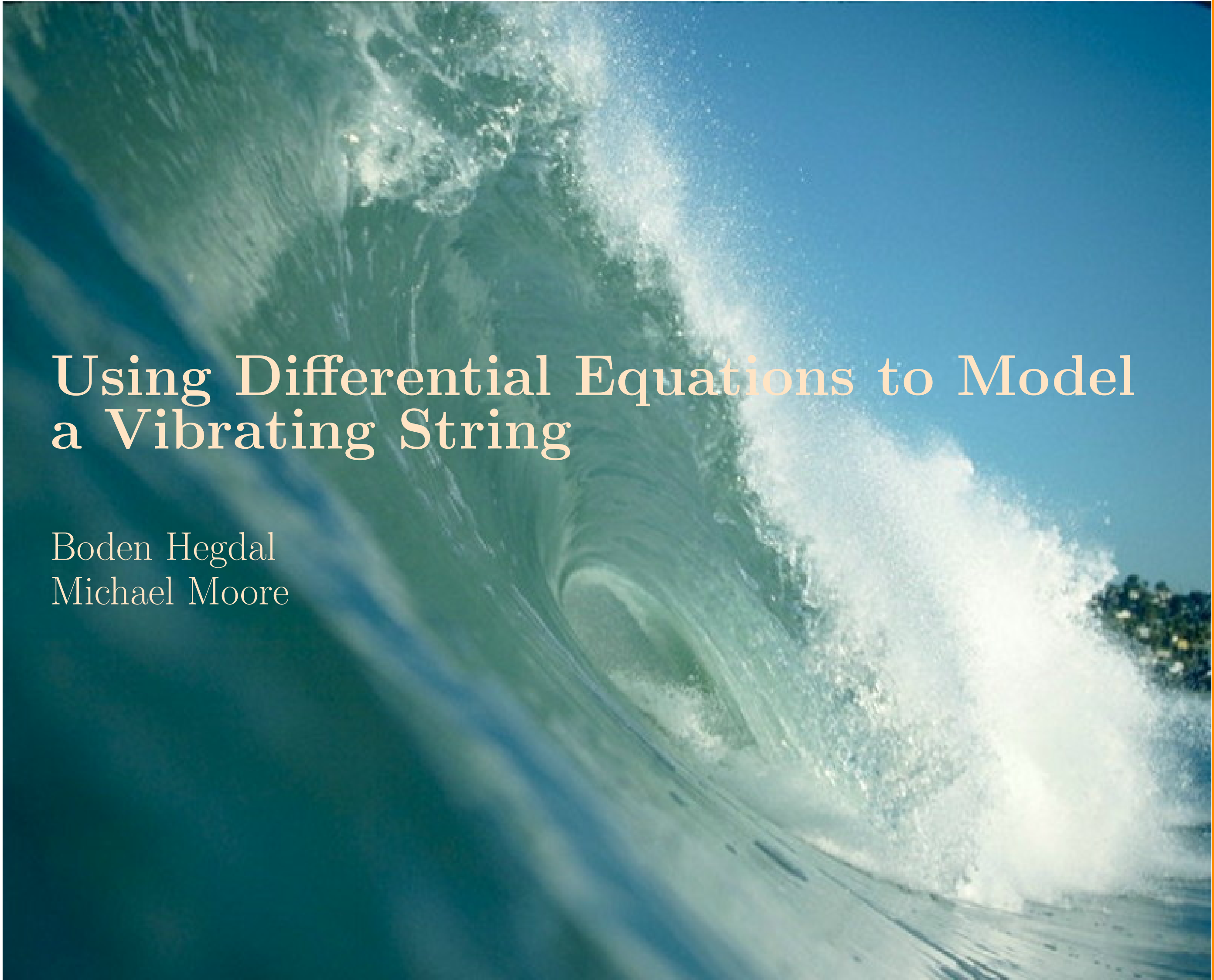

Using Differential Equations to Model a Vibrating String

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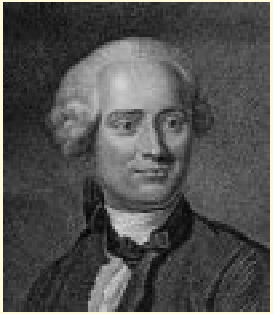
Pythagoras

The science of waves and wave motion is essential to a wide range of applications. In it's simplest form, a wave is a disturbance traveling through some medium.

In 550 B.C. the Pythagorians observed that vibrating strings produced sound and studied the mathematical relationship between the frequency of the sound and the length of the string. In the seventeenth century, the science of wave propagation received attention from Galileo Galilei, Robert Boyle, and Isaac Newton.

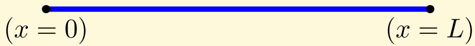
D'Alembert

It was not until the Eighteenth Century that French mathematician and scientist Jean Le Rond d'Alembert derived the wave equation.

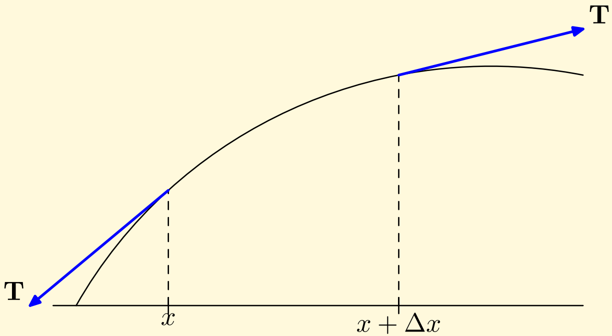


Jean le Rond D'Alembert's "Reflexions sur la cause generale des vents" ("on the general theory of the winds") is printed in 1747. It contained the first general use of partial differential equations in mathematical physics. That same year he published his theory of vibrating strings wherein he described and solved the wave equation in two dimensions.

Deriving the Wave Equation

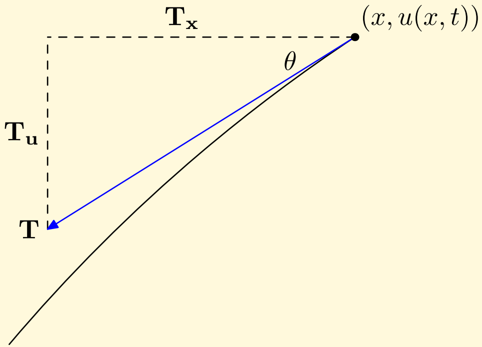


String of length L lays on the x axis



- We will consider the segment from x to $x + \Delta x$
- Ignore all energy losses due to stretching and bending
- Ignore all external forces

Left End



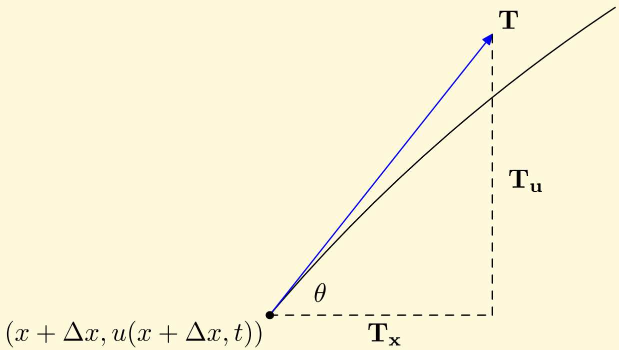
The Tension (T) is always tangential to the string, where θ is the angular displacement from the horizontal. The slope of the string at any point x can be described as $\partial u/\partial x = \tan(\theta)$. Since $u(x,t)$ is small compared to L, $\cos(\theta) \approx 1$ so $\tan(\theta) \approx \sin(\theta)$. Thus,

$$T_u = -T\sin\theta \approx -T\tan\theta = -T\,\frac{\partial u}{\partial x}(x,t)$$

and

$$T_x = -T\cos\theta \approx -T.$$

Right End



The Tension (T) is always tangential to the string, where θ is the angular displacement from the horizontal. The slope of the string at any point x can be described as $\partial u/\partial x = \tan(\theta)$. Since $u(x,t)$ is small compared to L, $\cos(\theta) \approx 1$ so $\tan(\theta) \approx \sin(\theta)$. Thus,

$$T_u = T \sin\theta \approx T \tan\theta = T\,\frac{\partial u}{\partial x}(x + \Delta x, t)$$

and

$$T_x = T\cos\theta \approx T.$$

Putting it together using $F = ma$

The total force in the horizontal direction is $F_x = -T + T = 0$ so there is no horizontal acceleration. The total force in the vertical direction is

$$F_u \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right).$$

The vertical acceleration is $\partial^2 u/\partial t^2$ and the mass per unit length is ρ. Putting it together using $F = ma$ we have

$$T\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right) = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

Now we can divide both sides by Δx and take the the limit as Δx approaches zero.

$$\rho\frac{\partial^2 u}{\partial t^2} = T \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right) \tag{1}$$

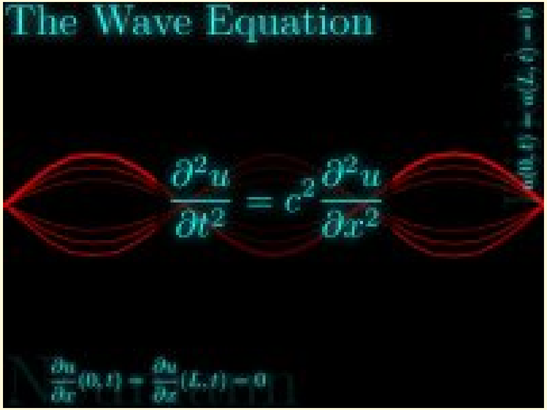
From the definition of a derivative

$$\lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right)}{\Delta x} = \frac{\partial^2 u}{\partial x^2} \tag{2}$$

Combining (2) and (3) and letting $c^2 = T/\rho$ gives us the wave equation as defined by d'Alembert.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

Now that we have derived the wave equation a general solution for $u(x,t)$ must be found.



The General Solution

Now that we have the wave equation a general solution for $u(x,t)$ can be obtained easily by separation of variables. First we will define $u(x,t)$ as a product of two independent functions.

$$u(x,t) = X(x)T(t)$$

Substituting this into (3) gives the following.

$$[X(x)T(t)]_{xx} = \frac{1}{c^2}[X(x)T(t)]_{tt}$$

Taking the second derivative and grouping variables gives

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)}$$

Setting both sides equal to a constant $-\lambda$ gives us a system of two second order ordinary differential equations.

$$X''(x) + \lambda X(x) = 0 \qquad \text{and} \qquad T''(t) + \lambda c^2 T(t) = 0 \tag{4}$$

Solving for $X(x)$

We will start with the first equation and solve for $X(x)$. There are three cases depending on the value of λ. These cases will be $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. By using our boundary conditions we will identify the appropriate case for the specific solutions.

$X(x)$:**Case 1,** $\lambda < 0$

Let $\lambda = -\omega^2, \omega > 0$. Substituting into (4) we have

$$X'' - \omega^2 X = 0$$

Now we choose an integrating factor $X(x) = e^{rx}$ giving us

$$r^2 e^{rx} - \omega^2 e^{rx} = 0$$

Diving both sides by e^{rx} and solving for r gives us

$$r = \pm\omega.$$

Now that we have two independent solutions they can be written as a linear combination.

$$X(x) = C_1 e^{\omega x} + C_2 e^{-\omega x} \tag{5}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_1 = -C_2$$

Substituting into (5) and using our second boundary condition $X(L) = 0$ gives us $e^{2\omega L} = 1$. This is a false statement since neither L nor ω can be zero. This means that λ is not less than zero.

$X(x)$:Case 2, $\lambda = 0$

When $\lambda = 0$, $X'' = 0$ and

$$X(x) = C_1 x + C_2 \tag{6}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_2 = 0$$

Substituting into (6) and using our second boundary condition $X(L) = 0$ gives us

$$C_1 = 0.$$

If both C_1 and C_2 then our string does not ever move. This is trivial and therefore λ cannot be equal to zero.

$X(x)$:**Case 3,** $\lambda > 0$

Let $\lambda = \omega^2, \omega > 0$. Substituting into (4) we have

$$X'' + \omega^2 X = 0$$

Now we choose an integrating factor $X(x) = e^{rx}$ giving us

$$r^2 e^{rx} + \omega^2 e^{rx} = 0$$

Diving both sides by e^{rx} and solving for r gives us $r^2 = -\omega^2$ or

$$r = \pm\omega i.$$

This means that $X(x) = e^{\pm\omega i}$. Using Euler's identity

$$e^{\omega x i} = [\cos \omega x + i \sin \omega x]$$

The general solution is a linear combination of the $\mathrm{Re}X(x)$ and $\mathrm{Im}X(x)$.

$$X(x) = C_1 \cos \omega x + C_2 \sin \omega x \tag{7}$$

Using the first boundary condition $X(0) = 0$ gives us

$$C_1 = 0$$

Substituting into (7) and using our second boundary condition $X(L) = 0$ gives us

$$0 = C_2 \sin \omega L$$

which is only satisfied when

$$\omega = \frac{n\pi}{L}, \qquad \text{for } n = 1, 2, 3, \ldots$$

Therefore our general solution for $X(x)$ is

$$X(x) = \sin \frac{n\pi x}{L}$$

Solving for $T(t)$

We take (4) and let $\omega^2 = \lambda c^2$.

$$T'' + \omega^2 T = 0$$

This equation is the same form as case 3 of the previous section replacing $X(x)$ with $T(t)$. Now we can see that the general solution of $T(t)$ will have the same form as the solution for $X(x)$ in equation (7) except now

$$\omega = \frac{n\pi c}{L}$$

Thus a fundamental set of solutions for $T'' + \omega^2 T = 0$ is

$$\sin\frac{n\pi ct}{L} \qquad \text{and} \qquad \cos\frac{n\pi ct}{L} \qquad \text{for } n = 1, 2, 3, \ldots$$

Linear Combinations

Now we can express the product of $X_n(x)$ and $T_n(t)$ as

$$X_n(x)T_n(t) = \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

and

$$X_n(x)T_n(t) = \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}.$$

Any linear combination of these fundamental solutions is also a solution of the wave equation.

$$u_n(x,t) = \sin\frac{n\pi}{L}x\left[a_n\cos\frac{n\pi c}{L}t + b_n\sin\frac{n\pi c}{L}t\right]$$

The Final Solution

Combining these in a linear combination we have the final solution for $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t \right]$$

Specific Solutions with Initial Conditions

We will define the initial shape of the string as $f(x)$ and the initial vertical velocity of the string as $g(x)$. To reduce the complexity of our solution we will only choose initial conditions where $g(x) = 0$ so $b_n = 0$. To find the fourier coefficients for a_n to satisfy the initial condition we define $f(x)$ as

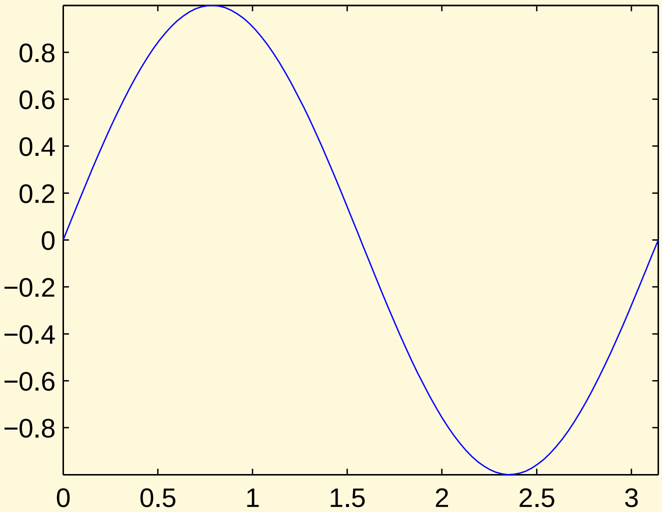
$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

It can be shown that

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin \frac{n\pi x}{L} dx \tag{8}$$

Standing Wave

Choose the initial condition $u(x, 0) = f(x) = \sin 2x$, for $0 < x < \pi$.



Shown above is the initial position of the string.

Computing the Coefficients

$$a_n = \frac{2}{L}\int_0^L \sin 2x \sin nx dx$$
$$= \frac{2\sin n\pi}{n^2-4}$$

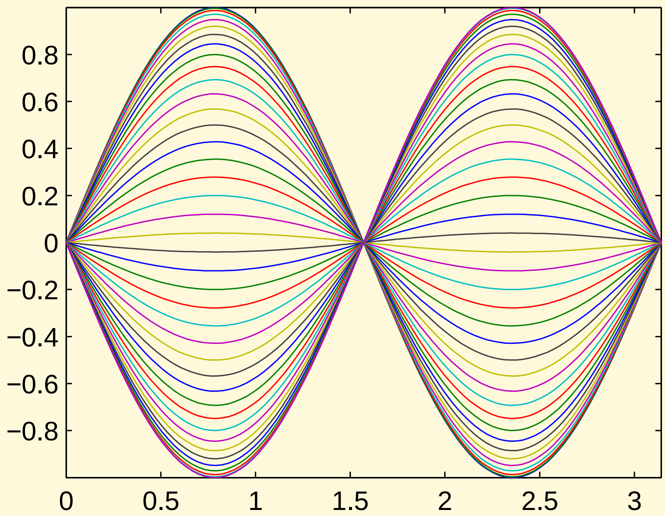
From this we can see that $a_n = 0$ for all values of n except $n = 2$. where the denominator is zero. This case must be done separately.

$$a_2 = \frac{2}{\pi}\int_0^\pi \sin 2x \sin 2x dx = 1$$

Thus;

$$u(x,t) = \sin(nx)\cos(nct)$$

The Solution at Various Times for One Period

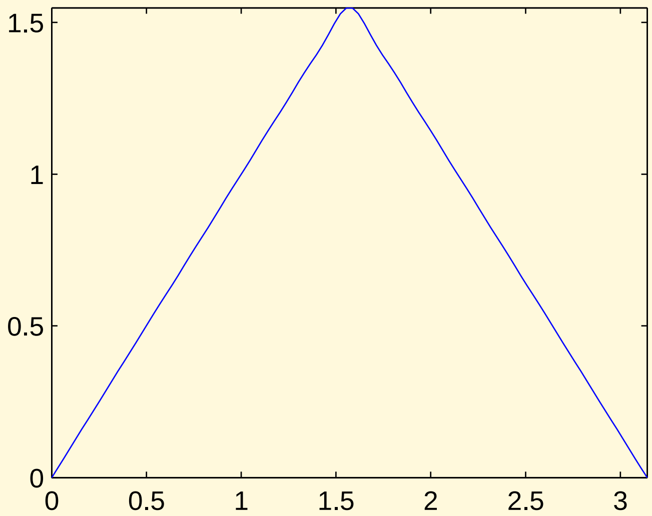


Displacement of the string at $L/2$

Choose the following initial conditions for $0 < x < \pi$.

$$u(x,0) = f(x) = \begin{cases} x, & \text{if } 0 \le x \le \pi/2 \\ \pi - x, & \text{if } \pi/2 \le x \le \pi \end{cases}$$

,



Shown above is the initial position of the string $u(x,0)$.

Computing the Coefficients

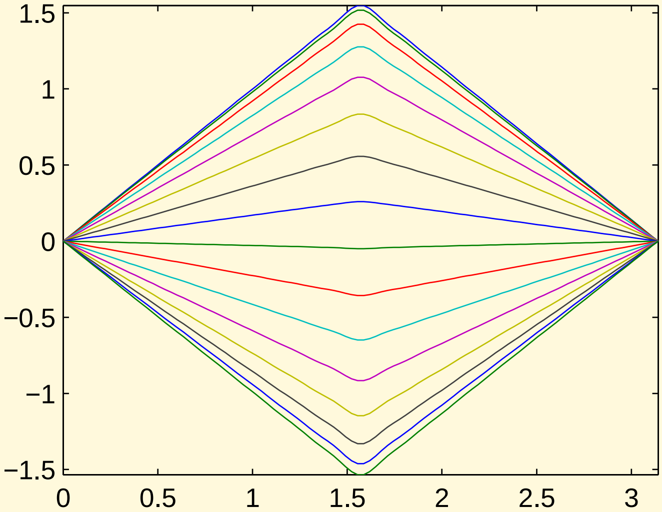
$$a_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx dx\right)$$

$$a_n = \frac{4 \sin(\frac{n\pi}{2})}{\pi n^2}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin(nx) \sin\left(\frac{n\pi}{2}\right) \cos(nct)$$

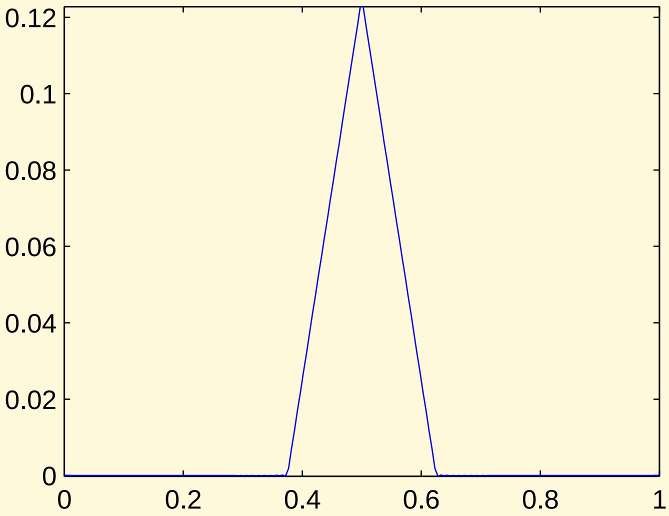
The Solution at Various Times for One Period



A More Interesting Example

Choose the following initial conditions for $0 < x < 1$

$$f(x) = \begin{cases} x - 3/8, & \text{if } 3/8 \le x \le 1/2 \\ 5/8 - x, & \text{if } 1/2 \le x \le 5/8 \\ 0, & \text{otherwise} \end{cases}$$



Computing the Coefficients

$$a_n = 2\int_0^1 f(x)\sin n\pi x dx$$

$$a_n = 2\left[\int_{\frac{3}{8}}^{\frac{1}{2}} \left(x - \frac{3}{8}\right)\sin n\pi x dx + \int_{\frac{1}{2}}^{\frac{5}{8}} \left(\frac{5}{8} - x\right)\sin n\pi x dx\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{8} - \sin\frac{5n\pi}{8}\right)$$

So the solution is,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\sin(n\pi x)\cos(n\pi ct) \left(2\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{8} - \sin\frac{5n\pi}{8}\right) \quad (9)$$

The Solution at Various Times for One Period