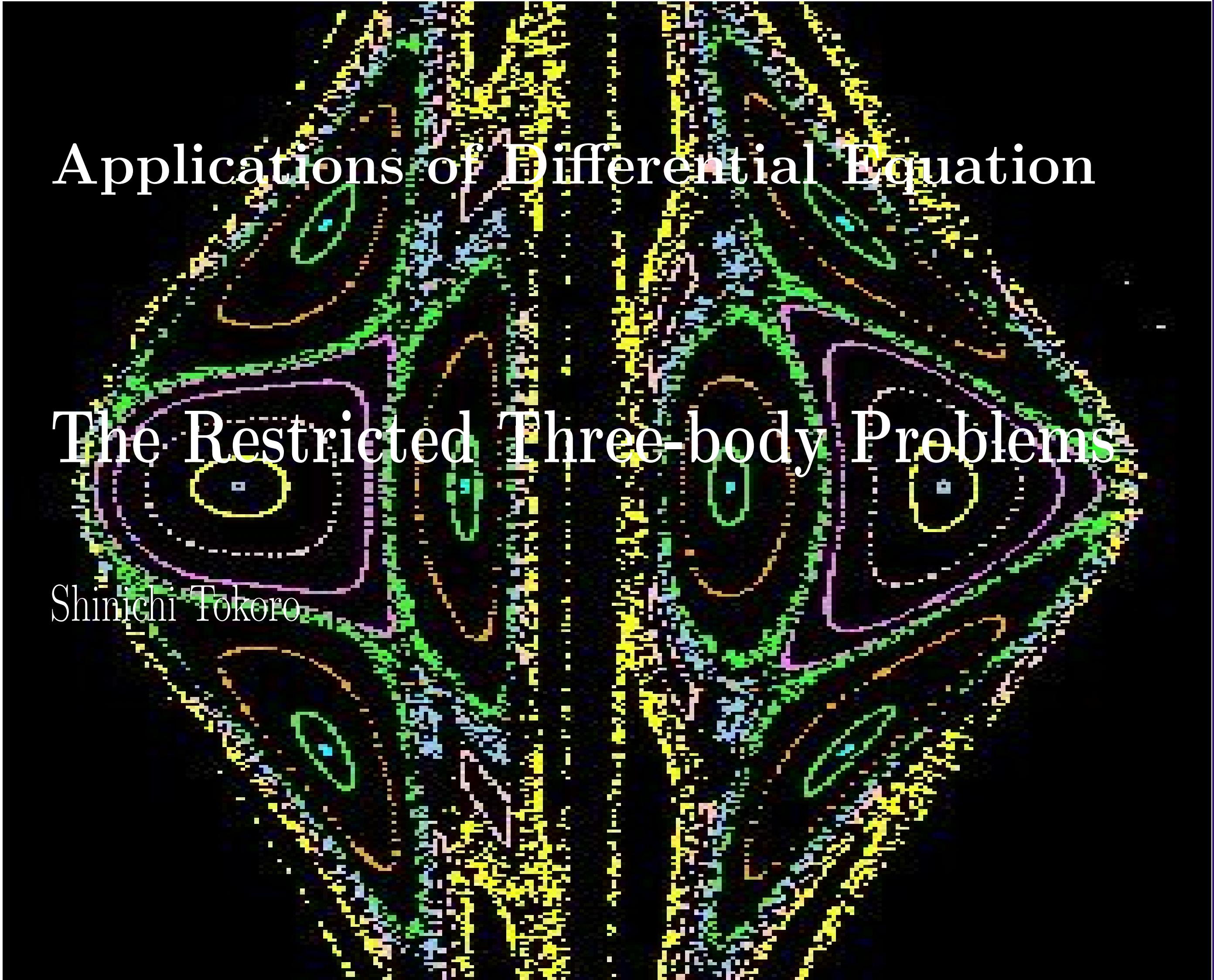

Applications of Differential Equation

The Restricted Three-body Problems

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Introduction

We will discus about the gravitational interaction among three bodies in the case of that third body has negligible mass compared with other two bodies. If the two bodies move around in circular, coplanar orbits about their common center of mass and the third mass is negligible so it doesn't affect the motion of the other two bodies, the problem of the third body is called the $circular, restricted, three-bodyproblems$. Although, real planetary orbitals and movements are not coplanar and circular, the restricted three-body problem give us reasonable approximation for certain systems. Mainly, we will discus the equations of the three-body problem, the location and stability of equilibrium points in the system.

Equations of Motion

We consider a tiny mass of particle move under the gravitational field of two masses m_1 and m_2, and, also two masses are creating circular orbit with their common center of mass. we use axes a, b, c as a frame referred to the center of the system and, the angular velocity and a constant separation are fixed about their common center of mass. If we chose the unit of mass such that $u = G(m_1 + m_2) = 1$, and we assume that $m_1 > m_2$ and define

$$\bar{u} = \frac{m_2}{m_1 + m_2}. \tag{1}$$

Then, in this system of units, the two masses are

$$u_1 = Gm_1 = 1 - \bar{u} \tag{2}$$

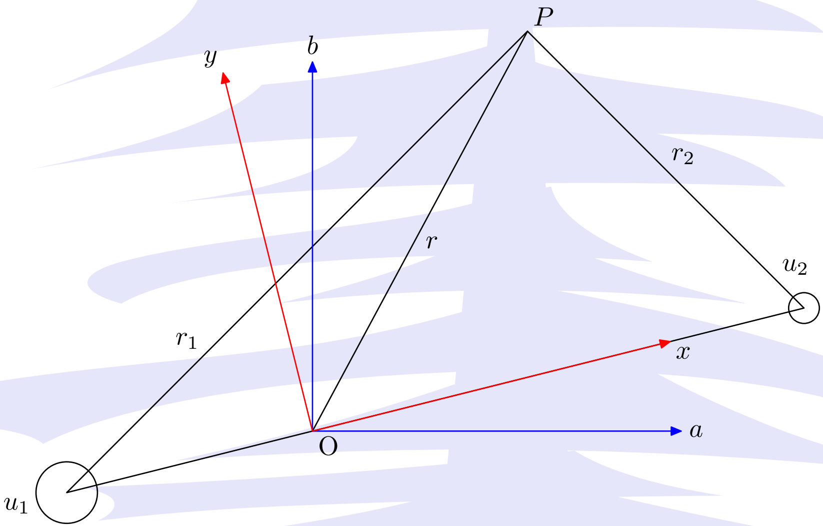


Figure 1: The relationship between the sidereal coordinates (a,b,c) and the synodic coordinates(x,y,z)

$$u_2 = Gm_2 = \bar{u} \tag{3}$$

where $\bar{u} < 1/2$. Let the coordinates of the particle in the $sidereal, system,$ be (a, b, c) and the equations of motion of the particle are

$$\ddot{a} = u_1 \frac{a_1 - a}{r_1^3} + u_2 \frac{a_2 - a}{r_2^3}, \tag{4}$$

$$\ddot{b} = u_1 \frac{b_1 - b}{r_1^3} + u_2 \frac{b_2 - b}{r_2^3}, \tag{5}$$

$$\ddot{c} = u_1 \frac{c_1 - c}{r_1^3} + u_2 \frac{c_2 - c}{r_2^3}, \tag{6}$$

where, from Fig.1

$$r_1^2 = (a_1 - a)^2 + (b_1 - b)^2 + (c_1 - c)^2, \tag{7}$$

$$r_2^2 = (a_2 - a)^2 + (b_2 - b)^2 + (c_2 - c)^2. \tag{8}$$

Then, we convert these to

$$r_1^2 = (x + u_2)^2 + y^2 + z^2, \tag{9}$$

$$r_2^2 = (x - u_2)^2 + y^2 + z^2, \tag{10}$$

where (x, y, z) are the coordinates of the particle with respect to the rotating, or $synodic, system$. There coordinates are related to the coordinates in the sidereal system by the simple rotation, and we can write this in the matrix form

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{11}$$

To obtain the equation of accelerations, we need take second derivertive of these equation.

$$\begin{pmatrix} \ddot{a} \\ \ddot{b} \\ \ddot{c} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2n\dot{y} - n^2 x \\ \ddot{y} + 2n\dot{x} - n^2 y \\ \ddot{z} \end{pmatrix}. \tag{12}$$

In the above matrix equation $n\dot{x}$ and $n\dot{y}$ are called *Corioli's acceleration* and n^2x and n^2y are called *the centrifugal acceleration*. Substitute for $a, b, c, \ddot{a}, \ddot{b}$, and $\ddot{c}$, (4),(5), and (6) become

$$(\ddot{x} - 2n\dot{y} - n^2x)\cos nt - (\ddot{y} + 2n\dot{x} - n^2y)\sin nt =$$

$$[u_1 \frac{x_1 - x}{r_1^3 + u_2} \frac{x_2 - x}{r_2^3}]\cos nt + [\frac{u_1}{r_1^3} + u_2 r_2^3]y \sin nt, \tag{13}$$

$$(\ddot{x} - 2n\dot{y} - n^2x)\sin nt + (\ddot{y} + 2n\dot{x} - n^2y)\cos nt =$$

$$[u_1 \frac{x_1 - x}{r_1^3 + u_2} \frac{x_2 - x}{r_2^3}]\sin nt - [\frac{u_1}{r_1^3} + u_2 r_2^3]y \cos nt, \tag{14}$$

$$\ddot{z} = -[\frac{u_1}{r_1^3} + \frac{u_2}{r_2^3}]z. \tag{15}$$

After the calculation, we can write these acceleration as the gradient of a scalar function U:

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}, \tag{16}$$

$$\ddot{y} - 2n\dot{x} = \frac{\partial U}{\partial y}, \tag{17}$$

$$\ddot{z} = \frac{\partial U}{\partial z}, \tag{18}$$

where $U = U(x, y, z)$ is given by

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{u_1}{r_1} + \frac{u_2}{r_2} \tag{19}$$

The Jacobi Integral

The Jacobi constant is determined by the particle's position and velocity. If we multiply Eq. (16) by $\dot{x}$, and Eq. (17) by $\dot{y}$, and Eq. (18) by $\dot{z}$ and adding them together, we get

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} = \frac{dU}{dt}, \tag{20}$$

and we integrate this and we get

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - C_j, \tag{21}$$

where C_j is the constant of integration. Since we know that $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$, the square of the velocity in the system is

$$v^2 = 2U - C_j \tag{22}$$

and from that we can write C_j by using equation (19),

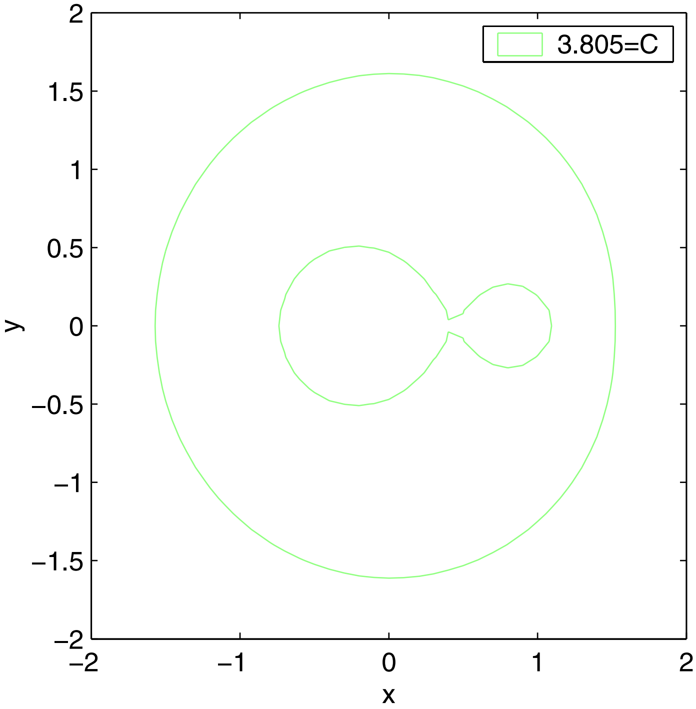
$$C_j = n^2(x^2 + y^2) + 2\left(\frac{u_1}{r_1} + \frac{u_2}{r_2}\right) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{23}$$

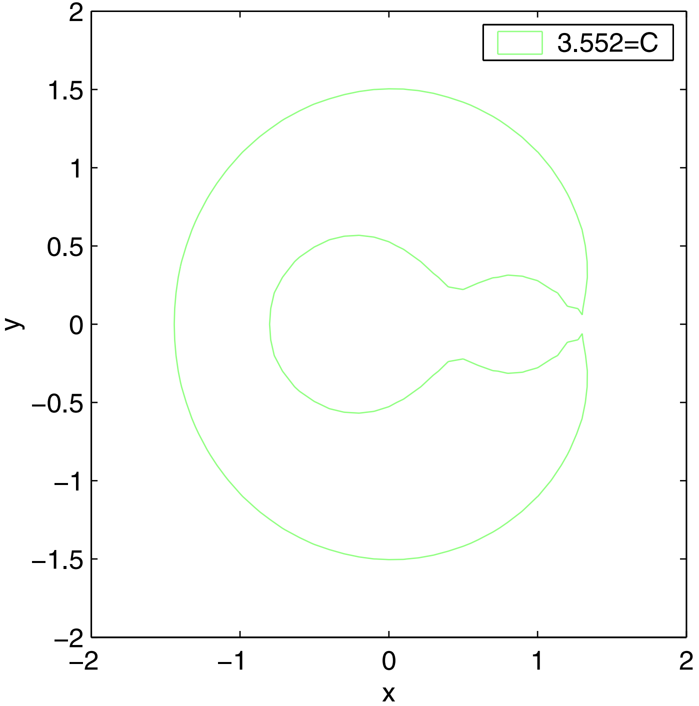
This means that the quantity $2U - v^2 = C_j$ is a constant of the motion and it is called *Jacobi constant*. Jacobi constant is only valid in the circular restricted three-body problem. However, it still useful tool to analyze the location of that the particle has zero velocity. In that case,

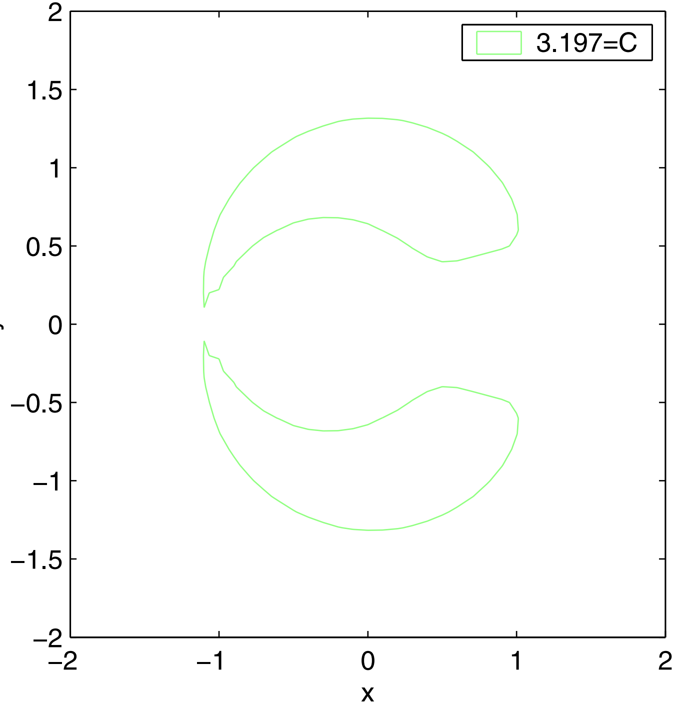
$$n^2(x^2 + y2) + 2\left(\frac{u_1}{r_1} + \frac{u_2}{r_2}\right) = C_j \tag{24}$$

The equation (24) defines boundary curves that particles cannot be. Jacobi constants permit us to find excluded region on the x-y plane, and some images are below with Jacobi constant $3.805, 3.552, 3.197$ where $u_2 = 0.2$.

Excluded Regions




3.552=C
y
x
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1
0
1
2


3.197=C
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
y
−2
−1
0
1
2
x

Lagrangian Equilibrium Points

Under the gravitational force by m_1 and m_2, the system has an equilibrium point at the apex of the equilateral triangle with a base formed of line go through two masses. From that we can assume there is another equilibrium point locates exactly opposite side of the line. They are called the *Lagrangian equilibrium points* L_4 and L_5.

Location of Equilibrium Points

Now we need find points where the particle has zero velocity and zero acceleration in the system. In this case we assume all events are happening on the $x-y$ plane. Also, we should state the unit of distance to be the constant separation of the two masses. To calculate the locations of the equilibrium points we rewrite U in a different form. From the equations of r_1 and r_2 in the equations (9) and (10), and using the fact $u_1 + u_2 = 1$, we get

$$U = u_1 \left(\frac{1}{r_1} + \frac{r_1^2}{2} \right) + u_2 \left(\frac{1}{r_2} + \frac{r_2^2}{2} \right) - \frac{1}{2} u_1 u_2. \tag{25}$$

Then, we need consider the equations (16) and (17) under the condition of $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$ that means both velocity and accelerations are zero. To find the locations of the equilibrium points we must solve simultaneous nonlinear equations

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x} = 0, \tag{26}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y} = 0 \tag{27}$$

by using the form $U = U(r_1, r_2)$ in (23) If we take the partial derivatives of these equation, we get

$$u_1\left(-\tfrac{1}{r_1^2} + r_1\right)\frac{x + u_2}{r_1} + u_2\left(-\tfrac{1}{r_2^2} + r_2\right)\frac{x - u_1}{r_2} = 0 \tag{28}$$

$$u_1\left(-\tfrac{1}{r_1^2} + r_1\right)\frac{y}{r_1} + u_2\left(-\tfrac{1}{r_2^2} + r_2\right)\frac{y}{r_2} = 0 \tag{29}$$

and if we look at equations (25),(26),(27),and (28) we see

$$\frac{\partial U}{\partial r_1} = u_1\left(-\tfrac{1}{r_1^2} + r_1\right) = 0, \tag{30}$$

$$\frac{\partial U}{\partial r_2} = u_2\left(-\frac{1}{r_2^2} + r_2\right) = 0. \tag{31}$$

so we can get $r_1 = r_2 = 1$ in our system, and then, Eq. (9) and (10) implies

$$(x + u_2)^2 + y^2 = 1 \tag{32}$$

$$(x - u_1)^2 + y^2 = 1 \tag{33}$$

with the two solutions

$$x = \frac{1}{2} - u_2$$

$$y = \pm\frac{\sqrt{3}}{2}.$$

According to above calculation we get $r_1 = r_2 = 1$, that means L_4 and L_5 are apex of an equilateral triangle with masses u_1 and u_2 and these points are called *the triangular Lagrangian equilibrium points*. Also, if we look at equations (27) easily we can see that $y = 0$ is one of solutions in Eq. (30), and that means other equilibrium points are on the x axis and satisfy Eq. (29). Actually, there are three more equilibrium points lie on the x axis, and we call them L_1, L_2, and L_3. L_1 locates at between u_1 and u_2. L_2 locates out side of u_2. L_3 locates outside of u_1. Now, we need analyze these points individually. After heavy calculation we get

$$r_2 = \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + O\left(\alpha^5\right) \tag{34}$$

for the point L_1. where

$$\alpha = \left(\frac{u_2}{3u_1}\right)^{1/3}, \tag{35}$$

and

$$r_2 = \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + O(\alpha^5) \tag{36}$$

for the point L_2.

$$\beta = -\frac{7}{12}\left(\frac{u_2}{u_1}\right) + \frac{7}{12}\left(\frac{u_2}{u_1}\right)^2 - \frac{13223}{20736}\left(\frac{u_2}{u_1}\right)^3 + O\left(\frac{u_2}{u_1}\right)^4 \tag{37}$$

for the point L_3 and we get r_1 and r_2 Then, if we plage in these values to the equation

$$n^2(x^2 + y^2) + 2\left(\frac{u_1}{r_1} + \frac{u_2}{r_2}\right) = C_j \tag{38}$$

where $n = 1$ and the value of u_2 is chosen by us, we obtain *Jacobi constant* for each points.

If we chose $u_2 = 0.2$ we get

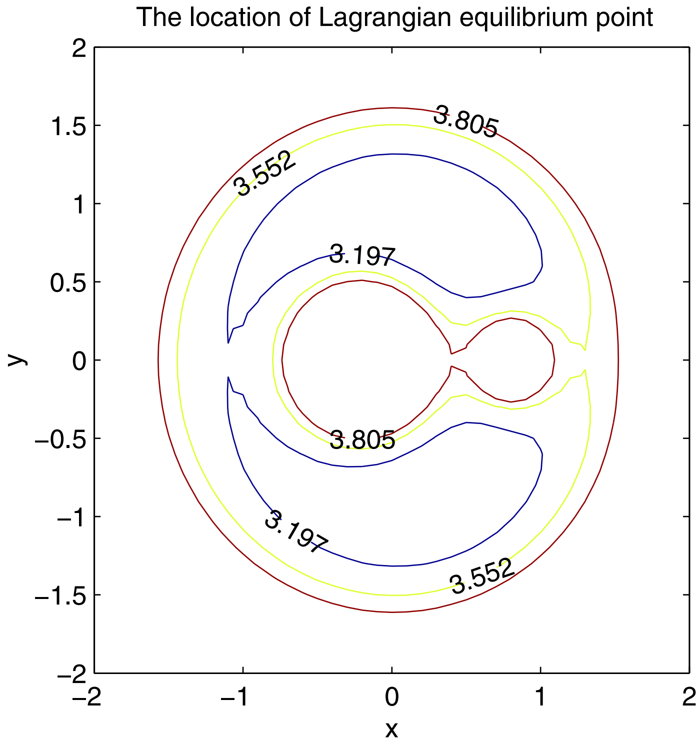
$$C_{L_1} = 3.805$$

$$C_{L_2} = 3.552$$

$$C_{L_3} = 3.197$$

$$C_{L_4} = 2.84$$

$$C_{L_5} = 2.84$$



For calculating above constants we use $u_2 = 0.2$, but in the solar system there are not such math ratio between planets and satellites. the largest value of u_2 is Pluto-Charon system where $u_2 = 10^{-1}$ and

Earth-Moon system has $u_2 = 10^{-2}$. so we need consider the shape of the zero-velocity curves and the positions of the Lagrangian equilibrium points for smaller than the value $u_2 = 0.2$. As $u_2 \to 0$ and $C_{L_1} \to C_{L_2}$ that means L_1 and L_2 had equal distant from u_2. The L_3 is at the distance $1 + \beta$ (where $\beta < 0$); thus, as $u_2 \to 0$ the L_3 approaches unit circle.

Conclusion

In conclusion, three-body problem is very complicative problem, and it hasn't discovered entirely. As we saw, we need restrict system all the way to x-y plan and it's also circular coplanar to deal with it relatively easily. We saw how to find equilibrium points and now we can move on to analyze the stability of these points, and still it gives us good approximation. Ultimately, if we could find these point in our solar system, we can place a satellite and it almost never moves.

References

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[3] Carl, D. Murray.**Solar System Dynamics.** Cambridge Universiy Press, 1999.