\Theta = \partial u/\partial x$ and multiply both sides by the tension. We see that

$$T\tan\Theta = T\left(\frac{\partial u}{\partial x}\right). \tag{6}$$

With this information we can rewrite the vertical force as:

$$\Sigma F_u = T\tan\Theta_2 - T\tan\Theta_1 = T\left(\frac{\partial u}{\partial x}\right)_B - T\left(\frac{\partial u}{\partial x}\right)_A \tag{7}$$



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Next we expand the slope at point B by using the Taylor series. The Taylor series expansion goes as follows:

$$f(x+dx) = f(x) + f'(x)dx + \frac{f''(x)}{2!}(dx)^2 + \cdots$$

We use the Taylor Series to expand the slope of the string at point B.

$$\left(\frac{\partial u}{\partial x}\right)_B = \left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx + \frac{1}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)_A (dx)^2 + \cdots$$

But the third term has a dx^2 in it, which is significantly small and therefore that term and every term following it can be ignored. Now by substituting into equation (7) and combining with equation (5) we obtain

$$T\tan\Theta_2 - T\tan\Theta_1 = T\left(\frac{\partial u}{\partial x}\right)_B + T\left(\frac{\partial u}{\partial x}\right)_A$$

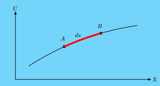
First we rewrite the right side of the equation by using the Taylor Series expansion,

$$\mu dx\frac{\partial^2 u}{\partial t^2} = T\left[\left(\frac{\partial u}{\partial x}\right)_A + \left(\frac{\partial^2 u}{\partial x^2}\right)_A dx + \cdots\right] - T\left(\frac{\partial u}{\partial x}\right)_A$$

$$\mu dx\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2}\right)_A dx.$$

Now divide both sides by μdx,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu}\left(\frac{\partial^2 u}{\partial x^2}\right)_A$$



We then obtain equation (8) by letting $a^2 = \frac{T}{\mu}$,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{8}$$

Equation (8) is the wave equation, which is the partial differential equation that models the motion of a vibrating string. In our future examples the string constant (a^2) will be set equal to 256. As we will soon see, this particular string constant will produce a period of vibration of 0.25 seconds. We will assume that the string is fixed at both ends and has a length of 2.

The Solution with Specific Boundary Conditions

Let $u(x,t)$ represent the vertical displacement of the string at a point x and at a specific time t. Since we assumed the string to have a fixed length of 2, the boundary conditions are

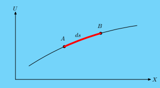
$$u(0,t) = u(2,t) = 0. \tag{9}$$

We are also assuming that the string is given an initial displacement and then released from rest; therefore, the initial conditions of the string will be

$$u(x,0) = \phi(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0, \tag{10}$$

where ϕ is a function in terms of x and is limited to the interval from [0,2]. Because the string is fixed at both ends $\phi(0) = \phi(2) = 0$. In other words the ends of the string will have no vertical displacement. We will be able to solve the initial boundary problem using the separation of variables technique. Furthermore, we will assume that the solution of equation (8) will have the form

$$u(x,t) = f(x)g(t). \tag{11}$$



By substituting equation (11) into equation (8) we obtain:

$$\frac{\partial^2}{\partial t^2} f(x)g(t) = a^2 \frac{\partial^2}{\partial x^2} f(x)g(t)$$

The derivative of a constant times a function is the constant times the derivative of the function

$$f(x)\frac{\partial^2}{\partial t^2} g(t) = a^2 g(t)\frac{\partial^2}{\partial x^2} f(x)$$

Because the partial derivative on the left is only in terms of t and the partial derivative on the right is only in terms of x we can rewrite the equation.

$$f(x)\frac{d^2}{dt^2} g(t) = a^2 g(t)\frac{d^2}{dx^2} f(x)$$

Next we will simplify both sides of the equation so that,

$$f(x)g''(t) = a^2 g(t) f''(x).$$

Now, by dividing both sides of the equation by $f(x)$ and by $a^2 g(t)$ we obtain

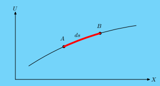
$$\frac{f''(x)}{f(x)} = \frac{1}{a^2}\frac{g''(t)}{g(t)}.$$

The only way that a function in terms of t can equal a function in terms of x is if both of the functions equal a constant, therefore,

$$\frac{f''(x)}{f(x)} = \frac{1}{a^2}\frac{g''(t)}{g(t)} = -\lambda,$$

where $-\lambda$ is equal to a constant. Now we can rewrite equation (**??**) as two separate equations,

$$\frac{f''(x)}{f(x)} = -\lambda \tag{12}$$



and

$$\frac{g''(t)}{a^2 g(t)} = -\lambda. \tag{13}$$

By first multiplying both sides of equation (12) by $f(x)$, and then adding $\lambda f(x)$ to each side we obtain

$$f''(x) + \lambda f(x) = 0. \tag{14}$$

Similarly, by first multiplying both sides of equation (13) by $a^2 g(t)$, and then adding $\lambda a^2 g(t)$ to each side we obtain

$$g''(t) + \lambda a^2 g(t) = 0. \tag{15}$$

When $\lambda = 0$ and $\lambda < 0$ the solution becomes trivial. We are not concerned with trivial solutions. Therefore we will assume that $\lambda > 0$.

Finding the Particular Solution $f''(x) + \lambda f(x) = 0$

To find a particular solution to equation (14) we will let

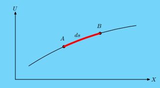
$$f(x) = e^{\omega x},$$

which yields the characteristic equation

$$\omega^2 + \lambda = 0.$$

Solving for ω, we can get the roots of the characteristic equation.

$$\omega = \pm i\sqrt{\lambda}.$$



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Using Euler's Identity we obtain,

$$f(x) = e^{\omega x} = e^{i\sqrt{\lambda}x} = \cos\sqrt{\lambda}x + i\sin\sqrt{\lambda}x.$$

This leads us to the general solution of equation (14).

$$f(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x \tag{16}$$

To find a particular solution of equation (14), we must recall our boundary conditions.

$$u(0,t) = u(2,t) = 0$$

This means that $0 = u(0,t) = f(0)g(t)$ and $0 = u(2,t) = f(2)g(t)$ for all t. This forces $f(0) = f(2) = 0$. By plugging in $f(0) = 0$ to equation (16) we discover that

$$0 = C_1\cos\sqrt{\lambda}(0) + C_2\sin\sqrt{\lambda}(0),$$

which simplifies to

$$C_1 = 0.$$

Since $C_1 = 0$, $f(x)$ can now be rewritten as

$$f(x) = C_2\sin\sqrt{\lambda}x. \tag{17}$$

Now, by plugging in $f(2) = 0$ into equation (17), we discover that

$$0 = C_2\sin\sqrt{\lambda}(2).$$

If $C_2 = 0$ then we will have a trivial solution, but we are not interested in the trivial solutions. Therefore we can assume that $C_2 \neq 0$. By dividing both sides by C_2 we obtain

$$0 = \sin\sqrt{\lambda}(2).$$

This implies that $\sqrt{\lambda}(2) = n\pi$, where $n = 1, 2, 3, \ldots$. Dividing both sides by 2 leads to the equation

$$\sqrt{\lambda} = \frac{n\pi}{2}, \tag{18}$$

which yields the particular solution to equation (14).

$$f(x) = C_2 \sin \frac{n\pi x}{2} \qquad n = 1, 2, \ldots \tag{19}$$

Finding the Particular Solution to $g''(t) + \lambda a^2 g(t) = 0$

To find the particular solution to $g''(t) + \lambda a^2 g(t) = 0$ we will let

$$g(t) = e^{\omega t}.$$

This yields the characteristic equation

$$\omega^2 + \lambda a^2 = 0.$$

We know that $\lambda = n^2\pi^2/4$ from equation (18); therefore, we can write the characteristic equation as follows.

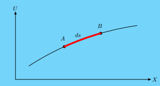
$$\omega^2 + \frac{n^2\pi^2 a^2}{4} = 0$$

The roots of the characteristic equation are

$$\omega = \pm \frac{n\pi a}{2} i.$$

Using Euler's Identity we discover that

$$g(t) = e^{n\pi a i t/2} = \cos \frac{n\pi a t}{2} + i \sin \frac{n\pi a t}{2},$$



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which leads to the general solution of equation (15).

$$g(t) = D_1 \cos \frac{n\pi at}{2} + D_2 \sin \frac{n\pi at}{2} \tag{20}$$

To find the particular solution of equation (15), we must recall that the initial velocity of the string is zero; that is,

$$\frac{\partial}{\partial t} u(x, 0) = 0.$$

This implies that $f(x)g'(0) = 0$ for all x and forces $g'(0) = 0$. The derivative of equation (20) is

$$g'(t) = -\frac{D_1 n\pi a}{2} \sin \frac{n\pi at}{2} + \frac{D_2 n\pi a}{2} \cos \frac{n\pi at}{2}.$$

We know that $g'(0) = 0$, therefore

$$g'(0) = \frac{D_2 n\pi a}{2} = 0.$$

Solving the above equation for D_2 we see that $D_2 = 0$. Now if we let $D_1 = D_n$ we can see that the particular solution of equation (15) is

$$g(t) = D_n \cos \frac{n\pi at}{2} \qquad n = 1, 2, \ldots \tag{21}$$

The Solution of the Wave Equation that models the Vibrating String

Using the fact that the function $u(x, t)$ is equal to $f(x)g(t)$, we can combine the particular solutions found in equations (19) and (21) so that,

$$u(x, t) = C_n \sin \frac{n\pi x}{2} D_n \cos \frac{n\pi at}{2}.$$

Combining the constants yields

$$u(x,t) = b_n \sin\frac{n\pi x}{2}\cos\frac{n\pi a t}{2},$$

and any linear combination must also be a solution. Therefore the solution to the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}\cos\frac{n\pi a t}{2}. \tag{22}$$

If we let $t = 0$, we see from equation (10) that $u(x,0) = \phi(x)$ is the initial condition of the strings position, and using this initial condition will help us determine the Fourier coefficients b_n. When $t = 0$ equation (22) becomes

$$\phi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}. \tag{23}$$

Using the fact that

$$\int_0^2 \sin\left(\frac{m\pi x}{2}\right)\sin\left(\frac{m\pi x}{2}\right)\,dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases} \tag{24}$$

we can solve for the Fourier coefficient (b_n). First we multiply equation (23) by $\sin(m\pi x/2)$ and then integrate from 0 to 2 to get the Fourier coefficient b_n.

$$\begin{aligned}
\int_0^2 \phi(x)\sin\frac{m\pi x}{2}dx &= \int_0^2 \sin\frac{m\pi x}{2}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}dx \\
&= \int_0^2 \sum_{n=1}^{\infty} b_n \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}dx \\
&= \sum_{n=1}^{\infty} b_n \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{n\pi x}{2}dx \\
&= b_n \int_0^2 \sin\frac{m\pi x}{2}\sin\frac{m\pi x}{2}dx \\
&= b_n \\
b_n &= \int_0^2 \phi(x)\sin\frac{m\pi x}{2}dx \qquad (25)
\end{aligned}$$

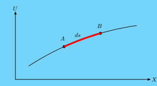
Examples

Example One

For our first example we will consider an initial position where

$$\phi(x) = \begin{cases} x/48, & 0 \le x \le 1, \\ (2-x)/48, & 1 < x \le 2, \end{cases}$$

The initial position of the string can be seen in Figure 4. The first step is to calculate the Fourier coefficient using equation (25). For the first half of the string we will integrate



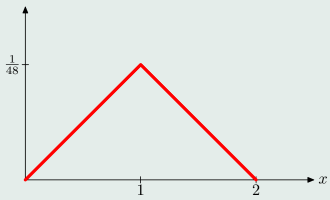


Figure 4: The Initial Position of the String (ex 1)

from 0 to 1, because the $\phi(x)$ is represented by a piecewise function. The function we will be integrating from 0 to 1 is

$$\int_0^1 \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_0^1 \frac{x}{48} \sin \frac{n\pi x}{2}\, dx$$

Integration by parts provides:

$$\begin{aligned} \int_0^1 \phi(x) \sin \frac{n\pi x}{2}\, dx &= \frac{-2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \Big|_0^1 \\ &= \frac{1}{24n^2\pi^2} \left[-n\pi \cos \frac{n\pi}{2} + 2 \sin \frac{n\pi}{2} \right]. \end{aligned} \tag{26}$$

We must now calculate for the second half of the string. To do this we must integrate $2 - x/48$ from 1 to 2.

$$\int_1^2 \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_1^2 \frac{2-x}{48} \sin \frac{n\pi x}{2}\, dx$$

By using tabular integration we obtain:

$$
\begin{aligned}
\int_1^2 \phi(x)\sin\frac{n\pi x}{2}\,dx &= \frac{x-2}{24n\pi}\cos\frac{n\pi x}{2} - \frac{1}{12n^2\pi^2}\sin\frac{n\pi x}{2}\Big|_1^2 \\
&= -\frac{1}{12n^2\pi^2}\sin(n\pi) + \frac{1}{24n\pi\pi}\cos\frac{n\pi}{2} - \frac{1}{12n^2\pi^2}\sin\frac{n\pi}{2} \\
&= \frac{1}{24n^2\pi^2}\left[n\pi\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2}\right] \qquad (27)
\end{aligned}
$$

Now to obtain b_n, we add the integrals; that is, add equation (26)to equation (27).

$$b_n = \frac{1}{6n^2\pi^2}\frac{\sin n\pi}{2}$$

To get our solution, we plug our b_n into equation (22).

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{6n^2\pi^2}\sin\frac{n\pi}{2}\sin\left(\frac{n\pi x}{2}\right)\cos(8n\pi t).$$

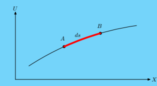
Snapshots of this particular vibrating string at times $t=0$, $t=\frac{1}{64}$, ... , $t=\frac{8}{64}$ can be seen in Figure 5.

Example Two

For our second example we will use the initial condition given by the function:

$$\phi(x) = \frac{2x-x^2}{48}, \qquad 0 \le x \le 2$$

The initial position of the string is parabolic and can be seen in Figure 6. The first step



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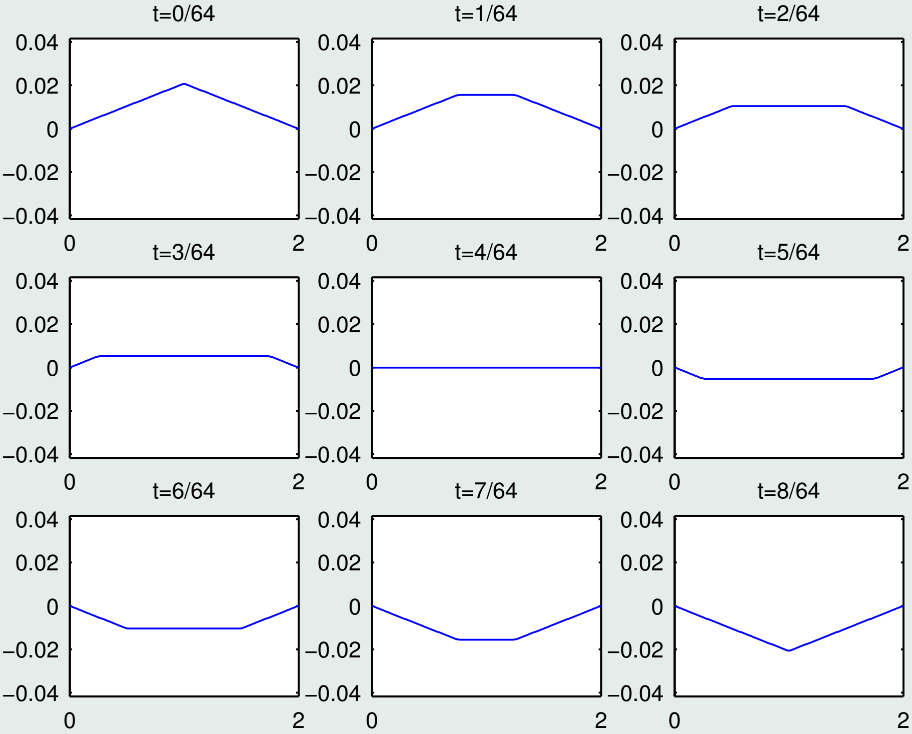
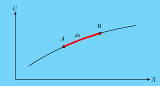


Figure 5: Example One



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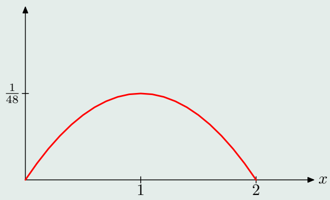


Figure 6: The Initial Position of the String (ex2)

is to calculate the Fourier coefficients using equation (25) just as we did in Example One. Now we will integrate over the entire length of our string:

$$b_n = \int_0^2 \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_0^2 \frac{2x - x^2}{48} \sin \frac{n\pi x}{2}\, dx$$

Using tabular integration provides us with,

$$\begin{aligned} b_n &= \frac{-4x + 2x^2}{48n\pi} \cos \frac{n\pi x}{2} + \frac{8 - 8x}{48n^2\pi^2} \sin \frac{n\pi x}{2} - \frac{16}{48n^3\pi^3} \cos \frac{n\pi x}{2} \Bigg|_0^2 \\ &= \frac{1}{3n^3\pi^3} [1 - \cos n\pi] && (28) \\ & && (29) \end{aligned}$$

To get our solution to Example Two, we plug our b_n into equation (22).

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{3n^3\pi^3} [1 - \cos n\pi] \sin\left(\frac{n\pi x}{2}\right) \cos(8n\pi t).$$

Snapshots of this particular vibrating string at times $t = 0$, $t = 1/64$, ..., $t = 8/64$ can be seen in Figure 7.

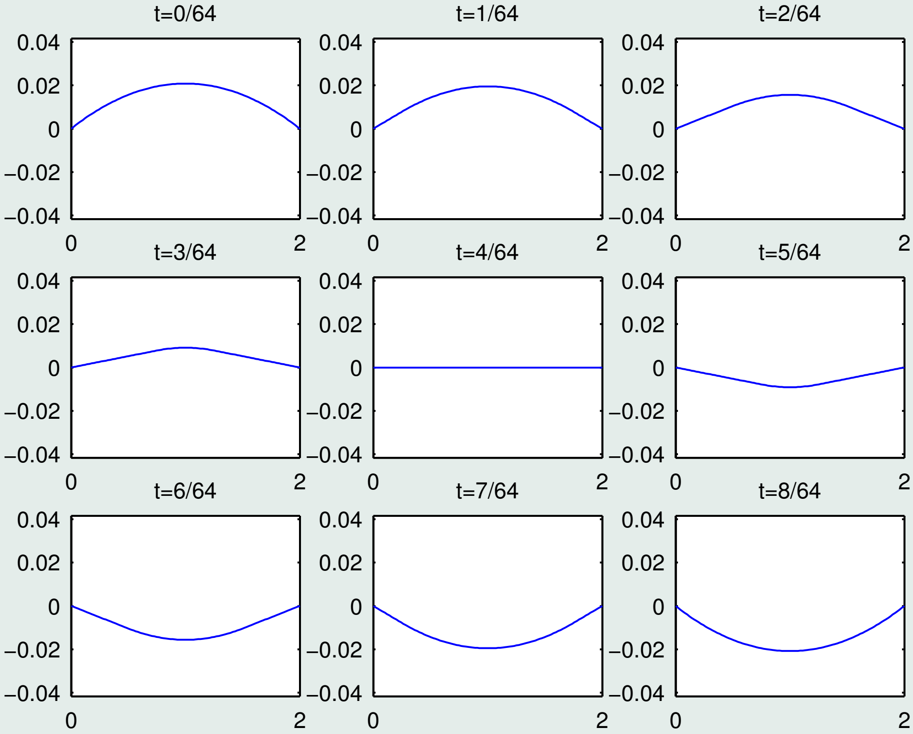
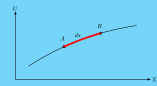


Figure 7: Example Two



Example Three

For our final example we will use the initial position given by $\phi(x)$ below.

$$\phi(x) = \begin{cases} x/24, & 0 \le x \le 1/2 \\ (x-1)/24, & 1/2 < x \le 1 \\ 0, & 1 < x \le 2. \end{cases}$$

Figure 8 shows a vibrating with string with these specific initial conditions. The first step is to calculate the Fourier coefficients using equation (25) just as we did in Example One and Two. However, for this example we will have to evaluate three separate integrals and add them together in order to obtain b_n.

The first integral covers one quarter of the string. Integrating on $[0, 1/2]$:

$$\int_0^{1/2} \phi(x) \sin \frac{n\pi x}{2}\, dx = \int_0^{1/2} \frac{x}{24} \sin \frac{n\pi x}{2}\, dx$$

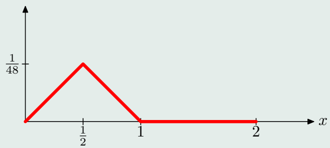
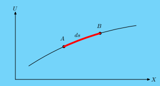


Figure 8: The Initial Position of the String (ex 3)



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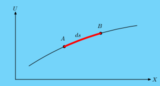

Integration by parts provides us with:

$$\int_0^{1/2} \frac{x}{24} \sin \frac{n\pi x}{2}\, dx = \frac{-2x}{24n\pi} \cos \frac{n\pi x}{2} + \frac{4}{24n^2\pi^2} \sin \frac{n\pi x}{2} \Bigg|_0^{1/2}$$

$$= \frac{1}{24n^2\pi^2} \left[-n\pi \cos \frac{n\pi}{4} + 4 \sin \frac{n\pi}{4} \right] \tag{30}$$

$$\tag{31}$$

Now we will integrate on the interval $[1/2, 1]$.

$$\int_{1/2}^{1} \sin \frac{n\pi x}{2}\, dx = \int_{1/2}^{1} \frac{x-1}{24} \sin \frac{n\pi x}{2}\, dx$$

Integration by parts provides us with:

$$\int_{1/2}^{1} \frac{x-1}{24} \sin \frac{n\pi x}{2}\, dx = \frac{x-1}{12n\pi} \cos \frac{n\pi x}{2} - \frac{1}{6n^2\pi^2} \sin \frac{n\pi x}{2} \Bigg|_{1/2}^{1}$$

$$= \frac{1}{24n^2\pi^2} \left[n\pi \cos \frac{n\pi}{4} + 4 \sin \frac{n\pi}{4} - 4 \sin \frac{n\pi}{2} \right] \tag{32}$$

$$\tag{33}$$

When we integrate on the interval $[1, 2]$ we get zero because the $\int_1^2 0\, dx = 0$. Therefore, we obtain b_n by adding equation (30) to equation (32).

$$b_n = \frac{1}{24n^2\pi^2} \left[8 \sin \frac{n\pi}{4} - 4 \sin \frac{n\pi}{2} \right] = \frac{1}{6n^2\pi^2} \left[2 \sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} \right]$$

To get our solution to Example Three, we plug our b_n into equation (22).

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{6n^2\pi^2} \left[2 \sin \frac{n\pi}{4} - \sin \frac{n\pi}{2} \right] \sin \left(\frac{n\pi x}{2} \right) \cos(8n\pi t).$$

Snapshots of this particular vibrating string at times $t = 0$, $t = 1/64$, $\ldots$, $t = 8/64$ can be seen in Figure 9.

Conclusion

By using skills gained through the study of differential equations, we were able to derive the wave equation and find its solution to model the behavior of a vibrating string. Once the behavior is graphed it can be studied closely for further understanding. Now you know the mathematics behind a simple vibrating string. I bet you never thought there was so much math behind a simple pluck of a string.

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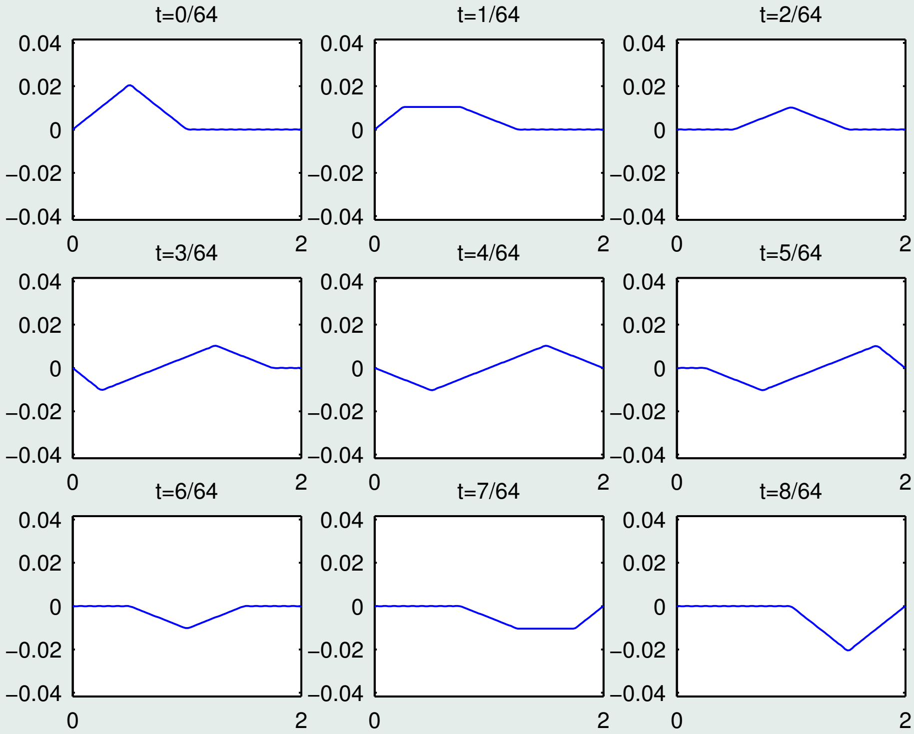

Figure 9: Example Three