15.5 Project
Surface Integrals and Rocket Nose Cones

The integrals involved in the problems for this project in the text are single-variable integrals that you should be able to evaluate manually. But the problems for Section 15.5 in the text provide examples of 2-dimensional surface integrals for which a computer algebra system is convenient (if not necessary). Here we illustrate typical CAS techniques using the Moebius strip of Problem 40. Given the parametrization $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$ described there, its surface area S is given by Equation (7) in Section 15.5,

$$S = \iint_D \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(z,x)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2}\, du\, dv \qquad (1)$$

where D denotes the uv-rectangle on which the Moebius strip is defined (as a parametric surface).

This Moebius strip is obtained by revolving (and twisting) a line segment of length 2 through a circle of radius 4. Thinking of the second theorem of Pappus (Section 14.5), does it seem plausible to you that the area of the Moebius strip might be close to $2\pi(4)\times(2) = 16\pi$, and that its centroid might be near the origin?

We set up and evaluate the surface integral (1) for the Moebius strip in the Maple, Mathematica, and MATLAB sections below. We also use the general surface area element $dS = \sqrt{[\cdots]^2 + [\cdots]^2 + [\cdots]^2}\, du\, dv$ in (1) to derive the 1-dimensional nose-cone integral

$$R = \int_0^1 \frac{2x\, dx}{1 + y'(x)^2} \qquad (2)$$

— which gives the drag coefficient R of a nose cone obtained by revolving the curve $y = y(x),\ 0 \le x \le 1$ about the y-axis — from the original nose-cone surface integral

$$R = \frac{1}{\pi} \iint_S \cos^3\phi\ dS. \qquad (3)$$

Using Maple

The Nose-Cone Integral For a nose cone S obtained by revolving the curve $y = y(x)$ around the y-axis, we obviously want to use polar coordinates r and θ in the plane. But for the sake of general notation that we can use in every problem, let's write $u = r$ and $v = \theta$. Then the nose cone is parameterized by

```
x := (u,v)->u*cos(v):
y := (u,v)->u*sin(v):
z := (u,v)->g(u):
```

First we calculate the partial derivatives that are needed.

```
xu := diff(x(u,v),u):
xv := diff(x(u,v),v):
yu := diff(y(u,v),u):
yv := diff(y(u,v),v):
zu := diff(z(u,v),u):
zv := diff(z(u,v),v):
```

Then our 2-by-2 Jacobian determinants are given by

```
with(linalg):
Jyz := det(array([[yu,yv],[zu,zv]])):
Jzx := det(array([[zu,zv],[xu,xv]])):
Jxy := det(array([[xu,xv],[yu,yv]])):
```

And the surface area element for S is

```
Sp :=
sqrt(simplify(Jyz^2 + Jzx^2 + Jxy^2));
        Sp := sqrt(diff(g(u),u)^2*u^2+u^2)

Sp := u*sqrt(1 + diff(g(u),u)^2):
```

Now $\cos\phi = dx/ds$ in the xy-plane. Here, the radial coordinate u plays the role of x and y is replaced with z, so

$$\cos\phi = \frac{du}{\sqrt{du^2 + dz^2}} = \frac{1}{\sqrt{1+(dz/du)^2}}.$$

Hence we define

```
cosphi := 1/sqrt(1 + diff(g(u),u)^2):
```

Consequently the integral of Eq. (3) above,

```
(1/Pi)*Int(Int(cosphi^3*Sp, u=0..1), v=0..2*Pi);
```

evaluates to the desired integral

```
R :=
Int(2*u/(1 + diff(g(u),u)^2), u=0..1);
```

of Eq. (2) in the introduction.

The Moebius Strip As described in Problem 40 of the text, our Moebius strip S with width 2 and center-line of radius 4 is parametrized by

```
x := (u,v) -> (u*cos(v/2) + 4)*cos(v):
y := (u,v) -> (u*cos(v/2) + 4)*sin(v):
z := (u,v) -> u*sin(v/2):
```

First we calculate the partial derivatives that are needed.

```
xu := diff(x(u,v),u):
xv := diff(x(u,v),v):
yu := diff(y(u,v),u):
yv := diff(y(u,v),v):
zu := diff(z(u,v),u):
zv := diff(z(u,v),v):
```

Then our 2-by-2 Jacobian determinants are given by

```
with(linalg):
Jyz := det(array([[yu,yv],[zu,zv]])):
Jzx := det(array([[zu,zv],[xu,xv]])):
Jxy := det(array([[xu,xv],[yu,yv]])):
```

And the surface area element for S is

```
Sp :=
sqrt(simplify(Jyz^2 + Jzx^2 + Jxy^2));
```

Sp := 1/2*(64+32*u*cos(1/2*v) + u^2 + 4*cos(1/2*v)^2*u^2)^(1/2)

Maple is unable to evaluate the surface integral symbolically, so we proceed numerically. The area of the Moebius strip is

```
S :=
evalf(Int(Int(Sp, u=-1..1), v=0..2*Pi, 6,_CCquad));
```

$$S := 50.3986$$

and the coordinates (x_c, y_c, z_c) of its centroid are given by

```
xc :=
(1/S)*evalf(Int(Int(x(u,v)*Sp, u=-1..1), v=0..2*Pi, 6,_CCquad));
```

$$xc := 0.0207775$$

```
yc :=
(1/S)*evalf(Int(Int(y(u,v)*Sp, u=-1..1), v=0..2*Pi, 6,_CCquad));
```

$$yc := -.593089*10^{-8}$$

```
zc :=
(1/S)*evalf(Int(Int(z(u,v)*Sp, u=-1..1), v=0..2*Pi ,6, _CCquad));
```

$$zc := -.2042477370\ 10^{-8}$$

Thus it appears that the centroid of the Moebius strip is approximately (0.0208, 0, 0), because symmetry suggests that the *y*- and *z*-coordinates of the centroid actually are zero. It may initially seem surprising that the *x*-coordinate of the centroid is not 0 also. But if you plot this Moebius strip, you may observe that its radial cross-section is horizontal on the right and vertical on the left. Does this suggest intuitively that the centroid should be displaced a bit to the right of the origin?

Using Mathematica

The Nose-Cone Integral For a nose cone S obtained by revolving the curve $y = y(x)$ around the *y*-axis, we obviously want to use polar coordinates r and θ in the plane. But for the sake of general notation that we can use in every problem, let's write $u = r$ and $v = \theta$. Then the nose cone is parameterized by

```
x = u*Cos[v];
y = u*Sin[v];
z = g[u];
```

First we calculate the partial derivatives that are needed.

```
xu = D[x, u];
xv = D[x, v];
yu = D[y, u];
yv = D[y, v];
```

```
zu = D[z, u];
zv = D[z, v];
```

Then our 2-by-2 Jacobian determinants are given by

```
Jyz = Det[{{yu, yv}, {zu, zv}}];
Jzx = Det[{{zu, zv}, {xu, xv}}];
Jxy = Det[{{xu, xv}, {yu, yv}}];
```

And the surface area element for S is

```
Sp = Sqrt[Simplify[Expand[Jxy^2 + Jyz^2 + Jzx^2]]]
Sqrt[u^2 (1 + g'[u]^2)]

Sp = u*Sqrt[1 + g'[u]^2];
```

Now $\cos\phi = dx/ds$ in the *xy*-plane. Here, the radial coordinate u plays the role of x and y is replaced with z, so

$$\cos\phi = \frac{du}{\sqrt{du^2+dz^2}} = \frac{1}{\sqrt{1+(dz/du)^2}}.$$

Hence we define

```
cosphi = 1/Sqrt[1 + g'[u]^2];
```

Consequently the integral of Eq. (3) in the introduction,

```
R =
(1/Pi) Integrate[Integrate[ cosphi^3 * Sp,
                     {u,0,1}], {v,0,2*Pi}]
2 Integrate[ u / (1 + g'[u]^2), {u, 0, 1}]
```

evaluates to the desired integral in Eq. (2)

The Moebius Strip As described in Problem 40 of the text, our Moebius strip with width 2 and center-line of radius 4 is parameterized by

```
x = (u*Cos[v/2] + 4)*Cos[v];
y = (u*Cos[v/2] + 4)*Sin[v];
z = u*Sin[v/2];
```

First we calculate the partial derivatives that are needed.

```
xu = D[x, u];
xv = D[x, v];
yu = D[y, u];
yv = D[y, v];
zu = D[z, u];
zv = D[z, v];
```

Then our 2-by-2 Jacobian determinants are given by

```
Jyz = Det[{{yu, yv}, {zu, zv}}];
Jzx = Det[{{zu, zv}, {xu, xv}}];
Jxy = Det[{{xu, xv}, {yu, yv}}];
```

And the surface area element for S is

```
Sp =
Sqrt[Simplify[Expand[Jxy^2 + Jyz^2 + Jzx^2]]]

Sqrt[16 + 3u^2/4 + 8u Cos[v/2] + u^2 Cos[v]/2
```

Mathematica is unable to evaluate the surface integral symbolically, so we proceed numerically. The area of the Moebius strip is

```
S =
NIntegrate[Sp, {u, -1,1}, {v, 0,2*Pi},
           AccuracyGoal -> 10]
50.3986
```

and the coordinates (x_c, y_c, z_c) of its centroid are given by

```
xc = 1/S*NIntegrate[x*Sp, {u, -1,1}, {v, 0,2*Pi},
          AccuracyGoal -> 10]
0.0207775

yc = 1/S* NIntegrate[y*Sp, {u, -1,1}, {v, 0,2*Pi},
          AccuracyGoal -> 10] // Chop
0

zc = 1/S* NIntegrate[z*Sp, {u, -1,1}, {v, 0,2*Pi},
          AccuracyGoal -> 10] // Chop
0
```

Thus it appears that the centroid of the Moebius strip is approximately (0.0208, 0, 0), because symmetry suggests that the *y*- and *z*-coordinates of the centroid actually are zero. It may initially seem surprising that the *x*-coordinate of the centroid is not 0 also. But if you plot this Moebius strip, you may observe that its radial cross-section is horizontal on the right and vertical on the left. Does this suggest intuitively that the centroid should be displaced a bit to the right of the origin?

Using MATLAB

The Nose-Cone Integral For a nose cone S obtained by revolving the curve $y = y(x)$ around the y-axis, we obviously want to use polar coordinates r and θ in the plane. But for the sake of general notation that we can use in every problem, let's write $u = r$ and $v = \theta$. Then the nose cone is parameterized by

```
syms u v
x = u*cos(v);
y = u*sin(v);
z = u^2;
```

First we calculate the partial derivatives that are needed.

```
xu = diff(x,u);
xv = diff(x,v);
yu = diff(y,u);
yv = diff(y,v);
zu = diff(z,u);
zv = diff(z,v);
```

Then our 2-by-2 Jacobian determinants are given by

```
Jyz = det([yu, yv; zu, zv]);
Jzx = det([zu, zv; xu, xv]);
Jxy = det([xu, xv; yu, yv]);
```

And the surface area element for S is

```
Sp = sqrt(simplify(Jyz^2 + Jzx^2 + Jxy^2))
Sp =
(4*u^4+u^2)^(1/2)

Sp = u*sqrt(1 + 4*u^2);
```

Now $\cos\phi = dx/ds$ in the xy-plane. Here, the radial coordinate u plays the role of x and y is replaced with z, so

$$\cos\phi = \frac{du}{\sqrt{du^2 + dz^2}} = \frac{1}{\sqrt{1 + (dz/du)^2}}.$$

Hence we define

```
cosphi = 1/sqrt(1 + zu^2);
```

Consequently the integral of Eq. (1) in the introduction gives

```
R = (1/pi)*int(int(cosphi^3*Sp, u,0,1), v,0,2*pi);
double(R)
ans =
0.4024
```

You need only edit the initial definition of $z = g(u)$ to calculate the air-resistance coefficient for another nose cone of different shape.

The Moebius Strip As described in Problem 40 of the text, our Moebius strip with width 2 and center-line of radius 4 is parametrized by

```
x = (u*cos(v/2) + 4)*cos(v);
y = (u*cos(v/2) + 4)*sin(v);
z = u*sin(v/2);
```

First we calculate the partial derivatives that are needed.

```
xu = diff(x,u);
xv = diff(x,v);
yu = diff(y,u);
yv = diff(y,v);
zu = diff(z,u);
zv = diff(z,v);
```

Then our 2-by-2 Jacobian determinants are given by

```
Jyz = det([yu, yv; zu, zv]);
Jzx = det([zu, zv; xu, xv]);
Jxy = det([xu, xv; yu, yv]);
```

And the surface area element for S is

```
Sp = sqrt(simplify(Jyz^2 + Jzx^2 + Jxy^2))
Sp =
1/2*(64+u^2+32*u*cos(1/2*v)+4*u^2*cos(1/2*v)^2)^(1/2)
```

In order to evaluate the needed surface integrals efficiently — numerically rather than symbolically — we define the area element dS and the area moments $x\,dS$, $y\,dS$, $z\,dS$ inline functions. Then the area of the Moebius strip is given by

```
dS = inline('(1/2)*(64+u.^2+32*u.*cos(v/2)+
               4*cos(v/2).^2*(u.^2)).^(1/2)');

S = dblquad(dS, -1,1, 0,2*pi)
S =
   50.3986
```

and the coordinates of its centroid (xc, yc, zc) are given by

```
xdS = inline('(u.*cos(v/2)+4).*cos(v).*
              (1/2).*(64+u.^2+32*u.*cos(v/2)+
               4*cos(v/2).^2*(u.^2)).^(1/2)');

xc = dblquad(xdS, -1,1, 0,2*pi)/S
xc =
    0.0208

ydS = inline('(u.*cos(v/2)+4).*sin(v).*
              (1/2).*(64+u.^2+32*u.*cos(v/2)+
               4*cos(v/2).^2*(u.^2)).^(1/2)');

yc = dblquad(xdS, -1,1, 0,2*pi)/S
yc =
     0

zdS = inline('u.*sin(v/2).*
              (1/2).*(64+u.^2+32*u.*cos(v/2)+
               4*cos(v/2).^2*(u.^2)).^(1/2)');

zc = dblquad(zdS, -1,1, 0,2*pi)/S
yc =
      4.4058e-018
```

Thus it appears that the centroid of the Moebius strip is approximately (0.0208, 0, 0), because symmetry suggests that the *y*- and *z*-coordinates of the centroid actually are zero. It may initially seem surprising that the *x*-coordinate of the centroid is not 0 also. But if you plot this Moebius strip, you may observe that its radial cross-section is horizontal on the right and vertical on the left. Does this suggest intuitively that the centroid should be displaced a bit to the right of the origin?