Modeling An Economy's Growth

Hans Parshall

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Some Assumptions

- The community is fully employed in producing a single commodity.
- Production is a function of two factors, capital and labor.
- The commodity's output is also the community's income.
- Savings are kept in financial institutions, which invest the money back into the production of the commodity.

Some Definitions

- K = Capital Stock
- L = Supply of Labor
- $L = L_0 e^{nt}$
- $Y(K, L)$ = Output & Income
- s = Fraction of Income Saved
- $sY(K, L)$ = Total Amount Saved
- K' = Investment
- $K' = sY(K, L)$

Capital-Labor Ratio

$$
\begin{aligned}
r &= \frac{K}{L} \\
K &= rL \\
K' &= r'L + rL' \\
r' &= \frac{sY(K, L) - rL'}{L} \\
r' &= \frac{sY(rL_0 e^{nt}, L_0 e^{nt}) - nrL_0 e^{nt}}{L_0 e^{nt}}
\end{aligned}
$$

Constant Returns to Scale

- Production that exhibits constant returns to scale implies that if each of the factors of production are increased by a factor of λ, production is likewise increased by a factor of λ.
- $Y(\lambda K, \lambda L) = \lambda Y(K, L)$
- The production function Y is assumed to have constant returns to scale.

Simplifying Further

$$r' = \frac{sY(rL_0e^{nt}, L_0e^{nt}) - nrL_0e^{nt}}{L_0e^{nt}}$$

$$r' = \frac{L_0e^{nt}sY(r,1) - nrL_0e^{nt}}{L_0e^{nt}}$$

$$r' = sY(r,1) - nr$$

Economic Equilibrium

- Economic equilibrium is defined to be a state of economic growth in which there are no induced changes in relative prices of the factors of production over time.
- Suppose $r' > 0$, implying the supply of capital is growing faster than the supply of labor.

A Change in Relative Prices

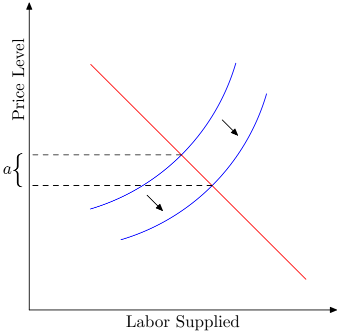


Figure 1: Changes in the Price Level by an Increasing Capital-Labor Ratio

Figure 1 shows an induced change in the relative prices of the factors of production; the price of capital has decreased more than the price of labor. Economic equilibrium cannot occur when $r' > 0$. A similar analysis of $r' < 0$ would result in a similar contradiction.

Constant Capital-Labor Ratio

- Economic equilibrium occurs when $r' = 0$, when r is a constant.
- $sY(r, 1) = nr$
- nr is a ray, but the shape of $sY(r, 1)$ is debatable.

Multiple Equilibriums

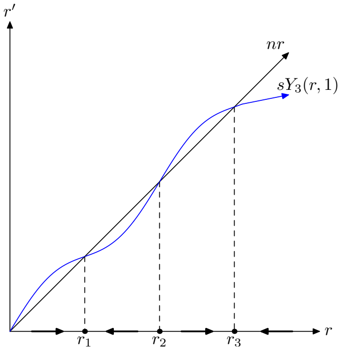


Figure 2: One Possibility

Figure 2 shows a scenerio with multiple equilibriums. r_1 and r_3 are asymptotically stable, while r_2 is unstable.

No Economic Equilibrium

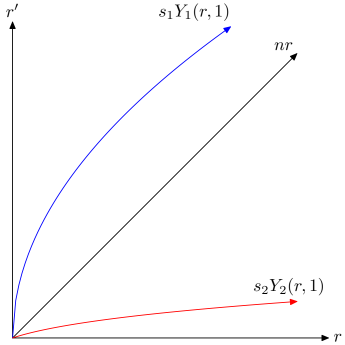


Figure 3: More Possibilities

Figure 3 shows two possibilities with diminishing marginal productivity, neither of which ever intersect with *nr*, resulting in no economic equilibrium.

The Cobb-Douglas Production Function

- (1) The marginal productivity of labor is proportional to the amount of production per unit of labor.

$$\frac{\partial P}{\partial L} = \beta \frac{P}{L}$$

- (2) The marginal productivity of capital is proportional to the amount of production per unit of capital.

$$\frac{\partial P}{\partial K} = \alpha \frac{P}{K}$$

- (3) If either labor or capital falls to zero, then production will fall to zero.

Assumption (1) & Assumption (2)

Assuming K is constant at K_0,

$$\frac{dP}{L} = \beta \frac{P}{L}$$

$$\int \frac{dP}{P} = \beta \int \frac{dL}{L}$$

$$\ln P = \beta \ln L + C$$

$$e^{\ln P} = e^{\ln L^{\beta} + C}$$

$$P(K_0, L) = C_1(K_0) L^{\beta}$$

Notice that C_1 has been written as a function of K_0, as its value may depend upon the value of K_0.
A similar procedure will show that $P(K, L_0) = C_2(L_0) K^{\alpha}$.

$P(K, L)$

- It's reasonable to hope that $P(K, L)$ would be some combination of $P(K_0, L)$ and $P(K, L_0)$.
- Cobb and Douglas picked $P(K, L) = bK^{\alpha}L^{\beta}$, and estimated that $b = 1.01$.
- This equation was justified by showing its accuracy in describing the United States economy from 1899 to 1923, and is still a commonly used production function in economics.

Assumption (3)

- If $\alpha = 0$, $P(K, L) = L^{\beta}$, and capital falling to zero would not have any effect on production.
- If $\alpha < 0$, $P(K, L) = \frac{L^{\beta}}{K^{-\alpha}}$, and capital falling to zero would cause production to rise to infinity.

$$\therefore \alpha > 0$$

- If $\beta = 0$, $P(K, L) = K^{\alpha}$, and labor falling to zero would not have any effect on production.
- If $\beta < 0$, $P(K, L) = \frac{K^{\alpha}}{L^{-\beta}}$, and labor falling to zero would cause production to rise to infinity.

$$\therefore \beta > 0$$

Constant Returns to Scale

Recall that the production function was assumed to have constant returns to scale, and must satisfy the following equation.

$$Y(\lambda K, \lambda L) = \lambda Y(K, L)$$

Attempting this for P,

$$\begin{aligned} P(\lambda K, \lambda L) &= (\lambda K)^{\alpha}(\lambda L)^{\beta} \\ &= \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} \\ &= \lambda^{\alpha+\beta} P(K, L) \end{aligned}$$

Note that if $\lambda^{\alpha+\beta} = \lambda$, $\alpha + \beta = 1$, and P can be simplified to

$$P(K, L) = K^{\alpha} L^{1-\alpha}$$

Shape of the Curve

- As the capital-labor ratio decreases, the marginal productivity of capital increases indefinitely, so $sP(r, 1)$ initially rises above the ray nr following the intersection at $(0, 0)$.
- As the capital-labor ratio increases, the marginal productivity of capital decreases, resulting in a curve tht is continuously concave down, asymptotically approaching a horizontal tracjectory.

The Graphical Representation

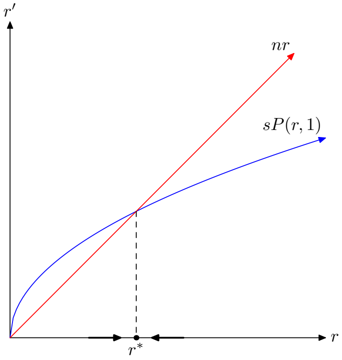


Figure 4: The Cobb-Douglas Production Function

The equilibrium point r^* shown in figure 4 can be found by expressing $P(K, L)$ as $P(r, 1)$, and returning to the previously found equation for r'.

The Equilibrium Capital-Labor Ratio

Finding $P(r, 1)$,

$$P(r, 1) = (r)^{\alpha}(1)^{1-\alpha} = r^{\alpha}$$

Plugging $P(r, 1)$ into the equation for r',

$$r' = sr^{\alpha} - nr$$

Setting $r' = 0$ and solving for r,

$$\begin{aligned} sr^{\alpha} - nr &= 0 \\ sr^{\alpha} &= nr \\ r^{1-\alpha} &= \frac{s}{n} \\ r &= \left(\frac{s}{n}\right)^{1/\beta} \end{aligned}$$

Finding K

Recall that K', investment, was assumed to equal the community's total income saved, $sP(K,L)$, and that $L = L_0 e^{nt}$.

$$
\begin{aligned}
K' &= sK^{\alpha}\left(L_0 e^{nt}\right)^{1-\alpha} \\
\int K^{-\alpha} dK &= sL_0^{\beta} \int e^{n\beta t} dt \\
\frac{1}{\beta} K^{\beta} &= \frac{sL_0^{\beta} e^{n\beta t}}{n\beta} + C \\
K &= \left(\frac{sL_0^{\beta} e^{n\beta t}}{n}\right)^{1/\beta} + C \\
&= \left(\frac{s}{n}\right)^{1/\beta} L_0 e^{nt} + C
\end{aligned}
$$

Solving for C

Assuming that at time $t = 0$, $K = K_0$,

$$K_0 = \left(\frac{s}{n}\right)^{1/\beta} L_0 + C$$

$$C = K_0 - \left(\frac{s}{n}\right)^{1/\beta} L_0$$

$$\Rightarrow K = K_0 + \left(\frac{s}{n}\right)^{1/\beta} L_0 \left(e^{nt} - 1\right)$$

Finding r

Dividing K through by L, and introducing the initial capital-labor ratio $r_0 = K_0/L_0$,

$$\frac{K}{L} = \frac{K_0 + \left(\frac{s}{n}\right)^{1/\beta} L_0 e^{nt} - \left(\frac{s}{n}\right)^{1/\beta} L_0}{L_0 e^{nt}}$$

$$r = r_0 e^{-nt} + \left(\frac{s}{n}\right)^{1/\beta} - \left(\frac{s}{n}\right)^{1/\beta} e^{-nt}$$

$$r = \left(\frac{s}{n}\right)^{1/\beta} + e^{-nt}\left(r_0 - \left(\frac{s}{n}\right)^{1/\beta}\right)$$

Looking closely at this equation confirms the previous results for r^*. As t grows infinitely large, e^{-nt} shrinks towards zero, and it can easily be seen that r approaches $(s/n)^{1/\beta}$, the previously calculated r^*.

Factors Affecting the Equilibrium

$$r^* = \left(\frac{s}{n}\right)^{1/\beta}$$

As β is a constant, the equilibrium capital-labor ratio, r^*, is based upon two factors.

- The fraction of income saved, s
- The natural growth rate of the labor population, n

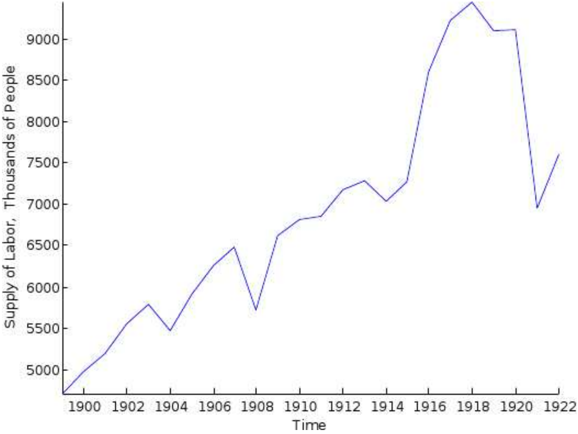
Reasonably enough, r^* is larger with a slow growing population that saves a large fraction of their income than a fast growing population that hardly saves any of their income.

Numerical Results

- $K_0 = 4449$, in millions of dollars
- $L_0 = 4713$, in thousands of people
- $\beta = .75$
- $n = .032$
- $s = .06$

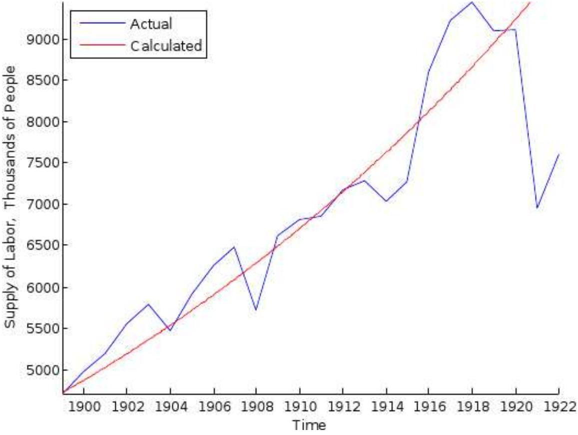
Where the n Came From...

$L = L_0 e^{nt}$



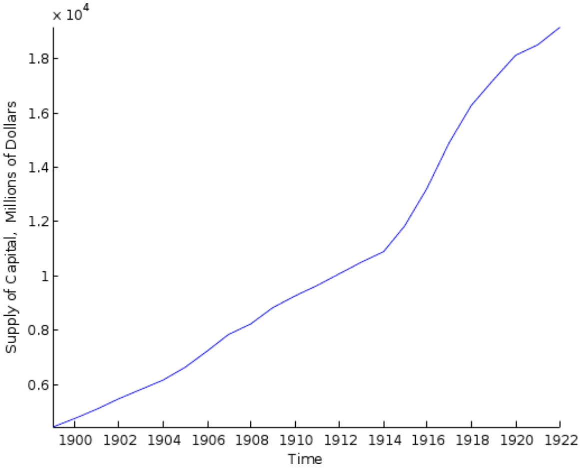
Where the n Came From...

$$L = L_0 e^{nt}$$



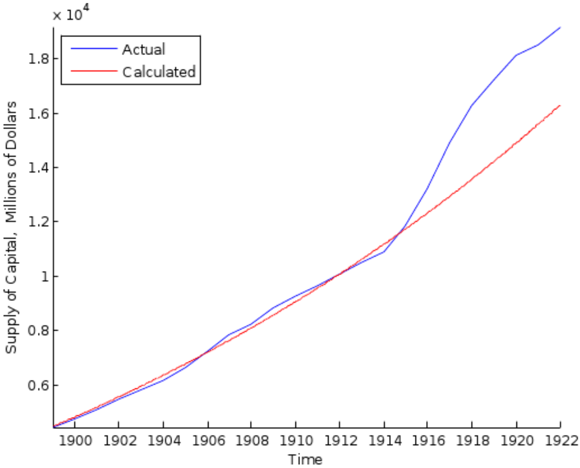
Where the s Came From...

$$K = K_0 + \left(\frac{s}{n}\right)^{1/\beta} L_0 \left(e^{nt} - 1\right)$$



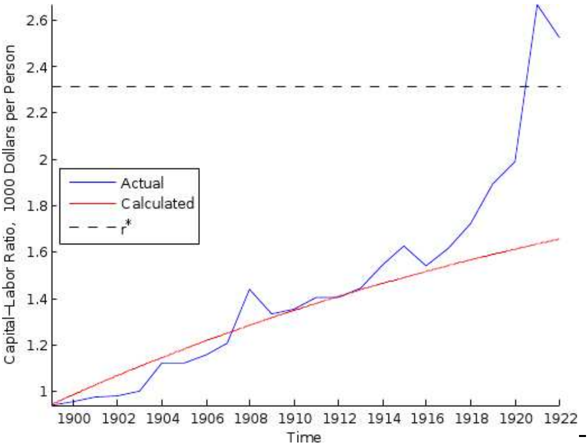
Where the s Came From...

$$K = K_0 + \left(\frac{s}{n}\right)^{1/\beta} L_0 \left(e^{nt} - 1\right)$$



The Capital-Labor Ratio Over Time

$$r = \left(\frac{s}{n}\right)^{1/\beta} + e^{-nt}\left(r_0 - \left(\frac{s}{n}\right)^{1/\beta}\right)$$



While this doesn't look good for the model, a larger sample size is truly needed.

Per-Capita Income

$P(r, 1)$ is the the community's per-capita income.

$$
\begin{aligned}
P(r, 1) &= r^{\alpha} \\
P(r^*, 1) &= \left(\frac{s}{n}\right)^{\alpha/\beta} \\
&= \left(\frac{.06}{.032}\right)^{.25/.75} \\
&= 1.233 \text{ Thousand Dollars per Person}
\end{aligned}
$$

$\Rightarrow$ The community's per-capita income will tend toward \$1233.

Comparing the Result to Reality

- \$1233 is in 1880 dollars, which are worth about \$19.61 each.
- $\$1233 * 19.61 = \$24,179$
- This is only 8% lower than the true per-capita income of \$26,352.

Improvements on the Model

- Most of the constants aren't really constants.
- s, n, β could all be expressed as a function of time.
- The supply of labor probably doesn't grow exogeneously.
- The Cobb-Douglas production function has even been modified to $A(t)K^{\alpha}L^{\beta}$, where A represents a function to represent technological progress.