



Modern Siege Weapons: Mechanics of the Trebuchet

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Abstract

The purpose of this project is to describe and analyze the motion of a trebuchet using differential equations. We will begin with the original model used historically and gradually make the model more realistic in accordance with the various modifications added to the trebuchet. Lagrangian mechanics will be used to determine the equations of motion and to find the release angle from vertical that maximizes the range of the projectile. By comparing the maximum range of each model with the maximum possible range for an idealized launcher, we hope to show how each addition to the trebuchet makes it more efficient.

1. History

The first trebuchets on record appeared in Asia in the midst of the 7th century. These crude siege weapons, today known as "traction trebuchets" used human power to launch projectiles hundreds of meters. The earliest trebuchets resembled catapults, simple structures with a projectile on one end and a counterweight or a series of ropes (as in the traction-type) on the other. As trebuchets were used generation after generation to wreak havoc upon villages and fortresses alike, they evolved, much as modern weaponry

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evolved from the revolver to the gatling gun. The next addition to the "seesaw"-type trebuchets was a hinged counterweight, which allowed more of the potential energy of the system to be utilized in the projectile. As trebuchets advanced through the ages, the most important addition was the use of a sling to launch the projectile. Engineers of the day discovered that they could improve the efficiency of their weapons many-fold by extending the throwing arm of the weapon using a rope and a sling. Similar to the way a baseball player will extend his bat at the last possible instant, a sling allows the projectile to reach a greater velocity before it leaves the machine by converting the kinetic energy of the throwing arm into the kinetic energy of the projectile itself. History shows the trebuchet to be one of the most efficient siege weapons of any era, and the ingenuity of thousands of archaic engineers led to its development.

2. Introduction

Today, we can mathematically model the trebuchet's motion. This motion is extremely complicated but can be broken down into several manageable steps. For this project, we have assumed that every beam is massless, which simplifies calculations, although it would not be difficult to add the weight of the beam to the differential equations. We have also assumed that no friction exists to damp the motion of the trebuchet and that the main beam, the hinged counter-weight, and the sling are perfectly rigid. For the first model, the so called "seesaw" model, Newtonian mechanics can be used to find the equations of motion. However, due to the complications introduced in the hinged counter-weight and sling models, it will be beneficial to model every system using Lagrangian mechanics.

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3. Lagrangian Mechanics and the Euler-Lagrange Equation

With even a slightly complicated mechanical system, it often becomes tedious, and at times impossible, to model the system using Newtonian mechanics. One alternative to the classical method is to analyze the system using Lagrangian mechanics. Developed by Joseph Louis Lagrange in 1788, Lagrangian mechanics utilizes a new unit, known as the Lagrangian, which for mechanics is given as $L = T - V$, where T is the kinetic energy and V is the potential energy of the system. Action is the integral over time of the Lagrangian. By minimizing the action, one can find which of an infinite choice of paths a system or particle is likely to take. A ball is thrown into the air; why does it follow the curve it does? One answer, the Lagrangian answer, is that the path is simply the path of smallest action.

Derived from minimizing the action, the Euler-Lagrange equation gives us an easy way to solve for the equations of motion of a system. Depending on the degrees of freedom of the system, that is, in how many ways can it move, the general Euler-Lagrange equation will appear as a function of the generalized coordinates of the system, q_i .

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad (1)$$

Thus, there will be an equation for each degree of freedom. In our systems, each degree of freedom will be an angle measure, and the x and y components are functions of these angles. Hence our final model, which will be described as a function of θ , ϕ , and ψ , will require three equations.

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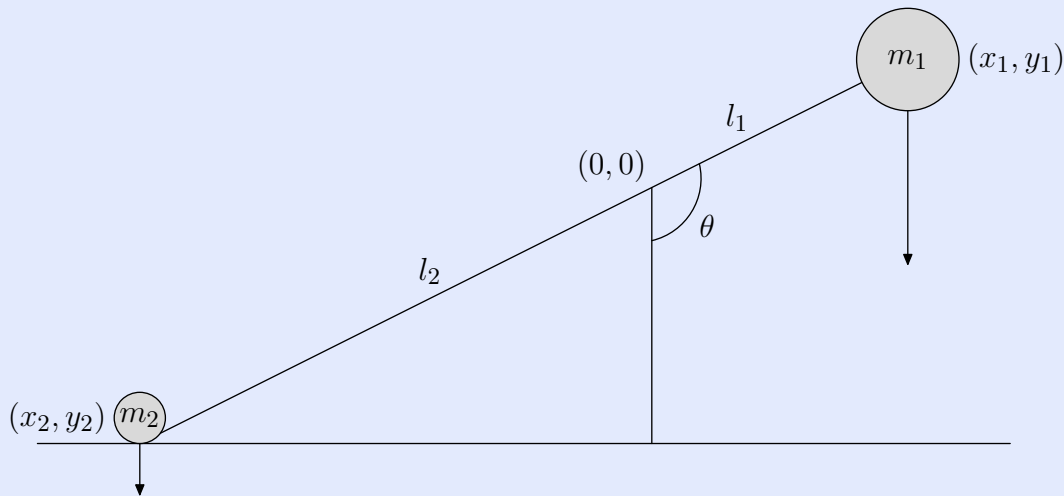
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Figure 1: Seesaw model of a trebuchet

4. The See-Saw Model

The simplest case to consider is that of the seesaw model. In this model, both the counterweight and the projectile are fixed to the rotating beam as depicted in Figure 1.

Physically, it is easy to see that the counterweight, which is much heavier than the projectile, will be driving the motion. Looking at the system long-term (as if the projectile were not released), it is easy to see that the motion of the masses is periodic. The system is simply a dual mass pendulum, much like a metronome. The counterweight will fall, raising the projectile into the air. The counterweight will carry on past the equilibrium position due to its momentum. It will rise until all of its kinetic energy becomes potential energy (the same height it was at initially), then falls



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once again towards the equilibrium. In a system with no friction, this motion repeats indefinitely.

However, because we are only interested in this system in so much as it can launch a projectile, the long term periodicity is not of interest. Because the equation of motion for the projectile after it is released is a simple case of kinematics, we will only explore the differential equations of motion for the projectile before release and to do this, we must apply the Euler-Lagrangian equation to the system. Finding these equations, we will use a numeric solver to find the motion over time and compare possible release angles of the projectile with their associated ranges to find the optimal release angle, should it prove to be other than the well known 45° .

4.1. Positions of the Masses

The first step is to find the positions of each of the masses. Because we want to limit the degrees of freedom, we'll find the x and y components of the positions as a function of θ which in turn is a function of time. The coordinates of m_1 can easily be seen to be

$$x_1 = l_1 \sin \theta(t) \text{ and } y_1 = -l_1 \cos \theta(t).$$

It serves us to vectorize the path traced by m_1 . The associated position vector, P_1 is defined as:

$$\vec{P}_1 = \langle l_1 \sin(\theta), -l_1 \cos(\theta) \rangle.$$

The coordinates of m_2 are

$$x_2 = -l_2 \sin(\theta) \text{ and } y_2 = l_2 \cos(\theta).$$

Similarly, the position vector describing the path of m_2 , P_2 , is defined as:

$$\vec{P}_2 = \langle -l_2 \sin(\theta), l_2 \cos(\theta) \rangle.$$



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4.2. The Kinetic and Potential Energy of the System

Kinetic energy for a mass is given by the equation $\frac{1}{2}mv^2$.

Velocity is the derivative of the position function with respect to time. To find the velocity of m_1 , we have to take the derivative of each component of P_1 .

For m_1 , this gives us a velocity (\vec{P}_1') of

$$\vec{P}_1' = \langle l_1 \dot{\theta} \cos(\theta), l_1 \dot{\theta} \sin(\theta) \rangle.$$

In a similar manner, we can calculate the velocity of m_2 .

$$\vec{P}_2' = \langle -l_2 \dot{\theta} \cos(\theta), -l_2 \dot{\theta} \sin(\theta) \rangle$$

Now we can see why it was useful to vectorize the motion of the masses, because the dot product of \vec{P}' with itself is the square of the magnitude. The kinetic energy (T) of the system, that is, the sum of the kinetic energies of both masses is shown by the following equation.

$$\begin{aligned} T &= \frac{1}{2}m_1(\vec{P}_1' \cdot \vec{P}_1') + \frac{1}{2}m_2(\vec{P}_2' \cdot \vec{P}_2') \\ &= \frac{1}{2}m_1(l_1^2 \dot{\theta}^2 \cos^2(\theta) + l_1^2 \dot{\theta}^2 \sin^2(\theta)) + \frac{1}{2}m_2(l_2^2 \dot{\theta}^2 \cos^2(\theta) + l_2^2 \dot{\theta}^2 \sin^2(\theta)) \\ &= \frac{1}{2}m_1(l_1^2 \dot{\theta}^2) + \frac{1}{2}m_2(l_2^2 \dot{\theta}^2) \\ &= \frac{1}{2}(m_1 l_1^2 + m_2 l_2^2) \dot{\theta}^2 \end{aligned}$$

The potential energy of the system is much easier to calculate. The potential energy (V) for each mass in a gravitational field is given by the equation mgy . When we put in our position functions, we obtain the following equation.

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$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g l_1 \cos(\theta) + m_2 g l_2 \cos(\theta) \end{aligned}$$

4.3. The Lagrangian and the Euler-Lagrange Equation

Now, we have enough information to set up the Lagrangian. Recall from earlier that $L = T - V$. So the Lagrangian for the seesaw model is:

$$\begin{aligned} L &= \frac{1}{2}(m_1 l_1^2 + m_2 l_2^2) \dot{\theta}^2 - g \cos(\theta)(-m_1 l_1 + m_2 l_2) \\ &= \frac{1}{2}(m_1 l_1^2 + m_2 l_2^2) \dot{\theta}^2 + g \cos(\theta)(m_1 l_1 - m_2 l_2). \end{aligned}$$

From here, we can use the Euler-Lagrange equation (ELE) to find the equation of motion (see Equation 1). With only one degree of freedom, θ , we only need to calculate

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0.$$



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We'll calculate each of these separately first.

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= -(m_1 l_1 - m_2 l_2)g \sin(\theta) \\ \frac{\partial L}{\partial \dot{\theta}} &= (m_1 l_1^2 + m_2 l_2^2) \dot{\theta} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) &= (m_1 l_1^2 + m_2 l_2^2) \ddot{\theta}\end{aligned}$$

Now we'll plug them back into the ELE. The equation for the motion of the masses is given by the solution to the ELE. Thus the equation of motion is given by the second order differential equation

$$Eq1 = -(m_1 l_1 - m_2 l_2)g \sin(\theta) - (m_1 l_1^2 + m_2 l_2^2) \ddot{\theta}. \quad (2)$$

4.4. Using a Numerical Solver

Due to the sine function on the right hand side, the differential equation is non-linear and must be solved numerically. The Maple programming language has an easy to use numerical solver in the form of `dsolve`. This function of Maple requires a differential equation (or a separable system of differential equations) and the associated initial values and returns a procedure which will calculate the various variables and their derivatives at a specific value of the independent variable. In our case, the independent variable is t and our dependent variables are $\theta(t)$ and $\dot{\theta}(t)$. The code for this and all other Maple functions can be found in Appendix I. It is not necessary to solve Equation 2 for $\ddot{\theta}$ in order to solve the differential equation, so long as it is possible to do so.

4.5. Parameters and Initial Conditions

Throughout the course of this paper, we will choose the same initial conditions and parameters when possible. The initial conditions for the seesaw model were chosen to be

$$\begin{aligned}\theta(0) &= 135^\circ \\ \dot{\theta}(0) &= 0.\end{aligned}$$

The parameters for the seesaw model were chosen to be

$$\begin{aligned}g &= 9.8 \\ l_1 &= 10 \\ l_2 &= 100 \\ m_1 &= 1000 \\ m_2 &= 1\end{aligned}$$

where the masses are measured in kilograms, the lengths in meters and g is the gravitational constant measured in m/s^2 . While these parameters are not exactly realistic, the range is proportionate to the scale of the model and all that is needed for realistic results is to scale down. The parameters chosen throughout were chosen to maximize the effectiveness of each model. With six parameters in the final solution, we did not think to take this on ourselves, but used those found in our source. (See References [3]).



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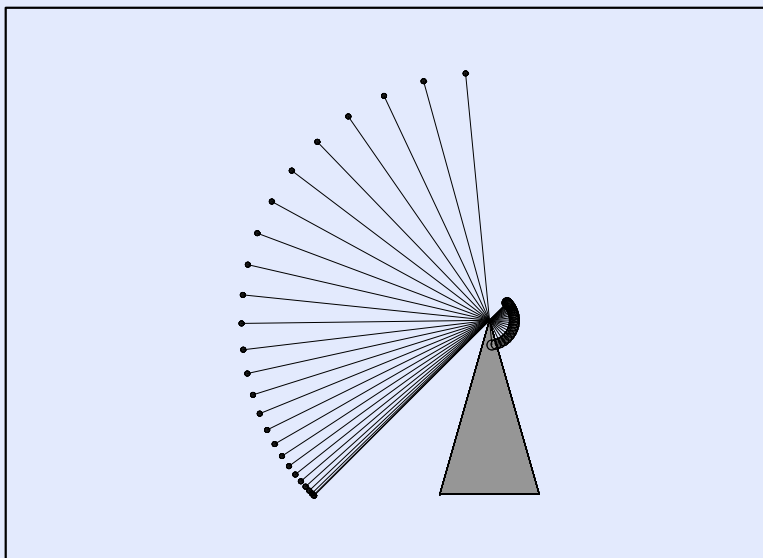


Figure 2: A Strobe-like Image of the Seesaw Model in Action

4.6. Visualizing the Solution

To make sure that this solution is correct, visualizing the result is helpful. Using the Maple command `odeplot`, it is easy to plot various visual aids.

Figure 2, a strobe-like plot of the positions of the masses at equally incremented time intervals, is all we need to verify that the solution matches the predicted result.

As the counterweight falls, $\dot{\theta}$ will increase until it should reach its maximum at $\theta = 0$. As we can see by Figure 2 the positions of the masses at each time interval are further and further apart, making it apparent that as θ decreases towards zero, $\dot{\theta}$ is increasing.

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4.7. Range of the Projectile and the Optimal Release Angle

Although the optimal release angle for a projectile in a gravitational field is 45° from horizontal, the kinetic energy of the projectile is greater further up in its arc, and therefore we expect the optimal angle of release to be somewhat less than 45° .

The motion of a projectile on Earth, ignoring air friction, is given by the kinematic equations

$$y = -4.9t^2 + \dot{y}_0 t + y_0$$
$$x = \dot{x}_0 t + x_0$$

where y_0 , \dot{y}_0 , x_0 , and \dot{x}_0 are initial conditions given by the system at the time of release and are functions of θ and $\dot{\theta}$. Using appropriate release times hinted at by Figure 3, we can set $y = 0$ (when the projectile hits the ground) and solve for t for our kinematic equations. Plugging this t into our equation for x will give us the ranges for the projectile given various release angles. A plot of these is given in Figure 4.

Remember, our purpose for finding the equations of motion was to maximize the range in order to compare the efficiencies of the different models. By looking at Figure 4, it appears that the optimal angle of release is 38.5° . This puts the maximum range at 2570 meters.

4.8. Range Efficiency

To calculate the range efficiency, we compared each model to an ideal projectile-launcher. In an ideal system, all of the potential energy of the counterweight would transfer into kinetic energy of the projectile. That is:

$$\frac{1}{2}m_2V_0^2 = m_1gh.$$



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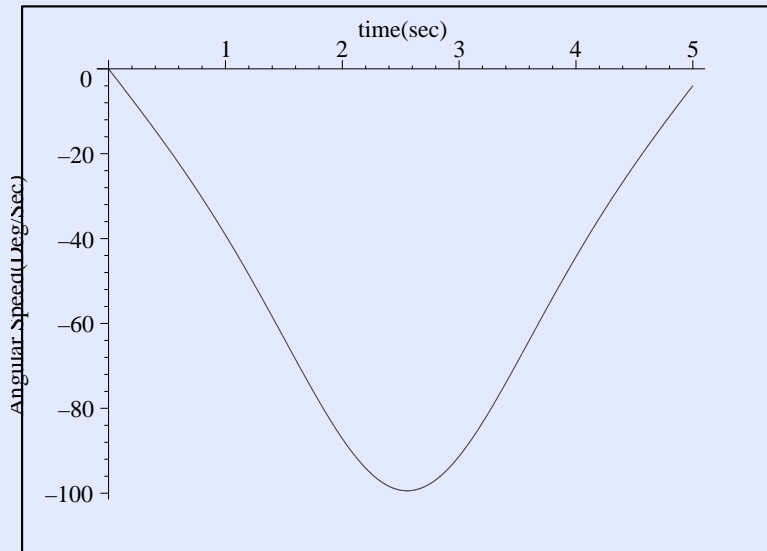


Figure 3: T vs $\dot{\theta}$



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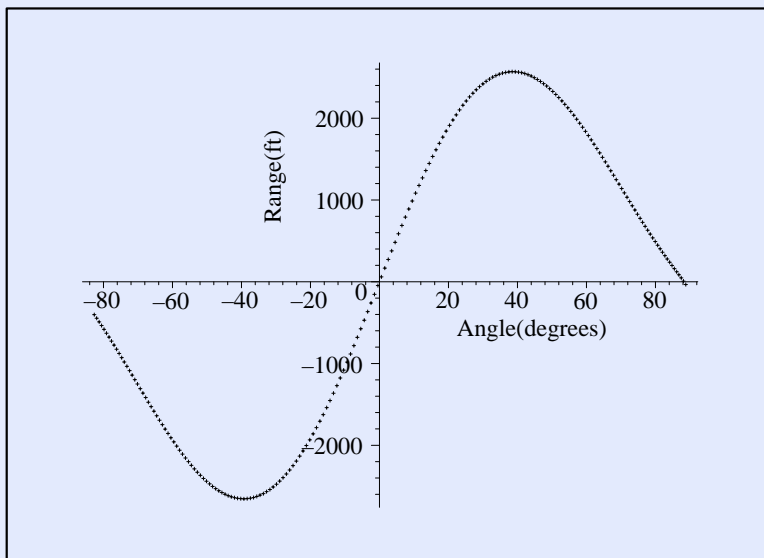


Figure 4: Release Angle vs. Range

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The optimal release angle for a projectile is known to be 45° , ignoring retarding forces. We'll use the change in height as the maximum possible change in height for our system, which can be seen to be

$$h = 5\sqrt{2} + 10$$

and the parameters for m_1 and m_2 are the same. Therefore,

$$\begin{aligned}\frac{1}{2}V_0^2 &= 9800(5\sqrt{2} + 10) \\ V_0 &= \pm\sqrt{19600(5\sqrt{2} + 10)} \\ V_0 &= 578.4\end{aligned}$$

measured in meters per second. We use the kinematic equations found in the last section and initial values of

$$\begin{aligned}y_0 &= 0 \\ x_0 &= 0 \\ \dot{y}_0 &= \cos(45^\circ)V_0 \\ \dot{y}_0 &= \frac{\sqrt{2}}{2}V_0 \\ \dot{y}_0 &= 409\end{aligned}$$

Using our parameters, and solving for x , we get a maximum range for the ideal launcher to be: 34142 meters. We'll use this range to calculate the efficiencies of the other models as well. Dividing the maximum range of the seesaw model by the maximum range of the ideal launcher gives us the range efficiency of the model. The seesaw model is only 8% efficient.

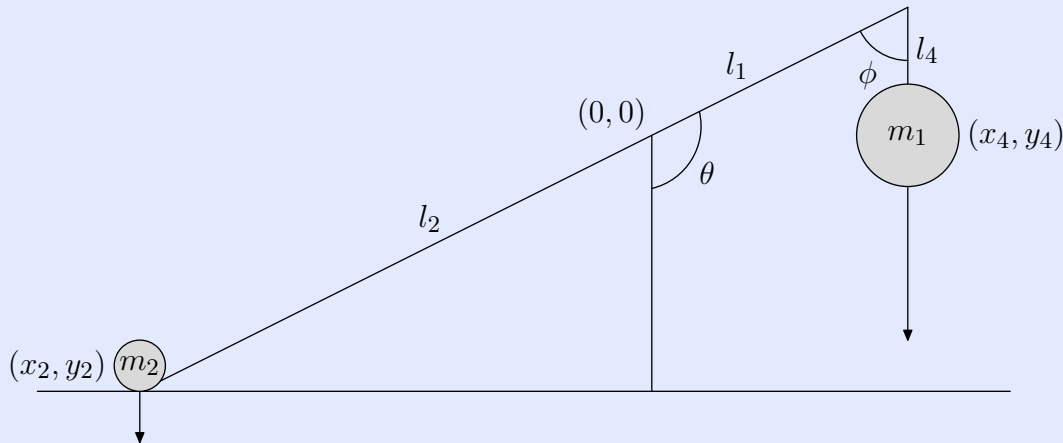


Figure 5: Trebuchet with a hinged counterweight

5. Hinged Counterweight Model

The next model we want to look at is the model of a trebuchet using a hinged counterweight. The counterweight is hung from a length which is free to rotate as shown in Figure 5. Again, this length is assumed to be rigid and massless to aid in calculations.

Notice that the angle ϕ adds another degree of freedom to the system. This means the ELE applied to this model will require two equations for motion: one with respect to θ and one with respect to ϕ .

This is where the use of Lagrangian mechanics will aid us most. The addition of another degree of freedom makes this model not only non-linear, but extremely sensitive to initial conditions. In fact, the hinged counterweight model is a version of the double pendulum, and is one of the simplest mechanical systems in which chaotic behavior

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occurs.

5.1. Positions of the Masses

The only thing that has changed in this system is the position of the counterweight, which can be seen as a geometric addition of the original position vector for m_1 and a new position vector from that point to the mass. As the position of m_2 has not changed, the position vectors for the masses are

$$\vec{Q}_2 = \langle -l_2 \sin(\theta), l_2 \cos(\theta) \rangle$$

$$\vec{Q}_1 = \langle l_1 \sin(\theta) - l_4 \sin(\theta + \phi), -l_1 \cos(\theta) + l_4 \cos(\theta + \phi) \rangle.$$

5.2. The Kinetic and Potential Energy of the System

Recall that we need the kinetic and potential energies of the system to solve the ELE. We start as before, with the kinetic energy. Again, we first need to calculate the derivative of each position vector.

$$\vec{Q}_1' = \langle l_1 \dot{\theta} \cos(\theta) - l_4 \cos(\theta + \phi)(\dot{\theta} + \dot{\phi}), l_1 \dot{\theta} \sin(\theta) - l_4 \sin(\theta + \phi)(\dot{\theta} + \dot{\phi}) \rangle$$

$$\vec{Q}_2' = \langle -l_2 \dot{\theta} \cos(\theta), -l_2 \dot{\theta} \sin(\theta) \rangle$$

Recall that the kinetic energy of the system will be:

$$T = \frac{1}{2} m_1 (\vec{Q}_1' \cdot \vec{Q}_1') + \frac{1}{2} m_2 (\vec{Q}_2' \cdot \vec{Q}_2').$$

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Plugging in our velocity vectors and evaluating the inner products we get

$$\begin{aligned}
 T &= \frac{1}{2}m_1[(l_1\dot{\theta}\cos(\theta))^2 - 2l_1l_1\cos(\theta)\cos(\theta+\phi)\dot{\theta}(\dot{\theta}+\dot{\phi}) \\
 &\quad + [l_4\cos(\theta+\phi)(\dot{\theta}+\dot{\phi})]^2 + (l_1\dot{\theta}\sin(\theta))^2 - 2l_1l_4\dot{\theta}\sin(\theta)\sin(\theta+\phi)(\dot{\theta}+\dot{\phi}) \\
 &\quad + [l_4\sin(\theta+\phi)(\dot{\theta}+\dot{\phi})]^2] + \frac{1}{2}m_2[(l_2\dot{\theta}\cos(\theta))^2 + (l_2\dot{\theta}\sin(\theta))^2] \\
 T &= \frac{1}{2}m_1[l_1^2\dot{\theta}^2 - 2l_1l_4\dot{\theta}(\dot{\theta}+\dot{\phi})\cos(\theta) + l_4^2(\dot{\theta}+\dot{\phi})^2] + \frac{1}{2}m_2l_2^2\dot{\theta}^2
 \end{aligned}$$

Now we calculate the potential energy. Because potential energy is only a function of position, it won't contain any $\dot{\theta}$ s or $\dot{\phi}$ s, so it will be much easier to calculate. All that is needed is the y component of each position vector.

$$\begin{aligned}
 U &= \sum U(m_1) + \sum U(m_2) \\
 &= g(m_1y_1 + m_2y_2) \\
 &= -m_1l_1g\cos(\theta) + m_1gl_4\cos(\theta+\phi) + m_2l_2g\cos(\theta)
 \end{aligned}$$

5.3. The Lagrangian and the Euler-Lagrange Equation

The next step is to calculate the Lagrangian of the system.

$$\begin{aligned}
 L &= T - V \\
 &= \frac{1}{2}m_1[l_1^2\dot{\theta}^2 - 2l_1l_4\dot{\theta}(\dot{\theta}+\dot{\phi})\cos(\theta) + l_4^2(\dot{\theta}+\dot{\phi})^2] \\
 &\quad + \frac{1}{2}m_2l_2^2\dot{\theta}^2 + m_1gl_1\cos(\theta) - m_1gl_4\cos(\theta+\phi) - m_2l_2g\cos(\theta)
 \end{aligned}$$



To apply the Euler-Lagrange Equation to the above, we again need to calculate $\frac{\partial L}{\partial \theta}$ and $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}})$. Due to the extra degree of freedom, however, we also need to calculate $\frac{\partial L}{\partial \phi}$ and $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\phi}})$.

$$\begin{aligned}\frac{\partial L}{\partial \theta} &= -m_1 g l_1 \sin(\theta) + m_1 g l_4 \sin(\theta + \phi) + m_2 g l_2 \sin(\theta) \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2} m_1 [2 l_1^2 \dot{\theta} - 2 l_1 l_4 \cos(\phi) (2 \dot{\theta} + \dot{\phi}) + 2 l_4^2 (\dot{\theta} + \dot{\phi})] + m_2 l_2^2 \dot{\theta} \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) &= \frac{1}{2} m_1 [2 l_1^2 \ddot{\theta} - (-2 l_1 l_4 \sin(\phi) \dot{\phi} (2 \dot{\theta} + \dot{\phi}) \\ &\quad + 2 l_1 l_4 \cos(\phi) (2 \ddot{\theta} + \ddot{\phi})) + 2 l_4^2 (\ddot{\theta} + \ddot{\phi})] + m_2 l_2^2 \ddot{\theta} \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) &= 0\end{aligned}$$

This gives our first equation of motion to be

$$\begin{aligned}Eq1 &= -m_1 g \sin(\theta) l_1 + m_2 g l_2 \sin(\theta) + g m_1 l_4 \cos(\theta) \sin(\phi) + g m_1 l_4 \sin(\theta) \cos(\phi) \\ &\quad - m_1 l_1^2 \ddot{\theta} + 2 \ddot{\theta} \cos(\phi) m_1 l_4 l_1 - 2 \dot{\theta} \sin(\phi) \dot{\phi} m_1 l_4 l_1 - m_1 l_4 l_1 \sin(\phi) (\dot{\phi})^2 \\ &\quad + m_1 l_4 l_1 \cos(\phi) \ddot{\phi} - m_2 l_2^2 \ddot{\theta} - m_1 l_4^2 \ddot{\theta} - m_1 l_4^2 \ddot{\phi}\end{aligned}\quad (3)$$

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The second equation of motion comes from applying the ELE to ϕ

$$\begin{aligned}\frac{\partial L}{\partial \dot{\phi}} &= m_1 g l_4 \sin \theta + \phi \\ \frac{\partial L}{\partial \dot{\phi}} &= -m_1 l_1 l_4 \dot{\theta} \cos \theta + m_1 l_4^2 (\dot{\theta} + \dot{\phi}) \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) &= -m_1 l_1 l_4 \ddot{\theta} \cos \theta + m_1 l_1 l_4 \dot{\theta}^2 \sin \theta + l_4^2 (\ddot{\theta} + \ddot{\phi}) \\ \frac{\partial L}{\partial \phi} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) &= 0\end{aligned}$$

Plugging these back in, we get our second equation for motion to be

$$\begin{aligned}Eq2 = & -m_1 l_4 (-l_1 (\dot{\theta})^2 \sin(\phi) - \sin(\phi) \cos(\theta) g - g \sin(\theta) \cos(\phi) \\ & - l_1 \ddot{\theta} \cos(\phi) + l_4 \ddot{\theta} + l_4 \ddot{\phi}).\end{aligned}\tag{4}$$

5.4. Parameters and Initial Conditions

Again, we'll use Maple to find a numeric solution to this system of non-linear differential equations. The initial conditions used to evaluate the solution are

$$\begin{aligned}\theta(0) &= 135^\circ \\ \dot{\theta}(0) &= 0 \\ \phi(0) &= 45^\circ \\ \dot{\phi}(0) &= 0.\end{aligned}$$

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We have the added parameter of l_4 , so now our parameters are

$$g = 9.8$$

$$l_1 = 10$$

$$l_2 = 100$$

$$l_4 = 21$$

$$m_1 = 1000$$

$$m_2 = 1.$$

Again, the code for this can be found in Appendix I.

5.5. Visualizing the Solution

Because this system is chaotic, it is difficult to predict what will occur. Luckily, chaotic behavior only seems to appear after our interest in the system has ended, and we can see in Figure 6 that the motion is basically the same as in the previous model. However, it seems that the angular velocity toward the top of the motion of the projectile is greater than in the seesaw model. We can verify this by comparing Figure 7 with Figure 3.

5.6. Range of the Projectile and the Optimal Release Angle

We saw from the last section that the angular speed is greater, but how much further can this model throw the projectile mass? Using the methods from the previous model, we can see from Figure 8, that the optimal angle of release is 19° . This gives us a maximum range of 16,050 meters, approximately six times the range of the seesaw model.



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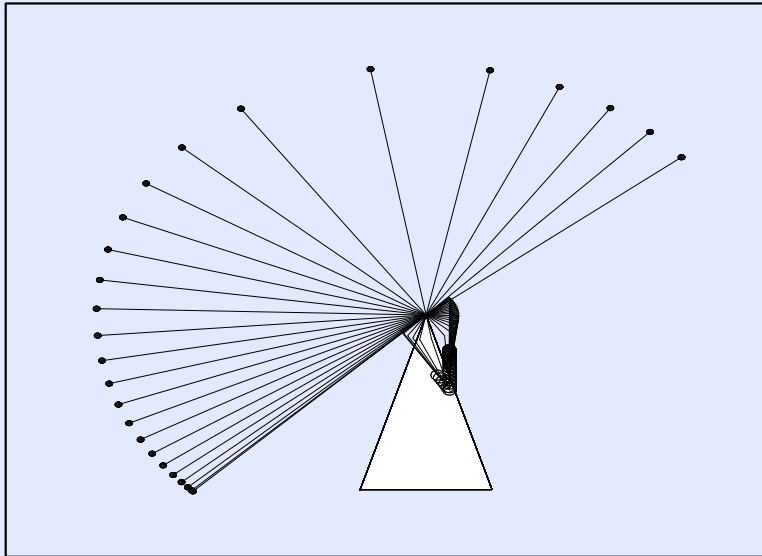


Figure 6: A strobe-like image of a trebuchet with a hinged counterweight



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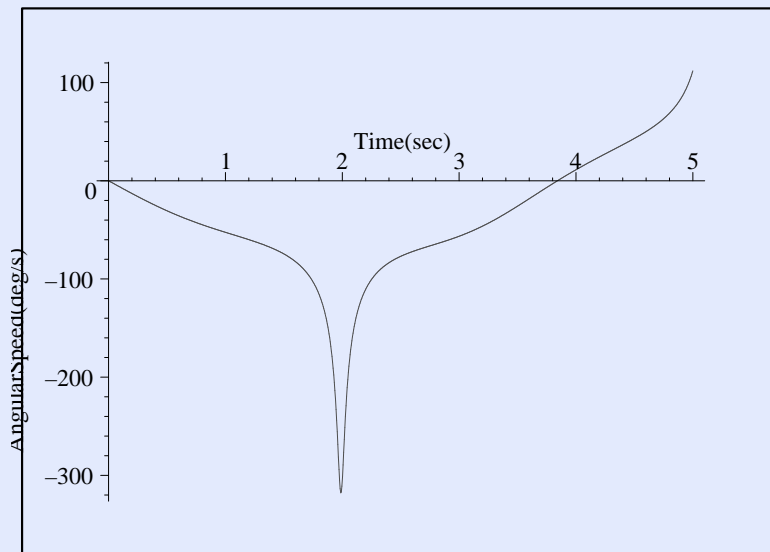


Figure 7: Time vs $\dot{\theta}$

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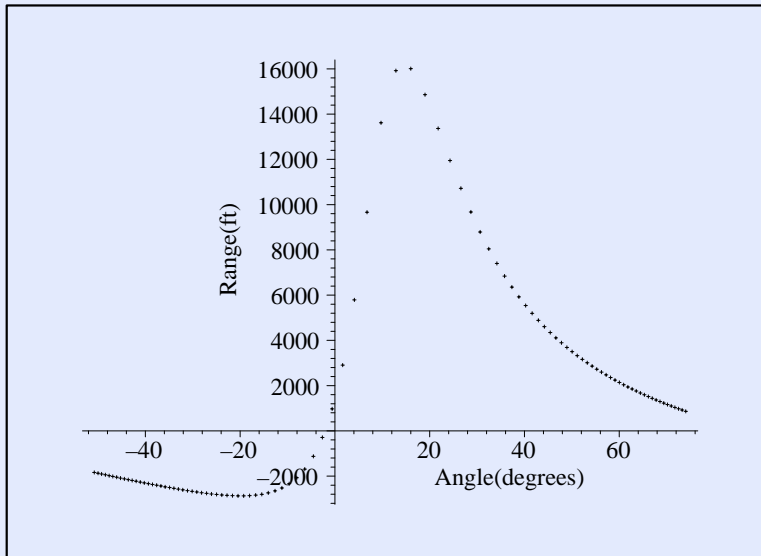


Figure 8: Release Angle vs. Range



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5.7. Range Efficiency

Calculating as before, this trebuchet model's range efficiency was calculated at 47%, a significant improvement over the last model. It is not easy to see why this might be so by looking at the equations, but it seems that the addition of a hinged and extended counterweight greatly increases the energy passed to the projectile at its release.

6. Trebuchet With A Hinged Counterweight and Sling

As you can see in Figure 9, this model adds another length between l_2 and m_2 . Again, this length will be considered to be both massless and perfectly rigid. The addition of a sling to the model greatly increases the range of the trebuchet. By keeping the throwing arm short until the moment of release, the rotational energy of the system is much greater than just extending the beam, as the system's moment of inertia is smaller. As the angular velocity first peaks and then decreases dramatically, as it did in the last model and can be expected to do in this one, the sling is spun around the end of the throwing arm and is released with much more speed than in the previous two models.

6.1. Positions of the Masses

The position of m_1 should be the same as that of the last model, for no additional modifications were made to the counterweight. Therefore, the position (\vec{R}_1) of m_1 is

$$\vec{R}_1 = \langle l_1 \sin(\theta) - l_4 \sin(\theta + \phi), -l_1 \cos(\theta) + l_4 \cos(\theta + \phi) \rangle .$$

However, the position of m_2 has changed. Now, the new position of m_2 is

$$\vec{R}_2 = \langle -l_2 \sin(\theta) - l_3 \sin(-\theta + \psi), l_2 \cos(\theta) - l_3 \cos(-\theta + \psi) \rangle .$$

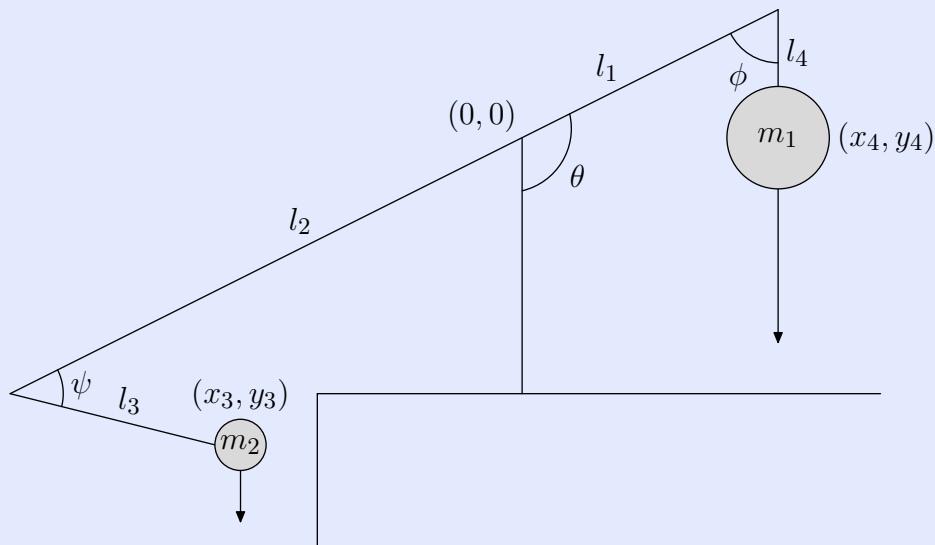


Figure 9: Trebuchet with a Hinged Counterweight and Sling

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6.2. Kinetic and Potential Energy of the System

The potential energy is fairly simple to calculate. The potential energy of the system is the sum of the potential energy of each mass.

$$V = -m_1 g l_1 \cos(\theta) + m_2 g l_2 \cos(\theta) + m_2 l_3 [-g \cos(\theta) \cos(\psi) - g \sin(\theta) \sin(\psi)] \\ + m_1 l_4 g [\cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi)]$$

The kinetic energy is, as usual, much more difficult to calculate. First, we need the velocity of each mass. The velocity of m_1 is just the derivative of the position vector \vec{R}_1 with respect to time.

$$\vec{R}_1' = \langle l_1 \cos(\theta) \dot{\theta} - l_4 \cos(\theta + \phi)(\dot{\theta} + \dot{\phi}), l_1 \sin(\theta) \dot{\theta} - l_4 \sin(\theta + \phi)(\dot{\theta} + \dot{\phi}) \rangle$$

Similarly, the velocity of m_2 is

$$\vec{R}_2' = \langle -l_2 \cos(\theta) \dot{\theta} - l_3 \cos(-\theta + \psi)(-\dot{\theta} + \dot{\psi}), -l_2 \sin(\theta) \dot{\theta} + l_3 \sin(-\theta + \psi)(-\dot{\theta} + \dot{\psi}) \rangle.$$

Evaluating and simplifying the kinetic energy, we get

$$T = \frac{1}{2} m_2 l_2^2 (\dot{\theta})^2 - m_2 l_3 l_2 (\dot{\theta})^2 \cos(\psi) + m_2 l_3 l_2 \dot{\theta} \dot{\psi} \cos(\psi) - m_1 l_4 l_1 \dot{\theta} \dot{\phi} \cos(\phi) \\ - m_1 l_4 l_1 (\dot{\theta})^2 \cos(\phi) + \frac{1}{2} (\dot{\theta})^2 m_1 l_1^2 + \frac{1}{2} m_1 l_4^2 (\dot{\phi})^2 + \frac{1}{2} m_1 l_4^2 (\dot{\theta})^2 + m_1 l_4^2 \dot{\theta} \dot{\phi} \\ - m_2 l_3^2 \dot{\theta} \dot{\psi} + \frac{1}{2} m_2 l_3^2 (\dot{\psi})^2 + \frac{1}{2} m_2 l_3^2 (\dot{\theta})^2.$$

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6.3. The Lagrangian and the Euler-Lagrange Equation

As before, we calculate the Lagrangian to be $L = T - V$.

$$\begin{aligned}
 L = & \frac{1}{2}(\dot{\theta})^2 m_1 l_1^2 - m_1 l_4 l_1 (\dot{\theta})^2 \cos(\phi) + m_1 g l_1 \cos(\theta) - m_1 l_4 l_1 \dot{\theta} \dot{\phi} \cos(\phi) \\
 & + \frac{1}{2}(\dot{\theta})^2 m_2 l_2^2 - m_2 g l_2 \cos(\theta) - m_2 l_3 l_2 (\dot{\theta})^2 \cos(\psi) + m_2 l_3 l_2 \dot{\theta} \dot{\psi} \cos(\psi) \\
 & + \frac{1}{2} m_2 l_3^2 (\dot{\psi})^2 - m_2 l_3^2 \dot{\theta} \dot{\psi} + \frac{1}{2} m_2 l_3^2 (\dot{\theta})^2 + m_2 g l_3 \cos(-\theta + \psi) + m_1 l_4^2 \dot{\theta} \dot{\phi} \\
 & + \frac{1}{2} m_1 l_4^2 (\dot{\phi})^2 + \frac{1}{2} m_1 l_4^2 (\dot{\theta})^2 - g m_1 l_4 \cos(\theta + \phi)
 \end{aligned}$$

Because the system has three degrees of freedom (θ , ϕ , and ψ), we will have three equations of motion derived from three applications of the ELE. Although these equations are not difficult to derive by hand, we chose to use Maple's Euler-Lagrange package to derive these equations. The code for this is again in Appendix I.

$$\begin{aligned}
 Eq1 = & -m_1 g l_1 \sin(\theta) + m_2 g l_2 \sin(\theta) - m_2 g l_3 \sin(\theta) \cos(\psi) + m_2 g l_3 \cos(\theta) \sin(\psi) \\
 & + m_1 g l_4 \sin(\theta) \cos(\phi) + m_1 g l_4 \cos(\theta) \sin(\phi) - \ddot{\theta} m_1 l_1^2 + 2 m_1 l_4 l_1 \ddot{\theta} \cos(\phi) \\
 & - 2 m_1 l_4 l_1 \dot{\theta} \sin(\phi) \dot{\phi} + m_1 l_4 l_1 \ddot{\phi} \cos(\phi) - m_1 l_4 l_1 (\dot{\phi})^2 \sin(\phi) - \ddot{\theta} m_2 l_2^2 \\
 & + 2 m_1 l_3 l_2 \ddot{\theta} \cos(\psi) - 2 m_2 l_3 l_2 \dot{\theta} \sin(\psi) \dot{\psi} - m_2 l_3 l_2 \ddot{\psi} \cos(\psi) \\
 & + m_2 l_3 l_2 (\dot{\psi})^2 \sin(\psi) + m_2 l_3^2 \ddot{\psi} - m_2 l_3^2 \ddot{\theta} - m_1 l_4^2 \ddot{\phi} - m_1 l_4^2 \ddot{\theta}
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 Eq2 = & m_2 l_3 l_2 (\dot{\theta})^2 \sin(\psi) - m_2 g l_3 \cos(\theta) \sin(\psi) + m_2 g l_3 \sin(\theta) \cos(\psi) \\
 & - m_2 l_3 l_2 \ddot{\theta} \cos(\psi) - m_2 l_3^2 \ddot{\psi} + m_2 l_3^2 \ddot{\theta}
 \end{aligned} \tag{6}$$

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$$Eq3 = m_1 l_4 l_1 (\dot{\theta})^2 \sin(\phi) + m_1 g l_4 \cos(\theta) \sin(\phi) + m_1 g l_4 \sin(\theta) \cos(\phi) \\ + m_1 l_4 l_1 \ddot{\theta} \cos(\phi) - m_1 l_4^2 \ddot{\theta} - m_1 l_4^2 \ddot{\phi} \quad (7)$$

6.4. Parameters and Initial Conditions

The only new parameters and initial conditions for this model are those dealing with m_2 .

$$l_3 = 100 \\ \psi(0) = 45^\circ \\ \dot{\psi}(0) = 0$$

6.5. Visualizing the Solution

Again, we want to be able to visualize our solution to verify that it makes sense with what we know about the system. Recall our hypothesis about the behavior of the trebuchet. We can see in Figure 10 that indeed, the beam slows dramatically and the sling whips around the end of the throwing arm right before release. We can verify this by studying Figures 11 and 12. As the angular speed of the beam falls off, the angular speed of the sling peaks.

6.6. Maximum Range and the Optimal Release Angle

From Figure 13, our optimal θ angle of release is -2.7° . Our optimal ψ angle of release is 144.5° . This makes the maximum range 29,030 meters. This is approximately twice the range of the trebuchet with just a hinged counterweight (and approximately fourteen times the range of the seesaw trebuchet).



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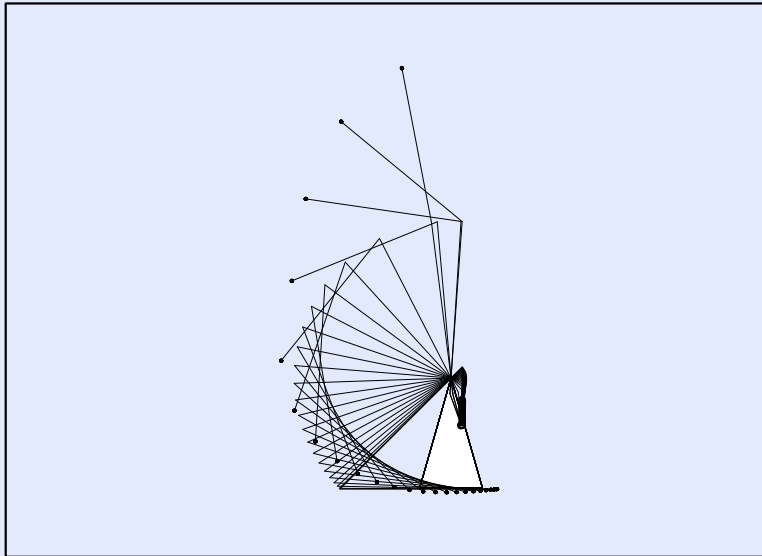


Figure 10: A strobe-like image of a trebuchet with a hinged counterweight and sling



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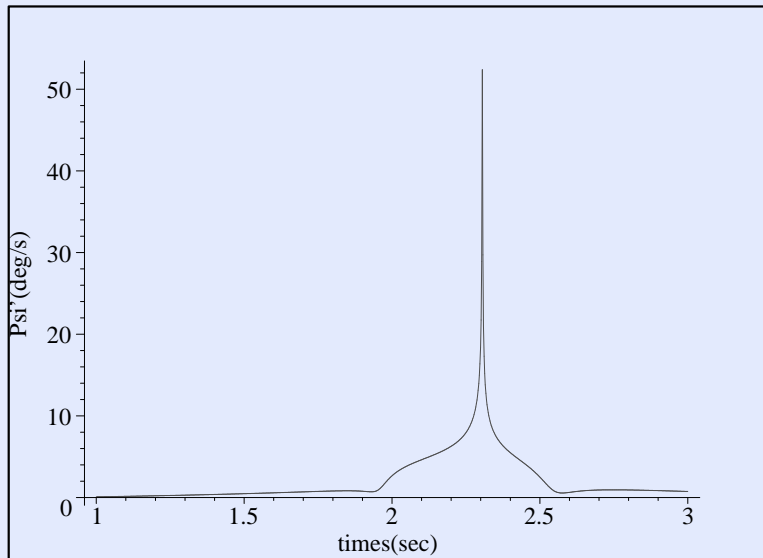


Figure 11: T vs. ψ



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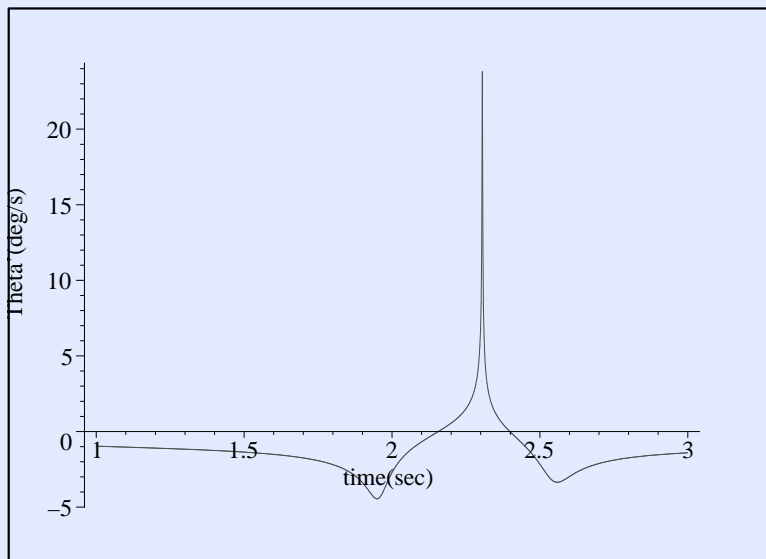


Figure 12: T vs. θ



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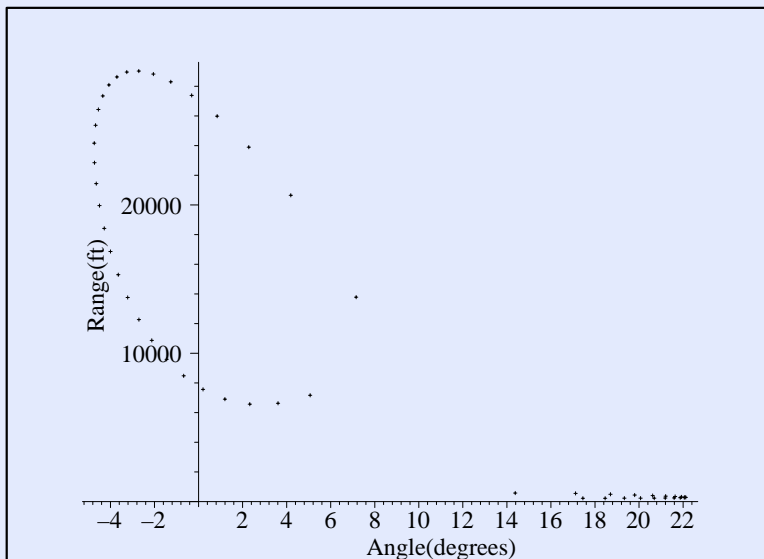


Figure 13: Release Angle (θ) vs. Range

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6.7. Range Efficiency

The range efficiency was 84% for this trebuchet model. That's only 16% shy of being a perfect launcher!

7. Conclusions

Amazingly, all of the major modifications to the trebuchet occurred before calculus was invented. Most were made by guess and check over many years. With the use of math and computer modeling technology, we can, and indeed have, shown how the simple and ineffective seesaw model of the trebuchet became the fearsome weapon that rose to popularity over 800 years ago!

References

- [1] Dave Arnold and Keith Level, for guidance and inspiration
- [2] Oprea, John, *Differential Geometry and its Applications*, Prentice Hall, Upper Saddle River, NJ, 1997.
- [3] [http : //www.algobeautytreb.com/trebmath35.pdf](http://www.algobeautytreb.com/trebmath35.pdf)
- [4] [http : //justin.turnthecrank.com/edu_treb.jsp](http://justin.turnthecrank.com/edu_treb.jsp)
- [5] [http : //www.eng.buffalo.edu/~chintang/school/MAE415_FinalProject.pdf](http://www.eng.buffalo.edu/~chintang/school/MAE415_FinalProject.pdf)
- [6] [http : //members.iinet.net.au/~rmine/tsims.html](http://members.iinet.net.au/~rmine/tsims.html)
- [7] [http : //www.ripcord.ws/theory.html](http://www.ripcord.ws/theory.html)

8. Appendix: Maple Code

8.1. Seesaw Trebuchet

First we initialize all of the packages we'll use in this study

```
> restart:
> with(VariationalCalculus):
> with(ODEtools):
> with(DEtools):
> with(plots):
> with(plottools):
> with(linalg):
> PDEtools[declare]((theta)(t),prime=t):
           theta(t) will now be displayed as theta
derivatives with respect to: t of functions of one variable will\
now be displayed with '
```

Here we define our position vectors and velocities.

```
> x[1]:=l[1]*sin(theta(t)):
> y[1]:=-l[1]*cos(theta(t)):
> x[2]:=-l[2]*sin(theta(t)):
> y[2]:=l[2]*cos(theta(t)):
> R[1]:=vector(2,[x[1],y[1]]):
> R[2]:=vector(2,[x[2],y[2]]):
> 'R[1]':='map(diff,R[1],t):
> 'R[2]':='map(diff,R[2],t):
```

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Next, we calculate the kinetic and potential energies of the system and find the Lagrangian.

```
> T:=factor(collect(simplify(m[1]/2*innerprod('R[1]','R[1]')+m[2]/2
    *innerprod('R[2]','R[2]')), [m[1],m[2],l[1],l[2],diff(theta(t),t)])):
> U:=simplify(m[1]*g*y[1]+m[2]*g*y[2]):
> L:=T-U:
```

Using the EulerLagrange package to find our equations for motion

```
> EuL:=EulerLagrange(L,t,theta(t)):
> eq:=op(remove(has,EuL,K[1])):
```

Setting up the initial conditions and Parameters.

```
> INITS:={theta(0)=3*Pi/4,D(theta)(0)=0}:
> PARAM:=[g=9.8,l[1]=10,l[2]=100,m[1]=1000,m[2]=1]:
```

Evaluating the general equation for motion with respect to our parameters.

```
> eq2:=eval(eq,PARAM):
```

We use a numeric ODE solver to approximate the change in theta and theta' over time.

```
> sol:=dsolve({eq2,op(INITS)},numeric,output=listprocedure):
```

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Plotting time versus angular velocity.

```
> odeplot(sol,[t,180/Pi*diff(theta(t),t)],0..5,numpoints=1000,  
  labels=["time(sec)","Angular Speed(Deg/Sec)"]):
```

The animation and/or the strobe plot for the model.

```
> noffm:=25:  
> divs:=10:  
> for i from 0 to noffm do  
> x1:=eval(1[1]*sin(rhs(sol[2](i/divs))),PARAM):  
> y1:=eval(-1[1]*cos(rhs(sol[2](i/divs))),PARAM):  
> x2:=eval(-1[2]*sin(rhs(sol[2](i/divs))),PARAM):  
> y2:=eval(1[2]*cos(rhs(sol[2](i/divs))),PARAM):  
> rod[i]:=line([x1,y1],[x2,y2]):  
> structure[i]:=polygon([[-20,-70],[0,0],[20,-70],[-20,-70]],colour=green):  
> ma1[i]:=disk([x1,y1],2,colour=red):  
> ma2[i]:=disk([x2,y2],1,colour=blue):  
> #grnd[i]:=polygon([[-100,-40],[100,-40],[100,-70],[-100,-70]],colour=brown):  
> anima[i]:=display({structure[i],rod[i],ma1[i],ma2[i]}):  
> end do:  
> display(seq(anima[i],i=0..noffm),scaling=constrained,axes=framed  
  ,title="See-Saw Model"):  
> divs:=100:
```

Calculating the optimal release angle and the associated range.

```
> x_0:=seq(eval(-1[2]*sin(rhs(sol[2](i/divs))),PARAM),i=150..350):
```

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```
> y_0:=seq(eval(1[2]*cos(rhs(sol[2](i/divs))),PARAM),i=150..350):
> Dx:=seq(eval(-1[2]*sin(rhs(sol[2](i/divs)))*rhs(sol[3](i/divs)),
  PARAM),i=150..350):
> Dy:=seq(eval(-1[2]*cos(rhs(sol[2](i/divs)))*rhs(sol[3](i/divs)),
  PARAM),i=150..350):
> y_proj:=seq(-4.9*(t)^2+Dy[i-149]*(t)+y_0[i-149],i=150..350):
> T:=seq(solvefor[t](y_proj[i]=0),i=1..200):
> Ti:=seq(op(select(type,[rhs(T[i][1]),rhs(T[i][2])],positive)),i=3..200):
> x_proj:=seq(Dx[i]*Ti[i]+x_0[i],i=1..197):
> theta_r:=seq(fnormal(180/evalf(Pi)*rhs(sol[2]((i+152)/divs)),4),i=1..197):
> rng:=seq(fnormal(x_proj[i],4),i=1..197):
> expr:=seq([theta_r[i],rng[i]],i=1..197):
> stuff:=seq(point([theta_r[i],rng[i]]),i=1..197):
```

Plotting the release angle versus the range to assist in determining the optimal release angle and range.

```
> display(seq(stuff[i],i=1..197),title="Release Angle Vs. Range",
  labels=["Angle(degrees)","Range(ft)"]):
```

8.2. Trebuchet with a Hinged Counterweight

First we initialize all of the packages we'll use in this study

```
> restart;
> with(VariationalCalculus):
> with(ODEtools):
> with(DEtools):
```



```
> with( plots ): with( plottools ):
> PDEtools[declare]( (theta,phi)(t), prime=t):
    derivatives with respect to: t of functions of one variable will\
        now be displayed with '
        theta(t) will now be displayed as theta
        phi(t) will now be displayed as phi
> with(linalg):
```

Here we define our position vectors and velocities.

```
> x[1]:=l[1]*sin(theta(t)):
> y[1]:=-l[1]*cos(theta(t)):
> x[2]:=-l[2]*sin(theta(t)):
> y[2]:=l[2]*cos(theta(t)):
> x[4]:=x[1]-l[4]*sin(theta(t)+phi(t)):
> y[4]:=y[1]+l[4]*cos(theta(t)+phi(t)):
> R[1]:=vector(2,[x[1],y[1]]):
> R[2]:=vector(2,[x[2],y[2]]):
> R[4]:=vector(2,[x[4],y[4]]):
> 'R[1]'' := map(diff,R[1],t):
> 'R[2]'' := map(diff,R[2],t):
> 'R[4]'' := map(diff,R[4],t):#the Velocity vectors
```

Next, we calculate the kinetic and potential energies of the system and find the Lagrangian.

```
> T := collect(simplify(m[1]/2*innerprod('R[4]','R[4]'),[m[1],l[1],l[4]]))+
> collect(simplify(m[2]/2*innerprod('R[2]','R[2]'),[m[1],m[2],l[2]]):
```

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```
> T:=combine(T,trig):
> #kinetic energy
> U:=m[1]*g*y[4]+m[2]*g*y[2]:
> U:=collect(expand(U,trig),[m[1],m[2],l[1],l[4],l[2]]):
> U:=collect(expand(U,trig),[m[1],m[2],l[1],l[4],l[2]]):
> #potential energy
> L:=collect(simplify(expand(T-U,trig)),[l[1],l[2],l[4],m[1],m[2],g,
    diff(theta(t),t),diff(diff(theta(t),t),t),t]):
> #lagrangian
```

Using the EulerLagrange package to find our equations for motion

```
> EuL:=EulerLagrange(L,t,[theta(t),phi(t)]):
> eqs:=remove(has,EuL,K[1]):
> eq1:=factor(simplify(op(select(has,eqs,l[2]^2)))):
> eq2:=factor(simplify(op(remove(has,eqs,l[2]^2)))):
> sys:={eq1,eq2}:
```

Setting up the initial conditions and Parameters.

```
> INITS:={theta(0)=3*Pi/4,D(theta)(0)=0,phi(0)=Pi/4,D(phi)(0)=0}:
> PARAM:=[g=9.8,l[1]=10,l[2]=100,l[4]=21,m[1]=1000,m[2]=1]:
> sys2:=subs(({theta(t)=th,diff(theta(t),t)=dth,diff(diff(theta(t),t),t)=ddth,
    phi(t)=ph,diff(phi(t),t)=dph,diff(diff(phi(t),t),t)=ddph},sys)):
> ddphi:=solvefor[ddph](op(op(remove(has,eliminate(sys2,ddth),ddth)))):
> ddthe:=solvefor[ddth](op(op(remove(has,eliminate(sys2,ddph),ddph)))):
> sys3:={ddphi,ddthe}:
> sys3:=subs({th=theta(t),dth=diff(theta(t),t),ddth=diff(diff(theta(t),t),t),
```

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```
ph=phi(t), dph=diff(phi(t),t),ddph=diff(diff(phi(t),t),t)},sys3):
```

Evaluating the general equation for motion with respect to our parameters.

```
> sys4:=eval(sys3,PARAM):
```

We use a numeric ODE solver to approximate the change in theta and theta' over time.

```
> sol:=dsolve(sys4 union INITS, numeric,output=listprocedure):
```

The animation and/or the strobe plot for the model.

```
> noffm:=50:
> divs:=10:
> for i from 0 to noffm do
> x1:=eval(l[1]*sin(rhs(sol[4](i/divs))),PARAM):
> y1:=eval(-l[1]*cos(rhs(sol[4](i/divs))),PARAM):
> x2:=eval(-l[2]*sin(rhs(sol[4](i/divs))),PARAM):
> y2:=eval(l[2]*cos(rhs(sol[4](i/divs))),PARAM):
> x4:=eval(x1-l[4]*sin(rhs(sol[4](i/divs))+rhs(sol[2](i/divs))),PARAM):
> y4:=eval(y1+l[4]*cos(rhs(sol[4](i/divs))+rhs(sol[2](i/divs))),PARAM):
  rod1[i]:=line([x2,y2],[x1,y1]):
> rod2[i]:=line([x1,y1],[x4,y4]):
> structure[i]:=polygon([-20,-70],[0,0],[20,-70],[-20,-70]):
> ma1[i]:=disk([x4,y4],2,colour=red):
> ma2[i]:=disk([x2,y2],1,colour=blue):
```


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```
> anima[i]:=display({structure[i],rod1[i],rod2[i],ma1[i],ma2[i]}):  
> end do:  
> display(seq(anima[i],i=0..noffm),insequence=true,scaling=constrained,  
    axes=framed):
```

Plotting time versus angular velocity.

```
> odeplot(sol,[t,180/Pi*diff(theta(t),t)],t=0..5,numpoints=1000,  
    labels=["Time(sec)","AngularSpeed(deg/s)  
> "],font=[TIMES,ROMAN,20],labeldirections=[horizontal,vertical]):  
> display(seq(anima[i],i=0..25),axes=framed,title="Hinged Counter-Weight"):
```

Calculating the optimal release angle and the associated range.

```
> divs:=100:  
> x_0:=seq(eval(-1[2]*sin(rhs(sol[4](i/divs))),PARAM),i=150..250):  
> y_0:=seq(eval(1[2]*cos(rhs(sol[4](i/divs))),PARAM),i=150..250):  
> Dx:=seq(eval(-1[2]*sin(rhs(sol[4](i/divs)))*rhs(sol[5](i/divs)),PARAM)  
    ,i=150..250):  
> Dy:=seq(eval(-1[2]*cos(rhs(sol[4](i/divs)))*rhs(sol[5](i/divs)),PARAM)  
    ,i=150..250):  
> y_proj:=seq(-4.9*(t)^2+Dy[i-149]*(t)+y_0[i-149],i=150..250):  
> T:=seq(solvefor[t](y_proj[i]=0),i=1..100):  
> Ti:=seq(op(select(type,[rhs(T[i][1]),rhs(T[i][2])],positive)),i=1..100):  
> x_proj:=seq(Dx[i]*Ti[i]+x_0[i],i=1..100):  
> theta_r:=seq(fnormal(180/evalf(Pi)*rhs(sol[4]((i+150)/divs)),4),i=1..100):  
> rng:=seq(fnormal(x_proj[i],4),i=1..100):  
> expr:=seq([theta_r[i],rng[i]],i=1..100):
```

8.3. Trebuchet with a Hinged Counterweight and Sling

First we initialize all of the packages we'll use in this study

```
> restart;
> with(VariationalCalculus):
> with(ODEtools):
> with(DEtools):
> with(plots): with(plottools):
> PDEtools[declare]((theta,phi,psi)(t), prime=t):
    theta(t) will now be displayed as theta
    phi(t) will now be displayed as phi
    psi(t) will now be displayed as psi
    derivatives with respect to: t of functions of one variable will\
    now be displayed with '
> with(linalg):
```

Here we define our position vectors and velocities.

```
> x[1]:=l[1]*sin(theta(t)):
> y[1]:=-l[1]*cos(theta(t)):
> x[2]:=-l[2]*sin(theta(t)):
> y[2]:=l[2]*cos(theta(t)):
> x[4]:=x[1]-l[4]*sin(theta(t)+phi(t)):
> y[4]:=y[1]+l[4]*cos(theta(t)+phi(t)):
> x[3]:=x[2]+l[3]*sin(theta(t)-psi(t)):
> y[3]:=y[2]-l[3]*cos(theta(t)-psi(t)):
```



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```
> R[1]:=vector(2,[x[1],y[1]]):
> R[2]:=vector(2,[x[2],y[2]]):
> R[4]:=vector(2,[x[4],y[4]]):
> R[3]:=vector(2,[x[3],y[3]]):
> 'R[1]'' := map(diff,R[1],t):
> 'R[2]'' := map(diff,R[2],t):
> 'R[4]'' := map(diff,R[4],t):#the Velocity vectors
> 'R[3]'':=map(diff,R[3],t):
```

Next, we calculate the kinetic and potential energies of the system and find the Lagrangian.

```
> T:=combine(simplify(collect(m[1]/2*innerprod('R[4]','R[4]')+m[2]/2*
  innerprod('R[3]','R[3]'),[l[1],l[2],l[3],l[4],m[1],m[2],
  diff(theta(t),t),diff(phi(t),t),diff(psi(t),t),cos(psi(t)),
  sin(psi(t))])),trig):
> U:=collect(simplify(expand(m[1]*g*y[4]+m[2]*g*y[3])),[l[1],l[2],l[3],l[4],
  m[1],m[2],diff(theta(t),t),diff(phi(t),t),diff(psi(t),t),cos(psi(t)),
  sin(psi(t)),g]):
> L:=simplify(collect(T-U,[l[1],l[2],l[3],l[4],m[1],m[2],diff(theta(t),t),
  diff(phi(t),t),diff(psi(t),t),cos(psi(t)),sin(psi(t))])):
```

Using the EulerLagrange package to find our equations for motion

```
> Eul:=remove(has,EulerLagrange(L,t,[theta(t),phi(t),psi(t)]),K[1]):
> eq1:=op(select(has,Eul,(diff(psi(t),t)^2))):
> eq2:=op(select(has,(remove(has,Eul,(diff(psi(t),t)^2)),sin(psi(t)))):
> eq3:=op(remove(has,(remove(has,Eul,(diff(psi(t),t)^2)),sin(psi(t)))):
```

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Setting up the initial conditions and Parameters.

```
> INITS:={theta(0)=3*Pi/4,D(theta)(0)=0,phi(0)=Pi/4,D(phi)(0)=0,psi(0)=Pi/4,  
  D(psi)(0)=0}:  
> PARAM:=[g=9.8,l[1]=10,l[2]=100,l[3]=100,l[4]=21,m[1]=1000,m[2]=1]:
```

Evaluating the general equation for motion with respect to our parameters.

```
> sys:=eval([eq1,eq2,eq3],PARAM):
```

We use a numeric ODE solver to approximate the change in theta and theta' over time.

```
> sol:=dsolve([op(sys),op(INITS)],numeric,output=listprocedure):
```

Plotting time versus angular velocity.

```
> odeplot(sol,[t,diff(psi(t),t)],1..3,numpoints=1000,labels=["times(sec)",  
  "Psi'(deg/s)"],labeldirections=[horizontal,vertical],  
  font=[TIMES,ROMAN,20]):  
> odeplot(sol,[t,diff(theta(t),t)],t=1..3,numpoints=1000,labels=["time(sec)",  
  "Theta'(deg/s)"],labeldirections=[horizontal,vertical],  
  font=[TIMES,ROMAN,20]):  
> noffm:=23:
```

The animation and/or the strobe plot for the model.

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```
> divs:=10:
> for i from 0 to noffm do
> x1:=eval(1[1]*sin(rhs(sol[6](i/divs))),PARAM):
> y1:=eval(-1[1]*cos(rhs(sol[6](i/divs))),PARAM):
> x2:=eval(-1[2]*sin(rhs(sol[6](i/divs))),PARAM):
> y2:=eval(1[2]*cos(rhs(sol[6](i/divs))),PARAM):
> x3:=eval(x2+1[3]*sin(rhs(sol[6](i/divs))-rhs(sol[4](i/divs))),PARAM):
> y3:=eval(y2-1[3]*cos(rhs(sol[6](i/divs))-rhs(sol[4](i/divs))),PARAM):
> x4:=eval(x1-1[4]*sin(rhs(sol[6](i/divs))+rhs(sol[2](i/divs))),PARAM):
> y4:=eval(y1+1[4]*cos(rhs(sol[6](i/divs))+rhs(sol[2](i/divs))),PARAM):
    rod1[i]:=line([x2,y2],[x1,y1]):
> rod3[i]:=line([x2,y2],[x3,y3]):
> rod2[i]:=line([x1,y1],[x4,y4]):
> structure[i]:=polygon([-20,-70],[0,0],[20,-70],[-20,-70]):
> ma1[i]:=disk([x4,y4],2,colour=red):
> ma2[i]:=disk([x3,y3],1,colour=blue):
> anima[i]:=display({structure[i],rod1[i],rod2[i],rod3[i],ma1[i],ma2[i]}):
> end do:
> display(seq(anima[i],i=0..noffm),scaling=constrained,axes=none):
```

Calculating the optimal release angle and the associated range.

```
> divs:=100:
> x_0:=seq(eval(-1[2]*sin(rhs(sol[6](i/divs)))+1[3]*sin(rhs(sol[6](i/divs))
    -rhs(sol[4](i/divs))),PARAM),i=200..250):
> y_0:=seq(eval(1[2]*cos(rhs(sol[6](i/divs)))-1[3]*cos(rhs(sol[6](i/divs))
    -rhs(sol[4](i/divs))),PARAM),i=200..250):
```

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```
> Dx:=seq(eval(-1[2]*cos(rhs(sol[6](i/divs)))*rhs(sol[7](i/divs))+1[3]
*cos(rhs(sol[6](i/divs))-rhs(sol[4](i/divs)))*(rhs(sol[7](i/divs))
-rhs(sol[5](i/divs))),PARAM),i=200..250):
> Dy:=seq(eval(-1[2]*sin(rhs(sol[6](i/divs)))*rhs(sol[7](i/divs))+1[3]
*sin(rhs(sol[6](i/divs))-rhs(sol[4](i/divs)))*(rhs(sol[7](i/divs))
-rhs(sol[5](i/divs))),PARAM),i=200..250):
> y_proj:=seq(-4.9*(t)^2+Dy[i-199]*(t)+y_0[i-199],i=200..250):
> T:=seq(solvefor[t](y_proj[i]=0),i=1..50):
> Ti:=seq(op(select(type,[rhs(T[i][1]),rhs(T[i][2])],positive)),i=1..50):
> x_proj:=seq(Dx[i]*Ti[i]+x_0[i],i=1..50):
> theta_r:=seq(fnormal(180/evalf(Pi)*rhs(sol[6]((i+199)/divs)),4),i=1..50):
> rng:=seq(fnormal(x_proj[i],4),i=1..50):
> psi_r:=seq(fnormal(190/evalf(Pi)*rhs(sol[4]((i+199)/divs)),4),i=1..50):
> expr:=seq([theta_r[i],psi_r[i],rng[i]],i=1..50):
> stuff:=seq(point([theta_r[i],rng[i]]),i=1..50):
> display(seq(stuff[i],i=1..50),labels=["Angle(degrees)","Range(ft)"],
font=[TIMES,ROMAN,20],labeldirections=[horizontal,vertical],
tickmarks=[100,3]):
> v_0:=evalf(sqrt(2000*9.8*(5*sqrt(2)+10))):
> vx:=evalf(sin(Pi/4)*v_0):
> x_0:=0:
> y_0:=200:
```

Calculating the maximum range for an idealized projectile launcher.

```
> y:=-4.9*t^2+vx*t+y_0:
> ti:=solvefor[t](y=0):
> tim:=op(select(type,[rhs(ti[1]),rhs(ti[2])],positive)):
```

```
> x:=evalf(vx*tim+x_0):  
> 29030/x:  
> 16010/x:  
> 2569/x:
```

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