## Application 7.5
# Defective Eigenvalues and Generalized Eigenvectors

The goal of this application is the solution of the linear systems like

$$\mathbf{x}' = \mathbf{A}\,\mathbf{x}, \tag{1}$$

where the coefficient matrix is the exotic 5-by-5 matrix

$$\mathbf{A} = \begin{bmatrix} -9 & 11 & -21 & 63 & -252 \\ 70 & -69 & 141 & -421 & 1684 \\ -575 & 575 & -1149 & 3451 & -13801 \\ 3891 & -3891 & 7782 & -23345 & 93365 \\ 1024 & -1024 & 2048 & -6144 & 24572 \end{bmatrix} \tag{2}$$

that is generated by the MATLAB command **`gallery(5)`**. What is so exotic about this particular matrix? Well, enter it in your calculator or computer system of choice, and then use appropriate commands to show that:

- First, the characteristic equation of $\mathbf{A}$ reduces to $\lambda^5 = 0$, so $\mathbf{A}$ has the single eigenvalue $\lambda = 0$ of multiplicity 5.
- Second, there is only a single eigenvector associated with this eigenvalue, which thus has defect 4.

To seek a chain of generalized eigenvectors, show that $\mathbf{A}^4 \neq \mathbf{0}$ but $\mathbf{A}^5 = \mathbf{0}$ (the $5 \times 5$ zero matrix). Hence *any* nonzero 5-vector $\mathbf{u}_1$ satisfies the equation

$$(\mathbf{A} - \lambda\mathbf{I})^5 \mathbf{u}_1 = A^5 \mathbf{u}_1 = \mathbf{0}.$$

Calculate the vectors $\mathbf{u}_2 = \mathbf{A}\mathbf{u}_1$, $\mathbf{u}_3 = \mathbf{A}\mathbf{u}_2$, $\mathbf{u}_4 = \mathbf{A}\mathbf{u}_3$, and $\mathbf{u}_5 = \mathbf{A}\mathbf{u}_4$ in turn. You should find that $\mathbf{u}_5$ is nonzero, and is therefore (to within a constant multiple) the unique eigenvector $\mathbf{v}$ of the matrix $\mathbf{A}$. But can this eigenvector $\mathbf{v}$ you find possibly be independent of your original choice of the starting vector $\mathbf{u}_1 \neq \mathbf{0}$? Investigate this question by repeating the process with several different choices of $\mathbf{u}_1$.

Finally, having found a length 5 chain $\{\mathbf{u}_5, \mathbf{u}_4, \mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_1\}$ of generalized eigenvectors based on the (ordinary) eigenvector $\mathbf{u}_5$ associated with the single eigenvalue $\lambda = 0$ of the matrix $\mathbf{A}$, write five linearly independent solutions of the 5-dimensional homogeneous linear system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$.

In the sections that follow we illustrate appropriate *Maple*, *Mathematica*, and MATLAB techniques to analyze the $4 \times 4$ matrix

$$\mathbf{A} = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} \tag{3}$$

of Problem 31 in Section 7.5 of the text. You can use any of the other problems there (especially Problems 23–30 and 32) to practice these techniques.

## Using *Maple*

First we enter the matrix in (3):

```
with(linalg):
A := matrix(4,4, [ 35, -12,  4,  30,
                   22,  -8,  3,  19,
                  -10,   3,  0,  -9,
                  -27,   9, -3, -23 ] ):
```

Then we explore its characteristic polynomial, eigenvalues, and eigenvectors:

```
charpoly(A,lambda);
```

$$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1$$

(that is, $(\lambda - 1)^4$ )

```
eigenvals(A);
```

$$1, 1, 1, 1$$

```
eigenvects(A);
```

$$[1, 4, \{[0\ \ 1\ \ 3\ \ 0], [-1\ \ 0\ \ 1\ \ 1]\}\,]$$

Thus *Maple* finds only the two independent eigenvectors

```
w1 := matrix(4,1, [ 0,  1,  3,  0]):
w2 := matrix(4,1, [-1,  0,  1,  1]):
```

associated with the multiplicity 4 eigenvalue $\lambda = 1$, which therefore has defect 2. To explore the situation we set up the $4 \times 4$ identity matrix and the matrix $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$:

```
Id := diag(1,1,1,1):
L = 1:
B := evalm( A - L*Id):
```

When we calculate $\mathbf{B}^2$ and $\mathbf{B}^3$,

```
B2 := evalm(B &* B);
B3 := evalm(B2 &* B);
```

we find that $\mathbf{B}^2 \neq 0$ but $\mathbf{B}^3 = 0$, so there should be a length 3 chain associated with $\lambda = 1$. Choosing

```
u1 := matrix(4,1,[1,0,0,0]);
```

we calculate the further generalized eigenvectors

```
u2 := evalm( B &* u1);
```

$$u2 := \begin{bmatrix} 34 \\ 22 \\ -10 \\ -27 \end{bmatrix}$$

and

```
u3 := evalm( B &* u2);
```

$$u3 := \begin{bmatrix} 42 \\ 7 \\ -21 \\ -42 \end{bmatrix}$$

Thus we have found the length 3 chain $\{\mathbf{u}_3,\ \mathbf{u}_2,\ \mathbf{u}_1\}$ based on the (ordinary) eigenvector $\mathbf{u}_3$. (To reconcile this result with *Maple*'s **`eigenvects`** calculation, you can check that $\mathbf{u}_3 + 42\mathbf{w}_2 = 7\mathbf{w}_1$.) Consequently four linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ are given by

$$\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{w}_1 e^t, \\
\mathbf{x}_2(t) &= \mathbf{u}_3 e^t, \\
\mathbf{x}_3(t) &= (\mathbf{u}_2 + \mathbf{u}_3 t)e^t, \\
\mathbf{x}_4(t) &= (\mathbf{u}_1 + \mathbf{u}_2 t + \tfrac{1}{2}\mathbf{u}_3 t^2)e^t.
\end{aligned}$$

## Using *Mathematica*

First we enter the matrix in (3):

```
A = { { 35, -12,  4,  30 },
      { 22,  -8,  3,  19 },
      {-10,   3,  0,  -9 },
      {-27,   9, -3, -23 } };
```

Then we explore its characteristic polynomial, eigenvalues, and eigenvectors:

```
CharacteristicPolynomial[A, r]
```

```
               2      3    4
1 - 4 r + 6 r  - 4 r  + r
```

(that is, $(r-1)^4$)

```
Eigenvalues[A]
```

```
{1, 1, 1, 1}
```

```
Eigenvectors[A]
```

```
{{-3,-1,0,3}, {0,1,3,0}, {0,0,0,0}, {0,0,0,0}}
```

Thus *Mathematica* finds only the two independent (nonzero) eigenvectors

```
w1 = {-3,-1, 0, 3};
w2 = { 0, 1, 3, 0};
```

associated with the multiplicity 4 eigenvalue $\lambda = 1$, which therefore has defect 2. To explore the situation we set up the $4\times 4$ identity matrix and the matrix $\mathbf{B} = \mathbf{A} - \lambda\mathbf{I}$:

```
Id = DiagonalMatrix[1,1,1,1];
L = 1;
B = A - L*Id;
```

When we calculate $\mathbf{B}^2$ and $\mathbf{B}^3$,

```
B2 = B.B
B3 = B2.B
```

we find that $\mathbf{B}^2 \neq 0$ but $\mathbf{B}^3 = 0$, so there should be a length 3 chain associated with $\lambda = 1$. Choosing

```
u1 = {{1},{0},{0},{0}}
```

we calculate

```
u2 = B.u1
```

{{34}, {22}, {-10}, {-27}}

```
u3 = B.u2
```

{{42}, {7}, {-21}, {-42}}

Thus we have found the length 3 chain $\{\mathbf{u}_3,\ \mathbf{u}_2,\ \mathbf{u}_1\}$ based on the (ordinary) eigenvector $\mathbf{u}_3$. (To reconcile this result with *Mathematica*'s **Eigenvectors** calculation, you can check that $\mathbf{u}_3 + 14\mathbf{w}_1 = -7\mathbf{w}_2$.) Consequently four linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ are given by

$$\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{w}_1 e^t, \\
\mathbf{x}_2(t) &= \mathbf{u}_3 e^t, \\
\mathbf{x}_3(t) &= (\mathbf{u}_2 + \mathbf{u}_3 t)e^t, \\
\mathbf{x}_4(t) &= (\mathbf{u}_1 + \mathbf{u}_2 t + \tfrac{1}{2}\mathbf{u}_3 t^2)e^t.
\end{aligned}$$

## Using MATLAB

First we enter the matrix in (3):

```
A = [ 35  -12   4   30
      22   -8   3   19
     -10    3   0   -9
     -27    9  -3  -23 ];
```

Then we proceed to explore its characteristic polynomial, eigenvalues, and eigenvectors.

```
poly(A)
ans =
    1.0000   -4.0000    6.0000   -4.0000    1.0000
```

These are the coefficients of the characteristic polynomial, which hence is $(\lambda - 1)^4$. Then

```
[V, D] = eigensys(A)
V =
[ 1, 0]
[ 0, 1]
[-1, 3]
[-1, 0]
```

```
D =
[1]
[1]
[1]
[1]
```

Thus MATLAB finds only the two independent eigenvectors

```
w1 = [1   0   -1   -1]';
w2 = [0   1    3    0]';
```

associated with the single multiplicity 4 eigenvalue $\lambda = 1$, which therefore has defect 2. To explore the situation we set up the $4 \times 4$ identity matrix and the matrix $\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}$:

```
Id = eye(4);
B = A - L*Id;
```

When we calculate $\mathbf{B}^2$ and $\mathbf{B}^3$,

```
B2 = B*B
B3 = B2*B
```

We find that $\mathbf{B}^2 \neq 0$ but $\mathbf{B}^3 = 0$, so there should be a length 3 chain associated with the eigenvalue $\lambda = 1$. Choosing the first generalized eigenvector

```
u1 = [1   0   0   0]';
```

we calculate the further generalized eigenvectors

```
u2 = B*u1
u2 =
    34
    22
   -10
   -27
```

and

```
u3 = B*u2
u3 =
    42
     7
   -21
   -42
```

Thus we have found the length 3 chain $\{\mathbf{u}_3,\ \mathbf{u}_2,\ \mathbf{u}_1\}$ based on the (ordinary) eigenvector $\mathbf{u}_3$. (To reconcile this result with MATLAB's **`eigensys`** calculation, you can check that $\mathbf{u}_3 - 42\mathbf{w}_1 = 7\mathbf{w}_2$.) Consequently four linearly independent solutions of the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$ are given by

$$
\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{w}_1 e^t,\\
\mathbf{x}_2(t) &= \mathbf{u}_3 e^t,\\
\mathbf{x}_3(t) &= (\mathbf{u}_2 + \mathbf{u}_3 t)e^t,\\
\mathbf{x}_4(t) &= (\mathbf{u}_1 + \mathbf{u}_2 t + \tfrac{1}{2}\mathbf{u}_3 t^2)e^t.
\end{aligned}
$$