Applications of Differential Equations

The Heat Equation with partial differential equations

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Introduction

- The purpose of our presentation is to derive the heat equation so that we are able to model the flow of heat through an object.
- But before we begin our discussion of the mathematics of the heat equation, we must first determine what is meant by the term **heat**?

  A common example of the misunderstanding of the term heat is the classic physics question of what contains more heat, a bathtub of warm water or a boiling cup of water?

  We all know the boiling cup of water has a higher temperature but contains less heat than the bathtub.

- Therefore, in calculating problems concerned with heat we must distinguish between the two types of measurements, the measurement of **temperature** and the measurement of the **quantity of heat contained in an object**. In this discussion when we mention the term **heat** we will be talking about the **quantity of heat in an object**.

Derivation of the heat equation

- There are two methods used to solve for the rate of heat flow through an object. The first method is derived from the properties of the object. The second method is derived by measuring the rate of heat flow through the boundaries of the object.

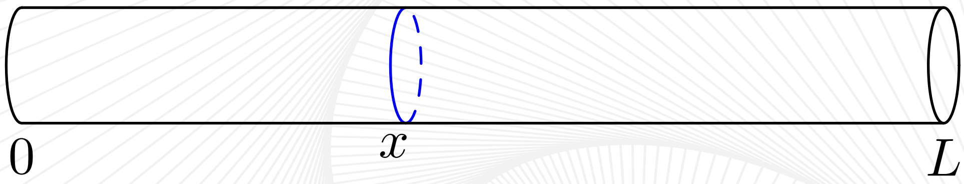
Method One

- Experimental calculations show that the heat (Q) in ΔV at time (t) can be defined by,

$$\Delta Q = c\rho u \Delta V. \tag{2.1}$$

- Where c is the specific heat, ρ is the density, u is the temperature, and ΔV is a small volume.

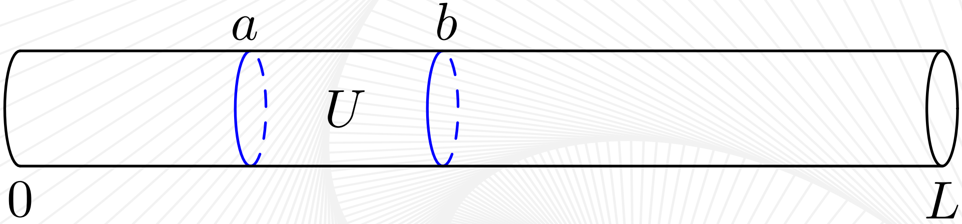
- Consider this thin rod, made of a homogenous material and perfectly insulated along its length so that heat can only flow through its ends. Any position along the rod is denoted as x, and the length of the rod is denoted as L such that $0 \leq x \leq L$.



- Therefore the temperature u is the only condition that depends on position(x) and time(t). Thus,

$$\Delta Q = c\rho u(x,t)\Delta V \tag{2.2}$$

- Consider this thin rod that is perfectly insulated along its length so that heat can only escape from its ends. Any position along the rod is denoted as x, and the length of the rod is denoted as L such that $0 \leq x \leq L$.



- Therefore the temperature u is the only condition that depends on position(x) and time(t). Thus,

$$\Delta Q = c\rho u(x,t)\Delta V \tag{2.3}$$

- Now consider a small section of the rod U defined as the interval from $x = a$ to $x = b$.

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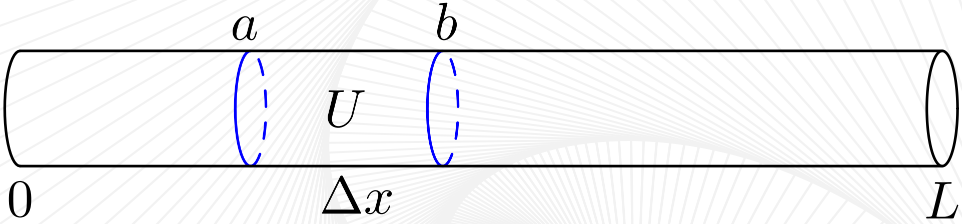


- Therefore the temperature u is the only condition that depends on position(x) and time(t). Thus,

$$\Delta Q = c\rho u(x,t)\Delta V \tag{2.4}$$

- Now consider a small section of the rod U defined as the interval from $x = a$ to $x = b$. The cross sectional area is defined as S, and the width of this section is Δx. This gives $\Delta V = S\Delta x$.

- We can now express the amount of heat in the cross-sectional area as,

$$\Delta Q = c\rho u(x,t)S\Delta x \tag{2.5}$$

- We find that the amount of heat in the section U is given by the integral.

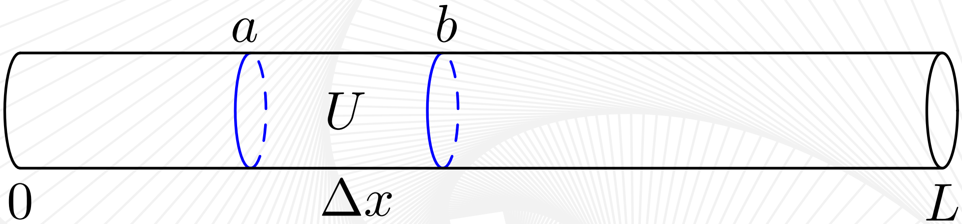
$$Q(t) = \int_a^b c\rho u(x,t)S dx. \tag{2.6}$$

- Since the rod has uniform thickness, S doesn't change with respect to time, and because we are dealing with homogenous materials c and ρ do not change with respect to time. Thus, by differentiating we take the partial of u to find the change in heat with respect to time equaling,

$$\frac{dQ}{dt} = \int_a^b c\rho \frac{\partial u}{\partial t} dx S. \tag{2.7}$$

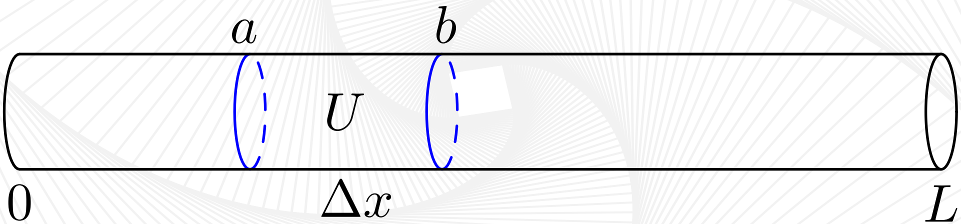
Method Two

- Our second method of finding the change in heat Q with respect to time is also determined experimentally, with a rod similar to that in method one.



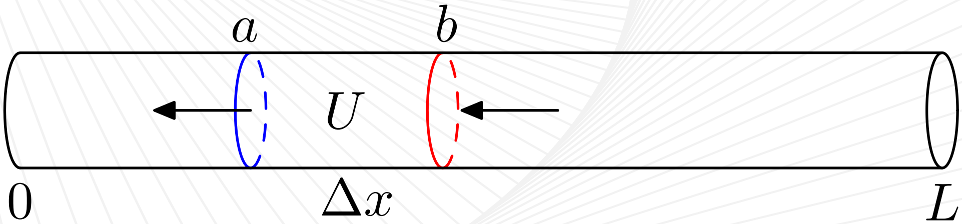
Method Two

- Our second method of finding the change in heat (Q) with respect to time is also determined experimentally, with a rod similar to that in method one.
- The rate of heat flow through U is inversely proportional to the width of U, and directly proportional to the cross-sectional area.



Method Two

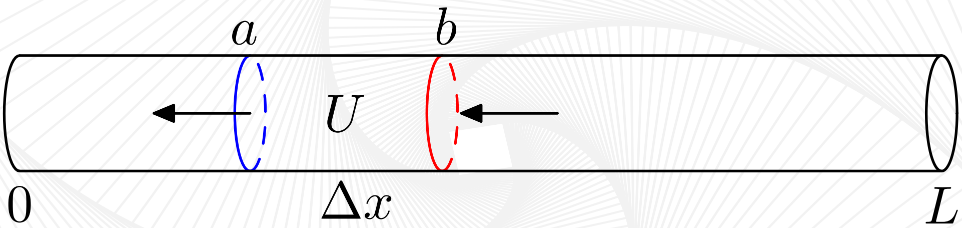
- Our second method of finding the change in heat (Q) with respect to time is also determined experimentally, with a rod similar to that in method one.
- The rate of heat flow through U is inversely proportional to the width of U, and directly proportional to the cross-sectional area.
- It is known that when two objects of different temperature are placed together (touching) heat will flow from the hotter object to the cooler one. If the temperature at $b > a$ then heat will flow from $b \longrightarrow a$.



- Combining these properties we find,

$$\Delta Q = -C \frac{u(a + \Delta x, t) - u(a, t)}{\Delta x} S \tag{2.8}$$

- The proportionality constant C is known as the thermal conductivity. This varies depending on the type of material being evaluated.

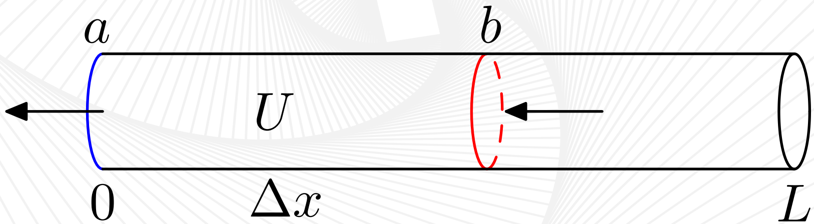


$$\Delta Q = -C\frac{u(a+\Delta x, t) - u(a,t)}{\Delta x}S \tag{2.9}$$

- To show this is true, we need to show the flow of heat through the section U where the boundaries of the section are defined as

$$a = x = 0;$$

$$b = a + \Delta x$$



- If the temperature at $u(a+\Delta x, t) > u(a,t)$, then ΔQ would be negative which is logical because the temperature at $a + \Delta x$ is greater than the temperature at a, so the heat would be flowing from $a + \Delta x$ to a, thus heat would be flowing out of the rod.

- Letting $\Delta x \to 0$ in equation (**??**), the difference quotient approaches $\dfrac{\partial u}{\partial x}$ then the rate of heat flow through (U) at $x = a$ is given by,

$$\Rightarrow -C\frac{\partial u}{\partial x}(a,t)S \tag{2.10}$$

- Following the same argument we can show that the rate of heat flow through U at b is defined as,

$$\Rightarrow C\frac{\partial u}{\partial x}(b,t)S \tag{2.11}$$

- Therefore, the amount of heat that U obtains at time t can be given by,

$$\begin{aligned} \frac{dQ}{dt} &= C[\text{rate in} - \text{rate out}]S \\ &= C\left[\frac{\partial u}{\partial x}(b,t) - \frac{\partial u}{\partial x}(a,t)\right] S \end{aligned} \tag{2.12}$$

- Using similar techniques as we used in method one we find the change in heat with respect to time to be,

$$\frac{dQ}{dt} = C \int_a^b \frac{\partial^2 u}{\partial x^2} dx S \tag{2.13}$$

Combining the two Methods

- Now that we have two equations for the rate of heat flow into and out of the section U,

Method 1

$$\frac{dQ}{dt} = \int_a^b c\rho \frac{\partial u}{\partial t} dx S \tag{3.1}$$

and,

Method 2

$$\frac{dQ}{dt} = C \int_a^b \frac{\partial^2 u}{\partial x^2} dx S. \tag{3.2}$$

We set these equations equal to each other.

$$c\rho \int_a^b \frac{\partial u}{\partial t} dx S = C \int_a^b \frac{\partial^2 u}{\partial x^2} dx S \tag{3.3}$$

- Doing some calculations we arrive at,

$$\frac{\partial u}{\partial t} - K \frac{\partial^2 u}{\partial x^2} = 0 \tag{3.4}$$

- More popularly written as,

$$u_t = K u_{xx} \tag{3.5}$$

- Where K is defined to be the thermal diffusivity.
- We have now developed the heat equation, also known as the diffusion equation.

Initial Boundary Value Problems (IBV)

- In order to solve the heat equation for many solutions or even just a single solution we must give the problem some initial conditions. So we must define the temperature of every point along the rod at time(t), which we can do with a function.

$$u(x,0) = u_0 x \qquad \text{for } 0 \le x \le L \tag{4.1}$$

- This is known as the initial temperature distribution.

- Since heat can only enter or exit the rod at its boundaries we must define some "boundary conditions," for the rod. Therefore we need to define the conditions of the rod at the boundaries of the rod $(0, L)$:

$$u(0,t) = T_0, u(L,t) = T_L; \qquad \text{for all } t > 0 \tag{4.2}$$

- For example if one end of the rod was submerged in a liquid that is a constant 0^o, and the other end in a liquid at 100^o, then,

$$u(0,t) = 0, \qquad \text{and } u(L,t) = 100 \qquad \text{for all } t > 0 \tag{4.3}$$

- This is called a Dirichlet condition.

- Now we can set up an initial boundary value (IBV) problem. Combining the heat equation with the initial conditions and boundary conditions. The problem is to find $u(x,t)$ such that:

$$
\begin{aligned}
u_t(x,t) &= K u_{xx}(x,t), \qquad \text{for } 0 \le x \le L \qquad \text{and } t > 0,\\
u(x,0) &= u_0(x), \qquad \text{for } 0 \le x \le L\\
u(0,t) &= T_0, \qquad \text{and } u(L,t) = T_L, \qquad \text{for } t > 0
\end{aligned}
$$

Examples

Example 1

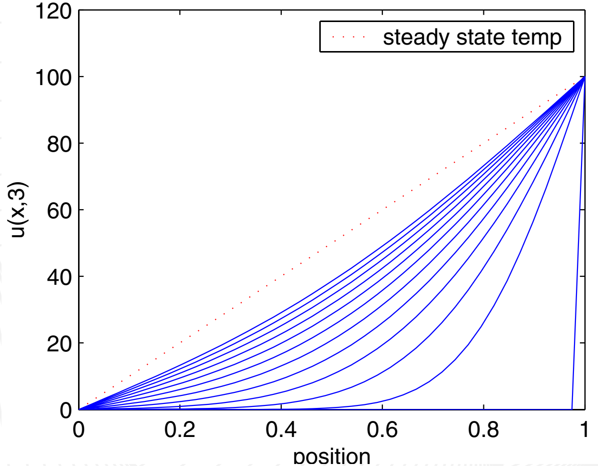
- For an example problem we will set some conditions. If we have a rod of length $L = 1$, and $K = 0.02$. Setting the initial temperature distribution, $u(x, 0) = 0$, and the boundary conditions defined with equation (**??**). We arrive with the problem

$$u_t(x,t) = 0.02u_{xx}(x,t), \qquad \text{for } 0 \leq x \leq 1 \qquad \text{and } t > 0,$$
$$u(x,0) = 0, \qquad \text{for } 0 \leq x \leq 1$$
$$u(0,t) = 0, \qquad \text{and } u(1,t) = 100, \qquad \text{for } t > 0.$$

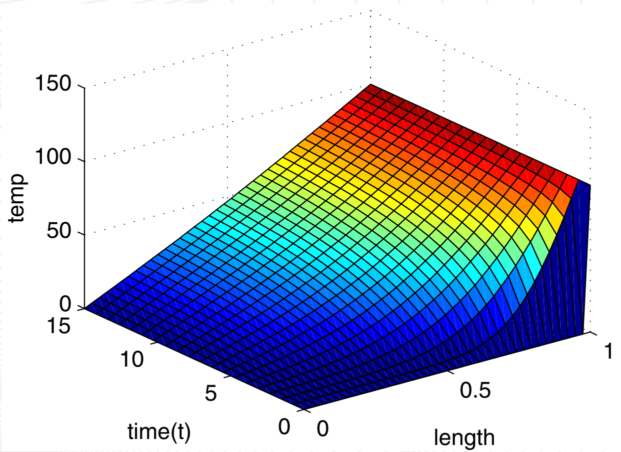
- Doing some math that involves using the linearity of the heat equation, the steady state solution, separation of variables, and Fourier series, we find the temperature along the length of the rod at various times can be modelled with the equation,

$$u(x,t) = 100x + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-0.02n^2\pi^2 t}\sin n\pi x.$$

- Of course we don't want to plot this equation by hand so we looked to Matlab to do the work for us. Conveniently enough, Matlab has a partial differential equation solver $(pdepe)$ built in which can solve IBV problems.



(a) Image 1



(b) Image 2

- Here are the solutions to the (IBV) problem of example 1.
- In image (1) we can see many solutions converging towards the Steady State Temperature.

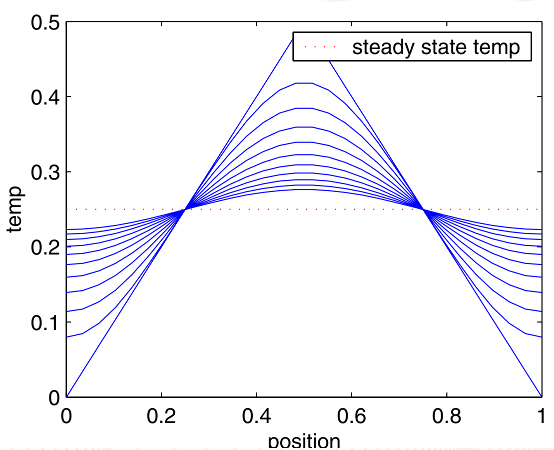
Example 2

- Next, an example with different initial conditions and boundary conditions, with $K = 1$, $L = 1$, the initial temperature distribution equal to $u_0(x)$, and insulated boundaries. These conditions are known as Neumann conditions.

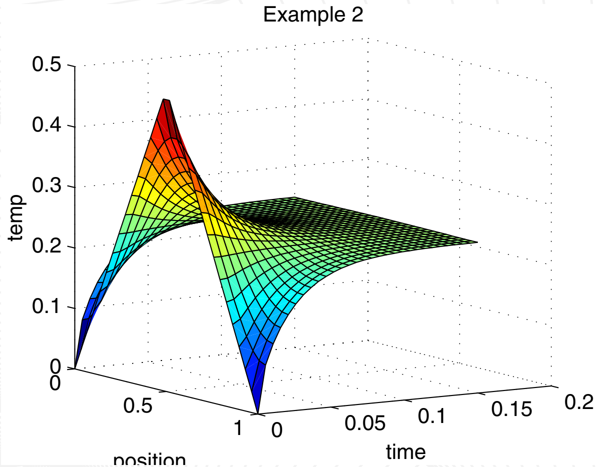
$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t), \qquad \text{for } 0 \le x \le 1 \qquad \text{and } t > 0,\\
u(x,0) &= u_0(x), \qquad \text{for } 0 \le x \le 1\\
u_x(0,t) &= 0, \qquad \text{and } u_x(1,t) = 0, \qquad \text{for } t > 0.
\end{aligned}
$$

- Where $u_0(x)$ is a piecewise function,

$$
u_0(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ (1-x), & 1/2 \le x \le 1 \end{cases}
$$



(c) Image 3



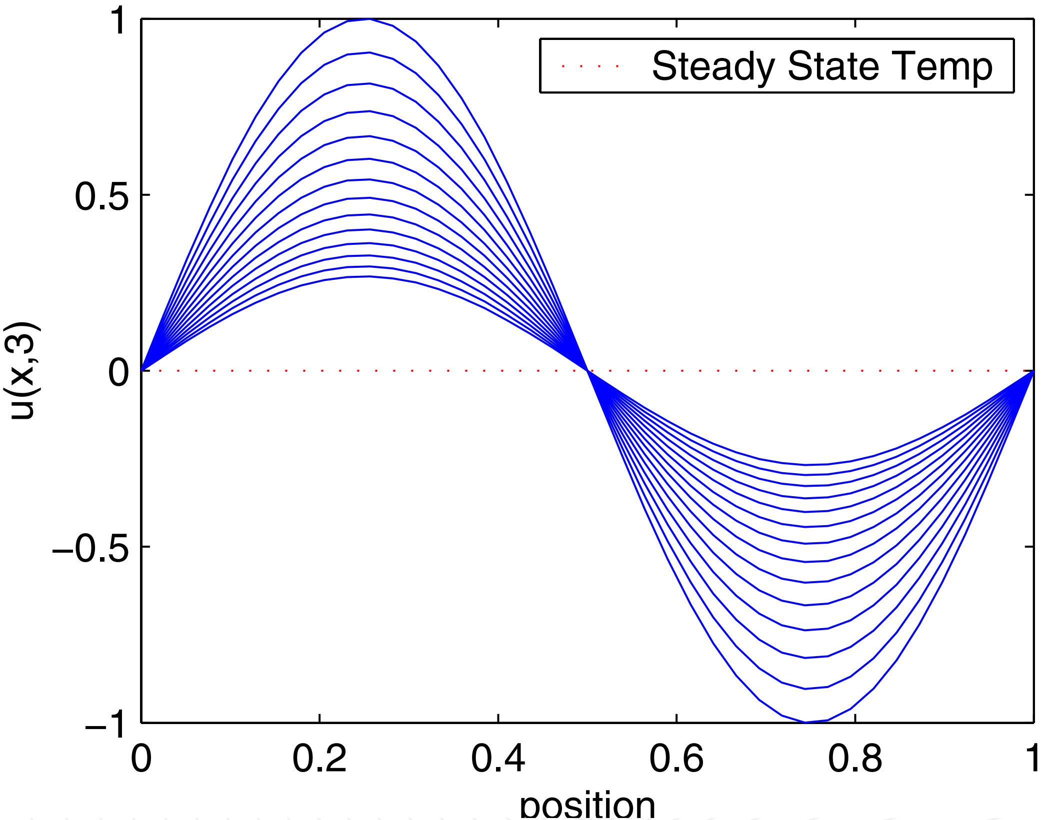
(d) Image 4

- The plots of the solutions to Example 2.
- With insulated boundaries no heat can escape from the rod.

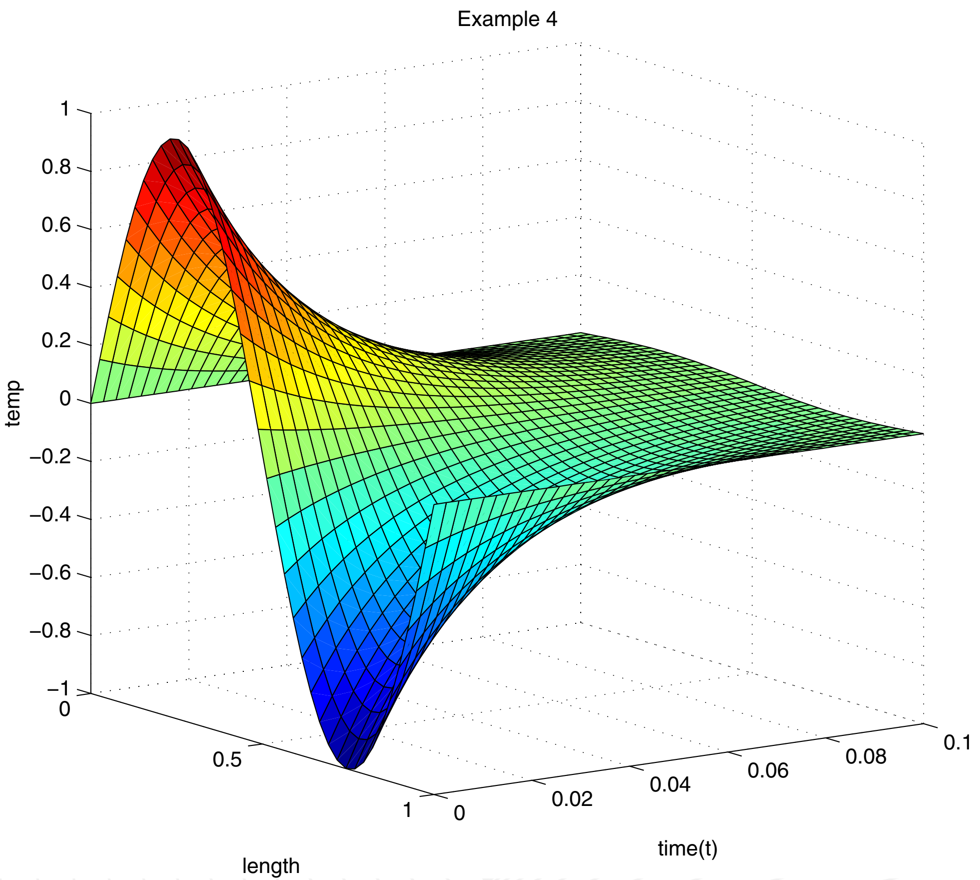
Example 3

- Another example with Dirichelet conditions,

$$u_t(x,t) = u_{xx}(x,t) + e^{-x}, \qquad \text{for } 0 \le x \le 1 \qquad \text{and } t > 0,$$
$$u(x,0) = sin(2\pi x), \qquad \text{for } 0 \le x \le 1$$
$$u(0,t) = 0, \qquad \text{and } u(1,t) = 0, \qquad \text{for } t > 0.$$



(e) Image 5



Example 4
temp
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.5
1
length
0
0.02
0.04
0.06
0.08
0.1
time(t)

(f) Image 6

Conclusion

In concluding we can imagine how the heat equation has many applications from engines to structural mechanics. The equation also has immense amounts of use in Biology where it is know as the **diffusion equation** and models the diffusion of a substance through a system. We have only examined the case where heat flows in one direction through a thin rod. You can imagine the complexity of heat moving in two dimensions or even three.

References

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[4] The Sauceman. His LaTeX expertise.