Rhythm of the Heart

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Abstract

An overlook of the heart and how it works as a pump with an explanation of how to build a mathematical model using differential equations.

1 The Heart

1.1 The Heart Beat

The heart beat is the only tangible, external sign we have that tells us our heart is beating in a steady rhythm all of the time. Each beat represents a pump of blood through the heart and out to other parts of the body. This process is driven by a patch of cells, in the upper part of the heart, called the Sinoatrial Node. These cells send waves of electrochemical activity over the heart which causes it to contract and relax over and over again.

1.2 Blood Flow

The blood flow is continuous and a cycle like the heart beat cycle. If we start where unoxygenated, blue blood enters the heart, we are at the vena cava. Blood goes through this to enter the right atrium, and then into the right ventricle, and then pumped out to the lungs through the pulmonary arteries. The lungs exchange oxygen for carbon dioxide and then the blood is sent back to the heart through the pulmonary veins. These veins take the blood to the left atrium, through to the left ventricle and pumped out of the heart to the rest of the body through the aorta. The pump needed to send blood to the lungs is weaker than the pump used to send blood to the rest of the body. Therefore, the right side of the heart is a low-pressure pump, while the left side of the heart is a high-pressure pump.

1.3 Equilibrium States

The heart has two different states of being where it is in equilibrium. Diastole is the relaxed state of the heart. It is a stable equilibrium point because the heart always returns to diastole, even after it stops beating. Systole is the contracted state.

1.4 Heart Beat Cycle

A logical place to start would be diastole, the stable equilibrium. The pacemaker sends a wave of electrochemical activity over the heart. First it goes over the atria, causing them to slowly contract. This is the first light pump that fills up the ventricles with blood. This is actually the atrium's job; to fill up the ventricle before the strong pump, so to get the most efficient amount of blood out to the rest of the body. There is a point where the slow wave of electricity changes to a rapid wave. This goes over the ventricles, causing them to quickly contract, which in turn pumps blood out of the heart. The point of the contraction is the systole equilibrium. After that, there is a rapid relaxation back to diastole. The cycle repeats itself, slowing down or speeding up in response to external or internal conditions, such as sleeping or running, anxiety or relaxation.

2 Derivation of the Differential Equations

2.1 Stipulations

1. The model must have an equilibrium point at diastole and at systole.
2. The model needs to show a threshold for triggering the electrochemical activity into changing from slow to rapid.
3. The model should reflect a rapid return to diastole equilibrium after reaching systole.

2.2 Variables

- x represents the length of muscle fiber.
- b represents the electrochemical activity.
- T represents the over-all tension of the system. This is also the main parameter for the system of differential equations.

2.3 Hypothesis

1. The rate of change of muscle fiber length, into and out of contraction, depends on tension, T, and electrochemical activity, b.
2. The chemical control, b, changes at a rate directly proportional to muscle fiber length, x.

2.4 Where to Begin?

The system of differential equations is

$$x' = f(x, b)$$
$$b' = g(x, b)$$

It has already been determined that diastole is an equilibrium point, so it will be denoted as (x_0, b_0). Systole will be denoted by the point (x_1, b_1). When the system is in diastole

$$f(x_0, b_0) = g(x_0, b_0) = 0.$$

If we linearize about the equilibrium point we will end up with these equations.

$$x' = a_{11}\ (x - x_0) + a_{12}\ (b - b_0)$$
$$b' = a_{21}\ (x - x_0) + a_{22}\ (b - b_0)$$

Higher order terms have been omitted because they are negligibly small near the equilibrium. In this equation a_{11} is the partial derivative of f with respect to x, a_{12} is the partial derivative of f with respect to b, a_{21} is the partial derivative of g with respect to x, and a_{22} is the partial derivative of g with respect to b. The Jacobian matrix is

$$J(x, b) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Diastole is a stable equilibrium point, therefore, the eigenvalues of matrix J need to be real, to avoid oscillatory behavior, and negative, so it acts like a sink. We also want to reflect in the equations hypotenuse 2, which says that b changes at a rate directly proportional to x. We can therefore say that $a_{22} = 0$ and $a_{21} = 1$. Now for the eigenvalues to be both real and negative, a_{11} and a_{12} must both be negative. The model needs to also show a rapid return to equilibrium, which can be accomplished by making a_{11} large and negative. a_{12} also needs to be large and negative, with a_{11} being larger relative to a_{12}. To make things easier to understand and write. I am going to introduce two new variables to help represent a_{11} and a_{12}.

$$a_{11} = \frac{-a}{\varepsilon} \qquad a_{12} = \frac{-1}{\varepsilon}$$

a and ε are both positive constants, with epsilon being small. Now the system can be written as

$$\varepsilon\ x' = -a(x - x_0) - (b - b_0)$$
$$b' = (x - x_0).$$

This system is a good model around the diastole equilibrium but it is a bad approximation away from this point. The "local" model does not reflect the triggering of electrochemical activity from slow to rapid. Researchers, like E.C. Zeeman, found a modified equation that satisfies the necessary triggering, through trial and error.

$$\begin{aligned}\varepsilon x' &= -a(x-x_0)-(b-b_0)-(x-x_0)^3-3x_0(x-x_0)^2\\ b' &= x-x_0\end{aligned}$$

I was able to simplify the equation for the derivative of x with these computations.

$$\begin{aligned}\varepsilon x' &= -[a(x-x_0)+(b-b_0)+(x-x_0)^3+3x_0(x-x_0)^2]\\ &= -[ax-ax_0+b-b_0+x^3-3x^2x_0+3xx_0^2-x_0^3+3x_0(x^2-2xx_0+x_0^2)]\\ &= -[x^3-3x^2x_0+3x^2-x_0+3xx_0^2-6xx_0^2+3x_0^3-x_0^3+ax-ax_0+b-b_0]\\ &= -[x^3-3xx_0^2+2x_0^3+ax-ax_0+b-b_0]\\ &= -[x^3-x(3x_0^2+a)+b+(2x_0^3-ax_0-b_0)]\end{aligned}$$

At this point it is good to define tension to be

$$T = 3x_0^2+a$$

and

$$2x_0^3-ax_0-b_0=0$$

so

$$b_0 = 2x_0^3-ax_0.$$

These equations are also true if we substitute the diastole equilibrium point for the systole equilibrium (x_1, b_1). The system is more simply written as

$$\varepsilon x' = -(x^3-Tx+b) \qquad (1)$$

$$b' = x-x_0. \qquad (2)$$

The system is a very good model of the heart beat except that it acts funny around systole equilibrium. The flow, in the phase plane, is going the wrong direction around systole. This problem occurs because the system was made to represent the biological phenomena around one equilibrium point, diastole. The model needs to be told when to change and what to change to when it is near systole. The problem in the direction of the flow is that it is going right instead of left, with respect to the b-axis. There is no problem with the up or down flow, therefore the only equation that needs to be changed is b'. When the flow is near systole the equation of b-prime needs to change to $b' = x - x_1$. If we incorporate a step function U.

$$b' = (x-x_0)+U(x_0-x_1)$$

When U is equal to one, x_0 cancels out and the equation becomes

$$b' = x-x_1.$$

The b-values where U is equal to one is when b is between b_0 and b_1, and also at the same time $x^3 - Tx + b > 0$. Actually, this just means that $x' < 0$ or the length of the muscle fiber is decreasing or otherwise put,contracting. If these two aren't true at the same time then U can also equal one if $b > b_1$, otherwise put, the amount of electricity is larger than that needed to be in systole. The step function is written

$$U = (b > b_1) + [(b > b_0) \cdot (b < b_1) \cdot (x^3 - Tx + b > 0)]$$

If the term inside the parenthesis is true, that term equals one and the other is zero, and $U = 1$ so $b' = x - x_1$. If the terms inside the brackets are all true that term equals one, the other term is zero, $U = 1$ and $b' = x - x_1$. At any other time $U = 0$ and $b' = x - x_0$. Now we have the complete system of equations.

$$\varepsilon x' = -(x^3 - Tx + b) \tag{3}$$

$$b' = (x - x_0) + U(x - x_1) \tag{4}$$

3 Modeling Beats

To model this system I used Matlab's ODE suite and found that ode23s, the stiff solver, worked best. The ode function file I wrote went like this,

```
function yprime=heart(t,y,flag,e,T,b0,x0,b1,x1)
yprime=zeros(2,1);
U=(y(1)>b1)+((y(1)>=b0)&(y(1)<=b1)&(y(2)^3-T*y(2)+y(1)>0));
yprime(1)=(y(2)-x0)+U*(x0-x1);
yprime(2)=(-1/e)*(y(2)^3-T*y(2)+y(1));.
```

To generate graphs of the model I wrote this small program,

```
options=odeset('refine',8);
[t,y]=ode23s('heart',[0,10],[0;0],options,e,T,b_0,x_0,b_1,x_1);
ax1=axes('position',[0.1,0.4,0.8,0.5]);
plot(y(:,1),y(:,2)),xlabel('b'),ylabel('x')
ax2=axes('position',[0.1,0.1,0.35,0.2]);
plot(t,y(:,1)),xlabel('t'),ylabel('b')
ax3=axes('position',[0.55,0.1,0.35,0.2]);
plot(t,y(:,2)),xlabel('t'),ylabel('x')
```

The first thing I did in the program, was to set the refinement of the numerical approximator higher than normal. This gave me a more accurate approximation. The axes command before each plot command is how I positioned the graphs of b versus x, t versus b, and t versus x. When graphing this phase plane, it is desirable to graph the electrochemical activity on the independent axes , because it better represents the flow from diastole to systole and back to diastole. This also affects how I wrote the ode function file. The first equation of the system is b'.

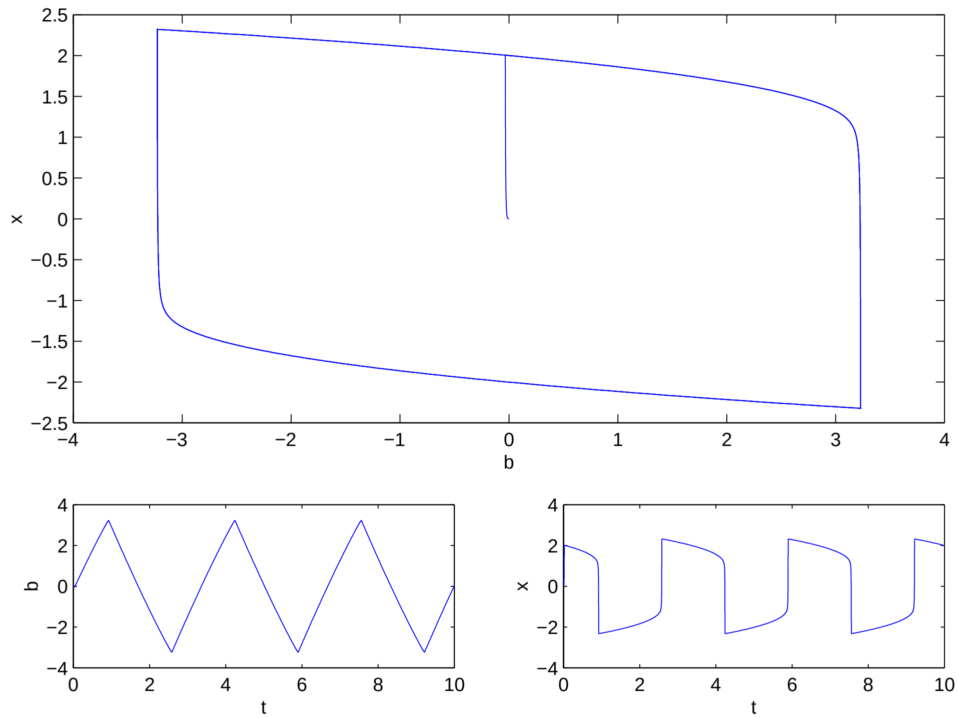

Figure 1: Tension equals four

A problem with the model is that it is hard to prove that there will be a closed trajectory, or limit cycle, that the graph goes to. If the tension is set too high, the heart will never be able to make it out of diastole, but if it is too low, the contraction into systole will be slow, and not give a hard enough pump to send blood to all parts of the body. A model with good results is presented in Figure 1 where I set $T = 4$ and (b_0, x_0) as $(-4, 2)$, and (b_1, x_1) as $(4, -2)$.

The solution to the system in Figure 2 has an initial condition of $(0, 0)$ If we change the initial condition to something outside of the trajectory cycle,like $(5, 1)$, it will still approach the limit cycle.

If I set the tension too high, $T = 50$, the system will not be able to leave diastole far enough to reach systole. Figure 3 shows this situation with an initial condition at $(0, 0)$.

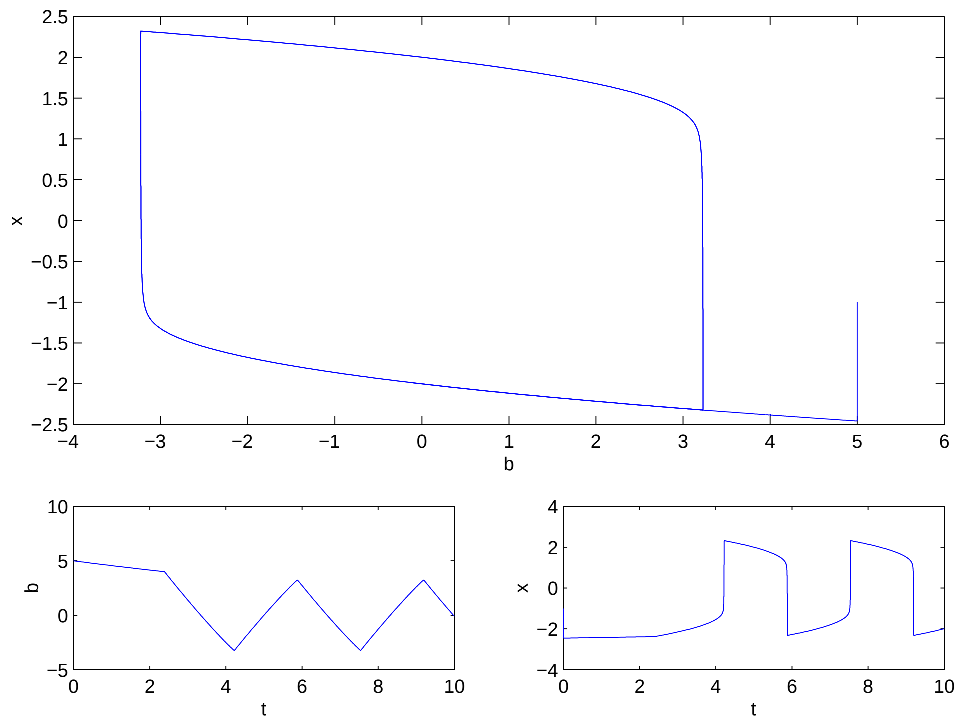

Figure 2: Initial Condition at $(5, 1)$

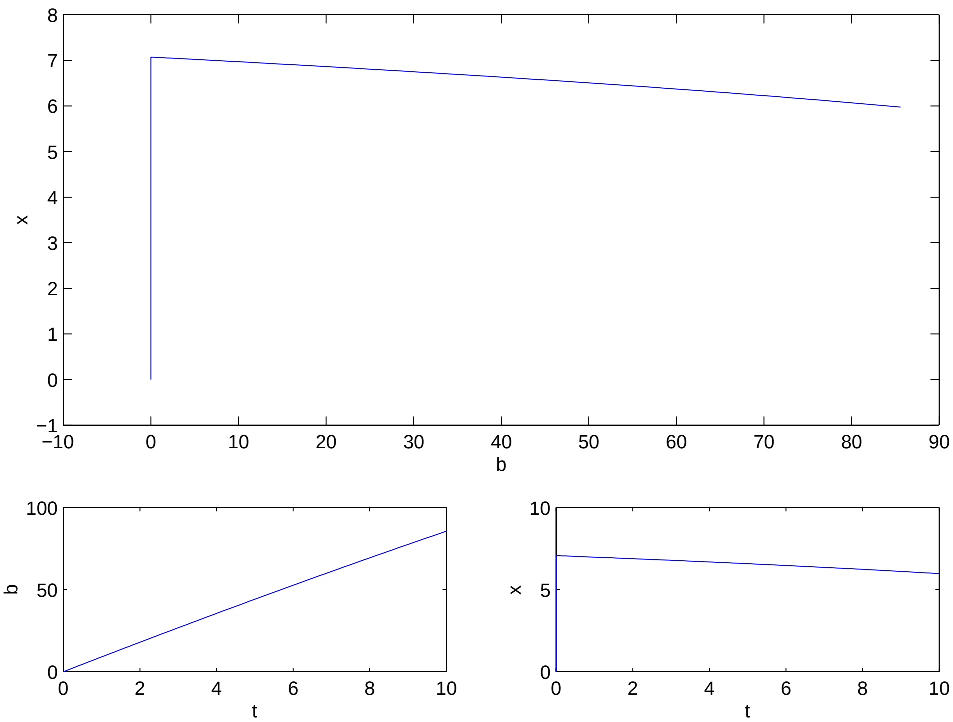

Figure 3: Tension is equal to 50

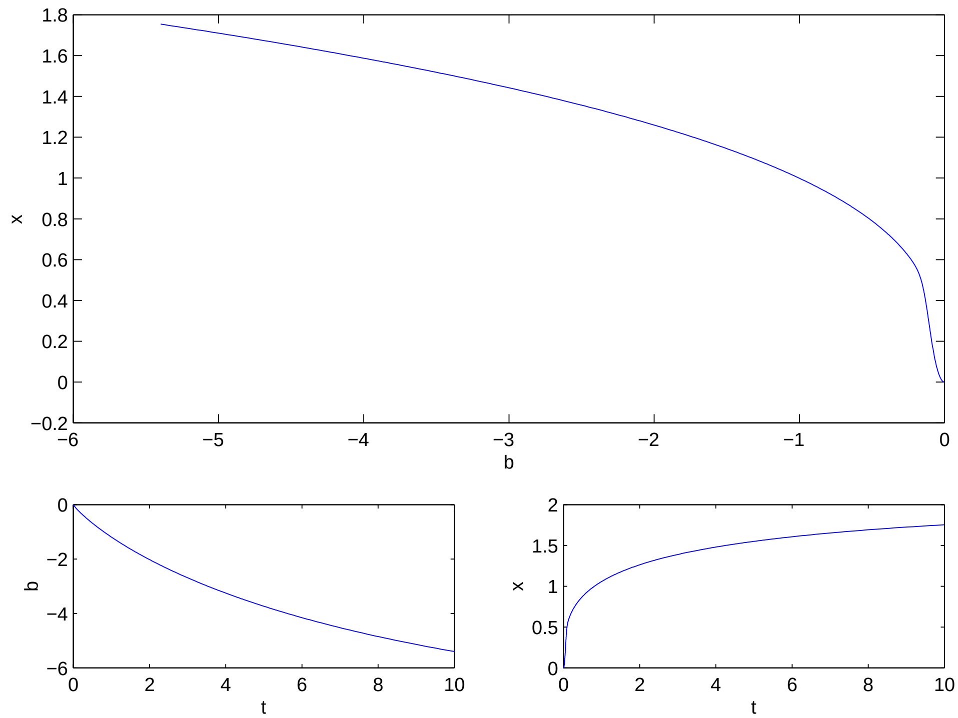

Figure 4: Tension equals zero

Another model to look at is when the tension is too low. Figure 4 has the tension set as $T = 0$. In this scenario the tension is so low it reaches systole but cannot leave it.

3.1 Conclusion

This is a good beginning for a model of the heart beat . Also, the processes involved are good training for modeling other biological phenomena in the future. There is one problem that I have noticed about this model. The heart consists of two pumps. The right hand side is a low-pressure pump that sends blood to the lungs. The high-pressure pump is on the left-side of the heart and it sends blood to the rest of the body. These two pumps do not beat simultaneously but one right after the other, and together they make one beat of the heart. The model does not represent two pumps of differing strengths. This is why the graph of the electro-chemical activity versus time has the same amplitude for every wave of electricity. It would look more like an electrocardiagram if the two pumps were both represented in the model. Over time and with research a better model will be found, hopefully I'll have something to do with it.

References

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