Lake Pollution Model

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Abstract

Using simple mixture problem techniques to derive a differentiable equation to model the pollution concentration of a lake.

1. Introduction

According to "Differential Equations", by Blanchard, Devaney, & Hall, mixture problems refer to a large collection of different problems where two or more substances are mixed together at various rates. Mixture problems range from mixing

chemicals in a vat to the diffusion of cigarette smoke in the air of a room. We did our model on the mixing of pollutants in a lake.

2. Mixing Equation

We start by using the basic mixture model,

$$\frac{\text{mass}}{\text{time}} = \text{rate in - rate out,}$$

and a list of variables:

$$
\begin{aligned}
C &= \text{concentration of the contaminant,} \\
Q &= \text{the volumetric flow rate through the lake,} \\
V &= \text{the volume of the lake,} \\
M &= \text{the mass of the contaminant,}
\end{aligned}
$$

where mass is defined by

$$M = C * V.$$

The rates are defined as

$$\frac{dM}{dt} = C * Q,$$

where

$$
\begin{aligned}
C &= \frac{\text{mass}}{\text{volume}}, \\
Q &= \frac{\text{volume}}{\text{time}},
\end{aligned}
$$

leaving

$$\frac{dM}{dt} = \frac{\text{mass}}{\text{time}}.$$

The mixture equation then becomes;

$$\frac{dM}{dt} = C_{in}\left(t\right) * Q_{in}\left(t\right) - C_{out}\left(t\right) * Q_{out}\left(t\right)$$

2.1. The k variable

Pollutants come in various chemical properties. Some pollutants are non-polar and do not react with water. This leaves the pollutant only one way to leave, through the outflow. Some pollutants are polar and react with water, causing either more pollutant, or less pollutant. k can be defined as:

$$k = 0; \text{ where there is no reaction}$$

We will not consider the case where pollutant is created, so k will always be non-negative.

$$k > 0; \text{ where contaminant is not generated}$$

3. The Lake Pollution Model

k is also used in the chemistry world and determines the order of the reaction (in this case 0th and 1st order reactions). Thus another term is added to our differential model, and it becomes:

$$\frac{dM}{dt} = C_{in}(t) * Q_{in(t)} - C_{out}(t) * Q_{out}(t) - k(t) * C(t) * V(t)$$

3.1. Assumptions

Assumptions are made for the model, to make it easier to apply. Without assuming too much and drastically affecting the accuracy of the model these assumptions were made:

1. The volume of the lake remains constant
2. The flow rate remains constant
3. The reaction rate remains constant
4. The lake is well mixed

Lakes do not usually fluctuate in volume over a short period of time, so assumption #1 seems plausible. In assuming that the volume is constant, the flow rate must also remain constant; therefore, assumption #2 is plausible. Assumption #3 is made to make the model easier to use but limited to only zeroth and first order reactions. Assuming the lake is well mixed means that the concentration of the pollutant inside the lake is the same as the concentration of the outflow. This assumption is also to make the model easier to manage and is also limited to its use.

3.2. Revamped Model

The lake pollution model is modified with these assumptions to become:

$$\frac{dM}{dt} = Q * C_{in}(t) - Q * C(t) - k * C(t) * V,$$

or,

$$V * \frac{dC(t)}{dt} = Q * C_{in}(t) - Q * C(t) - k * C(t) * V,$$

since mass is equal to volume times concentration. Now dividing the equation by the volume will give a nice robust equation.

$$\frac{dC(t)}{dt} = \frac{Q}{V} * C_{in}(t) - \frac{Q}{V} * C(t) - k * C(t)$$

Dividing the volume by the flow rate gives units of time. This also gives how long it would take to fill the lake, if there was no outflow, and how long to drain the lake if there was only outflow. The $\frac{V}{Q}$ term stands for *residence time*, and the symbol θ_h is used. The equation becomes:

$$\frac{dC(t)}{dt} = \frac{1}{\theta_h} * C_{in}(t) - \frac{1}{\theta_h} C(t) - k * C(t)$$

4. Input Pollution Models

There are three different types of input models used to monitor pollutants in a lake. These input models allow the pollution equation to be more accurate and respond more like the real world. The input models are the *impulse*, *step*, and *sinusoidal* input.

4.1. Impulse Input

The impulse input model is used for pollutants that have been released into the lake immediately. Impulse input functions have a spike, and everywhere else the function is zero. The spike represents the time at which the contaminant was dumped. Assuming the impulse is always at time zero, the model shows the contamination level after some time has passed. The significant points are that the lake starts with an initial concentration of pollutant, and that no pollutant enters the lake once the model has started.
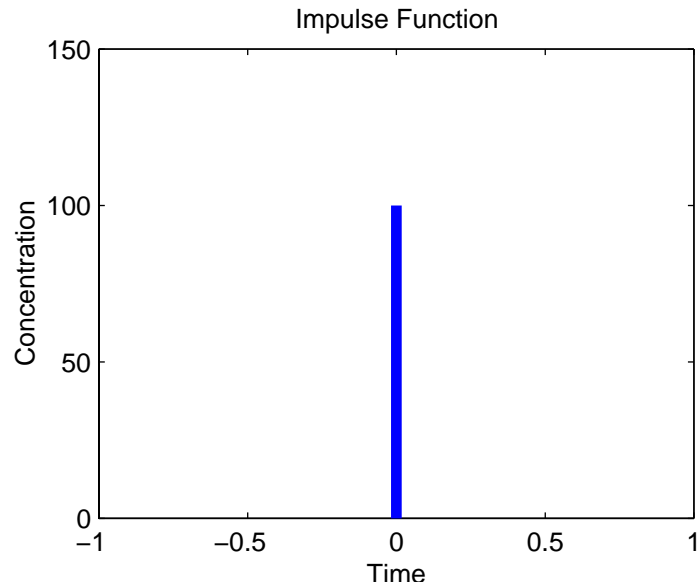
Figure 4.1: Impulse Input Graph.

The C_{in} term versus time in an impulse function is similar to Figure 4.1:
Using Matlab to graph the model, use a function m-file with the `dfield5` direction field routine to produce Figure 4.2:

```
function Cin=concentration(t)
Cin=100*(t<=10)+0*(t>10);
```

An example would be dumping a 55 gallon barrel of gasoline into a lake at time zero. One short heavy dumping of waste, with no reaction with the lake; therefore, C_{in} is equal to 100 units for a duration to 10 units of time. The contaminant is "washed away" after t units of time, with a k value of zero and *residence time* of 50.

4.2. Step Input

The step input model is used for pollutants that enter the lake at a steady concentration and rate and continues the same way indefinitely. The pollutant enters the system at time zero, and the concentration previous to time zero was nothing. The significant points for the step input are that the input increases suddenly at
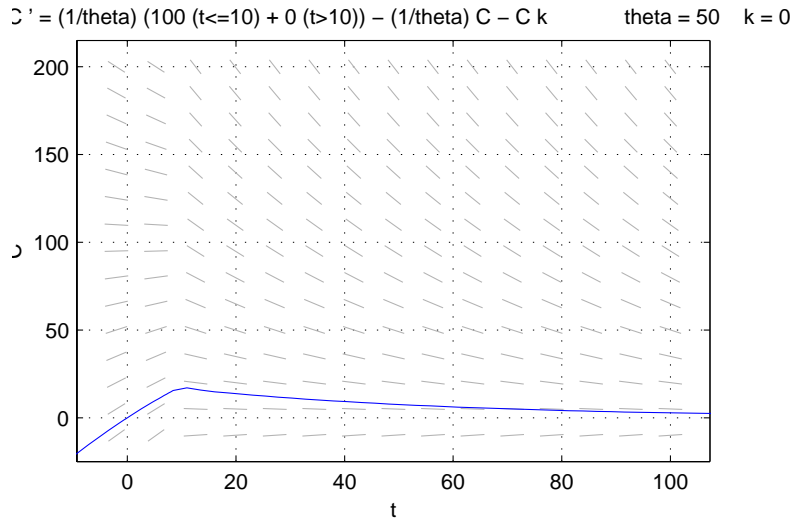
Figure 4.2: Impulse Input Model.

time zero and that the input remains essentially constant after time zero. C_{in} versus t graph is represented by the graph in Figure 4.3.

Use approximately the same procedure as the impulse input to obtain a graph in Matlab see Figure 4.4. Starting with writing the m-file and then using the function in `dfield5`.

```
function Cin=stepconc(t)
Cin=0*(t<0)+100*(t>=0);
```

An example would be when a manufacturing plant starts production at time zero and starts dumping raw sewage at a constant rate and concentration. Changing the time structure on the m-file gives an accurate model for the concentration of the lake being flooded with contaminant; continually and constantly. The k value is zero (the pollutant is not reacting with the lake), and the *residence time* is equal to 50.

4.3. Sinusoidal Input

The sinusoidal input model is used for pollutants that are introduced to the lake periodically(See Figure 4.5). The pollution enters the system with an average concentration it then varies periodically around that average. Sinusoidal input
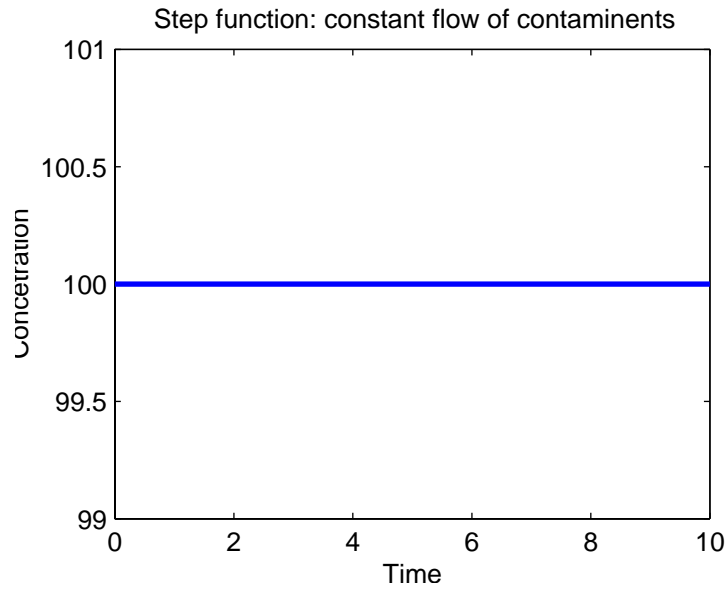
Step function: constant flow of contaminents

Figure 4.3: Step Input Graph.

$C' = (1/\text{theta})\ (100\ (t>=0) + 0\ (t>10)) - (1/\text{theta})\ C - C\ k$ theta = 50   k = 0
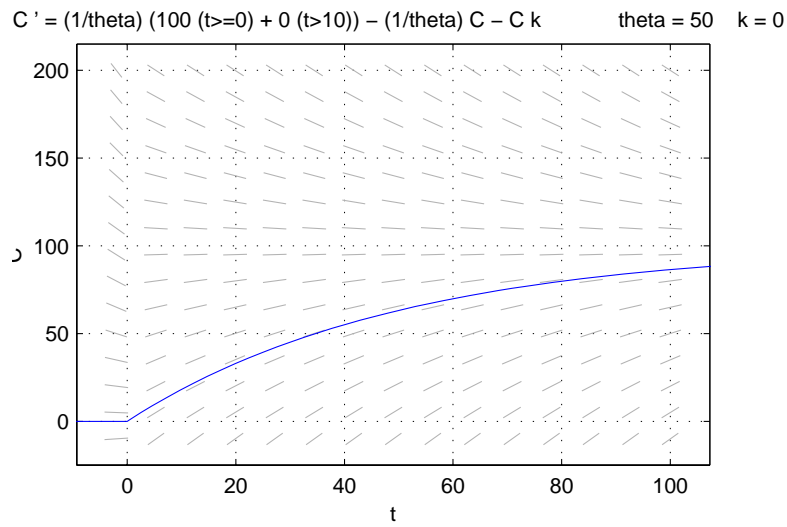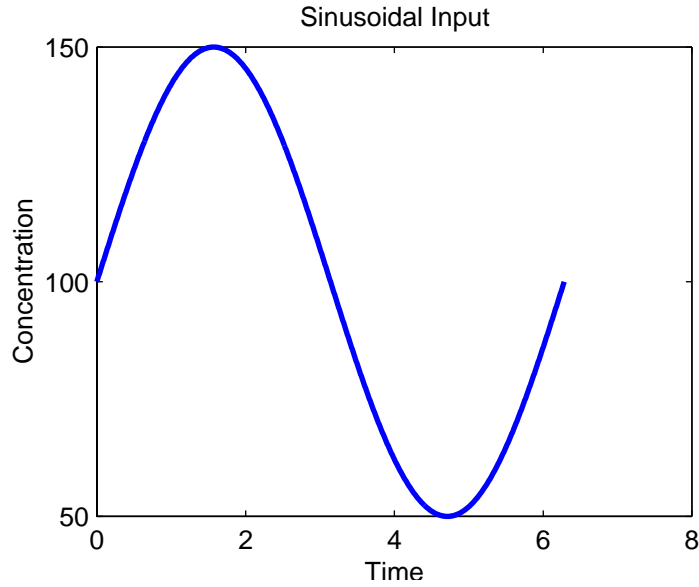

Figure 4.4: Step Input Model.

Figure 4.5: Sinusoidal Input Graph.

changes the input concentration variable to a more useful one:

$$C_{in}(t) = C_i * \left(1 + a * \sin \left(\frac{2\pi * t}{T} \right) \right)$$

The additional variables for the sinusoidal model are defined as follows:

$$
\begin{aligned}
a &= \text{The normalized amplitude (always inbetween 0 and 1)} \\
T &= \text{The period of the fluctuations} \\
C_i &= \text{The average input concentration of pollutant}
\end{aligned}
$$

The actual amplitude of the term is $C_i(1 + a)$ for the peaks and $C_i(1 - a)$ for the valleys. The normalized amplitude needs to stay in-between zero and one; otherwise, the concentration of the input will be negative and will make no sense. The significant points for a sinusoidal model are that the concentration changes over time, that the change in input concentration must be periodic, and that this change can be represented by a sine function added to the average concentration. The model used for Figure 4.6 is the same as the step and impulse for Matlab used earlier. The substitution for C_{in} is the sinusoidal model $C_i * (1 + a * \sin((2\pi * t)/T))$, where the average concentration is 50, k is equal to one, and *residence time* is
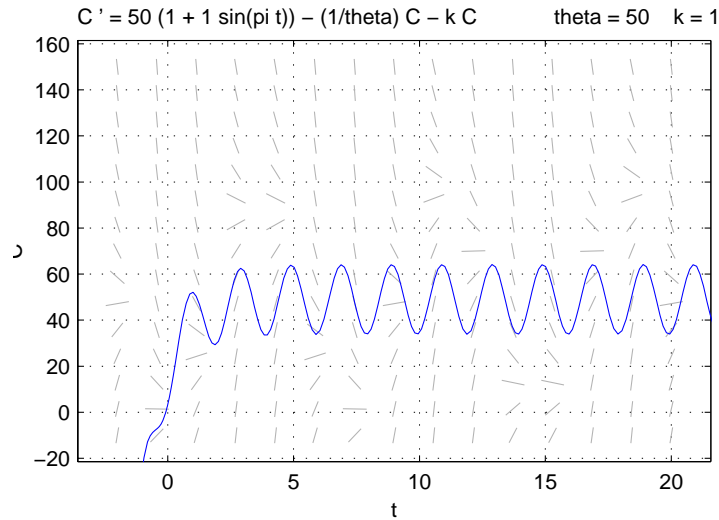
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Figure 4.6: Sinusoidal Input Model.

50. An example would be a manufacturing plant dumping waste producing more output during the day than at night because of hours of operation; hence a periodic input. The concentration of the lake eventually converges to the average input concentration of the contaminant.

4.4. Input Simulators

Most of the information gathered for this paper has been from Virginia Tech's Lake Pollution Modelling web site. For the convenience of simulators, Virginia did an excellent job of creating four of them (to see them just click on the hypertext link for the simulator you want).

- *Impulse input simulator*

    http://www.math.vt.edu/people/hoggard/links/new/Solutions
                    /foConstReactImp.html

- *Step input simulator*

    http://www.math.vt.edu/people/hoggard/links/new/Solutions
                    /foConstReactStep.html

- *Sinusoidal input simulator*

  ```
  http://www.math.vt.edu/people/hoggard/links/new/Solutions
                    /foConstReactSinIVP.html
  ```

- *Generic input simulator*

  ```
  http://www.math.vt.edu/people/hoggard/links/new
                    /Solutions/genIVP.html
  ```

Input simulators show what will happen to the lake if pollutant is released under any of these conditions. Simulators are a graphical representation of the values found over time and plotted on a graph for easier understanding.

5. Conclusion

Pollution has become a very serious threat to our environment. Through the use of differential equations, monitoring pollution has become a useful tool in saving the air, water, and soil. Although just a monitoring device, differential equations, used by people that care about what is happening to the environment, can be a solution to the problem. Showing the simplicity of the lake pollution model and it's three cases of input simulation, just about anyone can use this model with decent accuracy.

References

[1] Arnold, David. "Differential Equations Mixture Problem Theory." College of the Redwoods, Differential Equations Class – Spring 99.

[2] Blanchard, Devaney, & Hall. "Differential Equations." Brooks/Cole Publishing 1988.

[3] Hoggard, John, "Lake Pollution Modelling." Virginia Tech.

  ```
  http://www.math.vt.edu/people/hoggard/links/new/main.html
  ```