Basic Coupled Oscillator Theory Applied to the Wilberforce Pendulum

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Abstract

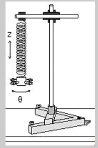
In this article, we outline some important characteristics of coupled oscillators that simplify the math used to set up and solve a system of second order homogeneous coupled differential equations describing the motion of the Wilberforce Pendulum.

1. Introduction

The Wilberforce pendulum is a vertical pendulum, where the mass at the end of a spring can oscillate not only in the vertical direction, but it can also rotate with oscillatory motion (see Figure 1). Also the energy of the system is transferred between the vertical and rotational oscillations. The amplitude and frequency of vertical oscillations are dependent on the amplitude and frequency of oscillation in the rotational direction, and visa-versa. Since the Wilberforce pendulum requires two independent coordinates to describe its motion, it is a two-degree-of-freedom-system.

2. One Degree of Freedom Systems

We are all familiar with the mass-spring problem. The pendulum bob with mass m is attached to a spring with spring constant k on a frictionless surface. Given an initial displacement x, the only force acting on the mass is the restoring force $-kx$ (see Figure 2). By Newton's second law of motion the force is equal to a mass times its acceleration, or $F = ma = m\ddot{x} = -kx$, where $\ddot{x}$ is the second derivative of



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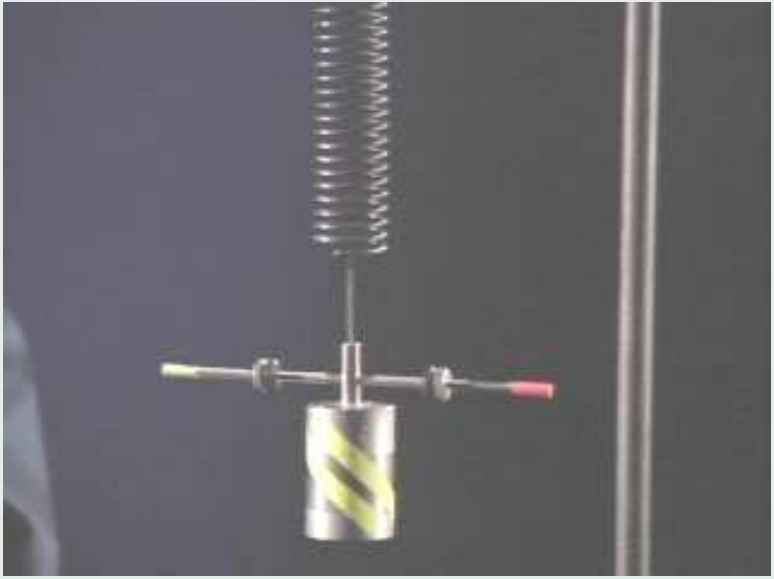
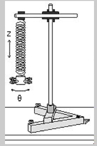


Figure 1: Wilberforce Pendulum.



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x with respect to time. Thus the differential equation describing the motion of the mass can be written as

$$m\ddot{x} + kx = 0.$$

Dividing through by m, this becomes

$$\ddot{x} + \omega_0^2 x = 0,$$

where we have set k/m equal to ω_0^2, It is well known that the solution to this equation is $x = A\cos(\omega_0 t + \phi)$, where ω_0 is called the "natural frequency" of the system. If left undisturbed, this

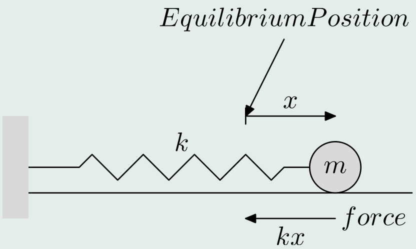


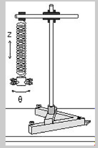
Figure 2: Mass on a spring.

oscillatory motion will continue on indefinitely. This spring-mass system may be described as a one-degree-of-freedom system because it requires only one independent coordinate to describe its motion.

3. Two-Degree-of-Freedom Systems

Let us turn our attention to two-degree-of-freedom system such as the Wilberforce pendulum. For any two-degree-of-freedom system, since there are two independent coordinates required to describe the motion, there are only two possibilities at which the frequencies of the two degrees of freedom can be equal. These are the normal frequencies for each coordinate. The cases at which this occurs are called normal modes of the system, and *any* combination of motion of the system can be written as a linear combination of these normal modes. We will explain normal frequencies and modes further in the following discussion.

Consider a system in which two masses connected to two separate springs with identical spring constants k are joined by a spring of constant k' (see Figure 3). Are there any ways in which these two masses can oscillate with the same frequency? Intuition tells us that if the displacement of the



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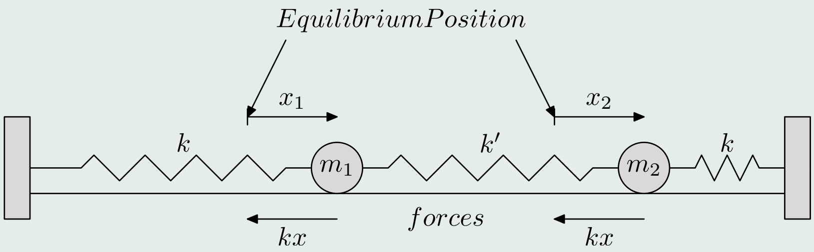


Figure 3: Horizontal mass-spring system: equal initial displacement.

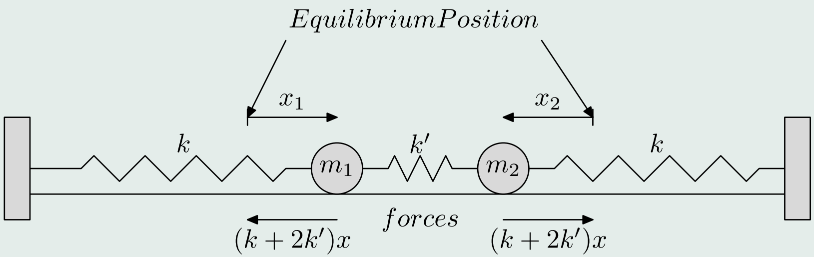


Figure 4: Horizontal mass-spring system: equal and opposite initial displacement.

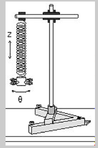
masses are exactly equal in direction and magnitude, then the joining spring will remain slack and can be ignored as both masses (being uncoupled) will oscillate with their natural frequencies $\sqrt{k/m}$. If the displacement of each mass is exactly equal and opposite (see Figure 4), then the force acting on each mass will be $(k+2k')x$. Setting this equal to $F = ma = m\ddot{x}$ we get the the differential equation

$$\ddot{x} + \left(\frac{k'+k}{m}\right)x = 0.$$

Then the frequency at which each mass oscillates is

$$\sqrt{\frac{k'+k}{m}}.$$

In these two scenarios in which the masses oscillate with the same frequency, they oscillate with the "normal frequencies" of the system. The motion that the masses exhibit when oscillating at these normal



frequencies are the normal modes of the system. Any possible combination of the motions of the two masses can be written as a linear combination of these normal modes.

4. Normal Modes

Applying these concepts to the Wilberforce pendulum, the natural response of a spring being stretched or compressed is a small unravelling or ravelling of the coils of the spring, inducing rotational motion of the mass attached to the end of the spring. We would expect that if the frequency of rotation forced by the inertia of the mass, matches the frequency of the "ravelling/unravelling" (the natural frequency of the system), then this would be a normal frequency. If the frequency of the rotation of the mass opposes the natural frequency of the spring, this could also be considered a normal frequency of the system. The inertia of the system is adjusted (by correctly positioning the nuts that are protruding from the vanes of the pendulum) in order to produce a normal frequency. Therefore, if we can analyze the system to

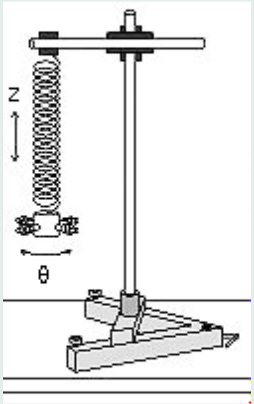
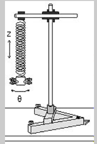


Figure 5: Wilberforce Pendulum.

find the motion of the mass as it oscillates at these normal frequencies then we would have found the normal modes of the system. Then, we will in fact have solved the system for any combination of the vertical and rotational motions of the pendulum. Now that we have conceptually identified the normal modes, let us find them with a little more rigor.



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In our system the vertical displacement is going to be measured from equilibrium in the z direction, and the displacement in rotation is going to be measured as an angle θ from equilibrium (see Figure 5). The Lagrangian for our system is the kinetic energy minus the potential energy plus a coupling term with coupling constant ϵ. This coupling term accounts for the fact that the energy in the z direction is dependent on the energy in the θ direction, and visa-versa, where ϵ is the magnitude of this effect. ϵ is dependent on the properties of the spring. The kinetic energy (K) of the mass in the vertical direction is defined to be

$$\frac{1}{2}mv^2 = \frac{1}{2}\dot{z}^2,$$

and the potential energy (U) is given by

$$\frac{1}{2}kz^2.$$

Likewise the kinetic and potential energies of the mass in rotation are

$$\frac{1}{2}I\dot{\theta}^2 \qquad \textit{and} \qquad \frac{1}{2}\delta\theta^2$$

respectively, where I is the inertia of the mass and δ is the rotational spring constant. The coupling term is a potential energy relation of z and θ, given by

$$\frac{1}{2}\epsilon z\theta.$$

Therefore the Lagrangian (L) for this system is

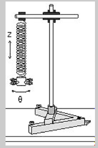
$$L = K - U = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\delta\theta^2 - \frac{1}{2}\epsilon z\theta. \tag{1}$$

We need to minimize the Lagrangian for our system. The well-known Euler-Lagrange equations will do this for us, in the same way that, to find the minimum of a function, we take its derivative and set it equal to zero.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \tag{2}$$

where the derivative with respect to time of the partial derivative of the Lagrangian with respect to $\dot{z}$ minus the partial derivative with respect to z equals zero. Also, the derivative with respect to time of the partial derivative with respect to $\dot{\theta}$ minus the partial derivative with respect to θ equals zero.

Applying the Euler-Lagrangian equation to Equation (1) returns



$$\frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \frac{\partial L}{\partial z} = -kz - \frac{1}{2}\epsilon\theta,$$

thus,

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} \\
0 &= \frac{d}{dt}\left(m\dot{z}\right) - \left(-kz - \frac{1}{2}\epsilon\theta\right) \\
0 &= m\ddot{z} + kz + \frac{1}{2}\epsilon\theta.
\end{aligned}$$

In the same manner,

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -\delta\theta - \frac{1}{2}\epsilon z,$$

thus,

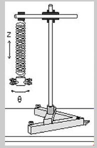
$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \\
0 &= \frac{d}{dt}\left(I\dot{\theta}\right) - \left(-\delta\theta - \frac{1}{2}\epsilon z\right) \\
0 &= I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z.
\end{aligned}$$

These two second-order, non-homogeneous differential equations describe the complete motion of the system.

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\theta = 0 \tag{3}$$

$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0 \tag{4}$$

The first of these two equations describes the vertical motion of the mass. The second describes its rotational motion. The terms $\frac{1}{2}\epsilon\theta$ and $\frac{1}{2}\epsilon z$ couple the two motions, thereby making each a function of both variables.



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Assuming the system to be in a normal mode, the masses can oscillate with the same frequency and phase angle but with different amplitudes. From well known oscillator theory, the solutions to Equations (3) and (4) can be assumed to be

$$z(t) = A_1 \cos(\omega t + \phi). \tag{5}$$
$$\theta(t) = A_2 \cos(\omega t + \phi) \tag{6}$$

We need to take the first and second derivatives of these equations with respect to t so that we can then substitute the result into Equations (3) and (4) to solve for the amplitudes of oscillation.

$$\dot{z}(t) = -A_1 \omega \sin(\omega t + \phi)$$
$$\dot{\theta}(t) = -A_2 \omega \sin(\omega t + \phi)$$

$$\ddot{z}(t) = -A_1 \omega^2 \cos(\omega t + \phi)$$
$$\ddot{\theta}(t) = -A_2 \omega^2 \cos(\omega t + \phi)$$

Substituting these equations back into Equations (3) and (4), we get

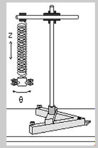
$$m(-A_1\omega^2 \cos(\omega t + \phi)) + kA_1 \cos(\omega t + \phi) + \frac{1}{2}\epsilon A_2 \cos(\omega t + \phi) = 0$$
$$I(-A_2\omega^2 \cos(\omega t + \phi)) + \delta A_2 \cos(\omega t + \phi) + \frac{1}{2}\epsilon A_1 \cos(\omega t + \phi) = 0.$$

Factoring out $\cos(\omega t + \phi)$ and dividing through by m, we get

$$\left(\frac{k}{m} - \omega^2\right) A_1 + \frac{\epsilon}{2m} A_2 = 0 \tag{7}$$
$$\frac{\epsilon}{2I} A_1 + \left(\frac{\delta}{I} - \omega^2\right) A_2 = 0. \tag{8}$$

We may set k/m and δ/I equal to ω_z^2 and ω_θ^2 respectively, because these are the squares of the natural frequencies of the vertical and rotational motions. Then grouping A_1 and A_2 terms yields a system for which we can solve A_1 and A_2,

$$(\omega_z^2 - \omega^2) A_1 + \frac{\epsilon}{2m} A_2 = 0 \tag{9}$$
$$\frac{\epsilon}{2I} A_1 + (\omega_\theta^2 - \omega^2) A_2 = 0. \tag{10}$$



The trivial solution to this system

$$\mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

produces a motionless system because the magnitude of the amplitudes are zero, so we need non-trivial solutions. To find non-trivial solutions of A_1 and A_2, the determinant of the coefficients of this system of linear equations must be equal to zero.

$$\begin{vmatrix} \omega_z^2 - \omega^2 & \frac{\epsilon}{2m} \\ \frac{\epsilon}{2I} & \omega_\theta^2 - \omega^2 \end{vmatrix} = 0.$$

Expanding the determinant and grouping like terms yields

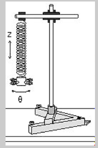
$$\omega^4 - (\omega_z^2 - \omega_\theta^2)\omega^2 + \left(\omega_z^2 \omega_\theta^2 - \frac{\epsilon^2}{4mI}\right) = 0.$$

Solving this equation for ω using a binomial expansion, we get the frequencies of the two normal modes.

$$\omega_1^2 = \frac{1}{2}\left\{\omega_\theta^2 + \omega_z^2 + \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\epsilon^2}{mI}}\right\}$$
$$\omega_2^2 = \frac{1}{2}\left\{\omega_\theta^2 + \omega_z^2 - \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\epsilon^2}{mI}}\right\}$$

Only when the frequency of oscillation of the mass is equal to the natural frequencies in z and θ can we observe the system to oscillate in a normal mode. Recall the two-mass system. The masses were only able to oscillate with the same frequency because the spring constants of the two outside springs were equal. Thus, when the natural frequency of the system in z equals the natural frequency of the system in θ, then we can make the substitution $\omega_\theta = \omega_z = \omega$, and the above equations can be reduced to

$$\omega_1^2 = \omega^2 + \frac{\epsilon}{\sqrt{mI}} \tag{11}$$
$$\omega_2^2 = \omega^2 - \frac{\epsilon}{\sqrt{mI}} \tag{12}$$



If we let $\omega_B = \epsilon/2\sqrt{mI}$. These equations become

$$\omega_1^2 = \omega^2 + \frac{\omega_B}{2} \tag{13}$$

$$\omega_2^2 = \omega^2 - \frac{\omega_B}{2} \tag{14}$$

Thus

$$\omega_B = \omega_1 - \omega_2,$$

where ω_B is the beat frequency produced by the interference of the two normal frequencies.

Now that we have the normal frequencies, we need to know the relation of the amplitudes of oscillation in z and θ. The ratio of those amplitudes is dependent on the frequency of the normal mode. Since we chose an ω that makes Equations (9) and (10) dependent, we only need to solve Equation (9) for the ratio of the amplitudes at the first normal frequency by substituting ω_1^2 for ω^2. Equation (9) becomes

$$(\omega_z^2 - \omega_1^2)A_1 + \frac{\epsilon}{2m}A_2 = 0,$$

but $\omega_1^2 = \omega^2 + \epsilon/\sqrt{4mI}$ from Equation (10) and $\omega_z^2 = \omega^2$. So the above equation becomes

$$-\frac{\epsilon}{\sqrt{4mI}}A_1 + \frac{\epsilon}{2m}A_2 = 0.$$

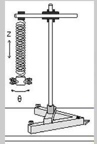
Solving for A_2/A_1 yields $\sqrt{m/I}$. Doing the same for the second normal frequency, we get the ratio at ω_2. It turns out that

$$\frac{A_2}{A_1} = r_1 = \sqrt{\frac{m}{I}} = -r_2,$$

where r_1 and r_2 are the ratios of the amplitudes at each normal frequency. Therefore, the amplitude vectors can be written as

$$\mathbf{A}^{(1)} = \begin{bmatrix} A_1^{(1)} \\ \\ A_2^{(1)} \end{bmatrix} = \begin{bmatrix} A_1^{(1)} \\ \\ r_1 A_1^{(1)} \end{bmatrix}$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} A_1^{(2)} \\ \\ A_2^{(2)} \end{bmatrix} = \begin{bmatrix} A_1^{(2)} \\ \\ r_2 A_1^{(2)} \end{bmatrix},$$



where superscript denotes the frequency at which the Amplitude was obtained. The solutions for the motions of system thus can be written as position vectors,

$$\mathbf{x}^{(1)} = \begin{bmatrix} z^{(1)}(t) \\ \\ \theta^{(1)}(t) \end{bmatrix} = \begin{bmatrix} A_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \\ r_1 A_1^{(1)} \cos(\omega_1 t + \phi_1) \end{bmatrix} = \text{first mode}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} z^{(2)}(t) \\ \\ \theta^{(2)}(t) \end{bmatrix} = \begin{bmatrix} A_1^{(2)} \cos(\omega_2 t + \phi_2) \\ \\ r_2 A_1^{(2)} \cos(\omega_2 t + \phi_2) \end{bmatrix} = \text{second mode}$$

Again we remind you that any motion of the pendulum can be written as a linear combination of its normal modes. Thus,

$$z(t) = z^{(1)}(t) + z^{(2)}(t) = A_1^{(1)} \cos(\omega_1 t + \phi_1) + A_1^{(2)} \cos(\omega_2 t + \phi_2) \tag{15}$$

$$\theta(t) = \theta^{(1)}(t) + \theta^{(2)}(t) = A_2^{(1)} r_1 \cos(\omega_1 t + \phi_1) + A_2^{(2)} r_2 \cos(\omega_2 t + \phi_2) \tag{16}$$

The initial conditions that we give our pendulum are a twist and a displacement in the z direction, but no initial velocity in either direction. Therefore, the initial conditions are

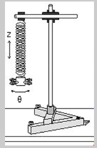
$$\begin{aligned} z(0) &= z_0 & \dot{z}(0) &= 0 \\ \theta(0) &= \theta_0 & \dot{\theta}(0) &= 0. \end{aligned}$$

Substituting these initial conditions into Equations (15) and (16), the equations that we will use to determine the amplitudes $A_1^{(1)}$ and $A_1^{(2)}$ and the phase angles ϕ_1 and ϕ_2 are

$$\begin{aligned} z_0 &= A_1^{(1)} \cos\phi_1 + A_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 A_1^{(1)} \sin\phi_1 - \omega_2 A_1^{(2)} \sin\phi_2 \\ \theta_0 &= r_1 A_1^{(1)} \cos\phi_1 + r_2 A_1^{(2)} \cos\phi_2 \\ 0 &= -r_1 \omega_1 A_1^{(1)} \sin\phi_1 - r_2 \omega_2 A_1^{(2)} \sin\phi_2. \end{aligned}$$

Solving this system for $A_1^{(1)}, A_1^{(2)}, \phi_1$, and ϕ_2

$$A_1^{(1)} = \frac{r_1\theta_0 + z_0}{r_1 - r_2}, \qquad A_1^{(2)} = \frac{r_1\theta_0 - z_0}{r_1 - r_2}, \qquad \phi_1 = \phi_2 = 0.$$



Substituting $\sqrt{m/I}$ for r_1, and $-\sqrt{m/I}$ for r_2, the general solution of the motion of the Wilberforce pendulum is

$$z(t) = \frac{\sqrt{\frac{m}{I}}\theta_0 + z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_1 t + \frac{\sqrt{\frac{m}{I}}\theta_0 - z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_2 t$$

$$\theta(t) = \sqrt{\frac{m}{I}}\frac{\sqrt{\frac{m}{I}}\theta_0 + z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_1 t - \sqrt{\frac{m}{I}}\frac{\sqrt{\frac{m}{I}}\theta_0 - z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_2 t.$$

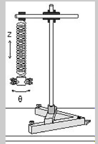
5. Conclusion

Let us look at the solution of the system of differential equations, Equations (3) and (4), plotted using Matlab's built-in ode15s m-file, so as to have something to compare our solutions to (see Figure 6). Our constants are set to those measured by Berg and Marshal [1, p.35], and tabulated in the table below.

Inertia (I)	$1.45E^{-4}$ Nm^2
Mass (m)	0.5164 kg
Epsilon (ϵ)	$9.28E^{-3}$ N
Torsional spring constant (δ)	$7.86E^{-4}$ Nm^2
Longitudinal spring constant (k)	2.69 N/m

Now let us plot our solutions using the same constants. See Figure 7 for a plot of our solutions versus time.

The first thing we notice is beating occurring, showing the complete transfer of kinetic energy from z to θ and back. The next thing we notice is that the displacement of the mass in the vertical direction is disproportional to the displacement of the mass' rotation. So the ratio of the amplitudes appear not to match our previous computations, but intuitively this is to be expected because of the small displacement in rotation that the mass experiences in relation to its large vertical displacement. To remedy this discrepancy, we need to put the displacement of rotation into units of the z displacement. If we change the θ displacement into arc length (s) by the relationship $s = r\theta$, we can obtain a plot



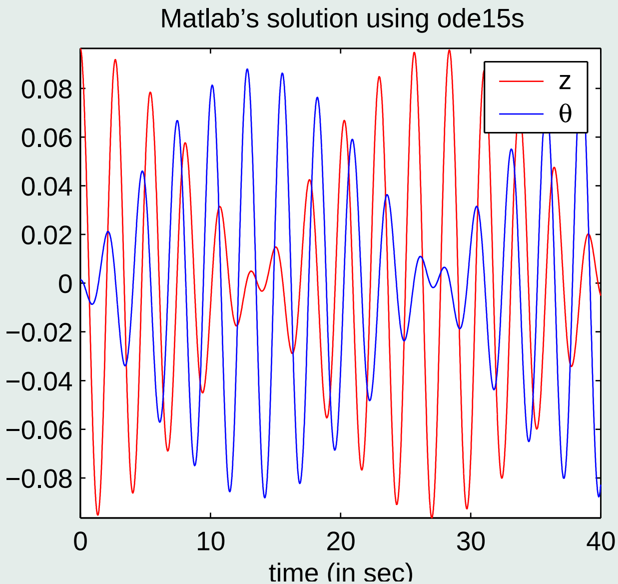


Figure 6: $z(t)$ and $\theta(t)$ vs. t

that represents the correct amplitude relationships (see Figure 8). Excitingly enough we can see that the solution that we obtained is in perfect agreement with the solution that Matlab's solver obtained. These solutions cover a region in the $z\theta$-space, that has boundaries defined by the initial conditions (see Figure 9). Every other solution to this system lies within this space.

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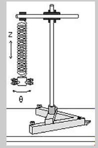

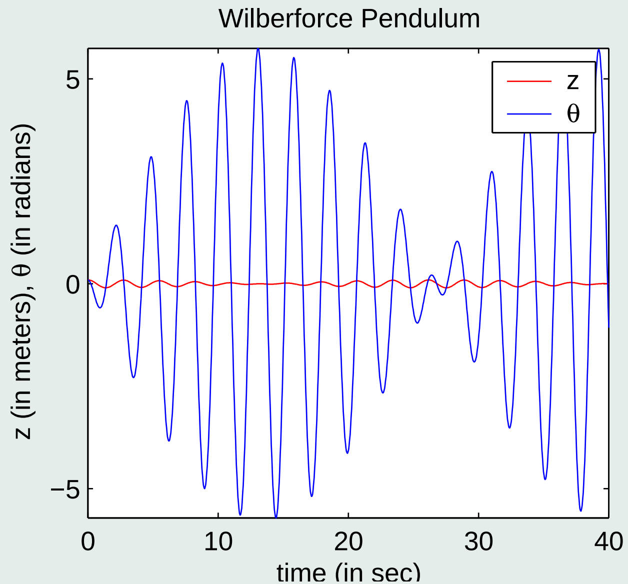

Figure 7: Plot of $z(t)$ v. time and $\theta(t)$ v. time

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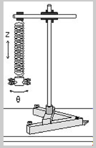


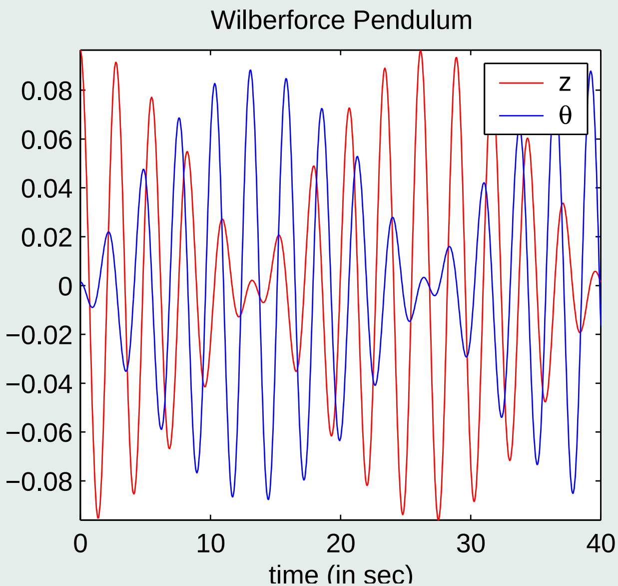
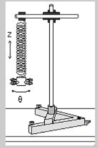


Figure 8: Plot of $z(t)$ v. time and $\theta(t)$ v. time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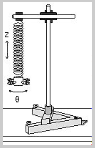

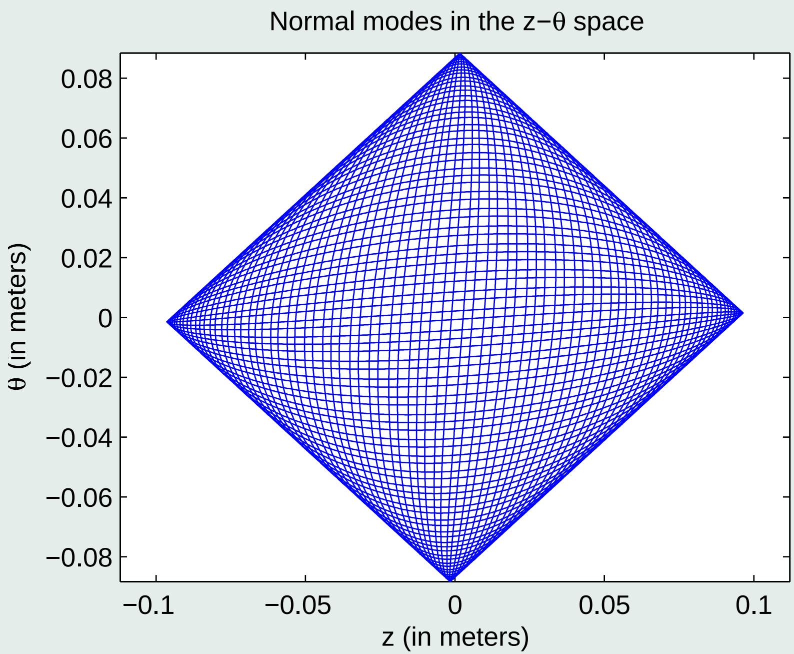

Figure 9: The Space Containing the Normal Modes