Application 7.4
Earthquake-Induced Vibrations of Multistory Buildings

In this application you are to investigate the response to transverse earthquake ground oscillations of a seven-story building like the one illustrated in Fig. 7.4.15 of the text. Suppose that each of the seven [aboveground] floors weighs 16 tons, so the mass of each is $m = 1000$ (slugs). Also assume a horizontal restoring force of $k = 5$ (tons per foot) between adjacent floors. That is, the internal forces in response to horizontal displacements of the individual floors are those shown in Fig. 7.4.16. It follows that the free transverse oscillations indicated in Fig. 7.4.15 satisfy the equation $\mathbf{Mx}'' = \mathbf{Kx}$ with $n = 7$ and $m_i = 1000$, $k_i = 10{,}000$ (lb/ft) for $1 \le i \le 7$. The system then reduces to the form $\mathbf{x}'' = \mathbf{Ax}$ with

$$\mathbf{A} = \begin{bmatrix} -20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 10 & -10 \end{bmatrix} \qquad (1)$$

Once the matrix **A** has been entered, the TI-86 command **`eigVl A`** takes only about 15 seconds to calculate the seven eigenvalues shown in the λ-row of the table below. Alternatively, you can use the *Maple* **`with(linalg)`** command **`eigenvals(A)`**, the *Mathematica* command **`Eigenvalues[A]`**, or the MATLAB command **`eig(A`**. Then the remaining entries $\omega = \sqrt{-\lambda}$ and $P = 2\pi/\omega$ showing the natural frequencies and periods of oscillation of the seven-story building are readily calculated. Note that a typical earthquake producing ground oscillations with a period of 2 seconds is uncomfortably close to the fifth natural period 1.9869 seconds of the building.

i	1	2	3	4	5	6	7
λ	–38.2709	-33.3826	–26.1803	–17.9094	10.0000	–3.8197	–0.4370
ω	6.1863	5.7778	5.1167	4.2320	3.1623	1.9544	0.6611
P (sec)	1.1057	1.0875	1.2280	1.4847	1.9869	3.2149	9.5042

A horizontal earthquake oscillation $E \cos \omega t$ of the ground, with amplitude E and acceleration $a = -E\omega^2 \cos \omega t$, produces an opposite inertial force $F = ma = mE\omega^2 \cos \omega t$ on each floor of the building. The resulting nonhomogeneous linear system is

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + (E\omega^2 \cos \omega t)\,\mathbf{b} \tag{2}$$

where $\mathbf{b} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T$ and $\mathbf{A}$ is the matrix of Eq. (1). Figure 7.4.18 in the text shows a plot of *maximal amplitude* (for the forced oscillations of any single floor) versus the *period* of the earthquake vibrations. The spikes correspond to the first six of the seven resonance frequencies. We see, for instance, that while an earthquake with period 2 seconds likely would produce destructive resonance vibrations in the building, it probably would be unharmed by an earthquake with period 2.5 seconds. Different buildings have different natural frequencies of vibration, and so a given earthquake may demolish one building, but leave untouched the one next door. This type of apparent anomaly was observed in Mexico City after the devastating earthquake of September 19, 1985.

Investigation
For your personal seven-story building to investigate, let the weight (in tons) of each story equal the largest digit of your student ID number and let k (in tons/ft) equal the smallest nonzero digit. Produce numerical and graphical results like those illustrated in Figs. 7.4.17 and 5.3.18 of the text. Is your building susceptible to likely damage from an earthquake with period in the 2 to 3 second range?

You might like to begin by working manually the following warm-up problems.

1. Find the periods of the natural vibrations of a building with two aboveground floors, with each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.

2. Find the periods of the natural vibrations of a building with three aboveground floors, with each weighing 16 tons and with each restoring force being $k = 5$ tons/ft.

3. Find the natural frequencies and natural modes of vibration of a building with three aboveground floors as in Problem 2, except that the upper two floors weigh 8 tons instead of 16 tons. Give the ratios of the amplitudes A, B, and C of the oscillations of the three floors in the form $A : B : C$ with $A = 1$.

4. Suppose that the building of Problem 3 is subject to an earthquake in which the ground undergoes horizontal sinusoidal oscillations with a period of 3 seconds and an amplitude of 3 inches. Find the amplitudes of the resulting steady periodic oscillations of the three above-ground floors. Assume the fact that a motion $E \cos \omega t$ of the ground, with amplitude E and acceleration $a = -E\omega^2 \cos \omega t$, produces an opposite inertial force $F = ma = mE\omega^2 \cos \omega t$ on a floor of mass m.

A Three-Mass Automobile Model

In the sections that follow, we illustrate appropriate *Maple*, *Mathematica*, and MATLAB techniques by analyzing the natural frequencies of vibration of a car that is modeled by a system of three masses and four springs. Suppose that

- mass m_1 is connected to the chassis by spring k_1;
- masses m_1 and m_2 are connected by spring k_2;
- masses m_2 and m_3 are connected by spring k_3; and
- masses m_1 and m_3 are connected by spring k_4.

The corresponding linear system $\mathbf{Mx}'' = \mathbf{Kx}$ has coefficient matrices

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{3}$$

and

$$\mathbf{K} = \begin{bmatrix} -(k_1+k_2+k_4) & k_2 & k_4 \\ k_2 & -(k_2+k_3) & k_3 \\ k_4 & k_3 & -(k_3+k_4) \end{bmatrix}. \tag{4}$$

The displacement vector $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^T$ of the three-mass system then satisfies the equation

$$\mathbf{x}'' = \mathbf{Ax} \tag{5}$$

with coefficient matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$.

We will use the numerical values $m_1 = 40$, $m_2 = 20$, $m_3 = 40$ (in slugs) and $k_1 = 5000$, $k_2 = 1000$, $k_3 = 2000$, $k_4 = 3000$ (in lbs/ft). We want to find the three natural frequencies $\omega_1, \omega_2, \omega_3$ of oscillation of this three-mass system modeling our car. If the car is driven with velocity v (ft/sec) over a washboard surface shaped like a cosine curve with a wavelength of $a = 30$ feet, then the result is a periodic force on the car with frequency $\omega = 2\pi v / a$. We would expect the car to experience resonance vibrations when this forcing frequency equals one of the car's natural frequencies.

Using *Maple*

First we define the masses

```
m1 := 40:   m2 := 20:   m3 := 40:
```

and the spring constants

```
k1 := 4000:   k2 := 1000:   k3 := 2000:   k4 := 3000:
```

Then the mass, stiffness, and coefficient matrices of our system are defined by

```
with(linalg):
M := diag(m1,m2,m3);
```

$$M := \begin{bmatrix} 40 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

```
K := matrix(3,3, [-(k1+k2+k4),    k2,         k4,
                        k2,      -(k2+k3),    k3,
                        k4,         k3,     -(k3+k4)] );
```

$$K := \begin{bmatrix} -8000 & 1000 & 3000 \\ 1000 & -3000 & 2000 \\ 3000 & 2000 & -5000 \end{bmatrix}$$

```
A := evalm( inverse(M) &* K );
```

$$A := \begin{bmatrix} -200 & 25 & 75 \\ 50 & -150 & 100 \\ 75 & 50 & -125 \end{bmatrix}$$

The eigenvalues of **A** are given by

```
eigs := evalf(eigenvals(A));
```

$$eigs := -200., -27.8129452, -247.1870548$$

and we record them in increasing order of magnitude:

```
L := matrix(1,3, [eigs[2],eigs[1],eigs[3]] );
```

$$L := \begin{bmatrix} -27.8129452 & -200. & -247.1870548 \end{bmatrix}$$

The corresponding natural frequencies of the system are then given by

```
w := matrix(1,3, [sqrt(-L[1,1]), sqrt(-L[1,2]),
                  sqrt(-L[1,3])] );
```

$$w := [5.273797986 \quad 14.14213562 \quad 15.72218353]$$

When the car is driven over a washboard surface with wavelength

```
a := 30:     # in feet
```

the resulting critical velocities are given (in ft/s) by

```
v := evalf(evalm( (w*a)/(2*Pi) )):
```

and — since 88 ft/sec corresponds to 60 miles per hour — by

```
mph := evalm( 60*v/88 );
```

$$mph := [17.16854355 \quad 46.03890251 \quad 51.18265687]$$

in miles per hour. If the car is accelerated from 0 to 60 mph, it would therefore experience resonance vibrations as it passes through the speeds of 17, 46, and 51 miles per hour.

Using *Mathematica*

First we define the masses

```
m1 = 40;   m2 = 20;   m3 = 40;
```

and the spring constants

```
k1 = 4000;   k2 = 1000;   k3 = 2000;   k4 = 3000;
```

Then the mass, stiffness, and coefficient matrices of our system are defined by

```
M = DiagonalMatrix[{m1,m2,m3}];
M // MatrixForm

40   0    0
0    20   0
0    0    40

K = { {-(k1+k2+k4),   k2,        k4   },
      {     k2,    -(k2+k3),     k3   },
      {     k4,        k3,   -(k3+k4)} };
K // MatrixForm

-8000   1000    3000
1000    -3000   2000
3000    2000    -5000
```

```
A = Inverse[M] . K;
A // MatrixForm

-200   25     75
50     -150   100
75     50     -125
```

The eigenvalues of **A** are given by

```
eigs = Eigenvalues[A] // N
{-200., -247.187, -27.8129}
```

and we sort them in increasing order of magnitude:

```
L = Reverse[Sort[eigs]]
{-27.8129, -200., -247.187}
```

The corresponding natural frequencies of the system are then given by

```
w = Sqrt[-L]
{5.2738, 14.1421, 15.7222}
```

When the car is driven over a washboard surface with wavelength

```
a = 30;        (* in feet  *)
```

the resulting critical velocities are given (in ft/s) by

```
v = w*a/(2*Pi) // N;
```

and — since 88 ft/sec corresponds to 60 miles per hour — by

```
mph = 60*v/88
{17.1685, 46.0389, 51.1827}
```

in miles per hour. If the car is accelerated from 0 to 60 mph, it would therefore experience resonance vibrations as it passes through the speeds of 17, 46, and 51 miles per hour.

Using MATLAB

First we define the masses

```
m1 = 40;   m2 = 20;   m3 = 40;
```

and the spring constants

```
k1 = 4000;   k2 = 1000;   k3 = 2000;   k4 = 3000;
```

Then the mass, stiffness, and coefficient matrices of our system are defined by

```
M = diag([m1  m2  m3])
M =
    40     0     0
     0    20     0
     0     0    40

K = [-(k1+k2+k4),      k2,         k4;
          k2,       -(k2+k3),      k3;
          k4,          k3,     -(k3+k4) ]
K =
       -8000        1000        3000
        1000       -3000        2000
        3000        2000       -5000

A = M\K
A =
  -200    25    75
    50  -150   100
    75    50  -125
```

The eigenvalues of **A** are given by

```
eigs = eig(A)'
eigs =
  -27.8129 -247.1871 -200.0000
```

and we sort them in increasing order of magnitude:

```
L = fliplr(sort(eigs))
L =
  -27.8129 -200.0000 -247.1871
```

The corresponding natural frequencies of the system are then given by

```
w = sqrt(-L)
w =
    5.2738   14.1421   15.7222
```

When the car is driven over a washboard surface with wavelength

```
a = 30;     % in feet
```

the resulting critical velocities are given by

```
v = w*a/(2*pi);
```

and — since 88 ft/sec corresponds to 60 miles per hour — by

```
mph = 60*v/88
mph =
   17.1685   46.0389   51.1827
```

in miles per hour. If the car is accelerated from 0 to 60 mph, it would therefore experience resonance vibrations as it passes through the speeds of 17, 46, and 51 miles per hour.

Resonance Vibrations of the 7-Story Building

We describe here how MATLAB was used to generate Figure 5.3.18 in the text, showing maximal amplitude of oscillations (for any single floor) as a function the period P of the earthquake. First, the commands

```
V = ones(1,6);
A = 10*diag(V,1) - 20*eye(7) + 10*diag(V,-1);
A(7,7) = -10
```

were entered to set up the coefficent matrix $\mathbf{A}$ of Eq. (1). If we substitute the trial solution

$$\mathbf{x} = \mathbf{v}\cos\omega t$$

(with undetermined coefficient vector $\mathbf{v}$) in Eq. (2), we get the matrix equation

$$(A+\omega^2 I)\mathbf{v} = -E\omega^2\mathbf{b}$$

that is readily solved numerically for the amplitude vector $\mathbf{v}$ of the resulting forced vibrations of the individual floors of the building. The following MATLAB function **amp** does this and then selects the maximal amplitude of forced vibration of any single floor of the building in response to an earthquake vibration with period P.

```
function  y = amp(A,P)
E = 0.25;            % earthquake amplitude
n = size(A);
n = n(1,1);          % dimension of system
Id = eye(n);         % n by n identity matrix
b = ones(n,1);       % constant vector
k = size(P);
```

```
k = k(1,2);           % length of input vector P
y = ones(1,k);        % initialize y
for j = 1:k
    w = 2*pi/P(j);
    V = (A + w*w*Id)\(-E*w*w*b);   % solution of
    y(j) = max( abs(V) );          % linear system
    end
```

To calculate the maximal response **y** to vibrations with periods 0.01, 0.02, 0.03, ..., 4.99, 5.00 we need only define the vector **P** of periods and invoke the function **amp**.

```
P = 0.01:0.01:5;
y = amp(A,P);
```

A modern microcomputer solves the 500 necessary 7-by-7 linear systems in a matter of seconds. Finally, we need only

```
plot(P, y)
```

to see our results on the screen.