2.3 Project
Numerical εδ-Limit Investigations

Figure 2.2.23 in the text shows a steadily rising graph $y = f(x)$ that passes through the point (a, L). Given a single numerical value $\varepsilon > 0$, we can illustrate the limit $\lim_{x \to a} f(x) = L$ by solving the equations $f(x) = L \pm \varepsilon$ graphically or numerically for the indicated values x_1 to the left of a such that $f(x_1) = L - \varepsilon$, and x_2 to the right of a such that $f(x_2) = L + \varepsilon$. If $\delta > 0$ is chosen smaller than either of the two indicated distances $\delta_1 = a - x_1$ and $\delta_2 = x_2 - a$, then the figure suggests that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |f(x) - L| < \varepsilon. \tag{1}$$

You should understand that an actual *proof* that $\lim_{x \to a} f(x) = L$ must show that, given *any* $\varepsilon > 0$ whatsoever, there exists a $\delta > 0$ that works for *this* ε — meaning that the implication in (1) holds.

Doing it for a single value of ε does not constitute a proof, but doing it for several successively smaller values of ε can be instructive and perhaps convincing. Suppose, for instance that

$$f(x) = x^3 + 5x^2 + 10x + 98, \qquad a = 3, \qquad \text{and} \qquad L = 200.$$

In the sections below we illustrate the use of a calculator or computer algebra system to solve the equations $x^3 + 5x^2 + 10x + 98 = 200 - \varepsilon$ and $x^3 + 5x^2 + 10x + 98 = 200 + \varepsilon$ numerically for the solutions x_1 and x_2 near 3. With $\varepsilon = 1$, $\varepsilon = 0.2$, and $\varepsilon = 0.04$ you yourself should complete the following table of results:

ε	x_1	x_2	δ_1	δ_2	δ
1	2.98503	3.01488	0.01497	0.01488	0.01
0.2	2.99701	3.00298	0.00299	0.00298	0.002
0.04	2.99940	3.00060	0.00060	0.00060	0.0005

In the final column, each δ-value is (for safety) chosen a bit smaller than either δ_1 or δ_2 to be sure that it works with the corresponding ε-value.

Then pick a (perhaps fairly exotic) limit of you own to investigate numerically in this manner. Continue until you feel certain that — given any $\varepsilon > 0$, you can find a value $\delta > 0$ that works for *this* ε.

Using a Graphing Calculator

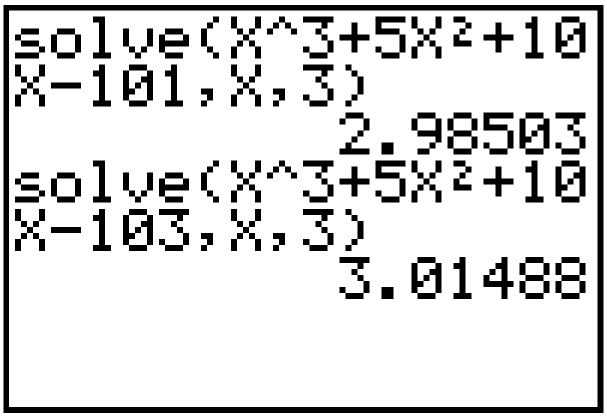
Starting with $\varepsilon = 1$, we use the TI-83 catalogue **solve** function to solve each of the equations

$$x^3 + 5x^2 + 10x + 98 = 199 \qquad \text{and} \qquad x^3 + 5x^2 + 10x + 98 = 201,$$

that is,

$$x^3 + 5x^2 + 10x - 101 = 0 \qquad \text{and} \qquad x^3 + 5x^2 + 10x - 103 = 0,$$

for its solution near $x = 3$.



Thus we get $x_1 \approx 2.98503$ and $x_2 = 3.01488$, so $\delta_1 = 3 - x_1 \approx 0.01497$ and $\delta_2 = x_2 - 3 \approx 0.01488$. Thus we see that the value $\delta = 0.01$, being less than either δ_1 or δ_2, works for $\varepsilon = 1$.

Using Maple

Starting with $\varepsilon = 1$, we use Maple's **`fsolve`** function to solve the equations

$$x^3 + 5x^2 + 10x + 98 \;=\; 199 \qquad \text{and} \qquad x^3 + 5x^2 + 10x + 98 \;=\; 201$$

for their solutions to the left of and to the right of $a = 3$.

```
f := x^3 + 5*x^2 + 10*x +98:

fsolve( f = 199, x, x=2..3 );
```

2.985027836

```
fsolve( f = 201, x, x=3..4 );
```

3.014879064

Thus we get $x_1 \approx 2.98503$ and $x_2 = 3.01488$, so $\delta_1 = 3 - x_1 \approx 0.01497$ and $\delta_2 = x_2 - 3 \approx 0.01488$. Thus we see that the value $\delta = 0.01$, being less than either δ_1 or δ_2, works for $\varepsilon = 1$.

Using Mathematica

Starting with $\varepsilon = 1$, we use Mathematica's **`FindRoot`** function to solve the equations

$$x^3 + 5x^2 + 10x + 98 = 199 \qquad \text{and} \qquad x^3 + 5x^2 + 10x + 98 = 201$$

for their solutions near $a = 3$.

```
f = x^3 + 5 x^2 + 10 x +98;

FindRoot[ f == 199, {x,3} ]

{x -> 2.98503}

FindRoot[ f == 201, {x,3} ]

{x -> 3.01488}
```

Thus we get $x_1 \approx 2.98503$ and $x_2 = 3.01488$, so $\delta_1 = 3 - x_1 \approx 0.01497$ and $\delta_2 = x_2 - 3 \approx 0.01488$. Thus we see that the value $\delta = 0.01$, being less than either δ_1 or δ_2, works for $\varepsilon = 1$.

Using MATLAB

Starting with $\varepsilon = 1$, we use MATLAB's **`fsolve`** function to solve each of the equations

$$x^3 + 5x^2 + 10x + 98 = 199 \qquad \text{and} \qquad x^3 + 5x^2 + 10x + 98 = 201,$$

that is,

$$x^3 + 5x^2 + 10x - 101 = 0 \qquad \text{and} \qquad x^3 + 5x^2 + 10x - 103 = 0,$$

for its solution near $x = 3$.

```
fsolve('x^3+5*x^2+10*x-101',3)
ans =
    2.9850

fsolve('x^3+5*x^2+10*x-103',3)
ans =
    3.0149
```

Thus we get $x_1 \approx 2.9850$ and $x_2 = 3.0149$, so $\delta_1 = 3 - x_1 \approx 0.0150$ and $\delta_2 = x_2 - 3 \approx 0.0149$. Thus we see that the value $\delta = 0.01$, being less than either δ_1 or δ_2, works for $\varepsilon = 1$.