Application 10.2
Transforms of Initial Value Problems

The typical computer algebra system knows Theorem 1 and its Corollary in Section 10.2 of the text, and hence can transform not only functions (as in the Section 10.1 application), but whole initial value problems. Here we illustrate this facility by applying *Maple*, *Mathematica*, and MATLAB to solve the initial value problem

$$x'' + 4x = \sin 3t, \qquad x(0) = x'(0) = 0 \tag{1}$$

of Example 2 in the text. You can try it for yourself with the initial value problems in Problems 1 through 16 there.

Using *Maple*

First we load the Laplace transforms package **inttrans** with the command

```
with(inttrans):
```

and define the differential equation and initial conditions that appear in (1):

```
de :=  diff(x(t),t$2)+4*x(t)=sin(3*t):
inits :=  {x(0)=0, D(x)(0)=0}:
```

The Laplace transform of the differential equation is given by

```
DE := laplace(de, t, s);
```

$$\mathrm{DE} := s\,(s\,\mathrm{laplace}(x(t),t,s) - x(0)) - D(x)(0) + 4\,\mathrm{laplace}(x(t),t,s) = 3\,\frac{1}{s^2+9}$$

The result of this command is a linear (algebraic) equation in the as yet unknown transform $\mathrm{laplace}(x(t),t,s)$. We proceed to solve for this transform $X(s)$ of the unknown function $x(t)$,

```
X(s) := solve(DE, laplace(x(t), t, s));
```

$$X(s) := \frac{x(0)\,s^3 + 9\,s\,x(0) + D(x)(0)\,s^2 + 9\,D(x)(0) + 3}{s^4 + 13s^2 + 36}$$

and substitute the initial conditions,

```
X(s) := subs(inits, X(s));
```

$$X(s) := 3\,\frac{1}{s^4 + 13s^2 + 36}$$

Finally we need only inverse transform to find the solution $x(t)$ of the initial value problem in (1).

```
x(t) := invlaplace(X(s), s, t);
```

$$x(t) := \frac{3}{10}\sin(2t) - \frac{1}{5}\sin(3t)$$

Of course we could probably get this result immediately with **`dsolve`**, but the intermediate output generated by the steps above can be quite instructive.

Using *Mathematica*

First we load the Laplace transforms package **`Calculus:LaplaceTransform`** with the command

```
Needs["Calculus`LaplaceTransform`"]
```

and define the differential equation and initial conditions that appear in (1):

```
de =    x''[t] + 4 x[t] == Sin[3t];
inits = {x[0]->0,  x'[0]->0};
```

Then the Laplace transform of the differential equation is given by

```
DE = LaplaceTransform[de, t, s]
```

$$\text{LaplaceTransform}(x(t),t,s)\,s^2 - x(0)\,s + 4\,\text{LaplaceTransform}(x(t),t,s) - x'(0) == \frac{3}{s^2+9}$$

The result of this command is a linear (algebraic) equation in the as yet unknown $\text{LaplaceTransform}(x(t),t,s)$. We proceed to solve for this transform $X(s)$ of the unknown function $x(t)$,

```
X =  Solve[DE, LaplaceTransform[x[t], t, s]]
```

$$\{\{\text{LaplaceTransform}(x(t),t,s) \to -\frac{-s\,x(0) - x'(0) - \frac{3}{s^2+9}}{s^2+4}\}\}$$

```
X =  X // Last // Last // Last
```

$$-\frac{-s\,x(0)-x'(0)-\frac{3}{s^2+9}}{s^2+4}$$

and substitute the initial conditions,

```
X =  X /. inits
```

$$\frac{3}{(s^2+4)(s^2+9)}$$

Finally we need only inverse transform to find $x(t)$.

```
x = InverseLaplaceTransform[X, s, t] // Expand
```

$$\frac{3}{10}\sin(2t)-\frac{1}{5}\sin(3t)$$

Of course we could probably get this result immediately with **DSolve**, but the intermediate output generated by the steps above can be quite instructive.

Using MATLAB

To use MATLAB it is convenient to rewrite equation (1) in the form

$$x''+4\,x-\sin 3t \;=\; 0\;.$$

Then we enter the symbolic expression on the left-hand side to define our differential equation:

```
syms s t x X
de = diff(sym('x(t)'),t,2)+4*sym('x(t)')-sin(3*t)
de =
diff(x(t),`$`(t,2)) + 4*x(t) - sin(3*t)
```

The Laplace transform of the differential equation is given by

```
DE = laplace(de)
DE =
s*(s*laplace(x(t),t,s) - x(0)) - D(x)(0) +
          4*laplace(x(t),t,s) - 3/(s^2+9)
```

At this point we substitute the initial conditions $x(0)=x'(0)=0$,

```
DE = subs(DE,{'x(0)','D(x)(0)'},{0,0})
DE =
s^2*laplace(x(t),t,s)+4*laplace(x(t),t,s)-3/(s^2+9)
```

The result (understood to be equated to 0) is a linear (algebraic) equation in the as yet unknown transform **`X = laplace(x(t),t,s)`**. We proceed to solve for this transform **`X`** of the unknown function **`x(t)`**,

```
DE = subs(DE,'laplace(x(t),t,s)','X')
DE =
s^2*X+4*X-3/(s^2+9)

X = solve(DE,X)
X =
3/(s^2+9)/(s^2+4)
```

Finally we need only find the inverse Laplace transform of $X(s)$ to obtain the solution $x(t)$ of the initial value problem in (1).

```
x = ilaplace(X,s,t)
x =
-1/5*sin(3*t)+3/10*sin(2*t)

pretty(x)
- 1/5 sin(3 t) + 3/10 sin(2 t)
```

Of course we could probably get this result immediately with **`dsolve`**, but the intermediate output generated by the steps above can be quite instructive.