The Restricted Three-Body Problem

Shinichi Tokoro

May, 22 2002

Abstract

The method to find gravitational equilibrium point and stability of them will be demonstrated.

1. Introduction

We will discuss about the gravitational interaction among three bodies in the case of that third body has negligible mass compared with the other two bodies. Many scientist has been analyzing three body problems and they are Euler, Lagrange, Laplace, Jacobi, Le Verrier, Hamiltion, Poincare, and Birkhoff. The general three body problems are still not discovered thoroughly in stead of contribution of those scientists. If the two bodies move around in circular, coplanar orbits about their common center of mass and the third mass is negligible so it doesn't affect the motion of the other two bodies, and the problem of the third body is called *the circular, restricted, three-body problems*. Although, real planetary orbital and movement are not coplanar and circular, the restricted three-body problem give us reasonable approximation for certain systems; also, the qualitative behaviour of the motion can be analyzed without too complicating method. Mainly, we will discus the equation of the three-body problem, the location and stability of equilibrium points in the case of circular restriction.

2. Equation of Motion

We consider a tiny particle move around gravitational field of two mass m_1 and m_2, and also two masses are creating circular orbit with their common center of mass. We use a set of axes a, b, c as a frame referred to the center of the system. Let the a axis lie along the line between m_1 and m_2 at time $t = 0$ and the c axis perpendicular to the $a - b$ plane; also, let the coordinates of the two masses in this frame be (a_1, b_1, c_1) and (a_2, b_2, c_2). Also, the angular velocity and a constant separation are fixed about their common center of mass. Let the unit of mass be chosen such that $u = G(m_1 + m_2) = 1$, and we assume that $m_1 > m_2$ and define

$$\bar{u} = \frac{m_2}{m_1 + m_2} \tag{1}$$

then, in this system of units, the two masses are

$$\begin{aligned} \bar{u} &= Gm_1 \\ Gm_1 &= \frac{1}{m_1 + m_2} m_1 \\ &= \frac{m_1 + m_2}{m_1 + m_2} - \frac{m_2}{m_1 + m_2} \\ &= 1 - \frac{m_2}{m_1 + m_2} \end{aligned}$$

$$u_1 = Gm_1 = 1 - \bar{u} \tag{2}$$

$$
\begin{aligned}
u_2 &= Gm_2 \\
&= \frac{1}{m_1 + m_2} m_2
\end{aligned}
$$

$$
u_2 = Gm_2 = \bar{u} \tag{3}
$$

where $\hat{u} < 1/2$. Let the coordinates of the particle in the *sidereal system*, be (a, b, c) and the equation of motion of the particle are

$$
\begin{aligned}
m\ddot{r} &= \Sigma F \\
&= \frac{Gmm_1}{r_1^2}\vec{u} + \frac{Gmm_2}{r_2^2}\vec{u} \\
\ddot{r} &= \frac{u_1}{r_1^3}\langle a - 1 - a, b_1 - b, c_1 - c\rangle + \frac{u_2}{r_2^3}\langle a_2 - a, b_2 - b, c_2 - c\rangle \\
< \ddot{a}, \ddot{b}, \ddot{c} > &= \frac{u_1}{r_1^3}\langle a_1 - a, b_1 - b, c_1 - c\rangle + \frac{u_2}{r_2^3}\langle a_2 - a, b_2 - b, c_2 - c\rangle
\end{aligned}
$$

$$
\ddot{a} = u_1 \frac{a_1 - a}{r_1^3} + u_2 \frac{a_2 - a}{r_2^3}, \tag{4}
$$

$$
\ddot{b} = u_1 \frac{b_1 - b}{r_1^3} + u_2 \frac{b_2 - b}{r_2^3}, \tag{5}
$$

$$
\ddot{c} = u_1 \frac{c_1 - c}{r_1^3} + u_2 \frac{c_2 - c}{r_2^3}, \tag{6}
$$

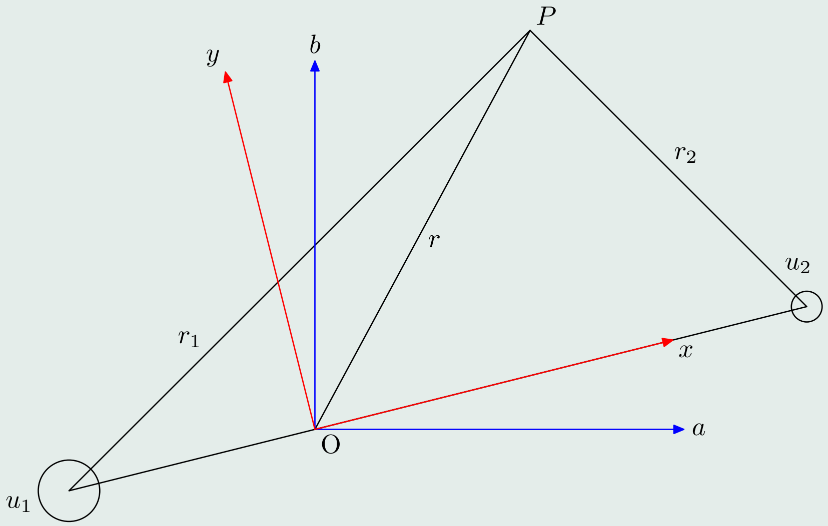

Figure 1: A planar view of the relationship between the sidereal coordinates (a, b, c) of the particle at the point P. O is the center of mass of two bodies

where, from Figure 1,

$$r_1^2 = (a_1 - a)^2 + (b_1 - b)^2 + (c_1 - c)^2, \tag{7}$$

$$r_2^2 = (a_2 - a)^2 + (b_2 - b)^2 + (c_2 - c)^2. \tag{8}$$

In the general three-body problem, these equations still work because they don't require any assumptions about the shape of two mass's orbital. We can rewrite these equations

$$r_1^2 = (x + u_2)^2 + y^2 + z^2, \tag{9}$$

$$r_2^2 = (x - u_2)^2 + y^2 + z^2, \tag{10}$$

where (x, y, z) are the coordinates of the particle with respect to the rotating, or *synodic system*. Their coordinates are related to the coordinates in the sidereal system by the simple rotation, so we write these in matrix form

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{11}$$

to obtain the equation of accelerations. We need to take the second derivertives of these equations by using the chain-rule.

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + \begin{pmatrix} -n\sin nt & -n\cos nt & 0 \\ n\cos nt & -n\sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \dot{x}\cos nt - \dot{y}\sin nt - xn\sin nt - yn\cos nt \\ \dot{x}\sin nt + \dot{y}\cos nt + xn\cos nt - ny\sin nt \\ \dot{z} \end{pmatrix}$$

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix}. \tag{12}$$

Similarly,

$$\begin{pmatrix} \ddot{a} \\ \ddot{b} \\ \ddot{c} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2n\dot{y} - n^2x \\ \ddot{y} + 2n\dot{x} - n^2y \\ \ddot{z} \end{pmatrix}. \tag{13}$$

In the above matrix equation $n\dot{x}$ and $n\dot{y}$ are called *Corioli's acceleration* and n^2x and n^2y are called the *centrifugal acceleration*. Substitute for $a, b, c, \ddot{a}, \ddot{b}$, and $\ddot{c}$, in Eqs. (4),(5), and (6) become

$$(\ddot{x}-2n\dot{y}-n^2x)\cos nt-(\ddot{y}+2n\dot{x}-n^2y)\sin nt = \left[u_1\frac{x_1-x}{r_1^3+u_2}\frac{x_2-x}{r_2^3}\right]\cos nt+\left[\frac{u_1}{r_1^3}+u_2r_2^3\right]y\sin nt, \tag{14}$$

$$(\ddot{x}-2n\dot{y}-n^2x)\sin nt+(\ddot{y}+2n\dot{x}-n^2y)\cos nt = \left[u_1\frac{x_1-x}{r_1^3+u_2}\frac{x_2-x}{r_2^3}\right]\sin nt-\left[\frac{u_1}{r_1^3}+u_2r_2^3\right]y\cos nt, \tag{15}$$

$$\ddot{z} = -\left[\frac{u_1}{r_1^3}+\frac{u_2}{r_2^3}\right]z. \tag{16}$$

After the calculation, we can write these acceleration as the gradient of a scalar function U:

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}, \tag{17}$$

$$\ddot{y} - 2n\dot{x} = \frac{\partial U}{\partial y}, \tag{18}$$

$$\ddot{z} = \frac{\partial U}{\partial z}, \tag{19}$$

where $U = U(x, y, z)$ is given by

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{u_1}{r_1} + \frac{u_2}{r_2}. \tag{20}$$

3. The Jacobi Integral

If we multiply Eq. (17) by $\dot{x}$, and Eq. (18) by $\dot{y}$, and Eq. (19) by $\dot{z}$ and adding them together, we get

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} = \frac{dU}{dt}, \tag{21}$$

and we integrate this equation, and we get

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - C_J, \tag{22}$$

where C_J is the constant of integration. Since we know that $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$, the square of the velocity in the system is

$$v^2 = 2U - C_J \tag{23}$$

and from that we can write C_J by using equation (20),

$$C_J = n^2(x^2 + y^2) + 2\left(\frac{u_1}{r_1} + \frac{u_2}{r_2}\right) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{24}$$

This means that the quantity $2U - v^2 = C_J$ is a constant of the motion and it is called the *Jacobi constant*. Jacobi constant is only valid in the circular restricted three-body problem. The C_J can be written in the nonrotating, sidereal frame. The equation (11) gives us the position vector.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{25}$$

Also, we can get the velocity vector from (12)

$$\begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} \tag{26}$$

However,

$$\begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + n \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{27}$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & cosnt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} - n \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{28}$$

Then, we let

$$A = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{29}$$

From Eq.(28),

$$
\begin{aligned}
\dot{x}^2+\dot{y}^2+\dot{z}^2 &= \begin{pmatrix}\dot{x} & \dot{y} & \dot{z}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\\
&= \begin{pmatrix}\dot{a} & \dot{b} & \dot{c}\end{pmatrix} A^T A \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} - n\begin{pmatrix}\dot{a} & \dot{b} & \dot{c}\end{pmatrix} A^T B \begin{pmatrix}a\\ b\\ c\end{pmatrix}\\
&\quad - n\begin{pmatrix}a & b & c\end{pmatrix} B^T A \begin{pmatrix}\dot{a}\\ \dot{b}\\ \dot{c}\end{pmatrix} + n^2\begin{pmatrix}a & b & c\end{pmatrix} B^T B \begin{pmatrix}a\\ b\\ c\end{pmatrix}\\
&= \dot{a}+\dot{b}+\dot{c}+n^2(a^2+b^2)+2n(\dot{a}b-\dot{b}a). \qquad (30)
\end{aligned}
$$

Because A and B are orthogonal matrices, A^T and B^T are their inverses. Distances are fixed by rotations because the determinants of orthogonal matrices are equal to 1; thus, we have $x^2+y^2+z^2=a^2+b^2+c^2$. From above calculations we can rewrite the Jacobi constant in the form of sidereal coordinates

$$C_J = 2\left(\frac{u_1}{r_1}+\frac{u_2}{r_2}\right) + 2n(a\dot{b}-b\dot{a}) - \dot{a}^2 - \dot{b}^2 - \dot{c}^2. \qquad (31)$$

Finding the velocity and position of particle gives us the value of Jacobi constant associated with the motion of the particle. The Jacobi constant is valid only in the restricted three-body problem, but still it helps us to determine the regions which particle has zero velocity. Since $v^2=0$, we have

$$C_J = n^2(x^2+y^2) + 2\left(\frac{u_1}{r_1}+\frac{u_2}{r_2}\right). \qquad (32)$$

Above equation gives us the value of C_J on the *zero-velocity surface*. To make a matter simple, we consider that all events are happening on the $x - y$ plane, and Eq.(32) determine the excluded area. In other words, Jacobi constant determine the area that a particle has zero-velocity. The area enclosed by the green line, in Figure 2, tells us that the particle cannot move in that region.

4. Lagrangian Equilibrium Points

Under the gravitational force by m_1 and m_2, the system has an equilibrium points. Let a, b and c indicate the location of two masses and center of system O respect to the position of the particle. Let F_1 and F_2 indicate the forces that is working on the particle toward two masses. Also, set particle's position is fixed in the system (the distance b from O never changes)

$$F = F_1 + F_2 \tag{33}$$

which has direction of b. The position of O is given by

$$m_1(a - b) = m_2(b - c). \tag{34}$$

Then, the vector product of $F_1 + F_2$ and Eq.(34) gives

$$\begin{aligned}
(F_1 + F_2) \times [m_1(a - b) - m_2(b - c)] &= 0 \\
m_1(F_1 + F_2) \times (a - b) - m_2(F_1 + F_2) \times (b - c) &= 0 \\
m_1\vec{F_2} \times \vec{a} &= -m_2\vec{F_1} \times \vec{c}
\end{aligned}$$

Since the angle F_1 and c is minus the angle betweenF_2 and a, we get

$$m_1 F_2 a \sin(\theta) = -m_2 F_1 c \sin(-\theta)$$

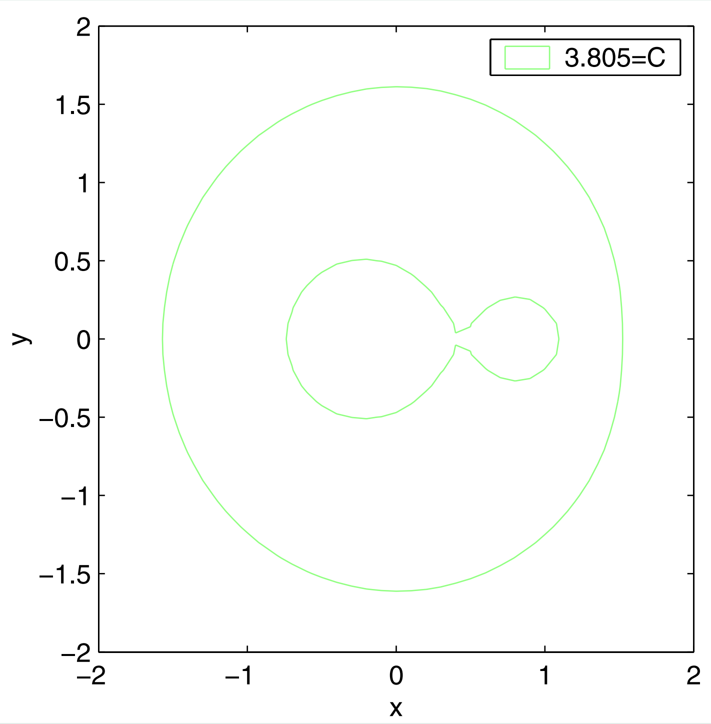

Figure 2: The area enclosed by green line is the excluded region

$$m_1 F_2 a = m_2 F_1 c. \tag{35}$$

Notice, the gravitational forces are$F_1 = Gm_1/a^2$ and $F_2 = Gm_2/c^2$, so we obtain $a = c$ from Eq.(35). Therefore, the triangle $m_1 P m_2$ must be an isosceles triangle, so that the locus of all points P will make the perpendicular bisector to the base line of the triangle Pm_1m_2. From, Figure 3. To balance the centrifugal acceleration of P we must have

$$n^2 b = F_1 \cos\beta + F_2 \cos\gamma. \tag{36}$$

Thus,

$$n^2 = \frac{G}{a^2} b^2 (m_1 b \cos\beta + m_2 b \cos\gamma). \tag{37}$$

Furthermore, if we look at the triangle, we have

$$b\cos\beta = a - g\cos\alpha, \qquad b\cos\gamma = a - (d-g)\cos\alpha, \tag{38}$$

where d is the distance between m_1 and m_2 and g is the distance between m_1 and O, and

$$\cos\alpha = \frac{d}{2a}. \tag{39}$$

From the definition of the center of mass, we have

$$g = \frac{m_2}{m_1 + m_2} d, \tag{40}$$

$$d - g = \frac{m_1}{m_1 + m_2} d, \tag{41}$$

and if we use these above conditions Eq.(37) becomes

$$n^2 = \frac{G(m_1 + m_2)}{a^3 b^2} \left(a^2 - \frac{m_1 m_2}{(m_1 + m_2)} d^2 \right). \tag{42}$$

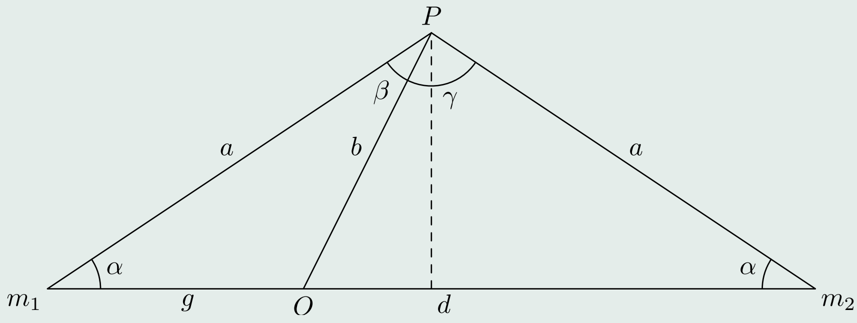

Figure 3: P denotes the location of an equilibrium point. The dashed line denotes perpendicular bisector of the line between two masses.

Also, from the cosine rule, we have

$$b^2 = a^2 + g^2 - 2ag\cos\alpha = a^2 + g^2 - gd. \tag{43}$$

Then if we put g into Eq.(43), we get

$$b^2 = a^2 - \frac{m_1 m_2}{(m_1 + m2)^2} d^2. \tag{44}$$

Finally, Eq.(42) becomes

$$n^2 = G(m_1 + m_2)/a^3 \tag{45}$$

However, the reference frame is rotating with angular velocity n, so

$$n^2 = G(m_1 + m_2)/d^3, \tag{46}$$

Hence we must have $a = d$. If we chose the direction of vector force to be perpendicular to the line between m_1 and m_2 then the result follows that. Thus, in the gravitational force created by m_1 and m_2, the system has an equilibrium point at the apex of an equilateral triangle with a base line go through two masses. Moreover, since the system is circular we could assume that there is another equilibrium point at exactly opposite side of the line. These are known as the *Lagranian equilibrium point*, and we label these L_4 and L_5.

5. Location of Equilibrium Points

Now we need find points where the particle has zero velocity and zero acceleration in the system, in order to find solutions. We need find where the particle has zero velocity and zero acceleration. In this case we assume all events are happening on the $x - y$ plane. Also, we should state the unit of distance is the constant separation of the two masses. Then we can state that $n = 0$. To calculate the locations of the equilibrium points we use the method introduced by Brouwer and Clemence (1961), and we rewrite U in a different form. From the equations of r_1 and r_2 in the Eqs. (7) and (8) and using the fact $u_1 + u_2 = 1$, we get

$$U = u_1\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) + u_2\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right) - \frac{1}{2}u_1u_2. \tag{47}$$

Then, we need consider the equations (17) and (18), in the case of $\ddot{x} = \ddot{y} = \dot{x} = \dot{y} = 0$. To find the locations of the equilibrium points we must solve simultaneous nonlinear equations

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x} = 0, \tag{48}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y} = 0, \tag{49}$$

by using the form $U = U(r_1, r_2)$. If we take the partial derivatives of these equations, we get

$$u_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{x+u_2}{r_1}+u_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{x-u_1}{r_2}=0 \tag{50}$$

$$u_1\left(-\frac{1}{r_1^2}+r_1\right)\frac{y}{r_1}+u_2\left(-\frac{1}{r_2^2}+r_2\right)\frac{y}{r_2}=0 \tag{51}$$

and if we look at equations (48) and (49) we see

$$\frac{\partial U}{\partial r_1}=u_1\left(-\frac{1}{r_1^2}+r_1\right)=0, \tag{52}$$

$$\frac{\partial U}{\partial r_2}=u_2\left(-\frac{1}{r_2^2}+r_2\right)=0, \tag{53}$$

so we can get $r_1 = r_2 = 1$ in our system. Eqs. (7) and (8) implies

$$(x+u_2)^2+y^2=1, \qquad (x-u_1)^2+y^2=1 \tag{54}$$

with the two solutions

$$x=\frac{1}{2}-u_2, \qquad y=\pm\frac{\sqrt{3}}{2}. \tag{55}$$

According to above calculation we get $r_1 = r_2 = 1$. If we look at Eq. (51) easily we can see that $y = 0$ is one of solutions in Eq. (49), and that means other equilibrium points are on the x- axis ,and that result satisfies Eq. (48). Actually, there are three more equilibrium points corresponding to the *collinear Lagrangian equilibrium point* which lie on the x-axis and are known as L_1, L_2, and L_3. L_1 locates at between u_1 and u_2. L_2 locates out side of u_2. L_3 locates outside of u_1. Now, we need analyze these points individually. The conditions at L_1 are

$$r_1+r_2=1, \qquad r_1=x+u_2, \qquad r_2=-x+u_1, \qquad \frac{\partial r_1}{\partial x}=-\frac{\partial r_2}{\partial x}=1 \tag{56}$$

and we substitute these into Eq. (50). Then we get

$$\frac{u_2}{u_1} = 3r_2^3 \frac{(1 - r_2 + r_2^2/3)}{(1 + r_2 + r_2^2)(1 - r_2)^3} \tag{57}$$

If we define

$$\alpha = \left(\frac{u_2}{3u_1}\right)^{1/3} \tag{58}$$

From Eq.(57) and α we get

$$\alpha = r_2 + \frac{1}{3}r_2^2 + \frac{1}{3}r_2^3 + \frac{53}{81}r_2^4 + O(r_2^5),$$

$$r_2 = \alpha - \frac{1}{3}r_2^2 - \frac{1}{2}r_2^3 - \frac{53}{81}r_2^4 - O(r_2^5),$$

$$r_2 = \alpha - \frac{1}{3}\left[r_2^2 + r_2^3 + \frac{53}{27}r_2^4 + O(r_2^5)\right], \tag{59}$$

$$r_2 = \alpha - \frac{1}{3}\Phi(r_2), \tag{60}$$

By using Taylor's series

$$r_2 = \alpha + \sum_{j=1}^{\infty} \frac{e^j}{j!} \frac{d^{j-1}}{d\alpha^{j-1}}[\Phi(\alpha)]^j \tag{61}$$

$$r_2 = \alpha + \frac{e}{1!}\frac{d^0}{d\alpha^0}[\Phi(\alpha)] + \frac{e^2}{2!}\frac{d}{d\alpha}[\Phi(\alpha)]^2 + \frac{e^3}{3!}\frac{d^2}{d\alpha^2}[\Phi(\alpha)]^3 + \frac{e^4}{4!}\frac{d^3}{d\alpha^3}[\Phi(\alpha)]^4 + \ldots \tag{62}$$

where, $e = -1/3$ and

$$\Phi(\alpha) = \alpha^2 + \alpha 3 + \frac{53}{27}\alpha^4 + O(\alpha^5). \tag{63}$$

Therefore, we end up with the equation

$$r_2 = \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + O(\alpha^5) \tag{64}$$

A Matlab code to compute above calculation is below.

```
close all

syms r a phi_a=a^2+a^3+53/27*a^4 e=sym(-1/3) r=a+...
    e*phi_a+...
    e^2*diff(phi_a^2,a)/2+...
    e^3*diff(diff(phi_a^3,a),a)/6+...
    e^4*diff(diff(diff(phi_a^4,a),a),a)/24
r=simple(r)
L1(r)
```

Similarly, for the point L_2, we have condition

$$r_1 - r_2 = 1, \qquad r_1 = x + u_2, \qquad r_2 = x - u_1, \qquad \frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = 1. \tag{65}$$

and we make substitution into Eq.(50), and we get

$$\frac{u_2}{u_1} = 3r_2^3 \frac{(1 + r_2 + r_2^2/3)}{(1 + r^2)^2(1 - r_2^3)}. \tag{66}$$

and by using Toylor's method, we get

$$\alpha = r_2 - \frac{1}{3}r_2^2 + \frac{1}{3}r_3^3 + \frac{1}{81}r_2^4 + O(r^5), \tag{67}$$

$$r_2 = \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + O(\alpha^5). \tag{68}$$

For the point L_3 we have

$$r_2 - r_1 = 1, \qquad r_1 = -x - u_2, \qquad r_2 = -x + u_1, \qquad \frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x} = -1. \tag{69}$$

From Eq.(50) we can get

$$\frac{u_2}{u_1} = \frac{(1 - r_1^3)(1 + r_1)}{r_1^3(r_1^2 + 3r_1 + 3).} \tag{70}$$

Now,we set $r_1 = 1 + \beta$ and $r_2 = 2 + \beta$, and by the Tylor's expansion, we obtain

$$\frac{u_2}{u_1} = -\frac{12}{7}\beta + \frac{144}{49}\beta^2 - \frac{1567}{343}\beta^3 + O(\beta^4), \tag{71}$$

$$\beta = -\frac{7}{12}\left(\frac{u_2}{u_1}\right) + \frac{7}{12}\left(\frac{u_2}{u_1}\right)^2 - \frac{13223}{20736}\left(\frac{u_2}{u_1}\right)^3 + O\left(\frac{u_2}{u_1}\right)^4. \tag{72}$$

From above calculations we determine r_1 and r_2 by choosing the value for u_2. Then, we calculate these values in series expansion of equation of Jacobi constant including terms up to $O(u_2)$, and that gives us $C_{L_1} = 3.805$, $C_{L_2} = 3.552$, $C_{L_3} = 3.197$, $C_{L_4} = 2.84$, and $C_{L_5} = 2.84$ where, $u_2 = 0.2$

$$C_{L_1} \approx 3 + 3^{4/3}u_2^{2/3} - 10u_2/3, \tag{73}$$

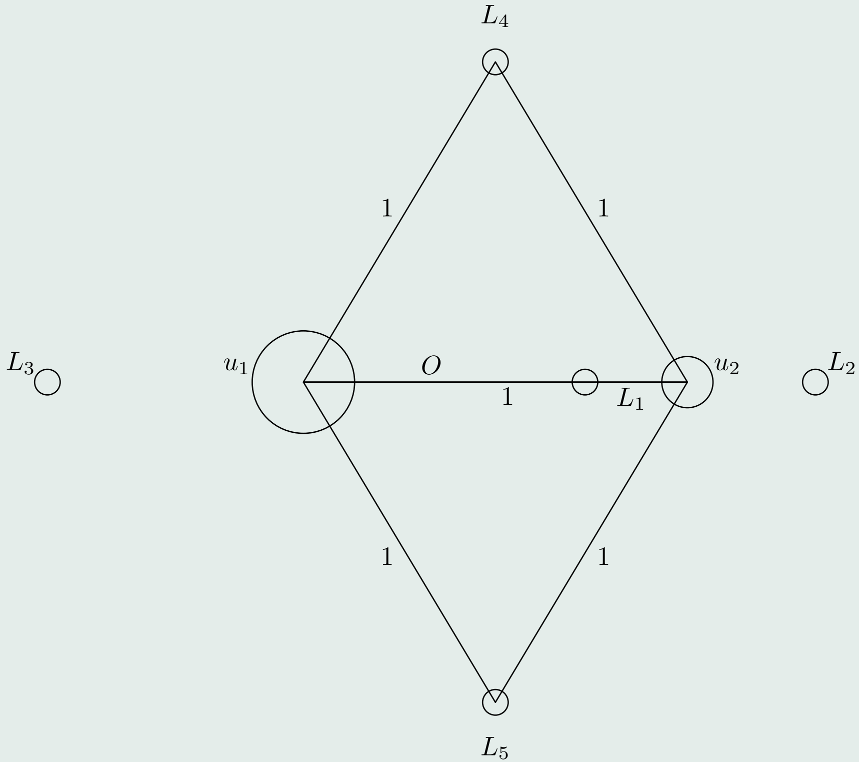

Figure 4: The location of the Lagrangian equilibrium points.

$$C_{L_2} \approx 3 + 3^{4/3} u_2^{2/3} - 14u_2/3, \tag{74}$$

$$C_{L_3} \approx 3 + u_2, \tag{75}$$

$$C_{L_4} \approx 3 - u_2, \tag{76}$$

$$C_{L_5} \approx 3 - u_2. \tag{77}$$

Jacobi constant at L_4 and L_5 are the same because Eqs. (7),(8), and (32) are idenical by substituting y and $-y$. From Eqs. (73) and (74) we get as $u_2 \to 0$, $C_{L_1} \to C_{L_2}$. Also, from Eqs. (65) and (68) as $u_2 \to 0$ the term of $O(\alpha^2)$ becomes neglegible as well as the higher term, so the distance from L_1 to m_2 and from L_2 to m_2 will be the same. The point L_3 lies at the distance $1+\beta$ since $\beta < 0$ as $u_2 \to 0$ the point L_3 gets close to the unit circle. On the unit circle, there lie triangular points that are also centered on the mass u_1 and are at a distance $r = (1 - u_2 + u_2^2)^{1/2}$ from the center of mass. As $u_2 \to 0$, the unit circle centered on u_1 approaches the unit circle that is centered on the center of mass. Ultimately, as $u_2 \to 0$, all equilibrium point approach to the unit circle, with the order of L_3, L_4 and L_5, and then L_1 and L_2.

6. Stability of Equilibrium Points

Now we move on to analyze how stable these points are. In a system of that a force derived from a potential, as we know in general, the sum of the kinetic energy and the potential energy remains constant. Then we are able to find "stable" equilibrium point in a system. As a particle move from away from the minimum, its potential energy must increase; at the same time, its kinetic energy must decrease until it reaches zero.

However, the sum of the two froms of energy has to be a constant, so at this point the motion has to reverse, leading to an increase in kinetic energy and a decrease in potential energy. In the case of L_4 the directions of the initial velocity vectors point away from the point L_4 and towards those regions associated with higher values of C_J, so we can assume that the point is unstable equilibrium point, but since we are not sure about that we need analyze the behaviour of the particle around an equilibrium point. We linearize the equations of motion and do a *linear stability analysis.* Let the location of an equilibrium point in the circular restricted problem be label as (x_0, y_0) and let consider a small displacement (X, Y), then we have $x = x_0 + X$ and $y = y_0 + Y$. Then, by substituting in Eqs.(17) and (18), and expanding in Taylor series, we have

$$\begin{aligned} \ddot{X} - 2n\dot{Y} &\approx \left(\frac{\partial U}{\partial x}\right)_0 + X\left(\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\right)_0 + Y\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)\right)_0 \\ &= X\left(\frac{\partial^2 U}{\partial x^2}\right)_0 + Y\left(\frac{\partial^2 U}{\partial x \partial y}\right)_0 \end{aligned} \tag{78}$$

and

$$\begin{aligned} \ddot{Y} - 2n\dot{X} &\approx \left(\frac{\partial U}{\partial y}\right)_0 + X\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)\right)_0 + Y\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)\right)_0 \\ &= X\left(\frac{\partial^2 U}{\partial x \partial y}\right)_0 + Y\left(\frac{\partial^2 U}{\partial y^2}\right)_0 \end{aligned} \tag{79}$$

We shall neglect the higher order terms because they will be small quantities; so we consider a small displacement from the point (x_0, y_0). We restrict ourselves to study the motion in close relation to the equilibrium point. The result is a set of linear differential equations of the form

$$\ddot{X} - 2\dot{Y} = XU_{xx} + YU_{xy}, \qquad \ddot{Y} + 2\dot{X} = XU_{xy} + YU_{yy} \tag{80}$$

where we have taken $n = 1$ and the quantities

$$U_{xx} = \left(\frac{\partial^2}{\partial x^2}\right)_0, \qquad U_{xy} = \left(\frac{\partial^2 U}{\partial x \partial y}\right)_0, \qquad U_{yy} = \left(\frac{\partial^2}{\partial y^2}\right)_0 \tag{81}$$

are constants, and we can rewrite these equations in matrix form

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \ddot{X} \\ \ddot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix}, \tag{82}$$

the point is switching from the solution of, simultaneous, second-order differential equation to the solution of four first-order differential equations, and we obtain

$$\dot{X} = AX, \tag{83}$$

To start with, we need fined the eigenvalues of A. and we have

$$(A - \lambda I)x = 0, \tag{84}$$

where I is the $n \times n$ matrix. Also, the determinant of the matrix $A - \lambda I$ is zero, so we have

$$det(A - \lambda I) = 0. \tag{85}$$

Therefore, we get *characteristic equation*, with degree n in λ with n possible complex roots. After getting λ we put them into $Ax = \lambda x$ and solve for the components of each x. The equation is coupled, meaning the time derivative of a variable depends on the product of a vector and a matrix. To solve this, we can use a technique that is known as a similarity transformation. Let the transformation form X to Y be represented by

$$Y = BX, \tag{86}$$

where the constant matrix B has to be determined.

$$X = B^{-1}Y, \qquad \dot{X} = B^{-1}\dot{Y}, \tag{87}$$

where B^{-1} is the inverse of the matrix B, so we can write Eq. (84) as

$$B^{-1}\dot{Y} = AB^{-1}Y, \tag{88}$$

or if we multiply both side by B, we get

$$\dot{Y} = BB^{-1}\dot{Y} = BAB^{-1}Y. \tag{89}$$

We want the new uncoupled differential equations given in Eq.(90). We can obtain that only if we have the matrix B such as

$$BAB^{-1} = \Lambda, \tag{90}$$

where Λ is a diagonal matrix. if the columns of the matrix B are constructed from the n eigenvectors of the matrix A, then the matrix Λ is a diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n. \end{pmatrix} \tag{91}$$

Because of Λ we have the transformed system of equation

$$\dot{Y} = \Lambda Y \tag{92}$$

or, in component form we can write it as

$$\dot{Y}_i = \lambda_i Y_i (i = 1, 2, ..., n). \tag{93}$$

The solutions of Eq.(94) are easily found as

$$Y_i = c_i e^{\lambda_i t} (i = 1, 2, ..., n), \tag{94}$$

where the c_i are n constants of integration. We must now transform back the solution of the equations in y_i in the form of the original variables, x_i.

$$X = B^{-1}Y = B^{-1} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix} \tag{95}$$

C_i(n constants of integration) will be found by solving them simultaneous linear equations given in Eq.(96). Now we use this method to solve the problem in the case of $n = 4$

$$det(A - \lambda I) = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ U_{xx} & U_{xy} & -\lambda & 2 \\ U_{xy} & U_{yy} & -2 & -\lambda \end{pmatrix} = 0, \tag{96}$$

which reduces to the polynomial equation

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0. \tag{97}$$

This is a quadratic equation in λ^2. This makes relatively easy to find the four roots, and they are

$$\lambda_{1,2} = \pm \left[\frac{1}{2}(U_{xx} + U_{yy} - 4) - \frac{1}{2} \left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2) \right]^{1/2} \right]^{1/2} \tag{98}$$

and

$$\lambda_{1,2} = \pm \left[\frac{1}{2}(U_{xx} + U_{yy} - 4) + \frac{1}{2} \left[(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2) \right]^{1/2} \right]^{1/2}. \tag{99}$$

It is clear from Eq.(96) that we can write the solution for X and $\dot{X}$ as

$$X = \sum_{j=1}^{4} \bar{\alpha}_j e^{\lambda_j t}, \qquad \dot{X} = \sum_{j=1}^{4} \bar{\alpha}_j \lambda_j e^{\lambda_j t}, \tag{100}$$

where the $\bar{\alpha}$ are the constants. Similarly we can derive Y and $\dot{Y}$ with using the constants $\bar{\beta}$

$$Y = \sum_{j=1}^{4} \bar{\beta}_j e^{\lambda_j t}, \qquad \dot{Y} = \sum_{j=1}^{4} \bar{\beta}_j \lambda_j e^{\lambda_j t}, \tag{101}$$

There are only four constants in the solution, and from the Eq. (81) we find the relationship between $\bar{\alpha_j}$ and $\bar{\beta}_j$. Then we make substitution for X, Y and $\dot{Y}$ into Eq. (81), and we get

$$\sum_{j=1}^{4} \left(\bar{\alpha_j}\lambda_j^2 - 2\bar{\beta}_j\lambda_j - U_{xx}\bar{\alpha_j} - U_{xy}\bar{\beta}_j\right)e^{\lambda_j t} = 0. \tag{102}$$

Then, we also get

$$\bar{\beta}_j = \frac{\lambda_j^2 - U_{xx}}{2\lambda_j + U_{xy}}\bar{\alpha_j}. \tag{103}$$

At time $t = 0$, there are boundary conditions $X = X_0, Y = Y_0, \dot{X} = \dot{X}_0$, and $\dot{Y} = \dot{Y}_0$, then, $\bar{\alpha}$ and $\bar{\beta}$ are determined by using the four simultaneous linear equations

$$\sum_{j=1}^{4} \bar{\alpha_j} = X_0, \qquad \sum_{j=1}^{4} \lambda_j\bar{\alpha_j} = \dot{X}_0, \qquad \sum_{j=1}^{4} \bar{\beta}_j = Y_0, \qquad \sum_{j=1}^{4} \lambda_j\bar{\alpha_j} = \dot{Y}_0. \tag{104}$$

As we saw, we can derive complete solution from Eqs. (101), (102), and(105). However, we can determine the stability of the equilibrium points from examing just eigenvalues. To investigate the nature of the eigenvalues we define the quantites

$$\bar{A} = \frac{u_1}{(r_1^3)_0} + \frac{u_2}{(r_2^3)_0}, \tag{105}$$

$$\bar{B} = 3\left[\frac{u_1}{(r_1^5)_0} + \frac{u_2}{(r_2^5)_0}\right] y_0^2, \tag{106}$$

$$\bar{C} = 3\left[u_1 \frac{(x_0 + u_2)}{(r_1^5)_0} + u_2 \frac{(x_0 - u_1)}{(r_2^5)_0}\right] y_0, \tag{107}$$

$$\bar{D} = 3\left[u_1 \frac{(x_0 + u_2)^2}{(r_1^5)_0} + u_2 \frac{(x_0 - u_1)^2}{(r_2^5)_0}\right]. \tag{108}$$

with these constant we write

$$U_{xx} = 1 - \bar{A} + \bar{D}, \tag{109}$$

$$U_{yy} = 1 - \bar{A} + \bar{B}, \tag{110}$$

$$U_{xy} = \bar{C}. \tag{111}$$

Since all these numbers are real, $X, Y, \dot{X}$, and $\dot{Y}$ must be real in stead of the fact that the constant $\bar{\alpha}_j$ and $\bar{\beta}_j$ and the eigenvalue λ_j can be complex numbers. The general form of the eigenvalues is

$$\lambda_{1,2} = \pm(j_1 + ik_1), \qquad \lambda_{3,4} = \pm(j_2 + ik_2), \tag{112}$$

where j_1, j_2, k_1 and k_2 are real quantities and $i = \sqrt{-1}$. The form of the general solution for the components of the position and velocity vectors around the equilibrium points involves a linear combination, linearly independent, of terms of $e^{\lambda_j t}$. That leads that $e^{+(j+ik)t}$ terms will be matched by an $e^{-(j+ik)t}$ term. When $j = 0$ we will get oscillatory solution because e^{+ikt} and e^{-ikt} become sines and cosines by *Euler's identity* ($e^{\pm i\theta} = \cos\theta \pm i\sin\theta$). However, if j is positive, the exponential function grow in at least one mode; therefore, the perturbed solution is unstable as well as in the case of that j is negative. Therefore the equilibrium point is stable if all the eigenvalues are purely imaginary.

7. Conclusion

In summary, the aim of this project was to study the restricted three-body problem. We then solved the equation of motion which lead us to solve the Jacobi Integral for this particular equation of motion. We then solved for Lagrangian equilibrium points. Once we idenifed the equilibrium points, we then carried out the process of locating them. At the end but not the least, we tested the stability of equilibrium points. Because of time constraints, I was unable to investigate further some more details of the equilibrium points. Further studies go into analyzing collinear points and triangular points. But overall, I have written amount in to understanding the three-body problem which is suitable for detailed.

References

[1] Danby, John. **Fundermentals of Celestial Mechanics.** The Macmillan Company, 1970.

[2] Arnold, David. His matlab and LaTeX expertise.

[3] Carl, D. Murray.**Solar System Dynamics.** Cambridge Universiy Press, 1999.

[4] Strang, Gilbert. **Introduction to Linear Algebra.** Sellesley-Cambridge Press, 1998.