Application 3.5
Automated Solution of Linear Systems

Suppose again that we want to solve the system

$$\begin{aligned}
322x_1 - 163x_2 + 231x_3 - 455x_4 &= 889 \\
107x_1 - 181x_2 + 428x_3 - 571x_4 &= 445 \\
351x_1 - 144x_2 + 421x_3 - 936x_4 &= 848 \\
111x_1 - 709x_2 + 484x_3 + 625x_4 &= 421.
\end{aligned} \tag{1}$$

of Application 3.3. If we define the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 322 & -163 & 231 & -455 \\ 107 & -181 & 428 & -571 \\ 351 & -144 & 421 & -936 \\ 111 & -709 & 484 & 625 \end{bmatrix} \tag{2}$$

and the constant and unknown vectors

$$\mathbf{b} = \begin{bmatrix} 889 \\ 445 \\ 848 \\ 421 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{3}$$

then the system in (1) takes the matrix form $\mathbf{Ax} = \mathbf{b}$ with unique solution given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{4}$$

provided the coefficient matrix $\mathbf{A}$ is nonsingular. If a computer algebra system is available to perform the matrix inversion and multiplication, we can then calculate the solution $\mathbf{x}$ automatically — as illustrated below using *Maple*, *Mathematica*, and MATLAB. We also illustrate more efficient linear solution methods that are based on Gaussian elimination rather than matrix inversion (which requires substantially more calculation than does Gaussian elimination if the dimension n of the system is large).

Use an available computer system to solve this way the linear systems in Problems 1–6 of Application 3.3. The applied problems below are elementary in character — resembling the "word problems" of high school algebra — but may illustrate the practical advantages of automated solutions.

1. You are walking down the street minding your own business when you spot a small but heavy leather bag lying on the sidewalk. It turns out to contain U. S. Mint American Eagle gold coins of the following types:

- One-half ounce gold coins that sell for $285 each,
- One-quarter ounce gold coins that sell for $150 each, and
- One-tenth ounce gold coins that sell for $70 each.

A bank receipt found in the bag certifies that it contains 258 such coins with a total weight of 67 ounces and a total value of exactly $40,145. How many coins of each type are there?

2. Now you really strike it rich! You find a bag containing one-ounce U.S. American Eagle gold coins valued at $550 each, together with half-ounce and quarter-ounce coins valued as in the preceding problem. If this bag contains a total of 365 coins with a total weight of exactly 11 pounds and a total value of $100,130, how many gold coins of each type are there?

3. A commercial customer orders 81 gallons of paint that contains equal amounts of red paint, green paint, and blue paint — and hence could be prepared by mixing 27 gallons of each. However, the store wishes to prepare this order by mixing three types of paint that are already available in large quantity:

- a *reddish* paint that is a mixture of 50% red, 25% green, and 25% blue paint;
- a *greenish* paint that is 12.5% red, 75% green, and 12.5% blue paint; and
- a *bluish* paint that is 20% red, 20% green, and 60% blue paint.

How many gallons of each are needed to prepare the customer's order?

4. Now the paint store receives a really big order — for 244 gallons of paint that is 1/2 red paint, 1/4 green paint, and 1/4 blue paint. The store has three already-mixed types of paint available in large quantity — the greenish paint and the bluish paint of the preceding problem, plus a reddish paint that is 2/3 red paint, 1/6 green paint, and 1/6 blue paint. How many gallons of each must be mixed in order to fill this order?

5. A tour busload of 45 people attended two Florida theme parks on successive days. On Day 1 the entrance fee was $15 per adult, $8 per child, $12 per senior citizen and the total charge was $558. On Day 2 the entrance fee was $20 per adult, $12 per child, $17 per senior citizen and the total charge was $771. How many adults, children, and senior citizens were on this tour bus?

6. For some crazy reason, the lunches bought at the first theme park were totaled separately for the adults, children, and seniors. The adults ordered 34 hot dogs, 15 French fries, and 24 soft drinks for a total bill of $70.85. The children ordered 20 hot dogs, 14 French fries, and 15 soft drinks for a total bill of $46.65. The senior citizens ordered 11 hot dogs, 10 French fries, and 12 soft drinks for a total bill of $30.05. What were the prices of a hot dog, an order of French fries, and a soft drink?

7. A fast food restaurant sells four types of sandwiches — hamburgers, cheeseburgers, roast beef, and chicken — and has four cash registers. At the end of each day, each cash register tallies the number of each type of sandwich sold, and the total sandwich receipts for the day. The four cash register operators work at different speeds, and one day's totals were as follows:

	Hamburgers	Cheeseburgers	Roast Beef	Chicken	Receipts
Register 1	37	44	17	23	$232.99
Register 2	28	35	13	17	$178.97
Register 3	32	39	19	21	$215.99
Register 4	47	51	25	29	$294.38

What was the price of each of the four types of sandwiches?

8. The fast food restaurant of the preceding problem adds a ham sandwich to its menu, and due to increased business it also adds a fifth cash register and reduces prices. After this expansion, one day's totals were as follows:

	Hamburgers	Cheeseburgers	Roast Beef	Chicken	Ham	Total
Register 1	41	49	22	26	19	$292.79
Register 2	34	39	18	20	16	$236.73
Register 3	36	43	23	24	18	$270.70
Register 4	49	52	26	31	24	$340.19
Register 5	52	55	24	28	25	$341.64

What were the new prices of the five types of sandwiches?

Using *Maple*

The matrices **A** and **b** in (2) and (3) can be entered with the *Maple* commands

```
with(linalg):
A := array( [[322, -163, 231, -455, 889],
              [107, -181, 428, -571, 445],
              [351, -144, 421, -936, 848],
              [111, -709, 484,  625, 421]] ):
b := array( [[889], [445], [848], [421]] ):
```

Then the inverse $\mathbf{A}^{-1}$ is given by

```
invA := inverse(A);
```

$$invA := \begin{bmatrix} \frac{15571825}{17086966} & \frac{23372979}{34173932} & -\frac{1292519}{1314382} & -\frac{6301457}{34173932} \\ \frac{16506551}{8543483} & \frac{49944881}{34173932} & -\frac{1375843}{657191} & -\frac{13447329}{34173932} \\ \frac{27711477}{17086966} & \frac{42007815}{34173932} & -\frac{2311085}{1314382} & -\frac{11261853}{34173932} \\ \frac{13224739}{17086966} & \frac{4993895}{8543483} & -\frac{1102257}{1314382} & -\frac{2679827}{17086966} \end{bmatrix}$$

This looks pretty bad, but if you've already studied determinants in Section 3.6, you may note that the denominator of each fraction here is a divisor of the determinant of **A**, given by

```
det(A);
```

$$68347864$$

At any rate, the desired solution vector is given by

```
x := multiply(invA, b);
```

$$x := \begin{bmatrix} 203 \\ 427 \\ 359 \\ 171 \end{bmatrix}$$

The more efficient direct solution by Gaussian elimination — bypassing the awkward inverse matrix — is given by

```
x := linsolve(A,b);
```

Using *Mathematica*

The matrices **A** and **b** in (2) and (3) can be entered with the *Mathematica* commands

```
A = {{322, -163, 231, -455},
     {107, -181, 428, -571},
     {351, -144, 421, -936},
     {111, -709, 484,  625}};
b = {{889}, {445}, {848}, {421}};
```

Then the inverse $\mathbf{A}^{-1}$ is given by

```
invA = Inverse[A]
```

$$\begin{pmatrix} \frac{15571825}{17086966} & \frac{23372979}{34173932} & -\frac{1292519}{1314382} & -\frac{6301457}{34173932} \\ \frac{16506551}{8543483} & \frac{49944881}{34173932} & -\frac{1375843}{657191} & -\frac{13447329}{34173932} \\ \frac{27711477}{17086966} & \frac{42007815}{34173932} & -\frac{2311085}{1314382} & -\frac{11261853}{34173932} \\ \frac{13224739}{17086966} & \frac{4993895}{8543483} & -\frac{1102257}{1314382} & -\frac{2679827}{17086966} \end{pmatrix}$$

This looks pretty bad, but if you've already studied determinants in Section 3.6, you may note that the denominator of each fraction here is a divisor of the determinant of **A**, given by

```
Det[A]
```

68347864

At any rate, the solution vector is then given by

```
x = invA . b
```

$$\begin{pmatrix} 203 \\ 427 \\ 359 \\ 171 \end{pmatrix}$$

The more efficient direct solution by Gaussian elimination — bypassing the awkward inverse matrix — is given by

```
x = LinearSolve[A,b];
```

Using MATLAB

The matrices **A** and **b** in (2) and (3) can be entered with the MATLAB commands

```
A = [322  -163  231  -455
     107  -181  428  -571
     351  -144  421  -936
     111  -709  484   625];
b = [889;  445;  848;  421];
```

Then the inverse $\mathbf{A}^{-1}$ is given by

```
invA = inv(A)
```

```
invA =
     0.9113    0.6839   -0.9834   -0.1844
     1.9321    1.4615   -2.0935   -0.3935
     1.6218    1.2292   -1.7583   -0.3295
     0.7740    0.5845   -0.8386   -0.1568
```

This doesn't look especially informative but, if you've already studied Section 3.6, then you'll know that multiplication by the determinant of **A**,

```
det(A)

detA = 68347864
```

should give a matrix with all-integer entries,

```
round(detA*invA)

ans =
      62287300    46745958   -67210988   -12602914
     132052408    99889762  -143087672   -26894658
     110845908    84015630  -120176420   -22523706
      52898956    39951160   -57317364   -10719308
```

At any rate, the desired solution vector is given by

```
x = invA*b
x =
   203.0000
   427.0000
   359.0000
   171.0000
```

The more efficient direct solution by Gaussian elimination — bypassing the awkward inverse matrix — is given by the special-notation MATLAB command

```
x = A\b
x =
   203.0000
   427.0000
   359.0000
   171.0000
```

This notation is intended to suggest division of the matrix equation $\mathbf{Ax} = \mathbf{b}$ on the *left* by the matrix $\mathbf{A}$, formally giving $\mathbf{A} \setminus \mathbf{Ax} = \mathbf{A} \setminus \mathbf{b}$, so $\mathbf{x} = \mathbf{A} \setminus \mathbf{b}$.