Tacoma Narrows and the Gradient Vector

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Abstract

The mathematical models that have been proposed to explain the collapse of the Tacoma Narrows Bridge are highly dependent on the initial conditions. This paper explores the use of a gradient vector to find initial conditions that lead to periodic solutions.

1. Introduction

On November 7, 1940 the Tacoma Narrows Bridge failed in spectacular fashion. After only being open to the public for five months. Since then the Tacoma Narrows Bridge has been the source of endless debate by both scientists and engineers. Even being compared to technology's version of the Kennedy assassination. "There's the grainy black-and-white film endlessly scrutinized frame by frame; the reams of expert analysis next to impossible for a layperson to evaluate; and, of course, the buffs who are convinced that only they know the real story"[5]. Many mathematical models have been posed to explain the collapse of the bridge. Ranging from trigonometric models, to complex nonlinear differential equation models. In this paper we will look at how the current nonlinear differential equation model was created. We will select appropriate constants, based on wind tunnel testing, and the original data collected after the bridge was built. We will find approximate initial conditions that yield periodic solutions. Refining those approximations with Newton's method. Finally, we will examine some possible solutions that might prevent this from happening in the future.

2. Constants

In order to make our calculations in the following sections, we will need to approximate some of the real world constants that occurred on the day of collapse. According to [2] the mass of one foot of the bridge weighed approximately 5000 lb, so we can approximate the mass per linear foot as 2500 kg. The width of the bridge was approximately 12 m. The bridge would deflect about a half a meter when loaded with 100 kg. Therefore, we can calculate the spring constants. Using Hooke's Law, we see that

$$2Ky = mg \tag{1}$$

$$2K(.5m) = (100kg)(9.8m/s^2) \tag{2}$$

$$K \approx 1000\frac{kg}{s^2}. \tag{3}$$

The bridge was observed to rotate torsionally at approximately 12 to 14 cycles per minute. This gives us a torsional forcing term $\lambda \sin \mu t$, with the value of μ between 1.2 and 1.6. Finally it is estimated that the dampening coefficient of the bridge was approximately $\delta = 0.01$. We will use these constants in our vertical and torsional models.

3. The Vertical Model

The idea behind the vertical model is that there are two restoring forces acting on the bridge. The first restoring force is caused by the stiffness of the bridge, and can be modelled as a spring. Using Hooke's Law the restoring force of a spring is $k_1 y$, where y is the displacement of the spring. The second restoring force is caused by the cables that hold the bridge up, and can be modelled as a rubber band. These restoring forces can be seen clearly in Figure 1.

If we let $y(t)$ be the vertical displacement. Then the restoring force of the spring is given by Newton's second Law.

$$\sum F = ma$$

Which can be rewritten as

$$my'' = -k_1 y. \tag{4}$$

The restoring force of the cable is a piecewise defined function. The cable resists displacement in the downward direction. However, it cannot resist displacement in the upward direction. Therefore, the resistance of the cable is given by

$$my'' = -k_2 y, \quad y = \begin{cases} y & y > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5}$$

As you can see there are two distinct time periods in which the restoring forces are different. If $y > 0$ there are two restoring forces acting on the bridge. Whereas, if $y < 0$ there is only one restoring force acting on the bridge. If we let a be the combination $k_1 + k_2$, and let $b = k_2$, we can combine Equations (4) and (5), giving us

$$my'' = -ay^+ + by^-, \quad y^+ = \begin{cases} y & y > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } y^- = \begin{cases} -y & y < 0 \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

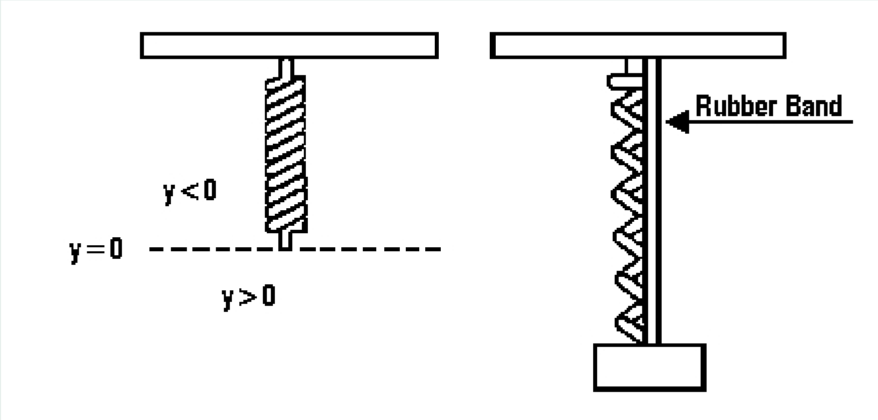

Figure 1: Vertical Model

The equation of a forced damped pendulum is given by

$$y'' + \delta y' + (g/l)y = \lambda \sin \mu t. \tag{7}$$

Where δ is the dampening constant, (g/l) is the restoring force, and $\lambda \sin \mu t$ is the forcing term. If we substitute our restoring force, and the dampening constant $\delta = 0.01$ into this equation, we obtain

$$y'' + 0.01y' + ay^+ - by^- = 10 + \lambda \sin \mu t. \tag{8}$$

Where a constant gravitational force 10 has been added. Giving us the model developed in [3]

4. The Torsional Model

In the torsional model we will assume that the bridge is suspended by two springs. As can be seen in Figure 2. With the bridge free to rotate around the center of the roadway.

The torsional model was developed in [2]. This model has studied as a past project, and the step-by-step derivation of the model can be obtained in [7]. Therefore, we will not spend a lot of time redeveloping this model.

According to [7] the angular velocity is given by the equation

$$\theta'' + \delta\theta' + \frac{6K}{m}\theta = \lambda \sin \mu t. \tag{9}$$

Substituting the constants $K = 1000\frac{kg}{s^2}$, $m = 2500kg$, and $\delta = 0.01$ into Equation (9), we obtain

$$\theta'' + 0.01\theta' + 2.4 \sin\theta = \lambda \sin \mu t. \tag{10}$$

5. The Gradient Vector

In the past, solutions for the vertical model and torsional model have been found using a variety of methods. The small amplitude solutions are fairly easy to find using the methods learned in an introductory differential equations course. However, the solutions that reproduce what was seen on the day of the collapse are not explained by these small amplitude solutions. It has been shown in several articles written by P.J. McKenna and Lisa Humphreys, that two more solutions exist. What is more interesting is that these two solutions match exactly the same amplitude and period observed at the bridge before it collapsed. In [3] a new method was proposed to help undergraduates find these solutions.

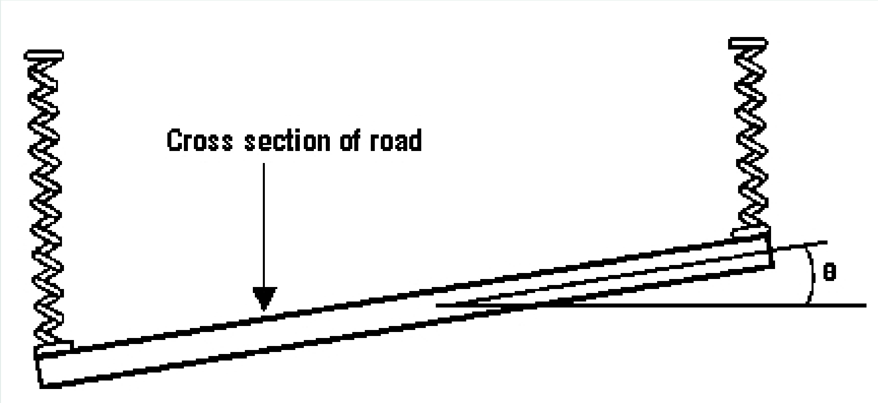

Figure 2: Torsional Model

We begin by looking at Equation (8). To find solutions to this problem we need to have an initial displacement and velocity. If we let c and d represent any initial displacement and velocity respectively. Then we have an initial value problem given by

$$y'' + 0.01y' + ay^{+} - by^{-} = 10 + \lambda \sin \mu t, \quad y(0) = c, \quad y'(0) = d. \tag{11}$$

We can replace a and b with the nonlinear constants 17 and 1 respectively, and we can substitute $\lambda = 0.1$ and $\mu = 4$, giving us

$$y'' + 0.01y' + 17y^{+} - y^{-} = 10 + 0.1 \sin 4t, \quad y(0) = c, \quad y'(0) = d. \tag{12}$$

To find solutions to this initial value problem we will use a gradient vector approach. If we begin at the point (c, d) and run and initial value solver for one period in time $T = 2\pi/\mu$. If this was a periodic solution to the initial value problem, we would end up back at the same point (c, d). However, the odds are not very good that this will be the case. After running the initial value solver for one period we can compute the error. Which is given by the function

$$E(c,d) = \big(c - y(T)\big)^2 + \big(d - y'(T)\big)^2. \tag{13}$$

If we were able to reduce the error, then we would be moving closer to a solution. In order to do this we must calculate the gradient vector.

$$\nabla E = \left\langle \frac{\partial E}{\partial c}, \frac{\partial E}{\partial d} \right\rangle \tag{14}$$

Evaluating the partial derivatives, and remembering that y and y' are functions of c and d, gives us

$$\frac{\partial E}{\partial c} = 2\big(c - y(T)\big)\left(1 - \frac{\partial y}{\partial c}(T)\right) + 2\big(d - y'(T)\big)\left(-\left(\frac{\partial y}{\partial c}\right)'(T)\right) \tag{15}$$

and

$$\frac{\partial E}{\partial d} = 2\big(c - y(T)\big)\left(-\frac{\partial y}{\partial d}(T)\right) + 2\big(d - y'(T)\big)\left(1 - \left(\frac{\partial y}{\partial d}\right)'(T)\right). \tag{16}$$

However, we still have not calculated $\frac{\partial y}{\partial c}$, $\frac{\partial y}{\partial d}$, $\left(\frac{\partial y}{\partial c}\right)'$, or $\left(\frac{\partial y}{\partial d}\right)'$. One way to calculate these partial derivatives is to compute the central difference for each partial derivative. For example, in the case of $\frac{\partial y}{\partial c}$ we compute the central difference by adding a tiny amount h to our initial displacement value c. Then we run the initial value solver for one period T and record the displacement value. Then we subtract the same amount from c and run the initial value solver again. The central difference is given by the formula.

$$\frac{\partial y}{\partial c}(T) = \frac{y(c+h,d) - y(c-h,d)}{2h}. \tag{17}$$

As you can see, we are taking the average of the differences. Similarly, we can compute the other partial derivatives that we need with the equations.

$$\frac{\partial y}{\partial d}(T) = \frac{y(c, d+h) - y(c, d-h)}{2h} \tag{18}$$

$$\left(\frac{\partial y}{\partial c}\right)'(T) = \frac{y'(c+h, d) - y'(c-h, d)}{2h} \tag{19}$$

$$\left(\frac{\partial y}{\partial d}\right)'(T) = \frac{y'(c, d+h) - y'(c, d-h)}{2h}. \tag{20}$$

Now we have all the information we need to compute the gradient.

The gradient vector may be of varying magnitude, and we will need to have a very fine control over how much we increment the initial conditions. Therefore, it makes sense to compute the gradient as a unit vector. In order to transform our gradient vector into a unit vector, we need to divide the gradient by its magnitude. The magnitude is given by

$$\sqrt{\left(\frac{\partial E}{\partial c}\right)^2 + \left(\frac{\partial E}{\partial d}\right)^2}. \tag{21}$$

However, the gradient points in the direction of steepest ascent and we need to minimize the error. Therefore, we can minimize our error by moving backwards a small step along our unit gradient vector. Giving us two new initial values. This can be repeated until you reach an acceptable level of error. We can formalize this into the equation

$$\begin{bmatrix} c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} c_n \\ d_n \end{bmatrix} - \epsilon \frac{\nabla E}{\|\nabla E\|}. \tag{22}$$

Where ϵ is a small number.

When performing these calculations with the computer, I found it most effective to write a nested loop. I began with a while loop which was controlled by the acceptable level of error that you want to achieve. The acceptable error levels are discussed in detail in Appendix A. One problem I encountered was if the ϵ values were not decreased throughout the calculations. Eventually the program would encounter a region where the error values stopped decreasing. I found that an efficient way to reduce the ϵ values was to nest an if loop into the main while loop. The if loop compares the error values to make sure that they were constantly decreasing.

Unfortunately the initial values obtained in this method require a lot of calculations, and are only accurate on the order of two to three significant figures. Since the solutions we are looking for are periodic over large time intervals, we need more accuracy.

6. Newton's Method

In order to refine our results we can use Newton's Method. Newton's Method takes our initial guess, which was obtained using the gradient vector approach. Approximates the tangent line at that point, and finds the point at which the tangent line equals zero. Then we use the new point as our next guess. Newton's method is very powerful, and converges very quickly to a high level of accuracy, if you begin with a good guess.

For our particular problem we are looking for periodic solutions to Equation (12). In order to refine our results obtained from the gradient method, we will use the definition of Newton's method as in [3]. The first step is to define a vector that will tell us how far off our initial guess was. This can be done by subtracting the position and velocity after one period T, from the results obtained by the gradient method. Giving us

$$F\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} y(T) \\ y'(T) \end{bmatrix}. \tag{23}$$

Next, we need to evaluate the Jacobian of the function F. The Jacobian is given by the matrix

$$\begin{bmatrix} 1-\dfrac{\partial y}{\partial c} & -\dfrac{\partial y}{\partial d} \\ -\left(\dfrac{\partial y}{\partial c}\right)' & 1-\left(\dfrac{\partial y}{\partial d}\right)' \end{bmatrix}. \tag{24}$$

Then, Newton's method is given by the equation

$$\begin{bmatrix} c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} c_n \\ d_n \end{bmatrix} - \begin{bmatrix} 1-\dfrac{\partial y}{\partial c} & -\dfrac{\partial y}{\partial d} \\ -\left(\dfrac{\partial y}{\partial c}\right)' & 1-\left(\dfrac{\partial y}{\partial d}\right)' \end{bmatrix}^{-1} F \begin{bmatrix} c_n \\ d_n \end{bmatrix}. \tag{25}$$

This is very easy to program, since the partial derivatives have already been calculated using central differences. I found that since Newton's method converges so quickly it was necessary to define a new error function. I calculated my error using the formula

$$E(c,d) = \big(c - y(T)\big) + \big(d - y'(T)\big). \tag{26}$$

7. Results

After running the program over a grid of initial values, equation (12) converged into three solutions. One was a small amplitude solution. Two were large amplitude solutions. Which can be seen in Figures 3, 4 , and 5.

It is obvious from these solutions, that the vertical model is highly dependent upon the initial conditions. Also it is interesting to note that a small forcing term can give rise to a very large amplitude wave. This is what was observed in the months leading up to the collapse of the Tacoma Narrows Bridge.

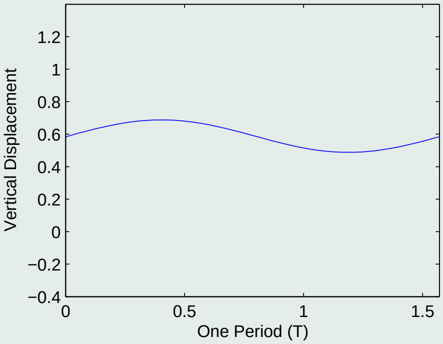

Figure 3: Small amplitude solution with initial conditions $c = 0.5842126367184$ and $d = 0.3993242265230$.

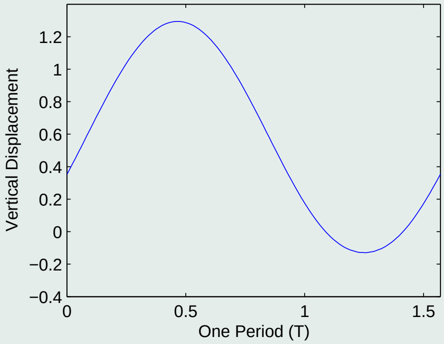

Figure 4: Large amplitude solution with initial conditions $c = 0.352529677237$ and $d = 2.731762638$.

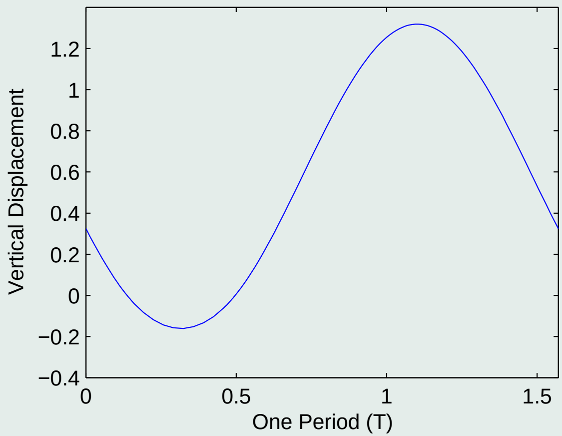

Figure 5: Large amplitude solution with initial conditions $c = 0.325$ and $d = -2.820$.

With the torsional model, I was also able to find three solutions. Using the same methods as for the vertical model, but replacing Equation (12) with Equation (10). Gives us the three solutions seen in Figures 6, 7, and 8. I also used the suggestion from McKenna and Humphreys in [3] to increase the amplitude of the forcing term from $\lambda = 0.02$ to $\lambda = 0.05$ in order to push the solutions further apart. I set the frequency of the forcing term to $\mu = 1.4$. It should be pointed out that the solutions seen in Figures 6 and 8, are not truly periodic solutions. It is obvious that the initial displacement and initial velocity do not match the position in velocity after one period of time T. I cannot explain this, because these initial conditions appear over a variety of starting points.

It is expected that a small amplitude forcing term would lead to a small amplitude oscillation. However, in Figures 7 and 8 we have a rotation of approximately one radian.

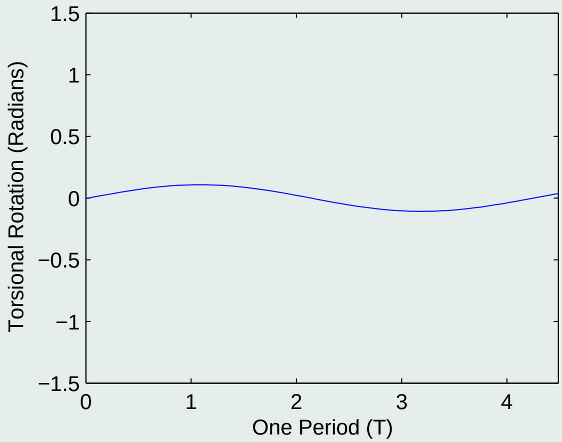

Figure 6: Small amplitude solution with initial conditions $c = -0.00369065407446$ and $d = 0.16039719801966$.

That is a rotation of approximately 57 deg, which is what was observed on the day of collapse.

8. Conclusion

According to [6] the collapse of the Tacoma Narrows Bridge has typically been attributed to "alternating vortices being shed by the bridge". With the frequency of these vortices just happening to match the resonant frequency of the bridge. However, later in the article it is pointed out that, "it was found that there is no sharp correlation between wind velocity and oscillation frequency" in suspension bridges. It also notes that a number of other bridges, such as the Golden Gate Bridge, "exhibit self-excitation

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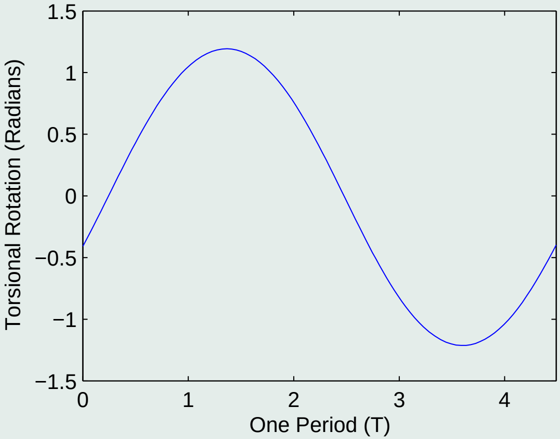

Figure 7: Large amplitude solution with initial conditions $c = -0.408357386906$ and $d = 1.5973907585260$.

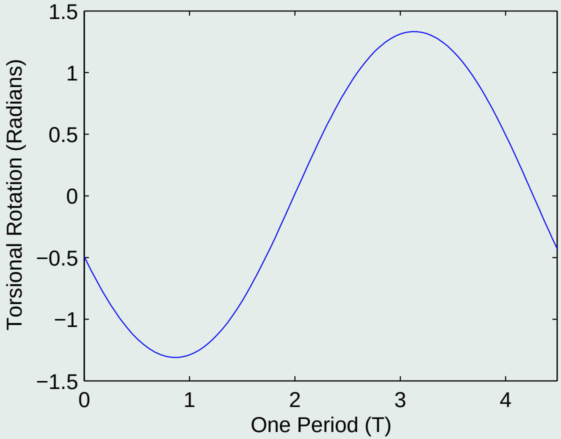

Figure 8: Large amplitude solution with initial conditions $c = -0.492611587706$ and $d = -1.7281509765743$.

under gusting". While there is no consensus on what caused the failure at the Tacoma Narrows Bridge, and is unlikely to ever be resolved. There are some interesting ideas on ways to prevent this from happening to bridges in the future.

The most logical place to begin is to look at the changes made to the new Tacoma Narrows Bridge. Since this bridge has not had the same problems with vertical and torsional oscillations. The designers must have fixed the problem. There is a great set of pictures in [6] that show the old Tacoma Narrows Bridge, contrasted with the new bridge. The old bridge had two lanes, and a flexible roadway. The new bridge is a larger four-lane highway, allowing it to resist greater torsional strain. The new bridge also has a large trussed section under the bridge. Significantly improving the stiffness of the roadway. These two factors, decrease the ratio of the cables restoring force versus the roadways. Another solution is discussed in [6], its authors believe that "A bridge less prone to oscillation would be created ... by having approximately equal restoring forces in the upward and downward direction." This would effectively stiffen the bridge. They propose putting a similar cabling structure to hold the bridge down, as is used to hold it up. Making the bridge act in a more linear fashion. While this would be ideal case, I find it hard to believe that we will see downward suspension cables on bridges in the near future. As this would severely limit shipping underneath such structures. Another solution that has been suggested, is to run two pipes underneath the bridge. These pipes would be empty for the majority of the time, but could be filled during extreme wind conditions. However, it has been pointed out, that while this might delay oscillations. If the oscillations began, the added mass would be detrimental to the bridge.

While the gradient method was extremely effective at finding periodic solutions for the vertical model. I am at a loss to explain its ineffectiveness with the torsional model. Unfortunately, there is still plenty of work to do. Given more time, I would like to explore the torsional model in further depth. Since, it is the torsional rotation that is credited with destroying the Tacoma Narrows Bridge.

9. Appendix A

9.1. Vertical Solution Finders

There are three vertical solution finders which can be obtained at http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Small_Amplitude_Vertical_Finder.m.
http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Large_Amplitude_Vertical_Finder_Pos.m.
http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Large_Amplitude_Vertical_Finder_Neg.m.
The user should be aware that not all random initial conditions converge to a solution. However, a large number of solutions will converge. The user will find it most effective to search a grid of initial conditions, Ex:($-3 \leq c \leq 3$ and $-3 \leq d \leq 3$). All three programs have the error tolerances set to the values that I found most efficient for the particular solutions I was looking for.

With the small amplitude solution finder. I found the most effective error tolerance values to be Error $= 7 \times 10^{-5}$ and NewtErr $= 1 \times 10^{-12}$. With these values I was able to converge to a solution with an accuracy of thirteen significant figures.

With the large amplitude positive initial velocity solution finder. I found the most effective error tolerance values to be Error $= 5 \times 10^{-7}$ and NewtErr $= 1 \times 10^{-12}$. With these values I was able to converge to a solution with an accuracy of ten significant figures.

With the large amplitude negative initial velocity solution finder. I found the most effective error tolerance values to be Error $= 5 \times 10^{-5}$ and NewtErr $= 1 \times 10^{-6}$. With these values I was able to converge to a solution with an accuracy of four significant figures.

9.2. Torsional Solution Finders

There are three torsional solution finders which can be obtained at http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Small_Amplitude_Torsional_Finder.m.
http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Large_Amplitude_Torsional_Finder_Pos.m.
http://online.redwoods.edu/instruct/danold/deproj/sp05/khuffman/Large_Amplitude_Torsional_Finder_Neg.m.
As before not all random initial conditions converge to a solution.

With the small amplitude solution finder. I found the most effective error tolerance values to be Error $= 1 \times 10^{-6}$ and NewtErr $= 1 \times 10^{-14}$. With these values I was able to converge to a solution with an accuracy of fourteen significant figures.

With the large amplitude positive initial velocity solution finder. I found the most effective error tolerance values to be Error $= 1 \times 10^{-7}$ and NewtErr $= 1 \times 10^{-12}$. With these values I was able to converge to a solution with an accuracy of twelve significant figures.

With the large amplitude negative initial velocity solution finder. I found the most effective error tolerance values to be Error $= 5 \times 10^{-5}$ and NewtErr $= 1 \times 10^{-6}$. With these values I was able to converge to a solution with an accuracy of twelve significant figures.

9.3. Using The Solution Finders

All programs are designed to be run using Matlab version 7. I am not an experienced programmer and I am sure there is a much more efficient way to program this. This said, I did not encounter any problems with this code. However, I take no responsibility for the use of this code.

1. Open Matlab version 7.
2. Open a new m-file environment.
3. Copy the desired solver into the m-file.
4. Set the initial values c and d.
5. Press the F5 key to execute the program.
6. Switch to the "Command Line" window.
7. The Error values will display in the "Command Line" window.
8. Allow the program to run for a reasonable amount of time. If the Error values stop decreasing for 30 seconds the initial conditions will probably not converge.
9. When the initial conditions that lead to a periodic solution have been found. The initial conditions will be displayed in the "Command Line" window.
10. The solution will be plotted for one time period in a Figure window.

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[8] Professor David Arnold, College of the Redwoods.