The Flight of a Discus

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Differential Equations

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1 Introduction

The focus of this project is to investigate the factors that effect the flight of a discus. When looking into references for this topic, we found a rather in depth article that explained extensively the influencing factors of the flight of a discus. This article was written by Cliff Frohlich*. Although some of the factors that influence the flight were well expected, other influences played surprising roles. For instance, experienced discus throwers claim that ideal throws may be achieved during a slight headwind and not necessarily with the help of a tail wind. This article will explain the effects of how the discuss travels with these ever changing conditions.

When a discus is along it's path of flight, certain forces play a role on the outcome of the discus. The force due to wind, and the force due to gravity are the two distinct forces that alter the discus flight. Wind forces may be separated into two distinct directional forces. Aerodynamic drag is the force against the discus in a direction opposite that of the oncoming air. Aerodynamic lift is the force perpendicular to the direction of the oncoming air. The lift force is created when the oncoming air (due to the discus) is separated into two separate regions. These two regions are divided by the plane of the discus, thus separating the air into the area above the discus, and below. Due to the inclination of the discus along it's trajectory, the air that collides with the bottom plane of the discus flows at a slower rate past the discus than the free-flowing air on the top half of the discus. The discus now acts as an air foil with an area of low pressure at the top. This low pressure enables the discus to be pushed up from below due to the force generated by this pressure vacancy as shown in Figure1.

Velocity of the discus is effected by the velocity of the wind. We may call the difference between these two velocities the relative velocities, given below

$$v_{rel} = v_{discus} - v_{wind} \tag{1}$$

Drag and lift forces are given in terms of the dimensionless drag and lift coeffi-

Figure 1: Diagram of Forces

cients represented in the following equations.

$$\begin{aligned} F_{drag} &= \frac{1}{2}\overrightarrow{c}_d \rho A \overrightarrow{v}^2_{rel} \\ F_{lift} &= \frac{1}{2}\overrightarrow{c}_L \rho A \overrightarrow{v}^2_{rel} \end{aligned} \tag{2}$$

Notice that these two equations are in two separate directions. The drag coefficient is in the horizontal x direction, while the lift coefficient is in the vertical y direction. The constants in equation 2 are listed as follows

$$\begin{aligned} \rho &= \text{air density} \\ A &= \text{maximum cross sectional area of discus} \\ M &= \text{mass of the discuss} \end{aligned}$$

In order to achieve the sum of these forces we may add these equations together to get

$$\frac{1}{2}\rho A \vec{v}^2_{rel}\overrightarrow{c}_d + \frac{1}{2}\rho A \vec{v}^2_{rel}\overrightarrow{c}_L$$

factoring yields

$$\frac{1}{2}\rho A \vec{v}^2_{rel}(\overrightarrow{c}_d + \overrightarrow{c}_L)$$

since $\overrightarrow{c}_d$ and $\overrightarrow{c}_L$ are vectors, we may look at the above equation in the following form

$$\frac{1}{2}\rho A\vec{v}_{rel}^2\left(\overrightarrow{r}\right) \tag{3}$$

We get their respective magnitudes by using Pythagorean's theorem to the right triangle determined by $\overrightarrow{r}$ and it's two vector components. Using Pythagorean's we get

$$\begin{aligned} \overrightarrow{r} &= \overrightarrow{c}_d + \overrightarrow{c}_L \\ \|\vec{r}\| &= \sqrt{\overrightarrow{c}_d^2 + \overrightarrow{c}_L^2} = \left(\overrightarrow{c}_d^2 + \overrightarrow{c}_L^2\right)^{\frac{1}{2}} \end{aligned}$$

substituting back into the equation 3, we now have

$$\frac{1}{2}\rho A\vec{v}_{rel}^2\left(\overrightarrow{c}_d^2 + \overrightarrow{c}_L^2\right)^{\frac{1}{2}} \tag{4}$$

This is essentially the relative force that contains both i and j directions. In order to get the relative direction, we divide equation 4 by the mass $(M)$of the discus. Below is the same equation Cliff Frohlich obtained in his calculations.

$$\frac{\frac{1}{2}\rho A\vec{v}_{rel}^2\left(\vec{c}_d^2 + \vec{c}_L^2\right)^{\frac{1}{2}}}{M} \tag{5}$$

Both the drag and lift coefficients depend heavily on the angle of attack ψ. The angle of attack is defined as the angle between the direction of $\overrightarrow{v}_{rel}$ and the plane of the discus. Although air density ρ, and $\overrightarrow{v}_{rel}$ are factors, we have set ρ to be constant and we may also omit $\overrightarrow{v}_{rel}$ as a constant along with the rotation vector $\overrightarrow{w}$. Setting these as constants greatly simplifies our system of equations. Refer back to Figure 1 for angle descriptions. β, the release angle, is that which lies between the horizontal plane and $\overrightarrow{v}_{rel}$, the relative (initial) velocity vector.

Assuming that our discus is thrown at time $t = 0$ into the first quadrant with an initial velocity denoted as $\overrightarrow{v}_0$, the discuss will make an angle α with the horizontal ground (the x-axis). The angle α is the inclination angle. We now have the following equation.

$$\overrightarrow{v}_0 = \left(\|\overrightarrow{v}_0\|\cos\alpha\right)i + \left(\|\overrightarrow{v}_0\|\sin\alpha\right)j \tag{6}$$

Here is when Newton's second law of motion plays a part into this objective. Newton's second law states that the force acting on our discus is it's mass (M) multiplied by it's acceleration.

$$\left(\frac{d^2\vec{r}}{dt^2}\right)M$$

$\vec{r}$ being the discus position vector. If the only force we were dealing with were gravity $(-Mgj)$, then

$$\left(\frac{d^2\vec{r}}{dt^2}\right) M = -Mgj \qquad = \left(\frac{d^2\vec{r}}{dt^2}\right) = -gj$$

However, because we are with both the force due to gravity and wind, we must make appropriate changes to our equation. Since we are working with horizontal and vertical positions, we now have a system of two second order differential equations.

$$\begin{aligned}
\left(\frac{d^2\vec{x}}{dt^2}\right) M &= j &= \ddot{x} = \frac{j}{M} = \ddot{x} = -\frac{1}{2}\left(\frac{\rho A\vec{v}_{rel}^2}{M}\right)(c_d\cos\beta + c_L\sin\beta) \\
\left(\frac{d^2\vec{y}}{dt^2}\right) M &= j &= \ddot{y} = -g + \frac{j}{M} = \ddot{y} = -g + \frac{1}{2}\left(\frac{\rho A\vec{v}_{rel}^2}{M}\right)(c_L\cos\beta - c_d\sin\beta)
\end{aligned}$$

Notice that with the horizontal x-direction their are no forces due to gravity while gravitational forces play a role in the vertical y-direction.

What we now have is a system of initial value problems that may be solved in the following manner.

$$\begin{aligned}
\ddot{x} &= -\frac{1}{2}\left(\frac{\rho A\vec{v}_{rel}^2}{M}\right)(c_d\cos\beta + c_L\sin\beta) \\
\ddot{y} &= -g + \frac{1}{2}\left(\frac{\rho A\vec{v}_{rel}^2}{M}\right)(c_L\cos\beta - c_d\sin\beta) \qquad (7)
\end{aligned}$$

Note that $\ddot{x}$ and $\ddot{y}$ are the accelerations in their respective directions with the discus The first integration gives

$$\begin{aligned}
\dot{x} &= -\frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T \vec{v}_{rel}^2 (c_d\cos\beta + c_L\sin\beta)\, dt' \\
\dot{y} &= -g + \frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T \vec{v}_{rel}^2 (c_L\cos\beta - c_d\sin\beta)\, dt'
\end{aligned}$$

We now have the relative velocities in each respective direction. Therefore we may write $\dot{x}, \dot{y} = v_d$ and recalling equation (1).

$$\begin{aligned}
v_{rel} &= v_{discus} - v_{wind} \\
\dot{x} &= v_{rel} + v_{wind} - \frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T \vec{v}_{rel}^2 (c_d\cos\beta + c_L\sin\beta)\, dt' \\
\dot{y} &= v_{rel} + v_{wind} - gT + \frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T \vec{v}_{rel}^2 (c_L\cos\beta - c_d\sin\beta)\, dt'
\end{aligned}$$

The second integration gives.

$$\begin{aligned}
x &= v_{rel}T + v_{wind}T - \frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T\int_0^t v_{rel}^2 (c_d\cos\beta + c_L\sin\beta)\, dt'dt \\
y &= -\frac{1}{2}gT^2 + v_{rel}T + \frac{1}{2}\left(\frac{\rho A}{M}\right)\int_0^T\int_0^t \vec{v}_{rel}^2 (c_L\cos\beta - c_d\sin\beta)\, dt'dt
\end{aligned}$$

In this final form, these equations show how gravity and wind forces play their own distinct roles. As stated before, we are not working with ideal projectile motion, therefor these results are expected. We find that x is not effected by the force due to gravity, g. Instead, x must contend with forces due to wind. Similarly y contends with gravity, while wind forces are not a factor.

In order to make the best use of our system of equations, one should take small time increments when solving for equation (7). A MATLAB function M-file has been created that makes use of the two second ordered differential equations to plot the discus trajectory. This function M-file reads the appropriate data and solves using an ODE45 solver. Below, we have the basic setup that enables MATLAB to read our equations in terms of x.

Recall equation (7)

$$\begin{aligned} \ddot{x} &= -\frac{1}{2}\left(\frac{\rho A \vec{v}_{rel}^2}{M}\right)(c_d \cos\beta + c_L \sin\beta) \\ \ddot{y} &= -g + \frac{1}{2}\left(\frac{\rho A v_{rel}}{M}\right)(c_L \cos\beta - c_d \sin\beta) \end{aligned}$$

To convert the two second order equations into two, first ordered systems, the following steps took place.

STEP #1

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ y_1 &= y \\ y_2 &= \dot{y} \end{aligned}$$

STEP #2

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{1}{2}\left(\frac{\rho A v_{rel}}{M}\right)(c_d \cos\beta + c_L \sin\beta) \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -g + \frac{1}{2}\left(\rho A v_{rel}\frac{\rho A v_{rel}}{M}\right)(c_L \cos\beta - c_d \sin\beta) \end{aligned}$$

We were able to generate graphs that plot the acceleration of the discus in both the x and y directions along with the x and y positions. Each graph is a representation of the flight of the discus in different wind conditions. By plotting many flights, we found that the discus will in fact travel farther against a head wind than with the help of a tail wind. Keep in mind that although we are capable of changing the release velocities to represent the strength of different throwers, we decided to keep the release velocities the same in order to show how the discus will travel when thrown in different conditions.

This first scenario shown, is a representation of a discus thrown against a headwind of -10 m/s. Through experimenting with various angles of attack and release angles, we have found that the farthest distances are achieved when the release angle is about 32° and the angle of attack is about 15°. A graph that represents accelerations and positions for the discus with a headwind of -10 m/s is shown below in figure 2.

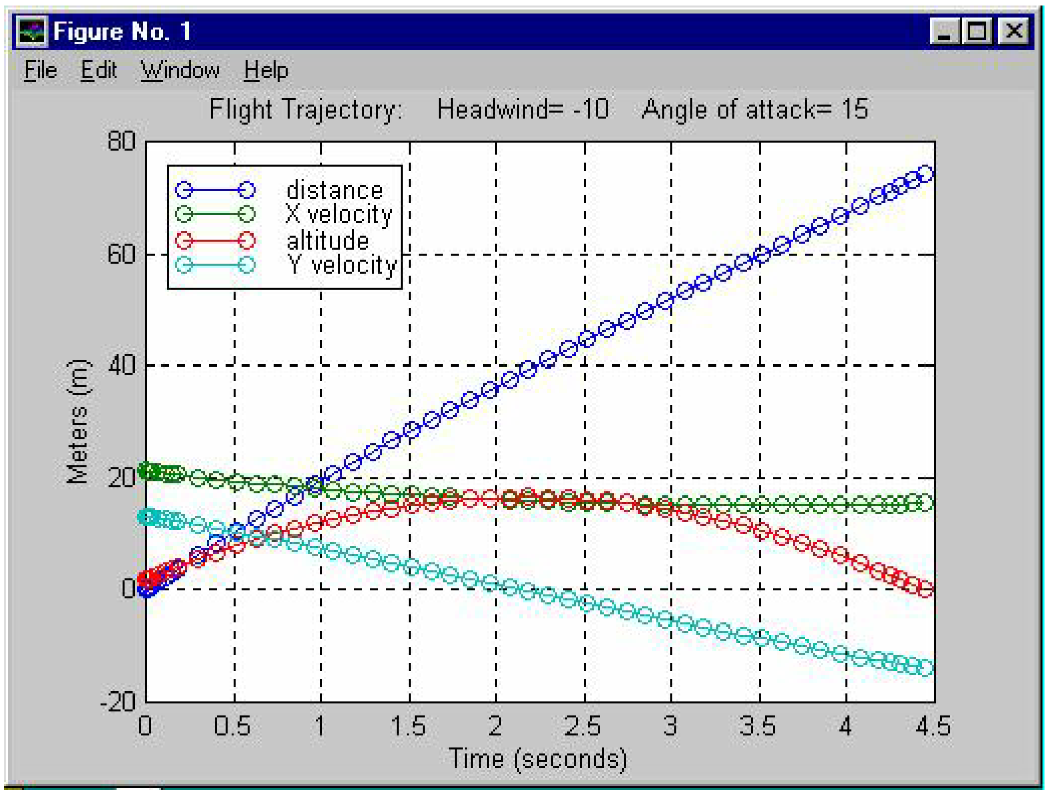

Figure 2.

This next graph is the actual trajectory of the discus. This single line plot represents accelerations in both the vertical and horizontal directions, along with the position of the discus after the initial release. In order to achieve this plot, we had to call MATLAB to plot the first row (the x-direction) against the third row (the y-direction), of the single column matrix represented below.

$$x = \begin{array}{ll} x_1 & ;\ x \text{ position} \\ x_2 & ;\ \text{acceleration in } x\text{-direction} \\ y_1 & ;\ y \text{ position} \\ y_2 & ;\ \text{acceleration in } y\text{-direction} \end{array}$$

The graph for the trajectory of the discus is represented below in figure3.

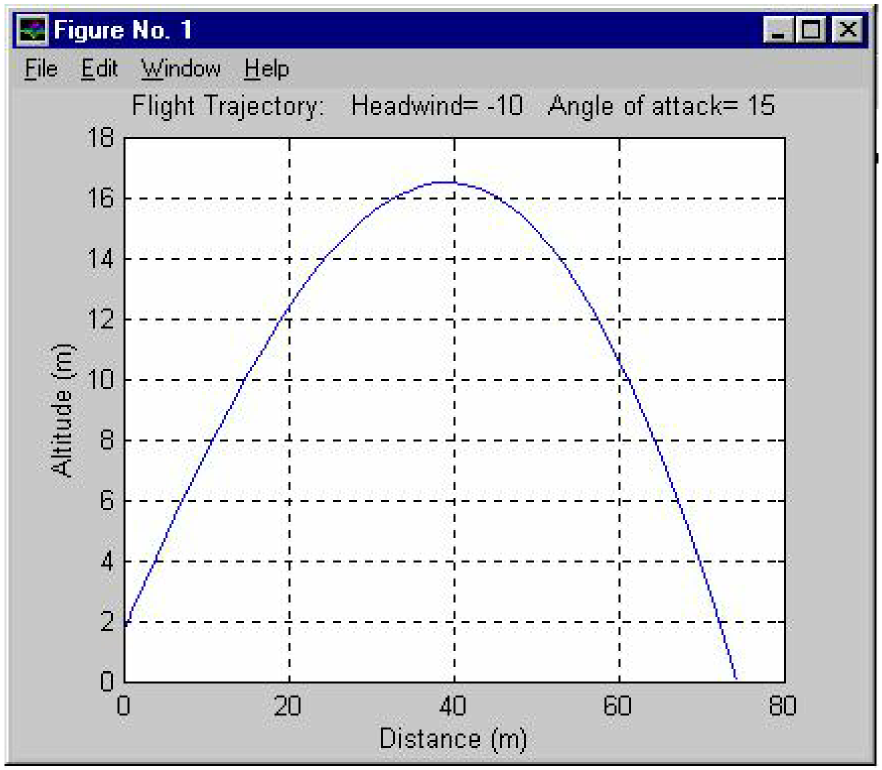

Figure 3.

In order to represent the trajectory that the discus would take if the angle of attack were altered, we decided to raise the angle of attack by about 15°. With the angle of attack at about 30° the drag coefficients increase as do the lift coefficients. With larger drag coefficients, we should expect the discus to be limited in the distance travelled. We can use the same rational with respect to our lift coefficients. The larger the lift coefficient, the larger the altitude the disc will obtain during it's flight. Evidence of this is shown below in figure 4.

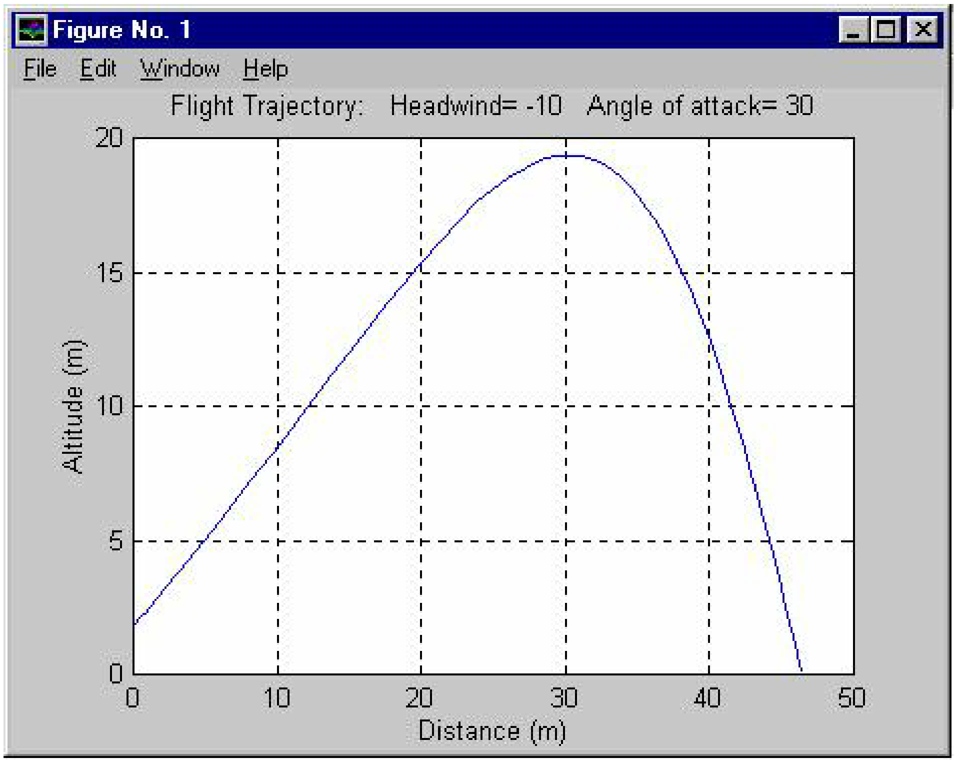

Figure 4.

Notice that the disc travelled well less in distance, and much higher in altitude than before. Referring back to figure1, we can see how the wind will collide on the underside of the disk when the angle of attack is at 30°. This is a good example of how the discus is acting as a wind foil that achieves an exaggerated lift, thus reducing it's overall distance travelled. As you can see, it is imperative that the discus thrower obtain the correct angle of attack in a head wind. Figure 5, is a representation of a discus thrown with the help of a tailwind at 20 m/s. Ones first assumption is that the discus will travel further with the tailwind, but such is not the case. As with figure 2, we used the method of trial and error when adjusting the release angle and angle of attack to achieve the furthest distance for the given wind condition.

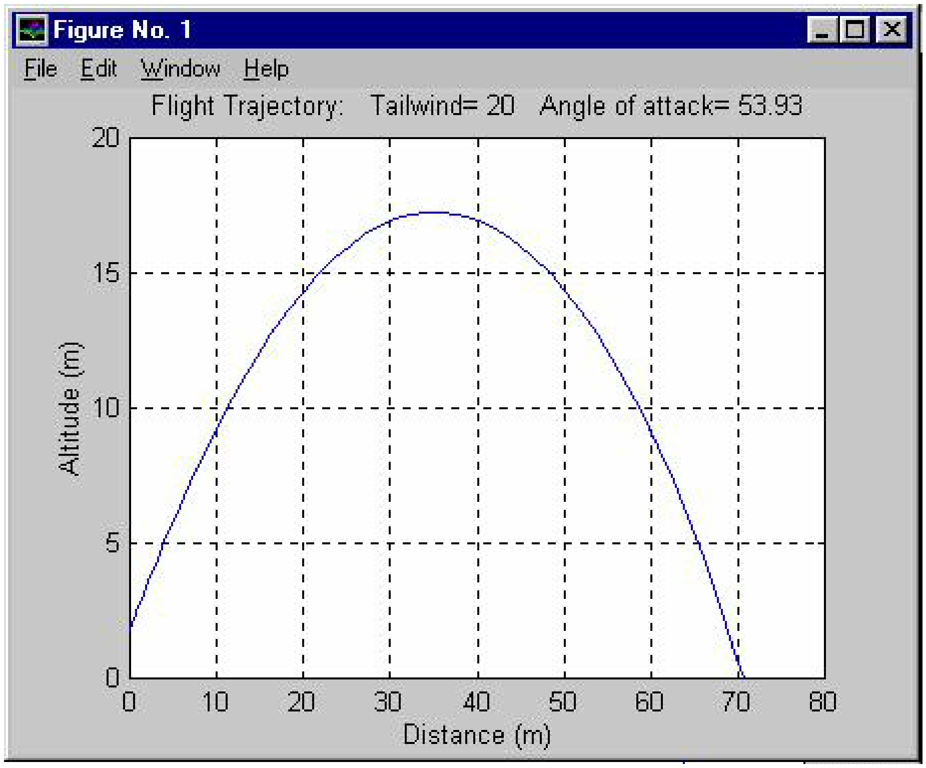

Figure 5.

To illustrate the importance of finding the proper angle of attack and angle of release with given wind conditions, We decided to create a graph for a headwind of -30 m/s. Deliberately using exaggerated angles, we received the following plot in figure 6.

Figure 6.

Figure6 above shows an extreme case of wind conditions along with an improper release angle and angle of attack. This is an example of how important it is for the thrower to achieve just the right angles upon release.

It's not difficult to see just how complex this subject really is. As stated earlier, we simply had to narrow it down to a few select factors, and examine discus behavior according to changes in those only. To tell the truth, we never imagined that there was so many things to consider when throwing a discus, that is, if you want the distance to be optimal. With all things considered, every aspect of our project was interesting and certainly educational. Lastly, since our areas of concentration solely included angles and aerodynamic forces, there are still many other areas that could be investigated and turned into exciting projects. That last sentence is in reference to all the students out there in search if an interesting subject from which to create an exciting, yet educational project.

2 References

- Wagon, Stan,"Predicting the Flight of a Discus", Differential Equations Resource Center, `www.pws.com/rcenters/diffeq/lab4/lab4.htm`, Thursday, July 27,1995.

- Frohlich,Cliff, "Aerodynamic effects on discus flight", American Journal of Physics, Vol.49, No.12, Dec. 1981, 1125-1132.