A Deep Thought on The Double Pendulum

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Abstract

Chaotic systems are strange. They are not periodic or convergent. One well-known chaotic system is the "Lorenz Attractor", first described by Edward Lorenz of the Massachusetts Institute of Technology. The Lorenz Attractor is very difficult to understand. However, we can understand the double pendulum.

A double pendulum consists of one pendulum attached to another. The double pendulum as an example of a simple physical system, which can exhibit chaotic behavior. Therefore, it leads to the goal of this paper, which is to solve the double pendulum problem. the double pendulum is a very common problem in a classical mechanics physics course at the college junior level. In order to present the beauty of the combination of physics and mathematics, which is what this paper is dedicated to, the paper starts with the development of Euler's equation and moves on to the development of the Lagrangian function. Finally we numerically solve the problem by using MATLAB, so that we can have a little understanding of the chaotic systems.

1. Euler Equation

This section pertains to the development of Euler Lagrangian Equation. In order to understand the motion of the double pendulum, we have to understand the Euler Lagrangian Equation. And in order to understand the Euler Lagrangian Equation, we have to start with the definition of admissible.

1.1. Admissible

Let $y(x)$ be a function such that (1) $f(x, y, y')$ has partial derivative of second order, (2) the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ is a well defined real number and (3) the function $y(x)$ has a continuous second derivative. Such a function is called *admissible*. At this point we are ready to go to the heart of this section.

1.2. The Minimizing Function

Assuming the there exists an admissible function $y(x)$ that minimizes the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')dx. \tag{1}$$

Let $\eta(x)$ be any function with the properties that $\eta''(x)$ is continuous and

$$\eta(x_1) = \eta(x_2) = 0. \tag{2}$$

If α is a small parameter, then

$$\bar{y}(x) = y(x) + \alpha\eta(x). \tag{3}$$

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And if the well-defined real number I is in terms of α, then

$$\begin{aligned} I(\alpha) &= \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')dx \\ &= \int_{x_1}^{x_2} f[x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x)]dx. \end{aligned}$$

When $\alpha = 0$, equation (3) yields $\bar{y}(x) = y(x)$. If we differentiate function I with respect to α[1], we have

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}')dx. \tag{4}$$

By using the chain rule, we obtain

$$\begin{aligned} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \alpha} \\ &= 0 + \frac{\partial f}{\partial \bar{y}}\eta(x) + \frac{\partial f}{\partial \bar{y}'}\eta'(x) \\ &= \frac{\partial f}{\partial \bar{y}}\eta(x) + \frac{\partial f}{\partial \bar{y}'}\eta'(x). \end{aligned}$$

Notice that for $\bar{y} = y + \alpha\eta(x)$,

$$\frac{\partial \bar{y}}{\partial \alpha} = \eta(x)$$

[1] We are trying to find the minimum point.

and

$$\begin{aligned}\frac{\partial \bar{y}'}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{d\bar{y}}{dx} \\ &= \frac{\partial}{\partial \alpha}(y' + \alpha\eta'(x)) \\ &= \eta'(x).\end{aligned}$$

Therefore equation (4) can be written as

$$I'(\alpha) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \bar{y}}\eta(x) + \frac{\partial f}{\partial \bar{y}'}\eta'(x)\right] dx. \tag{5}$$

Let equation (5) be zero and yield

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \bar{y}}\eta(x) + \frac{\partial f}{\partial \bar{y}'}\eta'(x)\right] dx = 0. \tag{6}$$

For $\int_{x_1}^{x_2} \frac{\partial f}{\partial \bar{y}'}\eta'(x)dx$, we have

$$\begin{aligned}\int_{x_1}^{x_2} \frac{\partial f}{\partial \bar{y}'}\eta'(x)dx &= \left[\eta(x)\frac{\partial f}{\partial \bar{y}'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}(\frac{\partial f}{\partial \bar{y}'})dx \\ &= -\int_{x_1}^{x_2} \eta(x)\frac{d}{dx}(\frac{\partial f}{\partial \bar{y}'})dx.\end{aligned}$$

Since $\eta(x_1) = \eta(x_2) = 0$, $\eta(x)\frac{\partial f}{\partial \bar{y}'}\Big|_{x_1}^{x_2} = 0$. From $\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial \bar{y}}\eta(x) + \frac{\partial f}{\partial \bar{y}'}\eta'(x)\right]dx = 0$, it becomes

$$\begin{aligned} \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial \bar{y}}\eta(x) - \frac{d}{dx}(\frac{\partial f}{\partial \bar{y}'})\eta(x)\right]dx &= 0 \\ \int_{x_1}^{x_2}\eta(x)\left[\frac{\partial f}{\partial \bar{y}} - \frac{d}{dx}(\frac{\partial f}{\partial \bar{y}'})\right]dx &= 0. \end{aligned}$$

Since $\eta(x)$ cannot always be zero, this yields

$$\frac{\partial f}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial f}{\partial \bar{y}'}\right) = 0.$$

Negate both sides and change the dummy variable, we obtain

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0, \tag{7}$$

which is Euler Lagrangian Equation.

2. The Semiperfect Physics

Physics is a perfect subject[2]. It has all the good things we want. In this section, we have to introduce the idea of generalized coordinates and constraints, so that we can proceed to the goal of this paper.

[2]The title of this section is from a classmate in my differential equation class, Adam Abrahamson, who said physics is not perfect. However, I think physics is great. It interprets the language of the universe.

2.1. Generalization

In a multivariable calculus class, we learned the Cartesian coordinates, cylindrical coordinates, spherical coordinates, and other coordinate system. If there are some restrictions or constraints[3] on the motion of the particle, we need less than three coordinates.

We are interested in finding the minimum number of coordinates needed to describe a system of N particles. Usually, the constraints on any given system are described by the number of equations. Suppose there are m number of such equations that describe the constraints[4]. The minimum number of coordinates, n, needed to completely describe the motion or the configuration of such a system at any given time is given by

$$n = 3N - m \tag{8}$$

where n is the number of *degrees of freedom* of the system. The number n could be any parameter, such as length, $(\text{length})^2$, angle, energy, a dimensionless quantity, or any other quantity, as long as it completely describes the configuration of the system. Think of the vector space, in order to span a vector space, we need certain vectors as basis. The number of these vectors has a minimum. We cannot go under that number. The name *generalized coordinates* is given to *any set of quantities that completely describes the state or configuration of a system*. These n generalized coordinates are customarily written as

$$q_1, q_2, q_3, \ldots, q_n$$

or

$$q_k, \quad \text{where } k = 1, 2, 3, \ldots, n \tag{9}$$

These n generalized coordinates are not restricted by any constraints.

[3] For example, the motion of a free falling object with initial velocity upward. The motion is in two dimensions.

[4] This needs some explanation, think of a function $f(x, y, z)$ constrained inside of a sphere, regardless of how the particle moves, the motion is inside the sphere.

In the same manner, we can define the derivatives of q_k with respect with time, t, that is $\dot{q}_1, \dot{q}_2, \ldots,$ or $\dot{q}_k$ as *generalized velocities.*[5]

Therefore, we can consider the the rectangular coordinates, $x, y,$ and z are a function of the generalized coordinates $q_1, q_2,$ and q_3; that is

$$\begin{aligned} x &= x(q_1, q_2, q_3) = x(\mathbf{q_k}) \\ y &= y(q_1, q_2, q_3) = y(\mathbf{q_k}) \\ z &= z(q_1, q_2, q_3) = z(\mathbf{q_k}). \end{aligned}$$

Suppose the system changes from an initial configuration given by (q_1, q_2, q_3) to a neighborhood configuration given by $(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3)$. We can express the corresponding changes in the Cartesian coordinates by the following relations:

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3 = \sum_{k=1}^{3} \frac{\partial x}{\partial q_k}\delta q_k, \tag{10}$$

with similar equation for δy and δz, where n is equal to three and the partial derivatives $\partial x/\partial q_k$, are functions of q's.

In a more general case, a mechanical system consists of a large number of particles having n degrees of freedom. The configuration of the system becomes

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial x_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k}\delta q_k \tag{11}$$

with similar expression for δy_i and δz_i.

[5] The physics geeks denote that $\dot{k}$ means the derivative of k with respect to time, for k is any function.

It is important to distinguish between two types of displacement: an actual displacement $d\mathbf{r}_i$ and a virtual displacement $\delta\mathbf{r}_i$. Actual displacements are consistent with both the equations of motion and the equations of constraints. Virtual displacement are consistent with the equations of the constraints but do not satisfy the equations of motion or time. Think of a particle moving on a plane. The constraint is the plane. If the actual displacement is a straight line, for example, from point A to point B, which satisfies the equations of the motion or time, the virtual displacement can be any curve starts from A and end with B. That means that the virtual displacement satisfies the constraints, but not the motion or time. If we have a force $\mathbf{F}$, we can define the virtual work,

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} = F_x \delta x + F_y \delta y + F_z \delta z, \tag{12}$$

where F_x, F_y, F_z are the rectangular components of $\mathbf{F}$. Also, we can express the displacements $\delta x, \delta y$ and δz in terms of the generalized coordinates q_k. From equation (11),

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3 = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\delta q_k,$$

we have

$$\begin{aligned}
\delta W &= \mathbf{F} \cdot \delta\mathbf{r} \\
&= F_x \delta x + F_y \delta y + F_z \delta z \\
&= F_x \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\delta q_k + F_y \sum_{k=1}^{n} \frac{\partial y}{\partial q_k}\delta q_k + F_z \sum_{k=1}^{n} \frac{\partial z}{\partial q_k}\delta q_k \\
&= \sum_{k=1}^{n} \left[\left(F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k} \right) \delta q_k \right] \\
&= \sum_{k=1}^{n} Q_k \delta q_k,
\end{aligned}$$

where

$$Q_k = F_x \frac{\partial x}{\partial q_k} + F_y \frac{\partial y}{\partial q_k} + F_z \frac{\partial z}{\partial q_k}. \tag{13}$$

Q_k is called the *generalized force* associated with the generalized coordinate q_k. If the generalized forces are conservative, it can be represented by a potential function $V = V(x, y, z)$. The rectangular components of a force acting on a particle are given by[6]

$$F_x = -\frac{\partial V}{\partial x}, \; F_y = -\frac{\partial V}{\partial y}, \; F_z = -\frac{\partial V}{\partial z}. \tag{14}$$

If we substitute equation (14) into the generalized force, we obtain

$$Q_k = -(\frac{\partial V}{\partial x}\frac{\partial x}{\partial q_k} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_k} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_k}). \tag{15}$$

This expression in the parentheses is the partial derivative of the function V with respect to q_k. That is,

$$Q_k = -\frac{\partial V}{\partial q_k}. \tag{16}$$

2.2. The Lagrange's Equations

Let the kinetic energy of a particle in Cartesian coordinates be

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{17}$$

[6]From basic physics, we know that the potential energy V of the system is equal to the negative of work done by the system. In addition, the work done by the system is $-V = W = Fd$, where F is the force excerted by the system. Take the partial derivative of V with respect with x, y and z to obtain what we have above.

for $x = x(q_1, q_2, \dots, q_n) = x(\mathbf{q})$, and also similarly for

$$y = y(\mathbf{q}), \quad z = z(\mathbf{q}).$$

Therefore, $\dot{x}$ can be evaluated as

$$\begin{aligned}
\dot{x} &= \frac{\partial x}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2}\frac{\partial q_2}{\partial t} + \dots + \frac{\partial x}{\partial q_n}\frac{\partial q_n}{\partial t} \\
&= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\frac{\partial q_k}{\partial t} \\
&= \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\dot{q_k} \\
&= \dot{x}(\mathbf{q}, \dot{\mathbf{q}}).
\end{aligned}$$

We can obtain $\dot{y}, \dot{z}$ in the same manner. Therefore, we have the velocity in Cartesian cordinates in terms of the generalized coordinates q_k and generalized velocities $\dot{q_k}$, that is,

$$\begin{aligned}
\dot{x} &= \dot{x}(\mathbf{q}, \dot{\mathbf{q}}), \\
\dot{y} &= \dot{y}(\mathbf{q}, \dot{\mathbf{q}}), \\
\dot{z} &= \dot{z}(\mathbf{q}, \dot{\mathbf{q}}).
\end{aligned}$$

Therefore, we can write the kinetic energy as

$$T = \frac{1}{2}m[\dot{x}^2(\mathbf{q}, \dot{\mathbf{q}}) + \dot{y}^2(\mathbf{q}, \dot{\mathbf{q}}) + \dot{z}^2(\mathbf{q}, \dot{\mathbf{q}})]. \tag{18}$$

Take the derivative with respect to the generalized velocity $\dot{q_k}$ of a specific coordinate,

$$\frac{\partial T}{\partial \dot{q_k}} = m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q_k}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q_k}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q_k}}\right) \tag{19}$$

Because[7] $\dot{x} = \dot{x}(q, \dot{q})$,

$$\frac{\partial \dot{x}}{\partial \dot{q_k}} = \frac{\partial x}{\partial q_k}, \tag{20}$$

therefore,

$$\frac{\partial T}{\partial \dot{q_k}} = m\left(\dot{x}\frac{\partial x}{\partial q_k} + \dot{y}\frac{\partial y}{\partial q_k} + \dot{z}\frac{\partial z}{\partial q_k}\right), \tag{21}$$

If we differentiate both side of this equation with respect to t:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) = m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right) \tag{22}$$

For the fourth terms on the right side, we have

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dx}{dt}\right) = \frac{\partial \dot{x}}{\partial q_k}$$

Thus the fourth terms can be written as

$$m\dot{x}\frac{d}{dt}\left(\frac{\partial}{\partial q_k}\right) = m\dot{x}\frac{\partial \dot{x}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{x}^2\right)$$

with the same manner expressions for other terms. Also note[8] that

$$F_x = m\ddot{x},\ F_y = m\ddot{y},\ F_z = m\ddot{z}$$

[7] I haven't figured this out yet. I think this may cause serious argument, but it works for every problem. For example, apply equation (20) to $x = q_1 + q_2 q_3^2$. $\dot{x} = 1\dot{q_1} + q_3^2\dot{q_2} + 2q_2q_3\dot{q_3}$, therefore, $\frac{\partial \dot{x}}{\partial \dot{q_3}} = 2q_2q_3$, and $\frac{\partial x}{\partial q_3} = 2q_2q_3$.

[8] One of Newton's Laws F=ma, for F is the force, m is the mass and a is the the second derivative of displacement.

According to what we have above, equation (22) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left[\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{x}^2)\right]. \tag{23}$$

From equation (13) and equation (18), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k}.$$

These differential equations are called *Lagrange's equations* of motion. If the motion is in a conservative force field, by using equation (16), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} \tag{24}$$

Therefore we define a *Lagrangian function* L as the difference between the kinetic energy and potential energy, that is,

$$L \equiv T - V \quad \text{or } L(q,\dot{q}) = T(q,\dot{q}) - V(q). \tag{25}$$

V is the potential energy and T is the kinetic energy of the system. If the potential energy V is not the function of generalized velocites[9], $V = V(\mathbf{q})$ and $\partial V/\partial \dot{q}_k = 0$. Thus we may write

$$\begin{aligned}\frac{\partial L}{\partial \dot{q}_k} &= \frac{\partial}{\partial \dot{q}_k}(T-V) = \frac{\partial T}{\partial \dot{q}_k}\\ \frac{\partial L}{\partial q_k} &= \frac{\partial}{\partial q_k}(T-V) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k}.\end{aligned}$$

[9] If $V = V(\mathbf{q}, \dot{\mathbf{q}})$, tensor force will be created.

Substitute these into equation (24), we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{26}$$

which are *Lagrange's equations describing the motion of a particle in a conservative force field.*

3. Apply The Theory

We have the math and the physics. Things are getting straightened out. In this section, we will show two approaches. They use the same math and physics, but use different argument.

3.1. One Approach

From Figure 1, we have a system

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ y_1 &= -l_1 \cos\theta_1 \\ y_2 &= -l_1 \cos\theta_1 - l_2 \cos\theta_2. \end{aligned}$$

Recall that, the Lagrange's function is defined as $L \equiv T - V$, where T is the kinetic energy and V is the potential energy. This equation is only good when the system is in a conservative force field. Thus,

$$\begin{aligned} T &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2). \end{aligned}$$

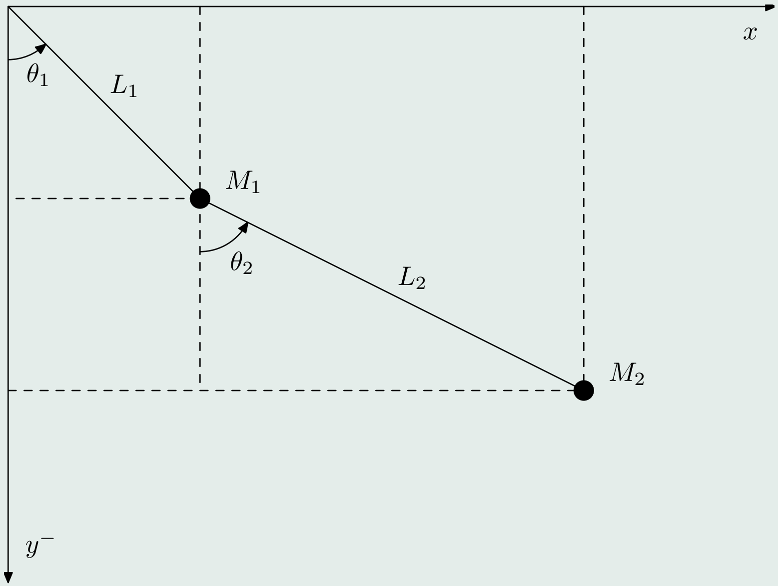

Figure 1: Diagram of a double pendulum

The velocities are

$$\begin{aligned}
\dot{x_1}^2 &= l_1^2 \cos^2\theta_1 \dot{\theta_1}^2 \\
\dot{x_2}^2 &= (l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 \\
&= l_1^2 \cos^2\theta_1 \dot{\theta_1}^2 + l_2^2 \cos^2\theta_2 \dot{\theta_2}^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 \\
\dot{y_1}^2 &= (l_1 \sin\theta_1 \dot{\theta}_1)^2 = l_1^2 \sin^2\theta_1 \dot{\theta_1}^2 \\
\dot{y_2}^2 &= (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2 \\
&= l_1^2 \sin^2\theta_1 \dot{\theta_1}^2 + l_2^2 \sin^2\theta_2 \dot{\theta_2}^2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2 .
\end{aligned}$$

Therefore,

$$\begin{aligned}
T &= \frac{1}{2} m_1 (l_1^2 \cos^2\theta_1 \dot{\theta_1}^2 + l_1^2 \sin^2\theta_1 \dot{\theta_1}^2) + \frac{1}{2} m_2 \\
&\quad (l_1^2 \cos^2\theta_1 \dot{\theta_1}^2 + l_2^2 \cos^2\theta_2 \dot{\theta_2}^2 + l_1^2 \sin^2\theta_1 \dot{\theta_1}^2 + \\
&\quad l_2^2 \sin^2\theta_2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) \\
&= \frac{1}{2} m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta_1}^2 + l_2^2 \dot{\theta_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) \\
&= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2),
\end{aligned}$$

and also,

$$\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 \\
&= -m_1 g (l_1 \cos\theta_1) + m_2 g (-l_1 \cos\theta_1 - l_2 \cos\theta_2) \\
&= -m_1 g l_1 \cos\theta_1 - m_2 g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \\
&= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 .
\end{aligned}$$

Therefore, the Lagrangian function is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2 l_2^2\dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\ &\quad +(m_1+m_2)gl_1\cos\theta_1 + m_2 g l_2 \cos\theta_2. \end{aligned}$$

Recall the Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0.$$

Thus, we obtain a system with two equations, they are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \tag{27}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{28}$$

To solve equation (27), we have

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}_1} &= (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2\cos(\theta_1-\theta_2), \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2\cos(\theta_1-\theta_2) \\ &\quad -m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2), \\ \frac{\partial L}{\partial \theta_1} &= -m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)gl_1\sin\theta_1. \end{aligned}$$

Therefore, from equation (27), we obtain

$$\begin{aligned}
0 &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\
&\quad + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g l_1 \sin\theta_1\\
0 &= (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_2{}^2 \sin(\theta_1-\theta_2)\\
&\quad + (m_1+m_2) g l_1 \sin\theta_1\\
0 &= (m_1+m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_2 \dot{\theta}_2{}^2 \sin(\theta_1-\theta_2)\\
&\quad + (m_1+m_2) g \sin\theta_1.
\end{aligned}$$

And also, to solve equation (28), we have

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2),\\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2)\\
&\quad - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2),\\
\frac{\partial L}{\partial \theta_2} &= m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 g l_1 \sin\theta_2.
\end{aligned}$$

Therefore, from equation (28), we obtain

$$\begin{aligned}
0 &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\\
&\quad - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2\\
0 &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1{}^2 \sin(\theta_1-\theta_2)\\
&\quad + m_2 g l_2 \sin\theta_2\\
0 &= m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 \dot{\theta}_1{}^2 \sin(\theta_1-\theta_2)\\
&\quad + m_2 g \sin\theta_2.
\end{aligned}$$

Thus, the system becomes

$$0 = (m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^{\,2} \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin\theta_1, \tag{29}$$

and

$$0 = m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^{\,2} \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2. \tag{30}$$

If we solve $\ddot{\theta}_2$ and $\ddot{\theta}_1$, and do a little bit of arrangement, we will have,

$$\mathbf{v} = \begin{bmatrix} Q_1 \\ \dot{Q}_1 \\ Q_2 \\ \dot{Q}_2 \end{bmatrix}$$

and,

$$\dot{\mathbf{v}} = \begin{bmatrix} \dot{Q}_1 \\ \ddot{Q}_1 \\ \dot{Q}_2 \\ \ddot{Q}_2 \end{bmatrix}.$$

Obtain these MATLAB codes,

```
function vprime=physics(t,v)
global l1 l2 m1 m2 g
l1=1;
l2=2;
```

```
m1=1;
m2=2;
g=9.8;
vprime=zeros(4,1);
A=cos(v(1)-v(3));
B=sin(v(1)-v(3));
C=sin(v(3));
D=m1+m2;
E=sin(v(1));
vprime(1)=v(2);
vprime(2)=(m2*g*C*A-m2*l2*(v(4))^2*B-D*g*E-m2*l1*(v(2))^2*B*A)/...
    (D*l1-m2*l1*A^2);
vprime(3)=v(3);
vprime(4)=(m2*l2*(v(4))^2*B*A+D*g*E*A+l1*(v(2))^2*B*D-g*C*D)/...
    (l2*D-m2*l2*A^2);
```

The function call is

```
close all
global l1 l2 m1 m2 g
l1=1;
l2=2;
m1=1;
m2=2;
g=9.8;
tspan=[0 10];
iniValue=[.1; .1; 0.01;0];
parameters=[g l1 l2 m1 m2];
step=0.1;
```

```
[t,x]=ode23s('physics',tspan,iniValue);
%[t, x]=rk4('physics',tspan,iniValue,step);
x1=l1*sin(x(:,1));
x2=l1*sin(x(:,1))+l2*sin(x(:,3));
y1=-l1*cos(x(:,1));
y2=-l1*cos(x(:,1))-l2*cos(x(:,3));
```

3.2. Another Approach

However, since the double pendulum is very sensitive, the program in MATLAB will crash very fast. The question that arises here is, do we have another idea to solve the system? Yes, there is another way to solve the system. Consider when the equilibrium point is at the point where all the angles are zeros and the angular momenta are zeros, see Figure 2. That is, our new system becomes

$$\begin{aligned} x_1 &= l_1 \sin\theta_1 \\ x_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ y_1 &= l_1 \cos\theta_1 \\ y_2 &= l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{aligned}$$

Therefore, the height of first mass can be written as

$$h_{\text{first mass}} = l_1(1 - \cos\theta_1)$$

The height of second mass can be written as

$$\begin{aligned} h_{\text{seond mass}} &= h_1 + h_2 - l_1 \cos\theta_1 - l_2 \cos\theta_2 \\ &= l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2). \end{aligned}$$

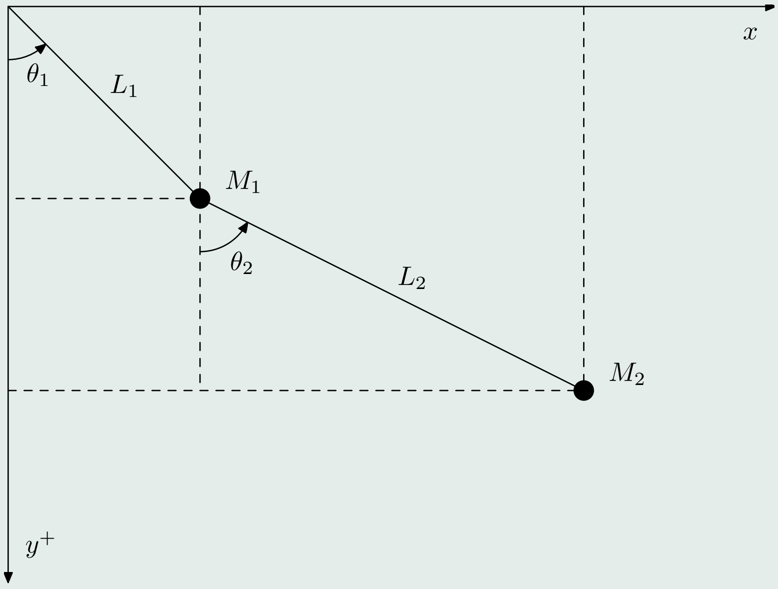

Figure 2: Diagram of a double pendulum

The potential energy can be written as

$$V = l_1(1-\cos\theta_1)m_1 g + [l_1(1-\cos\theta) + l_2(1-\cos\theta_1)]m_2 g.$$

Recall the kinetic energy

$$T = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^{\,2} + m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2).$$

If we do the same thing in the previous part, we will have

$$\begin{aligned}\ddot{\theta}_1 &= \frac{g(\sin\theta_2\cos(\triangle\theta) - u\sin\theta_1) - (l_2\dot{\theta}_2^{\,2} + l_1\dot{\theta}_1^{\,2}\cos(\triangle\theta))\sin(\triangle\theta)}{l_1(u-\cos^2(\triangle\theta)} \\ \ddot{\theta}_2 &= \frac{gu(\sin\theta_1\cos(\triangle\theta) - \sin\theta_2) + (ul_1\dot{\theta}_1^{\,2} + l_2\dot{\theta}_2^{\,2}\cos(\triangle\theta))\sin(\triangle\theta)}{l_2(u-\cos^2(\triangle\theta)},\end{aligned}$$

where $\triangle\theta = \theta_1 - \theta_2$ and $u = 1 + (m_1/m_2)$.

We repeat what we did for the system, that is

$$\mathbf{x} = \begin{bmatrix} Q_1 \\ \dot{Q}_1 \\ Q_2 \\ \dot{Q}_2 \end{bmatrix}$$

and,

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{Q}_1 \\ \ddot{Q}_1 \\ \dot{Q}_2 \\ \ddot{Q}_2 \end{bmatrix}.$$

In order to solve this system, the following MATLAB code is required,

```
function xprime=yorke(t,x,flag,g,l1,l2,m1,m2)
xprime=zeros(4,1);
dt=x(1)-x(3);
u=1+m1/m2;
xprime(1)=x(2);
xprime(2)=(g*(sin(x(3))*cos(dt)-u*sin(x(1)))-...
    (l2*x(4)^2+l1*x(2)^2*cos(dt))*sin(dt))/(l1*(u-(cos(dt))^2));
xprime(3)=x(4);
xprime(4)=(g*u*(sin(x(1))*cos(dt)-sin(x(3)))+...
    (u*l1*x(2)^2+l2*x(4)^2*cos(dt))*sin(dt))/(l2*(u-(cos(dt))^2));
```

And the function call is

```
close all
global l1 l2 m1 m2 g
l1=1;
l2=2;
m1=1;
m2=2;
g=9.8;
tspan=[0 500];
iniValue=[pi/3; pi/4; 0.01;0];
parameters=[g l1 l2 m1 m2];
step=0.1;
[t,x]=ode23s('yorke',tspan,iniValue,[],g,l1,l2,m1,m2);
%[t, x]=rk4('physics',tspan,iniValue,step);
x1=l1*sin(x(:,1));
x2=l1*sin(x(:,1))+l2*sin(x(:,3));
```

```
y1=l1*cos(x(:,1));
y2=l1*cos(x(:,1))+l2*cos(x(:,3));
```

This time, we are able to graph the paths of each bob. We will present them in the last section by using a MATLAB m-file called dpend.m. In the next section, we are going to show Hamilton's equations, which are difficult to solve with a normal solver. We will not present these calculations in the paper.

4. More About Classical Mechanic

In the previous section, we obtain a system drove from Lagrange's equation. However, in this section, it will better to show Hamilton's equation so that we will have a deeper understanding of the double pendulum.

4.1. The Development of Hamilton's Equations

If the Lagrangian were an explicit function of time, we write

$$L(q, \dot{q}, t) = L(\mathbf{q_i}; \dot{\mathbf{q}}_\mathbf{i}; t).$$

It follows that,

$$H(\mathbf{q_i}; \mathbf{p_i}; t),$$

which is called Hamilton's formalism. We need some explanation for this. First, we know that the kinetic energy is,

$$T = \frac{1}{2} m \dot{x}^2.$$

Therefore, from $\|\mathbf{p}\| = m\dot{x}$, for $p = \|\mathbf{p}\|$ is the magnitude of momentum of a partial, we have

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}} &= \frac{\partial}{\partial \dot{x}}(\frac{1}{2}m\dot{x}^2) \\ &= m\dot{x} \\ &= \|\mathbf{p}\|, \end{aligned}$$

which is

$$p_k = \frac{\partial T}{\partial \dot{q_k}}. \tag{31}$$

Recall the Lagrangian function, $L = T - V$. If V is independent[10] of $\dot{q_k}$, we can substitute L into equation (31), that is,

$$p = \frac{\partial L}{\partial \dot{q_k}}. \tag{32}$$

For a system described by a set of generalized coordinates, $p_k = \partial L / \partial \dot{q_k}$, where $\mathbf{q} = \mathbf{q}\,(q_1, q_2, \ldots, q_k, \ldots)$.

For a conservative system, Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0.$$

Therefore,

$$\dot{p_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right),$$

[10] V is defined to be the configuration of the system. Force is independent of velocity

where $p_k = \partial L/\partial \dot{q_k}$ Obtain,

$$\dot{p_k} - \frac{\partial L}{\partial q_k} = 0$$

$$\dot{p_k} = \frac{\partial L}{\partial q_k}. \tag{33}$$

Because

$$L = L(q_1, q_2, , \ldots, q_n; \dot{q_1}, \dot{q_2}, \ldots, \dot{q_n}; t).$$

The differential of L is

$$dL = \sum_{k=1}^{n} (\frac{\partial L}{\partial q_k} dq_k + \frac{\partial L}{\partial \dot{q_k}} d\dot{q_k}) + \frac{\partial L}{\partial t} dt.$$

Form equation (31) and equation (32), we obtain

$$dL = \sum_{k=1}^{n} (\dot{p_k} dq_k + p_k d\dot{q_k}) + \frac{\partial L}{\partial t} dt.$$

Therefore, we have the following,

$$\begin{aligned}
d(\sum_{k=1}^{n} p_k \dot{q_k} - L) &= \sum_{k=1}^{n} (p_k d\dot{q_k} + \dot{q_k} dp_k) - dL \\
&= \sum_{k=1}^{n} (p_k d\dot{q_k} + \dot{q_k} dp_k) - \sum_{k=1}^{n} (\dot{p_k} dq_k + p_k d\dot{q_k}) - \frac{\partial L}{\partial t} dt \\
&= \sum_{k=1}^{n} (p_k d\dot{q_k} + \dot{q_k} dp_k - \dot{p_k} dq_k - p_k d\dot{q_k}) - \frac{\partial L}{\partial t} dt \\
&= \sum_{k=1}^{n} (\dot{q_k} dp_k - \dot{p_k} dq_k) - \frac{\partial L}{\partial t} dt.
\end{aligned}$$

At this point, we defined the Hamiltonian function H to be

$$H = \sum_{k=1}^{n} p_k \dot{q_k} - L. \tag{34}$$

That is,

$$dH = \sum_{k=1}^{n} (\dot{q_K} dp_k - \dot{p_k} dq_k) - \frac{\partial L}{\partial t} dt$$

L is an explicit function of $(\mathbf{q_k}; \dot{\mathbf{q_k}}; t)$. In many cases it is possible to express H as an explicit function of $(\mathbf{q_k}; \mathbf{p_k}; t)$. This can be done by using the relation defining generalized momentum, that is, $\partial L / \partial \dot{q_k} = p_k$. Therefore, H can be written as

$$H = H(q_1, \ldots, q_n; p_1, \ldots, p_n; t).$$

We may write the differential of H as

$$dH = \sum_{k=1}^{n} (\frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial p_k} dq_k) + \frac{\partial H}{\partial t} dt$$

Thus, we obtain

$$\begin{aligned} \dot{q_k} &= \frac{\partial H}{\partial p_k} \\ -\dot{p_k} &= \frac{\partial H}{\partial q_k} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}$$

These equations are Hamilton's equations of motion.

4.2. Hamilton's Equations of The Double Pendulum

Recall the Lagrangian for the double pendulum,

$$\begin{aligned} L &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2l_2^2\dot{\theta_2}^2 + m_2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2) \\ &\quad +(m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2. \end{aligned}$$

From equation (32), the angular momenta of the the double pendulum is,

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{35}$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2l_2^2\dot{\theta}_2 + m_2l_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2) \tag{36}$$

According to equation (34), the Hamiltonian function for the double pendulum is,

$$\begin{aligned} H &= (m_1+m_2)l_1^2\dot{\theta_1}^2 + m_2l_1l_2\dot{\theta_2}\dot{\theta_1}\cos(\theta_1-\theta_2) \\ &\quad + m_2l_2^2\dot{\theta_2}^2 + m_2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2) \\ &\quad - \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta_1}^2 - \frac{1}{2}m_2l_2^2\dot{\theta_2}^2 - m_2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2) \\ &\quad - (m_1+m_2)gl_1\cos\theta_1 - m_2gl_2\cos\theta_2. \end{aligned}$$

After arranging the terms, we have,

$$\begin{aligned} H &= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2l_2^2\dot{\theta_2}^2 + m_2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2) \\ &\quad - (m_1+m_2)gl_1\cos\theta_1 - m_2gl_2\cos\theta_2. \end{aligned}$$

Therefore, by solving the system of equations (35) and (36) for θ_1 and θ_2, then plug into H, obtain,

$$H = \frac{l_2^2 m_2 p_{\theta_1}^2 + l_2^2(m_1+m_2)p_{\theta_2}^2 - 2m_2 l_1 l_2 p_{\theta_1} p_{\theta_2}\cos(\theta_1-\theta_2)}{2l_1^2 l_2^2 m_2[m_1+\sin^2(\theta_1-\theta_2)m_2]} - m_2 g l_2\cos\theta_2 - (m_1+m_2)g l_1\cos\theta_1$$

By using the Hamilton's equation, we have

$$\dot{\theta_1} = \frac{\partial H}{\partial p_{\theta_1}} = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2}\cos(\theta_1-\theta_2)}{l_1^2 l_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{37}$$

$$\dot{\theta_2} = \frac{\partial H}{\partial p_{\theta_2}} = \frac{l_1(m_1+m_2)p_{\theta_2} - l_2 m_2 p_{\theta_1}\cos(\theta_1-\theta_2)}{l_1 l_2^2 m_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{38}$$

$$\dot{p_{\theta_1}} = -\frac{\partial H}{\partial \theta_1} = -(m_1+m_2)g l_1\sin\theta_1 - C_1 + C_2 \tag{39}$$

$$\dot{p_{\theta_2}} = -\frac{\partial H}{\partial \theta_2} = -m_2 g l_2\sin\theta_2 + C_1 - C_2. \tag{40}$$

Where C_1 and C_2 are,

$$C_1 = \frac{p_{\theta_1} p_{\theta_2}\sin(\theta_1-\theta_2)}{l_1 l_2[m_1+m_2\sin^2(\theta_1-\theta_2)]}$$

$$C_2 = \frac{l_2^2 m_2 p_1^2 + l_1^2(m_1+m_2)p_2^2 - l_1 l_2 m_2 p_1 p_2\cos(\theta_1-\theta_2)}{2l_1^2 l_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]^2}\sin[2(\theta_1-\theta_2)]$$

These equations can be solved numerically. The MATLAB codes are

```
function xprime=dp(t,x,flag,g,L1,L2,M1,M2)
C1=(x(3).*x(4).*sin(x(1)-x(2)))./...
    (L1*L2*(M1+M2*(sin (x(1)-x(2))).^2));
```

```
C2=((L2^2*M2*(x(3)).^2+L1^2*(M1+M2)*...
    (x(4)).^2-L1*L2*M2*x(3).*x(4).*cos(x(1)-x(2)))...
    ./(2*L1^2*L2^2*(M1+M2*sin(x(1)-x(2))).^2).^2)...
    .*sin(2*(x(1)-x(2)));
xprime=zeros(4,1);
xprime(1)=(L2*x(3)-L1*x(4).*cos(x(1)-x(2)))...
    ./(L1^2*L2*(M1+M2*(sin(x(1)-x(2))).^2));
xprime(2)=(L1*(M1+M2)*x(4)-L2*M2*x(3).*cos(x(1)-x(2)))...
    ./(L1*L2^2*M2*(M1+M2*sin(x(1)-x(2))).^2);
xprime(3)=-(M1+M2)*g*L1*sin(x(1))-C1+C2;
xprime(4)=-M2*g*L2*sin(x(2))+C1-C2;
```

And the function call is

```
close all
A1=2;
A2=3;
P1=1;
P2=2;
L1=1;
L2=2;
M2=3;
M1=1;
ti=3;
tf=5;
tspan=[ti tf]
InitialValue=[A1;A2;P1;P2];
[t,x]=ode45('doublependulum',tspan,InitialValue,[],9.8,L1,L2,M1,M2);
xplacement=L1*sin(x(1))+L2*sin(x(2));
```

```
yplacement=-L1*cos(x(1))-L2*cos(x(2));
plot(xplacement,yplacement)
```

Since the system is very sensitive, we could not use ode45 to solve these equations, unless we could pick some good initial conditions. However, since we know that, we are able to solve the system numerically by using the second argument in section 3.2, we don't have to use Hamilton's equations to show the chaotic motion of the double pendulum.

5. Numerical Solving

It is difficult to solve the equations we obtained in the previous sections. In addition, we can not view the chaotic motion of the double pendulum. In order to see the chaotic motion, we need software to plot the trace of the motion of the pendulum. There are tons of programs on the internet we can find to plot the motion of the pendulum[11]. However, at the time we wrote this paper, we also created a m-file, called `dpend.m`. We will use this m-file to explain a little bit of the chaotic motion.

There are three types of motion for the double pendulum. The first one is periodic motion, the second one is quasiperiodic motion, and the third one is chaotic motion.

5.1. Periodic Motion

If the energy of the double pendulum is low enough, it would appear to be periodic motion. In Figure 3, we can see the periodic motion of the double pendulum. The second bob moves back and forth in a certain path, and we can predict the position of

[11]For example, http://www.maths.tcd.ie/~plynch/SwingingSpring/doublependulum.html and http://www.zfm.ethz.ch/meca/applets/doppelpendel/dPendulum.html

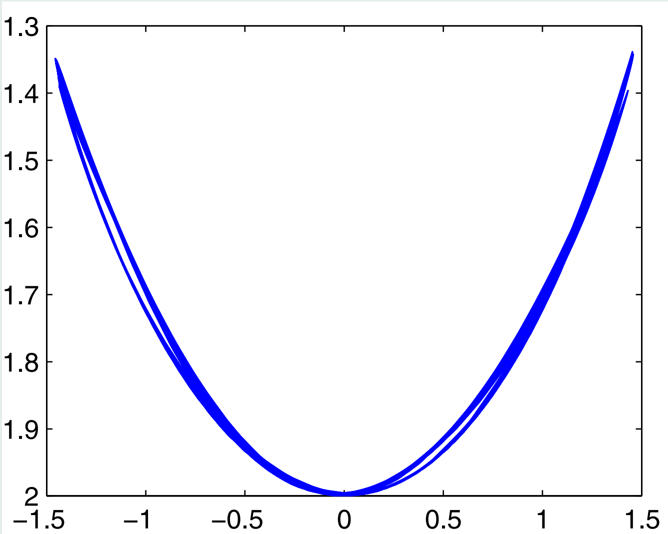

Figure 3: Periodic motion

the pendulum at every moment. In `dpend`, the input is

$$\begin{aligned} M1 &= 3 \\ M2 &= 3 \\ L1 &= 4 \\ L2 &= 3 \\ \text{AnVe1} &= 0 \\ \text{AnVe2} &= 0 \\ \text{Angle1} &= pi/4 \\ \text{Angle2} &= pi/4 \\ \text{tolerance} &= 1e-006 \\ \text{power} &= 1/3 \end{aligned}$$

5.2. Quasiperiodic Motion

If the energy of the double pendulum is at some threshold, where it is at the point between periodic and chaotic, we will see the quasiperiodic motion of the double pendulum in Figure 4. The motion has a certain period, but the bob can not repeat the previous path. The input is

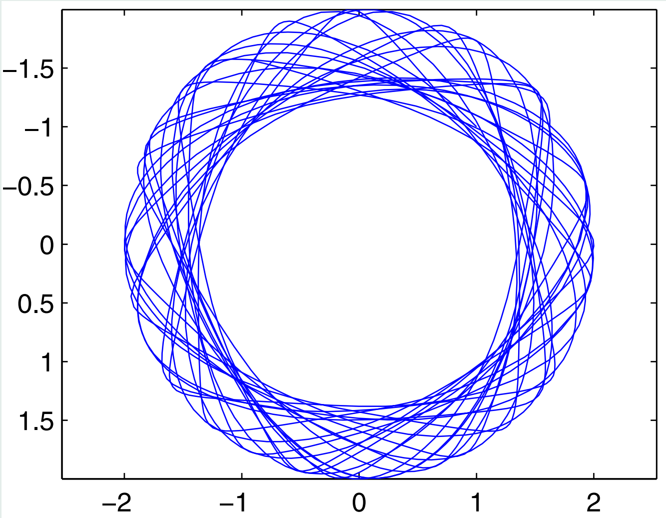

Figure 4: Quasiperiodic motion

$$M1 = 3$$
$$M2 = 3$$
$$L1 = 4$$
$$L2 = 3$$
$$\text{AnVe1} = 0$$
$$\text{AnVe2} = 10$$
$$\text{Angle1} = pi/2$$
$$\text{Angle2} = pi/2$$
$$\text{tolerance} = 1e - 006$$
$$\text{power} = 1/4$$

5.3. Chaotic motion

If the energy of the double pendulum is high enough, but not too high, the motion will be chaotic. In Figure 5, we can not predict the motion of the pendulum. The input is

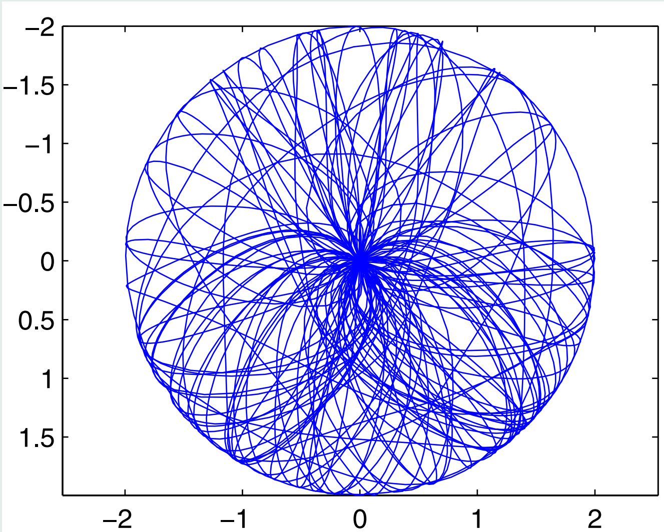

Figure 5: Chaotic motion

$$
\begin{aligned}
M1 &= 10\\
M2 &= 1\\
L1 &= 3\\
L2 &= 3\\
\mathrm{AnVe1} &= 2\\
\mathrm{AnVe2} &= 10\\
\mathrm{Angle1} &= pi\\
\mathrm{Angle2} &= pi\\
\mathrm{tolerance} &= 1e-006\\
\mathrm{power} &= 1/3
\end{aligned}
$$

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