1.5 Application
Linear Equations and Temperature Oscillations

Section 1.5 of the text describes the following 4-step algorithm for solving the linear first-order differential equation

$$\frac{dy}{dx} + p(x)\ y = q(x). \tag{1}$$

1. Begin by calculating the *integrating factor*

$$\rho(x) = e^{\int p(x)\,dx}. \tag{2}$$

2. Then multiply both sides of the differential equation by $\rho(x)$.

3. Next, recognize the left-hand side of the resulting equation as the derivative of a product, so it takes the form

$$D_x[\rho(x)\ y(x)] = \rho(x)\ q(x). \tag{3}$$

4. Finally, integrate this last equation to get

$$\rho(x)\ y(x) = \int \rho(x)\ q(x)\ dx + C, \tag{4}$$

and then solve for $y(x)$ to obtain the general solution of the original differential equation in (1).

This algorithm is well-adapted to automatic symbolic computation. In the technology-specific sections of this project, we illustrate its implementation using *Maple*, *Mathematica*, and MATLAB to solve the initial value problem

$$\frac{dy}{dx} - 3y = e^{2x}, \qquad y(0) = 3. \tag{5}$$

In each case, the commands shown here constitute a "template" that you can apply to any given linear first-order differential equation. First redefine the coefficient functions $p(x)$ and $q(x)$ as specified by *your* linear equation, then work through the subsequent steps. You may apply whatever computer algebra system is available to carry out this algorithmic process for the initial value problem in (5), and then apply it to a selection of other examples and problems in Section 1.5 of the text.

For an applied problem to solve in this manner, consider indoor temperature oscillations that are "driven" by outdoor temperature oscillations of the form

$$A(t) \;=\; a_0 + a_1 \cos\frac{\pi t}{12} + b_1 \sin\frac{\pi t}{12} \tag{6}$$

having a period of 24 hours (so the cycle of outdoor temperatures repeats itself daily). For instance, for a typical July day in Athens, GA with a minimum temperature of 70°F when $t = 4$ (4 am) and a maximum of 90°F when $t = 16$ (4 pm), we would take

$$A(t) \;=\; 80 - 10\cos\omega(t-4) \;=\; 80 - 5\cos\omega t - 5\sqrt{3}\sin\omega t \tag{7}$$

with $\omega = \pi/12$, using the trigonometric identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to get $a_0 = 80$, $a_1 = 5$, and $b_1 = -5\sqrt{3}$ in (22).

If we write Newton's law of cooling (Eq. (3) of Section 1.1 of the text) for the corresponding indoor temperature $u(t)$ at time t, but with the outside temperature $A(t)$ given by (6) instead of a constant ambient temperature A, we get the linear first-order differential equation

$$\frac{du}{dt} \;=\; -k(u - A(t)) \tag{8}$$

with coefficient functions $P(t) = k$ and $Q(t) = k\,A(t)$. Typical values of the proportionality constant k range from 0.2 to 0.5 (though k might be greater than 0.5 for a poorly insulated building with open windows, or less than 0.2 for a well-insulated building with tightly sealed windows).

Scenario

Suppose our air conditioner fails at time $t_0 = 0$ one midnight, and we cannot afford to get it repaired until payday at the end of the month. We therefore want to investigate the resulting indoor temperatures that we must endure for the next several days.

Begin your investigation by using a computer algebra system to solve Eq. (8) with $A(t)$ given by (7) and with the initial condition $u(0) = u_0$ (the indoor temperature at the time of the air conditioner failure). (Remember that the independent and dependent variables are t and u instead of x and y.) You should get the solution

$$u(t) \;=\; 80 + c_0 e^{-kt} + c_1\cos\omega t + d_1 \sin\omega t \tag{9}$$

where

$$c_0 \;=\; u_0 - \frac{75k^2 + 5\sqrt{3}\,k\omega + 80\omega^2}{k^2 + \omega^2},$$

$$c_1 \;=\; \frac{-5k^2 + 5\sqrt{3}\,k\omega}{k^2 + \omega^2}, \qquad d_1 \;=\; \frac{-5\sqrt{3}\,k^2 - 5k\omega}{k^2 + \omega^2}$$

with $\omega = \pi/12$. With $k = 0.2$ (for instance), this solution reduces (approximately) to

$$u(t) = 80 + e^{-t/5}(u_0 - 82.3351) + 2.3351\cos\frac{\pi t}{12} - 5.6036\sin\frac{\pi t}{12}. \qquad (10)$$

Observe first that the damped exponential term in (10) approaches 0 as $t \to \infty$, leaving the long-term "steady periodic" solution

$$u_{sp}(t) = 80 + 2.3351\cos\frac{\pi t}{12} - 5.6036\sin\frac{\pi t}{12}. \qquad (11)$$

Consequently, the long-term indoor temperatures oscillate every 24 hours about the same average temperature of 80°F as the outdoor average temperature.

Fig. 1.5.10 in the text shows a number of solution curves corresponding to possible initial u_0 temperatures ranging from 65°F to 95°F. Observe that — whatever the initial temperature — the indoor temperature "settles down" within about 18 hours to a periodic daily oscillation. However, the "amplitude" of temperature variation is less indoors than outdoors. Indeed, using the trigonometric identity mentioned previously, Eq. (11) can be rewritten (verify this!) as

$$u(t) = 80 - 6.0707\cos\left(\frac{\pi t}{12} - 1.9656\right) = 80 - 6.0707\cos\frac{\pi}{12}(t - 7.5082). \qquad (12)$$

Do you see that this implies that the indoor temperature varies between a minimum of about 74°F and a maximum of about 86°F?

Finally, comparison of (7) and (12) indicates that the indoor temperature lags behind the outdoor temperature by about $7.5082 - 4 \approx 3.5$ hours, as illustrated in Fig. 1.5.11. Thus the temperature inside the house continues to rise until about 7:30 pm each evening, so the hottest part of the day inside is early evening, rather than late afternoon (as outside).

For a personal problem to investigate, carry out a similar analysis using average July daily maximum/minimum figures for your own locale, and a value of k appropriate to your own home. You might also consider a winter day instead of a summer day. (What is the winter-summer difference for the indoor temperatures problem?)

Using *Maple*

We first define the coefficient functions and initial values for the initial value problem specified in (5).

```
p := -3:
q := exp(2*x):
```

```
x0 := 0:
y0 := 3:
```

Then we calculate the integrating factor defined in (2),

```
rho := exp( int(p,x) );
```

$$\rho := e^{(-3x)}$$

Equation (4) above now gives the general solution

```
yg := (1/rho)*(c + int(rho*q, x));
```

$$yg := \frac{c - \frac{1}{e^x}}{e^{(-3x)}}$$

```
yg := simplify(%);
```

$$yg := e^{(3x)}(c - e^{(-x)})$$

If we want a particular solution, we need only substitute the given initial values and solve for the constant c.

```
c := eval( solve(subs(x=x0,yg)=y0,c) );
```

$$c := 4$$

Now that $c = 4$, we need only evaluate our general solution to get the desired particular solution.

```
yp := eval(yg);
```

$$yp := e^{(3x)}(4 - e^{(-x)})$$

Frequently it is necessary to expand and simplify an expression to get the "best" form.

```
yp := simplify(expand( yp ));
```

$$yp := 4e^{(3x)} - e^{(2x)}$$

Using *Mathematica*

We first define the coefficient functions and initial values for the initial value problem specified in (5).

```
p = -3;
q =  Exp[2x];
x0 = 0
y0 = 3;
```

Then we calculate the integrating factor defined in (2),

```
rho = Exp[ Integrate[p, x] ]
```

e^{-3x}

Equation (4) above now gives the general solution

```
yg = (1/rho)(c + Integrate[rho*q, x])
```

$e^{3x}\left(c-e^{-x}\right)$

If we want a particular solution, we need only substitute the given initial values and solve for the constant c.

```
Solve[(yg /. x -> x0) == y0, c];
c = c /. First[%]
```

4

Now that $c = 4$, we need only evaluate our general solution to get the desired particular solution

```
yp = yg // Expand
```

$-e^{2x}+4e^{3x}$

Frequently it is necessary to expand and/or simplify an expression to get the "best" form.

Using MATLAB

We first define the coefficient functions and initial values for the initial value problem specified in (5).

```
syms x y c
p = -3;
q = exp(2*x);
x0 = 0;
y0 = 3;
```

Then we calculate the integrating factor defined in (2),

```
rho = exp(int(p,x))

rho =
exp(-3*x)
```

Equation (4) above now gives the general solution

```
yg = (1/rho)*(c + int(rho*q, x))

yg =
1/exp(-3*x)*(c-1/exp(x))
```

That is, $y(x) = e^{3x}(c - e^{-x})$. To find the desired particular solution, we substitute $x = x_0$ and solve the equation $y(x) = x_0$ for the numerical value of the arbitrary constant c.

```
solve( subs(yg, x,x0)-y0, c)

ans =
4
```

Finally, we get the desired particular solution by substituting $c = 4$ in the general solution.

```
yp = expand(subs(yg, c,4))

yp =
4*exp(x)^3-exp(x)^2
```

That is, $y(x) = 4e^{3x} - e^{2x}$.