2.3 Inverses in Matlab

In this section we will discuss the inverse of a matrix and how it relates to solving systems of equations. Not all matrices have inverses and this leads seamlessly to the discussion of the determinant. Finally, Matlab has some powerful built-in routines for solving systems of linear equations and we will investigate these as well.

Let's begin with a discussion of the identity matrix.

The Identity Matrix

In this section we will restrict our attention to *square matrices*; i.e., matrices of dimension $n \times n$, i.e., matrices having an equal number of rows and columns. A square matrix having ones on its main diagonal and zeros in all other entries is called an *identity matrix*. For example, a 3×3 identity matrix is the matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To see why I is called the identity matrix, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Note that A times the first column of I is

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}.$$

Multiplying matrix A times $[1, 0, 0]^T$ simply strips off the first column of matrix A. In similar fashion, it is not hard to show that multiplying matrix A times $[0, 1, 0]^T$ and $[0, 0, 1]^T$, the second and third columns of I, strips off the second and third columns of matrix A.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Hence,

[1] Copyrighted material. See: http://msenux.redwoods.edu/Math4Textbook/
[2] Copyrighted material. See: http://msenux.redwoods.edu/IntAlgText/

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = A.$$

Thus, $AI = A$. We'll leave it to our readers to check that $IA = A$. This result should make it clear why matrix I is called the identity matrix. If you multiply any matrix by I and you get the identical matrix back.

You use Matlab's **eye** command to create an identity matrix.

```
>> I=eye(3)
I =
     1     0     0
     0     1     0
     0     0     1
```

Checking the result above is a simple matter of entering the matrix A and performing the multiplications AI.

```
>> A=[1 2 3;4 5 6;7 8 9];
>> A*I
ans =
     1     2     3
     4     5     6
     7     8     9
```

The matrix I commutes with any matrix A.

```
>> I*A
ans =
     1     2     3
     4     5     6
     7     8     9
```

Note that $AI = A$ and $IA = A$.

Identity Property. Let A be a square matrix; i.e., a matrix of dimension $n \times n$. Create a square matrix I of equal dimension $(n \times n)$ which has ones on the main diagonal and all other entries are zero. Then,

$$AI = IA = A$$

The matrix I is called the **identity matrix**.

The Inverse of a Matrix

In the real number system, the number 1 acts as the multiplicative identity. That is,

$$a \cdot 1 = 1 \cdot a = a.$$

Then, for any nonzero real number a, there exists another real number, denoted by a^{-1}, such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

The number a^{-1} is called the *multiplicative inverse* of the number a. For example, $5 \cdot (1/5) = 1$, so the multiplicative inverse of 5 is $1/5$. That is, $5^{-1} = 1/5$.

Zero, however, has no multiplicative inverse, because there is no number whose product with zero will equal 1.

The situation with square matrices is similar. We have already established that I is the muliplicative identity; that is,

$$AI = IA = A$$

for all square matrices A. The next question to ask is this: given a square matrix A, can we find another square matrix A^{-1}, such that

$$AA^{-1} = A^{-1}A = I.$$

The answer is "Sometimes."

To find out when a square matrix has an inverse, we must first introduce the concept of the *determinant*. Every square matrix has a unique number associated with it that is called the determinant of the matrix. Finding the determinant of a 2×2 matrix is simple.

Determinant of a 2×2 Matrix. Let A be a 2×2 matrix, then the determinant of A is given by the following formula.

$$\det(A) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Let's look at an example.

▶ **Example 1.** *Compute the determinant of*

$$A = \begin{bmatrix} 3 & 3 \\ 5 & -8 \end{bmatrix}.$$

Using the formula,

$$\det\left(\begin{bmatrix} 3 & 3 \\ 5 & -8 \end{bmatrix}\right) = (3)(-8) - (5)(3) = -39.$$

Matlab calculates determinants with ease.

```
>> A=[3 3;5 -8]
A =
     3     3
     5    -8
>> det(A)
ans =
   -39
```

Finding the determinants of higher order square matrices is more difficult. We encourage you to take a good linear algebra class to find our how to find higher ordered determinants. However, in this class, we'll let Matlab do the job for us.

▶ **Example 2.** *Find the determinant of the matrix*

$$A = \begin{bmatrix} 1 & -2 & 2 & 0 & -2 \\ -1 & -1 & -1 & -2 & 0 \\ 1 & -2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & -2 \\ -1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

We simply load the matrix into Matlab, then use Matlab's determinant operator **det**.

```
>> A=[1 -2 2 0 -2;-1 -1 -1 -2 0;1 -2 0 0 0;-1 -1 1 0 -2;-1 1 0 1
0]
A =
     1    -2     2     0    -2
    -1    -1    -1    -2     0
     1    -2     0     0     0
    -1    -1     1     0    -2
    -1     1     0     1     0
>> det(A)
ans =
     4
```

There is a simple test to determine whether a square matrix has an inverse.

Determining if an Inverse Exists. If the determinant of a square matrix is nonzero, then the inverse of the matrix exists.

Let's put this to the test.

▶ **Example 3.** *Determine if the inverse of the matrix*

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 1 & -4 & 1 \\ 0 & 6 & -2 \end{bmatrix}$$

exists. If it exists, find the inverse of the matrix.

First, determine the determinant of the matrix A.

```
>> A=[2 -3 2;1 -4 1;0 6 -2];
>> det(A)
ans =
    10
```

The determinant of A is 10, hence nonzero. Therefore, the inverse of matrix A exists. It is easily found with Matlab's **inv** command.

```
>> B=inv(A)
B =
    0.2000    0.6000    0.5000
    0.2000   -0.4000   -0.0000
    0.6000   -1.2000   -0.5000
```

We can easily check that matrix B is the inverse of matrix A. First, note that $AB = I$.

```
>> A*B
ans =
     1     0     0
     0     1     0
     0     0     1
```

Second, note that $BA = I$.

```
>> B*A
ans =
    1.0000         0         0
         0    1.0000    0.0000
         0    0.0000    1.0000
```

There is a little bit of roundoff error present here, but still enough evidence to see that $BA = I$. Hence, B is the inverse of matrix A.

Let's look at another example.

▶ **Example 4.** *Determine if the inverse of the matrix*

$$A = \begin{bmatrix} 5 & 0 & 5 \\ -5 & 3 & -8 \\ 2 & 0 & 2 \end{bmatrix}$$

exists. If it exists, find the inverse of the matrix.

Load the matrix into Matlab and calculate its determinant.

```
>> A=[5 0 5;-5 3 -8;2 0 2];
>> det(A)
ans =
     0
```

The determinant is zero, therefore the matrix A has no inverse. Let's see what happens when we try to find the inverse with Matlab's **inv** command.

```
>> B=inv(A)
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 2.467162e-18.
B =
   1.0e+15 *
  -3.60287970189640                  0   9.00719925474099
   3.60287970189640   0.00000000000000  -9.00719925474099
   3.60287970189640                  0  -9.00719925474099
```

A singular matrix is one that has determinant zero, or equivalently, one that has no inverse. Due to roundoff error, Matlab cannot determine exactly if the matrix is singular, but it suspects that this is the case. A check reveals that $AB \neq I$, evidence that matrix B is not the inverse of matrix A.

```
>> A*B
ans =
     1     0     0
     0     1     0
     0     0     0
```

Singular Versus Nonsingular. If the determinant of a matrix is zero, then we say that the matrix is **singular**. A singular matrix does not have an inverse. If the determinant of a matrix is nonzero, then we say that the matrix is **nonsingular**. A nonsingular matrix is invertible; i.e., the nonsingular matrix has an inverse.

Solving Systems of Linear Equations

Consider the system of linear equations

$$\begin{aligned} 2x + y + 3z &= -2 \\ x - 2y + 5z &= -13 \\ 3x + 4y - 2z &= 15 \end{aligned} \tag{2.1}$$

We can place this system into matrix-vector form

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 5 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -13 \\ 15 \end{bmatrix}. \tag{2.2}$$

System (2.2) is now in the form

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 5 \\ 3 & 4 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -2 \\ -13 \\ 15 \end{bmatrix}.$$

Let's calculate the determinant of the coefficient matrix A.

```
>> A=[2 1 3;1 -2 5;3 4 -2];
>> det(A)
ans =
    15
```

Thus, $\det(A) = 15$. Because the determinant of the matrix is nonzero, matrix A is nonsingular and A^{-1} exists. We can multiply both sides of equation $A\mathbf{x} = \mathbf{b}$ *on the left* (remembering that matrix multiplication is not commutative) to obtain

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

Matrix multiplication is associative, so we can change the grouping on the left-hand side of this last result to obtain

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}.$$

But multiplying inverses together produces the identity matrix I. Therefore,

$$I\mathbf{x} = A^{-1}\mathbf{b}.$$

Finally, because I is the identity matrix, $I\mathbf{x} = \mathbf{x}$, and we obtain the solution to the system $A\mathbf{x} = \mathbf{b}$.

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Using Matlab and the solution $\mathbf{x} = A^{-1}\mathbf{b}$, let's find the solution to the system (**2.1**. First, enter $\mathbf{b}$, the vector of the right-hand side of system (**2.1**) (or the equivalent system (**2.2**)).

```
>> b=[-2;-13;15]
b =
    -2
   -13
    15
```

Use Matlab to calculate the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

```
>> x=inv(A)*b
x =
    1.0000
    2.0000
   -2.0000
```

Thus, the solution of system (**2.1**) is $[x, y, z]^T = [1, 2, -2]^T$. This solution can be checked manually by substituting $x = 1$, $y = 2$, and $z = -2$ into each equation of the original system (**2.1**).

$$\begin{aligned} 2(1) + (2) + 3(-2) &= -2 \\ (1) - 2(2) + 5(-2) &= -13 \\ 3(1) + 4(2) - 2(-2) &= 15 \end{aligned}$$

Note that the solution satisfies each equation of system (**2.1**). However, we can also use Matlab to check that the result $\mathbf{x}$ satisfies the system's matrix-vector form (**2.2**) with the following calculation.

```
>> A*x
ans =
   -2.0000
  -13.0000
   15.0000
```

Note that the result $A\mathbf{x}$ equals the vector $\mathbf{b}$ (to roundoff error) and thus is a solution of system (**2.2**).

Matlab's backslash operator \ (also used for left division) can be used to solve systems of the form $A\mathbf{x} = \mathbf{b}$. If we might be allowed some leeway, the following abuse in notation outlines the key idea. We first "left-divide" both sides of the equation $A\mathbf{x} = \mathbf{b}$ on the left by the matrix A.

$$A\backslash A\mathbf{x} = A\backslash\mathbf{b}$$

On the left, $A\backslash A$ is again the identity, so this leads to the solution

$$\mathbf{x} = A\backslash\mathbf{b}.$$

Enter the following in Matlab.

```
>> x=A\b
x =
    1.0000
    2.0000
   -2.0000
```

Note that this matches the previous result that was found with the computation $\mathbf{x} = A^{-1}\mathbf{b}$.

One needs an introductory linear algebra course to fully understand the use of Matlab's backslash operator. For example, it's possible that a system has no solutions.

► **Example 5.** *Solve the system*

$$\begin{aligned} x - y &= 1 \\ -x + y - z &= 1 \\ 5x - 5y + 3z &= 1. \end{aligned}$$

This system can be written in matrix-vector form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 5 & -5 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Enter the matrix A in Matlab and check the value of its determinant.

```
>> A=[1 -1 0;-1 1 -1;5 -5 3];
>> det(A)
ans =
     0
```

Because the determinant is zero, matrix A is singular and has no inverse. Hence, we will not be able to solve this system using $\mathbf{x} = A^{-1}\mathbf{b}$.

Let's find out what will happen if we try to solve the system using Matlab's backslash operator. Enter the vector **b** in Matlab and execute **x=A\b**.

```
>> b=[1;1;1];
>> x=A\b
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 6.608470e-18.
x =
   1.0e+15 *
  -7.20575940379279
  -7.20575940379279
  -0.00000000000000
```

Note that Matlab thinks that the coefficient matrix is nearly singular, but cannot decide due to roundoff error. So the backslash operator attempts to find a solution but posts a pretty serious warning that the solution may not be accurate. Therefore, we should attempt to check. The result satisfies the equation if and only if $A\mathbf{x} = \mathbf{b}$.

```
>> A*x
ans =
     1
     1
     0
```

Note that the vector this computation returns is not the vector $\mathbf{b} = [1, 1, 1]^T$, so the solution found by Matlab's backslash operator is not a solution of $A\mathbf{x} = \mathbf{b}$.

It's also possible that a system could have an infinite number of solutions. For example, in a system of three equations in three unknowns, the three planes represented by the equations in the system could intersect in a line of solutions.

▶ **Example 6.** *Solve the system*

$$\begin{aligned} x - y &= 1 \\ -x + y - z &= 1 \\ 5x - 5y + 3z &= -1. \end{aligned}$$

This system can be written in matrix-vector form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 5 & -5 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Note that the coefficient matrix of this system is identical to that of the system in **Example 5**, but the vector $\mathbf{b}$ is slightly different. Because the determinant of the coefficient matrix is zero, matrix A is singular and has no inverse. Hence, we will not be able to solve this system using $\mathbf{x} = A^{-1}\mathbf{b}$.

What will happen if we try to solve the system using Matlab's backslash operator? Enter the vector $\mathbf{b}$ in Matlab and execute **x=A\b**.

```
>> x=A\b
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 6.608470e-18.
x =
    -1
    -2
    -2
```

Note that Matlab still thinks that the coefficient matrix is nearly singular, but cannot decide due to roundoff error. So the backslash operator attempts to find a solution but posts a pretty serious warning that the solution may not be accurate. Therefore, we should attempt to check. The result satisfies the equation if and only if $A\mathbf{x} = \mathbf{b}$.

```
>> A*x
ans =
     1
     1
    -1
```

The vector this computation returns does equal the vector $\mathbf{b} = [1, 1, -1]^T$, so the solution found by Matlab's backslash operator is a solution of $A\mathbf{x} = \mathbf{b}$.

However, this one calculation does not reveal the whole picture. Indeed, one can check that the vector $\mathbf{x} = [2, 1, -2]^T$ is also a solution.

```
>> x=[2;1;-2]; A*x
ans =
     1
     1
    -1
```

Note that $A\mathbf{x} = \mathbf{b}$, so $\mathbf{x} = [2, 1, -2]^T$ is a solution.

Indeed, one can use Matlab's *Symbolic Toolbox* to show that

$$\mathbf{x} = \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2.3}$$

is a solution of the system for all real values of α. First, declare **alpha** to be a symbolic variable.

```
>> syms alpha
```

Enter $\mathbf{x}$, as defined in **equation (2.3)**.

```
>> x=[-1;-2;-2]+alpha*[1;1;0]
x =
 -1+alpha
 -2+alpha
       -2
```

Now, calculate $A\mathbf{x}$ and compare the answer to the vector $\mathbf{b}$.

```
>> A*x
ans =
     1
     1
    -1
```

Thus, the vector $\mathbf{x} = [-1+\alpha, -2+\alpha, -2]^T$ is a solution of the system for all real values of α. For example, by letting $\alpha = 0$, we produce the solution produced by Matlab's backslash operator, $\mathbf{x} = [-1, -2, -2]^T$. By varying α, you can produce all solutions of the system. As a final example, if $\alpha = 10$, then we get the solution $\mathbf{x} = [9, 8, -2]^T$. Readers should use Matlab to check that this is actually a solution.

Explaining why this works is beyond the scope of this course. If you want to enter the fascinating world of solving systems of equations, make sure that you take a good introductory course in linear algebra. However, here are the pertinent facts.

Solving Systems of Equations. If you have a system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns, there are three possible solutions scenarios.

1. The system has exactly one unique solution.

2. The system has no solutions.

3. The system has an infinite number of solutions.

When working with square systems $A\mathbf{x} = \mathbf{b}$, that is, n equations in n unknowns, here are the facts.

1. If $\det(A) \neq 0$, then matrix A is nonsingular and A^{-1} exists. In this case, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. You can also find this solution with Matlab's backslash operator. That is, perform the calculation **x=A\b**.

2. If $\det(A) = 0$, then matrix A is singular and A^{-1} does not exist. In this case, the system $A\mathbf{x} = \mathbf{b}$ either has no solutions or an infinite number of solutions. If you use Matlab's backslash operator to calculate **x=A\b**, be sure to check that your solution satisfies $A\mathbf{x} = \mathbf{b}$.

2.3 Exercises

1. For each of the following matrices, form the appropriate identity matrix with Matlab's **eye** command, then use Matlab to show that $AI = IA = A$.

a)

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 4 \\ 0 & 0 & -5 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & 2 \\ 3 & -1 & -2 & 0 \end{bmatrix}$$

2. If you reorder the rows or columns of the identity matrix, you obtain what is known as a *permuation* matrix.

a) Form a 3×3 identity matrix with the Matlab command **I=eye(3)**. Swap the first and third columns of I with **P=I(:,[3,2,1])** to form a permutation matrix P. Now, enter

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Multiply A on the right by P. Explain the result. Next, multiply A on the left by P and explain the result.

b) Form a 4×4 identity matrix with the Matlab command **I=eye(4)**. Reorder the rows of I with the command **P=I([4,2,1,3],:)** to form a permutation matrix P. Now, enter

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

Multiply A on the left by P. Explain the result. Next, multiply A on the right by P and explain the result.

3. Use hand calculations to determine the determinant of the matrix

$$A = \begin{bmatrix} -3 & 1 \\ -2 & -4 \end{bmatrix}.$$

Use Matlab to verify your result.

4. Calculate the determinant of each of the following matrices, then classify the given matrix as *singular* or *non-singular*.

a)

$$A = \begin{bmatrix} 5 & -3 & -2 \\ -2 & 2 & 0 \\ -5 & 2 & 3 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 0 & 2 & 2 \\ -3 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 0 & -2 & 2 \\ 0 & -4 & -1 & 3 \\ 1 & -4 & 1 & 3 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

5. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then following formula shows how to find the inverse of a 2×2 matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Use hand calculations and the formula to determine the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ -4 & 6 \end{bmatrix}.$$

Use Matlab's **inv** command to verify your result. *Hint: You might want to use* **format rat** *to view your results.*

6. Use Matlab's **inv** command to find the inverse B of each of the following matrices. In each case, check your result by calculating AB and BA. Comment on each result. *Hint: Use the determinant to check if each matrix is singular or nonsingular.*

a)

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 1 & -5 & -1 \\ -5 & 1 & 5 \\ -4 & -2 & 4 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} -1 & 2 & 3 & 2 \\ -3 & 1 & -2 & 3 \\ 0 & 1 & -1 & 3 \\ 2 & -1 & 2 & -2 \end{bmatrix}$$

7. Set

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}.$$

Calculate $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$. Which two of these three expressions are equal?

8. Set

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}.$$

Use Matlab commands to determine if $(A^T)^{-1} = (A^{-1})^T$.

9. Set

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}.$$

Use Matlab commands to determine if $(A+B)^{-1} = A^{-1} + B^{-1}$.

In **Exercises 10-13**, place the given system in matrix form $A\mathbf{x} = \mathbf{b}$. Check the determinant of the coefficient matrix, then solve the system with $x = A^{-1}\mathbf{b}$. Check your result by showing that $A\mathbf{x} = \mathbf{b}$.

10.

$$\begin{aligned} 4y - 2z &= 1 \\ -x + y + z &= -1 \\ 4x - 2y &= 3 \end{aligned}$$

11.

$$\begin{aligned} x + z &= 3 \\ 4x + 2y + 2z &= 5 \\ 3x + 2y + 2z &= -1 \end{aligned}$$

12.

$$\begin{aligned} -w - 3x - y &= 2 \\ 3x + 2y - 2z &= 1 \\ 4w - 2y - 2z &= 10 \\ -2w + 2y &= 1 \end{aligned}$$

13.

$$\begin{aligned} w + 5x + 2y + z &= 4 \\ -4w - 2x - 2y + z &= 0 \\ -5w + x - 4y - z &= 3 \\ -4w - y + 2z &= 5 \end{aligned}$$

In **Exercises 14-20**, place the given system in matrix form $A\mathbf{x} = \mathbf{b}$. Check the determinant of the coefficient matrix, then solve the system with Matlab's backslash operator **x=A\b**. Check your result by showing that $A\mathbf{x} = \mathbf{b}$. Classify the system as having a unique solution, no solutions, or an infinite number of solutions.

14.

$$\begin{aligned} -x + y &= 3 \\ y - z &= 2 \\ x - y - 3z &= 4 \end{aligned}$$

15.

$$\begin{aligned} -5x - y + z &= 7 \\ x + 3y - 3z &= 0 \\ -5x + y - z &= 8 \end{aligned}$$

16.

$$\begin{aligned} 5x + 3y - 5z &= 11 \\ 2x - y - 2z &= 0 \\ -2x - y + 2z &= -5 \end{aligned}$$

17.

$$\begin{aligned} -2x - 2y - 2z &= 12 \\ x - 3y - z &= 4 \\ 4x + 3z &= 12 \end{aligned}$$

18.

$$\begin{aligned} -w + x &= 4 \\ w + x - 3z &= 12 \\ -3x - 2y + z &= 0 \\ -3w + 3x - 2y + 2z &= 5 \end{aligned}$$

19.

$$\begin{aligned} w + y - z &= 0 \\ 2w + x + y + z &= 1 \\ -2w - x - y - z &= -1 \\ 2w + 2x &= 0 \end{aligned}$$

20.

$$\begin{aligned} w + y - z &= 0 \\ 2w + x + y + z &= 2 \\ -2w - x - y - z &= -1 \\ 2w + 2x &= 0 \end{aligned}$$

2.3 Answers

1.

a) Enter matrix A and the identity for 2×2 matrices.

```
>> A=[1 2;4 8]; I=eye(2);
```

Note that AI and IA both equal A.

```
>> A*I
ans =
       1               2
       4               8
>> I*A
ans =
       1               2
       4               8
```

b) Enter matrix A and the identity for 3×3 matrices.

```
>> A=[1 2;4 8]; I=eye(2);
```

Note that AI equals A.

```
>> A*I
ans =
     1          0          2
     3         -2          4
     0          0         -5
```

Note that IA equals A.

```
>> I*A
ans =
     1          0          2
     3         -2          4
     0          0         -5
```

c) Enter matrix A and the identity for 4×4 matrices.

```
>> A=[1 2 0 0;1 -1 1 -1;
0 0 2 2;3 -1 -2 0];
>> I=eye(4);
```

Note that AI equals A.

```
>> A*I
ans =
  1       2       0       0
  1      -1       1      -1
  0       0       2       2
  3      -1      -2       0
```

Note that IA equals A.

```
>> I*A
ans =
  1       2       0       0
  1      -1       1      -1
  0       0       2       2
  3      -1      -2       0
```

3.

$$\begin{vmatrix} -3 & 1 \\ -2 & -4 \end{vmatrix} = (-3)(-4) - (-2)(1)$$
$$= 14$$

Enter the matrix in Matlab and calculate its determinant.

```
>> A=[-3 1;-2 -4];
>> det(A)
ans =
      14
```

5. If

$$A = \begin{bmatrix} 1 & 3 \\ -4 & 6 \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{18}\begin{bmatrix} 6 & -3 \\ 4 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 & -1/6 \\ 2/9 & 1/18 \end{bmatrix}.$$

Enter the matrix in Matlab, select rational format, then use Matlab's **inv** command to calculate the inverse.

```
>> A=[1 3;-4 6];
>> format rat
>> inv(A)
ans =
       1/3            -1/6
       2/9             1/18
```

7. Enter matrices A and B and calculate $(AB)^{-1}$.

```
>> format rat
>> A=[1 2;3 4]; B=[1 1;3 5];
>> inv(A*B)
ans =
     -23/4            11/4
      15/4            -7/4
```

Calculate $A^{-1}B^{-1}$.

```
>> inv(A)*inv(B)
ans =
     -13/2             3/2
       9/2            -1
```

Calculate $B^{-1}A^{-1}$.

```
>> inv(B)*inv(A)
ans =
     -23/4            11/4
      15/4            -7/4
```

Note that $(AB)^{-1} \neq A^{-1}B^{-1}$. However, $(AB)^{-1} = B^{-1}A^{-1}$.

9. Enter matrices A and B and calculate $(A+B)^{-1}$.

```
>> format rat
>> A=[1 1;2 -4];
>> B=[3 -1;2 5];
>> inv(A+B)
ans =
       1/4             0
      -1               1
```

Calculate $A^{-1} + B^{-1}$.

```
>> format rat
>> inv(A)+inv(B)
ans =
       49/51              23/102
       11/51               1/102
```

Therefore, $(+B)^{-1} \neq A^{-1} + B^{-1}$.

11. In matrix form,

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}.$$

Enter the matrix A and check the determinant of A.

```
>> A=[1 0 1;4 2 2;3 2 2];
>> det(A)
ans =
     2
```

Hence, A is nonsingular and the system has a unique solution. Enter the vector $\mathbf{b}$, then compute the solution $\mathbf{x}$ with $x = A^{-1}\mathbf{b}$.

```
>> b=[3; 5; -1];
>> x=inv(A)*b
x =
    6.0000
   -6.5000
   -3.0000
```

13. In matrix form,

$$\begin{bmatrix} 1 & 5 & 2 & 1 \\ -4 & -2 & -2 & 1 \\ -5 & 1 & -4 & -1 \\ -4 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 5 \end{bmatrix}$$

Enter the matrix A and check the determinant of A.

```
>> A=[1 5 2 1;-4 -2 -2 1;
-5 1 -4 -1;-4 0 -1 2];
>> det(A)
ans =
```

Hence, A is nonsingular and the system has a unique solution. Enter the vector $\mathbf{b}$, then compute the solution $\mathbf{x}$ with $x = A^{-1}\mathbf{b}$.

```
>> b=[4;0;3;5];
>> x=inv(A)*b
x =
   60.5000
   10.5000
  -93.0000
   77.0000
```

15. In matrix form,

$$\begin{bmatrix} -5 & -1 & 1 \\ 1 & 3 & -3 \\ -5 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 8 \end{bmatrix}.$$

Enter the matrix A and check the determinant of A.

```
>> A=[-5 -1 1;1 3 -3;-5 1 -1];
>> det(A)
ans =
     0
```

Hence, A is singular and the system either has no solutions or an infinite number of solutions. Enter the vector $\mathbf{b}$, then compute the solution $\mathbf{x}$ with

x=A\b.

```
>> b=[7;0;8];
>> x=A\b
Warning: Matrix is close to
singular or badly scaled.
Results may be inaccurate.
RCOND = 3.280204e-18.
x =
   -1.5000
    1.9231
    1.4231
```

Let's see if the answer checks.

```
>> A*x
ans =
    7.0000
    0.0000
    8.0000
```

Thus, it would appear that it is not the case that the system has no solutions. Therefore, the system must have an infinite number of solutions.

17. In matrix form,

$$\begin{bmatrix} -2 & -2 & -2 \\ 1 & -3 & -1 \\ 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \\ 12 \end{bmatrix}.$$

Enter the matrix A and check the determinant of A.

```
>> A=[-2 -2 -2;1 -3 -1;4 0 3];
>> det(A)
ans =
     8
```

Hence, A is nonsingular and the system either a unique solution. Enter the vector **b**, then compute the solution **x** with **x=A\b**.

```
>> b=[12;4;12];
>> x=A\b
x =
  -16.5000
  -15.5000
   26.0000
```

Let's see if the answer checks.

```
>> A*x
ans =
    12
     4
    12
```

This checks and the system has a unique solution.

19. In matrix form,

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 2 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 \\ 2 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Enter the matrix A and calculate the determinant.

```
>> A=[1 0 1 -1;2 1 1 1;
-2 -1 -1 -1;2 2 0 0];
>> det(A)
ans =
     0
```

Hence, A is singular and the system

either has no solutions or an infinite number of solutions. Enter the vector **b**, then compute the solution **x** with **x=A\b**.

```
>> b=[0;1;-1;0];
>> x=A\b
Warning: Matrix is singular
to working precision.
x =
       NaN
       NaN
       NaN
    0.5000
```

Unfortunately, the presence of **NaN** ("not a number") will prevent us from futher analysis. In a later chapter, we'll learn how to determine a solution with a technique called *Gaussian Elimination*.