Double Spring Pendulum

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Abstract

I will model the motion of a two spring-mass system. The first spring is located at the origin, with a mass at the other end. Off of this mass is another spring with a mass on its end. I will be modelling this in 3-D space using vectors and forces, and then use the Lagrangian to solve the equations.

1. Getting Started

Figure 1 shows the system. Spring 1 has spring constant k_1 and un-stretched length L_1. Spring 2 has spring constant k_2 and un-stretched length L_2. Spring 1 is fixed from the origin to m_1, and spring 2 is fixed from m_1 to m_2. In modelling this system, I will come up with some impossibilities, such as the masses passing through each other or the springs, but this can be ignored.

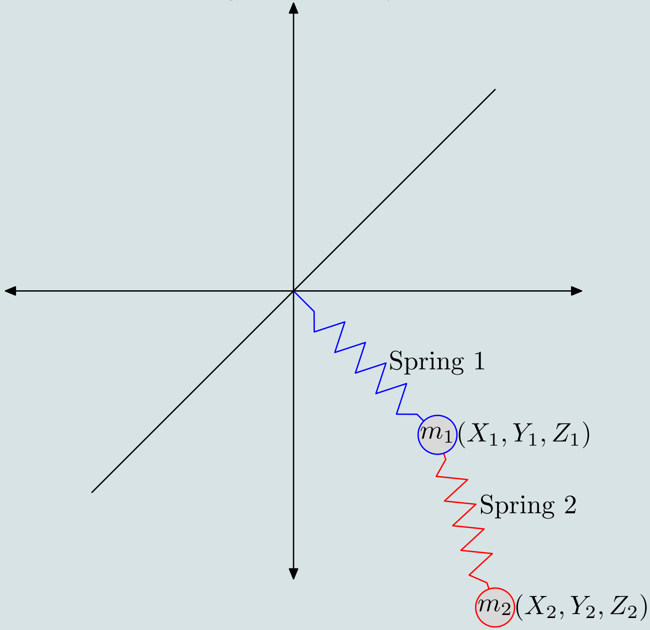
The position of the first mass is

$$(X_1, Y_1, Z_1) \tag{1}$$

And the second mass is located at

$$(X_2, Y_2, Z_2) \tag{2}$$

Figure 1: The System



I need to find a few things. First, the unit vector that points from the first mass to the origin, which is the vector that is its position divided by its magnitude.

$$\boldsymbol{r}_1 = \langle X_1, Y_1, Z_1 \rangle \tag{3}$$

$$\|\boldsymbol{r}_1\| = \sqrt{X_1^2 + Y_1^2 + Z_1^2} \tag{4}$$

$$\hat{\boldsymbol{r}}_1 = \frac{\langle X_1, Y_1, Z_1 \rangle}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} \tag{5}$$

This is the direction of the force that spring 1 applies on m_1. Spring 2 also applies a force on m_1. To find the direction, I have to subtract the position of m_2 from the position of m_1.

$$\boldsymbol{r}_2 = \langle X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2 \rangle \tag{6}$$

$$\|\boldsymbol{r}_2\| = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} \tag{7}$$

$$\hat{\boldsymbol{r}}_2 = \frac{\langle X_1 - X_2, Y_1 - Y_2, Z_1 - Z_2 \rangle}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} \tag{8}$$

Now I must find the direction of the force that spring 2 applies to m_2.

$$\boldsymbol{r}_3 = \langle X_2 - X_1, Y_2 - Y_1, Z_2 - Z_1 \rangle \tag{9}$$

$$\|\boldsymbol{r}_3\| = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{10}$$

$$\hat{\boldsymbol{r}}_3 = \frac{\langle X_2 - X_1, Y_2 - Y_1, Z_2 - Z_1 \rangle}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} \tag{11}$$

Now I need to find the forces acting on all of the masses. I'll start with m_1, there are 4 forces acting on it, the force of gravity, the force of spring 1, the force of spring 2, and the force of air resistance. The force a spring applies is $-kx$, where x is the displacement from the resting point of the spring. The lengths of spring 1 and spring 2 at rest are L_1 and L_2, respectively. So to find the force of spring 1 on m_1, I must find its displacement, which is R_1, and subtract the springs natural length, which is L_1. This is x in the formula $-kx$. This is the magnitude of the force, but to get the force as a vector, I must multiply this by $\hat{\boldsymbol{r}}_1$.

$$
\begin{aligned}
\boldsymbol{F}_{s1} =& -k_1(\|\boldsymbol{r}_1\| - L_1)\boldsymbol{r}_1 & (12)\\
\boldsymbol{F}_{s1} =& -k_1\left(\sqrt{X_1^2+Y_1^2+Z_1^2} - L_1\right)\frac{\langle X_1, Y_1, Z_1\rangle}{\sqrt{X_1^2+Y_1^2+Z_1^2}} & (13)\\
\boldsymbol{F}_{s1} =& k_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}} - 1\right)\langle X_1, Y_1, Z_1\rangle & (14)
\end{aligned}
$$

Now I must model the force spring 2 applies on m_1, which will follow the same steps.

$$
\begin{aligned}
\boldsymbol{F}_{s2} =& -k_2\left(\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2} - L_2\right)\\
&\times\left(\frac{\langle X_1-X_2, Y_1-Y_2, Z_1-Z_2\rangle}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}\right) & (15)\\
\boldsymbol{F}_{s2} =& k_2\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}} - 1\right)\\
&\times\langle X_1-X_2, Y_1-Y_2, Z_1-Z_2\rangle & (16)
\end{aligned}
$$

And now to find the force spring 2 applies on m_2.

$$
\begin{aligned}
\boldsymbol{F}_{s3} =& -k_2\left(\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2} - L_2\right)\\
&\times\left(\frac{\langle X_2-X_1, Y_2-Y_1, Z_2-Z_1\rangle}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2}}\right) & (17)\\
\boldsymbol{F}_{s3} =& k_2\left(\frac{L_2}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2}} - 1\right)\\
&\times\langle X_2-X_1, Y_2-Y_1, Z_2-Z_1\rangle & (18)
\end{aligned}
$$

Now I must find the force due to air resistance. I will assume in this project that the force of air resistance is proportional to the velocity, and for sake of experiment, I will use two different proportionality constants, one for each mass. The force from air resistance is always opposite velocity. I will now write the vectors in $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{\kappa}}$ form. This will make the analysis easier, and gravity is always in the $-\hat{\boldsymbol{\kappa}}$ direction. The velocities of the masses are

$$\boldsymbol{V}_1 = X_1^{'}\hat{\boldsymbol{\imath}} + Y_1^{'}\hat{\boldsymbol{\jmath}} + Z_1^{'}\hat{\boldsymbol{\kappa}} \tag{19}$$

$$\boldsymbol{V}_2 = X_2^{'}\hat{\boldsymbol{\imath}} + Y_2^{'}\hat{\boldsymbol{\jmath}} + Z_2^{'}\hat{\boldsymbol{\kappa}} \tag{20}$$

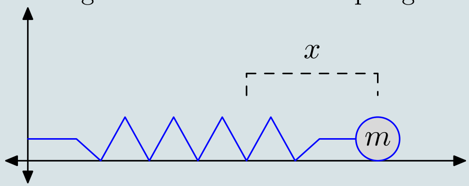
So the force due to air resistance (F_{1_r} for m_1 and F_{2_r} for mass two) is

$$\boldsymbol{F}_{1R} = -\,R_1(X_1^{'}\hat{\boldsymbol{\imath}} + Y_1^{'}\hat{\boldsymbol{\jmath}} + Z_1^{'}\hat{\boldsymbol{\kappa}}) \tag{21}$$

$$\boldsymbol{F}_{2R} = -\,R_2(X_2^{'}\hat{\boldsymbol{\imath}} + Y_2^{'}\hat{\boldsymbol{\jmath}} + Z_2^{'}\hat{\boldsymbol{\kappa}}) \tag{22}$$

2. Modelling the forces

Figure 2: The force of a spring



Now I can model all forces acting on m_1 and m_2. The force from a spring is $-kx$, where x is the displacement from un-stretched, as seen in figure 2. The force on m_1 is

$$
\begin{aligned}
\boldsymbol{F}_1 = & -m_1 g\hat{\boldsymbol{\kappa}} + k_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right)(X_1\hat{\boldsymbol{\imath}}+Y_1\hat{\boldsymbol{\jmath}}+Z_1\hat{\boldsymbol{\kappa}}) \\
& + k_2\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right) \\
& \times((X_1-X_2)\hat{\boldsymbol{\imath}}+(Y_1-Y_2)\hat{\boldsymbol{\jmath}}+(Z_1-Z_2)\hat{\boldsymbol{\kappa}}) \\
& - R_1(X_1'\hat{\boldsymbol{\imath}}+Y_1'\hat{\boldsymbol{\jmath}}+Z_1'\hat{\boldsymbol{\kappa}})
\end{aligned} \tag{23}
$$

And I can also model all forces acting on m_2.

$$
\begin{aligned}
\boldsymbol{F}_2 = -m_2 g\hat{\boldsymbol{\kappa}} + k_2 & \left(\frac{L_2}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2+(Z_2-Z_1)^2}}-1\right) \\
& \times((X_2-X_1)\hat{\boldsymbol{\imath}}+(Y_2-Y_1)\hat{\boldsymbol{\jmath}}+(Z_2-Z_1)\hat{\boldsymbol{\kappa}}) \\
& - R_2(X_2'\hat{\boldsymbol{\imath}}+Y_2'\hat{\boldsymbol{\jmath}}+Z_2'\hat{\boldsymbol{\kappa}})
\end{aligned} \tag{24}
$$

Please remember that $F = ma$, I know the force, and I want to find the acceleration. The division of m is not shown from this last step to the next, but it is trivial.

Now that I've got the forces acting on the objects, I will separate the forces into the vector components, and by dividing by m_1, I get the accelerations.

$$
\begin{aligned}
X_1'' = & \frac{k_1}{m_1}X_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right) \\
& + \frac{k_2}{m_1}(X_1-X_2)\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right) \\
& - \frac{R_1}{m_1}X_1'
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
Y_1'' = & \frac{k_1}{m_1} Y_1 \left(\frac{L_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} - 1 \right) \\
& + \frac{k_2}{m_1} (Y_1 - Y_2) \left(\frac{L_2}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} - 1 \right) \\
& - \frac{R_1}{m_1} Y_1'
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
Z_1'' = & \frac{k_1}{m_1} Z_1 \left(\frac{L_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} - 1 \right) \\
& + \frac{k_2}{m_1} (Z_1 - Z_2) \left(\frac{L_2}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} - 1 \right) \\
& - \frac{R_1}{m_1} Z_1' - g
\end{aligned} \tag{27}
$$

And similarly, I find the accelerations for the second mass.

$$
\begin{aligned}
X_2'' = & \frac{k_2}{m_2} (X_2 - X_1) \left(\frac{L_2}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} - 1 \right) \\
& - \frac{R_2}{m_2} X_2'
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
Y_2'' = & \frac{k_2}{m_2} (Y_2 - Y_1) \left(\frac{L_2}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} - 1 \right) \\
& - \frac{R_2}{m_2} Y_2'
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
Z_2'' =& \frac{k_2}{m_2}(Z_2 - Z_1)\left(\frac{L_2}{\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2}} - 1\right) \\
& - \frac{R_2}{m_2} Z_2' - g
\end{aligned}
\tag{30}
$$

3. The Lagrangian

Now the question arises, Can this be done another way? The answer is yes. This can also be done using Lagrangian. The Lagrangian basically says that the actual path an object takes is the minimum of a certain quantity. What the Lagrangian ends up with is the equation

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right) - \frac{\partial L}{\partial q} = 0, \tag{31}
$$

where q is any variable of differentiation, and L is defined to be kinetic energy (T) minus potential energy (V).

$$
L = T - V \tag{32}
$$

So I need to find T and V, I'll start with T. Kinetic energy is $\frac{1}{2}mv^2$, so using this formula, I come up with

$$
T = \frac{1}{2}m_1\left((X_1')^2 + (Y_1')^2 + (Z_1')^2\right) + \frac{1}{2}m_2\left((X_2')^2 + (Y_2')^2 + (Z_2')^2\right) \tag{33}
$$

And the Potential energy is

$$
\begin{aligned}
V =& \frac{1}{2}k_1\left(\sqrt{X_1^2 + Y_1^2 + Z_1^2} - L_1\right)^2 \\
& + \frac{1}{2}k_2\left(\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2} - L_2\right)^2 \\
& + m_1 g Z_1 + m_2 g Z_2
\end{aligned}
\tag{34}
$$

So to put this into equation (31), I have to take the Lagrange of each variable separately, meaning q in equation (31) is going to be X_1, Y_1, Z_1, X_2, Y_2, and Z_2. And putting T and V from before into equation (32), I get

$$\begin{aligned} L = \frac{1}{2}m_1\left((X_1')^2 + (Y_1')^2 + (Z_1')^2\right) + \frac{1}{2}m_2\left((X_2')^2 + (Y_2')^2 + (Z_2')^2\right) \\ - \frac{1}{2}k_1\left(\sqrt{X_1^2 + Y_1^2 + Z_1^2} - L_1\right)^2 \\ - \frac{1}{2}k_2\left(\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 - (Z_1 - Z_2)^2} - L_2\right)^2 \\ - m_1 g Z_1 - m_2 g Z_2 \end{aligned} \tag{35}$$

Notice that equation (31) can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q'}\right) = \frac{\partial L}{\partial q} \tag{36}$$

So first I'll use equation (36) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial X_1'}\right) = \frac{\partial L}{\partial X_1} \tag{37}$$

So I must first find $\partial L/\partial X_1'$, which is

$$\frac{\partial L}{\partial X_1'} = m_1 X_1' \tag{38}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial X_1'}\right) = m_1 X_1'' \tag{39}$$

Putting this into equation (37) and solving for $\partial L/\partial X_1$, I get

$$\begin{aligned} m_1 X_1'' = k_1 X_1\left(\frac{L_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} - 1\right) \\ + k_2(X_1 - X_2)\left(\frac{L_2}{\sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}} - 1\right) \end{aligned} \tag{40}$$

This is the equation for the path that an object will take. Now that I have this equation, I can add a few parameters. I know that the air resistance is $-X_1^{'}R_1$, so I can just add this to equation (40) and solve for $X_1^{''}$

$$\begin{aligned} X_1^{''} =& \frac{k_1}{m_1}X_1\left(\frac{L_1}{\sqrt{X_1^2+Y_1^2+Z_1^2}}-1\right) \\ &+\frac{k_2}{m_1}(X_1-X_2)\left(\frac{L_2}{\sqrt{(X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2}}-1\right) \\ &-\frac{R_1}{m_1}X_1^{'} \end{aligned} \tag{41}$$

Look familiar? This looks like equation (25) of the force on m_1 in the $\hat{\imath}$ direction. Thats right, this is the force from before. And notice that if I plug this into equation (36), I get $F = ma$, and when I divide by the mass, I get exactly what I got for $X_1^{''}$ earlier. And when I do this for the other 5 variables, I will get what I already had for each of those also. Other more complex models can be modelled using the Lagrangian, an example of one is the spring pendulum on Peter Selinger's page [1]

4. Using ode45

OK, I've got the accelerations of each mass, now what? I can graph the actual path of each mass now using MATLAB's numerical solver ode45. ode45 is a variable step solver that approximates the actual path, but the error is very small.

How well does ode45 work? To show this, I've come up with a simple demonstration. Lets take our double mass-spring system and change it a little. Instead of spring 1 being attached to mass one and pulling on it, lets just have a spring between the masses, and no spring elsewhere. To do this, I just set the k_1 value equal to zero. Lets think of what would happen if gravity were zero, and the masses were set to start spinning around each other slowly. With no gravity, and with no friction, the total energy of the system will remain constant, and the masses will spin around each other. The maximum distance the springs will get from the center will always be the same, because the energy is conserved. Putting this into ode45, I get figure 3. The mass starts at the red dot, and as it moves around, it starts getting towards the center. But the energy must be conserved because there is no friction. This graph is a result of ode45's default precision, or relative error. The default relative error is 10^{-3}, and by changing to 10^{-4}, I get figure 4. This shows that there is some considerable error going on in figure 3,

Figure 3: Default Settings

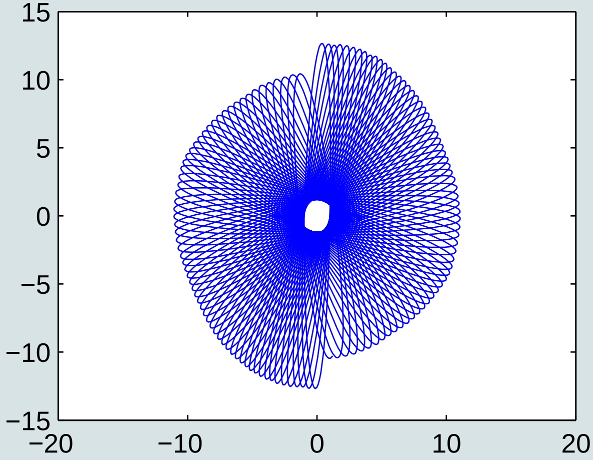
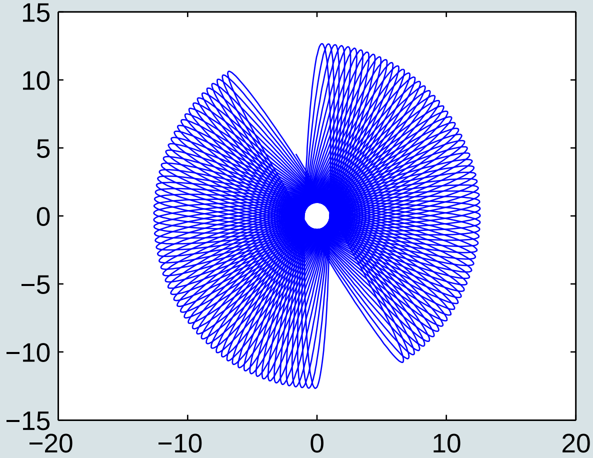


Figure 4: Refined Settings



and when I get my results, I don't want high error. To check how much error there is, lets think back to conservation of energy. With no outside forces such as friction, the system's energy must be constant, which means the kinetic energy plus the potential energy must be constant. Remember back to T and V from the Lagrangian, where T is the kinetic energy, and V is the potential energy. From this I can find the error,

$$E = \left| \frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I} \right| \tag{42}$$

Where T_F is the final kinetic energy calculated from ode45, and T_I is the initial kinetic energy. V_F and V_I are the final and initial potential energies, respectively.

Applying this to the equation for graphs 3 and 4, both of which have the same initial conditions, I get

$$\left| \frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I} \right| = 0.3235$$

for figure 3. This is 32.35% error, which is a lot. However, with figure 4, which had 10 times less relative error from ode45, the error is

$$\left| \frac{(T_F + V_F) - (T_I + V_I)}{T_I + V_I} \right| = 0.0057$$

Which is only 0.57% error, much better than before. I will have all of the figures from now on be less than 1 percent error, for analysis sake. But this error analysis is just looking at the beginning and end values for the energy. Figure 5 shows the position of the two masses for certain initial conditions. Figure 6 shows the total energy vs. time graph for the masses in figure 5, which should remain constant. As you can see, the graph is fairly constant until about 9.5 seconds, when it takes a huge dive down. Using the method for error analysis from before, the error of this system would be about 0.28%, which is small, but the average error should be much less, because most of the time there was much less error. To account for this, I used a method for error analysis that takes the average of the error. To do this, in matlab I used the commands

```
ErrorSize=size(Energy)
ErrorSize=ErrorSize(1);
Percent_Error=(sum(Energy)/ErrorSize-Energy(1))/Energy(1)
```

Getting Started
Modelling the forces
The Lagrangian
Using ode45
Different Models

Home Page

Title Page

◀◀ ▶▶

◀ ▶

Go Back

Full Screen

Close

Quit

Figure 5: Position of the masses

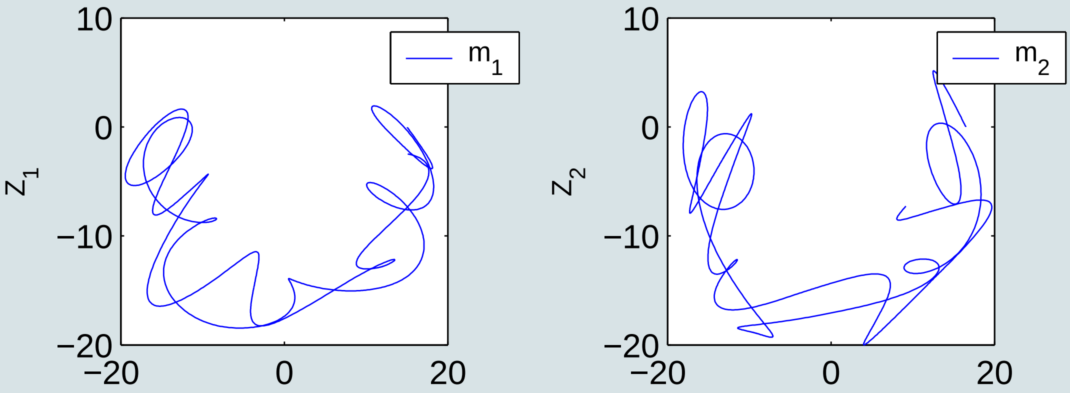


Figure 6: Total Energy vs. Time

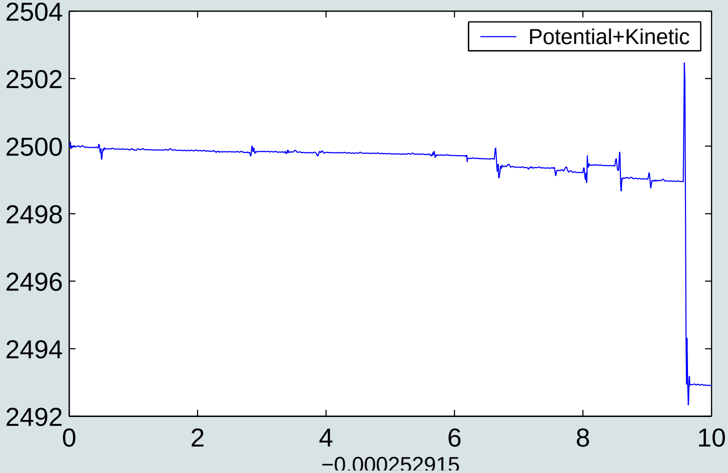
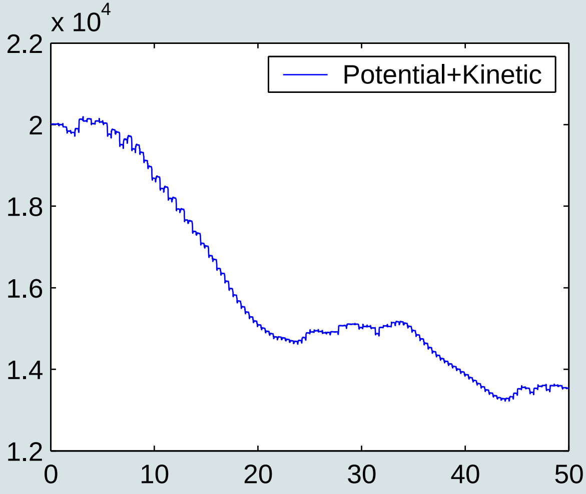


Figure 7: Total Energy vs. Time



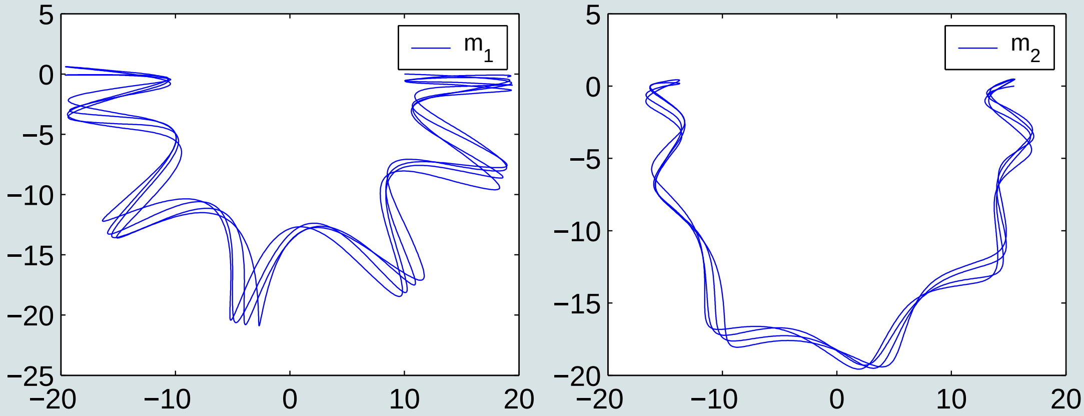
Where Energy has the values for total energy vs. time. Using this method, the error turned out to be .0253%, which is, as expected, much smaller. Lets look at figure 7, the graph of the total energy vs. time for figure 3. Remember that before I got 32.35% error. Using the new method of averaging the error, I get 19.765% error.

5. Different Models

If you notice, with the error analysis last section, I made a big assumption, that with the first spring constant zero, the system would act like two independent masses connected by a spring. Taking this further, then there really can be three different models here. One is the normal two spring and two mass, the other is what we had last section, and the last is one spring and one mass. To get this last model, all I have to do is set the constant for spring 2 equal to zero.

From figure 4, we see what type of motion occurs with two independent masses in space connected by a spring. But what about the other two models we can get from the equations we have? First lets look at what would happen if there was only one mass and one spring. Logically, the mass would bounce

Figure 8: Position of the masses



around without friction in some periodic way. Lets see for ourselves, figure 9 shows a single spring-mass system. As you can see, the movement is periodic. The motion of this system will always be periodic if there is no air resistance.

Now, for the motion of the double spring-mass system. The motion of this system depends on the initial conditions, but it can be periodic, or it can be chaotic, meaning that it follows no specific pattern. An example of this system being periodic is figure 8. An example of chaotic motion is figure 5. With air resistance, the motion of the system will always and eventually stop. the springs stretching only enough to account for gravity.

References

[1] Peter Selinger: Lagrange Applet http://quasar.mathstat.uottawa.ca/~selinger/lagrange/mechsystem.html

[2] Polking, Boggess, Arnold. *Ordinary Differential Equations Using MATLAB*

Figure 9: Simple Periodic Motion