Poe's Pendulum

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Abstract

This project is based on a story in Edgar Allan Poe's literary classic, *The Pit and the Pendulum*, written in 1842. In this story Poe tells of a prisoner tied on the floor,facing a sharp-edged pendulum descending toward him. Poe describes that the sweep of the descending pendulum increases as the velocity goes faster. In this activity, we will examine this model from a mathematical standpoint, and discover whether Poe's description of the pendulum's motion is accurate.

1. Poe's Description of the Pendulum's Motion

We begin our project with a quotation directly from Poe's work, *The Pit and the Pendulum* describing the pendulum.

> ...Looking upward, I surveyed the ceiling of my prison. It was some thirty or forty feet overhead, and constructed much as the side walls. In one of its panels a very singular figure riveted my whole attention. It was the painted figure of Time as he is commonly represented, save that, in lieu of a scythe, he held what, at a casual glance. I supposed to to be the pictured image of a huge pendulum, such as we see on a antique clocks. There was something, however, in the appearance of this machine which caused me to regard it more attentively. While I gazed directly upward at it...I fancied that I saw it in motion. In a instant afterward and fancy was confirmed. *Its sweep was brief, and of course slow*...It might has been half an hour, perhaps even an hour...before I again cast my eyes upward. What I then saw confounded and amazed me. *The sweep of the pendulum had increased in extent by nearly a yard. As a natural consequence its velocity was also much greater.*

Figure 1: Edgar Allan Poe

> But what mainly disturbed me was the idea that it had perceptibly descended. I now observed–with what horror it is needless to say–that its nether extremity was formed of a crescent of glittering steel, about a foot in length from horn to horm; the horns upward, and the under edge as keen as that of a razor...and the whole hissed as it swung through the air..long, long hours of horror more than mortal during which I counted the rushing oscillations of the steel! Inch by inch–line by line–which a descent only appreciable at intervals that seemed ages–down and still down it came!...The vibration of the pendulum was at right angles to my length. I saw that the crescent was designed to cross the region of my heart...its terrifically wide sweep (some thirty feet or more)... Down–steadily down it crept. I took a frenzied pleasure in contrasting its downward with its lateral velocity. To the right to the left–far and wide–with the shriek of a damned spirit!... Down–certainly, relentlessly down!...

In his story, Poe claims that the pendulum's *sweep was brief and of course slow*, but that as the pendulum descended, *the sweep...had increased by nearly a yard...its velocity also was much greater*, until finally it had a *terrifically wide sweep some thirty feet or more*. We will set up a model of a descending pendulum and see if it agrees with Poe's pendulum.

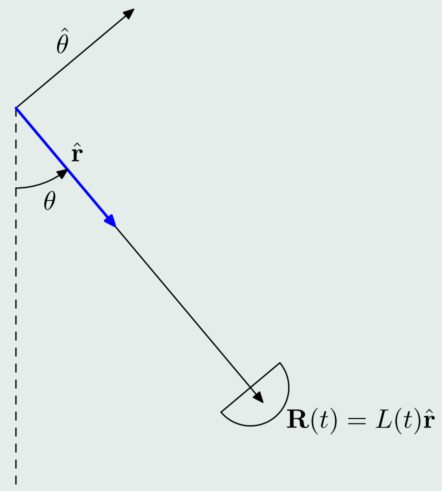


Figure 2: Poe's Pendulum

2. The Mathematical Model of a descending Pendulum

Let the pendulum be supported by a wire which is descending at a constant(steady) rate. The length of the wire is a function of time, $L(t)$. The angle that the wire makes with the downward vertical from the point of support can also be expressed as a function of time, $\theta(t)$. Assuming that $\hat{r}$ and $\hat{\theta}$ are unit vectors starting at the point of support, with $\hat{r}$ parallel to the wire and $\hat{\theta}$ perpendicular, the position vector for the pendulum bob is $\mathbf{R} = L\hat{r}$.

We obtain two more equations, $\hat{r}'$=$\theta'\hat{\theta}$ and $\hat{\theta}'$=-$\theta'\hat{r}$, by using the following procedures: From the unit circle, we know that any point on the circle has the coordinates $(\cos\alpha, \sin\alpha)$, where α is the angle between the vector going to the point from the origin and the vector going from the origin in the positive direction of the horizontal axis. Thus:

$$\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

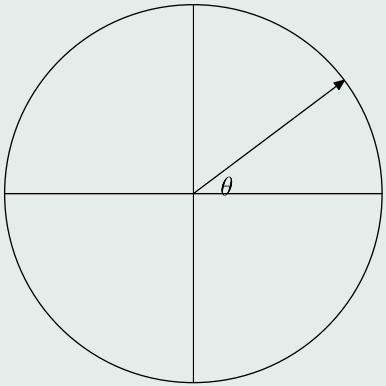


Figure 3: Unit Circle

and

$$\hat{\theta} = \begin{pmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

From the above two equations, we can easily derive the following:

$$\frac{d\hat{r}}{d\theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\theta}$$

and

$$\frac{d\hat{\theta}}{d\theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{r}$$

Then

$$\hat{r}' = \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \hat{\theta}\theta'$$

$$\hat{\theta}' = \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\hat{r}\theta'$$

Since $\mathbf{R} = L\hat{r}$,$\hat{r}' = \hat{\theta}\theta'$ and $\hat{\theta}' = -\hat{r}\theta'$, the velocity and acceleration vectors are:

$$\begin{aligned} \mathbf{R}' &= (L\hat{r})' \\ &= L'\hat{r} + L\hat{r}' \\ &= L'\hat{r} + L\theta'\hat{\theta}, \\ \mathbf{R}'' &= L''\hat{r} + L'\hat{r}' + L'\theta'\hat{\theta} + L'\theta''\hat{\theta} + L\theta'\hat{\theta}' \\ &= \hat{r} + L'\hat{r}' + L'\theta'\hat{\theta} + L'\theta''\hat{\theta} - L\theta'\theta'\hat{r} \\ &= (L'' - L\theta'\theta')\hat{r} + (2L'\theta' + L\theta'')\hat{\theta} \end{aligned} \tag{1}$$

Three forces are affecting the pendulum: the force of gravity, the tension of the wire, and the frictional force due to air. Since the magnitude of the friction force is small, we can ignore it. We shall also ignore the tension force because we only need to consider the forces in the $\hat{\theta}$ direction perpendicular to the wire. Supposed that the pendulum's mass is m, the component in the $\hat{\theta}$ direction of the gravitational force is $-mg\sin\theta$. From equation (1), we have the component of the acceleration vector along $\hat{\theta}$ being $2L'\theta' + L\theta''$. Taking the mass of the pendulum bob to be m, and g being the gravitational constant, the component of the gravitational force in the $\hat{\theta}$ direction is $-mg\sin\theta$. By using Newton's second law, we have the equation of the pendulum's angular motion: $m(2L'\theta' + L\theta'')$ Setting our 2 equations equal to each other:

$$-mg\sin\theta = m(2L'\theta' + L\theta'') \tag{2}$$

Since the angle θ is relatively small, we use θ to replace $\sin\theta$ in our equation (2). After cancelling the mass factor, m, we have the equation of the pendulum's linear motion:

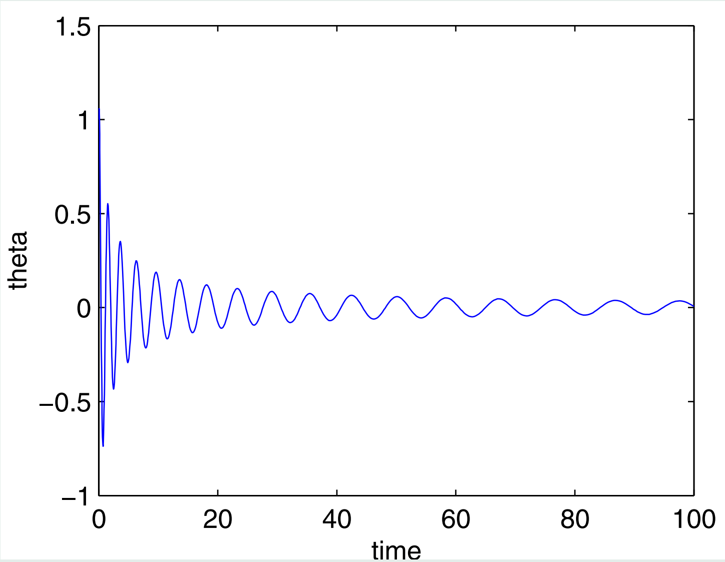
$$2L'\theta' + L\theta'' + g\theta = 0. \tag{3}$$

We will compare the sweep of this pendulum model, $L\theta$, and the curvilinear velocity, $(L\theta)'$, with Poe's description of his pendulum.

3. Matlab ODE45 results

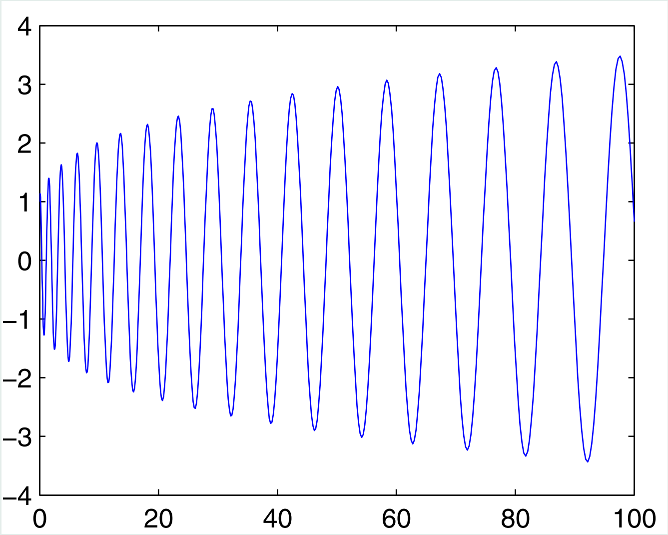
By breaking the second order differential equation (3) into a system of first order differential equations, we can use Matlab's ODE45 solver to get numerical results for the equation's solutions and graph them. Taking $L = 1 + t$, here are the graphs:

3.1. The decaying angle of a steadily descending pendulum

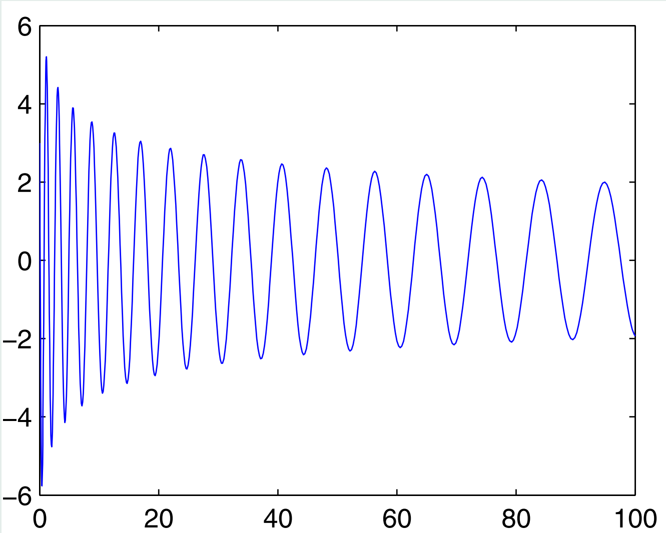


- Here is a graph drawn with the help of Matlab's ode45 solver of θ versus time.

3.2. The Sweep of the Pendulum



- This is the graph of $L\theta$, the pendulum's sweep.



- Lastly, this graph shows $(L\theta)'$ over time which is the velocity of the pendulum.

4. The Differential Equation Transformed

Assuming that the pendulum descending at a constant rate, the length of the wire is expressed as

$$L(t) = a + bt, \tag{4}$$

where a and b are positive constants. Using equation (4) for L in $2L'\theta' + L\theta'' + g\theta = 0$, we get the equation of the pendulum's angular motion:

$$(a + bt)\theta'' + 2b\theta' + g\theta = 0. \tag{5}$$

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In order to fully understand properties of the angular motion of the steadily descending pendulum, we transform equation (5) to a Bessel equation. Some intense work is required to figure out what new variables will transform our original equation. But we will simply introduce the two new variables x and y:

$$x = \frac{2}{b}\sqrt{(a+bt)g} \qquad \text{and} \qquad y = \theta\sqrt{a+bt}.$$

We convert equation (5) to the Bessel equation of order one,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y = 0, \tag{6}$$

by using the chain rule:

Step One Calculating $a + bt$. From the new variable $x == \frac{2}{b}\sqrt{(a+bt)g}$ we get the following:

$$\begin{aligned} a + bt &= \frac{b^2x^2}{4g} \\ bdt &= \frac{2b^2x}{4g}dx \\ &= \frac{b^2x}{2g}dx \\ dt &= \frac{bx}{2g}dx \end{aligned}$$

The next two equations derived from $dt = \frac{bx}{2g}dx$ are important for upcoming calculations.

$$\frac{dt}{dx} = \frac{bx}{2g}$$

$$\frac{dx}{dt} = \frac{2g}{bx}$$

Step Two Finding θ and $g\theta$. From the new variable $y = \theta\sqrt{a+bt}$ we get the following:

$$\begin{aligned}
y &= \frac{b}{2\sqrt{g}}\theta x \\
&= \frac{bx}{2\sqrt{g}}\theta \\
\theta &= \frac{2\sqrt{g}}{bx}y \\
&= \frac{2\sqrt{g}}{b}\left(\frac{1}{x}y\right) \\
g\theta &= \frac{2g\sqrt{g}}{b}\left(\frac{1}{x}y\right)
\end{aligned}$$

Step Three Differentiating θ.

$$\theta' = \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{2\sqrt{g}}{bx}y\right) = \frac{2\sqrt{g}}{b}\frac{d}{dt}\left(\frac{1}{x}y\right)$$

Using the chain rule developed in step one and the product rule:

$$\begin{aligned}
&= \frac{2\sqrt{g}}{b}\left(\frac{1}{x}\frac{dy}{dx}\frac{2g}{bx} - \frac{1}{x^2}\frac{2g}{bx}y\right) \\
&= \frac{2\sqrt{g}}{b}\frac{2g}{b}\left(\frac{1}{x}\frac{dy}{dx}\frac{1}{x} - \frac{1}{x^2}\frac{1}{x}y\right) \\
&= \frac{4g\sqrt{g}}{b^2}\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right)
\end{aligned}$$

Then,

$$\begin{aligned}
2b\theta' &= 2b\left(\frac{4g\sqrt{g}}{b^2}\right)\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right) \\
&= \frac{8g\sqrt{g}}{b}\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right)
\end{aligned}$$

Step Four Finding the second derivative of θ.

$$
\begin{aligned}
\theta'' &= \frac{d}{dt}\theta' \\
&= \frac{d}{dt}\left[\frac{4g\sqrt{g}}{b^2}\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right)\right] \\
&= \frac{4g\sqrt{g}}{b^2}\frac{d}{dt}\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right)
\end{aligned}
$$

Further work with the chain rule and the product rule:

$$
\begin{aligned}
\theta'' &= \frac{4g\sqrt{g}}{b^2}\left[\frac{1}{x^2}\frac{d^2y}{dx^2}\frac{2g}{bx} - \frac{2}{x^3}\frac{2y}{bx}\frac{dy}{dx} - \left(\frac{1}{x^3}\frac{dy}{dx}\frac{2g}{bx} - \frac{3}{x^4}\frac{2g}{bx}y\right)\right] \\
&= \frac{4g\sqrt{g}}{b^2}\frac{2g}{b}\left(\frac{1}{x^2}\frac{d^2y}{dx^2}\frac{1}{x} - \frac{2}{x^3}\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^3}\frac{dy}{dx}\frac{1}{x} + \frac{3}{x^4}\frac{1}{x}y\right) \\
&= \frac{8g^2\sqrt{g}}{b^3}\left(\frac{1}{x^3}\frac{d^2y}{dx^2} - \frac{2}{x^4}\frac{dy}{dx} - \frac{1}{x^4}\frac{dy}{dx} + \frac{3}{x^5}y\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
(a+bt)\theta'' &= \frac{b^2}{4g}\left[\frac{8g^2\sqrt{g}}{b^3}\left(\frac{1}{x^3}\frac{d^2y}{dx^2} - \frac{3}{x^4}\frac{dy}{dx} + \frac{3}{x^5}y\right)\right] \\
&= \frac{2g\sqrt{g}x^2}{b}\left(\frac{1}{x^3}\frac{d^2y}{dx^2} - \frac{3}{x^4}\frac{dy}{dx} + \frac{3}{x^5}y\right)
\end{aligned}
$$

Step Five Combining all the terms.

$$
\begin{aligned}
&(a+bt)\theta'' + 2b\theta' + g\theta \\
&= \frac{2g\sqrt{g}x^2}{b}\left(\frac{1}{x^3}\frac{d^2y}{dx^2} - \frac{3}{x^4}\frac{dy}{dx} + \frac{3}{x^5}y\right) + \frac{8g\sqrt{g}}{b}\left(\frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^3}y\right) + \frac{2g\sqrt{g}}{b}\left(\frac{1}{x}y\right) \\
&= \frac{2g\sqrt{g}}{b}\left(\frac{1}{x}\frac{d^2y}{dx^2} - \frac{3}{x^2}\frac{dy}{dx} + \frac{3}{x^3}y + \frac{4}{x^2}\frac{dy}{dx} - \frac{4}{x^3}y + \frac{1}{x}y\right) \\
&= \frac{2g\sqrt{g}}{b}\left[\frac{1}{x}\frac{d^2y}{dx^2} + \frac{1}{x^2}\frac{dy}{dx} + \left(\frac{1}{x} - \frac{1}{x^3}\right)y\right] \\
&= \frac{2g\sqrt{g}}{bx^3}\left[x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-1)y\right]
\end{aligned}
$$

Any solution $y(x)$ of equation (6) determines a solution of equation (5):

$$\theta(t) = \frac{1}{\sqrt{a+bt}} y\left(\frac{2}{b}\sqrt{(a+bt)g}\right), \tag{7}$$

and properties of the steadily descending pendulum's angular motion can be translated by properties of Bessel's equation (6)'s solutions.

5. A Bessel's Equation

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

This is a Bessel's equation of order 1! The general form of the Bessel's equation of order n is:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Bessel equations are special differential equations with solutions known as Bessel functions. The fact that our original equation can be transformed into a Bessel's equation is very good. Bessel equations and their solutions, Bessel functions, have been widely studied because of their various applications to such areas as physics, mechanics, and engineering. So there are some very useful, general properties that are known of Bessel functions.

5.1. The Solutions:Bessel Functions

The solutions to the bessel equations have a rather daunting form. Solutions of the Bessel equations exist in the form of a series of ascending powers of x, called Bessel functions of the first kind. Here is the *nth* order Bessel function of the First kind:

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\{1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - ...\}$$

where $\Gamma(x)$ is known as the $gammafunction$ and has the form $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $x > 0$. For integral orders of the Bessel function, $\Gamma(n+1) = n!$. To find the general solution of the Bessel equation we must have two independent solutions. Generally, $J_n(x)$ and $J_{-n}(x)$ gives two solutions independent of one another. But when n is an integer, $J_n(x)$ and $J_{-n}(x)$ are merely equal, at least in absolute value. Because our specific Bessel equation is of the first order, an integral order, $J_n(x)$ gives us only one independent solution and we must search further for a second. This is found in the Bessel function of the Second Kind, given by

$$Y_n(x) = J_n(x) \int \frac{dx}{x\{J_n(x)\}^2}$$

which involves logarithms, a finite series of negative powers, and an infinite series of positive powers. Therefore the general solution is written as a linear combination of these two independent solution:

$$y_n(x) = AJ_n(x) + BY_n(x)$$

where A and B are arbitrary constants, and no solution can have any other form but this. For our Bessel equation of order one the general solution is given by:

$$y(x) = AJ_1(x) + BY_1(x).$$

We could do work to solve these Bessel functions, finding series solutions and getting numerical estimates for say the first 20 terms or so. But we propose a whole different approach. Properties of the solutions to Bessel's equations translate into properties of the angular motion of the steadily descending pendulum. Any solution $y(x)$ dictates a solution of (5)

$$\theta(t) = \frac{1}{\sqrt{a+bt}} y\left(\frac{2}{b}\sqrt{(a+bt)g}\right)$$

So we are going to examine the properties of Bessel's Equations and see what they can tell us about the descending pendulum.

5.2. Cylinder Functions

Before proceeding further, let us introduce some special functions called *CylinderFunctions* which are functions of x and involve a parameter n and satisfy the following two recurrence formulas:

$$C_{n-1}(x) + C_{n+1}(x) = \frac{2n}{x} C_n(x) \tag{8}$$

$$C_{n-1}(x) - C_{n+1}(x) = 2\frac{d}{dx} C_n(x). \tag{9}$$

We will be working with n being real and x as being real and positive. From now on we will write $C_n(x)$ as C_n and $\frac{d}{dx}C_n(x)$ as C'_n By adding and subtracting (8) and (9), we find:

$$xC_{n-1} = nC_n + xC'_n \tag{10}$$

and

$$xC_{n+1} = nC_n - xC'_n \tag{11}$$

This function C_n satisfies Bessel's equation which is shown from the following: Differentiating (10) and multiplying by x

$$x^2C''_n + x(n+1)C'_n = x^2C'_{n-1} + xC_{n-1} \tag{12}$$

Multiplying (10) by n gives

$$nxC_{n-1} = n^2C_n + nxC'_n \tag{13}$$

Subtracting (13) from (12) we get

$$x^2C''_n + xC'_n - n^2C_n = x^2C'_{n-1} + (1-n)xC_{n-1}$$

From (11)

$$x^2C'_{n-1} + (1-n)xC_{n-1} = -x^2C_n$$

Therefore

$$x^2C''_n + xC'_n - n^2C_n = -x^2C_n$$

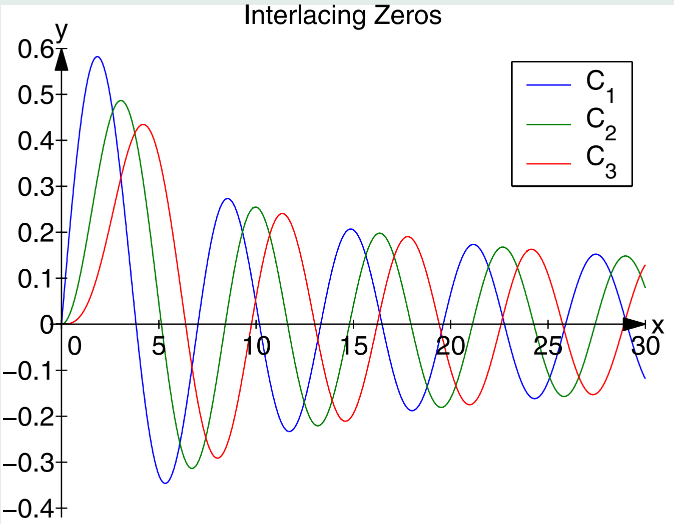
which comes to:

$$x^2C''_n + xC'_n + (x^2 - n^2)C_n = 0.$$

C_n satisfies the Bessel equation! So Cylinder functions are Bessel functions. Any properties found of Cylinder functions are also properties of Bessel functions.

5.3. Side-note

One important fact that is crucial to upcoming statements, but which we will not give vigorous proof for is that of interlacing zeros of the Cylinder functions. Between two consecutive positive zeros of C_n is a zero of C_{n-1} and between two consecutive zeros of C_{n-1} is a zero of C_n. (This is from (10) and (11) and Rolle's theorem.) Therefore the conclusion is that the zeros of C_n and C_{n-1} interlace.



6. Normal Form

We will compare Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{n^2}{x^2})y = 0,$$

with,

$$y'' + y'f(x) + yg(x) = 0.$$

We then have:

$$f(x) = \frac{1}{x}, \qquad g(x) = 1 - \frac{n^2}{x^2}$$

With the substitution $y = uv$

$$v(u'' + u'f + ug) + v'(2u' + uf) + uv'' = 0.$$

Choosing u such that$(2u' + uf) = 0$, our coefficient of v' disappears and we have:

$$v(u'' + u'f + ug) + uv'' = 0$$

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$(2u' + uf) = 0$ leads to $u = e^{-\frac{1}{2}\int f(x)dx}$ which for our value of $f(x)$ equals $x^{-\frac{1}{2}}$

$$v'' + \frac{u'' + u'f + ug}{u} v = 0$$

goes to:

$$v'' + (1 + \frac{1 - 4n^2}{4x^2})v = 0, \tag{14}$$

the *normal form* of Bessel's equation with

$$uv = y = C_n \qquad\qquad v = \frac{y}{u} = u^{-1}y = x^{\frac{1}{2}}C_n$$

6.1. The First Property

The special case where $n = \frac{1}{2}, -\frac{1}{2}$, gives us the simple oscillation equation $v'' + v = 0$, which has solution $v = a\cos x + b\sin x = R\sin(x + \phi)$.

$$\begin{aligned}
n &= \frac{1}{2}, \frac{-1}{2} \\
v'' + v &= 0 \\
\lambda^2 + 1 &= 0 \\
\lambda &= \pm i \\
z = e^{ix} &= \cos x + i\sin x \\
y(x) &= A\cos x + B\sin x
\end{aligned}$$

$$\begin{aligned}
y(x) &= \sqrt{A^2 + B^2}(\frac{A}{\sqrt{A^2 + B^2}}\cos x + \frac{B}{\sqrt{A^2 + B^2}}\sin x) \\
&= \sqrt{A^2 + B^2}(\cos\phi\cos X + \sin\phi\sin x) \\
&= \sqrt{A^2 + B^2}\sin(x + \phi)
\end{aligned}$$

Thus Bessel functions are connected to trigonometric functions and $x^{\frac{1}{2}}C_{\frac{1}{2}}$ and $x^{\frac{1}{2}}C_{-\frac{1}{2}}$ have the form $a\cos x + b\sin x$.The function $a\cos x + b\sin x$ has an infinity of zeros spaced π units apart. From (14), we see that the coefficient of the v term is positive if $4n^2 < 1$ or $4x^2 + 1 > 4n^2$, which for any n value is ultimately the case for large values of x. We let $I = (u'' + u'f + ug)/u$ so that $v'' + Iv = 0$. If $4n^2 < 1$, then $I > 1$ and decreasing monotonically to 1. If $4n^2 > 1$, $I < 1$ and increasing monotonically to 1. We now want to show that for $4n^2 < 1$, $x^{\frac{1}{2}}C_n$ has at least one zero in any range $0 < \phi < x < \phi + \pi$. This is done by showing that the assumption that this is not true leads to a contradiction.

Assuming that the function $x^{\frac{1}{2}}C_n$ has no zero in the range, we will take the function to be positive throughout the range. Let us consider the function $W = \sin(x - \phi)$, which has consecutive zeros at ϕ and $\phi + \pi$. Between these zeros the function is positive and it satisfies the equation $w'' + w = 0$. On the left, w' is positive at ϕ, and on the right it is negative at $\phi + \pi$. Doing some combinatory work with the normal form $v'' + vI = 0$, we get:

$$vw'' - v''w = (I - 1)vw$$

thus,

$$\int_\phi^{\phi+\pi} (I - 1)vwdx = [vw' - v'w]_\phi^{\phi+\pi} = [vw']_\phi^{\phi+\pi} = v(\phi + \pi)w'(\phi + \pi) - v(\phi)w(\phi)$$

$w(\phi) = 0$, so we have that $\int_\phi^{\phi+\pi}(I - 1)vwdx = v(\phi + \pi)w'(\phi + \pi)$. Of course the right-hand side is negative, $v(\phi + \pi$ being positive by assumption and $w'(\phi + \pi)$ being negative. But the integrand is not negative anywhere in the range. This gives the contradiction we were looking for and we can conclude that $x^{\frac{1}{2}}C_n$ has a zero in the given range and further that C_n has an infinite number of positive zeros at intervals of not more than π. Remember, though, that this is only proved for $4n^2 < 1$. But because of the interlacing of zeros, we also know that C_n has an infinite number of positive zeros at intervals of not more than π for any n value. This is the first important property.

6.2. The Second Property

Consider the zero order Bessel's equation:

$$C_0'' + \frac{1}{x}C_0' + C_0 = 0$$

Multiplying by $2C_0'$ and integrating between as yet unspecific limits a and b, we get:

$$[C_0'^2]_a^b + 2\int_a^b \frac{1}{x}C_0'^2 dx + [C_0^2]_a^b = 0$$

Taking a and b to be positions of stationary value, not necessarily consecutive, we have:

$$2\int_a^b \frac{1}{x}C_0'^2 dx = C_0^2(a) - C_0^2(b).$$

The integral is obviously positive if we are working on the right side of the origin so $|C_0(a)| > |C_0(b)|$. More generally, we can multiply Bessel's equation of order n by $2C'_n$ and write:

$$\frac{2x^2}{x^2-n^2}C''_n C'_n + \frac{2x}{x^2-n^2}C'^{\,2}_n + 2C_n C'_n = 0$$

We know that:

$$\frac{d}{dx}\left(\frac{x^2}{x^2-n^2}\right) = \frac{-2xn^2}{(x^2-n^2)^2}$$

and integration by parts gives:

$$\left[\frac{x^2}{x^2-n^2}C'^{\,2}_n\right]_a^b + \int_a^b \left[\frac{2x}{x^2-n^2} + \frac{2xn^2}{(x^2-n^2)^2}\right] C'^{\,2}_n dx + [C_n^2]_a^b = 0$$

Once again taking a and b to be positions of stationary value, not necessarily consecutive, we have:

$$\int_a^b \frac{2x^3}{(x^2-n^2)^2}C'^{\,2}_n dx = C_n^2(a) - C_n^2(b)$$

Again, the integrand is necessarily positive if we are working to the right of the origin,thus, the absolute value of any maximum or minimum of the function C_n is greater than the absolute value of any maximum or minimum after it. We can also deduce from this that there is only one zero of C'_n between two consecutive zeros of C_n.

6.3. What we now know

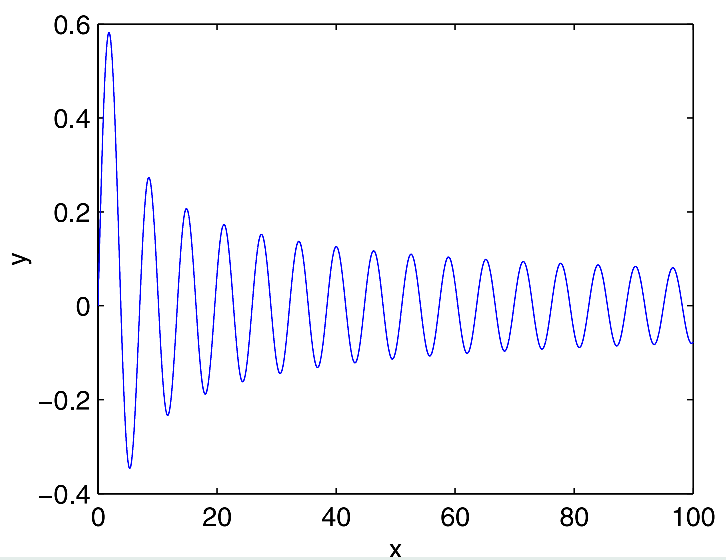
$$y_n(x) = AJ_n(x) + BY_n(x)$$

We have found the following two properties of the Bessel Functions:

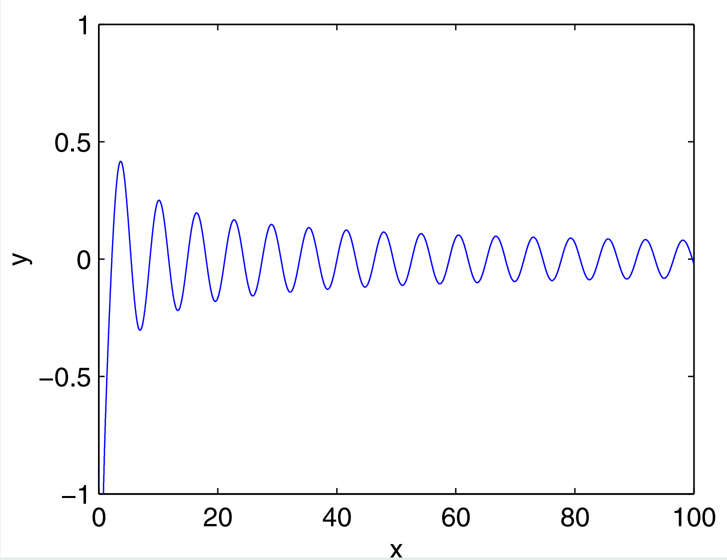
- $y_n(x)$ has an infinite number of positive zeros at intervals of not more than π for any n value. In other words, the function oscillates.
- The absolute value of any maximum or minimum value of $y_n(x)$ is greater than than the absolute value of any maximum or minimum after it.

These properties translate into properties of the solution to our original differential equation (5). So we know that the amplitude of the function $\theta(t)$ is decaying in an oscillatory manner. Here are the graphs drawn by Matlab of the first order Bessel functions of the first and second kinds.

6.4. $J_1(x)$: The Bessel Equation of the First kind



6.5. $Y_1(x)$: The Bessel Equation of the Second kind



Both these graphs show the expected properties. The functions decay sinusoidally.

7. Conclusion

Has our mathematical model fit Poe's description? Well, the sweep of our steadily descending pendulum was seen to increase over time, but this seemed to happen much more slowly than Poe's story suggested. As for the velocity, our model showed the pendulum decayed in velocity as it descended, which is contrary to Poe's comments about *rushing oscillations of steel* and the change from *its sweep was brief and of course slow* to *its velocity also was much greater*. Obviously, our model is not the pendulum of Poe's story.

This project shows that it is possible and sometimes very useful to find and analyze properties of solutions without even knowing the actual solutions. Without even solving for the Bessel functions, we could know for sure that the angle θ decays over time in an oscillatory manner.

8. Acknowledgements

1. David Arnold, Knowledge of LaTeX and Mathematics.

2. F.E. Relton *Applied Bessel functions* 1946.

3. Robert Borrelli, Courtney Coleman, and Dana Hobson. Poe's Pendulum, *Mathematics Magazine* Vol.58, No.2,(March 1985).

4. John Polking, Albert Boggess, and David Arnold, *Differential Equations.* 2001.

5. Greg Brown, Help with figures, computer issues, and LaTeX