Differential Equations Student Projects

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The Wilberforce Pendulum

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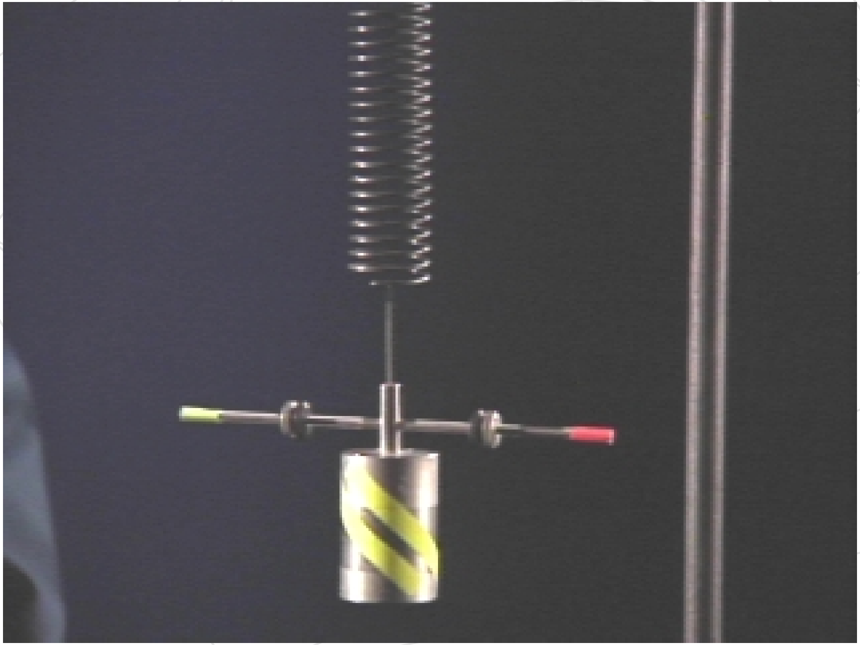
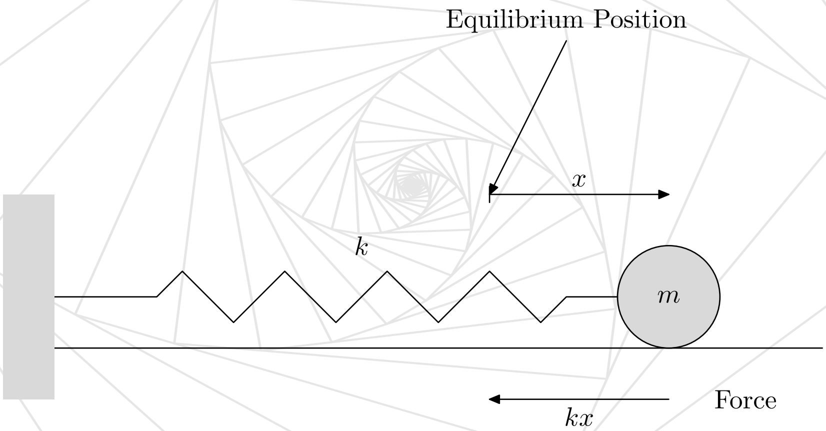
The Wilberforce Pendulum

The Wilberforce Pendulum is a pendulum that couples longitudinal and torsional oscillations transferring energy between the vertical and the rotational motions.



One Degree of Freedom

A simple oscillator has one degree of freedom.



It requires only one independent coordinate to describe its motion. That coordinate is a displacement from equilibrium in the above system.

Newton's Second Law

Newton's Second Law $F = ma$ applied to springs is

$$-kx = m\ddot{x}.$$

Dividing by m and letting k/m equal to ω_0^2 yields

$$\ddot{x} + \omega_0^2 = 0.$$

A well known solution to the spring problem is

$$x = A\cos(\omega_0 t + \phi),$$

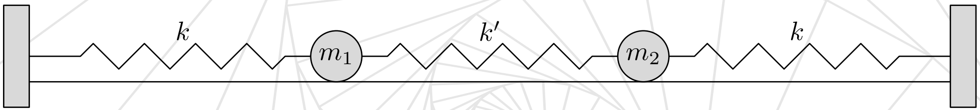
where ω_0 is the "natural frequency."

Natural Frequency

- Natural frequency is a constant frequency of a non-driven, undamped harmonic oscillator.
- Frequency is dependent on the spring constant k and mass m.
- Frequency is not dependent on initial displacement of mass or initial velocity.

Two Degrees of Freedom

Two degree of freedom systems require two independent coordinates that describe their motion.



Two degrees of freedom:

1. x_1 is the displacement of mass m_1 from its equilibrium position.
2. x_2 is the displacement of mass m_2 from its equilibrium position.

Normal Frequency

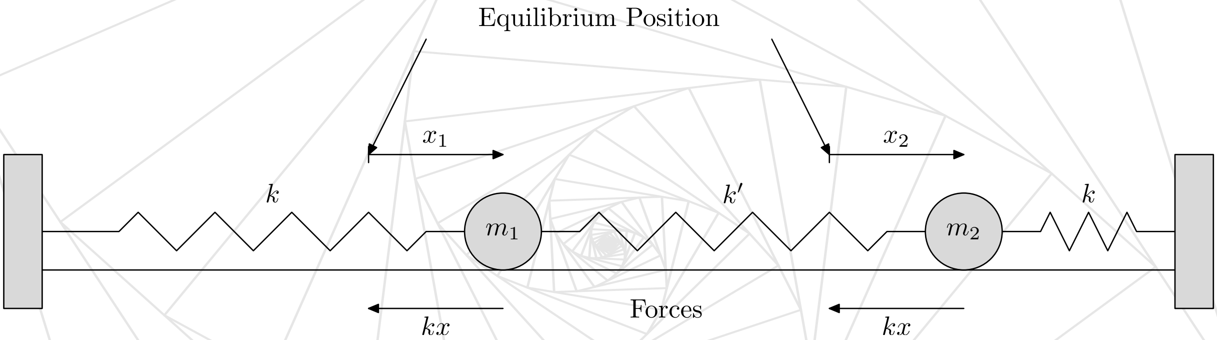
- Are there any configurations that in which the two masses can oscillate with the same frequency? These are the normal frequencies of the system

- If the masses are oscillating with the same frequency and phase angle, then they can be said to be oscillating at a normal frequency.

- If the masses are oscillating at equal frequencies that are 180^o out of phase, then they also can be said to be oscillating at a normal frequency.

First Normal Frequency

Spring k' will remain slack and can be ignored.



Therefore, the mass will oscillate with their natural frequencies

$$\omega_1 = \sqrt{\frac{k}{m}}.$$

First Normal Mode

The equations of motion describing the first normal mode are

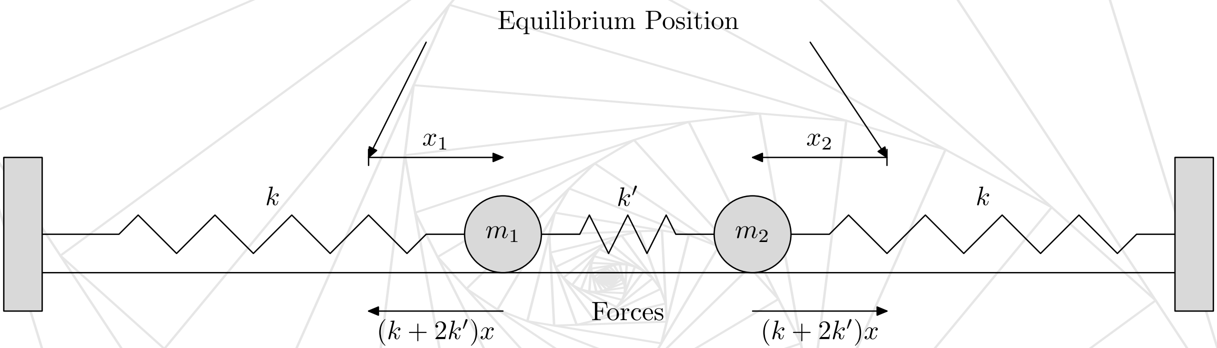
$$x_1 = A\cos(\omega_1 t + \phi_1)$$
$$x_2 = A\cos(\omega_1 t + \phi_1).$$

In vector form,

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A\cos(\omega_1 t + \phi_1) \\ A\cos(\omega_1 t + \phi_1) \end{bmatrix}.$$

Second Normal Frequency

If the displacement of the mass are equal and opposite



then the force acting on each mass will be $-(k+2k')x$, then

$$m\ddot{x} = (-kx - 2k')x$$

$$\ddot{x} + \frac{k+2k'}{m}x = 0$$

$$w_2^2 = \frac{k+2k'}{m}.$$

Second Normal Mode

The equations of motion describing the second normal mode are

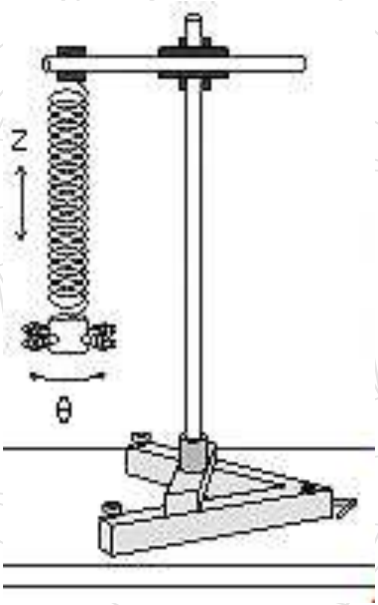
$$x_1 = A\cos\left(\omega_2 t + \phi_2\right)$$

$$x_2 = -A\cos\left(\omega_2 t + \phi_2\right).$$

These can be expressed inn vector form by,

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A\cos\left(w_2 t + \phi_2\right) \\ -A\cos\left(\omega_2 t + \phi_2\right) \end{bmatrix}.$$

Analyzing The Wilberforce Pendulum



In the following discussion, z will be measured from equilibrium in the vertical direction, and θ will be measured from equilibrium in rotation.

The Lagrangian

The Lagrangian is the kinetic energy minus the potential energy of the system,

$$L = K - U = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\delta\theta^2 - \frac{1}{2}\epsilon z\theta$$

Where

$$K = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2,$$

and

$$U = \frac{1}{2}kz^2 + \frac{1}{2}\delta\theta^2 + \frac{1}{2}\epsilon z\theta.$$

Euler-Lagrange Equations

The Euler-Lagrange Equations that minimize the Lagrangian of the system are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0,$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$$

Applying the Euler-Lagrange Equations

Recall our Lagrangian.

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}kz^2 - \frac{1}{2}\delta\theta^2 - \frac{1}{2}\epsilon z\theta$$

Applying the Euler-Lagrange equation,

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z}$$

$$0 = \frac{d}{dt}(m\dot{z}) - \left(-kz - \frac{1}{2}\epsilon\theta\right)$$

$$0 = m\ddot{z} + kz + \frac{1}{2}\epsilon\theta.$$

Applying the Euler-Lagrange Equations

Similarly,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

becomes

$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0.$$

Assumed Solutions for z and θ

Thus, the following system describes the motion.

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\theta = 0$$
$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

We will assume solutions

$$z(t) = A_1 \cos(\omega t + \phi)$$
$$\theta(t) = A_2 \cos(\omega t + \phi),$$

since they are well known solutions for harmonic oscillators.

Differential Equations

Taking first derivatives

$$\dot{z}(t) = -A_1\omega \sin(\omega t + \phi)$$
$$\dot{\theta}(t) = -A_2\omega \sin(\omega t + \phi)$$

and second derivatives,

$$\ddot{z}(t) = -A_1\omega^2 \cos(\omega t + \phi)$$
$$\ddot{\theta}(t) = -A_2\omega^2 \cos(\omega t + \phi).$$

By substituting these into

$$m\ddot{z} + kz + \frac{1}{2}\epsilon\theta = 0$$
$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

we produce the results on the next slide.

Solving the Differential Equation

These results are

$$m(-A_1\omega^2\cos(\omega t+\phi)) + kA_1\cos(\omega t+\phi) + \frac{1}{2}\epsilon A_2\cos(\omega t+\phi) = 0$$
$$I(-A_2\omega^2\cos(\omega t+\phi)) + \delta A_2\cos(\omega t+\phi) + \frac{1}{2}\epsilon A_1\cos(\omega t+\phi) = 0.$$

Factoring out $\cos(\omega t+\phi)$ and dividing the first equation by m and the second by I produces

$$\left(\frac{k}{m}-\omega^2\right)A_1 + \frac{\epsilon}{2m}A_2 = 0$$
$$\frac{\epsilon}{2I}A_1 + \left(\frac{\delta}{I}-\omega^2\right)A_2 + = 0.$$

Because these are the natural frequencies of the uncoupled system, we may set

$$k/m = \omega_z^2 \quad \text{and} \quad \delta/I = \omega_\theta^2.$$

Finding Non-Trivial Solutions

The equations for which we can solve for A_1 and A_2 are

$$(\omega_z^2 - \omega^2)A_1 + \frac{\epsilon}{2m}A_2 = 0$$
$$\frac{\epsilon}{2I}A_1 + (\omega_\theta^2 - \omega^2)A_2 + = 0.$$

In order to find non-trivial solutions, the determinant of the coefficient matrix must be equal to zero.

$$\begin{vmatrix} \omega_z^2 - \omega^2 & \frac{\epsilon}{2m} \\ \frac{\epsilon}{2I} & \omega_\theta^2 - \omega^2 \end{vmatrix} = 0.$$

Normal Frequencies

Expanding the determinant and grouping like terms yields

$$\omega^4 - (\omega_z^2 - \omega_\theta^2)\omega^2 + \left(\omega_z^2\omega_\theta^2 - \frac{\epsilon^2}{4mI}\right) = 0.$$

Compare this to

$$A\omega^2 + B\omega + C = 0.$$

Using the quadratic formula yields the normal frequencies

$$\omega_1^2 = \frac{1}{2}\left\{\omega_\theta^2 + \omega_z^2 + \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\epsilon^2}{mI}}\right\}$$

$$\omega_2^2 = \frac{1}{2}\left\{\omega_\theta^2 + \omega_z^2 - \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\epsilon^2}{mI}}\right\}.$$

Amplitudes

We need to know the relation of the amplitudes in both θ and z directions when the system is in a normal mode. Since we have chosen ω's that make

$$(\omega_z^2 - \omega^2)A_1 + \frac{\epsilon}{2m}A_2 = 0$$
$$\frac{\epsilon}{2I}A_1 + (\omega_\theta^2 - \omega^2)A_2 + = 0$$

dependent, we only need to solve one for the ratio of the amplitudes.

Finding the First Amplitude Ratio

To solve,

$$(\omega_z^2 - \omega^2)A_1 + \frac{\epsilon}{2m}A_2 = 0 \tag{1}$$

for the ratio of the amplitudes at the first normal frequency, we substitute

$$\omega_z^2 = \omega_\theta^2 = \omega^2$$

in

$$\omega_1^2 = \frac{1}{2}\left\{\omega_\theta^2 + \omega_z^2 + \sqrt{(\omega_\theta^2 - \omega_z^2)^2 + \frac{\epsilon^2}{mI}}\right\}$$

to get

$$\omega_1^2 = \omega + \frac{\epsilon}{\sqrt{4mI}}.$$

We then substitute this result into Equation (1) for ω^2 to get,

$$-\frac{\epsilon}{\sqrt{4mI}}A_1 + \frac{\epsilon}{2m}A_2 = 0.$$

Finding the Amplitude ratios

Solving

$$-\frac{\epsilon}{\sqrt{4mI}}A_1 + \frac{\epsilon}{2m}A_2 = 0$$

for A_2/A_1, we find that the ratio of the amplitudes at ω_1 to be

$$r_1 = \frac{A_2}{A_1} = \sqrt{\frac{m}{I}}.$$

Second Amplitude Ratio

Using the same procedure, we get

$$r_2 = \frac{A_2}{A_1} = -\sqrt{\frac{m}{I}}.$$

Therefore, the amplitude vectors can be written as

$$\mathbf{A}^{(1)} = \begin{bmatrix} A_1^{(1)} \\ \\ A_2^{(1)} \end{bmatrix} = \begin{bmatrix} A_1^{(1)} \\ \\ r_1 A_1^{(1)} \end{bmatrix}$$
$$\mathbf{A}^{(2)} = \begin{bmatrix} A_1^{(2)} \\ \\ A_2^{(2)} \end{bmatrix} = \begin{bmatrix} A_1^{(2)} \\ \\ r_2 A_1^{(2)} \end{bmatrix},$$

where the superscript denotes the normal frequency at which the amplitude was obtained.

Normal Modes

The solution vectors can be written as

$$\mathbf{x}^{(1)} = \begin{bmatrix} z^{(1)}(t) \\ \\ \theta^{(1)}(t) \end{bmatrix} = \begin{bmatrix} A_1^{(1)} \cos\left(\omega_1 t + \phi_1\right) \\ \\ r_1 A_1^{(1)} \cos\left(\omega_1 t + \phi_1\right) \end{bmatrix} = \text{first mode}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} z^{(2)}(t) \\ \\ \theta^{(2)}(t) \end{bmatrix} = \begin{bmatrix} A_1^{(2)} \cos\left(\omega_2 t + \phi_2\right) \\ \\ r_2 A_1^{(2)} \cos\left(\omega_2 t + \phi_2\right) \end{bmatrix} = \text{second mode}$$

General Solutions

The general solution can be written as a linear combination of the normal modes.

$$z(t) = z^{(1)}(t) + z^{(2)}(t) = A_1^{(1)} \cos\left(\omega_1 t + \phi_1\right) + A_1^{(2)} \cos\left(\omega_2 t + \phi_2\right)$$

$$\theta(t) = \theta^{(1)}(t) + \theta^{(2)}(t) = A_1^{(1)} r_1 \cos\left(\omega_1 t + \phi_1\right) + A_1^{(2)} r_2 \cos\left(\omega_2 t + \phi_2\right)$$

Initial Conditions

Giving our mass an initial twist and pull, the initial conditions are

$$z(0) = z_0 \qquad \dot{z}(0) = 0$$
$$\theta(0) = \theta_0 \qquad \dot{\theta}(0) = 0.$$

Substituting these into the general solution yields

$$z_0 = A_1^{(1)} \cos\phi_1 + A_1^{(2)} \cos\phi_2$$
$$0 = -\omega_1 A_1^{(1)} \sin\phi_1 - \omega_2 A_1^{(2)} \sin\phi_2$$
$$\theta_0 = r_1 A_1^{(1)} \cos\phi_1 + r_2 A_1^{(2)} \cos\phi_2$$
$$0 = -r_1\omega_1 A_1^{(1)} \sin\phi_1 - r_2\omega_2 A_1^{(2)} \sin\phi_2.$$

Coefficients

Solving this system for $A_1^{(1)}, A_1^{(2)}, \phi_1$ and ϕ_2

$$A_1^{(1)} = \frac{r_1\theta_0 + z_0}{r_1 - r_2},$$

$$A_1^{(2)} = \frac{r_1\theta_0 - z0}{r_1 - r_2},$$

$$\phi_1 = \phi_2 = 0.$$

Solutions

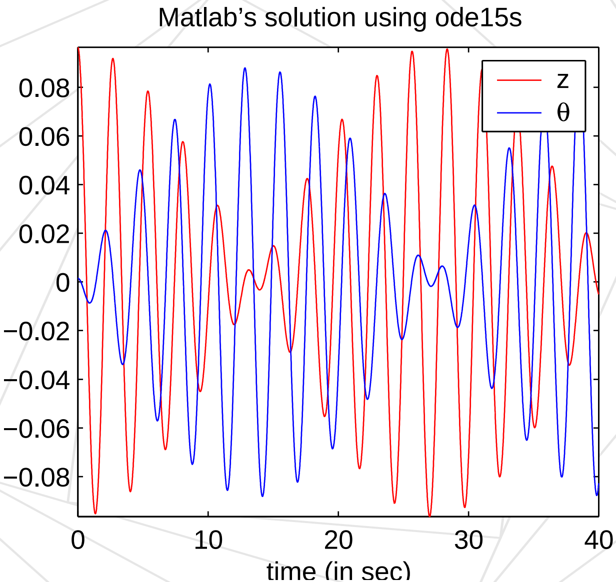
Substituting $\sqrt{m/I}$ for r_1, and $-\sqrt{m/I}$ for r_2, the general solution for the motion of the Wilberforce pendulum is

$$z(t) = \frac{\sqrt{\frac{m}{I}}\theta_0 + z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_1 t + \frac{\sqrt{\frac{m}{I}}\theta_0 - z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_2 t$$

$$\theta(t) = \sqrt{\frac{m}{I}}\frac{\sqrt{\frac{m}{I}}\theta_0 + z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_1 t - \sqrt{\frac{m}{I}}\frac{\sqrt{\frac{m}{I}}\theta_0 - z_0}{2\sqrt{\frac{m}{I}}}\cos\omega_2 t.$$

$$r_1 = \sqrt{\frac{m}{I}} \qquad r_2 = -\sqrt{\frac{m}{I}}$$

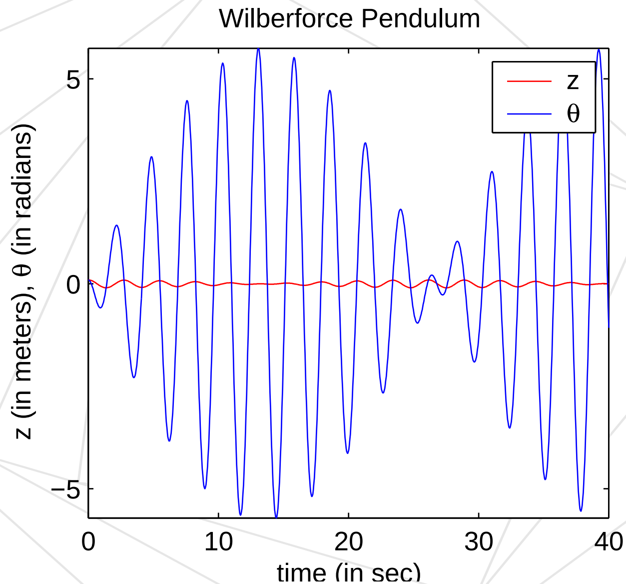
Matlab Solution



$$m\ddot{z} + kz + \frac{1}{2}\epsilon\theta = 0$$

$$I\ddot{\theta} + \delta\theta + \frac{1}{2}\epsilon z = 0$$

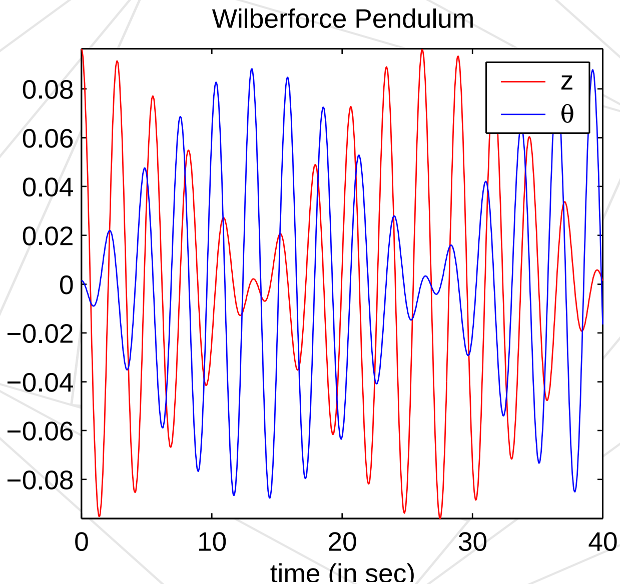
Initial Figure



Notice that the displacement of the mass in the vertical direction is disproportional to the displacement of the mass' rotation.

Compensated Figure

We needed to convert θ from radians into meters by $s = r\theta$.



Now our graph matches Matlab's identically.