# De-constructing the Deltoid

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# A Brief History of the Deltoid:

The deltoid (sometimes referred to by its other name, the tricuspoid) is one specific curve in the family of hypocycloid curves. It is generally noted that the deltoid's name stems from its shape, which is said to be reminiscent of the Greek letter Delta: $\Delta$. Compare $\Delta$ to the deltoid below and you can come to your own conclusions.
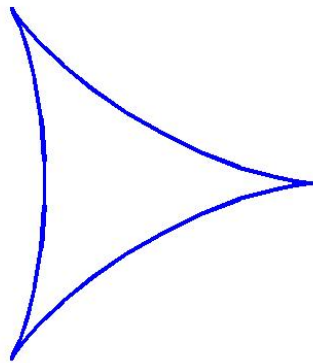


The two primary individuals involved in the initial study of the Deltoid were Leonhard Euler (1707-1783) and Jakob Steiner (1796-1863), both of which were Swiss mathematicians. Euler was also active in other areas as well, including physics (which consequently related to his studies in math) and Steiner was a fan of geometric thinking. Euler first encountered the deltoid curve while working in the field of optics, but a more rigorous study was not made until a century later when Steiner revisited the deltoid. Steiner's work with the deltoid has led some to refer to the curve as Steiner's Hypocycloid.



Steiner          Euler
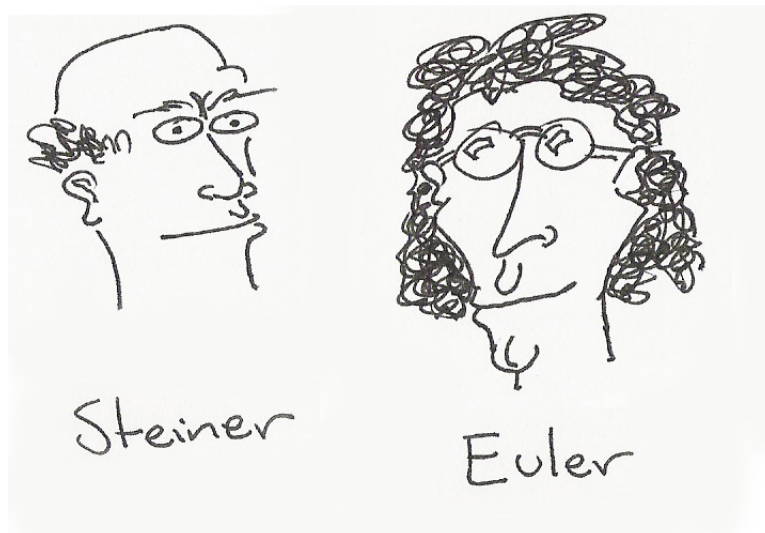
# The Deltoid Defined:

To understand the deltoid, an understanding of the hypocycloid is required. A hypocycloid is the trace of a point on a small circle situated inside a larger circle. The smaller circle rolls like a wheel along the inside circumference of the larger circle without slipping. As the center of the inner circle rotates through an angle $\theta$ about the origin, the point on the edge of the inner circle also rotates through a certain angle, about the center of inner circle. The
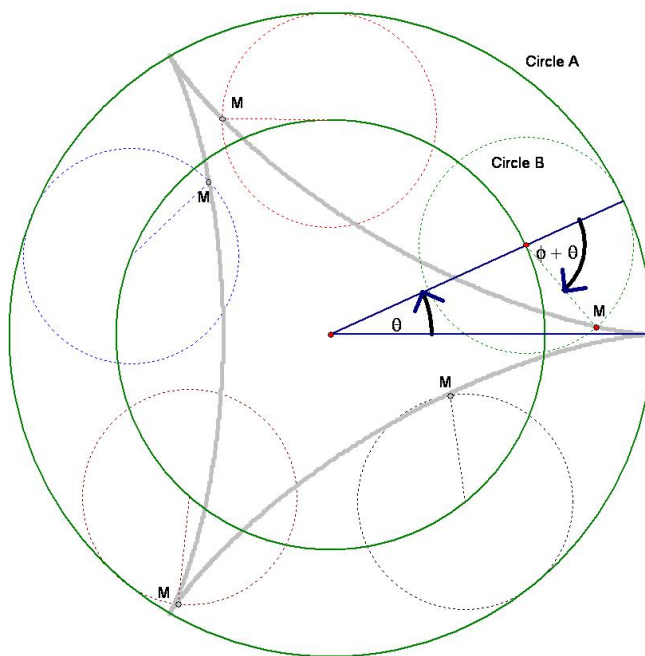


Figure 1: The Trace of the Deltoid

radius of the inner circle with relation to the outer circle (i.e. the ratio between the two) is what makes each hypocylcoid curve different. The deltoid is the hypocycloid in which the inner turning circle has a radius that is $\frac{1}{3}$ the radius of the outer circle. When a point on the edge of the inner circle is traced, a triangular-like shaped curve with concave sides is drawn. This curve is the deltoid curve.

# General Parametrization: The Generic Hypocycloid

Since the deltoid is just one specific case of the hypocycloid, its parametrization is inherently similar to other cases of the hypocycloid. By parameterizing the hypocycloid in general, not only can the equation of the deltoid be found, but the equations of all the variations of the hypocycloid can easily be found using the ratio of the radii of the two circles. To develop the equations, however, a diagram with the geometry involved is needed:
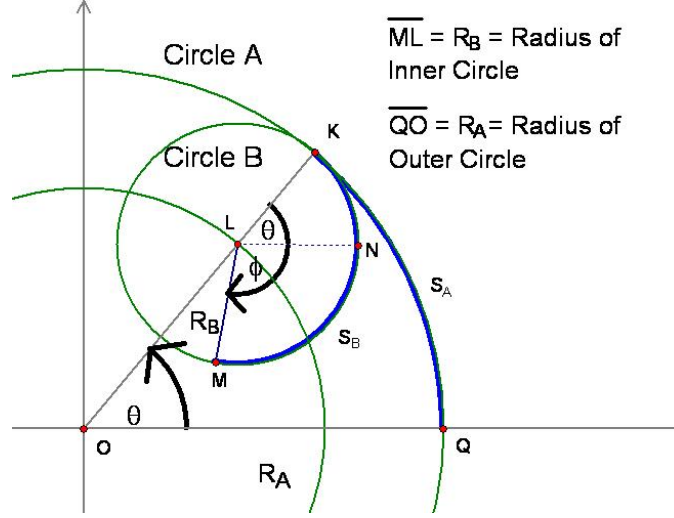
Figure 2: Hypocycloid Geometry

Figure 2 shows the various circles, angles, points, and directions of movement utilized in the parametrization. As noted before, the curves derived from the hypocycloid are drawn by letting the center of the inner circle (point L) rotate an angle of $\theta$ about the origin and also letting the point on the inner circle (point M) rotate a certain angle in the opposite direction to the $\theta$ about the center of the inner circle. Positive angles are angles that increase in the counter-clockwise direction and negative angles are angles that increase in the clockwise direction. The following parametrization assumes that $\theta$ increases in the positive direction, so the center of circle B will rotate in the counter-clockwise direction, while point M will rotate in the clockwise direction. The angle that point M rotates through depends on the radius of the inner circle relative to the outer circle's radius. Regardless of the ratio between the radii, the arc lengths $\overgroup{QK}$ and $\overgroup{KM}$ must be equal. So $S_A = S_B$. The curve is to be parameterized in terms of the angle $\theta$. The parametrization can be separated into two parts: the parametrization of the rotational movement of point L about point O and the parametrization of the rotational movement of point M about point L, depending on the position of point L.

## Parametrization of the Rotational Component of Point L

To parameterize the center of circle B, one can see from Figure 2 that circle B travels about the inner boundary of circle A (an analogy could be made to a wheel turning inside a larger, cylindrical shell), such that the center of circle B, pint L, travels along a circular path. The x and y components can be shown as:
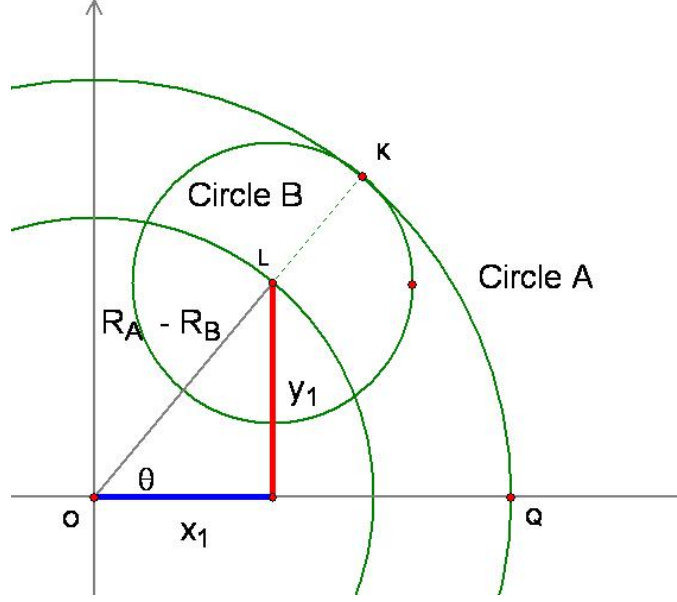
Figure 3: x, y Components of the Movement of Point L

Figure 3 shows the x and y components of the movement of point L about point O. The circular path that point L travels on has a radius of $R_A - R_B$ from the origin, point O. Where $R_A$ is the radius of circle A and $R_B$ is the radius of circle B. By utilizing the relation between cartesian and parametric coordinates

$$x_1 = r \cos \theta$$
$$y_1 = r \sin \theta$$

where $r = R_A - R_B$ and $\theta$ is the angle $\angle KOQ$

the parametrization of the point L becomes:

$$x_1 = (R_A - R_B) \cos \theta$$
$$y_1 = (R_A - R_B) \sin \theta$$

The equations $x_1 = (R_A - R_B) \cos \theta$ and $y_1 = (R_A - R_B) \sin \theta$ explain the first part of the parametrization: the rotational motion of circle B as the angle $\theta$ changes.

4

# Parametrization of the Rotational Component of Point M

Now the second part of the parametrization, the rotational component of the point M on the circle B about point L, must be factored into the equation. The x and y components can be shown as:
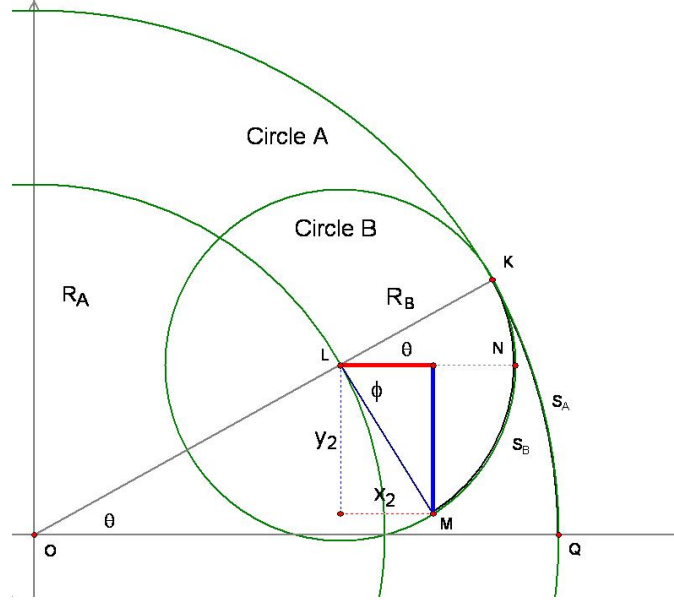


Figure 4: x, y Components of Point M's Motion About Point L

Figure 4 shows the x and y components of the movement of point M about point L. As the angle $\theta$ increases in the counter-clockwise direction, point M travels in the clockwise direction along the perimeter of circle B. The arc length traveled by point K is $S_A$ and the arc length $S_A$ is equal to $S_B$. Arc length is defined as $S = r\theta$.

$$S_A = S_B$$
$$R_A\theta = R_B(\theta + \phi)$$

Since $\phi$ is increasing in the opposite direction compared to $\theta$, $\phi$ will be negative.

$$R_A\theta = R_B(\theta + (-\phi))$$
$$R_A\theta = R_B\theta - R_B\phi$$
$$R_A\theta - R_B\theta = -R_B\phi$$
$$(R_B - R_A)\theta = R_B\phi$$
$$\frac{(R_B - R_A)\theta}{R_B} = \phi$$

Again using the relationship between cartesian and parametric coordinates

$$x_2 = r\cos\phi$$
$$y_2 = r\sin\phi$$
$$\text{where } r = R_B \text{ and } \phi \text{ is } \angle\,\text{MLN}$$

the parametric equation for point M rotating about point L (neglecting the rotational motion of point L) is:

$$x_2 = R_B \cos\left(\frac{(R_B - R_A)\theta}{R_B}\right)$$
$$y_2 = R_B \sin\left(\frac{(R_B - R_A)\theta}{R_B}\right)$$

## The Full Parametrization

The full parametrization for the point M can be obtained by combining the parametrization of the rotational movement of point L about point O and the rotational movement of point M about point L. Figure 5 shows the x-components and y-components for the rotational
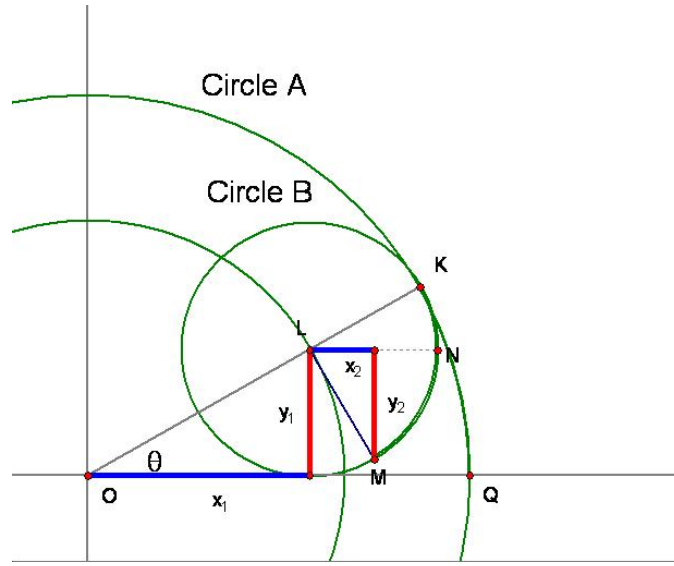


Figure 5: Complete (x,y) Components of Point M

movement of point L about point O and the rotational movement of point M about point L. The x-components add to each other. The x-component for the rotational motion of point L is initially positive and the rotational motion of point M, is initially negative due to the ratio $(\frac{(R_B - R_A)\theta}{R_B})$ being a negative angle. Using our knowledge of even and odd functions, we can simplify this ratio to a positive fraction:

$$\cos(-\phi) = \cos(\phi)$$
$$\sin(-\phi) = -\sin(\phi)$$

This convention allows a negative one to be factored out of each term. Factoring out a negative from the $\phi$ term gives:

$$\phi = \left(\frac{R_B - R_A}{R_B}\theta\right)$$
$$-\phi = \left(\frac{R_A - R_B}{R_B}\theta\right)$$

6

Putting this together with with the previous parameterizations gives:

$$x = (R_A - R_B)\cos\theta + R_B\cos\left(\frac{R_A - R_B}{R_B}\theta\right)$$

$$y = (R_A - R_B)\sin\theta - R_B\sin\left(\frac{R_A - R_B}{R_B}\theta\right)$$

The negative $R_B\sin\left(\frac{R_A-R_B}{R_B}\theta\right)$ accounts for the negative component of point M. This equation fully explains all the hypocycloids if the ratio between the circle radii are known.

## Specific Parametrization: The Deltoid

Given that the general equation of a hypocycloid curve is

$$x = (R_A - R_B)\cos\theta + R_B\cos\left(\frac{R_A - R_B}{R_B}\theta\right)$$

$$y = (R_A - R_B)\sin\theta - R_B\sin\left(\frac{R_A - R_B}{R_B}\theta\right)$$

the parametrization of the deltoid can be easily calculated. The special condition that makes the deltoid a unique curve is that the radius of the inner, rolling circle is $\frac{1}{3}$ the radius of the larger curve. This is the only difference that separates the deltoid from other hypocycloids, like the asteroid. In our diagram, Figure 1, the radius $R_B$ would be $\frac{1}{3}R_A$. The parametric
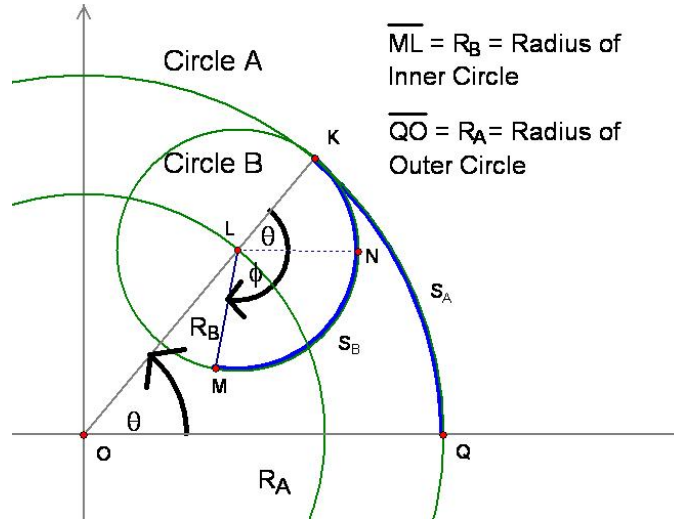


Figure 6: Figure 1 Reprised

equation is obtained simply by replacing the generic $R_B$ radius with the special condition

radius of $\frac{1}{3}R_A$.

$$x = \left(R_A - \frac{1}{3}R_A\right)\cos\theta + \frac{1}{3}R_A\cos\left(\frac{R_A - \frac{1}{3}R_A}{\frac{1}{3}R_A}\theta\right)$$

$$y = \left(R_A - \frac{1}{3}R_A\right)\sin\theta - \frac{1}{3}R_A\sin\left(\frac{R_A - \frac{1}{3}R_A}{\frac{1}{3}R_A}\theta\right)$$

Which simplifies to:

$$x = \frac{2}{3}R_A\cos\theta + \frac{1}{3}R_A\cos 2\theta$$

$$y = \frac{2}{3}R_A\sin\theta - \frac{1}{3}R_A\sin 2\theta$$

The deltoid will be drawn in full if $\theta$ rotates through $0 \le \theta \le 2\pi$. Since all of the hypocycloids depend on the ratio between the inner circle radius and the outer circle radius, any of the specific hypocycloids parameterizations can be found with the general equation of hypocycloid if the ratio is known. The deltoid's ratio is $\frac{1}{3}$, and the asteroid's is $\frac{1}{4}$.

# References

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