Low Pass Filters, Sinusoidal Input, and Steady State Output

Jaimie Stephens and Michael Bruce

5-20-04

Abstract

Discussion of applications and behaviors of low pass filters when presented with a steady-state sinusoidal input voltage. A specific example is chosen and the steady-state behavior is examined first in the general sense, then again in greater detail with much attention paid to the gain of the circuit in response to input signals of varying frequency.

1. Introduction

A filter is generally used to separate one unwanted quantity from another. In the case of electrical circuits, it is sometimes required that an unwanted signal be eliminated. This process requires a filter. Filters in electrical circuits are used in much the same way as the mechanical filters that we are all familiar with. That is, they eliminate one or more unwanted quantities while preserving the integrity of a desired signal. An ideal filter will separate and pass sinusoidal input signals based upon their frequency. There are many types of filters, and they are classified based upon which signals it is desired to effectively eliminate.

2. Types of Electrical Filters

An important quantity to consider when analyzing electronic filters is that of gain. Gain is a dimensionless quantity that simply represents the ratio of the amplitude of output signal to that of the input signal:

$$Gain = \left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| \tag{1}$$

A low pass filter will effectively pass signals of frequency below a certain cutoff, that is to say that the signals of high frequency exhibit very low gain (see Figure 1) while signals of lower, desired, frequency exhibit a gain of nearly one or greater than one (see Figure 2). High pass filters will pass signals of a higher frequency (see Figure 3) while virtually eliminating those whose signals are of lower frequency (see Figure 4). Another type of electrical filter is the band pass, which passes only frequencies in a range above a lower cutoff and below a higher cutoff. Lastly, notch filters attenuate signals of frequency over a specific range.

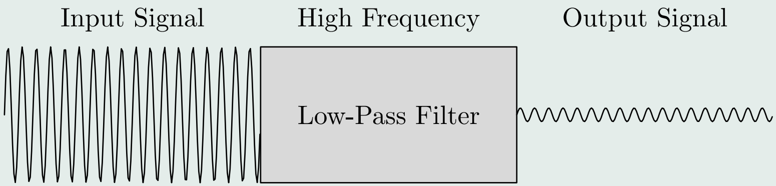

Figure 1: Low Pass filter presented with a high frequency input signal.

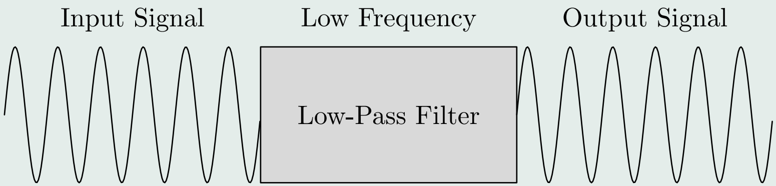

Figure 2: Low Pass filter presented with a low frequency input signal.

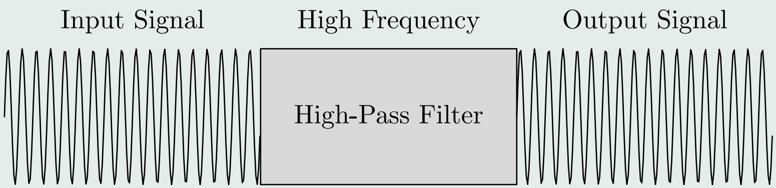

Figure 3: High Pass filter presented with a high frequency input signal.

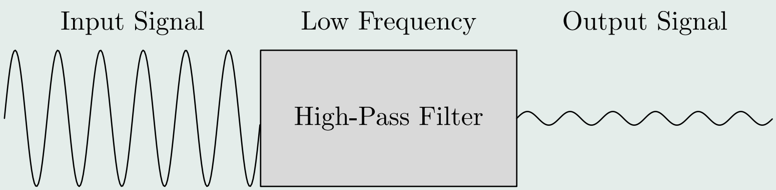

Figure 4: High Pass filter presented with a low frequency input signal.

Home Page
Title Page
Page 3 of 31
Go Back
Full Screen
Close
Quit

3. Low Pass Filter Applications

Low pass filters are used in many applications within the field of electrical engineering. In broadcast engineering, an oscillator is used within FM radio transmitters to generate signals of specific frequency. These signals lie within the FM broadcast band which begins at 88.1 MHz and ends at 107.9 MHz. These oscillators also produce what are called harmonic oscillations, or unintended signals of frequency at integer multiples of the desired signal's frequency. Since many different bands are allotted at frequencies above FM, including aircraft and emergency communications, these harmonic oscillations must be eliminated. A low pass filter is used within the transmitter's circuit for this purpose. Low pass filters are also used in the car audio industry to eliminate higher frequency audio signals. Leaving the signals of lower frequency (bass) unchanged. The bass is then sent to the sub-woofer, which is designed specifically to radiate audio signals of lower frequencies.

4. Steady State Solutions

It should be clear now that we are interested in the response of the circuit as $t \to \infty$. This is known as the steady-state response. We would also hope that when this steady-state response is analyzed, signals of undesired frequency will experience a gain of nearly zero (severe attenuation). If we were to model the output of some low pass filter as a function of voltage versus time, we will see that we obtain a response for each of the input signals. Now, given a specific design for a low pass filter, how do we obtain information about the gain of the steady-state output?

5. Periodic Steady State Theorems

Since we are interested in the behavior of the low pass filter when given a sinusoidal input (as in the case of an audio or radio signal), we wish to describe the output in terms of gain versus frequency (ω) of input signal. That is, we wish to see the effect of the input frequency on the gain. We start with circuit analysis and obtain a system of ordinary differential equations, which model the behavior of the circuit. This linear system will take the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t),$$

where $\mathbf{F}(t)$ is the driving force (input voltage), A is a square matrix of real constants, and $\mathbf{x}$ is a vector having the same number of rows as A. We state Theorem 1 without proof.

Theorem 1 ***Periodic Steady State***: *Suppose all eigenvalues of the constant matrix A have negative real parts and that $\mathbf{F}(t)$ is periodic with period T. Then the system $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$ has a unique steady-state, which is periodic of period T.*

Thus we must consider the location of the eigenvalues of A in the complex plane. If the eigenvalues of A have negative real parts in the complex plane, then we know that the Low Pass Filter under consideration will pass a signal whose period will be unchanged which implies that the angular frequency has not been changed. After which we may examine the effects of the filter on the amplitude of the output signal, in regard to ω.

6. Circuit Analysis

We begin by defining two of the fundamental laws of electrical engineering,

Theorem 2 ***Kirchhoff's Voltage Law***: *The sum of the voltage drops around any closed loop of a circuit is equal to zero.*

Theorem 3 ***Kirchhoff's Current Law****: The sum of the current entering a node, or junction, is equal to the sum of the current leaving the node.*

The specific low pass filter that we will be working with is a Two-Loop Low-Pass filter. Figure 5 shows the Two-Loop Low-Pass filter which consists of 2 capacitors, an inductor and one resistor. The voltage drop across a capacitor having capacitance C and charge q can be calculated by Coulombs Law $V_C = q/C$, where $q = \int i\,dt$ and i is current. The voltage drop, by Ohm's Law, across a resistor with resistance R can be calculated by $V_R = iR$, and the voltage drop across an inductor, by Faraday's Law, is $V_L = Li'$.

Our main objective is to find the steady-state output voltage as a function of the input voltage. We assume V_1 to be a fixed sinusoidal AC Voltage source, and R, L, C_1, C_2 are fixed values, while I_1 (the current in the left branch, through the resistor, R), I_2 (the current through C_1 in the middle branch), I_3 (the current in the right top branch through L), and V_2 (output voltage) are variable and dependent on the fixed values (see Figure 5 for locations). Using Theorem 3, we analyze node a in Figure 5 and get

$$I_1 = I_3 + I_2$$

However, the voltage through the inductor and the resistor depend on I_1 and I_2, so we want to solve for I_3 in terms of I_1 and I_2.

$$I_3 = I_1 - I_2$$

Using Coulombs Law, the voltage across C_2, the output voltage, can be written as

$$V_2 = \frac{1}{C_2}\int I_2\,dt,$$

and by differentiating both sides we get

$$V_2' = \frac{I_2}{C_2}. \tag{2}$$

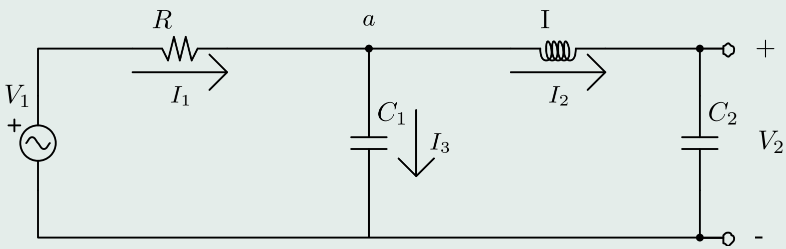

Figure 5: Two-Loop Low-Pass filter where V_1 is the input voltage and V_2 is the output voltage.

If we can now find expressions for I_1' and I_2' we will have a system of three equations and three unknowns.

Using Theorem 2 around the outer loop of the circuit gives us

$$V_1 = RI_1 + LI_2' + V_2,$$

and solving for I_2' we have

$$I_2' = \frac{V_1}{L} - \frac{RI_1}{L} - \frac{V_2}{L}. \tag{3}$$

Now we just need an expression for I_1'. Using Theorem 2 around the left loop of Figure 5, we get

$$V_1 - RI_1 - \frac{1}{C_1}\int I_3 dt = 0.$$

Therefore,

$$V_1 = RI_1 + \frac{1}{C_1}\int I_3 dt.$$

Differentiating both sides,

$$V_1' = RI_1' + \frac{I_3}{C_1}.$$

Replacing I_3 with $(I_1 - I_2)$, we have

$$V_1' = RI_1' + \frac{I_1 - I_2}{C_1}.$$

Solving for I_1', we have

$$I_1' = -\frac{I_1}{RC_1} + \frac{I_2}{RC_1} + \frac{V_1'}{R}. \tag{4}$$

Using Equations (2), (3), and (4) which are restated here for convenience,

$$V_2' = \frac{I_2}{C_2},$$

$$I_1' = -\frac{I_1}{RC_1} + \frac{I_2}{RC_1} + \frac{V_1'}{R},$$

$$I_2' = \frac{V_1}{L} - \frac{RI_1}{L} - \frac{V_2}{L},$$

we now have enough information to put together our linear system of first order differential equations which will take this form:

$$\begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & \frac{1}{C_2} \\ 0 & -\frac{1}{RC_1} & \frac{1}{RC_1} \\ -\frac{1}{L} & -\frac{R}{L} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{V_1{}'}{R} \\ \frac{V_1}{L} \end{bmatrix}.$$

Note that the system takes the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t), \tag{5}$$

where

$$\mathbf{x} = \begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & \frac{1}{C_2} \\ 0 & -\frac{1}{RC_1} & \frac{1}{RC_1} \\ -\frac{1}{L} & -\frac{R}{L} & 0 \end{bmatrix}, \quad \mathbf{F}(t) = \begin{bmatrix} 0 \\ \frac{V_1{}'}{R} \\ \frac{V_1}{L} \end{bmatrix}$$

7. Applying the Periodic Steady State Theorem

Now back to our main objective, to find the output voltage V_2 of the circuit with a fixed sinusoidal voltage source V_1. Before we go any further however, let us look to see if the eigenvalues of our system will have negative, real parts, otherwise there will not be a steady-state solution based on Theorem 3. To do this we must first find the characteristic polynomial, $p(\lambda) =\mid \lambda I - A \mid$, where I represents the identity matrix and λ represents the eigenvalues of matrix A. Looking at the determinant of $\lambda I - A$, we have

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 & -\frac{1}{C_2} \\ 0 & \lambda + \frac{1}{RC_1} & -\frac{1}{RC_1} \\ \frac{1}{L} & \frac{R}{L} & \lambda \end{vmatrix}.$$

Expanding across the top,

$$p(\lambda) = \lambda \begin{vmatrix} \lambda + \frac{1}{RC_1} & -\frac{1}{RC_1} \\ \frac{R}{L} & \lambda \end{vmatrix} - \frac{1}{C_2} \begin{vmatrix} 0 & \lambda + \frac{1}{RC_1} \\ \frac{1}{L} & \frac{R}{L} \end{vmatrix}.$$

Simplifying,

$$p(\lambda) = \lambda\left[\left(\lambda^2 + \frac{\lambda}{RC_1}\right) - \left(-\frac{1}{LC_1}\right)\right] - \frac{1}{C_2}\left[(0) - \left(\frac{\lambda}{L} + \frac{1}{LRC_1}\right)\right],$$

and the characteristic polynomial therefore looks like

$$p(\lambda) = \left[\lambda^3 + \frac{1}{RC_1}\lambda^2 + \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\lambda + \frac{1}{LRC_1C_2}\right]. \tag{6}$$

If we wanted to find the actual eigenvalues at this point we could, though it would be a lot of unnecessary work. Note that we only need to know the sign of the real part of the eigenvalues and not the actual values in order to satisfy Theorem 1. For this reason we are going to use something called the Routh Test.

8. The Routh-Hurwitz Table and Stability Criterion

Let

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

be a polynomial such that all the coefficients are real and a_0 is non-zero. To create a Routh-Hurwitz Table, put all the coefficients with even subscripts into the first row and all the coefficients with odd subscripts into the second row as follows:

$$\begin{array}{c|cccc} x^n & a_0 & a_2 & a_4 & \dots \\ x^{n-1} & a_1 & a_3 & a_5 & \dots \end{array}$$

Then to find the remaining rows refer to the following procedure:

$$\begin{array}{c|cccc} x^n & a_0 & a_2 & a_4 & \dots \\ x^{n-1} & a_1 & a_3 & a_5 & \dots \\ x^{n-2} & b_1 & b_2 & b_3 & \dots \\ x^{n-2} & c_1 & c_2 & c_3 & \dots \end{array}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \dots,$$

This can also be written as

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \dots,$$

Note that to find the rest of the rows, use a similar method. As an additional example, c_1 and c_2 would take the form of

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1}, \quad c_2 = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1}, \dots,$$

Definition 1 *If all the roots of a polynomial $p(x)$ have negative real parts, we shall say that the polynomial is* ***stable*** *(sometimes the term* ***Hurwitz*** *is used.)*

Definition 2 *The* ***Pivot*** *column is the first column to the right of the x's.*

Elements in the pivot columns we call "pivot elements." Now we will state without proof the following proposition.

Proposition 1 *The polynomial $p(x)$ is stable if and only if the first pivot element of the Routh-Hurwitz Table is non-zero and all of the elements have the same sign.*

We now pause to examine some examples.

Example 1 *Create the Routh Table for the Cubic Polynomial $p(x) = a_0x^3 + a_1x^2 + a_2x + a_3$.*

To create the Routh Table, start as before:

$$\begin{array}{c|ccc} x^3 & a_0 & a_2 & 0 \\ x^2 & a_1 & a_3 & 0 \\ x^1 & & & \\ x^0 & & & \end{array}$$

Calculating b_1 and b_2 we get,

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & 0 \\ a_1 & 0 \end{vmatrix}}{a_1}.$$

Note that $b_2 = 0$, as is b_3. The third row of the Routh Table then becomes:

$$\begin{array}{c|ccc} x^3 & a_0 & a_2 & 0 \\ x^2 & a_1 & a_3 & 0 \\ x^1 & b_1 & 0 & 0 \\ x^0 & & & \end{array}$$

Calculating c_1 and c_2,

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & 0 \end{vmatrix}}{b_1}, \quad c_2 = \frac{-\begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1}$$

Note that $c_2 = 0$, the last row of Routh Table becomes:

$$\begin{array}{c|ccc} x^3 & a_0 & a_2 & 0 \\ x^2 & a_1 & a_3 & 0 \\ x^1 & b_1 & 0 & 0 \\ x^0 & c_1 & 0 & 0 \end{array}$$

Now by Proposition 1, we know that if a_0, a_1, b_1, and c_1 all have the same sign and $a_0 \neq 0$, our polynomial will have real negative parts.

Example 2 *Create a Routh Table for the polynomial* $p(x) = x^3 + 9x^2 + 26x + 24$.

Create the Routh Table.

$$\begin{array}{c|ccc} x^3 & 1 & 26 & 0 \\ x^2 & 9 & 24 & 0 \\ x^1 & \frac{70}{3} & 0 & 0 \\ x^0 & 24 & 0 & 0 \end{array}$$

Note that all the coefficients in the first column all have the same sign. Therefore we know that all the roots of $p(x)$ have negative real parts. When we factor $p(x)$, it becomes $p(x) = (x + 2)(x + 3)(x + 4)$. It can then easily be seen that the roots are $x = -2, -3$, and -4, all of which have negative real parts. This is illustrated in Figure 6.

Example 3 *Determine if the roots of* $p(x)$ *all have negative real parts where* $p(x) = x^3 + 5x^2 - 2x - 24$.

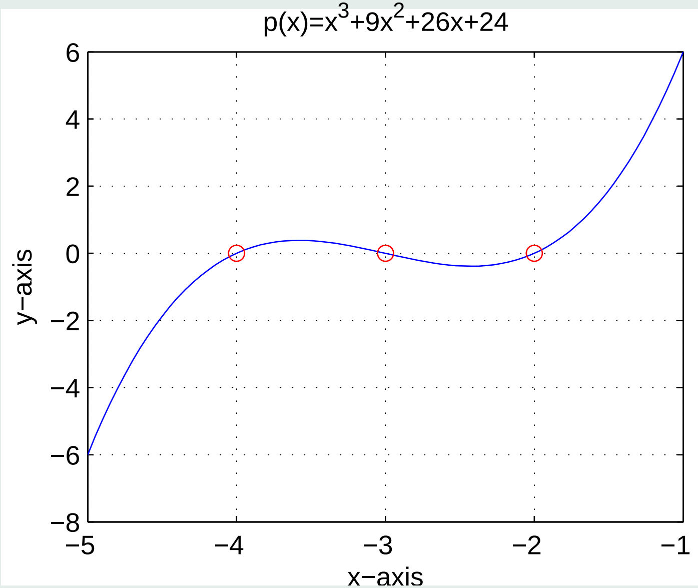

Figure 6: All the roots are negative and real, $x = -2, -3, -4$.

Creating Routh table we get:

$$\begin{array}{c|ccc} x^3 & 1 & -2 & 0 \\ x^2 & 5 & -24 & 0 \\ x^1 & \frac{14}{5} & 0 & 0 \\ x^0 & -24 & 0 & 0 \end{array}$$

Note that all the coefficients in the first column do not have the same sign. Therefore the hypothesis of Proposition 1 is not satisfied and it is not necessary that all roots have negative real parts. Factoring, $p(x) = (x-2)(x+3)(x+4)$ and it can be seen that $p(x)$ has roots of $x = 2, -3$, and -4. Indeed not all the roots have negative real parts. This is illustrated in Figure 7.

Example 4 *Create a Routh Table for $p(x) = x^3 + 7x^2 + 15x + 25$ and analyze the nature of the roots.*

Making a Routh table we get,

$$\begin{array}{c|ccc} x^3 & 1 & 15 & 0 \\ x^2 & 7 & 25 & 0 \\ x^1 & \frac{80}{7} & 0 & 0 \\ x^0 & 25 & 0 & 0 \end{array}$$

Note that all the values in the first column do have the same sign. This says that all the roots will have negative real parts. Factoring $p(x)$ we get $[x-(-1+2\imath)][x-(-1-2\imath)][x+5]$, which has complex roots $x = -1 + 2\imath$, $-1 - 2\imath$, and a real root of -5. Notice that all the roots have negative real parts. Figure 8 shows the roots in the real plane and since there is only one purely real root, that is all we see.

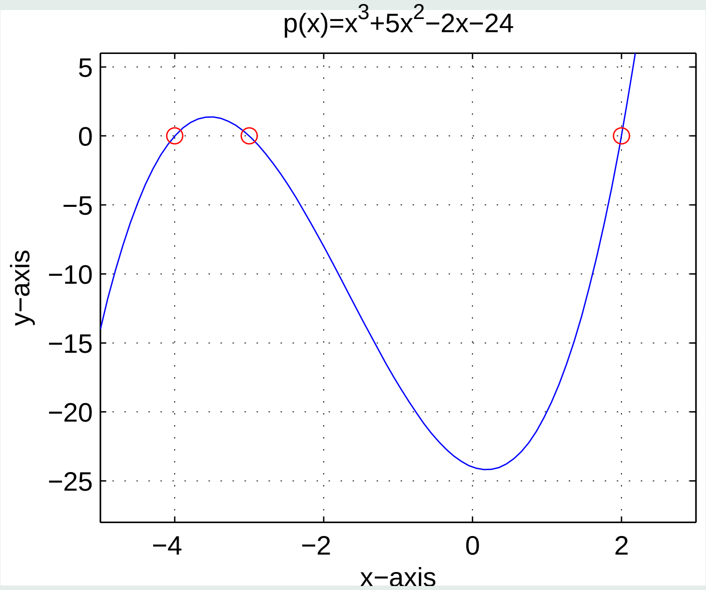

Figure 7: All the roots are real, but they are not all negative. There is one positive root at $x = 2$.

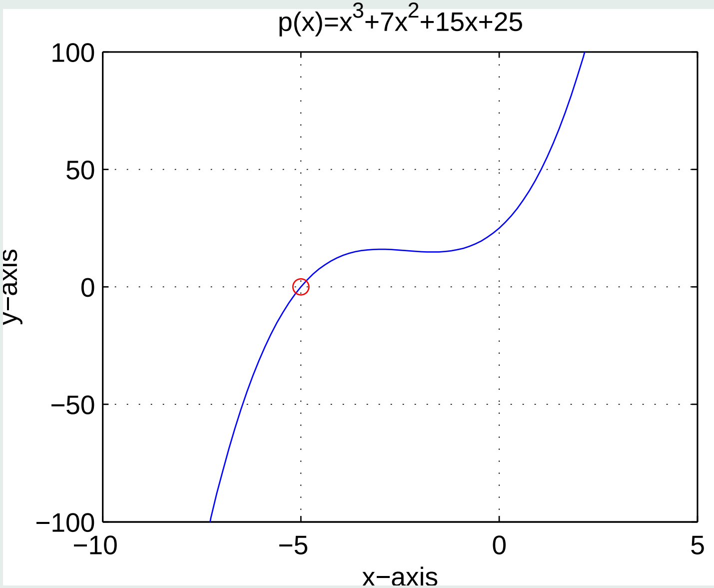

Figure 8: All the roots have negative real parts.

Introduction
Types of Electrical . . .
Low Pass Filter . . .
Steady State Solutions
Periodic Steady . . .
Circuit Analysis
Applying the . . .
The Routh-Hurwitz . . .
Analyzing our . . .
Analyzing Steady- . . .
Calculating the . . .
Gain In Relation to . . .
Analysis of Specific . . .
Conclusion
Home Page
Title Page
Page 17 of 31
Go Back
Full Screen
Close
Quit

9. Analyzing our Characteristic Polynomial

Now let us analyze our characteristic cubic polynomial,

$$p(\lambda) = \left[\lambda^3 + \frac{1}{RC_1}\lambda^2 + \frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\lambda + \frac{1}{LRC_1C_2}\right].$$

We begin by creating a Routh Table (Table 1). Looking at Table 1, all of the pivot coefficients are possitive. Therefore by the Routh-Hurwitz test, there are no unstable roots and all the eigenvalues have real, negative parts. We may then conclude by Theorem 1 that there is a unique steady-state solution for our specific circuit.

λ^3	1	$\frac{1}{L}\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$	0
λ^2	$\frac{1}{RC_1}$	$\frac{1}{LRC_1C_2}$	0
λ^1	$\frac{1}{LC_1}$	0	0
λ^0	$\frac{1}{LRC_1C_2}$	0	0

Table 1: Routh Test.

10. Analyzing Steady-State Output

Now that we have applied the Routh-Hurwitz test to our polynomial and discovered that it does indeed have a unique steady-state solution, we wish to examine the output in greater detail. We are most interested in the gain of the circuit. Let's begin by choosing an arbitrary sinusoidal input function of the form $V_1 = a_0 e^{\imath\omega t}$. With this choice of input signal,

$$\mathbf{F} = \begin{bmatrix} 0 \\ V_1'/R \\ V_1/L \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 \imath\omega/R \\ a_0/L \end{bmatrix} e^{\imath\omega t} = \boldsymbol{\alpha} e^{\imath\omega t}, \quad \text{where } \boldsymbol{\alpha} = \begin{bmatrix} 0 \\ a_0 \imath\omega/R \\ a_0/L \end{bmatrix}.$$

From our analysis using Theorem 1, we know that there will be a unique steady-state output, whose period, and therefore frequency, is identical to that of the input. We may then assume a steady-state output of the form $\mathbf{x}^{\mathbf{s}} = \boldsymbol{\beta} e^{\imath\omega t}$, which we substitute into Equation (5), which you will recall is $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$, to produce

$$\imath\omega e^{\imath\omega t} \boldsymbol{\beta} = e^{\imath\omega t} A\boldsymbol{\beta} + e^{\imath\omega t} \boldsymbol{\alpha}.$$

Dividing through by $e^{\imath\omega t}$,

$$\imath\omega\boldsymbol{\beta} = A\boldsymbol{\beta} + \boldsymbol{\alpha},$$

then manipulating,

$$[\imath\omega I - A]\boldsymbol{\beta} = \boldsymbol{\alpha}.$$

Now we solve for $\boldsymbol{\beta}$,

$$\boldsymbol{\beta} = [\imath\omega I - A]^{-1}\boldsymbol{\alpha}.$$

Recall that

$$A = \begin{bmatrix} 0 & 0 & 1/C_2 \\ 0 & -1/RC_1 & 1/RC_1 \\ -1/L & -R/L & 0 \end{bmatrix}.$$

So;

$$[\imath\omega I - A] = \begin{bmatrix} \imath\omega & 0 & -1/C_2 \\ 0 & \imath\omega + 1/RC_1 & -1/RC_1 \\ 1/L & R/L & \imath\omega \end{bmatrix}.$$

Now, to find $[\imath\omega I - A]^{-1}$, we could augment $[\imath\omega I - A]$ with the identity matrix and reduce into row echelon form, but this is too much work. Instead, we are going to use the fact that we can calculate the inverse of any matrix A with

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A),$$

where adj (A) denotes the adjoint of matrix A. Thus,

$$[\imath\omega I - A]^{-1} = \frac{1}{\det([\imath\omega I - A])} \operatorname{adj}([\imath\omega I - A]).$$

Recall that $p(\lambda) = \det[\lambda I - A]$, so $\det([\imath\omega I - A]) = p(\imath\omega)$. Thus we may now write

$$[\imath\omega I - A]^{-1} = \frac{1}{p(\imath\omega)} \operatorname{adj}([\imath\omega I - A]).$$

11. Calculating the Inverse Using the Adjoint

To find the adjoint of an arbitrary matrix, we replace each term with its cofactor, and then transpose the resultant matrix. For example, the adjoint of the matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

is

$$\operatorname{adj}(B) = \begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}^T.$$

We must pause here and recall that we are only interested in V_2, the output of the circuit. Looking back to Equation (5) we are reminded of the fact that

$$\mathbf{x}^{\mathbf{s}} = \begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix},$$

and because we are only interested in V_2 (not I_1 or I_2), we only need to expand the adjoint down the first column, for use in our calculation of $x^s = \boldsymbol{\beta} e^{\imath\omega t}$. For this reason, we have

$$\operatorname{adj}\left([\imath\omega I - A]\right) = \begin{bmatrix} \begin{vmatrix} (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ -\begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} & * & * \end{bmatrix}^T.$$

Now,

$$\boldsymbol{\beta} = [\imath\omega I - A]^{-1}\boldsymbol{\alpha} = \frac{1}{p(\imath\omega)} \begin{bmatrix} \begin{vmatrix} (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ -\begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} & * & * \end{bmatrix}^T \begin{bmatrix} 0 \\ a_0\imath\omega/R \\ a_0/L \end{bmatrix}.$$

Home Page
Title Page
◀◀ ▶▶
◀ ▶
Page 21 of 31
Go Back
Full Screen
Close
Quit

Next,

$$\mathbf{x}^{\mathbf{s}} = \boldsymbol{\beta} e^{\imath\omega t}$$

$$\begin{bmatrix} V_2 \\ I_1 \\ I_2 \end{bmatrix} = \frac{1}{p(\imath\omega)} \begin{bmatrix} \begin{vmatrix} (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ -\begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} & * & * \\ \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} & * & * \end{bmatrix}^T \begin{bmatrix} 0 \\ a_0\imath\omega/R \\ a_0/L \end{bmatrix} e^{\imath\omega t}$$

Because we are only interested in the output voltage V_2,

$$V_2 = \frac{1}{p(\imath\omega)} \left\{ 0 - \frac{a_o\imath\omega}{R} \begin{vmatrix} 0 & -\frac{1}{C_2} \\ \frac{R}{L} & \imath\omega \end{vmatrix} + \frac{a_o}{L} \begin{vmatrix} 0 & -\frac{1}{C_2} \\ (\imath\omega + \frac{1}{RC_1}) & -\frac{1}{RC_1} \end{vmatrix} \right\} e^{\imath\omega t},$$

and when the determinants are calculated,

$$V_2 = \frac{1}{p(\imath\omega)} \left\{ -\frac{a_o\imath\omega}{R} \left(\frac{R}{LC_2} \right) + \frac{a_o}{L} \left(\frac{\imath\omega}{C_2} + \frac{1}{RC_1C_2} \right) \right\} e^{\imath\omega t},$$

then multiplying through

$$\frac{1}{p(\imath\omega)} \left\{ -\frac{a_o\imath\omega}{LC_2} + \frac{a_o\imath\omega}{LC_2} + \frac{a_o}{LRC_1C_2} \right\} e^{\imath\omega t}.$$

Finally, when the first two terms are cancelled, we get V_2, the output voltage,

$$V_2(t) = \frac{a_o}{p(\imath\omega)LRC_1C_2} e^{\imath\omega t}.$$

12. Gain In Relation to Frequency

Now that we have $V_2(t)$ as a function of ω, L, R, C_1 and C_2, we wish to find the gain of the circuit, let us compare $V_2(t)$ against $V_1(t)$ according to Equation (1). We must recall that we originally chose $V_1 = a_0 e^{\imath\omega t}$. So

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{\frac{a_0 e^{\imath\omega t}}{p(\imath\omega)LRC_1C_2}}{a_0 e^{\imath\omega t}}\right|,$$

and with some simplification

$$\left|\frac{V_2}{V_1}\right| = \frac{1}{|p(\imath\omega)LRC_1C_2|}. \tag{7}$$

Now, to examine the gain's response to frequency in more depth let us substitute the characteristic polynomial, Equation 6, into the gain, Equation (7) and reduce,

$$Gain = \left|-\left((\imath\omega)^3 + \frac{1}{RC_1}(\imath\omega)^2 + \frac{1}{L}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)(\imath\omega) + \frac{1}{LRC_1C_2}\right)(LRC_1C_2)\right|^{-1}$$

$$Gain = \left|1 - LC_2\omega^2 + \imath R\omega(C_1 + C_2 - \omega^2 LC_1C_2)\right|^{-1} \tag{8}$$

Because we are looking at the magnitude of a vector in the imaginary plane, we can rewrite Equation (8) as

$$Gain = \frac{1}{\sqrt{(1 - LC_2\omega^2)^2 + R^2\omega^2\left(C_1 + C_2 - \omega^2 LC_1C_2\right)^2}}.$$

In general, if the denominator of the gain is large, the value of the gain will be small, and if the denominator is small then the value of the gain will be large. Looking closely at what changes the value of the gain, we can see that if ω is large then the denominator will be large and therefore the gain will be small and if ω is small, then the denominator will be small and therefore the gain will be large. This is why we call this circuit a Low-Pass filter. The low frequencies are passed relatively unchanged while the high frequencies are almost completely attenuated. However, because there is some ω^2 terms in the denominator, the maximum value of the gain usually does not occur at the lowest value of ω. In fact, there is usually a point where ω is amplified, depending on the values of the components. As ω approaches ∞, the gain goes to zero.

13. Analysis of Specific Component Sets

Exactly as we desired, we may analyze the output of our circuit in response to a sinusoidal input voltage of varying frequency. For example, when we choose component values of $R = 20$, $L = 1$, $C_1 = 0.002$, $C_2 = 0.004$, and an input source with $\omega = 200$, (let us assume from this point forward that we will use input signals of amplitude one) we are able to produce a plot of V_1 and V_2 versus time. The result is shown in Figure 9. It is obvious that the circuit is totally attenuating the input source of $\omega = 200$ (high frequency).

We can also observe how this same circuit behaves with different choices of V_1. When we plot gain versus ω over the interval $0 \leq \omega \leq 150$ we arrive at Figure 10. The result is of a form that intuitively makes sense from what we understand of low pass filters. The gain is ≈ 1 when the frequency is extremely low, however, as the frequency increases, the gain begins to drop, with the rate of change becoming extremely large when $\omega \approx 20$.

Next we choose a value for L that is a bit more reasonable. Let us set $R = 100$, $L = .001$, $C_1 = 0.6$, $C_2 = 0.3$, and $\omega = 70.71$. Which provides some extremely interesting results. The circuit behaves like a band-pass filter, where a very small frequency range is actually amplified, this occurs at $\omega \approx 70.71$. This response of the output voltage V_2 to the input voltage V_1 is shown in Figure 11. Signals with frequencies in the ranges

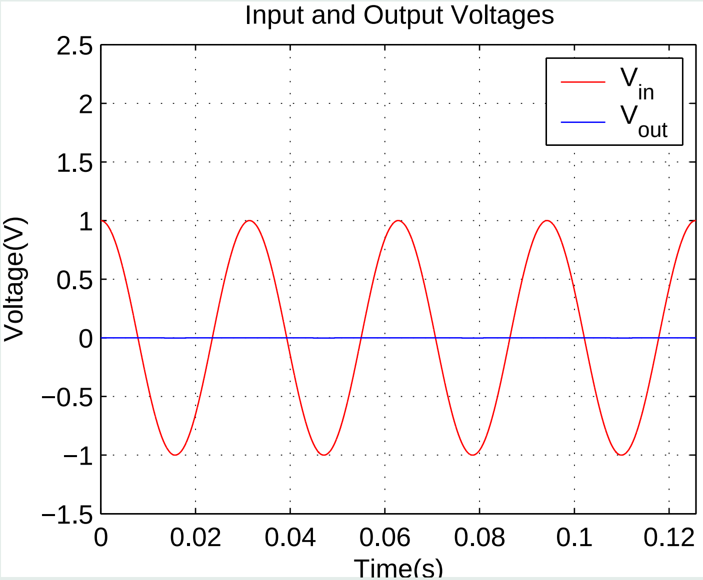

Figure 9: Plot of V_1 and V_2 versus time for $R = 20$, $L = 1$, $C_1 = 0.002$, $C_2 = 0.004$, and $\omega = 200$.

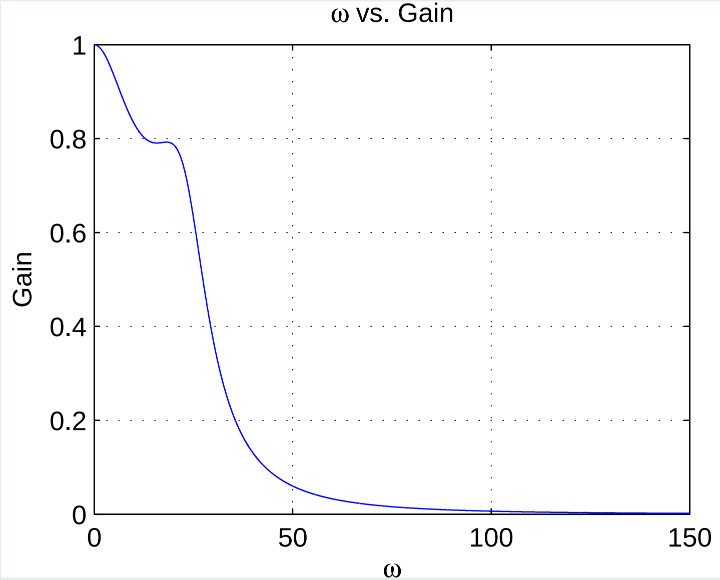

Figure 10: Gain as a function of ω for $R = 20$, $L = 1$, $C_1 = 0.002$, $C_2 = 0.004$, and $\omega = 200$.

$10 \leq \omega \leq 60$ and $80 \leq \omega \leq \infty$ passed to the circuit experience a gain of nearly zero. This result is observed in Figure 12.

An even more interesting example occurs with some additional changes in component values. Let us set $R = 1$, $L = .1$, $C_1 = 0.004$, $C_2 = 0.004$. The circuit behaves mostly as expected (see Figure 13) , signals with frequencies over the range $0 \leq \omega \leq 70$ experience a gain of one or greater, and there is some very obvious amplification exhibited, with the peak occuring at $\omega \approx 50.5$, this maximum gain being about 5. The response of the output voltage V_2 to the input voltage V_1 at $\omega \approx 50.5$ is shown in Figure 14.

14. Conclusion

In summary, we have constructed a model of a low pass filter circuit, which enables us to observe it's behavior when presented with a sinusoidal input signal. We are now able to adjust component values and predict the effects this will have upon our output signal. In addition, the methods used to predict a steady state output, including the use of the Routh-Hurwitz test, have been thoroughly examined. As well the in-depth examination of circuit gain has been described. It should be possible now, from viewing of this document, and careful examination of the example presented, for others to provide the same sort of circuit analysis of an electrical filter.

References

[1] Steven J. Leon. *Linear Algebra With Applications.*

[2] Robert L. Borrelli and Courtney S. Coleman. *Differential Equations A Modeling Perspective.*

[3] Polking, Boggess, Arnold. *Differential Equations With Boundry Value Problems.*

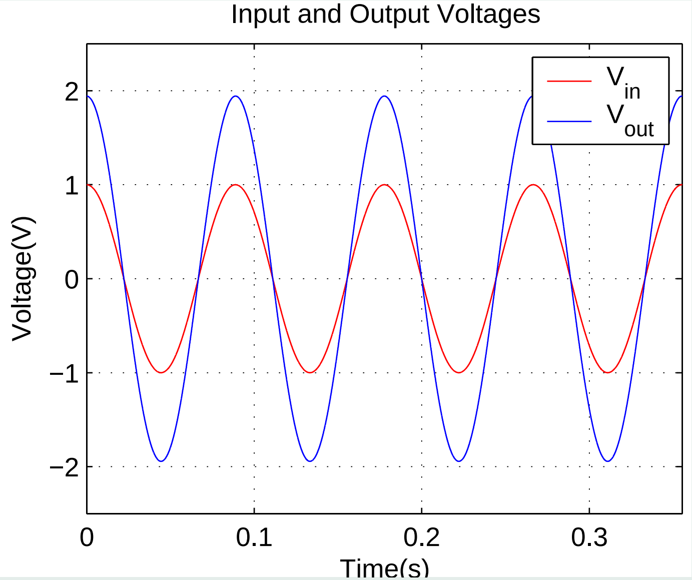

Figure 11: Plot of V_1 and V_2 versus time for $R = 100$, $L = .001$, $C_1 = 0.6$, $C_2 = 0.3$, and $\omega = 70.71$.

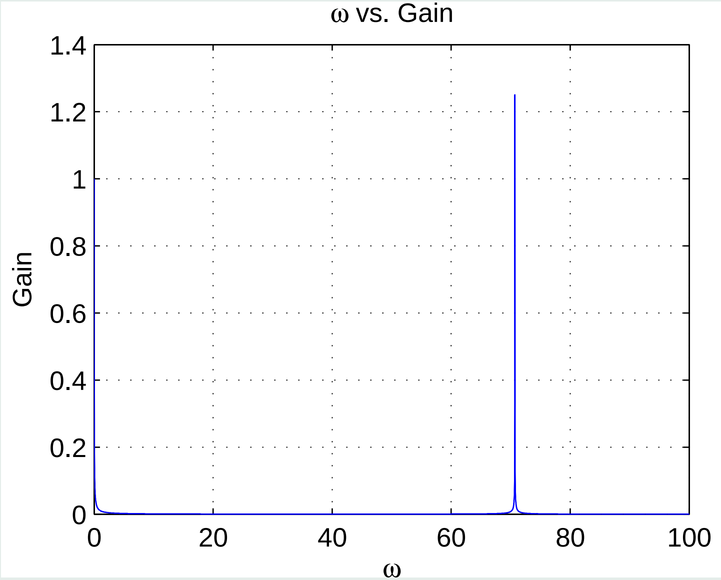

Figure 12: Gain as a function of ω, for $R = 100$, $L = .001$, $C_1 = 0.6$, $C_2 = 0.3$, and $\omega = 70.71$.

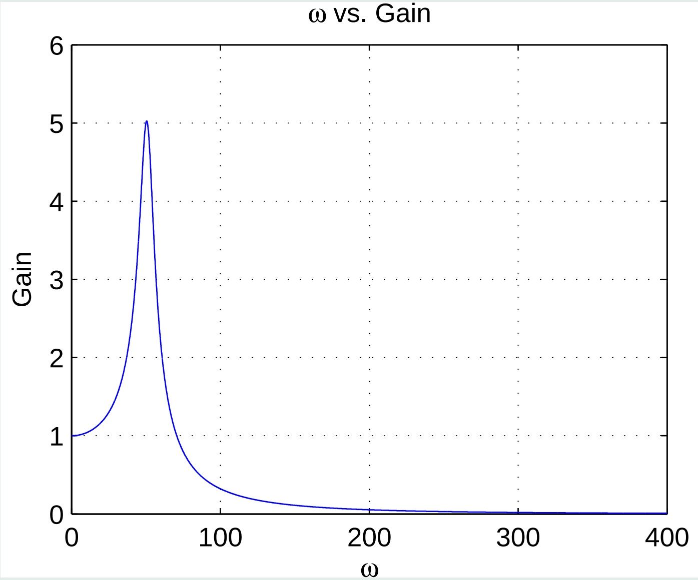

Figure 13: Gain as a function of ω, for $R = 1$, $L = .1$, $C_1 = 0.004$, $C_2 = 0.004$.

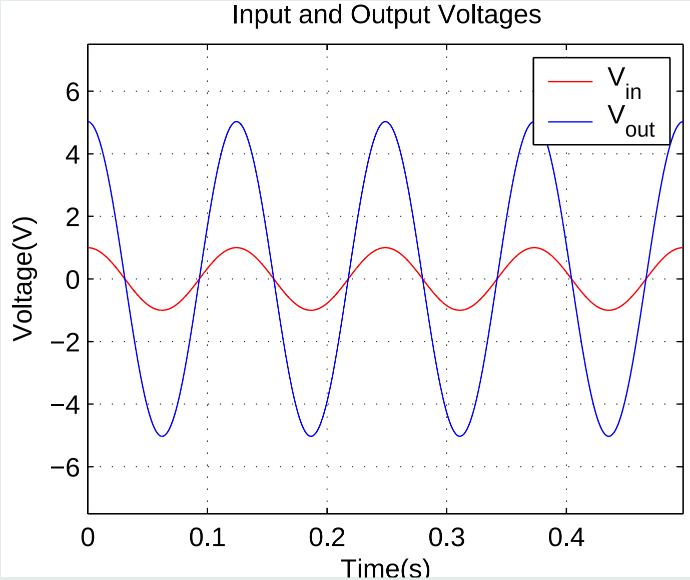

Figure 14: Plot of V_1 and V_2 versus time for $R = 1$, $L = .1$, $C_1 = 0.004$, $C_2 = 0.004$ and $\omega = 50.5$.