Application 5.6

Forced Vibrations and Resonance

Here we investigate forced vibrations of the mass-spring-dashpot system with equation

$$m x'' + c x' + k x = F(t) \tag{1}$$

To simplify the notation, let's take $m = p^2$, $c = 2p$, and $k = p^2q^2 + 1$ where $p, q > 0$. Then the complementary function of Eq. (1) is

$$x_c(t) = e^{-t/p}\left(c_1 \cos qt + c_2 \sin qt\right). \tag{2}$$

We will take $p = 5$, $q = 3$ and thus investigate the transient and steady periodic solutions corresponding to

$$25x'' + 10x' + 226x = F(t), \quad x(0) = 0, \quad x'(0) = 0 \tag{3}$$

with several illustrative possibilities for the external force $F(t)$. For your personal investigations to carry out similarly, you might select integers p and q with $6 \le p \le 9$ and $2 \le q \le 5$.

Investigation 1

With periodic external force $F(t) = 901\cos 3t$ the graph of the solution $x(t)$ of the resulting initial value problem in (3) is shown in Fig. 5.6.13 in the text. There we see the (transient plus steady periodic) solution

$$x(t) = \cos 3t + 30\sin 3t + e^{-t/5}\left[-\cos 3t - (451/15)\sin 3t\right]$$

rapidly "building up" to the steady periodic oscillation $x_{sp}(t) = \cos 3t + 30\sin 3t$.

Investigation 2

With damped oscillatory external force $F(t) = 900e^{-t/5}\cos 3t$ we have duplication with the complementary function in (2). The graph of the solution $x(t)$ of the resulting initial value problem in (3) is shown in Fig. 5.6.14 in the text. There we see the solution

$$x(t) = 6t\,e^{-t/5}\sin 3t$$

oscillating up-and-down between the envelope curves $x = \pm 6t\,e^{-t/5}$. (Note the t-factor that betokens a resonance situation.)

Investigation 3

With damped oscillatory external force $F(t) = 2700t\,e^{-t/5}\cos 3t$ we have a still more complicated resonance situation. The graph of the solution $x(t)$ of the resulting initial value problem in (3) is shown in Fig. 5.6.15 in the text. There we see the solution

$$x(t) = e^{-t/5}\left[3t\cos 3t + \left(9t^2 - 1\right)\sin 3t\right]$$

oscillating up-and-down between the envelope curves $x = \pm e^{-t/5}\sqrt{(3t)^2 + (9t^2+1)^2}$.

Using *Maple*

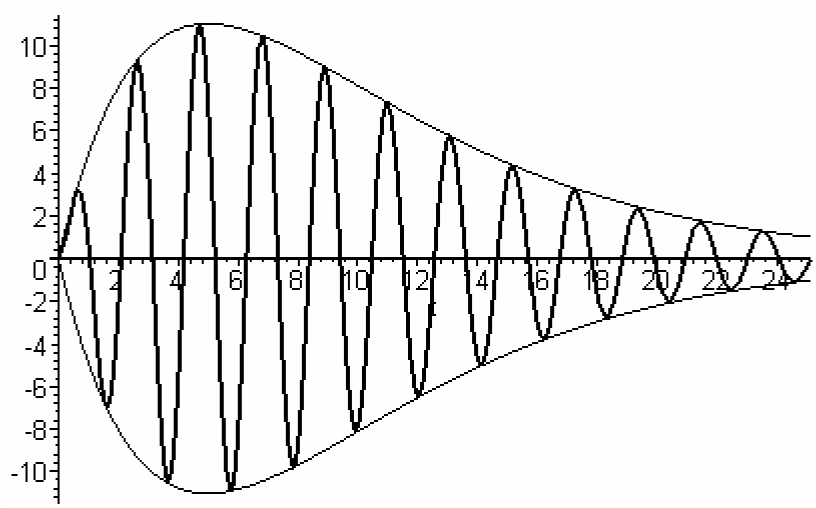
With the damped oscillatory external force

```
F := 900*exp(-t/5)*cos(3*t):
```

of Investigation 2 we have the differential equation

```
de :=
25*diff(x(t),t,t)+10*diff(x(t),t)+226*x(t) = F;
```

$$de := 25\left(\frac{\partial^2}{\partial t^2}x(t)\right) + 10\left(\frac{\partial}{\partial t}x(t)\right) + 226\,x(t) = 900\,e^{\left(-\frac{1}{5}t\right)}\cos(3\,t)$$



The solution of the initial value problem in (3) is then given by

```
dsolve({de,x(0)=0,D(x)(0)=0}, x(t)):
x := simplify(combine(rhs(%),trig));
```

$$x := 6e^{\left(-\frac{1}{5}t\right)}\sin(3t)\,t$$

Thus we have a damped oscillation with amplitude function

```
C :=  6*t*exp(-t/5):
```

The command

```
plot([x,C,-C],t=0..8*Pi);
```

then produces the figure shown on the preceding page.

Using *Mathematica*

With the damped oscillatory external force

```
F = 2700t Exp[-(t/5)] Cos[3t];
```

of Investigation 3 we have the differential equation

```
de =   25 x''[t] + 10 x'[t] + 226 x[t] == F
```

$$226\,x(t)+10\,x'(t)+25\,x''(t) == 2700\,e^{-t/5}t\cos(3t)$$

The solution of the initial value problem in (3) is then given by

```
soln =
DSolve[{de, x[0]==0, x'[0]==0}, x[t], t];

x = First[x[t] /. soln] // Simplify
```

$$e^{-t/5}\left(3t\cos(3t)+(9t^2-1)\sin(3t)\right)$$

Thus we have a damped oscillation with amplitude function

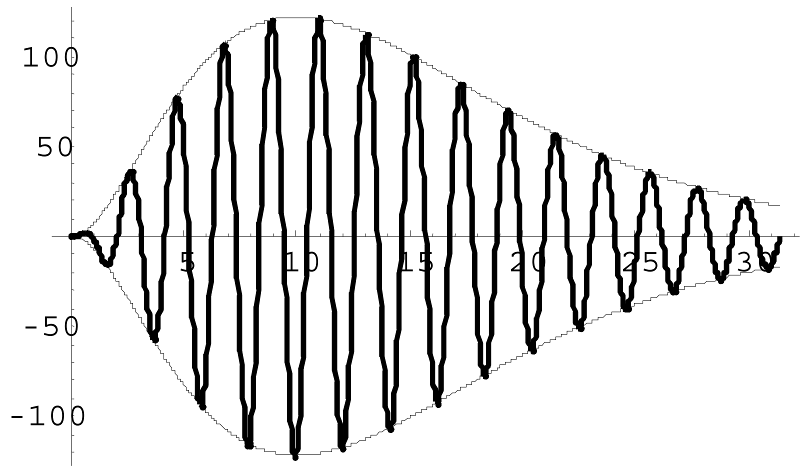
```
amp = Exp[-(t/5)] Sqrt[(3t)^2 + (9t^2-1)^2]
```

$$e^{-t/5}\sqrt{9t^2+(9t^2-1)^2}$$

The command

```
Plot[{x, amp,-amp}, {t, 0,10 Pi}];
```

then produces the figure shown at the top of the next page.



Using MATLAB

With the periodic external force $F(t) = 901\cos 3t$ of Investigation 1 our initial value problem is

$$25x'' + 10x' + 226x = 901\cos 3t, \qquad x(0) = 0, \quad x'(0) = 0.$$

We proceed to solve this problem using MATLAB's symbolic **dsolve** function.

```
dsolve('25*D2x+10*Dx+226*x=901*cos(3*t)',
          'x(0)=0, Dx(0)=0');
x = simple(ans)
x =
cos(3*t)+30*sin(3*t)-exp(-1/5*t)*cos(3*t)-
     451/15*exp(-1/5*t)*sin(3*t)
```

We see that this particular solution is the sum of the steady periodic solution

```
t = 0 : pi/100 : 6*pi;
xsp = cos(3*t)+30*sin(3*t);
```

and the transient solution

```
xtr = -exp(-t/5).*(cos(3*t)+(451/15)*sin(3*t));
```

The plot commands

```
plot(t, xsp),
axis([0 6*pi -40 40])
hold on
plot(t, xsp + xtr)
```

finally produce the plot shown below. We see the (transient plus steady periodic) solution

$$x(t) = \cos 3t + 30\sin 3t + e^{-t/5}\left[-\cos 3t - (451/15)\sin 3t\right]$$

rapidly "building up" to the steady periodic oscillation $x_{sp}(t) = \cos 3t + 30\sin 3t$.