Tacoma Narrows and the Gradient Vector

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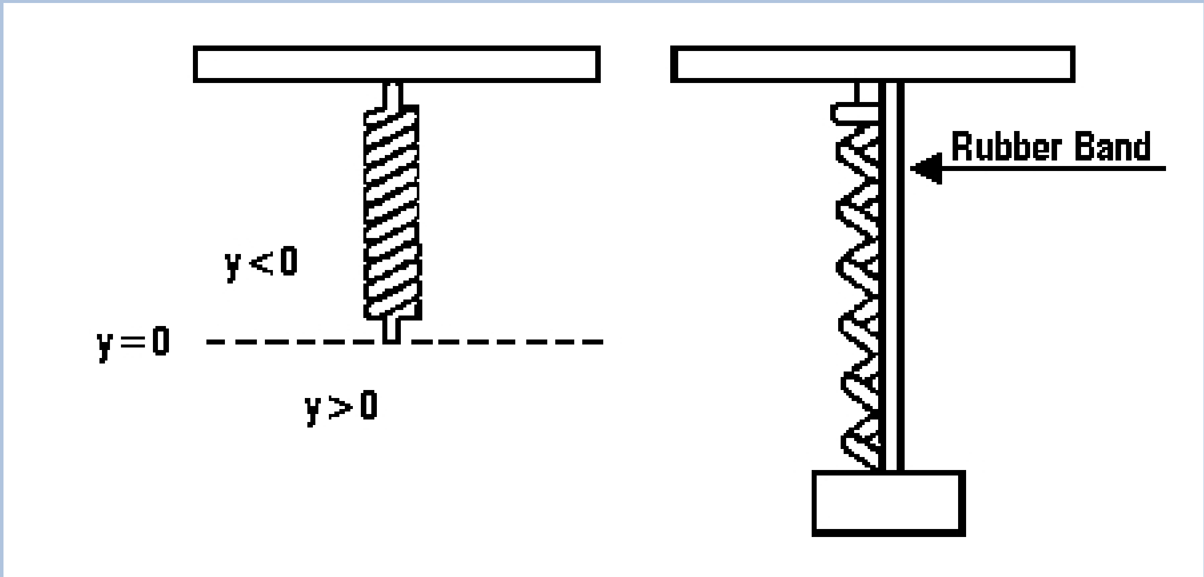
Introduction

The mathematical models that have been proposed to explain the collapse of the Tacoma Narrows Bridge are highly dependent on the initial conditions. Tacoma Narrows and the Gradient Vector introduces the vertical and torsional models used to approximate the behavior of the Tacoma Narrows Bridge. We will use a gradient vector to find initial conditions that lead to periodic solutions. We will look at refining our results using Newton's method. Finally, we will look at some interesting ideas concerning future bridge building.

Constants

- Bridge weighs $\approx 2500\frac{kg}{ft}$
- Width of bridge $12m$
- Spring constants of the cables $1000\frac{kg}{m^2}$
- Torsional oscillations $12 - 14$ per Min.
- Damping constant ≈ 0.01

Vertical Model



Choosing positive values of $y(t)$ to be in the downward direction.

Two Restoring Forces

The first restoring force comes from the structure of the bridge itself, and can be modelled as a simple spring. Newton second law gives us,

$$\sum F = ma.$$

Which can be rewritten as

$$my'' = -k_1 y. \tag{1}$$

The second restoring force comes from the cables of the bridge. We will model this as a rubber band. Its equation is

$$my'' = -k_2 y, \quad y = \begin{cases} y & y > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

As you can see there are two distinct time periods in which the restoring forces are different. If $y > 0$ there are two restoring forces acting on the bridge.

$$my'' = -k_1 y - k_2 y \tag{3}$$

$$my'' = -(k_1 + k_2)y \tag{4}$$

Whereas, if $y < 0$ there is only one restoring force acting on the bridge.

$$my'' = -k_1 y \tag{5}$$

If we let a be the combination $k_1 + k_2$, and let $b = k_1$, we can combine Equations (4) and (5), giving us

$$my'' = -ay^+ + by^-, \quad y^+ = \begin{cases} y & y > 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } y^- = \begin{cases} -y & y < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

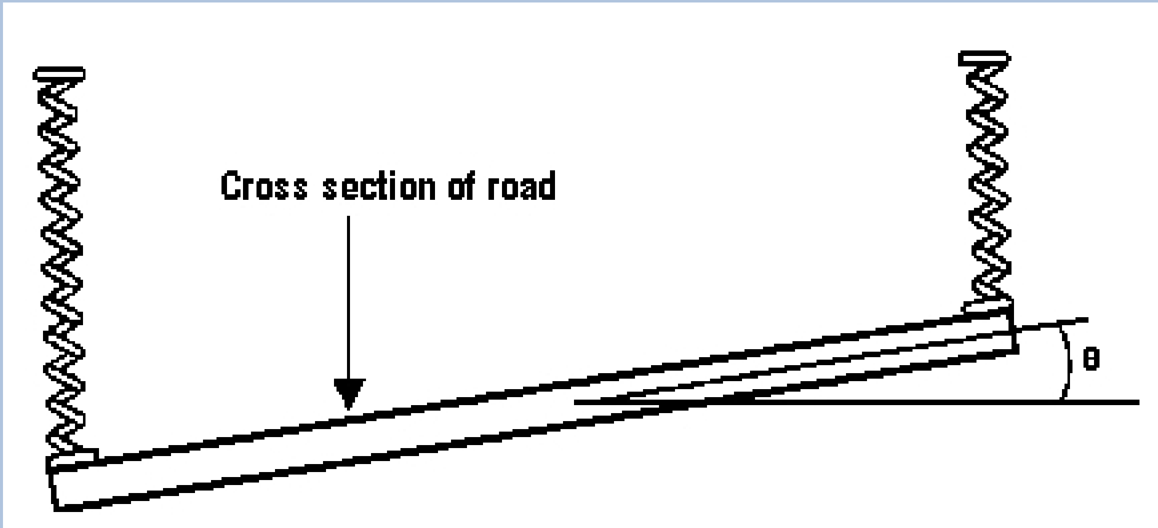
The equation of a forced pendulum is given by

$$y'' + \delta y' + (g/l)y = \lambda \sin \mu t \tag{7}$$

Where δ is the dampening constant, (g/l) is the restoring force, and $\lambda \sin \mu t$ is the forcing term. If we substitute our restoring force into the equation of a forced pendulum, we obtain

$$y'' + 0.01y' + ay^{+} - by^{-} = 10 + \lambda \sin \mu t \tag{8}$$

Torsional Model



- Modelled by two springs
- Piecewise functions

Given by the equation

$$\theta'' + \delta\theta' + \frac{6K}{m}\theta = \lambda \sin \mu t \tag{9}$$

Substituting the appropriate constants into this equation, we obtain

$$\theta'' + 0.01\theta' + 2.4 \sin \theta = \lambda \sin \mu t. \tag{10}$$

Gradient

Beginning with the equation

$$y'' + 0.01y' + ay^+ - by^- = 10 + \lambda \sin \mu t$$

If we let $a = 17$ and $b = 1$, with the initial conditions

$$y(0) = c$$

and

$$y'(0) = d$$

and we let a numeric solver run for one period

$$T = \frac{2\pi}{\mu}$$

If we have a periodic solution the start and end points will be the same.

If they are not we can calculate the error with the function.

$$E(c,d) = \big(c - y(T)\big)^2 + \big(d - y'(T)\big)^2. \tag{11}$$

If we were able to minimize the error, then we would be moving closer to a solution. In order to do this we must calculate the gradient vector.

$$\nabla E = \left\langle \frac{\partial E}{\partial c}, \frac{\partial E}{\partial d} \right\rangle \tag{12}$$

Evaluating the partial derivatives, and remembering that y and y' are functions of c and d, gives us

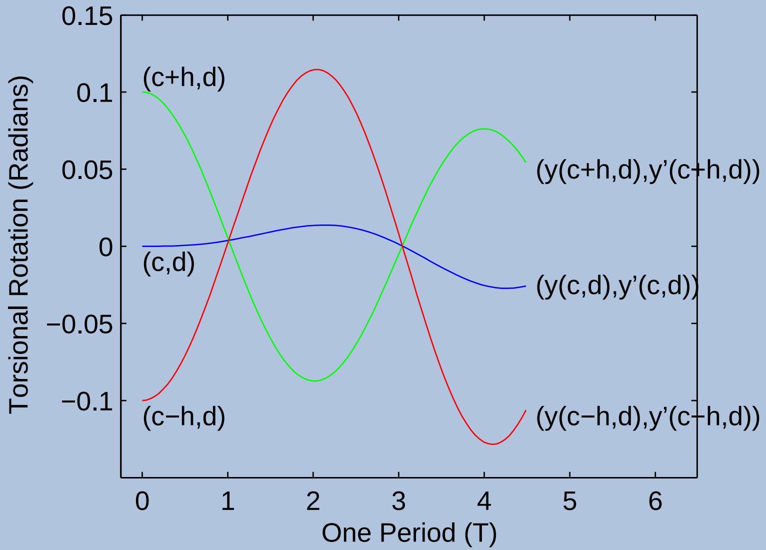
$$\frac{\partial E}{\partial c} = 2\big(c - y(T)\big)\left(1 - \frac{\partial y}{\partial c}(T)\right) + 2\big(d - y'(T)\big)\left(-\left(\frac{\partial y}{\partial c}\right)'(T)\right) \tag{13}$$

and

$$\frac{\partial E}{\partial d} = 2\big(c - y(T)\big)\left(-\frac{\partial y}{\partial d}(T)\right) + 2\big(d - y'(T)\big)\left(1 - \left(\frac{\partial y}{\partial d}\right)'(t)\right). \tag{14}$$

But, how do we calculate $\frac{\partial y}{\partial c}$, $\frac{\partial y}{\partial d}$, $\left(\frac{\partial y}{\partial c}\right)'$, and $\left(\frac{\partial y}{\partial d}\right)'$?

Central Differences



$$\frac{\partial y}{\partial c}(T) = \frac{y(c+h,d) - y(c-h,d)}{2h}. \tag{15}$$

Similarly

$$\frac{\partial y}{\partial d}(T) = \frac{y(c, d+h) - y(c, d+h)}{2h} \tag{16}$$

$$\left(\frac{\partial y}{\partial c}\right)'(T) = \frac{y'(c+h, d) - y'(c-h, d)}{2h} \tag{17}$$

$$\left(\frac{\partial y}{\partial d}\right)'(T) = \frac{y'(c, d+h) - y'(c, d-h)}{2h}. \tag{18}$$

Now we can compute the gradient.

Controlling the Gradient Vector

The gradient vector may be of varying magnitude, depending on the circumstances. We don't want that so we will normalize the vector. The magnitude is given by

$$\sqrt{\left(\frac{\partial E}{\partial c}\right)^2 + \left(\frac{\partial E}{\partial d}\right)^2}. \tag{19}$$

Choosing New Initial Conditions

To minimize the error we have to make a better guess of the proper initial conditions.

$$\begin{bmatrix} c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} c_n \\ d_n \end{bmatrix} - \epsilon \frac{\nabla E}{\|\nabla E\|} \tag{20}$$

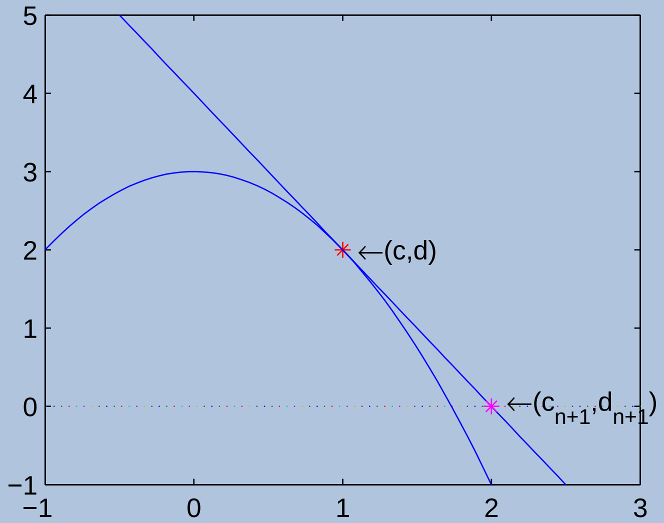
Refining the Results

The gradient method has some limitations.

- It is slow!
- Results are relatively inaccurate 3 sig. fig.

This is a good start but we need another method to get better results.

Newton's Method



We need to define a vector F so that

$$F\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} - \begin{bmatrix} y(T) \\ y'(T) \end{bmatrix}. \tag{21}$$

The Jacobian of F is given by

$$\begin{bmatrix} 1 - \frac{\partial y}{\partial c}, & -\frac{\partial y}{\partial d} \\ -\left(\frac{\partial y}{\partial c}\right)' & 1 - \left(\frac{\partial y}{\partial d}\right)' \end{bmatrix} \tag{22}$$

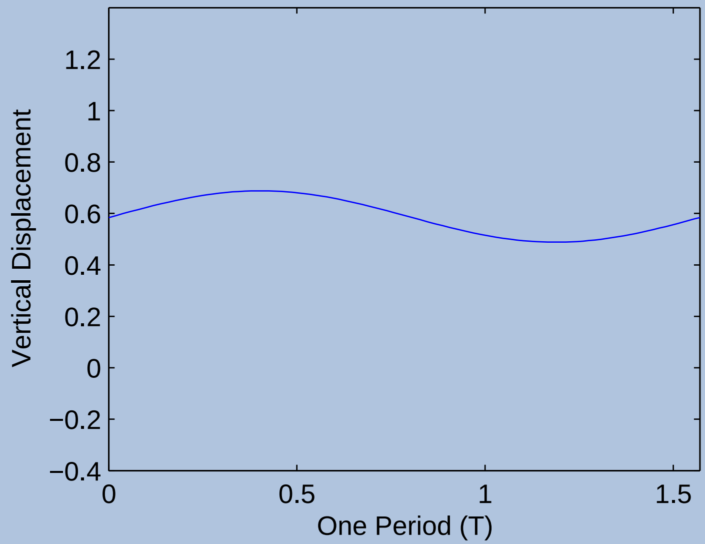
Then, Newton's method is given by the equation

$$\begin{bmatrix} c_{n+1} \\ d_{n+1} \end{bmatrix} = \begin{bmatrix} c_n \\ d_n \end{bmatrix} - \begin{bmatrix} 1 - \frac{\partial y}{\partial c}, & -\frac{\partial y}{\partial d} \\ -\left(\frac{\partial y}{\partial c}\right)' & 1 - \left(\frac{\partial y}{\partial d}\right)' \end{bmatrix}^{-1} F\begin{bmatrix} c \\ d \end{bmatrix}. \tag{23}$$

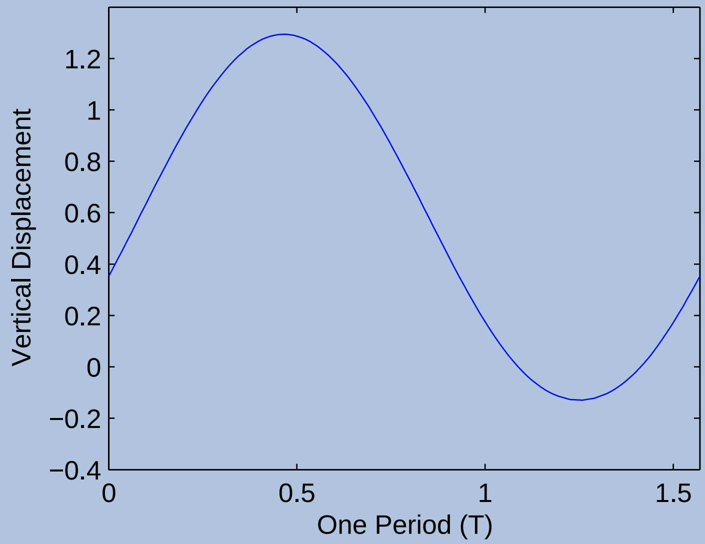
Vertical Model Results

After running these algorithms over a grid of initial conditions I obtained three periodic solutions.

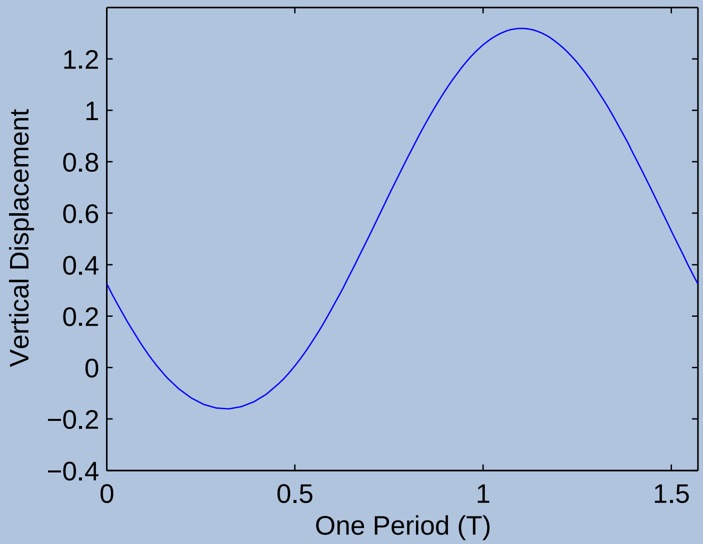
Small amplitude solution with initial conditions $c = 0.5842126367184$ and $d = 0.3993242265230$.



Large amplitude solution with initial conditions $c = 0.352529677237$ and $d = 2.731762638$.



Large amplitude solution with initial conditions $c = 0.325$ and $d = -2.820$.



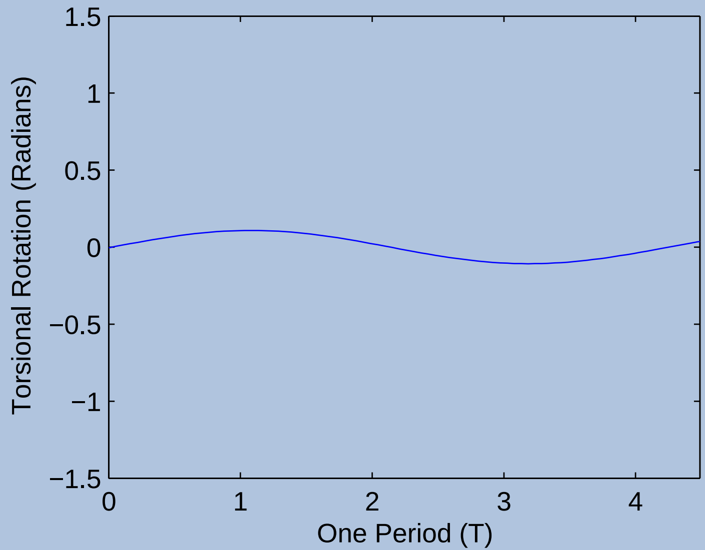
Torsional Results

Using the Equation (10) with $\lambda = 0.05$ and $\mu = 1.4$

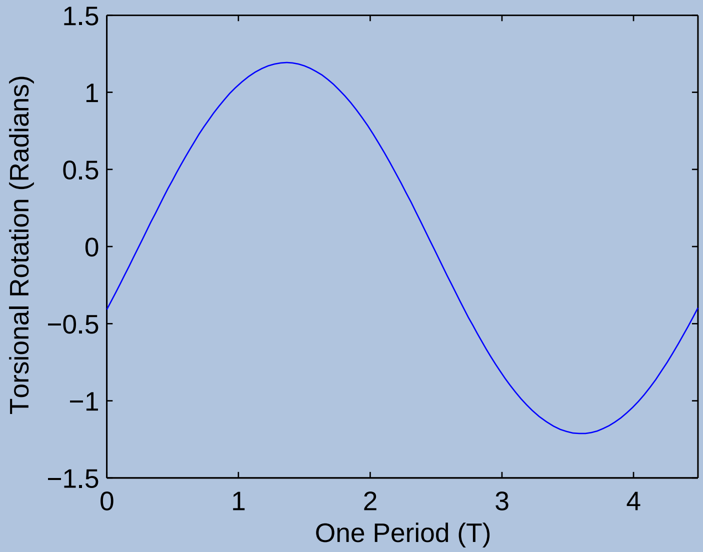
$$\theta'' + 0.01\theta' + 2.4\sin\theta = 0.05\sin(1.4t).$$

I was able to find three solutions.

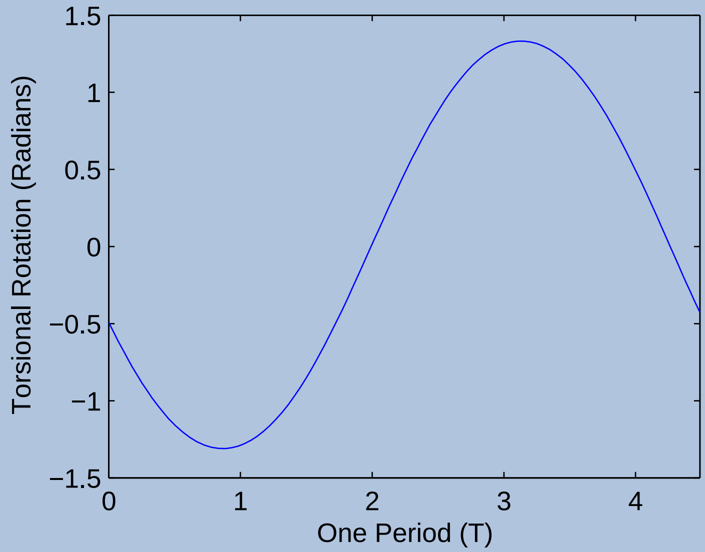
Small amplitude solution with initial conditions $c = -0.00369065407446$ and $d = 0.16039719801966$.



Large amplitude solution with initial conditions $c = -0.408357386906$ and $d = 1.5973907585260$



Large amplitude solution with initial conditions $c = -0.492611587706$ and $d = -1.7281509765743$.



Summary

- The results of a small forcing term are completely dependent on the initial conditions.
- Resonant Frequency
- What can be done?

  Wider is better

  Truss structures

  Under-cabling

Conclusion

While the gradient method was extremely effective at finding periodic solutions for the vertical model. I am at a loss to explain its ineffectiveness with the torsional model. Unfortunately, there is still plenty of work to do. Given more time, I would like to explore the torsional model in further depth. Since, it is the torsional rotation that is credited with destroying the Tacoma Narrows Bridge.