Application 5.3
Approximate Solutions of Linear Equations

Section 5.3 in the text shows that the problem of solving a homogeneous linear differential equation reduces to solving its characteristic (polynomial) equation. For this and similar purposes, polynomial-solving utilities are now a common feature of calculator and computer systems, and can be used to solve a characteristic equation numerically even when no simple and explicit factorization is evident or even possible. For instance, suppose that we want to solve the homogeneous linear differential equation

$$y''' - 3y'' + y = 0 \tag{1}$$

with characteristic equation

$$r^3 - 3r^2 + 1 = 0. \tag{2}$$

A computer algebra system provides the three solutions of this cubic equation in the form

$$r_1 = 1 + u + \frac{1}{u}, \quad r_2 = 1 - u^2 - uv^3, \quad r_3 = 1 - u^4 - \frac{v^3}{u}$$

where

$$u = \sqrt[3]{\frac{1+i\sqrt{3}}{2}} \quad \text{and} \quad v = \sqrt[3]{\frac{1-i\sqrt{3}}{2}}$$

involving cube roots of complex numbers. For instance,

$$r_1 = 1 + \sqrt[3]{\frac{1+i\sqrt{3}}{2}} + \sqrt[3]{\frac{2}{1+i\sqrt{3}}}.$$

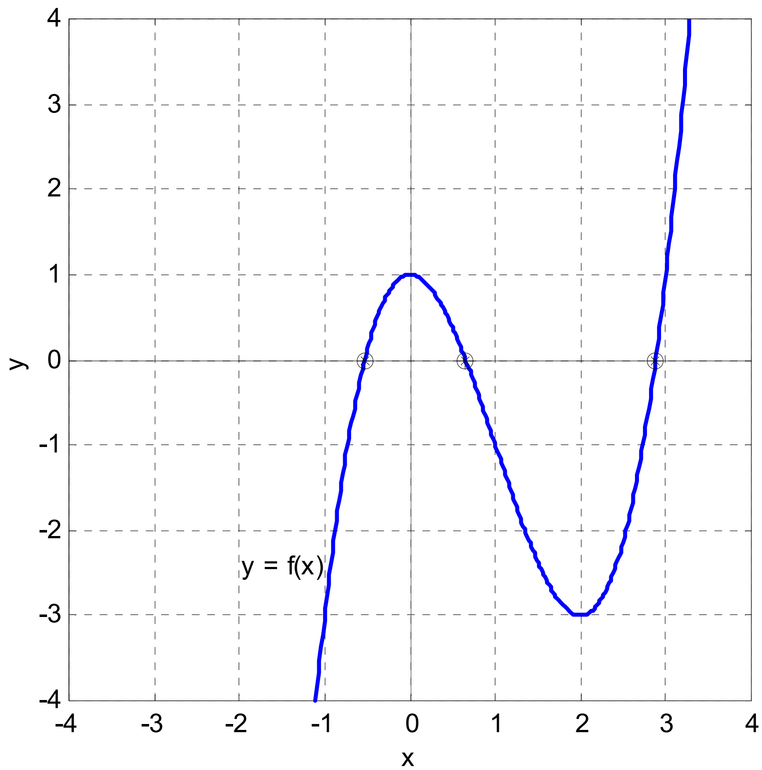
One could write a general solution of (1) in the form

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}. \tag{3}$$

However, this general solution would seem needlessly complex, inasmuch as the graph of the function $f(x) = x^3 - 3x^2 + 1$ shows that all three roots of (2) actually are real numbers!

Indeed, with a calculator (like the TI-89) having a built-in polynomial solver, we can simply enter the coefficients $1, -3, 0, 1$ of this cubic polynomial and get the three (approximate) roots $r = -0.5321, 0.6527, 2.8794$ at the press of a key. Computer systems like *Maple*, *Mathematica*, and MATLAB also have built-in polynomial solvers that can provide such numerical solutions.

5.



However we find these numerical roots of (2), it follows by substitution in (3) that a general solution of the differential equation in (1) is given (approximately) by

$$y(x) \;=\; c_1 e^{-0.5321x} + c_2 e^{0.6527x} + c_3 e^{2.8794x}. \tag{4}$$

Computer algebra systems also offer simple DE solver commands for the explicit solution of differential equations. It is interesting to compare the symbolic solutions produced by such DE solvers with explicit numerical solutions of the form in (4).

Use calculator or computer methods to find general solutions (in approximate numerical form) of the following differential equations. Compare the results obtained using a polynomial solver and using a DE solver.

1. $y''' - 3y' + y \;=\; 0$

2. $y''' + 3y'' - 3y \;=\; 0$

3. $y''' + y' + y \;=\; 0$

4. $y''' + 3y' + 5y \;=\; 0$

5. $y^{(4)} + 2y''' - 3y = 0$

6. $y^{(4)} + 3y' - 4y = 0$

Using *Maple*

The characteristic equation in (2) is defined by

```
charEq := r^3 - 3*r^2 + 1 = 0:
```

The command

```
soln := solve(charEq, r);
```

yields complex "Cardan formula expressions" like those exhibited previously, but the floating point evaluation

```
soln := [evalf(soln,5)];
```

$$soln := [2.8794+.00001\,I,\ -.53209+.00002\,I,\ .65271-.00002\,I]$$

after deletion of the imaginary round-off errors,

```
soln := map(Re,soln);
```

$$soln := [2.8794,\ -.53209,\ .65271]$$

gives the three approximate characteristic roots mentioned before. We can now assemble the approximate solution

```
y = sum(c[i]*exp(soln[i]*x),i=1..3);
```

$$y = c_1\,e^{(2.8794\,x)} + c_2\,e^{(-.53209\,x)} + c_3\,e^{(.65271\,x)}$$

Alternatively, we can first define the differential equation in (1) by entering the command

```
diffEq := diff(y(x),x$3)-3*diff(y(x),x$2)+y(x) = 0;
```

$$diffeq := \left(\frac{\partial^3}{\partial^3 x}\,y(x)\right) - 3\left(\frac{\partial^2}{\partial^2 x}\,y(x)\right) + y(x) = 0$$

and then ask for its exact symbolic solution:

```
soln := dsolve( diffEq, y(x) ):
```

This gives the complicated form with the complex exponents referred to in Eq. (3), but the floating point evaluation

```
soln := evalf(soln);
```

gives the approximate solution obtained above (though with the imaginary round-off errors still visible).

Using *Mathematica*

The characteristic equation in (2) is defined by

```
charEq =   x^3 - 3x^2 + 1 == 0;
```

The polynomial solve command

```
Solve[ charEq, r]
```

yields the complex "Cardan formula expressions" for the three roots exhibited previously, but the numerical solve command

```
soln = NSolve[ charEq, r ]
```

$\{\{r \to -0.532089\}, \{r \to 0.652704\}, \{r \to 2.87939\}\}$

```
roots = r /.{{r->0.532089},{r->0.652704},{r->2.87939}}
```

$\{-0.532089,\ 0.652704, 2.87939\}$

gives the approximate roots of the characteristic equation (2) mentioned earlier. We can now assemble an approximate solution of the differential equation in (1):

```
c = {c1,c2,c3};
y == Apply[Plus, c Exp[roots x]]
```

$$y == c_1\, e^{-0.532089\, x} + c_2\, e^{0.652704\, x} + c_3\, e^{2.97939\, x}$$

Alternatively, we can first define the differential equation in (1) by entering the command

```
diffEq = y'''[x] - 3 y''[x] + y[x] == 0
```

and then ask for its solution by means of the DE solve command

```
DSolve[diffEq, y[x], x]
```

$$\left\{\left\{y(x) \to c_1\, e^{x\,\mathrm{Root}(\#1^3 - 3\#1^2 + 1\&,1)} + c_2\, e^{x\,\mathrm{Root}(\#1^3 - 3\#1^2 + 1\&,2)} + c_3\, e^{x\,\mathrm{Root}(\#1^3 - 3\#1^2 + 1\&,3)}\right\}\right\}$$

This expresses the general solution in terms of the three symbolic roots of the characteristic equation, but the numerical evaluation

```
y[x] /. N[soln] // First
```

$$2.71828^{-0.532089x}c_1+2.71828^{0.652704x}c_2+2.71828^{2.97939x}c_3$$

of the result gives the approximate numeric form (4) of the solution — provided that we recognize the three appearances of the exponential base $e \approx 2.71828$.

Using MATLAB

We can work either in a purely numeric mode or in a symbolic mode. For a numerical approach, the characteristic polynomial in (2) is defined by the vector **`[1 -3 0 1]`** listing its coefficients in order of descending powers. Then the command

```
roots([1 -3 0 1])
ans =

    2.8794
    0.6527
   -0.5321
```

yields the three approximate characteristic roots that appear in the approximate general solution (4). The symbolic command

```
soln = solve('x^3 - 3*x^2 + 1 = 0')
```

yields "Cardan formula expressions" for these three roots similar to symbolic expressions exhibited originally. But then the command

```
numeric(soln)
ans =
   2.8794 + 0.0000i
  -0.5321 + 0.0000i
   0.6527 - 0.0000i
```

reproduces the approximate roots obtained previously. With a bit of work we can assemble the corresponding approximate solution:

```
syms C1 C2 C3 x
y1 = ['C1*e^',num2str(real(numeric(soln(1)))),'x'];
y2 = ['C2*e^',num2str(real(numeric(soln(2)))),'x'];
y3 = ['C3*e^',num2str(real(numeric(soln(3)))),'x'];
soln = [y1,'+',y2,'+',y3]
```

```
soln =
C1*e^2.8794x+C2*e^-0.53209x+C3*e^0.6527x
```

Alternatively, we can first define the differential equation in (1) by entering the symbolic command

```
diffeq = 'D3y - 3*D2y + y = 0'
diffeq =
D3y - 3*D2y + y = 0
```

and then ask for its solution by means of the command

```
y = dsolve(diffeq)
```

We can extract three independent particular solutions y_1, y_2, and y_3 from the resulting symbolic general solution by isolating the coefficients of the arbitrary constants C_1, C_2, and C_3:

```
y1 = subs(y,{C1,C2,C3},{1,0,0})
y1 =
exp((-299539413723655/562949953421312+
                    1/9007199254740992*i)*x)

y2 = subs(y,{C1,C2,C3},{0,1,0})
y2 =
exp((6483799150499469/2251799813685248-
                    1/9007199254740992*i)*x)

y3 = subs(y,{C1,C2,C3},{0,0,1})
y3 =
exp((1469757945450895/2251799813685248-
                    1/9007199254740992*i)*x)
```

Finally, we cut/paste/evaluate the rational fractions that appear as real parts in the exponents here, so as to verify that they correspond to the three approximate characteristic roots found previously.

```
-299539413723655/562949953421312
ans =
   -0.5321

6483799150499469/2251799813685248
ans =
    2.8794

1469757945450895/2251799813685248
ans =
    0.6527
```