Note about : A short introduction to the Lindblad master equation

Orkesh Nurbolat

September 28, 2020

Glossary :

- $\mathcal{H}$ represents a Hilber space
- $|\psi\rangle \in \mathcal{H}$ represents a vector of Hilbert space $\mathcal{H}$ (a column vector)
- $\langle\psi| \in \mathcal{H}$ represents a vector of the dual Hilbert space of $\mathcal{H}$ (a row vector) .
- $\langle\psi|\phi\rangle \in \mathbb{C}$ is the scakar product of vectors $|\psi\rangle$ and $|\phi\rangle$
- $||\,|\psi\rangle||$ is the norm of the vector $|\psi\rangle$ where there is $||\,|\psi\rangle|| = \sqrt{\langle\psi|\psi\rangle}$
- $B(\mathcal{H})$ represents the space of bounded operatores : $B : \mathcal{H} \mapsto \mathcal{H}$
- $\mathbb{1}_{\mathcal{H}} \in B(\mathcal{H})$ is the identity operator of the Hilber space $\mathcal{H}$ .s.t. $\mathbb{1}|\psi\rangle = |\psi\rangle, \forall|\psi\rangle \in \mathcal{H}$
- $|\psi\rangle\langle\phi| \in B(\mathcal{H})$ is the operator such tath $(|\psi\rangle\langle\phi|)|\varphi\rangle = \langle\phi|\varphi\rangle|\psi\rangle, \forall|\varphi\rangle \in \mathcal{H}$
- $\hat{O}^\dagger$ is the hermitian conjugate of operator : $\hat{O} \in B(\mathcal{H})$
- $\hat{U} \in B(\mathcal{H})$ is the unitary operator iff $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1}$
- $\hat{H} \in B(\mathcal{H})$ is a Hermitian operator iff $\hat{H} = \hat{H}^\dagger$
- $\hat{A} \in B(\mathcal{H})$ is a positive operator $(A > 0)$, iff $\langle\phi|\hat{A}|\phi\rangle > 0, \forall|\phi\rangle \in \mathcal{H}$
- $\hat{P} \in B(\mathcal{H})$ is a projector iff $\hat{P}\hat{P} = \hat{P}$
- $\mathrm{Tr}\,[B]$ represents the trace of operator B
- $\rho(\mathcal{L})$ repressets the space of density matrices, meaning the space of bounded operators action on $\mathcal{H}$ with trace 1 and positive
- $|\rho\rangle\rangle$ is a vector in the Fock-Liouville space

- $\langle\langle \hat{A}|\hat{B}\rangle\rangle = \mathrm{Tr}\left[\hat{A}^\dagger \hat{B}\right]$ is the scalar product of operators $\hat{A}, \hat{B} \in B(\mathcal{H})$ in the Fock-Liouville space
- $\widetilde{\mathcal{L}}$ is the matrix representation of a super operator in the Fock-Liuville space

1 Looking back to quantum mechanics

1.1 density matrix

the density matrix is like : (L:151)

$$\hat{\rho} \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1}$$

and this p_i coefficient are non-negative and $\sum p_i = 1$ which is a thing called probability , I belive there is $p_i \in \mathbb{R}$ as well . it means that p_i is the probablity that system is in the pure state $|\psi_j\rangle$ so there is : (L:168)

$$\mathrm{Tr}\left[\hat{\rho}\right] = \sum p_j = 1 \tag{2}$$

what the trace does is to take the diogonal elements and put them at the place needed all the time this holds and there is aways positive , aka (L:178)

$$\hat{\rho} > 0 \tag{3}$$

there is $\mathrm{Tr}\left[\hat{\rho}^2\right]$ called the purity of the state . somehow it measures something like $\frac{1}{d} \leq \mathrm{Tr}\left[\hat{\rho}^2\right] \leq 1$
now, given arbitary basis like : $|i\rangle_{i=1}{}^N$ which is ofcourse in the Hilbert space , then the density matrix will be looking like : (L:195)

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} & \cdots\cdots & \rho_{0N} \\ \rho_{10} & \rho_{11} & \cdots\cdots & \rho_{0N} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \rho_{N0} & \rho_{N1} & \cdots\cdots & \rho_{NN} \end{pmatrix} \tag{4}$$

populations means the diagonal element of the density matrix , and there is $\rho_{ii} \in \mathbb{R}_0^+$ and also $\sum_i \rho_{i,i} = 1$
as we know there is $\mathcal{H}_2 = \mathcal{H} \otimes \mathcal{H}$ A pure state of the system would be any unit vector of $\mathcal{H}_2$ we can say : $|\psi\rangle = a|0\rangle + b|1\rangle$ and $a, b \in \mathbb{C}$ s.t $|a|^2 + |b|^2 = aa^* + bb^* = 1$ so : $\hat{\rho} \in O(\mathcal{H})$ (L:231)

$$\hat{\rho} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1| \tag{5}$$

and here we have (L:246)

$$\rho_{00} + \rho_{11} = |\langle 0|\psi\rangle|^2 + |\langle 1|\psi\rangle|^2 = 1 \tag{6}$$

and there is as well : (L:254)

$$\rho_{01} = \rho_{10}^* \tag{7}$$

1.2 about other operators

an operator is in the form of (L:261)

$$\hat{O} = \sum_i a_i |a_i\rangle\langle a_i| \tag{8}$$

where this $a_i \in \mathbb{R}$ is in the operator $\hat{O}$'s space so here we can write that : (L:273)

$$\begin{aligned}
\langle \hat{O} \rangle &= \sum_i p_i \mathrm{Tr}\left[\hat{O}\right] \\
&= \sum_i p_i \langle \psi_i | \hat{O} | \psi_i \rangle \\
&= \sum_i \mathrm{Tr}\left[p_i |\psi_i\rangle\langle\psi_i| \hat{O}\right] \\
&= \mathrm{Tr}\left[\sum_i p_i |\psi_i\rangle\langle\psi_i| \hat{O}\right] \\
&= \mathrm{Tr}\left[\hat{\rho}\hat{O}\right] = \mathrm{Tr}\left[\hat{O}\hat{\rho}\right]
\end{aligned} \tag{9}$$

if this operator $\hat{O}$ has an spectral resolution like : (L:307)

$$\begin{aligned}
\hat{O} &= \sum_i a_i |a_i\rangle\langle a_i| \\
&= \sum_i a_i P_i
\end{aligned} \tag{10}$$

now after the measurement it writes that : (L:316)

$$P(a_i) = |\langle \phi | a_i \rangle|^2 \tag{11}$$

why there is an : a_i instead of just i ? I am confued , here what it would like to say might be a_i is constant mapped from i , and it can degenerate anyway : (L:333)

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle \tag{12}$$

and (L:342)

$$\begin{aligned}
P(a_i) &= |\langle \phi | a_i \rangle|^2 \\
&= \mathrm{Tr}\left[\hat{\rho} |a_i\rangle\langle a_i|\right]
\end{aligned} \tag{13}$$

and for example it can be writen as : (L:351)

$$\begin{aligned}
&\mathrm{Tr}\left[[\rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|]|a_i\rangle\langle a_i|\right] \\
&= (\rho_{00}|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|)|a_i\rangle\langle a_i| \\
&= \rho_{00}|\langle 0|a_i\rangle|^2 + \rho_{11}|\langle 1|a_i\rangle|^2
\end{aligned} \tag{14}$$

and then here it writes that : (L:373)

$$\langle \hat{O} \rangle = \mathrm{Tr}\left[\hat{O}\hat{\rho}\right] \tag{15}$$

a minimal Hamiltonina looks like : (L:380)

$$\hat{H} = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1| \tag{16}$$

and we say that $\psi = a|0\rangle + b|1\rangle$ so here comes that : (L:390)

$$\begin{aligned} P(E_0) &= |\langle 0|\psi\rangle|^2 = |a|^2 \\ P(E_1) &= |\langle 1|\psi\rangle|^2 = |b|^2 \end{aligned} \tag{17}$$

so there we have that : (L:394)

$$\langle \hat{H} \rangle = E_0|a|^2 + E_1|b|^2 \tag{18}$$

in the language of density matrix : (L:402)

$$\hat{\rho} = \rho_{00}|0\rangle\langle 0| + \rho_{01}|0\rangle\langle 1| + \rho_{10}|1\rangle\langle 0| + \rho_{11}|1\rangle\langle 1| \tag{19}$$

now here the writer changed the language back to that : $P(0) = \mathrm{Tr}\left[|0\rangle\langle 0|\hat{\rho}\right] = \rho_{00}$ and we have that: (L:416)

$$\langle \hat{H} \rangle = \mathrm{Tr}\left[\hat{H}\hat{\rho}\right] = E_0\rho_{00} + E_1\rho_{11} \tag{20}$$

so we can know that here we have : $\rho_{00} = |a|^2$ and $\rho_{11} = |b|^2$ and we know that : (L:430)

$$\frac{\mathrm{d}}{\mathrm{dt}}|\psi(t)\rangle = -\mathbb{i}\hbar\hat{H}|\psi(t)\rangle \tag{21}$$

this papers like to set that : $\hbar = 1$ and we wil have the time independent $\hat{H}$ causing : (L:441)

$$|\psi(t)\rangle = \mathbb{e}^{-\mathbb{i}\hat{H}t}|\psi(0)\rangle \tag{22}$$

which is as well as : (L:447)

$$|\psi(t)\rangle = \hat{U}|\psi(0)\rangle \tag{23}$$

here we know that : $\hat{U} \in B(\mathcal{H})$ s.t. $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1}$ so here also have something speaking that : (L:465)

$$\frac{\mathbf{d}\hat{\rho}}{\mathbf{d}t} = -\mathbb{i}[\hat{H}, \hat{\rho}] \equiv \mathcal{L}\hat{\rho} \tag{24}$$

which was called von Neumann equation and then here goes like (L:475)

$$\begin{aligned}
\frac{\mathbf{d}}{\mathbf{d}t}\mathrm{Tr}\left[\hat{\rho}\right]^2 &= \mathrm{Tr}\left[\frac{\mathbf{d}\hat{\rho}^2}{\mathbf{d}t}\right] \\
&= \mathrm{Tr}\left[2\hat{\rho}\frac{\mathbf{d}\hat{\rho}}{\mathbf{d}t}\right] \\
&= -2\mathbb{i}\mathrm{Tr}\left[\hat{\rho}[\hat{H},\hat{\rho}]\right] \\
&= 0
\end{aligned} \tag{25}$$

now we can write that : (L:488)

$$\hat{H}_{free} = E_0|0\rangle\langle 0| + E_1|1\rangle\langle 1| \tag{26}$$

why wont one start with $|\psi(0)\rangle = |1\rangle$ so that we have $|\psi(t)\rangle = \mathbb{e}^{-\mathbb{i}\hat{H}t} = \mathbb{e}^{-\mathbb{i}E_1 t}|1\rangle$
I dont know if we can actually do this without losing any generality : (L:499)

$$\hat{H}_{free} = E|1\rangle\langle 1| \tag{27}$$

and then there goes that : (L:504)

$$\hat{H} = E|1\rangle\langle 1| + \Omega(|0\rangle\langle 1| + |1\rangle\langle 0|) \tag{28}$$

and in the ine it has something to say that : (L:512)

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N \\
|\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_N\rangle \\
\hat{\rho} &= \hat{\rho}_1 \otimes \hat{\rho}_2 \otimes ... \otimes \hat{\rho}_N
\end{aligned} \tag{29}$$

so for simpler use we just say that : $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ for seperatable varaibales
ofcouse here we say that : $|\psi\rangle = \sum_{i,j} |\psi_i\rangle \otimes |\psi_j\rangle$
so now if we have that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and the density matrix could be expressed as (L:537)

$$\hat{\rho}_a = \mathrm{Tr}_b[\hat{\rho}] \tag{30}$$

because (L:545)

$$\mathrm{Tr}_b\left[\sum_{i,j,k,l}|a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|\right] = \sum_{i,j}|a_i\rangle\langle a_j|\mathrm{Tr}\left[\sum_{k,l}|b_k\rangle\langle b_l|\right] \tag{31}$$

1.3 2 level atioms example

say : (L:562)

$$\begin{aligned}
|00\rangle &= |0\rangle_1 \otimes |0\rangle_2 \\
|01\rangle &= |0\rangle_1 \otimes |1\rangle_2 \\
|10\rangle &= |1\rangle_1 \otimes |0\rangle_2 \\
|11\rangle &= |1\rangle_1 \otimes |1\rangle_2
\end{aligned} \tag{32}$$

is better with : (L:570)

$$\begin{aligned}
|\psi\rangle_G &= |00\rangle \\
|\psi\rangle_S &= \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1) \otimes \frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_2) \\
&= \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle + |11\rangle) \\
|\psi\rangle_E &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)
\end{aligned} \tag{33}$$

so here it can write that (L:585)

$$\hat{\rho}_E = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \tag{34}$$

and lets say there : (L:592)

$$\begin{aligned}
\hat{\rho}_E^{(1)} &= \langle 0|_2 \hat{\rho}_E |0\rangle_2 + \langle 1|_2 \hat{\rho}_E |1\rangle_2 \\
&= \frac{1}{2}(|00\rangle\langle 00|_1 + |11\rangle\langle 11|_2)
\end{aligned} \tag{35}$$

1.4 Fock-Liouvile Hlbert space (FLS)

defineing the thing like : $\hat{\rho} \to |\hat{\rho}\rangle\rangle$ so there is a realation that writes : (L:612)

$$|\hat{\rho}\rangle\rangle = \begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{01} \\ \rho_{11} \end{pmatrix} \tag{36}$$

and then it says that it has the Liouvillian superoperator can bow be expressed as a matrix (L:625)

$$\widetilde{\mathcal{L}} = \begin{pmatrix} 0 & \mathbb{i}\Omega & -\mathbb{i}\Omega & 0 \\ \mathbb{i}\Omega & \mathbb{i}E & 0 & -\mathbb{i}\Omega \\ -\mathbb{i}\Omega & 0 - \mathbb{i}E & \mathbb{i}\Omega & \\ 0 & -\mathbb{i}\Omega & \mathbb{i}\Omega & 0 \end{pmatrix} \tag{37}$$

this is kindof like a Schrodingers equation thoght : $\dfrac{\mathbf{d}|\hat{\rho}\rangle\rangle}{\mathbf{d}t} = \widetilde{\mathcal{L}}|\hat{\rho}\rangle\rangle$ (L:638)

$$\frac{\mathbf{d}}{\mathbf{d}t}\begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{01} \\ \rho_{11} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{i}\Omega & -\mathbb{i}\Omega & 0 \\ \mathbb{i}\Omega & \mathbb{i}E & 0 & -\mathbb{i}\Omega \\ -\mathbb{i}\Omega & 0 - \mathbb{i}E & \mathbb{i}\Omega & \\ 0 & -\mathbb{i}\Omega & \mathbb{i}\Omega & 0 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{10} \\ \rho_{01} \\ \rho_{11} \end{pmatrix} \tag{38}$$

2 CPT-MAPS and the Lindblad Master Equation

2.1 what is Lindblad master equation

(L:665)

$$\frac{\mathbf{d}\hat{\rho}}{\mathbf{d}t} = -\frac{\mathbb{i}}{\hbar}\left[\hat{H}\ ,\ \hat{\rho}\right] + \sum_{n,m=1}^{N^{2-1}} h_{nm\left(\hat{A}_n\hat{\rho}\hat{A}_m^{\dagger} - \frac{1}{2}\{\hat{A}_m^{\dagger}\hat{A}_n,\hat{\rho}\}\right)} \tag{39}$$

it gives :

- $\{\hat{A}_m\}$: arbitary orthonormal basis that satisfies : $||\hat{A}||^2_{HS} = \sum_{i\in I}||\hat{A}e_i||^2$ or Hilbert-Schmidt operator $\hat{A}_{N^2}$ is proportional to the identiy operator
- $h_{[\text{Escaped } nm]}$: must be positive semidefinite with it being all zero, there it back to $\frac{\mathbf{d}\hat{\rho}}{\mathbf{d}t} = -(\mathbb{i}/\hbar)\left[\hat{H}\ ,\ \hat{\rho}\right]$

2.2 Completely positve maps (CPT)

so there need a kind of map (L:698)

$$\mathcal{V} : \hat{\rho}(\mathcal{H}) \mapsto \hat{\rho}(\mathcal{H}) \tag{40}$$

it will need : (L:701)

$$\mathrm{Tr}\left[\mathcal{V}\hat{A}\right] = \mathrm{Tr}\left[\hat{A}\right] \qquad \forall A \in O(\mathcal{H}) \tag{41}$$

and it need to be completely positive , which means : (L:709)

$$\begin{aligned} &\mathcal{V} \text{ is positive} \quad \text{iff } \ \forall A \in B(\mathcal{H}) \text{ s.t. } A \geq 0 \Rightarrow \mathcal{V}A \geq 0 \\ &\mathcal{V} \text{ is completly positive} \quad \text{iff } \ \forall n \in \mathbb{N} \text{ s.t. } , \mathcal{V} \otimes \mathbb{1}_n \text{ is positive} \end{aligned} \tag{42}$$

so the positive means that it has all the eigen values positive not all maps are completely positive , for example : (L:720)

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{43}$$

so there is its density matrix : (L:728)

$$
\begin{aligned}
\hat{\rho}_B &= \frac{1}{2}(|0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1|) \\
&= \frac{1}{2}\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\right\} \\
&= \frac{1}{2}\left\{\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right\} \\
&= \frac{1}{2}\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}
\end{aligned}
\tag{44}
$$

that looks positive , but : given that $\hat{\hat{T}}_2$ is a map called transformation which works on matrix (aka operators)like $\mathcal{V}$ means that we transpose the matrxi of the secound subsystem now it gives that (L:814)

$$
\begin{aligned}
(\mathbb{1} \otimes \hat{\hat{T}}_2)\hat{\rho}_B &= \frac{1}{2}\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\right\} \\
&= \begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}
\end{aligned}
\tag{45}
$$

and its eigen value has -1 in it ,

2.3 Drivation of the Lindblad eqiation from microscopic dynamics

(L:864)

$$\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t} = -\mathbb{i}[\hat{H}_T, \hat{\rho}_T(t)] \tag{46}$$

T is short for total , E is short for Enviroment, non mean system . (L:872)

$$\hat{\rho}(t) = \mathrm{Tr}_E[\hat{\rho}_T(t)] \tag{47}$$

so now the environment is cut out with trace . set this to make things simplier : (L:879)

$$\hat{H}_T = \hat{H}_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \hat{H}_E + \alpha\hat{H}_I \tag{48}$$

($\hat{H} \in \mathcal{H}$, $\hat{H}_E \in \mathcal{H}_E$, $\hat{H}_I \in \mathcal{H}_T$) (L:891)

$$\hat{H}_I = \sum_i \hat{S}_i \otimes \hat{E}_i \tag{49}$$

($\hat{S}_i \in B(\mathcal{H})$, $\hat{E}_i \in B(\mathcal{H}_E)$) for $\hat{O} \in B(\mathcal{H}_T)$ there is (L:907)

$$\hat{O}(t) = \mathbb{e}^{\mathbb{i}(\hat{H}+\hat{H}_E)t}\hat{O}\mathbb{e}^{-\mathbb{i}(\hat{H}+\hat{H}_E)t} \tag{50}$$

(L:916)

$$\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t} = -\mathbb{i}\alpha[\hat{H}_I(t), \hat{\rho}_T(t)] \tag{51}$$

α comes from the interaction coefficinet ? integrate by time gives : (L:927)

$$\int \frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t}\mathbf{d}t = -\mathbb{i}\alpha \int [\hat{H}_I(t), \hat{\rho}_T(t)]\mathbf{d}t \tag{52}$$

(L:939)

$$\begin{aligned} \hat{\rho}_T(t) &= -\mathbb{i}\alpha \int [\hat{H}_I(t), \hat{\rho}_T(t)]\mathbf{d}t \\ &= \hat{\rho}_T(0) - \mathbb{i}\alpha \int_0^t [\hat{H}_I(s), \hat{\rho}_T(s)]\mathbf{d}s \end{aligned} \tag{53}$$

skadoodling around gives : (L:956)

$$\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t} = -\mathbb{i}\alpha[\hat{H}_I(t), \hat{\rho}_T(0)] - \alpha^2 \int_0^t [\hat{H}_I(t), [\hat{H}_I(s), \hat{\rho}_T(s)]]\mathbf{d}s \tag{54}$$

lets do this again , this gives : (L:967)

$$\begin{aligned} \int_0^t \frac{\mathbf{d}\hat{\rho}_T(t')}{\mathbf{d}t'}\mathbf{d}t' &= \hat{\rho}_T(t) - \hat{\rho}_T(0) \\ &= \int_0^t \mathbf{d}t' \left(-\mathbb{i}\alpha[\hat{H}_I(t'), \hat{\rho}_T(0)] - \alpha^2 \int_0^{t'} \mathbf{d}s[\hat{H}_I(t'), [\hat{H}_I(s), \hat{\rho}_T(s)]] \right) \\ &= -\mathbb{i}\alpha \int_0^t \left[\hat{H}_I(t'), \hat{\rho}_T(0)\right] \mathbf{d}t' - \alpha^2 \int_0^t \mathbf{d}t' \int_0^{t'} \mathbf{d}s[\hat{H}_I(t'), [\hat{H}_I(s), \hat{\rho}_T(s)]] \end{aligned} \tag{55}$$

and do derivation over the result gives : so there is (L:990)

$$\hat{\rho}_T(t) = \hat{\rho}_T(0) - \mathbb{i}\alpha \int_0^t \left[\hat{H}_I(t'), \hat{\rho}_T(0)\right] \mathbf{d}t' - \alpha^2 \int_0^t \mathbf{d}t' \int_0^{t'} \mathbf{d}s[\hat{H}_I(t'), [\hat{H}_I(s), \hat{\rho}_T(s)]] \tag{56}$$

do another commutationover this yields (L:1003)

$$\begin{aligned}\frac{\mathbf{d}\hat{\rho}_T}{\mathbf{d}(t)}t &= -\mathbb{i}\alpha\left[\hat{H}_I(t), \hat{\rho}_T(t)\right] \\ &= -\mathbb{i}\alpha\left[\hat{H}_I(t), \hat{\rho}_T(0) - \mathbb{i}\alpha\int_0^t \left[\hat{H}_I(t'), \hat{\rho}_T(0)\right]\mathbf{d}t' - \alpha^2\int_0^t \mathbf{d}t'\int_0^{t'}\mathbf{d}s[\hat{H}_I(t'), [\hat{H}_I(s), \hat{\rho}_T(s)]]\right] \\ &= -\mathbb{i}\alpha\left[\hat{H}_I(t), \hat{\rho}_T(0)\right] - \alpha^2\left[\hat{H}_I(t), \int_0^t \mathbf{d}t'\left[\hat{H}_I(t'), \hat{\rho}_T(0)\right]\right] + O(\alpha^3)\end{aligned} \tag{57}$$

and we got : (L:1035)

$$\begin{aligned}\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t} &= -\mathbb{i}\alpha[\hat{H}_I(t), \hat{\rho}_t(t)] \\ &= -\mathbb{i}\alpha\left[\hat{H}_I(t) \quad , \quad \hat{\rho}_T(0) + \int_0^t\left[-\mathbb{i}\alpha[\hat{H}_I(s), \hat{\rho}_T(0)]\right]\mathbf{d}s\right] + O(\alpha^3) \\ &= -\mathbb{i}\alpha\left[\hat{H}_I(t)\ ,\ \hat{\rho}_T(0)\right] - \alpha^2\int_0^t\left[\hat{H}_I(t)\left[\hat{H}_I(s)\ ,\ \hat{\rho}_T(0)\right]\ ,\ \mathbf{d}\right]s + O(\alpha^3)\end{aligned} \tag{58}$$

so we know that (L:1066)

$$\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t} = -\mathbb{i}\alpha\left[\hat{H}_I(t)\ ,\ \hat{\rho}_T(0)\right] - \alpha^2\int_0^t\left[\hat{H}_I(t)\ ,\ \left[\hat{H}_I(s)\ ,\ \hat{\rho}_T(0)\right]\right]\mathbf{d}s \tag{59}$$

with the $\hat{\rho}_T$ we could have the $\hat{\rho}$ so : (L:1092)

$$\begin{aligned}\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} &= \mathrm{Tr}_E\left[\frac{\mathbf{d}\hat{\rho}_T(t)}{\mathbf{d}t}\right] \\ &= -\mathbb{i}\alpha\mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \hat{\rho}_T(0)\right]\right] - \alpha^2\int_0^t \mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \left[\hat{H}_I(s)\ ,\ \hat{\rho}_T(0)\right]\right]\right]\mathbf{d}s\end{aligned} \tag{60}$$

$\hat{\rho}$ is dependnet on $\hat{\rho}_T$ and for start condition , it can have sperable state $\hat{\rho}_T(0) = \hat{\rho}(0) \otimes \hat{\rho}_E(0)$ here assumes that the environtment is thermal , whichmeans (L:1126)

$$\hat{\rho}_E(0) = \frac{\exp\left(-\frac{\hat{H}_E}{T}\right)}{\mathrm{Tr}\left[\exp\left(-\frac{\hat{H}_E}{T}\right)\right]} \tag{61}$$

(with respection $k_B = 1$) and this gives : (L:1136)

$$\langle\hat{E}_i\rangle = \mathrm{Tr}\left[\hat{E}_i\hat{\rho}_E(0)\right] \tag{62}$$

for $t = 0$ we can have that : (L:1145)

$$\mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \hat{\rho}_T(0)\right]\right] = \sum_i \left(\hat{S}_i(t)\hat{\rho}(0)\mathrm{Tr}_E\left[\hat{E}_i(t)\hat{\rho}_E(0)\right] - \hat{\rho}(0)\hat{S}_i(t)\mathrm{Tr}_E\left[\hat{\rho}_E(0)\hat{E}_i(t)\right]\right) \tag{63}$$

so here I can infer that : (L:1164)

$$\left[\hat{H}_I(t)\ ,\ \hat{\rho}_T(0)\right] = \hat{H}_I(t)\hat{\rho}_T(0) - \hat{\rho}_T(0)\hat{H}_I(t) \tag{64}$$

what evein is this index i doing here ? what is the constraints ?
any way , lets just assume that $\langle\hat{E}_i\rangle = \mathrm{Tr}\left[\hat{E}_i\hat{\rho}_E(0)\right] = 0$ independt pf i and it gives anyway , before , we have that (L:1176)

$$\hat{H}_T = \hat{H}_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes \hat{H}_E + \alpha\hat{H}_I \tag{65}$$

but now lets write it this way : (L:1183)

$$\hat{H}_T = (\hat{H}_S + \alpha\sum_i \langle\hat{E}_i\rangle\hat{S}_i) + \hat{H}_E + \alpha\hat{H}'_i \tag{66}$$

reminding : (L:1190)

$$\hat{H}'_i = \sum_i \hat{S}_i \otimes (\hat{E}_i - \langle\hat{E}_i\rangle) \tag{67}$$

now with bold assumption $\langle\hat{E}_i\rangle = 0$ writing that : $\hat{E}' = \hat{E}_i - \langle\hat{E}_i\rangle$ **the cyclic property of trace** means what, any way ,assumptions ends up in : $\forall i$ (L:1204)

$$\begin{aligned} \mathrm{Tr}_E\left[\hat{E}_i(t)\hat{\rho}_E(0)\right] &= 0 \\ \mathrm{Tr}_E\left[\hat{\rho}_E(0)\hat{E}_i(t)\right] &= 0 \end{aligned} \tag{68}$$

there took main attention at the : (L:1218)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\alpha^2\int_0^t \mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \left[\hat{H}_I(s)\ ,\ \hat{\rho}_T(0)\right]\right]\right]\mathbf{d}s \tag{69}$$

and here we have : (L:1236)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\alpha^2\int_0^t \mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \left[\hat{H}_I(s)\ ,\ \hat{\rho}(t)\otimes\hat{\rho}_E(0)\right]\right]\right]\mathbf{d}s \tag{70}$$

why would here suddenly has the time in $\hat{\rho}(t)\otimes\hat{\rho}_E(0)$ and now replacing the variable to : $s \to t - s$ then we have that **Redfield equation** (L:1261)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\alpha^2\int_0^\infty \mathrm{Tr}_E\left[\left[\hat{H}_I(t)\ ,\ \left[\hat{H}_I(s-t)\ ,\ \hat{\rho}(t)\otimes\hat{\rho}_E(0)\right]\right]\right]\mathbf{d}s \tag{71}$$

lets say that here we have **rotating wave approximation** (L:1284)

$$\widetilde{H}\hat{A} \equiv \left[\hat{H}\ ,\ A\right] \forall A \in B(\mathcal{H}) \tag{72}$$

then we have (L:1290)

$$\hat{S} = \sum_{\omega} \hat{S}_i(\omega) \tag{73}$$

there is : (L:1297)

$$\begin{aligned}\left[\hat{H}\ ,\ \hat{S}_i(\omega)\right] &= -\omega\hat{S}_i(\omega)\\ \left[\hat{H}\ ,\ \hat{S}_i^{\dagger}(\omega)\right] &= \omega\hat{S}_i(\omega)\end{aligned} \tag{74}$$

then we roll the sheet up with : (L:1307)

$$\begin{aligned}\left[\hat{H}, \hat{S}_i(\omega)\right] &= -\omega\hat{S}_i(\omega)\\ \left[\hat{H}, [\hat{H}, \hat{S}_i(\omega)]\right] &= (-\omega)^2\hat{S}_i(\omega)\\ &...\end{aligned} \tag{75}$$

so here we would like to see what is the (L:1317)

$$\hat{S}_k(t,\omega) = \mathbb{e}^{\mathbb{i}\hat{H}t}\hat{S}_k(\omega)\mathbb{e}^{-\mathbb{i}\hat{H}t} \tag{76}$$

we remember the CBH-forula again : (L:1325)

$$\mathbb{e}^{\lambda t}\mu\mathbb{e}^{-\lambda t} = \mu + [\lambda, \mu]t + \frac{1}{2!}[\lambda, [\lambda, \mu]] + ... \tag{77}$$

so there is : (L:1331)

$$\begin{aligned}\hat{S}_k(t,\omega) &= \mathbb{e}^{\mathbb{i}\hat{H}t}\hat{S}_k(\omega)\mathbb{e}^{-\mathbb{i}\hat{H}t}\\ &= \hat{S}_k(\omega)\sum_n \frac{(-\mathbb{i}\omega t)^n}{n!}\\ &= \hat{S}_k(\omega)\mathbb{e}^{-\mathbb{i}\omega t}\end{aligned} \tag{78}$$

the Environment sims like have nothing to do with the time , so we have that interaction hamiltonian : (L:1346)

$$\hat{H}_I(t) = \sum_{k,\omega} \mathbb{e}_k^{-\mathbb{i}\omega t\hat{S}}(\omega) \otimes \hat{E}_k(t) \tag{79}$$

where this $\hat{E}_k(t)$ has time with it and yet not spreaded and it is hermitian the paper described this as : (L:1358)

$$\begin{aligned}\widetilde{H}_i(t) &= \sum_{k,\omega} \mathbb{e}^{-\mathbb{i}\omega t}\hat{S}_k(\omega) \otimes \widetilde{E}_k(t)\\ &= \sum_{k,\omega} \mathbb{e}^{\mathbb{i}\omega t}\hat{S}_k^{\dagger}(\omega) \otimes \widetilde{E^{\dagger}}_k(t)\end{aligned} \tag{80}$$

and threading open the Redfield Equation (71) we havee (L:1371)

$$\frac{\mathbf{d}\hat{\rho}(t)}{\mathbf{d}t} = -\alpha^2 \int_0^\infty \mathbf{d}s \mathrm{Tr}_E \left[\hat{H}_I(t), \left[\hat{H}_I(s-t)\ ,\ \hat{\rho}(t) \otimes \hat{\rho}_E(0) \right] \right] \tag{81}$$

(L:1388)

$$\begin{aligned}
\left[\hat{H}_I(t), \left[\hat{H}_I(s-t)\ ,\ \hat{\rho}(t) \otimes \hat{\rho}_E(0) \right] \right] &= \left[\hat{H}_I(t), \hat{H}_I(s-t)\hat{\rho}(t) \otimes \hat{\rho}_E(0) - \hat{\rho}(t) \otimes \hat{\rho}_E(0)\hat{H}_I(s-t) \right] \\
&= \hat{H}_I(t)\hat{H}_I(s-t)\hat{\rho}(t) \otimes \hat{\rho}_E(0) \\
&- \hat{H}_I(t)\hat{\rho}(t) \otimes \hat{\rho}_E(0)\hat{H}_I(s-t) \\
&- \hat{H}_I(s-t)\hat{\rho}(t) \otimes \hat{\rho}_E(0)\hat{H}_I(t) \\
&+ \hat{\rho}(t) \otimes \hat{\rho}_E(0)\hat{H}_I(s-t)\hat{H}_I(t)
\end{aligned} \tag{82}$$

this paper is full of error