The Ovals of Cassini

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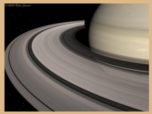

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Introduction

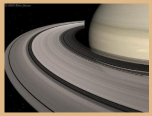
We will be looking at a curve that was developed by Giovanni Domenico Cassini in 1680. He had believed that the motion of the Earth and Sun followed a lobe of one of these ovals. If we are given two fixed points F_1 and F_2 and a constant c the Ovals of Cassini are defined as the locus of points that are $|PF_1| \cdot |PF_2| = c$. This is given to us in Cartesian coordinates as:

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = b^4$$

Polar Form

This form can be quite cumbersome so we will convert the equation to one in polar coordinates.

$$\text{Let } x = r\cos\theta \text{ and } y = r\sin\theta \text{ Then,}$$



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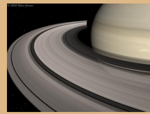
$$
\begin{aligned}
[(x-a)^2+y^2][(x+a)^2+y^2] &= b^4 \\
[(r\cos\theta-a)^2+(r\sin\theta)^2][(r\cos\theta+a)^2+(r\sin\theta)^2] &= \\
[r^2\cos^2\theta-2ar\cos\theta+a^2+r^2\sin^2\theta][r^2\cos^2\theta+2ar\cos\theta+a^2+r^2\sin^2\theta] &= \\
r^4\sin^4\theta+r^4\cos^4\theta-2a^2r^2\cos^2\theta+2a^2r^2\sin^2\theta+2r^4\sin^2\theta\cos^2\theta+a^4 &= \\
r^4\sin^4\theta+r^4\cos^4\theta-2a^2r^2(\cos^2\theta-\sin^2\theta)+2r^4\sin^2\theta\cos^2\theta+a^4 &= \\
r^4\sin^4\theta+r^4\cos^4\theta-2a^2r^2\cos 2\theta+2r^4\sin^2\theta\cos^2\theta+a^4 &= \\
r^4(\sin^4\theta+2\sin^2\theta\cos^2\theta+\cos^4\theta)-2a^2r^2\cos 2\theta+a^4 &= \\
r^4(\sin^2\theta+\cos^2\theta)^2-2a^2r^2\cos 2\theta+a^4 &= \\
r^4-2a^2r^2\cos 2\theta+a^4 &=
\end{aligned}
$$

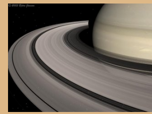
Parametric Form

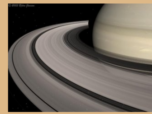
Now that we have our equation in polar coordinates we can try to get a parametric form as well. To do this we will solve for r using the quadratic formula.



$$
\begin{aligned}
r^2 &= \frac{2a^2\cos 2\theta \pm \sqrt{4a^4\cos^2 2\theta - 4(a^4 - b^4)}}{2} \\
&= a^2\cos 2\theta \pm \sqrt{a^4\cos^2 2\theta - a^4 + b^4} \\
&= a^2\cos 2\theta \pm \sqrt{a^4(\cos^2 2\theta - 1) + b^4} \\
&= a^2\cos 2\theta \pm \sqrt{-a^4(\sin^2 2\theta) + b^4} \\
&= a^2\left[\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}\right] \\
r &= \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}}
\end{aligned}
$$

Now substitute this r into our earlier $x = r\cos\theta$ and $y = r\sin\theta$,





$$x = \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}}\,[\cos\theta]$$

$$y = \pm a\sqrt{\cos 2\theta \pm \sqrt{\left(\frac{b}{a}\right)^4 - \sin^2 2\theta}}\,[\sin\theta]$$

While these may be our parametric equations they inherently have a problem, there will be times when we will receive complex numbers which are totally useless for us. To remedy this we will have to restrict θ so that the radicand will be greater than or equal to zero. To find this we begin with:

$$\left[\left(\frac{b}{a}\right)^4 - \sin^2 2\theta\right] = 0$$

$$\left(\frac{b}{a}\right)^2 = \sin 2\theta$$

$$t = \frac{1}{2}\arcsin\left(\frac{b}{a}\right)^2$$

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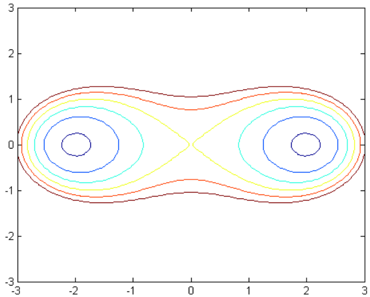
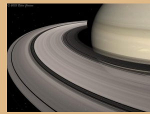
Therefore our θ values must run from $\frac{1}{2}\arcsin\left(\frac{b}{a}\right)^2$ to $-\frac{1}{2}\arcsin\left(\frac{b}{a}\right)^2$. Now we have several useful versions for the Ovals of Cassini.

Representations

The Cassinian Ovals exhibit peculiar behavior, at times becoming two separate curves! Let's take a look at this with *Matlab*®.



As you may have noticed the yellow curve here is the Lemniscate of Bernoulli, a special case of the Cassini Ovals equation. Our Cartesian equation allows a and b to be modified, where a is the distance the foci are placed from the origin and b^2 is our earlier stated constant c. Depending on the ratio of numbers chosen different behaviors are exhibited. If $b = a$



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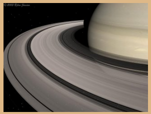
then one can conceptually come to the conclusion that the Lemniscate is the result, as the distance from the foci to a point P is the same as the distance a focus is from the origin resulting in a common meeting point at the origin. There are still two more ratios of b and a that need to be covered. If $\frac{b}{a} > 1$ one loop is created, but if $\frac{b}{a} < 1$ two separate curves result. One can imagine this as the constant from b is small enough in comparison to a that a point P can't be found at the origin as with the Lemniscate case but rather are within a distance $b < a$ from a focus, tracing out a region around each focus.

Experimenting With Variables

Once again emphasizing the Cartesian equation discussion, the only variables we can modify are a and b. We will start with a varying a, take note that these images are of the same curves as earlier, we only modify a single variable. This is demonstrated in the first four images.

Understandably in this case the foci merely separate from each other with increasing a values. We know modifying b will increase the constant of where the curve is defined at.



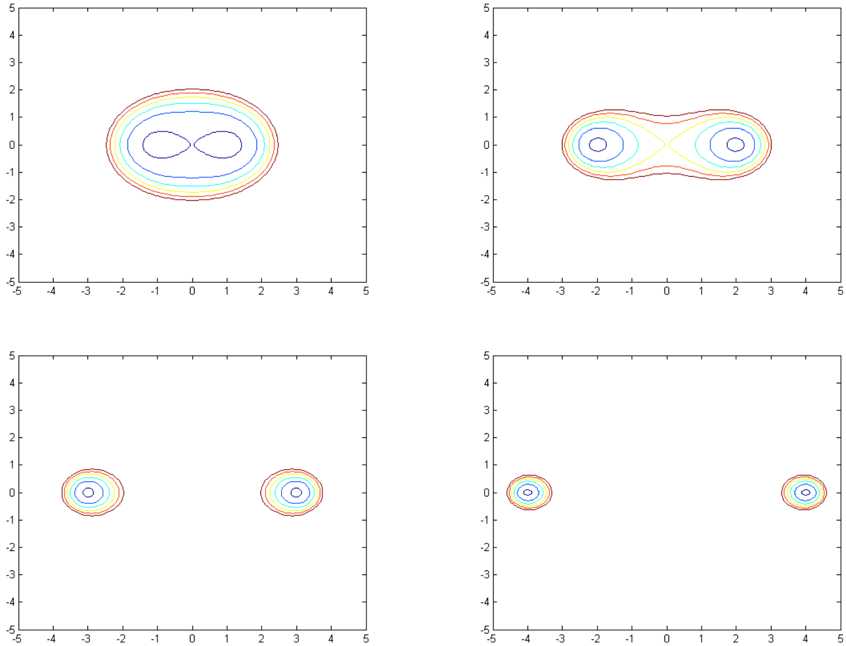
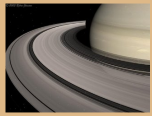


Figure 1: Varying a

Conclusion

While it has been established that planetary bodies do not actually travel on these curves but rather on elliptical paths it is still an interesting curve



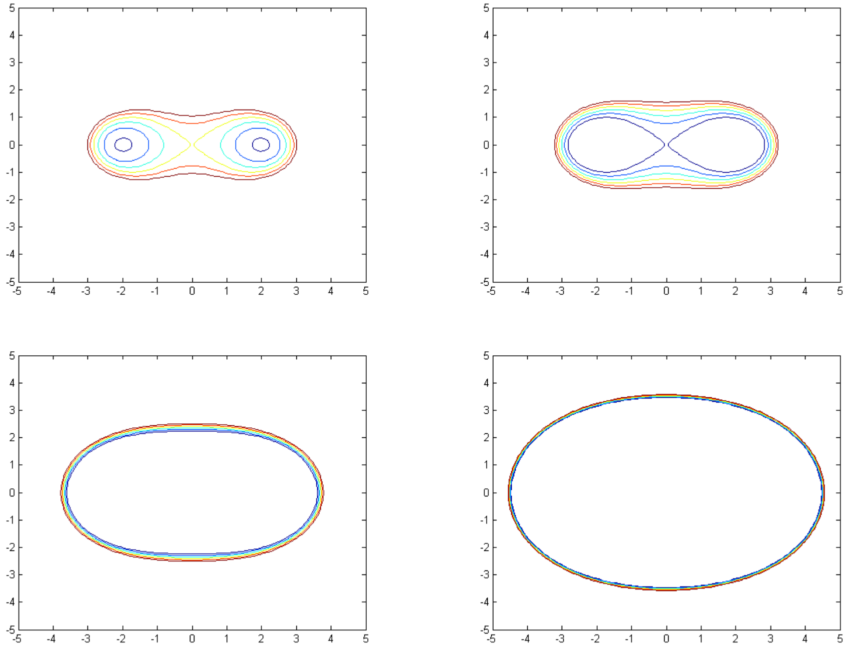
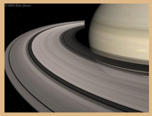
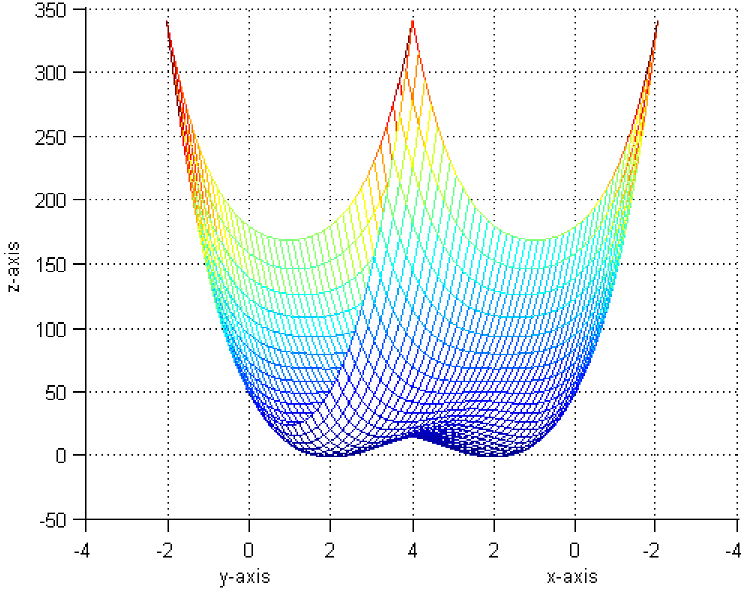


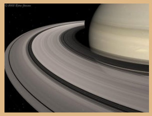
Figure 2: Varying b

to experiment with. For further consideration one can delve deeper into the curve and will undoubtedly discover that this curve is a toric section and



the three dimensional representation of the special case of the Lemniscate is the Möbius Strip. This will be an interesting thing for another time. For example I have included the Cassini Ovals plotted as a surface, the viewing angle is not standard but it emphasizes the features of the surface.





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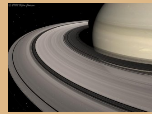

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