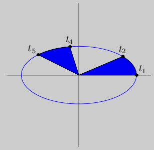

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Celestial Orbits

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1. History

Tycho Brache, a Danish astronomer of the late 1500s, had collected large amounts of raw data concerning planetary motion. After his death in 1601, Johannes Kepler, Brache's assistant, struggled for twenty years to develop his three laws of planetary motion. It was not until 1687 when these laws were mathematically proven by Newton in his *Principia*.

2. The Physics

We begin with a mass M and consider it the origin of our coordinate system. This paper discusses planetary motion and therefore we can think of M as the sun and m as a satellite. There is a second object, a satellite, with mass m somewhere in space. Understanding the relationships between these two masses is the first step.

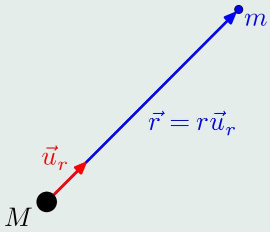

Figure 1: The position vector $\vec{r}$ of the mass m from the central mass M.

Mass m exists some distance from M. This distance has a length, we'll call it r, and a direction and is therefore denoted $\vec{r}$. There is a vector $\vec{u}_r$ that points in the same direction as $\vec{r}$, but has a length of 1. We can rewrite the position vector as the unit vector multiplied by the magnitude, r, of the position vector.

$$\vec{r} = r\vec{u}_r$$

We now have the following diagram. The vector $\vec{u}_r$ can be broken down into its components,

$$\vec{u}_r = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta.$$

We can make a vector perpendicular to $\vec{u}_r$. To be perpendicular, we add $\pi/2$ to the angle.

$$\begin{aligned}
\cos(\theta + \pi/2) &= \cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2} \\
&= -\sin\theta \\
\sin(\theta + \pi/2) &= \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} \\
&= \cos\theta
\end{aligned}$$

Thus, our perpendicular vector has coordinates of

$$\vec{u}_\theta = (-\sin\theta, \cos\theta),$$

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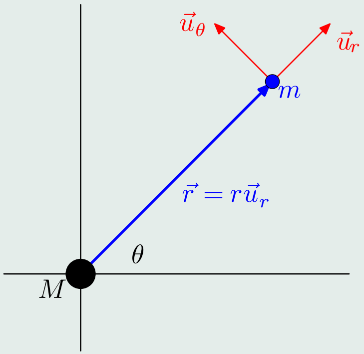


Figure 2: The relationships between the masses and the unit vectors for the r and θ directions.

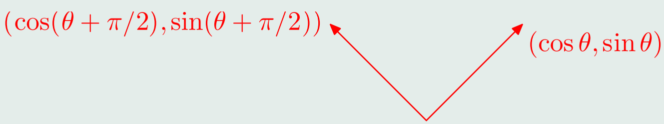


Figure 3: Two perpendicular vectors.

and components of

$$\vec{u}_\theta = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta.$$

The derivatives, with respect to θ of the unit position and unit angle vectors are

$$\frac{d\vec{u}_r}{d\theta} = \vec{u}_\theta \quad \text{and} \quad \frac{d\vec{u}_\theta}{d\theta} = -\vec{u}_r.$$

Both the position r and the unit vector $\vec{u}_r$ are functions of time. So, when calculating the derivative, it is necessary to use the product rule.

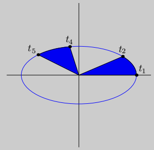
$$\begin{aligned}
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \frac{d}{dt}(r\vec{u}_r) \\
&= r\frac{d\vec{u}_r}{dt} + \frac{dr}{dt}\vec{u}_r \\
&= r\frac{d\theta}{dt}\vec{u}_\theta + \frac{dr}{dt}\vec{u}_r
\end{aligned}$$

It follows, then that the acceleration is

$$\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d}{dt}\left(\frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta\right) \\
&= \frac{d^2r}{dt^2}\vec{u}_r + \frac{d\vec{u}_r}{dt}\frac{dr}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\left(\frac{d\theta}{dt}\right)^2\vec{u}_\theta + r\frac{d\vec{u}_\theta}{dt}\frac{d\theta}{dt}.
\end{aligned}$$

Therefore, the equation for the acceleration is

$$\vec{a} = \left(r\frac{d^2\theta}{dt^2}\vec{u}_\theta + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\vec{u}_\theta + \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r. \tag{1}$$



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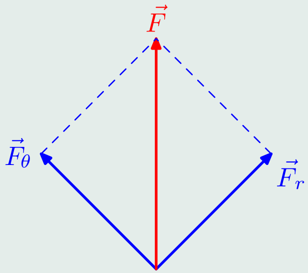


Figure 4: Summing the forces in the r and θ directions will lead to the resultant vector, $\vec{F}$.

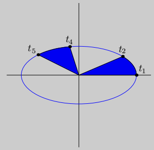
3. The Case of the Central Force

Two bodies interacting apply equal and opposite forces on each other by Newton's law of gravity and third law. It is important to point out that the sun's mass is so large in comparison to its satellites that the force from orbiting objects is negligible. The force acting on m from M can simply be called $\vec{F}$. And, like the other vectors, this force vector can be broken down into r and θ components.

$$\vec{F} = F_\theta \vec{u}_\theta + F_r \vec{u}_r$$

From Newton's second law, the sum of the forces is equal to the mass times the acceleration, $\vec{F} = m\vec{a}$. We will define a force and acceleration in the r direction and a force and acceleration in the θ direction. The equations for the acceleration come from equation (1). They can be written as two distinct equations,

$$\begin{aligned} \vec{F}_r &= m\vec{a}_r \\ \vec{F}_r &= m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) \end{aligned} \tag{2}$$



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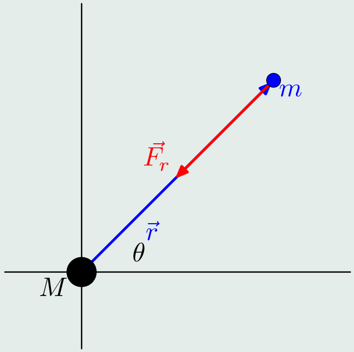


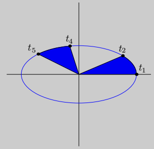
Figure 5: The only force action on mass m is the force towards M.

and

$$\begin{aligned} \vec{F}_\theta &= m\vec{a}_\theta \\ \vec{F}_\theta &= m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \end{aligned} \tag{3}$$

However, in the case of orbits, we can assume that when two objects are interacting, the forces are directed towards each other and have no perpendicular components. Thus, $\vec{F}_\theta = 0$. The remaining force $\vec{F}_r$ is in the direction of (and directly opposite of) the position vector and is commonly referred to as the *Central Force*. The result is as follows.

$$\begin{aligned} \vec{F}_\theta &= m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \\ 0 &= m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \end{aligned}$$



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If we multiply through by r, the derivative of $r^2 d\theta/dt$ emerges.

$$
\begin{aligned}
0 &= r^2 \frac{d^2\theta}{dt^2} + 2r \frac{dr}{dt}\frac{d\theta}{dt} \\
0 &= \frac{d}{dt}\left(r^2 \frac{d\theta}{dt} \right)
\end{aligned}
$$

Now, we see that the derivative of $r^2 d\theta/dt$ is equal to zero meaning it must be a constant. So

$$
h = r^2 \frac{d\theta}{dt}, \tag{4}
$$

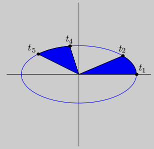
where h is a positive constant; indicating that m is moving in a counterclockwise direction.

4. Kepler's Second

Kepler's second law states that the radius vector $\vec{r}$ from the sun to a planet sweeps out equal areas in equal intervals of time.

Begin with the following figure. The shaded area is a portion of the total area of the path as m completes one rotation. Therefore, it can be calculated geometrically. The angle θ swept out can be seen as the percent of the amount possible, 2π.

$$
\begin{aligned}
\text{Area of swept region } = A_r &= \frac{\theta}{2\pi}\pi r^2 \\
A_r &= \frac{1}{2}\theta r^2 \\
dA_r &= \frac{1}{2} r^2 \, d\theta \\
dA_r &= \left(\frac{1}{2} r^2 \frac{d\theta}{dt} \right) dt
\end{aligned}
$$



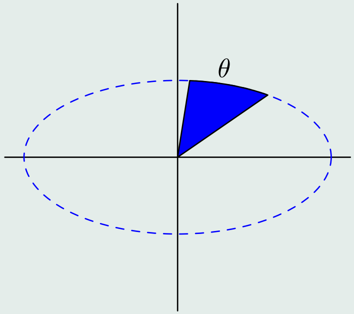


Figure 6: The area swept out over a time $t_2 - t_1$.

We have a value for $r^2 d\theta/dt$ from (4).

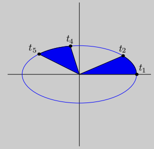
$$\begin{aligned} dA_r &= \frac{1}{2} h\, dt \\ \int_{t_1}^{t_2} dA_r &= \int_{t_1}^{t_2} \frac{1}{2} h\, dt \end{aligned}$$

Performing this integration will produce Kepler's second law.

$$A_r(t_2) - A_r(t_1) = \frac{1}{2} h(t_2 - t_1) \tag{5}$$

5. Kepler's First

Remember that $\vec{F}_r$ is the force acting on m. Newton's law of gravitation states that the magnitude of this force is directly proportional to the square of the distance between them.



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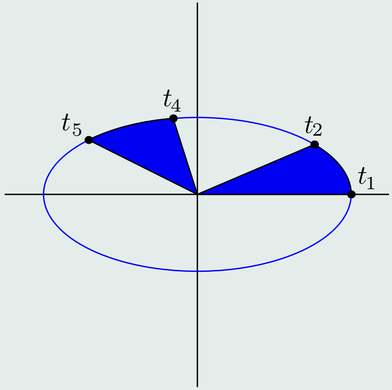


Figure 7: Kepler's second law: equal areas swept out over equal time intervals.

$$\vec{F}_r = -G\frac{Mm}{r^2} \tag{6}$$

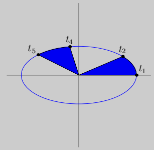
G is the *gravitational constant*. Now let $k = GM$ and simplify.

$$\vec{F}_r = -\frac{km}{r^2} \tag{7}$$

Retrieve the value of F_r from equation (2) and equate the two expressions for $\vec{F}_r$.

$$-\frac{km}{r^2} = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

$$-\frac{k}{r^2} = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$



In order to simplify the proceeding calculations, let

$$z = \frac{1}{r^2}$$

and use θ in place of t as the independent variable. Solve for dr/dt with the new substitutions and a variation of equation (4).

$$\begin{aligned}
\frac{dr}{dt} &= \frac{d}{dt}\left(\frac{1}{z}\right) \\
&= -\frac{1}{z^2}\frac{dz}{d\theta}\frac{d\theta}{dt} \\
&= -\frac{1}{z^2}\frac{dz}{d\theta}\frac{h}{r^2} \\
&= -r^2\frac{dz}{d\theta}\frac{d}{r^2} \\
&= -h\frac{dz}{d\theta}
\end{aligned}$$

This is the velocity and the acceleration follows.

$$\begin{aligned}
\frac{d^2r}{dt^2} &= \frac{dr}{dt}\frac{d}{dt} \\
&= -h\frac{dz}{d\theta} \\
&= -h\frac{d}{d\theta}\left(\frac{dz}{d\theta}\right)\frac{d\theta}{dt} \\
&= -h\frac{d^z}{d\theta^2}\frac{h}{r^2} \\
&= -h^2z^2\frac{d^2z}{d\theta^2}
\end{aligned}$$

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We now have an expression for d^2r/dt^2 to put into our previous equation.

$$
\begin{aligned}
-\frac{k}{r^2} &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \\
-kz^2 &= -h^2z^2\frac{d^z}{d\theta^2} - \frac{1}{z}h^2z^4 \\
\frac{k}{h^2} &= \frac{d^z}{d\theta^2} + z
\end{aligned}
$$

Solve the ordinary differential equation as follows:

1. Find the homogeneous solution.

$$
\begin{aligned}
\lambda^2 + 1 &= 0 \\
\lambda &= \pm i
\end{aligned}
$$

   Therefore,

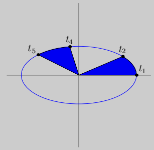
$$z_h = C_1 \cos\theta + C_2 \sin\theta.$$

2. Find the particular solution. We will guess $z_p = k/h^2$.

$$
\begin{aligned}
z_p &= \frac{k}{h^2} \\
z_p' &= 0 \\
z_p'' &= 0
\end{aligned}
$$

   Therefore,

$$z_p = \frac{k}{h^2}.$$



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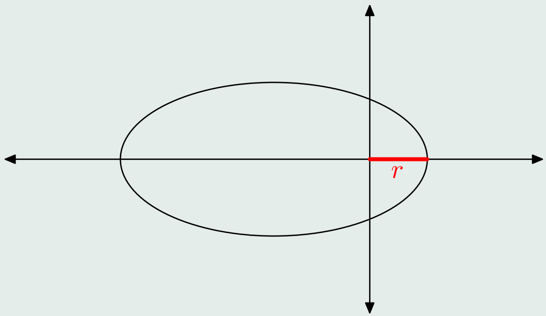


Figure 8: The mass is closest to the focus in this position.

The general solution is the sum of the homogeneous and particular solutions.

$$z = C_1 \cos\theta + C_2 \sin\theta + \frac{k}{h^2}$$

To simplify calculations, shift the polar axis so that r is smallest when $\theta = 0$. Because $z = 1/r$, z is largest when $\theta = 0$, implying

$$\begin{aligned} \frac{dz}{d\theta} &= 0 & \qquad \frac{d^2 z}{d\theta^2} &< 0, \\ z'(0) &= 0 & z''(0) &< 0. \end{aligned}$$

From the general solution,

$$z' = -C_1 \sin\theta + C_2 \cos\theta.$$

From the initial condition $z'(0) = 0 \Rightarrow C_2 = 0$ and $C_1 > 0$. We replace z with $1/r$ from the general solution, taking into account that $C_2 = 0$.

$$
\begin{aligned}
z &= C_1 \cos\theta + \frac{k}{h^2} \\
\frac{1}{r} &= C_1 \cos\theta + \frac{k}{h^2} \\
r &= \frac{1}{C_1 \cos\theta + \frac{k}{h^2}} \\
r &= \frac{h^2/k}{1 + (C_1 h^2/k)\cos\theta}
\end{aligned}
$$

Define eccentricity as $e = c_1 h^2/k$ and a constant p as $p = 1/C_1$. Using these in the equation, we get Kepler's first law.

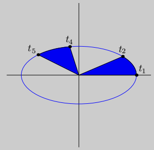
$$
r = \frac{h^2/k}{1 + e\cos\theta} r = \frac{pe}{1 + e\cos\theta} \tag{8}
$$

This law states that orbit of each planet is an ellipse with the sun at one focus. This is the equation for all conic sections.

6. Kepler's Third

Remember that velocity equation? Well, here it is anyway...

$$
\vec{v} = r\frac{d\theta}{dt}\vec{u}_\theta + \frac{dr}{dt}\vec{u}_r
$$



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We want to find the total energy of the mass m, which is the sum of the kinetic and potential energies. Begin with kinetic energy, KE.

$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right) \end{aligned}$$

Potential energy, PE, is defined as the negative of the work required to move m to infinity. Work is the integral of force; force is defined in equation (7).

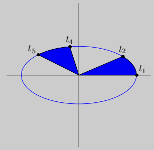
$$\begin{aligned} PE &= -\int_r^\infty \frac{km}{r^2}\,dr \\ &= -\frac{km}{r} \end{aligned}$$

Now we can state

$$\begin{aligned} E &= PE + KE \\ &= \frac{1}{2}m\left(r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right) - \frac{km}{r} \end{aligned}$$

If we now look at the situation when $\theta = 0$ and recall Kepler's third law from equation (5),

$$\begin{aligned} r &= \frac{h^2/k}{1 + e\cos\theta} \\ &= \frac{h^2/k}{1 + e}. \end{aligned}$$



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Therefore,

$$
\begin{aligned}
E &= \frac{1}{2}m\left[r^2\left(\frac{h}{r^2}\right)^2 + (0)\right] - \frac{km}{r^2} \\
E &= \frac{mh^2}{2\left(\frac{h2/k}{1+e}\right)^2} - \frac{km}{\frac{h^2}{1+e}} \\
\left(\frac{h^2/k}{1+e}\right)E &= \frac{mh^2}{2} - km\left(\frac{h^2/k}{1+e}\right) \\
2E\frac{h^4}{k^2} &= (1+e)^2mh^2 - 2(1+e)mh^2 \\
0 &= (1+e)^2 - 2(1+e) - \frac{2Eh^2}{mk^2}.
\end{aligned}
$$

Use the quadratic formula to solve for e.

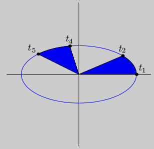
$$e = \sqrt{1 + E\frac{2h^2}{mk^2}}$$

This is the eccentricity. Geometrically it is the length c divided by a.

$$
\begin{aligned}
c^2 &= \frac{a^2 - b^2}{a^2} \\
a^2e^2 - a^2 &= -b^2 \\
b^2 &= a^2(1 - e^2)
\end{aligned}
$$

From geometry, the equation of an ellipse in rectangular coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



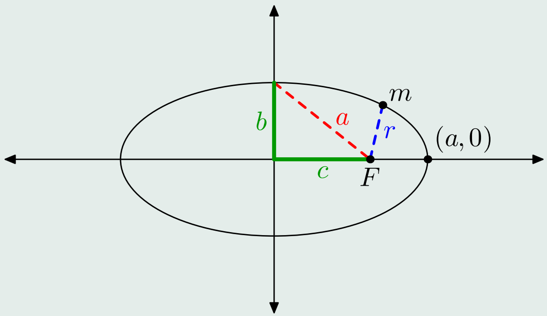


Figure 9: The eccentricity e is equal to c/a.

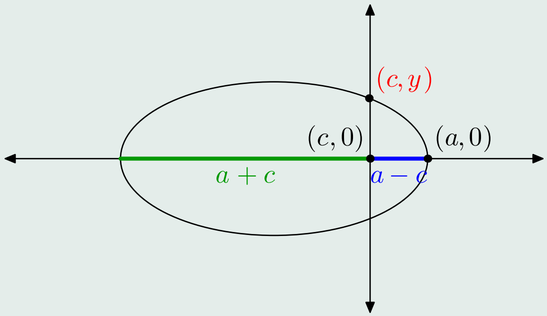
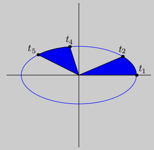


Figure 10: The coordinate system where the focus is the origin.



Now, shift the coordinate axes so the origin is at the same point as the focus. It is unclear as to whether the coordinate $(a, 0)$ or (c, y) is the shortest distance from the origin. We need to find the y coordinate when we are at (c, y).

$$\begin{aligned} \frac{c^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ e^2 + \frac{y^2}{b^2} &= 1 \\ y^2 &= b^2(1 - e^2) \\ y^2 &= a^2(1 - e^2)(1 - e^2) \\ y &= \pm a(1 - e^2) \end{aligned}$$

To find the shortest distance, compare $a(1 - e^2)$ with $a - c$.

$$\begin{aligned} a(1 - e^2) &\text{ vs } a - c \\ a(1 - e^2) &\; a(1 - e) \end{aligned}$$

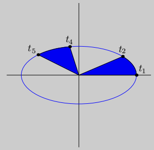
Because $e < 1$ for an ellipse, $a(1 - e^2) > a - c$. This means that the shortest distance is when the mass is directly to the right of the focus, in the $\hat{\imath}$ direction.

The average length of the mass from the focus is half the sum of the shortest and longest rs. Symbolically,

$$\text{average length} = a = \frac{1}{2}(r_{\max} + r_{\min}).$$

Find the maximum and minimum values of r. These occur at $\theta = \pi$ and $\theta = 0$, respectively.

$$\begin{aligned} r_{\max} &= \frac{h^2/k}{1 + e\cos\pi} = \frac{h^2}{k(1 - e)} \\ r_{\min} &= \frac{h^2/k}{1 + e\cos 0} = \frac{h^2}{k(1 + e)} \end{aligned}$$



Now,

$$a = \frac{1}{2}\left(\frac{h^2}{k(1-e)} + \frac{h^2}{k(1+e)}\right)$$

$$a = \frac{h^2}{k(1-e^2)}$$

Put this equation in terms of b and a and eliminate e using

$$1 - e^2 = 1 - \frac{c^2}{a^2}$$

$$1 - e^2 = \frac{b^2}{a^2}\,\frac{1}{1-e^2} = \frac{a^2}{b^2}$$

So, by plugging this into

$$a = \frac{h^2}{k(1-e^2)} \quad = \frac{h^2 a^2}{kb^2}$$

Solve for b^2. This will be used later.

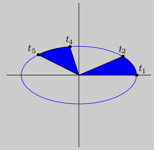
$$b^2 = \frac{h^2 a}{k} \tag{9}$$

The area of an ellipse is commonly known to be πab. We can equate this to the area equation (5) from Kepler's second law.

$$\pi ab = \frac{1}{2}h(t_2 - t_1)$$

If we consider the initial time as $t_1 = 0$ and the time when the mass m completes one rotation as the period $t_2 = T$, we can simplify this equation.

$$\pi ab = \frac{hT}{2}$$



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Square both sides of this equation because we found a value for b^2 in equation (9).

$$\pi^2 a^2 b^2 = \frac{h^2 T^2}{4}$$

Use the value for b^2 and solve for the period T.

$$\pi^2 a^2 \left(\frac{h^2 a}{k}\right) = \frac{h^2 T^2}{4}$$
$$T = \frac{4\pi^2}{k} a^3.$$

This is Kepler's third law which states the squares of the periods of revolution of the planets are proportional to the cubes of their mean distances.

7. Graphing the Orbits with Matlab

Most of the math involved thus far has used a polar coordinate system. To be able to linearize the equations and graph the orbits in Matlab, we convert to rectangular coordinates. We begin with the position unit vector and break it into its x, $\hat{i}$, and y, $\hat{j}$, coordinates.

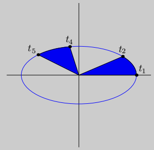
$$\vec{u}_r = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$$

Then, we look at the force and acceleration vectors and repeat the procedure. We rewrote the position of $\vec{r}$ as $r\vec{u}_r$ and we can do the same for $\vec{F}$.

$$\vec{F} = -F\vec{u}_r$$

Now, sub in the components for $\vec{u}_r$.

$$\begin{aligned}\vec{F} &= -F\left(\frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}\right)\\ &= -F_x\hat{i} - F_y\hat{j}\end{aligned}$$



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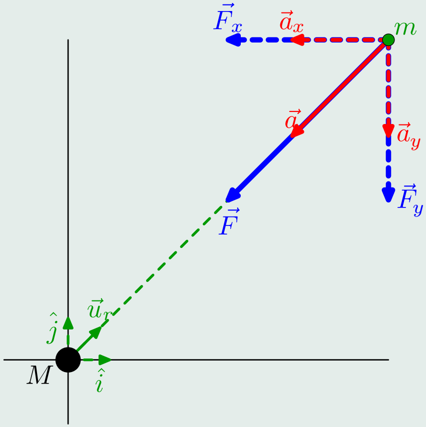
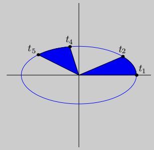


Figure 11: The force and acceleration vector and their components.



From Newton's second law, we have equations relating the force in the x direction with the acceleration in the x direction.

$$F_x = ma_x$$
$$F_x = mx''$$

The same follows for the components of force and acceleration in the y direction.

The position and velocity variables are set equal to the elements of a vector, $\vec{X}$.

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= y \\ x_4 &= y' \end{aligned}$$

Look familiar? Now, take the derivative of $\vec{X}$.

$$\vec{X}' = \begin{cases} x_1' &= x_2 \\ x_2' &= F_x/m \\ x_3' &= x_4 \\ x_4' &= F_y/m \end{cases}$$

By Newton's law of gravitation,

$$F = \frac{GMm}{r^2}.$$

But, we are converting to rectangular coordinates and $r^2 = x^2 + y^2$. Therefore the equation becomes

$$F = \frac{GMm}{x^2 + y^2}.$$

Substitute this into our system of equations and we get

$$\vec{X}' = \begin{cases} x_1' &= x_2 \\ x_2' &= \dfrac{-GM}{(x(1)^2 + x(3)^2)^{3/2}} x_1 \\ x_3' &= x_4 \\ x_4' &= \dfrac{-GM}{(x(1)^2 + x(3)^2)^{3/2}} x_3 \end{cases}$$

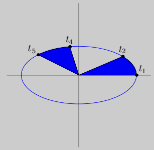
This last set of equations will be used in a Matlab m-file. The initial conditions were chose carefully to produce the following graphs.

References

[1] Chandrasekhar, S, *Newton's Principia for the Common Reader*, Oxford University Press, 1995

[2] Danby, J. M., *Fundamentals of Celestial Mechanics*, McMillan, 1962

[3] Simmons, G. F., *Differential Equations with Applications and Historical Notes*, McGraw Hill, 1972



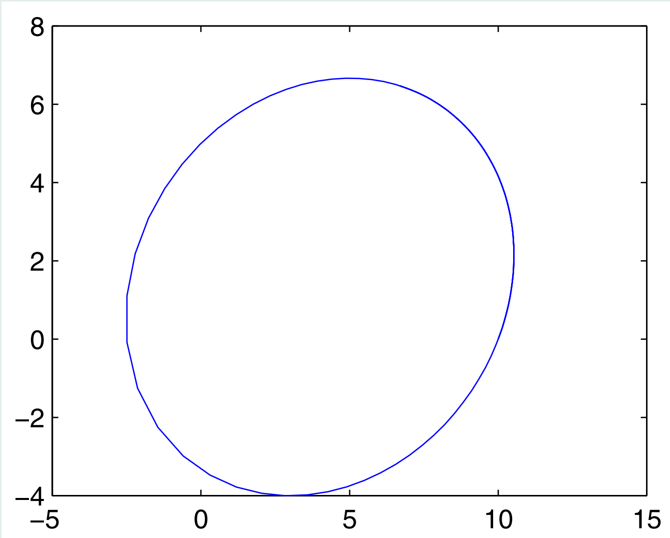


Figure 12: An elliptical orbit. The object will remain along this path.

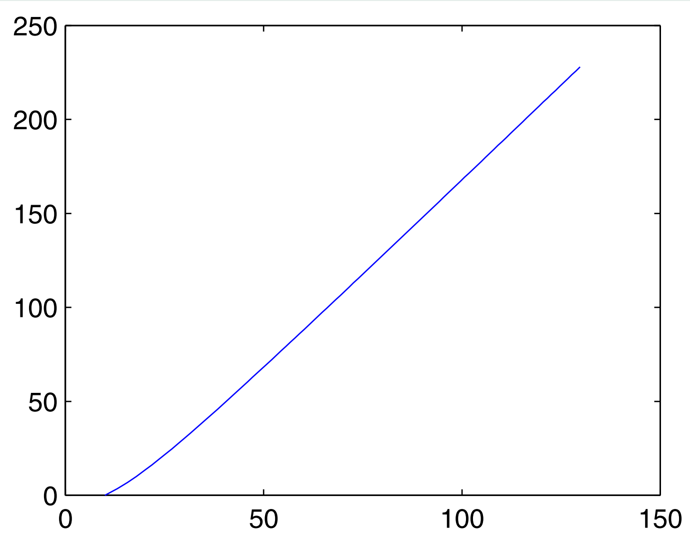


Figure 13: The hyperbolic orbit. It demonstrates that an object with enough energy will escape the gravitational pull.