Buckling of a Yardstick and Boundary Value Problems

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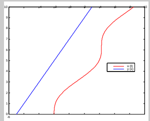
1. Introduction

The most commonly introduced problems to students beginning an exploration into Differential Equations are initial value problems. There are however scenarios were initial values are not enough to model the situation and one of these is something called a border value problem.

For instance, take a wooden yardstick and press it strait against the wall as illustrated in Figure 1. Increase the pressure on it being careful to keep applied pressure directed horizontally along the axis of the yardstick.

While initially nothing may appear to happen the yardstick is compressing in the axial direction. Imagine that the yardstick was replaced by a piece of foam rubber and the compression would be much more visible. If you continue to apply force however something interesting happens. At a certain instant the yardstick will bow slightly upward or downward and as we increase the force the yardstick bows more a behavior called buckling.



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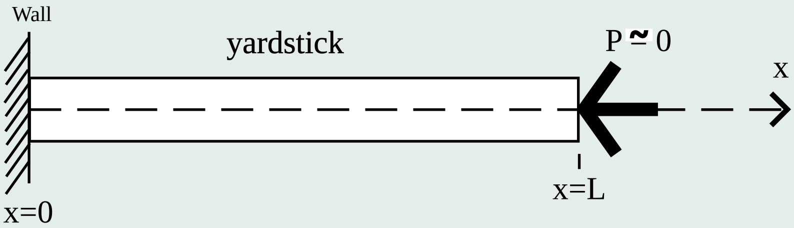


Figure 1: a yardstick prepped from pressing

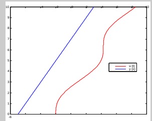
We begin to analyze the behavior of the yardstick by looking at the various forces that affect the it. The only forces are those from gravity and the pressure you exert on the yardstick with your hand (and the equal and opposite force exerted by the wall). If we compare the magnitude of the force you exert to make the yardstick bend with the force due to the weight of yardstick it seems reasonable to ignore gravity.

As a you apply increasing amounts of pressure, but not enough to cause buckling, the yardstick undergoes axial compression as illustrated in Figure 2 (the axial compression has been greatly exaggerated):

When p is low there is no buckling and only a small axial compression that will be omitted from further calculations. The illustrations will also be simplified by omitting the yardstick and only displaying the center line of the stick. After these simplification we can represent the problem as shown in Figure 3.

1.1. Radius of Curvature

We will need to gather some formulas from physics to assist us in modelling the behavior of the yard stick. The first is the radius of curvature of a plane which might (perhaps oversimplifying) be defined as the radius of a circle or sphere that would fit into the curve



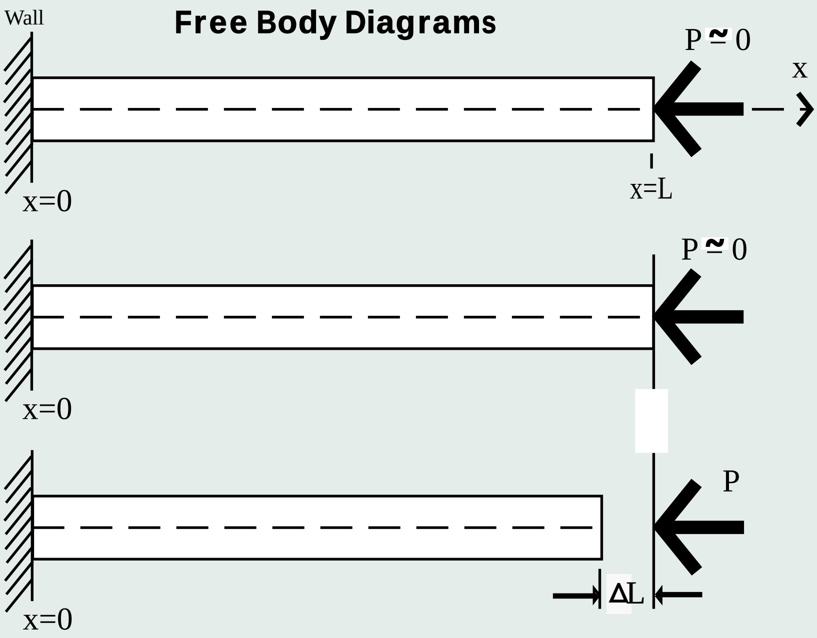
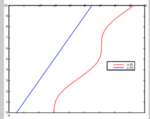


Figure 2: axial compression of yardstick



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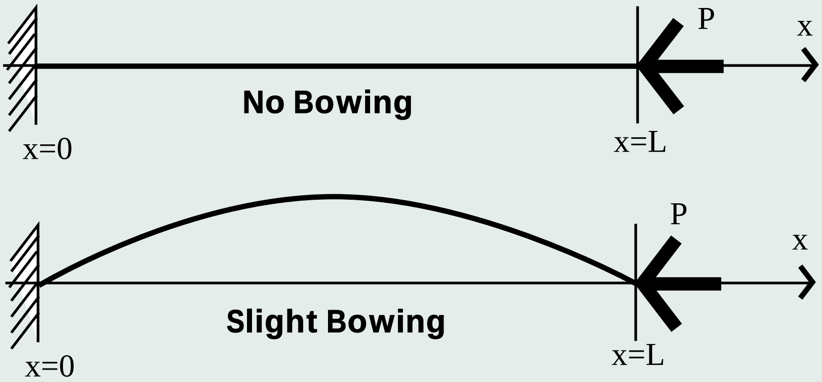
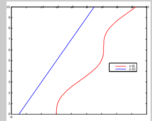


Figure 3: bowing of yardstick under pressure



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The formula below is the parameterization of the simpler form $R = 1/k$ where k is the curvature. The curvature (again perhaps over-simplified) is the measure of how much a curve bends at each point. It is an alternative method of describing the structure of a curve. (Don't worry about it, just know that it relates the bend in the stick with other factors).

If you have a plane curve (in the xy- plane) represented by the equation $y = u(x)$ the the radius of curvature is:

$$\frac{1}{r} = \frac{\dfrac{d^2u}{dx^2}}{\left[1 + \left(\dfrac{du}{dx}\right)^2\right]^{3/2}} \tag{1}$$

1.2. Torque / Moment / Moment of Force

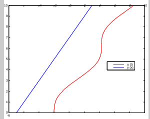
The term most familiar to most students is probably torque but all three terms mean the same thing. The formula here is expressed as a moment to better mesh with later formulas. The moment of a force about a point in a given plane is the product of the magnitude of the force and the lever arm from the line action of the force to the point. As illustrated in Figure 4, the moment M about the point $(x, u(x))$ is

$$M = -P\,u(x), \tag{2}$$

the minus sign that appears follows the convention that moments are positive if counter-clockwise.

1.3. Moment of Inertia

The moment of inertia can be though of as a rotational analogue to mass. It is the measurement of an object's resistance to change of angular acceleration around some rotational point due to the acceleration of a torque.



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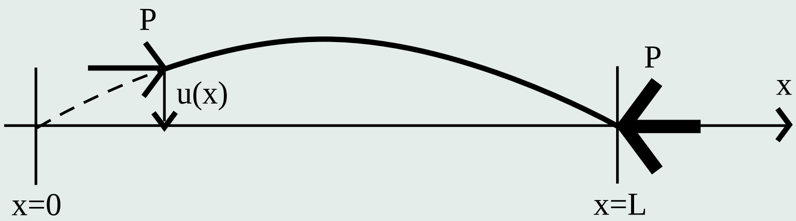


Figure 4: the more pressure the more buckle

The moment of inertia of a plane region R in the yz plane about the z axis is

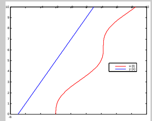
$$I = \iint_R y^2 \, dz \, dy,$$

or in the case of a rectangular region about its centerline (as shown in Figure 5), this works out to a moment of inertia that is

$$I = \int_{-b/2}^{b/2} \left\{ \int_{-b/2}^{b/2} y^2 \, dy \right\} ds = \frac{1}{12} b h^3. \tag{3}$$

1.4. Elementary Beam Theory

The *Bernoulli-Euler Law of Elementary Beam Theory* is an advanced theory detailing the relationship between the radius of the curvature and the moment. It does so by making the assumption that deformations due to transverse shear forces are small and can be neglected with respect to deformations by pure bending. Appropriate for slender beams. Knowing exactly what how it works isn't necessary for this exercise, its ability to relate



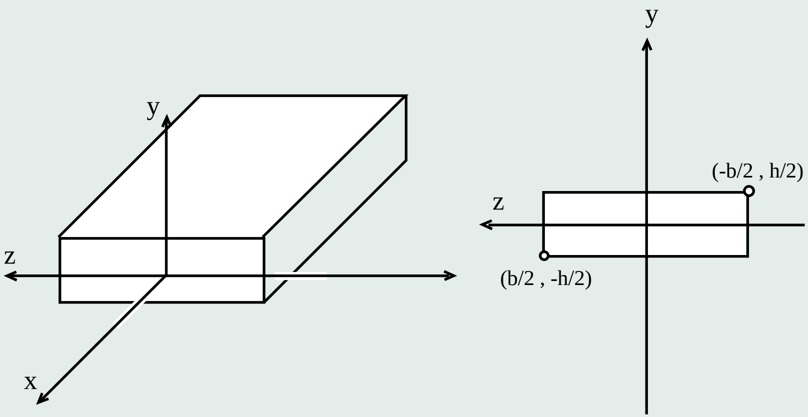
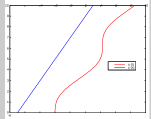


Figure 5: the blocks moment of inertia will resist rotation



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the (bending) moment of our yardstick to the radius of the curvature is all we need. The simplified formula for our circumstance is

$$M = \frac{E\,I}{r}, \tag{4}$$

where I is the moment of inertia of the cross-section, E is Young's modulus or modulus of elasticity of the material (wood in our specific case) and r is the radius of curvature.

The modulus of elasticity is defined as $E = \text{Stress/Strain}$ and, as can clearly be seen, it relates the stress to strain.. The value of E for Douglas fir is about 13×10^9 newtons per square meter (N/m^2). For comparison steel is about 200×10^9 (N/m^2).

2. Compilation of Different Elements

We can now solve for u by using the functions we have assembled so far, repeated here for emphasis.

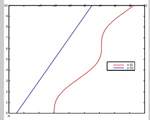
$$\frac{1}{r} = \frac{\dfrac{d^2u}{dx^2}}{\left[1 + \left(\dfrac{du}{dx}\right)^2\right]^{3/2}} \tag{5}$$

$$M = -P\,u(x) \tag{6}$$

$$M = \frac{E\,I}{r} \tag{7}$$

Substituting equation (6) into equation (7) yields

$$-P\,u = \frac{E\,I}{r},$$



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and substituting equation (5) for $1/r$,

$$-P\,u = E\,I\,\frac{\dfrac{d^2u}{dx^2}}{\left[1+\left(\dfrac{du}{dx}\right)^2\right]^{3/2}}.$$

Dividing both sides by $E\,I$,

$$-\frac{P}{E\,I}\,u = \frac{\dfrac{d^2u}{dx^2}}{\left[1+\left(\dfrac{du}{dx}\right)^2\right]^{3/2}},$$

then multiplying both sides by

$$\left[1+\left(\frac{du}{dx}\right)^2\right]^{3/2},$$

we get

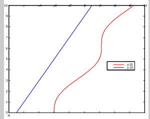
$$-\frac{P}{E\,I}\,u\left[1+\left(\frac{du}{dx}\right)^2\right]^{3/2} = \frac{d^2u}{dx^2}.$$

Subtracting

$$-\frac{P}{E\,I}\,u\left[1+\left(\frac{du}{dx}\right)^2\right]^{3/2}$$

from both sides results in the nonlinear second order DE,

$$\frac{d^2u}{dx^2} + \frac{P}{E\,I}\,u\left[1+\left(\frac{du}{dx}\right)^2\right]^{3/2} = 0. \tag{8}$$



For slight bowing, where du/dx is small, the DE simplifies to

$$\frac{d^2u}{dx^2} + \frac{P}{E\,I}\,u = 0. \tag{9}$$

This step is perhaps the most worrisome in the entire process regarding the accuracy of the final outcome as in order to understand the accuracy of the simplification it would be necessary to evaluate the second order non-linear differential equation which is beyond the scope of this paper. That is not the only cause for concern however, we are evaluating the behavior on an interval and therefore need side conditions or subsidiary conditions to further specify the process.

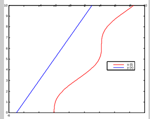
The first is rather obvious at$u(0) = 0$ since there is no vertical motion at the wall. And our second condition $u(L) = 0$ is true when the bar is in its strait position or if we neglect the slight compression.

These conditions are the side conditions that we have been looking for. They are not the initial conditions that one often sees in the study of second order DEs.

$$u(0) = u_o, \frac{du}{dt}(0) = u_1$$

In the case of the IVP the side conditions are two initial conditions (ICs) are specified at a single point $t = 0$. In the case of the BVP the side conditions are two boundary conditions (BCs), one specified at $x = 0$ the other at a different point $x = L$. These conditions are specified on the unknown function u at two different points. These conditions are specified at the boundary of the region we are considering (the endpoints of the interval $(0, L)$. We refer to them as boundary conditions and sum up the problem as a boundary value problem.

$$(BVP)\begin{cases} (DE)\dfrac{d^2u}{dx^2} + \lambda u = 0, \quad 0 < x < L, \\ (BC)\begin{cases} u(0) = 0 \\ u(L) = 0 \end{cases} \end{cases} \tag{10}$$



Further,

$$\lambda = \frac{P}{EI}.$$

3. Boundary Value Problems

As mentioned before the difference between Boundary Value Problems (BVP) and Initial Value Problems (IVP) is that the two initial conditions (ICs) are specified at a single point $t = 0$. In the BVP the Boundary Conditions (BC) are specified at two different points. The implications of this might not seem immediately apparent but they are important. Take these two problems,

$$(BVP) \begin{cases} (DE) y'' + y = x, \qquad 0 < x < L, \\ (BC) \begin{cases} y(0) = 0, \\ y(L) = 0, \end{cases} \end{cases} \tag{11}$$

and

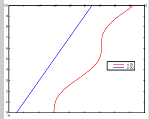
$$(BVP) \begin{cases} (DE) u'' + u = t, \qquad t > 0, \\ (IC) \begin{cases} u(0) = 0, \\ \dot{u}(L) = 0. \end{cases} \end{cases} \tag{12}$$

Both these systems have a general solution of the form

$$g(x) = -c_1 \sin(x) + c_2 \cos(x) + x$$

but the differences in the ways c_1 and c_2 are calculated lead to rather different results as will be seen in the next section. The general solution of the IVP is

$$u(t) = t - \sin(t)$$



and the solution of the BVP is

$$y(t) = x - = \left(\frac{L}{\sin(L)}\right)\sin(x)$$

and while that particular example of a solution to the BVP contains a single solution, BVPs can contain more than one solution as demonstrated in the solution of the Yardstick BVP

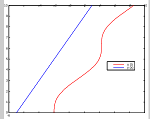
The differences between these solutions can clearly be seen when graphed in Figure 6.

```
diff.m
t=linspace(0,10,1000);
L=5;
u=t-sin(t);
y=x-(L./(sin(t))).*sin(t);
plot(u,t,'r',y,x,'b')
legend('u(t)','y(x)')
```

4. Solution of the Yardstick BVP

Looking at our function we can see that this BVP is homogeneous. The linear DE is homogeneous and the BCs are homogeneous. An IVP involving a linear homogeneous DE with homogeneous ICs has exactly one solution (for continuous coefficients for the DE written in normal form) and we might expect that the BVP will also only have one solution under these conditions.

$$(BVP)\begin{cases}(DE)\dfrac{d^2u}{dx^2} + \lambda u = 0, \quad 0 < x < L, \\ (BC)\begin{cases} u(0) = 0 \\ u(L) = 0 \end{cases}\end{cases} \tag{13}$$



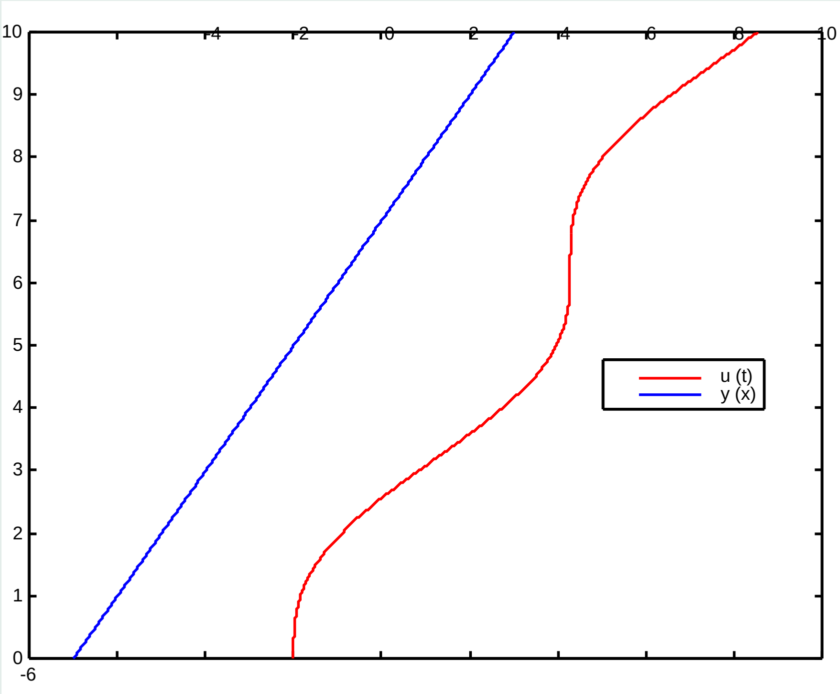
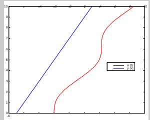


Figure 6: Notice the instability of the $u(x)$ line.



Looking at our function again we see that we need to find values of λ for which there exists a nontrivial solution (by finding λs that work we are also finding Ps that work because P is divided by constants). We can do this the same way we find solutions to IVPs. We get a general solution from the DE and then plug in the BCs.

λ is positive since P, E, and I are positive. Therefore the general solution of the BVP can be written, with $v = \sqrt{\lambda}$ for convenience,

$$u(x) = c_1 \cos vx + c_2 \sin vx. \tag{14}$$

Now, if the BC at $x = 0$ is 0,

$$0 = u(0) = c_1$$

and $c_1 = 0$. Then evaluating for the BC at $x = L$

$$0 = u(L) = c_2 \sin(v\,L).$$

If $c_2 = 0$, you would arrive at the trivial solution, so we can assume that $c_2 \neq 0$ and write

$$\sin(v\,L) = 0. \tag{15}$$

Which means that

$$v = \frac{n\,\pi}{L},\ n = 1, 2, 3, \ldots \tag{16}$$

Substituting $c_1 = 0$ and (15) with v as shown in (16) into the general form of the solution

$$u(x) = c_1\ \cos v\,x + c_2\ \sin v\,x$$

yields,

$$u(x) = 0\ \cos\left(\frac{n\,\pi}{L}\right) + c_2\ \sin\left(\frac{n\,\pi}{L}\right)$$

or

$$u(x) = c_2\ \sin\left(\frac{n\,\pi}{L}\right).$$

Because c_2 is arbitrary, we can set $c_2 = 1$. Then we can write that the non-trivial solutions of u_n are non-zero multiples of

$$\phi_n(x) = \sin\left(\frac{n\,\pi}{L}x\right). \tag{17}$$

We can then derive the values of P for which we are looking. First comparing (15) and (17) we can see that

$$v = \frac{n\,\pi}{L}.$$

Solving $v = \sqrt{\lambda}$ as set in (14) for λ yields

$$\lambda = \lambda_n = \left(\frac{n\,\pi}{L}\right)^2,\ n = 1, 2, 3, \ldots$$

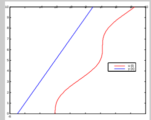
Solving $\lambda = (P/E\,I)$ as set in (10) for P yields

$$P = E\,I\,\lambda$$

or

$$P = P_n = E\,I\left(\frac{n\,\pi}{L}\right)^2,\ n = 1, 2, 3\ldots \tag{18}$$

Our eigenvalues are the values of λ for which there exist non-trivial solutions (which depends on P as is shown in the next section) and the corresponding ϕ_n (or any non-zero multiple of ϕ_n). We call ϕ_n an eigenfunction belonging to that eigenvalue. The eigenvalues and eigenfunctions play a tremendous role in the study of both homogeneous and non-homogeneous BVPs.



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5. Results & Analysis

Our solution of tells us something about the buckling of the yardstick. Specifically if the force satisfies the following relationship

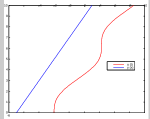
$$0 < P < E\,I\left(\frac{\pi}{L}\right)^2$$

then the stick will not buckle. We know this because the values of P that we have solved for in (18) have a minimum value of $E\,I\,(\pi/L)^2$ and those are the values for which a solution exists. Anything smaller than this is not a solution and will therefore not cause the yardstick to buckle. This is as we hoped, for small values of P the stick will not buckle until a certain force P is applied at which point it will. The model does predict however that the yardstick will buckle in the shape of a sine curve

$$u(x) = k_1 \sin\left(\frac{\pi}{L}x\right)$$

which does not seem to fit the experimental data and we are unable to determine the amplitude k_1 from equation (10).

We have met with some success. The model does correctly predict that for small compressive forces there will be no buckling and once a compressive force P was reached the yardstick would begin to buckle. These are qualitative results however and we know little about the amplitude of the buckle. Our model predicts a sine wave for the pattern of buckle which seems to defy the physical evidence. This is likely a result of simplifying the second order non-linear system. A comparison between the solution of that second order non-linear system and the simplified second order linear system would yield insight into the accuracy of the simple model but that is a topic beyond the scope of this paper.



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